Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

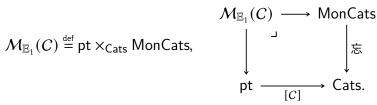
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| 13.1 | Moduli Categories of Monoidal Structures | | |
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| 13.1. | The Moduli Category of Monoidal Structures on a Category | ry | |
| Let C | be a category. | | |

DEFINITION 13.1.1.1.1 ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATE-

The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C)\stackrel{\scriptscriptstyle\mathsf{def}}{=}\mathsf{pt} imes_{\mathsf{Cats}}\mathsf{MonCats},$$



REMARK 13.1.1.1.2 ➤ Unwinding Definition 13.1.1.1.1, I

In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- · Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C,$ α^{C} , λ^{C} , ρ^{C}) whose underlying category is C.
- · Morphisms. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^C, \lambda^C, \rho^C)$ $\alpha^{C,\prime}$, $\lambda^{C,\prime}$, $\,\rho^{C,\prime}\big)$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C : C \to C$ of C.

· Identities. For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{B}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \stackrel{\text{def}}{=} \left(\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes}\right),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of ??.

· Composition. For each $M, N, P \in \mathsf{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_1}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,P)$$
 of $\mathcal{M}_{\mathbb{B}_1}(C)$ at (M,N,P) is defined by

$$\left(\operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime} \right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \left(\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes} \right) \stackrel{\operatorname{def}}{=} \left(\operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime} \circ \operatorname{id}_{\mathbb{1}|C}^{\otimes} \right).$$

REMARK 13.1.1.1.3 ► Unwinding Definition 13.1.1.1.1, II

In particular, a morphism in $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

1. Naturality. For each pair $f:A\to X$ and $g:B\to Y$ of morphisms of C, the diagram

$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow \operatorname{id}_{A,B}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{X,Y}^{\otimes}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

commutes.

2. Monoidality. For each $A, B, C \in \mathsf{Obj}(C)$, the diagram

$$(A \boxtimes_{C} B) \boxtimes_{C} C$$

$$(A \otimes_{C} B) \boxtimes_{C} C$$

$$(A \otimes_{C} B) \boxtimes_{C} C$$

$$id_{A \otimes_{C} B, C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C$$

$$id_{A \otimes_{C} B, C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C$$

$$A \boxtimes_{C} (B \otimes_{C} C)$$

$$id_{A \otimes_{C} G, C}^{\otimes}$$

$$id_{A \otimes_{C} G, C}^{\otimes}$$

$$id_{A, B \otimes_{C} C}^{\otimes}$$

$$id_{A, B \otimes_{C} C}^{\otimes}$$

commutes.

3. Left Monoidal Unity. For each $A \in Obj(C)$, the diagram

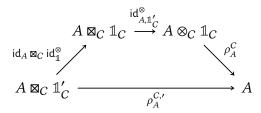
$$\mathbb{1}_{C}\boxtimes_{C}A\overset{\operatorname{id}_{\mathbb{1}'_{C},A}^{\otimes}}{\longrightarrow}\mathbb{1}_{C}\otimes_{C}A$$

$$\operatorname{id}_{\mathbb{1}}^{\otimes}\boxtimes_{C}\operatorname{id}_{A}\overset{\lambda_{A}^{C}}{\longrightarrow}A$$

$$\mathbb{1}'_{C}\boxtimes_{C}A\overset{\lambda_{A}^{C,\prime}}{\longrightarrow}A$$

commutes.

4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram



commutes.

PROPOSITION 13.1.1.1.4 ➤ PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUC-TURES ON A CATEGORY

Let C be a category.

- 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

$$A\boxtimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\boxtimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\boxtimes_{C}(B\otimes_{C}C)$$

$$\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}\downarrow \qquad \qquad \downarrow\operatorname{id}_{A,B\otimes_{C}C}^{\otimes}$$

$$A\otimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\otimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\otimes_{C}(B\otimes_{C}C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. Mixed Associators. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathsf{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \longrightarrow -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes} \colon (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} \mathcal{C} \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} \mathcal{C})$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \text{id}_{A \otimes_C B,C}^{\otimes} & & \text{id}_{A \otimes_C \text{ id}_{B,C}^{\otimes}} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A \boxtimes_C (B \boxtimes_C C) \\ \operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A\boxtimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A\boxtimes_C (B\otimes_C C) \\ \operatorname{id}_{A,B}^{\otimes}\otimes_C \operatorname{id}_C & & & & & \operatorname{id}_{A,B\otimes_C C}^{\otimes} \\ (A\otimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A\otimes_C (B\otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A\boxtimes_{C}B)\boxtimes_{C}C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A\boxtimes_{C}(B\boxtimes_{C}C) \\ \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} & & & & & \operatorname{id}_{A\boxtimes_{C}}\operatorname{id}_{B,C}^{\otimes} \\ (A\boxtimes_{C}B)\otimes_{C}C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A\boxtimes_{C}(B\otimes_{C}C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{AB,C}^{\boxtimes,\otimes} \colon (A\boxtimes_C B) \otimes_C C \to A \otimes_C (B\boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A\boxtimes_{C}B)\otimes_{C}C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A\otimes_{C}(B\boxtimes_{C}C) \\ & \mathrm{id}_{A,B}^{\otimes}\otimes_{C}\mathrm{id}_{C} & & & & \mathrm{id}_{A}\otimes_{C}\mathrm{id}_{B,C}^{\otimes} \\ & & (A\otimes_{C}B)\otimes_{C}C & \xrightarrow{\alpha_{A,B,C}^{C}} A\otimes_{C}(B\otimes_{C}C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

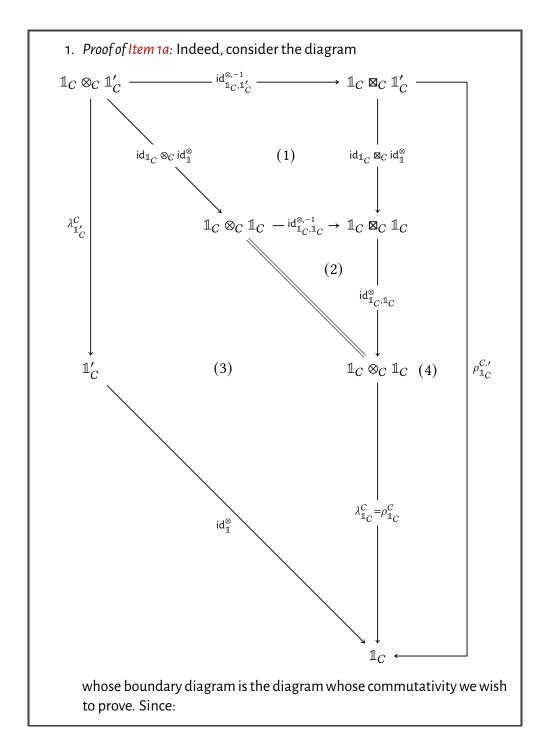
We claim that Items 1a and 1b are indeed true:

- 1. Proof of Item 1a: This follows from the naturality of id^{\otimes} with respect to the morphisms id_{AB}^{\otimes} and id_{C} .
- 2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and id_{BC}^{\otimes} .

This finishes the proof.

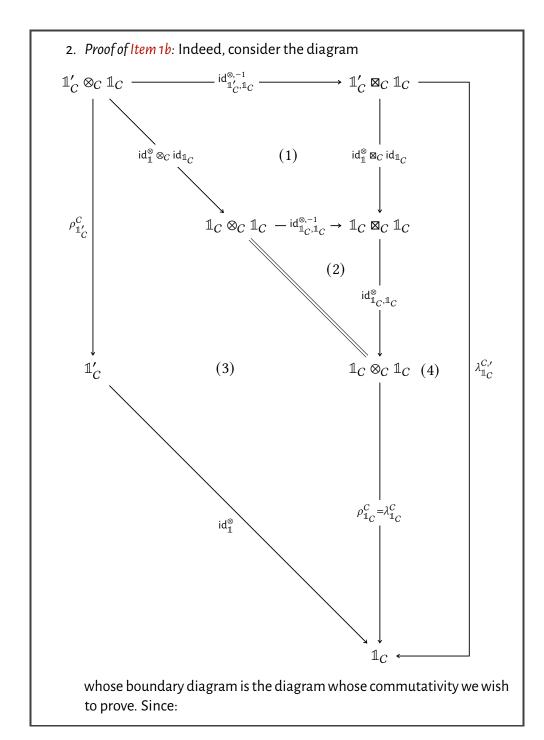
Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:



- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- · Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:composition}$;
- · Subdiagram (4) commutes by the right monoidal unity of $\left(\mathrm{id}_C,\mathrm{id}_C^\otimes,\mathrm{id}_{C|\mathbb{1}}^\otimes\right);$

so does the boundary diagram, and we are done.



- · Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- · Subdiagram (2) commutes trivially;
- · Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:condition}$;
- · Subdiagram (4) commutes by the left monoidal unity of $\left(\mathrm{id}_{C},\mathrm{id}_{C}^{\otimes},\mathrm{id}_{C|\mathbb{I}}^{\otimes}\right);$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- · The boundary diagram commutes trivially;
- · Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- · The boundary diagram commutes trivially;
- · Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \alpha_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}_{C}$$

commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

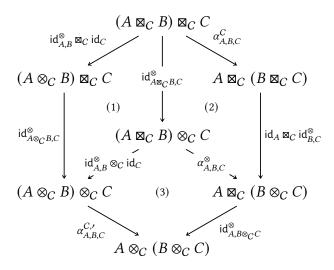
This finishes the proof.

Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for id^{\otimes} of

Item 2 of Remark 13.1.1.1.3 as follows:



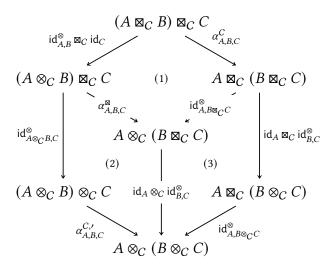
Since:

- · Subdiagram (1) commutes by Item 1a of Item 1.
- · Subdiagram (2) commutes by assumption.
- · Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for id^{\otimes} of

Item 2 of Remark 13.1.1.1.3 as follows:



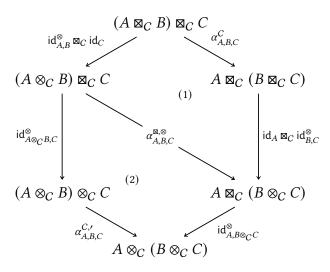
Since:

- · Subdiagram (1) commutes by assumption.
- · Subdiagram (2) commutes by assumption.
- · Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. *Proof of Item* 3c: We may partition the monoidality diagram for id^{\otimes} of





Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- Monoidal Structures on the Category of Sets
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- 8. Relations
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- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes