Sets

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This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

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0002 3.1.1 Functions

0001

- **Definition 3.1.1.1.1.** A **function** is a functional and total relation.
- **Notation 3.1.1.1.2.** Throughout this work, we will sometimes denote a function $f: X \to Y$ by

 $f \stackrel{\text{def}}{=} [\![x \mapsto f(x)]\!].$

01Y2 1. For example, given a function

$$\Phi \colon \mathsf{Hom}_{\mathsf{Sets}}(X,Y) \to K$$

taking values on a set of functions such as $\mathsf{Hom}_\mathsf{Sets}(X,Y)$, we will sometimes also write

$$\Phi(f) \stackrel{\text{def}}{=} \Phi(\llbracket x \mapsto f(x) \rrbracket).$$

01Y3 2. This notational choice is based on the lambda notation

$$f \stackrel{\text{def}}{=} (\lambda x. f(x)),$$

but uses a " \mapsto " symbol for better spacing and double brackets instead of either:

- 01Y4 (a) Square brackets $[x \mapsto f(x)]$;
- 01Y5 (b) Parentheses $(x \mapsto f(x))$;

hoping to improve readability when dealing with e.g.:

- 01Y6 (a) Equivalence classes, cf.:
- 01Y7 i. $\llbracket [x] \mapsto f(\llbracket x]) \rrbracket$
- 01Y8 ii. $[[x] \mapsto f([x])]$
- 01Y9 iii. $(\lambda[x], f([x]))$
- **01YA** (b) Function evaluations, cf.:
- 01YB i. $\Phi(\llbracket x \mapsto f(x) \rrbracket)$
- 01YC ii. $\Phi((x \mapsto f(x)))$
- 01YD iii. $\Phi((\lambda x. f(x)))$
- 01YE 3. We will also sometimes write -, -₁, -₂, etc. for the arguments of a function. Some examples include:
- **01YF** (a) Writing f(-1) for a function $f: A \to B$.
- **01YG** (b) Writing f(-1, -2) for a function $f: A \times B \to C$.
- **01YH** (c) Given a function $f: A \times B \rightarrow C$, writing

$$f(a,-): B \to C$$

for the function $[\![b\mapsto f(a,b)]\!]$.

01YJ (d) Denoting a composition of the form

$$A \times B \xrightarrow{\phi \times id_B} A' \times B \xrightarrow{f} C$$

by
$$f(\phi(-1), -2)$$
.

01YK 4. Finally, given a function $f: A \rightarrow B$, we will sometimes write

$$ev_a(f) \stackrel{\text{def}}{=} f(a)$$

for the value of f at some $a \in A$.

For an example of the above notations being used in practice, see the proof of the adjunction

$$(A \times - \dashv \mathsf{Hom}_{\mathsf{Sets}}(A, -))$$
: Sets $\underbrace{ \overset{A \times -}{\bot}}_{\mathsf{Hom}_{\mathsf{Sets}}(A, -)} \mathsf{Sets},$

stated in Constructions With Sets, Item 2 of Definition 4.1.3.1.3.

0005 3.2 The Enrichment of Sets in Classical Truth Values

- **0006 3.2.1** (-2)-Categories
- **Definition 3.2.1.1.1.** A (-2)-category is the "necessarily true" truth value.^{1,2,3}
- **0008 3.2.2** (-1)-Categories
- **Definition 3.2.2.1.1.** A (-1)-category is a classical truth value.
- **Remark 3.2.2.1.2.** $^{4}(-1)$ -categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object Hom(x, y) that is a (-2)-category (i.e. trivial).

As a result, a (-1)-category C is either:⁵

¹Thus, there is only one (-2)-category.

²A (-n)-category for $n=3,4,\ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

³For motivation, see [BS10, p. 13].

⁴For more motivation, see [BS10, p. 13].

⁵See [BS10, pp. 33–34].

- 01D2 1. *Empty*, having no objects.
- 01D3 2. Contractible, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathsf{Hom}_C(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathsf{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

Thus there are only two (-1)-categories up to equivalence:

- 01D4 1. The (-1)-category false (the empty one);
- 01D5 2. The (-1)-category true (the contractible one).
- **Definition 3.2.2.1.3.** The **poset of truth values**⁶ is the poset ($\{\text{true}, \text{false}\}, \leq \}$) consisting of:
 - The Underlying Set. The set {true, false} whose elements are the truth values true and false.
 - · The Partial Order. The partial order

$$\preceq$$
: {true, false} \times {true, false} \rightarrow {true, false}

on {true, false} defined by⁷

false
$$\leq$$
 false $\stackrel{\text{def}}{=}$ true,
true \leq false $\stackrel{\text{def}}{=}$ false,
false \leq true $\stackrel{\text{def}}{=}$ true,
true \prec true $\stackrel{\text{def}}{=}$ true.

- **Notation 3.2.2.1.4.** We also write $\{t, f\}$ for the poset $\{true, false\}$.
- **Proposition 3.2.2.1.5.** The poset of truth values {t, f} is Cartesian closed with product given by⁸

$$t \times t = t$$
, $f \times t = f$,
 $t \times f = f$, $f \times f = f$,
$$t \times f = f$$

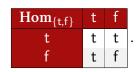
⁶ Further Terminology: Also called the **poset of** (-1)-categories.

⁷This partial order coincides with logical implication.

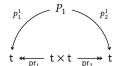
⁸ Note that \times coincides with the "and" operator, while $\mathbf{Hom}_{\{t,f\}}$ coincides with the logical implication operator.

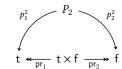
and internal Hom $Hom_{\{t,f\}}$ given by the partial order of $\{t,f\}$, i.e. by

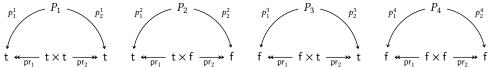
$$Hom_{\{t,f\}}(t,t) = t, \quad Hom_{\{t,f\}}(f,t) = t,$$
 $Hom_{\{t,f\}}(t,f) = f, \quad Hom_{\{t,f\}}(f,f) = t,$
 t

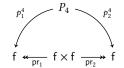


Proof. Existence of Products: We claim that the products $t \times t$, $t \times f$, $f \times t$, and $f \times f$ satisfy the universal property of the product in $\{t, f\}$. Indeed, suppose we have diagrams of the form

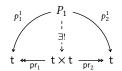


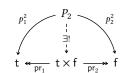


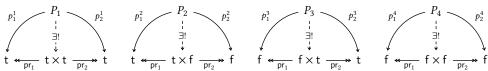


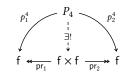


where the pr_1 and pr_2 morphisms are the only possible ones (since $\{t, f\}$ is posetal). We claim that there are unique morphisms making the diagrams









commute. Indeed:

- 1. If $P_1 = t$, then $p_1^1 = p_2^1 = id_t$, so there's a unique morphism from P_1 to t 01YL making the diagram commute, namely idt.
- 2. If $P_1 = f$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, **01YM** so there's a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t.
- 3. If $P_2 = t$, then there is no morphism p_2^2 . **01YN**
- 4. If $P_2={\sf f}$, then p_1^2 is the unique morphism from ${\sf f}$ to t while $p_2^2={\sf id}_{\sf f}$, so there's **01YP** a unique morphism from P_2 to f making the diagram commute, namely id_f .
- 5. The proof for P_3 is similar to the one for P_2 . 01YQ
- 6. If $P_4 = t$, then there is no morphism p_1^4 or p_2^4 . 01YR
- 7. If $P_4 = f$, then $p_1^4 = p_2^4 = id_f$, so there's a unique morphism from P_4 to f 01YS making the diagram commute, namely id_f .

This finishes the existence of products part of the proof. *Cartesian Closedness*: We claim there's a bijection

$$\operatorname{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \operatorname{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A, \operatorname{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)),$$

natural in $A, B, C \in \{t, f\}$. Indeed:

· For (A, B, C) = (t, t, t), we have

$$\begin{split} \text{Hom}_{\{t,f\}}(t\times t,t) &\cong \text{Hom}_{\{t,f\}}(t,t) \\ &= \{\text{id}_{true}\} \\ &\cong \text{Hom}_{\{t,f\}}(t,t) \\ &\cong \text{Hom}_{\{t,f\}}(t,\textbf{Hom}_{\{t,f\}}(t,t)). \end{split}$$

· For (A, B, C) = (t, t, f), we have

$$\begin{split} \text{Hom}_{\{t,f\}}(t\times t,f) &\cong \text{Hom}_{\{t,f\}}(t,f) \\ &= \emptyset \\ &\cong \text{Hom}_{\{t,f\}}(t,f) \\ &\cong \text{Hom}_{\{t,f\}}(t,\text{Hom}_{\{t,f\}}(t,f)). \end{split}$$

· For (A, B, C) = (t, f, t), we have

$$\begin{split} \text{Hom}_{\{t,f\}}(t\times f,t) &\cong \text{Hom}_{\{t,f\}}(f,t) \\ &\cong \text{pt} \\ &\cong \text{Hom}_{\{t,f\}}(f,t) \\ &\cong \text{Hom}_{\{t,f\}}(f,\text{Hom}_{\{t,f\}}(f,t)). \end{split}$$

· For (A, B, C) = (t, f, f), we have

$$\begin{split} \mathsf{Hom}_{\{t,f\}}(t\times f,f) &\cong \mathsf{Hom}_{\{t,f\}}(f,f) \\ &\cong \{\mathsf{id}_{\mathsf{false}}\} \\ &\cong \mathsf{Hom}_{\{t,f\}}(f,f) \\ &\cong \mathsf{Hom}_{\{t,f\}}(t,\mathbf{Hom}_{\{t,f\}}(f,f)). \end{split}$$

· For
$$(A, B, C) = (f, t, t)$$
, we have

$$\mathsf{Hom}_{\{t,f\}}(f\times t,t)\cong \mathsf{Hom}_{\{t,f\}}(f,t)$$

$$\begin{split} &\cong \text{ pt} \\ &\cong \text{ Hom}_{\{t,f\}}(f,t) \\ &\cong \text{ Hom}_{\{t,f\}}(f,\text{Hom}_{\{t,f\}}(t,t)). \end{split}$$

· For (A, B, C) = (f, t, f), we have

$$\begin{split} \text{Hom}_{\{t,f\}}(f\times t,f) &\cong \text{Hom}_{\{t,f\}}(f,f) \\ &\cong \{\text{id}_{false}\} \\ &\cong \text{Hom}_{\{t,f\}}(f,f) \\ &\cong \text{Hom}_{\{t,f\}}(f,\text{Hom}_{\{t,f\}}(t,f)). \end{split}$$

· For (A, B, C) = (f, f, t), we have

$$\begin{split} \text{Hom}_{\{t,f\}}(f\times f,t) &\cong \text{Hom}_{\{t,f\}}(f,t) \\ &\cong \text{pt} \\ &\cong \text{Hom}_{\{t,f\}}(f,t) \\ &\cong \text{Hom}_{\{t,f\}}(f,\text{Hom}_{\{t,f\}}(f,t)). \end{split}$$

· For (A, B, C) = (f, f, f), we have

$$\begin{split} \mathsf{Hom}_{\{t,f\}}(f\times f,f) &\cong \mathsf{Hom}_{\{t,f\}}(f,f) \\ &= \{\mathsf{id}_{\mathsf{false}}\} \\ &\cong \mathsf{Hom}_{\{t,f\}}(f,f) \\ &\cong \mathsf{Hom}_{\{t,f\}}(f,\pmb{Hom}_{\{t,f\}}(f,f)). \end{split}$$

Since $\{t, f\}$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

000E 3.2.3 0-Categories

Definition 3.2.3.1.1. A 0-category is a poset.⁹

Definition 3.2.3.1.2. A 0**-groupoid** is a 0-category in which every morphism is invertible.¹⁰

⁹ Motivation: A 0-category is precisely a category enriched in the poset of (-1)-categories.

¹⁰That is, a set.

OOOH 3.2.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. The analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.

Remark 3.2.4.1.1. The basic analogies between set theory and category theory are summarised in the following table:

| Set Theory | Category Theory |
|--|--|
| Enrichment in {true, false} | Enrichment in Sets |
| $\operatorname{Set} X$ | Category C |
| Element $x \in X$ | $ObjectX \in Obj(\mathcal{C})$ |
| Function $f: X \to Y$ | Functor $F \colon \mathcal{C} \to \mathcal{D}$ |
| Function $X \to \{\text{true}, \text{false}\}$ | Copresheaf $C \rightarrow Sets$ |
| Function $X \to \{\text{true}, \text{false}\}$ | Presheaf $C^{op} \rightarrow Sets$ |

Q1D7 Remark 3.2.4.1.2. The category of presheaves PSh(C) and the category of copresheaves CoPSh(C) on a category C are the 1-categorical counterparts to the powerset $\mathcal{P}(X)$ of subsets of a set X. The further analogies built upon this are summarised in the following table:

| Set Theory | Category Theory |
|---|--|
| Powerset $\mathcal{P}(X)$ | Presheaf category $PSh(\mathcal{C})$ |
| Characteristic function $\chi_{\{x\}} \colon X \to \{t,f\}$ | Representable presheaf $h_X \colon C^{op} \hookrightarrow Sets$ |
| Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ | Yoneda embedding $\mathcal{L}: C^{\mathrm{op}} \hookrightarrow PSh(C)$ |
| Characteristic relation $\chi_X(-1,-2): X \times X \to \{t,f\}$ | Hom profunctor $Hom_{\mathcal{C}}(1,2) \colon \mathcal{C}^{op} \times \mathcal{C} \to Sets$ |
| The Yoneda lemma for sets $\operatorname{Hom}_{\mathcal{P}(X)}(\chi_x,\chi_U)=\chi_U(x)$ | The Yoneda lemma for categories $\operatorname{Nat}(h_X,\mathcal{F})\cong\mathcal{F}(X)$ |
| The characteristic embedding is fully faithful, $\operatorname{Hom}_{\mathcal{P}(X)}(\chi_x,\chi_y)=\chi_X(x,y)$ | The Yoneda embedding is fully faithful, $\operatorname{Nat}(h_X,h_Y)\cong\operatorname{Hom}_{\mathcal{C}}(X,Y)$ |
| Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathcal{P}(U)} (\chi_x)$ | Presheaves are colimits of representables, $\mathcal{F}\cong \operatorname*{colim}_{h_X\in\int_{\mathcal{C}}\mathcal{F}}(h_X)$ |

Remark 3.2.4.1.3. We summarise the analogies between un/straightening in set theory and category theory in the following table:

| SET THEORY | CATEGORY THEORY |
|---|---|
| Assignment $U\mapsto \chi_U$ | Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ |
| Un/straightening isomorphism $\mathcal{P}(X) \cong Sets(X, \{t, f\})$ | Un/straightening equivalence $DFib(C) \stackrel{eq.}{\cong} PSh(C)$ |

Q1D9 Remark 3.2.4.1.4. We summarise the analogies between functions $\mathcal{P}(X) \to \mathcal{P}(Y)$ and functors $\mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$ in the following table:

| SET THEORY | CATEGORY THEORY |
|--|--|
| Direct image function $f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ | Left Kan extension functor $F_! \colon PSh(C) \to PSh(\mathcal{D})$ |
| Inverse image function $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ | Precomposition functor $F^* \colon PSh(\mathcal{D}) \to PSh(\mathcal{C})$ |
| Codirect image function $f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ | Right Kan extension functor $F_* \colon PSh(\mathcal{C}) \to PSh(\mathcal{D})$ |

Q1DA Remark 3.2.4.1.5. We summarise the analogies between functions, relations and profunctors in the following table:

| SET THEORY | CATEGORY THEORY |
|--|--|
| Relation $R: X \times Y \rightarrow \{t, f\}$ | Profunctor $\mathfrak{p} \colon \mathcal{D}^{op} \times \mathcal{C} \to Sets$ |
| Relation $R: X \to \mathcal{P}(Y)$ | $Profunctor \mathfrak{p} \colon \mathcal{C} \to PSh(\mathcal{D})$ |
| Relation as a cocontinuous morphism of posets $R \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$ | Profunctor as a colimit-preserving functor $\mathfrak{p} \colon PSh(C) \to PSh(\mathcal{D})$ |

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets

- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

Categories

11. Categories

References 11

12. Presheaves and the Yoneda Lemma

14. Types of Morphisms in Bicategories

Monoidal Categories

13. Constructions With Monoidal Categories **Extra Part**

Bicategories

15. Notes

References

[BS10] John C. Baez and Michael Shulman. "Lectures on *n*-Categories and Cohomology". In: *Towards higher categories*. Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 1–68. DOI: 10.1007/978-1-4419-1524-5_1. URL: https://doi.org/10.1007/978-1-4419-1524-5_1 (cit. on p. 3).