# Pointed Sets

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## July 22, 2025

0098 This chapter contains some foundational material on pointed sets.

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### 009A 6.1.1 Foundations

#### 009B **DEFINITION 6.1.1.1.1** ► POINTED SETS

A **pointed set**<sup>1</sup> is equivalently:

- An  $\mathbb{E}_0$ -monoid in  $(N_{\bullet}(Sets), pt)$ .
- A pointed object in (Sets, pt).

#### 009C REMARK 6.1.1.1.2 ► Unwinding Definition 6.1.1.1.1

In detail, a **pointed set** is a pair  $(X, x_0)$  consisting of:

- *The Underlying Set.* A set X, called the **underlying set of**  $(X, x_0)$ .
- The Basepoint. A morphism

$$[x_0]: pt \to X$$

in Sets, determining an element  $x_0 \in X$ , called the **basepoint of** X.

#### EXAMPLE 6.1.1.1.3 ► THE ZERO SPHERE 009D

The 0-**sphere**<sup>I</sup> is the pointed set  $(S^0, 0)^2$  consisting of:

• The Underlying Set. The set  $S^0$  defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

<sup>&</sup>lt;sup>1</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, pointed sets are viewed as  $\mathbb{F}_1$ -modules.

• *The Basepoint.* The element 0 of  $S^0$ .

<sup>1</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

<sup>2</sup> Further Notation: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras,  $S^0$  is also denoted  $(\mathbb{F}_1, 0)$ .

#### 009E EXAMPLE 6.1.1.1.4 ► THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set  $(pt, \star)$  consisting of:

- *The Underlying Set.* The punctual set pt  $\stackrel{\text{def}}{=} \{ \star \}$ .
- *The Basepoint.* The element ★ of pt.

#### **O1QB EXAMPLE 6.1.1.1.5** ► THE STANDARD POINTED SET WITH n + 1 ELEMENTS

The **standard pointed set with** n + 1 **elements** is the pointed set  $\langle n \rangle$  consisting of

• *The Underlying Set.* The set  $\langle n \rangle$  defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

• *The Basepoint.* The element \* of  $\langle n \rangle$ .

### 009H 6.1.2 Morphisms of Pointed Sets

#### 009J DEFINITION 6.1.2.1.1 ► MORPHISMS OF POINTED SETS

A morphism of pointed sets<sup>1,2</sup> is equivalently:

- A morphism of  $\mathbb{E}_0$ -monoids in  $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$ .
- A morphism of pointed objects in (Sets, pt).

<sup>&</sup>lt;sup>1</sup>Further Terminology: Also called a **pointed function**.

<sup>&</sup>lt;sup>2</sup> Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, morphisms of pointed sets are also called **morphism of**  $\mathbb{F}_1$ -**modules**.

009K

#### REMARK 6.1.2.1.2 ► Unwinding Definition 6.1.2.1.1

In detail, a **morphism of pointed sets**  $f:(X,x_0) \to (Y,y_0)$  is a morphism of sets  $f:X \to Y$  such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & [y_0] \\
X & \xrightarrow{f} Y
\end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

## 009L 6.1.3 The Category of Pointed Sets

009M

#### **DEFINITION 6.1.3.1.1** ► THE CATEGORY OF POINTED SETS

The category of pointed sets is the category Sets\* defined equivalently as:

- The homotopy category of the  $\infty$ -category  $\mathsf{Mon}_{\mathbb{E}_0}\big(N_{\bullet}(\mathsf{Sets}),\mathsf{pt}\big)$  of ??,??.
- The category Sets\* of Constructions With Categories, ??.

009N

### REMARK 6.1.3.1.2 ► Unwinding Definition 6.1.3.1.1

In detail, the  $category\ of\ pointed\ sets$  is the  $category\ Sets_*$  where:

- *Objects*. The objects of Sets\* are pointed sets.
- Morphisms. The morphisms of Sets\* are morphisms of pointed sets.
- *Identities.* For each  $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$ , the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets<sub>\*</sub> at  $(X, x_0)$  is defined by

$$id_{(X,x_0)}^{Sets_*} \stackrel{\text{def}}{=} id_X$$
.

• *Composition.* For each  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0) \in Obj(Sets_*)$ , the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*\big(\big(Y,y_0\big),(Z,z_0)\big) \times \mathsf{Sets}_*\big((X,x_0),\big(Y,y_0\big)\big) \to \mathsf{Sets}_*\big((X,x_0),(Z,z_0)\big)$$

of Sets\* at 
$$((X, x_0), (Y, y_0), (Z, z_0))$$
 is defined by<sup>2</sup>

$$g \circ_{(X,x_0),(Y,y_0),(Z,z_0)}^{\mathsf{Sets}_*} f \stackrel{\text{def}}{=} g \circ f.$$

<sup>2</sup>Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

### 09P 6.1.4 Elementary Properties of Pointed Sets

#### 0090 PROPOSITION 6.1.4.1.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let  $(X, x_0)$  be a pointed set.

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- I. *Completeness.* The category Sets\* of pointed sets and morphisms between them is complete, having in particular:
  - (a) Products, described as in Definition 6.2.3.1.1.
  - (b) Pullbacks, described as in Definition 6.2.4.I.I.
  - (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets\* of pointed sets and morphisms between them is cocomplete, having in particular:
  - (a) Coproducts, described as in Definition 6.3.3.1.1.
  - (b) Pushouts, described as in Definition 6.3.4.1.1;

<sup>&</sup>lt;sup>1</sup>Note that  $id_X$  is indeed a morphism of pointed sets, as we have  $id_X(x_0) = x_0$ .

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(c) Coequalisers, described as in Definition 6.3.5.1.1.

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3. Failure To Be Cartesian Closed. The category Sets\* is not Cartesian closed. 

1. Cartesi

00A0

4. Morphisms From the Monoidal Unit. We have a bijection of sets<sup>2</sup>

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in  $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$ , internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in  $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

00A1

5. Relation to Partial Functions. We have an equivalence of categories<sup>3</sup>

$$\mathsf{Sets}_* \stackrel{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

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(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

- i. A pointed set  $(X, x_0)$  to X.
- ii. A pointed function

$$f: (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f: X \to Y$$

defined on  $f^{-1}(Y \setminus y_0)$  and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in f^{-1}(Y \setminus y_0)$ .

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(b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1}$$
: Sets<sup>part.</sup>  $\stackrel{\cong}{\to}$  Sets<sub>\*</sub>

sends:

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024Z

- i. A set X is to the pointed set  $(X, \star)$  with  $\star$  an element that is not in X.
- ii. A partial function

$$f: X \to Y$$

defined on  $U \subset X$  to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each  $x \in X$ .

<sup>1</sup>The category Sets<sub>\*</sub> does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

<sup>2</sup>In other words, the forgetful functor

defined on objects by sending a pointed set to its underlying set is corepresentable by  $S^0$ . Warning: This is not an isomorphism of categories, only an equivalence.

#### PROOF 6.1.4.1.2 ► PROOF OF PROPOSITION 6.1.4.1.1

#### Item 1: Completeness

This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

#### Item 2: Cocompleteness

This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

#### Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

### Item 4: Morphisms From the Monoidal Unit

Since a morphism from  $S^0$  to a pointed set  $(X, x_0)$  sends  $0 \in S^0$  to  $x_0$  and then can send  $1 \in S^0$  to any element of X, we obtain a bijection between pointed maps  $S^0 \to X$  and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*\big(S^0,X\big)\cong(X,x_0)$$

follows by noting that  $\Delta_{x_0} : S^0 \to X$ , the basepoint of **Sets**<sub>\*</sub>( $S^0, X$ ), corresponds to the pointed map  $S^0 \to X$  picking the element  $x_0$  of X, and thus we see that the bijection between pointed maps  $S^0 \to X$  and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

#### Item 5: Relation to Partial Functions

See [MSE 884460].

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## 01QC 6.1.5 Active and Inert Morphisms of Pointed Sets

#### 010D DEFINITION 6.1.5.1.1 ► ACTIVE AND INERT MORPHISMS OF POINTED SETS

Let  $f: (X, x_0) \to (Y, y_0)$  be a morphism of pointed sets.

- 1. The morphism f is **active** if  $f^{-1}(y_0) = x_0$ .
- 2. The morphism f is **inert** if, for each  $y \in Y$ , the set  $f^{-1}(y)$  has exactly one element.

### 01QG NOTATION 6.1.5.1.2 ► THE CATEGORY OF POINTED SETS AND ACTIVE MORPHISMS

We write  $\mathsf{Sets}^{\mathsf{actv}}_*$  for the wide subcategory of  $\mathsf{Sets}_*$  spanned by pointed sets and the active maps between them.

#### 01QH EXAMPLE 6.1.5.1.3 ► EXAMPLES OF ACTIVE AND INERT MAPS OF POINTED SETS

Here are some examples of active and inert maps of pointed sets.

01QJ I. The map  $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$  given by

$$1 \longrightarrow 2$$

is active but not inert.

01QK

01QL

2. The map  $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$  given by

$$* \longmapsto *$$

is inert but not active.

3. The map  $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$  given by

$$1 \longmapsto 1$$

is neither inert nor active. However, it factors as  $f = a \circ i$ , where

$$i\colon \langle 3\rangle \to \langle 2\rangle,$$

$$a: \langle 2 \rangle \rightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by

with *i* being inert and *a* being active.

#### **PROPOSITION 6.1.5.1.4** ► PROPERTIES OF ACTIVE AND INERT MAPS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

1. Active-Inert Factorisation. Every morphism of pointed sets  $f:(X,x_0) \to (Y,y_0)$  factors uniquely as

$$f = a \circ i$$

where:

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- (a) The map  $i: (X, x_0) \to (K, k_0)$  is an inert morphism of pointed sets
- (b) The map  $a: (K, k_0) \rightarrow (Y, y_0)$  is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets<sub>\*</sub>.

#### PROOF 6.1.5.1.5 ► PROOF OF PROPOSITION 6.1.5.1.4

Item 1: Active-Inert Factorisation

Let  $f\colon X\to Y$  be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\longrightarrow} K \stackrel{a}{\longrightarrow} Y$$

where:

• *K* is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$
  
=  $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$ 

•  $i: X \to K$  is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each  $x \in X$ ;

•  $a: K \to Y$  is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in K$ .

Next, let

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & & \downarrow g \\
A & \xrightarrow{} & B
\end{array}$$

be a commutative diagram in Sets\*. Consider the morphism  $\phi\colon Y\to A$  given by

$$\phi(y) = f(i^{-1}(y))$$

for each  $y \in Y$  (which is well-defined since, as i is inert,  $i^{-1}(y)$  is a singleton for all  $y \in Y$ ). We claim that  $\phi$  is the unique diagonal filler in the diagram

$$X \xrightarrow{i} Y$$

$$f \downarrow \exists ! \qquad f \downarrow g$$

$$A \xrightarrow{g} B.$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\stackrel{\text{def}}{=} \phi(i(x)) \\ &\stackrel{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each  $x \in X$  and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each  $y \in Y$ . Moreover, given another morphism  $\psi$  such that the diagram

$$X \xrightarrow{i} Y$$

$$f \downarrow \qquad \qquad \downarrow \qquad \downarrow g$$

$$A \xrightarrow{d} B$$

commutes, it follows that we must have  $\psi = \phi$ , since, given  $y \in Y$ , there exists a unique  $x \in X$  such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

## **00A2 6.2** Limits of Pointed Sets

### 00A3 6.2.1 The Terminal Pointed Set

### 00A4 DEFINITION 6.2.1.1.1 ► THE TERMINAL POINTED SET

The **terminal pointed set** is the terminal object of Sets<sub>\*</sub> as in Limits and Colimits, ??.

### 0250 CONSTRUCTION 6.2.1.1.2 ➤ CONSTRUCTION OF THE TERMINAL POINTED SET

Concretely, the **terminal pointed set** is the pair  $(pt, \star)$ ,  $\{!_X\}_{(X,x_0) \in Obj(Sets_*)}$  consisting of:

- *The Limit.* The pointed set  $(pt, \star)$ .
- The Cone. The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathsf{pt}, \star)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each  $x \in X$  and each  $(X, x_0) \in Obj(Sets)$ .

#### PROOF 6.2.1.1.3 ► PROOF OF CONSTRUCTION 6.2.1.1.2

We claim that  $(pt, \star)$  is the terminal object of Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt,  $\star$ )

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathsf{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists !}} (\text{pt}, \star)$$

commute, namely  $!_X$ .

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#### **Products of Families of Pointed Sets** 00A5 **6.2.2**

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

#### 00A6 **DEFINITION 6.2.2.1.1** ► THE PRODUCT OF A FAMILY OF POINTED SETS

The **product of**  $\{(X_i, x_0^i)\}_{i \in I}$  is the product of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets $_*$  as in Limits and Colimits, ??.

# CONSTRUCTION 6.2.2.1.2 ► CONSTRUCTION OF THE PRODUCT OF A FAMILY OF POINTED

Concretely, the **product of**  $\{(X_i, x_0^i)\}_{i \in I}$   $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$  consisting of:

• *The Limit*. The pointed set  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ . pair

• *The Cone.* The collection

$$\left\{ \operatorname{pr}_{i} : \left( \prod_{i \in I} X_{i}, \left( x_{0}^{i} \right)_{i \in I} \right) \to \left( X_{i}, x_{0}^{i} \right) \right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_{i}\left(\left(x_{j}\right)_{j\in I}\right)\stackrel{\operatorname{def}}{=}x_{i}$$

for each  $(x_j)_{j \in I} \in \prod_{i \in I} X_i$  and each  $i \in I$ .

#### PROOF 6.2.2.1.3 ► PROOF OF CONSTRUCTION 6.2.2.1.2

We claim that  $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$  is the categorical product of  $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$  in Sets<sub>\*</sub>. Indeed, suppose we have, for each  $i\in I$ , a diagram of the form

$$(P, *) \xrightarrow{p_i} (X_i, (x_0^i)_{i \in I}) \xrightarrow{\operatorname{pr}_i} (X_i, x_0^i)$$

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$$

making the diagram

$$(P, *)$$

$$\phi \mid_{\exists \exists !}$$

$$\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right) \xrightarrow{\operatorname{pr}_i} (X_i, x_0^i)$$

commute, being uniquely determined by the condition  $\operatorname{pr}_i \circ \phi = p_i$  for each  $i \in I$  via

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$$\phi(x) = (p_i(x))_{i \in I}$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= (x_0^i)_{i \in I},$$

where we have used that  $p_i$  is a morphism of pointed sets for each  $i \in I$ .

PROPOSITION 6.2.2.1.4 ➤ PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

1. Functoriality. The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$  defines a functor

 $\prod_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$ 

PROOF 6.2.2.1.5 ► PROOF OF PROPOSITION 6.2.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

00A9 6.2.3 Products

00A8

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00AA DEFINITION 6.2.3.1.1 ▶ PRODUCTS OF POINTED SETS

The **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the product of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.

6.2.3 Products

#### 0252 CONSTRUCTION 6.2.3.1.2 ➤ CONSTRUCTION OF PRODUCTS OF POINTED SETS

Concretely, the **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pair consisting of:

- *The Limit.* The pointed set  $(X \times Y, (x_0, y_0))$ .
- *The Cone.* The morphisms of pointed sets

$$\operatorname{pr}_1: (X \times Y, (x_0, y_0)) \to (X, x_0),$$
  
 $\operatorname{pr}_2: (X \times Y, (x_0, y_0)) \to (Y, y_0)$ 

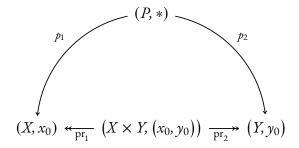
defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$
  
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$ 

for each  $(x, y) \in X \times Y$ .

#### PROOF 6.2.3.1.3 ► PROOF OF CONSTRUCTION 6.2.3.1.2

We claim that  $(X \times Y, (x_0, y_0))$  is the categorical product of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

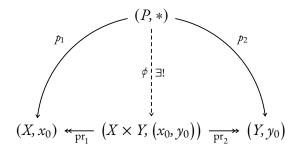


in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

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making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
$$\operatorname{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*)) 
= (x_0, y_0),$$

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets.

00AB

### PROPOSITION 6.2.3.1.4 ► PROPERTIES OF PRODUCTS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

00AC 1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$A \times -:$$
 Sets<sub>\*</sub>  $\rightarrow$  Sets<sub>\*</sub>,  
 $- \times B:$  Sets<sub>\*</sub>  $\rightarrow$  Sets<sub>\*</sub>,  
 $-_1 \times -_2:$  Sets<sub>\*</sub>  $\times$  Sets<sub>\*</sub>  $\rightarrow$  Sets<sub>\*</sub>,

defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

- 01QR
- 2. *Lack of Adjointness.* The functors  $X \times -$  and  $\times Y$  do not admit right adjoints.
- 00AD
- 3. Associativity. We have an isomorphism of pointed sets  $((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$

natural in  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

- 00AE
- 4. Unitality. We have isomorphisms of pointed sets

$$(pt, \star) \times (X, x_0) \cong (X, x_0),$$
  
$$(X, x_0) \times (pt, \star) \cong (X, x_0),$$

natural in  $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

- 00AF
- 5. Commutativity. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in  $(X, x_0)$ ,  $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

- 00AG
- 6. *Symmetric Monoidality*. The triple (Sets\*\*,  $\times$ \*, (pt,  $\star$ )) is a symmetric monoidal category.

#### PROOF 6.2.3.1.5 ▶ PROOF OF PROPOSITION 6.2.3.1.4

#### Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2: Lack of Adjointness

See [MSE 2855868].

#### Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

#### Item 4: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.

#### Item 5: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.

#### Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.

#### 00AH 6.2.4 Pullbacks

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (X, x_0) \to (Z, z_0)$  and  $g: (Y, y_0) \to (Z, z_0)$  be morphisms of pointed sets.

#### 00AJ DEFINITION 6.2.4.1.1 ▶ PULLBACKS OF POINTED SETS

The **pullback of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along** (f, g) is the pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) in Sets<sub>\*</sub> as in Limits and Colimits, ??.

#### 0253 CONSTRUCTION 6.2.4.1.2 ➤ CONSTRUCTION OF PULLBACKS OF POINTED SETS

Concretely, the **pullback of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set  $(X \times_Z Y, (x_0, y_0))$ .
- *The Cone.* The morphisms of pointed sets

$$\operatorname{pr}_{1} : (X \times_{Z} Y, (x_{0}, y_{0})) \to (X, x_{0}),$$
  
$$\operatorname{pr}_{2} : (X \times_{Z} Y, (x_{0}, y_{0})) \to (Y, y_{0})$$

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$
  
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$ 

for each  $(x, y) \in X \times_Z Y$ .

#### PROOF 6.2.4.1.3 ► PROOF OF CONSTRUCTION 6.2.4.1.2

We claim that  $X \times_Z Y$  is the categorical pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to (f, g) in Sets<sub>\*</sub>. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$(X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad \operatorname{pr}_{1} \downarrow \qquad \qquad \downarrow^{g}$$

$$(X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given  $(x, y) \in X \times_Z Y$ , we have

$$[f \circ \operatorname{pr}_{1}](x, y) = f(\operatorname{pr}_{1}(x, y))$$

$$= f(x)$$

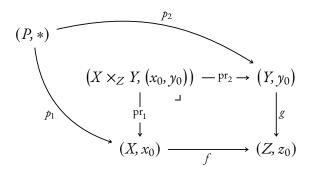
$$= g(y)$$

$$= g(\operatorname{pr}_{2}(x, y))$$

$$= [g \circ \operatorname{pr}_{2}](x, y),$$

where f(x) = g(y) since  $(x, y) \in X \times_Z Y$ . Next, we prove that  $X \times_Z Y$  satisfies the universal property of the pullback. Suppose we have a diagram

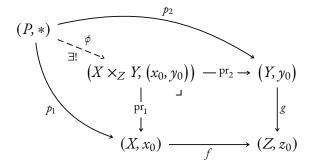
of the form



in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
  
 $\operatorname{pr}_2 \circ \phi = p_2$ 

via

$$\phi(x) = \big(p_1(x), p_2(x)\big)$$

for each  $x \in P$ , where we note that  $(p_1(x), p_2(x)) \in X \times Y$  indeed lies in  $X \times_Z Y$  by the condition

$$f \circ p_1 = g \circ p_2,$$

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which gives

$$f(p_1(x)) = g(p_2(x))$$

for each  $x \in P$ , so that  $(p_1(x), p_2(x)) \in X \times_Z Y$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
  
=  $(x_0, y_0),$ 

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets.

00AK

#### PROPOSITION 6.2.4.1.4 ► PROPERTIES OF PULLBACKS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

00AL

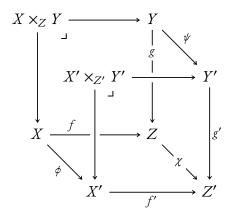
I. Functoriality. The assignment  $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$  defines a functor

$$-_1 \times_{-_3} -_1 : \operatorname{\mathsf{Fun}}(\mathcal{P}, \operatorname{\mathsf{Sets}}_*) \to \operatorname{\mathsf{Sets}}_*,$$

where  $\mathcal P$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \times_{-3} -1$  is given by sending a morphism



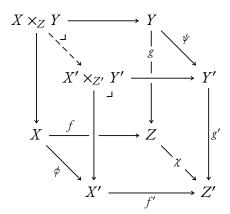
in  $\operatorname{Fun}(\mathcal{P},\operatorname{\mathsf{Sets}}_*)$  to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

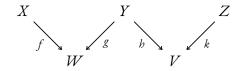
$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each  $(x, y) \in X \times_Z Y$ , which is the unique morphism of pointed sets making the diagram



commute.

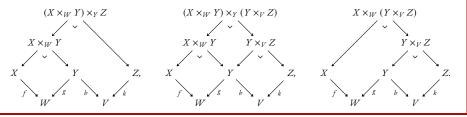
2. Associativity. Given a diagram



in Sets\*, we have isomorphisms of pointed sets

$$(X\times_WY)\times_VZ\cong (X\times_WY)\times_Y(Y\times_VZ)\cong X\times_W(Y\times_VZ),$$

where these pullbacks are built as in the diagrams



00AM

00AN

3. Unitality. We have isomorphisms of pointed sets

$$A = \longrightarrow A$$

$$f \downarrow \qquad \qquad X \times_X A \cong A, \qquad A \xrightarrow{f} X$$

$$X \times_X X \cong A, \qquad X \xrightarrow{f} X.$$

00AP

4. Commutativity. We have an isomorphism of pointed sets

$$A \times_X B \longrightarrow B$$

$$\downarrow \qquad \qquad \downarrow g \qquad A \times_X B \cong B \times_X A$$

$$A \xrightarrow{f} X, \qquad B \xrightarrow{g} X.$$

$$B \times_X A \longrightarrow A$$

$$\downarrow \qquad \qquad \downarrow f$$

$$B \xrightarrow{g} X.$$

00A0

5. Interaction With Products. We have an isomorphism of pointed sets

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \xrightarrow{!_{X}} \text{pt.}$$

00AR

6. *Symmetric Monoidality*. The triple (Sets<sub>\*</sub>,  $\times_X$ , X) is a symmetric monoidal category.

#### PROOF 6.2.4.1.5 ► PROOF OF PROPOSITION 6.2.4.1.4

#### Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of

the cube pullback diagram.

#### Item 2: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.4.1.7.

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### Item 3: Unitality

This follows from Constructions With Sets, Item 6 of Proposition 4.1.4.1.7.

#### Item 4: Commutativity

This follows from Constructions With Sets, Item 7 of Proposition 4.1.4.1.7.

#### Item 5: Interaction With Products

This follows from Constructions With Sets, Item 10 of Proposition 4.1.4.1.7.

### Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 11 of Proposition 4.1.4.1.7.

### 00AS 6.2.5 Equalisers

Let  $f, g: (X, x_0) \Rightarrow (Y, y_0)$  be morphisms of pointed sets.

#### 00AT DEFINITION 6.2.5.1.1 ► EQUALISERS OF POINTED SETS

The **equaliser of** (f, g) is the equaliser of f and g in Sets<sub>\*</sub> as in Limits and Colimits, ??.

#### 0254 CONSTRUCTION 6.2.5.1.2 ► CONSTRUCTION OF EQUALISERS OF POINTED SETS

Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set  $(Eq(f,g), x_0)$ .
- The Cone. The morphism of pointed sets

$$\operatorname{eq}(f,g): (\operatorname{Eq}(f,g),x_0) \to (X,x_0)$$

given by the canonical inclusion  $eq(f, g) \rightarrow Eq(f, g) \rightarrow X$ .

#### PROOF 6.2.5.1.3 ► PROOF OF CONSTRUCTION 6.2.5.1.2

We claim that  $(\text{Eq}(f,g), x_0)$  is the categorical equaliser of f and g in Sets<sub>\*</sub>. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \operatorname{eq}(f, g) = g \circ \operatorname{eq}(f, g),$$

which indeed holds by the definition of the set Eq(f, g). Next, we prove that Eq(f, g) satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\left(\operatorname{Eq}(f,g),x_{0}\right) \xrightarrow{\operatorname{eq}(f,g)} (X,x_{0}) \xrightarrow{f} \left(Y,y_{0}\right)$$

$$(E,*)$$

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\text{Eq}(f, g), x_0)$$

making the diagram

commute, being uniquely determined by the condition

$$\operatorname{eq} \big( f, g \big) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in A$  indeed lies in Eq(f, g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f,g)$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that *e* is a morphism of pointed sets.

#### 00AU

#### PROPOSITION 6.2.5.1.4 ▶ PROPERTIES OF EQUALISERS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \to (Y, y_0)$  be morphisms of pointed sets.

00AV

I. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\operatorname{Eq}(f \circ \operatorname{eq}(g,h), g \circ \operatorname{eq}(g,h))}_{=\operatorname{Eq}(f \circ \operatorname{eq}(g,h), h \circ \operatorname{eq}(g,h))} \cong \operatorname{Eq}(f,g,h) \cong \underbrace{\operatorname{Eq}(f \circ \operatorname{eq}(f,g), h \circ \operatorname{eq}(f,g))}_{=\operatorname{Eq}(g \circ \operatorname{eq}(f,g), h \circ \operatorname{eq}(f,g))}$$

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f \atop -g \xrightarrow{f}} (Y, y_0)$$

in Sets\*, being explicitly given by

$$\operatorname{Eq}(f,g,h)\cong\big\{a\in A\,\big|\, f(a)=g(a)=h(a)\big\}.$$

00AW

2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,f) \cong X.$$

00AX

3. Commutativity. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,g) \cong \operatorname{Eq}(g,f).$$

#### Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.1.5.1.4.

### Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.1.5.1.4.

#### Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.1.5.1.4.

## **6.3** Colimits of Pointed Sets

### **00AZ 6.3.1** The Initial Pointed Set

#### 00B0 DEFINITION 6.3.1.1.1 ► THE INITIAL POINTED SET

The **initial pointed set** is the initial object of Sets\* as in Limits and Colimits, ??.

### 0255 CONSTRUCTION 6.3.1.1.2 ➤ CONSTRUCTION OF THE INITIAL POINTED SET

Concretely, the **initial pointed set** is the pair  $(pt, \star)$ ,  $\{\iota_X\}_{(X,x_0) \in Obj(Sets_*)}$  consisting of:

- *The Limit.* The pointed set  $(pt, \star)$ .
- The Cone. The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathsf{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star)\stackrel{\mathrm{def}}{=} x_0.$$

#### PROOF 6.3.1.1.3 ► PROOF OF CONSTRUCTION 6.3.1.1.2

We claim that  $(pt, \star)$  is the initial object of Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
  $(X, x_0)$ 

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow{-\frac{\phi}{\exists !}} \to (X, x_0)$$

commute, namely  $\iota_X$ .

0256

### Coproducts of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

#### 00B2 **DEFINITION 6.3.2.1.1** ► COPRODUCTS OF FAMILIES OF POINTED SETS

The coproduct of the family  $\{(X_i, x_0^i)\}_{i \in I}$  is the coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.

<sup>1</sup>Further Terminology: Also called the **wedge sum of the family**  $\{(X_i, x_0^i)\}_{i \in I}$ .

# **CONSTRUCTION 6.3.2.1.2** ► CONSTRUCTION OF COPRODUCTS OF FAMILIES OF POINTED

Concretely, the **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $((\bigvee_{i \in I} X_i, p_0), \{\inf_i\}_{i \in I})$  consisting of:

• *The Colimit.* The pointed set  $(\bigvee_{i \in I} X_i, p_0)$  consisting of:

- The Underlying Set. The set  $\bigvee_{i \in I} X_i$  defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left( \coprod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\coprod_{i\in I} X_i$  given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each  $i, j \in I$ .

- *The Basepoint.* The element  $p_0$  of  $\bigvee_{i \in I} X_i$  defined by

$$p_0 \stackrel{\text{def}}{=} \left[ \left( i, x_0^i \right) \right]$$
$$= \left[ \left( j, x_0^j \right) \right]$$

for any  $i, j \in I$ .

• *The Cocone*. The collection

$$\left\{\operatorname{inj}_i\colon \left(X_i,x_0^i\right)\to \left(\bigvee_{i\in I}X_i,p_0\right)\right\}_{i\in I}$$

of morphism of pointed sets given by

$$inj_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each  $x \in X_i$  and each  $i \in I$ .

#### PROOF 6.3.2.1.3 ► PROOF OF CONSTRUCTION 6.3.2.1.2

We claim that  $(\bigvee_{i \in I} X_i, p_0)$  is the categorical coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub>. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$(X_i, x_0^i) \xrightarrow[\inf_i]{inj_i} \left(\bigvee_{i \in I} X_i, p_0\right)$$

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i\in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$(X_{i}, x_{0}^{i}) \xrightarrow{\underset{i \in I}{t_{i}}} \left( \bigvee_{i \in I} X_{i}, p_{0} \right)$$

commute, being uniquely determined by the condition  $\phi \circ \operatorname{inj}_i = \iota_i$  for each  $i \in I$  via

$$\phi([(i,x)]) = \iota_i(x)$$

for each  $[(i, x)] \in \bigvee_{i \in I} X_i$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$
  
= \*,

as  $\iota_i$  is a morphism of pointed sets.

00B3

#### PROPOSITION 6.3.2.1.4 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

00B4

I. Functoriality. The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$  defines a functor

$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

#### PROOF 6.3.2.1.5 ► PROOF OF PROPOSITION 6.3.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

## 00B5 6.3.3 Coproducts

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00B6

#### **DEFINITION 6.3.3.1.1** ► COPRODUCTS OF POINTED SETS

The **coproduct of**  $(X, x_0)$  **and**  $(Y, y_0)^T$  is the coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.

<sup>1</sup>Further Terminology: Also called the **wedge sum of**  $(X, x_0)$  **and**  $(Y, y_0)$ .

0257

#### CONSTRUCTION 6.3.3.1.2 ► CONSTRUCTION OF COPRODUCTS OF POINTED SETS

Concretely, the **coproduct of**  $(X, x_0)$  **and**  $(Y, y_0)$ , also called their **wedge sum**, is the pair consisting of:

• *The Colimit.* The pointed set  $(X \vee Y, p_0)$  consisting of:

- *The Underlying Set.* The set  $X \vee Y$  defined by

$$(X \lor Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \qquad X \lor Y \longleftarrow Y$$

$$\cong (X \coprod_{\text{pt}} Y, p_0) \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \downarrow [y_0]$$

$$\cong (X \coprod Y/\sim, p_0), \qquad X \longleftarrow_{[x_0]} \text{pt,}$$

where  $\sim$  is the equivalence relation on  $X \coprod Y$  obtained by declaring  $(0, x_0) \sim (1, y_0)$ .

- *The Basepoint.* The element  $p_0$  of  $X \vee Y$  defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$
$$= [(1, y_0)].$$

• *The Cocone.* The morphisms of pointed sets

$$inj_1: (X, x_0) \to (X \lor Y, p_0), 
inj_2: (Y, y_0) \to (X \lor Y, p_0),$$

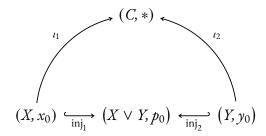
given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)], 
inj_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each  $x \in X$  and each  $y \in Y$ .

#### PROOF 6.3.3.1.3 ► PROOF OF CONSTRUCTION 6.3.3.1.2

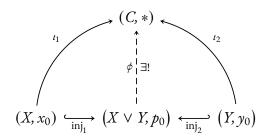
We claim that  $(X \lor Y, p_0)$  is the categorical coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in Sets\*. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$
  
$$\phi \circ \operatorname{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each  $z \in X \vee Y$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

$$= \iota_Y([(1, y_0)])$$

$$= *,$$

as  $\iota_X$  and  $\iota_Y$  are morphisms of pointed sets.

#### 00B7

#### PROPOSITION 6.3.3.1.4 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00B8

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
  
 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$   
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$ 

00B9

2. Associativity. We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0) \in \mathsf{Sets}_*$ .

00BA

3. Unitality. We have isomorphisms of pointed sets

$$(\operatorname{pt}, *) \vee (X, x_0) \cong (X, x_0),$$
  
$$(X, x_0) \vee (\operatorname{pt}, *) \cong (X, x_0),$$

natural in  $(X, x_0) \in \mathsf{Sets}_*$ .

00BB

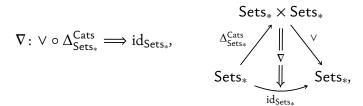
4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$

natural in  $(X, x_0)$ ,  $(Y, y_0) \in Sets_*$ .

00BC

- 5. *Symmetric Monoidality*. The triple (Sets∗, ∨, pt) is a symmetric monoidal category.
- 00BD
- 6. The Fold Map. We have a natural transformation



called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each  $p \in X \vee X$ .

#### PROOF 6.3.3.1.5 ► PROOF OF PROPOSITION 6.3.3.1.4

### Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

### Item 5: Symmetric Monoidality

Omitted.

#### Item 6: The Fold Map

Naturality for the transformation  $\nabla$  is the statement that, given a morphism of pointed sets  $f: (X, x_0) \to (Y, y_0)$ , we have

$$\begin{array}{ccc}
X \lor X & \xrightarrow{\nabla_X} X \\
\nabla_Y \circ (f \lor f) = f \circ \nabla_X, & f \lor f \downarrow & \downarrow f \\
Y \lor Y & \xrightarrow{\nabla_Y} Y.
\end{array}$$

Indeed, we have

$$\begin{split} \big[\nabla_Y \circ \big(f \vee f\big)\big] \big(\big[(i,x)\big]\big) &= \nabla_Y \big(\big[\big(i,f(x)\big)\big]\big) \\ &= f(x) \\ &= f(\nabla_X (\big[(i,x)\big])) \\ &= \big[f \circ \nabla_X\big] \big(\big[(i,x)\big]\big) \end{split}$$

for each  $[(i, x)] \in X \vee X$ , and thus  $\nabla$  is indeed a natural transformation.

### 00BE 6.3.4 Pushouts

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (Z, z_0) \to (X, x_0)$  and  $g: (Z, z_0) \to (Y, y_0)$  be morphisms of pointed sets.

#### 00BF DEFINITION 6.3.4.1.1 ▶ PUSHOUTS OF POINTED SETS

The **pushout of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along** (f, g) is the pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) in Sets<sub>\*</sub> as in Limits and Colimits, ??.

### 0258 CONSTRUCTION 6.3.4.1.2 ➤ CONSTRUCTION OF PUSHOUTS OF POINTED SETS

Concretely, the **pushout of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set  $(X \coprod_{f,Z,g} Y, p_0)$ , where:
  - The set  $X \coprod_{f,Z,g} Y$  is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have  $p_0 = [x_0] = [y_0]$ .
- *The Cocone*. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$
  
 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$ 

given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)] 
inj_2(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each  $x \in X$  and each  $y \in Y$ .

#### PROOF 6.3.4.1.3 ▶ PROOF OF ??

Firstly, we note that indeed  $[x_0] = [y_0]$ , as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation  $\sim$  on  $X \coprod_Z Y$  then identifying  $x_0 = f(z_0) \sim g(z_0) = y_0$ .

We now claim that  $(X \coprod_Z Y, p_0)$  is the categorical pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to (f, g) in Sets<sub>\*</sub>. First we need to check

that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \xleftarrow{\inf_{2}} (Y, y_{0})$$

$$\inf_{1} \circ f = \inf_{2} \circ g, \qquad \inf_{1 \in J_{1}} \left( X, x_{0} \right) \xleftarrow{f} (Z, z_{0})$$

Indeed, given  $z \in Z$ , we have

$$[\inf_{1} \circ f](z) = \inf_{1} (f(z))$$

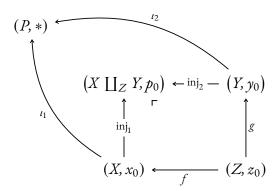
$$= [(0, f(z))]$$

$$= [(1, g(z))]$$

$$= \inf_{2} (g(z))$$

$$= [\inf_{2} \circ g](z),$$

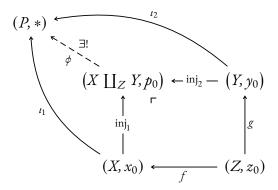
where [(0, f(z))] = [(1, g(z))] by the definition of the relation  $\sim$  on  $X \coprod Y$  (the coproduct of unpointed sets of X and Y). Next, we prove that  $X \coprod_Z Y$  satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1,$$
  
$$\phi \circ \operatorname{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each  $p \in X \coprod_Z Y$ , where the well-definedness of  $\phi$  is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.I.I. Finally, we show that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$

$$= \iota_1(x_0)$$

$$= *,$$

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$

$$= \iota_2(y_0)$$

$$= *,$$

where we use that  $\iota_1$  (resp.  $\iota_2$ ) is a morphism of pointed sets.

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#### 00BG PROPOSITION 6.3.4.1.4 ▶ PROPERTIES OF PUSHOUTS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

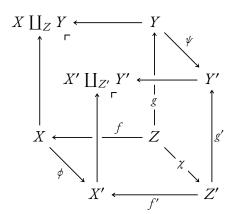
I. Functoriality. The assignment  $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$  defines a functor

$$-_1 \coprod_{-_2} -_1 : \operatorname{\mathsf{Fun}}(\mathcal{P}, \operatorname{\mathsf{Sets}}) \to \operatorname{\mathsf{Sets}}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \coprod_{-3} -1$  is given by sending a morphism



in  $\operatorname{Fun}(\mathcal{P},\operatorname{\mathsf{Sets}}_*)$  to the morphism of pointed sets

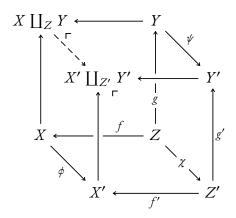
$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

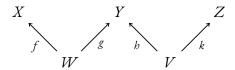
00BH

for each  $p \in X \coprod_Z Y$ , which is the unique morphism of pointed sets making the diagram



commute.

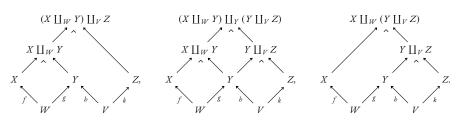
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams



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3. Unitality. We have isomorphisms of sets



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4. Commutativity. We have an isomorphism of sets

00BM

5. Interaction With Coproducts. We have

$$X \coprod_{\mathsf{pt}} Y \cong X \vee Y, \qquad \bigwedge^{\mathsf{r}} \qquad \bigwedge^{\mathsf{r}} \qquad \bigwedge^{\mathsf{pt}} [y_0]$$

$$X \longleftarrow_{[x_0]} \mathsf{pt}.$$

00BN

6. *Symmetric Monoidality*. The triple (Sets\*\*,  $\coprod_X$ , (X,  $x_0$ )) is a symmetric monoidal category.

#### PROOF 6.3.4.1.5 ► PROOF OF PROPOSITION 6.3.4.1.4

#### Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of

**??**, with the explicit expression for  $\xi$  following from the commutativity of the cube pushout diagram.

#### Item 2: Associativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.

### Item 3: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.

### Item 4: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.

#### Item 5: Interaction With Coproducts

Omitted.

### Item 6: Symmetric Monoidality

Omitted.

### 00BP 6.3.5 Coequalisers

Let  $f, g: (X, x_0) \Rightarrow (Y, y_0)$  be morphisms of pointed sets.

#### 00BQ DEFINITION 6.3.5.1.1 ► COEQUALISERS OF POINTED SETS

The **coequaliser of** (f, g) is the pointed set  $(CoEq(f, g), [y_0])$ .

### 0259 CONSTRUCTION 6.3.5.1.2 ► CONSTRUCTION OF COEQUALISERS OF POINTED SETS

The **coequaliser of** (f, g) is the pair  $((CoEq(f, g), [y_0]), coeq(f, g))$  consisting of:

• *The Colimit.* The pointed set  $(CoEq(f, g), [y_0])$ , where CoEq(f, g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.I.I.

• The Cocone. The map

$$coeq(f,g): Y \rightarrow (CoEq(f,g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Construction 4.2.5.1.2.

#### PROOF 6.3.5.1.3 ► PROOF OF CONSTRUCTION 6.3.5.1.2

We claim that  $(CoEq(f, g), [y_0])$  is the categorical coequaliser of f and g in Sets<sub>\*</sub>. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f, g) \circ f = coeq(f, g) \circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g) \circ g](x)$$

for each  $x \in X$ . Next, we prove that CoEq(f, g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each  $a \in A$ , it follows from Conditions on Relations, Items 4 and 5 of Proposition 10.6.2.1.3 that there

exists a unique map  $\phi \colon \operatorname{CoEq}(f,g) \xrightarrow{\exists !} C$  making the diagram

commute, where we note that  $\phi$  is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \operatorname{coeq}(f, g)]([y_0])$$

$$= c([y_0])$$

$$= *.$$

where we have used that *c* is a morphism of pointed sets.

#### 00BR

#### PROPOSITION 6.3.5.1.4 ▶ PROPERTIES OF COEQUALISERS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \to (Y, y_0)$  be morphisms of pointed sets.

00BS

I. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f,g) \circ f, \text{coeq}(f,g) \circ h)}_{=\text{CoEq}(\text{coeq}(f,g) \circ b, \text{coeq}(f,g) \circ h)} \cong \text{CoEq}(f,g,h) \cong \underbrace{\text{CoEq}(\text{coeq}(g,h) \circ f, \text{coeq}(g,h) \circ g, \text{coeq}(g,h) \circ h)}_{=\text{CoEq}(\text{coeq}(g,h) \circ f, \text{coeq}(g,h) \circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets\*.

00BT

2. Unitality. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

00BU

3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f,g) \cong CoEq(g,f).$$

#### Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.2.5.1.7.

### Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.2.5.1.7.

### Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.5.1.7.

### **6.4** Constructions With Pointed Sets

### 00BW 6.4.1 Free Pointed Sets

Let *X* be a set.

#### 00BX

#### **DEFINITION 6.4.1.1.1** ► FREE POINTED SETS

The **free pointed set on** X is the pointed set  $X^+$  consisting of:

• *The Underlying Set.* The set  $X^+$  defined by

$$X^{+} \stackrel{\text{def}}{=} X \coprod \text{pt}$$
$$\stackrel{\text{def}}{=} X \coprod \{ \star \}.$$

• *The Basepoint.* The element  $\star$  of  $X^+$ .

#### 00BY

### PROPOSITION 6.4.1.1.2 ► PROPERTIES OF FREE POINTED SETS

Let *X* be a set.

00BZ

I. *Functoriality*. The assignment  $X \mapsto X^+$  defines a functor

$$(-)^+$$
: Sets  $\rightarrow$  Sets<sub>\*</sub>,

where:

<sup>&</sup>lt;sup>1</sup> Further Notation: We sometimes write  $\star_X$  for the basepoint of  $X^+$  for clarity, specially when there are multiple free pointed sets involved in the current discussion.

• *Action on Objects.* For each  $X \in \text{Obj}(\mathsf{Sets})$ , we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where  $X^+$  is the pointed set of Definition 6.4.I.I.

• *Action on Morphisms.* For each morphism  $f: X \to Y$  of Sets, the image

$$f^+\colon X^+\to Y^+$$

of f by  $(-)^+$  is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. Adjointness. We have an adjunction

$$((-)^+ \dashv \overline{\bowtie})$$
: Sets  $\underbrace{\bot}_{\overline{\bowtie}}$  Sets<sub>\*</sub>,

witnessed by a bijection of sets

$$\mathsf{Sets}_* ((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in  $X \in \text{Obj}(\mathsf{Sets})$  and  $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+,(-)^{+,\coprod},(-)_1^{+,\coprod}\right)\colon(\mathsf{Sets},\coprod,\varnothing)\to\big(\mathsf{Sets}_*,\vee,\mathsf{pt}\big),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod}: X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod}: \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in  $X, Y \in \text{Obj}(\mathsf{Sets})$ .

00C0

00C1

00C2

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)_1^+) : (\operatorname{Sets}, \times, \operatorname{pt}) \to (\operatorname{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_1^+ \colon S^0 \xrightarrow{\sim} \operatorname{pt}^+,$$

natural in  $X, Y \in \text{Obj}(\mathsf{Sets})$ .

#### PROOF 6.4.1.1.3 ► PROOF OF PROPOSITION 6.4.1.1.2

### Item 1: Functoriality

We claim that  $(-)^+$  is indeed a functor:

• *Preservation of Identities.* Let  $X \in Obj(Sets)$ . We have

$$\operatorname{id}_{X}^{+}(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_{X} & \text{if } x = \star_{X}, \end{cases}$$

for each  $x \in X^+$ , so  $id_X^+ = id_{X^+}$ .

• Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$
,  $g: Y \to Z$ ,

we have

$$[g^+ \circ f^+](x) \stackrel{\text{def}}{=} g^+(f^+(x))$$
$$\stackrel{\text{def}}{=} g^+(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^+(x)$$

for each  $x \in X$  and

$$[g^{+} \circ f^{+}](\star_{X}) \stackrel{\text{def}}{=} g^{+}(f^{+}(\star_{X}))$$

$$\stackrel{\text{def}}{=} g^{+}(\star_{Y})$$

$$\stackrel{\text{def}}{=} \star_{Z}$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(\star_{X}),$$

so 
$$(g \circ f)^+ = g^+ \circ f^+$$
.

This finishes the proof.

### Item 2: Adjointness

We proceed in a few steps:

025A

• Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger} : X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X$ .

025B

• *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function  $\xi \colon X \to Y$  to the morphism of pointed sets

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, \gamma_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each  $x \in X^+$ .

• Invertibility I. Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, \gamma_0),$$

we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} (\Phi_{X,Y} (\xi)) \\ &= \Psi_{X,Y} \Big( \xi^{\dagger} \Big) \\ &\stackrel{\text{def}}{=} \left[ x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \\ &= \left[ x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[ \text{id}_{\mathsf{Sets}_*(X^+,Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}.$$

• Invertibility II. Given a map of sets  $\xi \colon X \to Y$ , we have

$$\begin{split} \left[ \Phi_{X,Y} \circ \Psi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big( \Psi_{X,Y} (\xi) \big) \\ &= \Phi_{X,Y} \Big( \xi^{\dagger} \Big) \end{split}$$

025C

025D

$$= \Phi_{X,Y} \left( \begin{bmatrix} x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right)$$

$$= \begin{bmatrix} x \mapsto \xi(x) \end{bmatrix}$$

$$= \xi$$

$$\stackrel{\text{def}}{=} \left[ \text{id}_{\mathsf{Sets}(X,Y)} \right] (\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}$$
.

• Naturality for  $\Phi$ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\begin{aligned} \mathsf{Sets}_*(X'^{,+},Y) & \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}(X',Y) \\ & f^* \middle\downarrow & & \downarrow f^* \\ \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{XY}} \mathsf{Sets}(X,Y) \end{aligned}$$

commutes. Indeed, given a morphism of pointed sets  $\xi\colon X'^{,+}\to Y$ , we have

$$\begin{split} \left[ \Phi_{X,Y} \circ f^* \right] (\xi) &= \Phi_{X,Y} \big( f^*(\xi) \big) \\ &= \Phi_{X,Y} \big( \xi \circ f \big) \\ &= \xi \circ f \\ &= \Phi_{X',Y} (\xi) \circ f \\ &= f^* \big( \Phi_{X',Y} (\xi) \big) \\ &= f^* \big( \Phi_{X',Y} (\xi) \big) \\ &= \left[ f^* \circ \Phi_{X',Y} \right] (\xi). \end{split}$$

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Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

• Naturality for  $\Phi$ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\mathsf{Sets}_*(X^+,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X,Y)$$

$$\downarrow^{g_*} \qquad \qquad \downarrow^{g_*}$$

$$\mathsf{Sets}_*(X^+,Y'), \xrightarrow{\Phi_{X,Y'}} \mathsf{Sets}(X,Y')$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger} \colon X^+ \to Y,$$

we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'} \left(g_*(\xi)\right) \\ &= \Phi_{X,Y'} \left(g \circ \xi\right) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_* \left(\Phi_{X,Y'}(\xi)\right) \\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'}\circ g_*=g_*\circ \Phi_{X,Y'}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

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• Naturality for  $\Psi$ . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\Psi$  is also natural in each argument.

This finishes the proof.

### Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on  $(-)^+$  with respect to  $\coprod$  and  $\vee$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^{+,\coprod}: X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each  $z \in X^+ \vee Y^+$ , with inverse

$$(-)^{+,\coprod,-1}_{X,Y}\colon (X\coprod Y)^+\stackrel{\sim}{\dashrightarrow} X^+\vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each  $z \in (X \mid \mid Y)^+$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,1}$$
: pt  $\stackrel{\sim}{\longrightarrow} \emptyset^+$ 

is given by sending  $\star_X$  to  $\star_\varnothing$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

### Item 4: Symmetric Strong Monoidality With Respect to Smash Products

We construct the strong monoidal structure on  $(-)^+$  with respect to  $\times$  and  $\wedge$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{YY}^+: X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each  $x \land y \in X^+ \land Y^+$ , with inverse

$$(-)^{+,-1}_{X,Y} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each  $z \in (X \times Y)^+$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+$$

is given by sending 0 to  $\star_{pt}$  and 1 to  $\star$ , where  $pt^+ = \{\star, \star_{pt}\}$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

### 01QS 6.4.2 Deleting Basepoints

Let  $(X, x_0)$  be a pointed set.

### 010T DEFINITION 6.4.2.1.1 ► SETS WITH DELETED BASEPOINTS

The **set with deleted basepoint associated to** X is the set  $X^-$  defined by

$$X^- \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

### 01QU PROPOSITION 6.4.2.1.2 ➤ PROPERTIES OF SETS WITH DELETED BASEPOINTS

Let  $(X, x_0)$  be a pointed set.

1. Functoriality. The assignment  $(X, x_0) \mapsto X^-$  defines a functor

$$X^-: \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets},$$

where:

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• *Action on Objects.* For each  $X \in Obj(Sets^{actv}_*)$ , we have

$$[(-)^{-}](X) \stackrel{\text{def}}{=} X^{-},$$

where  $X^-$  is the set of Definition 6.4.2.I.I.

• *Action on Morphisms*. For each morphism  $f: X \to Y$  of  $Sets^{actv}_*$ , the image

$$f^-\colon X^-\to Y^-$$

of f by  $(-)^-$  is the map defined by

$$f^{-}(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in X^-$ .

2. Adjoint Equivalence. We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+)$$
: Sets<sub>\*</sub>  $\xrightarrow{(-)^+}$  Sets,

witnessed by a bijection of sets

$$Sets(X^-, Y) \cong Sets_*(X, Y^+),$$

natural in  $X \in \text{Obj}(\mathsf{Sets}_*)$  and  $Y \in \text{Obj}(\mathsf{Sets})$ , and by isomorphisms

$$(X^{-})^{+} \cong X,$$
$$(Y^{+})^{-} \cong Y,$$

once again natural in  $X \in \text{Obj}(\mathsf{Sets}_*)$  and  $Y \in \text{Obj}(\mathsf{Sets})$ .

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\vee},(-)_{1}^{-,\vee}\right)\colon \left(\mathsf{Sets}^{\mathsf{actv}}_{*},\vee,\mathsf{pt}\right),\to \left(\mathsf{Sets},\sqsubseteq,\varnothing\right),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{\neg,\vee}: X^{-} \coprod Y^{-} \xrightarrow{\sim} (X \vee Y)^{-},$$
$$(-)_{1}^{\neg,\vee}: \varnothing \xrightarrow{\sim} \mathsf{pt}^{-},$$

natural in  $X, Y \in \text{Obj}(\mathsf{Sets})$ .

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)_1^{-,\times}): (\mathsf{Sets}^{\mathsf{actv}}_*, \wedge, S^0), \to (\mathsf{Sets}, \times, \mathsf{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-},$$
$$(-)_{1}^{-} \colon \operatorname{pt} \xrightarrow{\sim} (S^{0})^{-},$$

natural in  $X, Y \in \text{Obj}(\mathsf{Sets})$ .

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#### PROOF 6.4.2.1.3 ► PROOF OF PROPOSITION 6.4.2.1.2

#### Item 1: Functoriality

We claim that  $(-)^-$  is indeed a functor:

• Preservation of Identities. Let  $X \in \text{Obj}(\mathsf{Sets})$ . We have

$$id_X^-(x) \stackrel{\text{def}}{=} x$$

for each  $x \in X^-$ , so  $id_X^- = id_{X^-}$ .

• Preservation of Composition. Given morphisms of pointed sets

$$f: (X, x_0) \to (Y, y_0),$$
  
 $g: (Y, y_0) \to (Z, z_0),$ 

we have

$$\begin{bmatrix} g^- \circ f^- \end{bmatrix}(x) \stackrel{\text{def}}{=} g^- (f^-(x)) \\
\stackrel{\text{def}}{=} g^- (f(x)) \\
\stackrel{\text{def}}{=} g(f(x)) \\
\stackrel{\text{def}}{=} [g \circ f]^- (x)$$

for each  $x \in X$ , so  $(g \circ f)^- = g^- \circ f^-$ .

This finishes the proof.

### Item 2: Adjoint Equivalence

We proceed in a few steps:

I. Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-,Y) \to \mathsf{Sets}^{\mathsf{actv}}_* \big(X,Y^+\big)$$

by sending a map  $\xi\colon X^-\to Y$  to the active morphism of pointed sets

$$\xi^{\dagger} \colon X \to Y^{+}$$

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given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each  $x \in X$ , where this morphism is indeed active since  $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$  for all  $x \in X^-$ .

2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$

given by sending an active morphism of pointed sets  $\xi \colon X \to Y^+$  to the map

$$\xi^{\dagger} \colon X^{-} \to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X^-$ , which is indeed well-defined (in that  $\xi(x) \in Y$  for all  $x \in X^-$ ) since  $\xi$  is active.

3. *Invertibility I.* Given a map of sets  $\xi: X^- \to Y$ , we have

$$\begin{split} \left[ \Psi_{X,Y} \circ \Phi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left( \Phi_{X,Y} (\xi) \right) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \left( \left[ x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^{-} \\ \star_{Y} & \text{if } x = x_{0} \end{cases} \right] \right) \\ &= \left[ \left[ x \mapsto \xi(x) \right] \right] \\ &= \xi \\ &= \left[ \text{id}_{\mathsf{Sets}(X^{-},Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}.$$

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4. Invertibility II. Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= \left[ \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} \right](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)}.$$

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5. Naturality for  $\Phi$ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\mathsf{Sets}\big(X^{',-},Y\big) \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X',Y^+)$$

$$f^* \hspace{1cm} \downarrow f^* \hspace{1cm} \downarrow f^*$$

$$\mathsf{Sets}_*(X^-,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

commutes. Indeed, given a map of sets  $\xi\colon X'\to Y$ , we have

$$\begin{split} \left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}\big(f^*(\xi)\big) \\ &= \Phi_{X,Y}\big(\xi \circ f\big) \end{split}$$

$$= \begin{bmatrix} x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases} \end{bmatrix}$$

$$= f^* \left( \begin{bmatrix} x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x'_0 \end{cases} \right) \right)$$

$$= f^* \left( \Phi_{X',Y}(\xi) \right)$$

$$= \left[ f^* \circ \Phi_{X',Y} \right] (\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

6. Naturality for  $\Phi$ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\operatorname{Sets}(X^{-}, Y) \xrightarrow{\Phi_{X,Y}} \operatorname{Sets}^{\operatorname{actv}}_{*}(X, Y^{+})$$

$$\downarrow^{g_{*}} \qquad \qquad \downarrow^{g_{*}}$$

$$\operatorname{Sets}(X^{-}, Y') \xrightarrow{\Phi_{X,Y'}} \operatorname{Sets}^{\operatorname{actv}}_{*}(X, Y'^{,+})$$

commutes. Indeed, given a map of sets  $\xi \colon X^- \to Y$ , we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}\left(g_*(\xi)\right) \\ &= \Phi_{X,Y'}\left(g \circ \xi\right) \\ &= \left[\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}\right]\!\right] \end{split}$$

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$$= g_* \left( \begin{bmatrix} x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{bmatrix} \right)$$
$$= g_* \left( \Phi_{X,Y'}(\xi) \right)$$
$$= \left[ g_* \circ \Phi_{X,Y'} \right] (\xi).$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

- 7. Naturality for  $\Psi$ . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\Psi$  is also natural in each argument.
- 8. Fully Faithfulness of  $(-)^-$ . We aim to show that the assignment  $f \mapsto f^-$  sets up a bijection

$$(-)_{X,Y}^- \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y) \xrightarrow{\sim} \mathsf{Sets}(X^-,Y^-).$$

Indeed, the inverse map

$$(-)^{\neg,-1}_{X,Y} \colon \mathsf{Sets}(X^{\neg},\,Y^{\neg}) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X,\,Y)$$

is given by sending a map of sets  $f:X^-\to Y^-$  to the active morphism of pointed sets  $f^\dagger\colon X\to Y$  defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_{0} & \text{if } x = x_{0} \end{cases}$$

for each  $x \in X$ .

9. Essential Surjectivity of  $(-)^-$ . We need to show that, given an object  $X \in \text{Obj}(\mathsf{Sets})$ , there exists some  $X' \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}})$  such that  $(X')^- \cong X$ . Indeed, taking  $X' = X^+$ , we have

$$(X^+)^- \stackrel{\mathrm{def}}{=} (X \cup \{\star_X\})^-$$

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$$\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\}$$

$$= X,$$

and thus we have in fact an *equality*  $(X^+)^- = X$ , showing  $(-)^-$  to be essentially surjective.

10. The Functor (-) Is an Equivalence. Since (-) is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Proposition 11.6.7.1.2.

This finishes the proof.

### Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on  $(-)^-$  with respect to  $\vee$  and  $\coprod$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{X,Y}^{-,\vee}\colon X^-\coprod Y^-\xrightarrow{\sim} (X\vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each  $z \in X^- \coprod Y^-$ , with inverse

$$(-)_{XY}^{-,\vee,-1} \colon (X \vee Y)^{-} \xrightarrow{\sim} X^{-} \coprod Y^{-}$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each  $z \in (X \vee Y)^-$ .

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• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\vee,1} \colon \overset{\sim}{\mathcal{O}} \xrightarrow{\sim} \mathsf{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^-$  into a symmetric strong monoidal functor is omitted.

### Item 4: Symmetric Strong Monoidality With Respect to Smash Products

We construct the strong monoidal structure on  $(-)^+$  with respect to  $\wedge$  and  $\times$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^- \colon X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{XY}(x,y)=x\wedge y$$

for each  $(x, y) \in X^- \times Y^-$ , with inverse

$$(-)^{-,-1}_{X,Y}\colon (X\wedge Y)^{-}\stackrel{\sim}{\dashrightarrow} X^{-}\times Y^{-}$$

given by

$$(-)^{-,-1}_{X,Y}(x \wedge y) \stackrel{\text{def}}{=} (x,y)$$

for each  $x \land y \in (X \land Y)^-$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{-,1} \colon \operatorname{pt} \xrightarrow{\sim} (S^0)^{-}$$

is given by sending  $\star$  to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

# **Appendices**

## A Other Chapters

#### **Preliminaries**

- I. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

#### Relations

- 8. Relations
- 9. Constructions With Relations

#### 10. Conditions on Relations

### Categories

- II. Categories
- 12. Presheaves and the Yoneda Lemma

### **Monoidal Categories**

13. Constructions With Monoidal Categories

#### **Bicategories**

 Types of Morphisms in Bicategories

### Extra Part

15. Notes

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