Pointed Sets

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0098   This chapter contains some foundational material on pointed sets.

Contents

6.1   Pointed Sets

6.1.1   Foundations

DEFINITION 6.1.1.1.1 ▶ POINTED SETS

A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

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[1]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$-**modules**.

REMARK 6.1.1.1.2 ▶ UNWINDING DEFINITION 6.1.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).

- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

EXAMPLE 6.1.1.1.3 ▶ THE ZERO SPHERE

The 0-**sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

---

[1]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[2]*Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

009E

EXAMPLE 6.1.1.1.4 ▶ THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.

- *The Basepoint.* The element $\star$ of pt.

01QB

EXAMPLE 6.1.1.1.5 ▶ THE STANDARD POINTED SET WITH $n + 1$ ELEMENTS

The **standard pointed set with $n + 1$ elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \overset{\mathrm{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n \rangle$.

009H **6.1.2   Morphisms of Pointed Sets**

009J

DEFINITION 6.1.2.1.1 ▶ MORPHISMS OF POINTED SETS

A **morphism of pointed sets**[1,2] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

---

[1]*Further Terminology:* Also called a **pointed function**.
[2]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

009K

REMARK 6.1.2.1.2 ▶ UNWINDING DEFINITION 6.1.2.1.1

In detail, a **morphism of pointed sets** $f \colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f \colon X \to Y$ such that the diagram

$$
\begin{array}{ccc}
 & \mathrm{pt} & \\
{\scriptstyle [x_0]}\swarrow & & \searrow{\scriptstyle [y_0]} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

commutes, i.e. such that
$$f(x_0) = y_0.$$

009L   ## 6.1.3   The Category of Pointed Sets

009M   **DEFINITION 6.1.3.1.1 ▶ THE CATEGORY OF POINTED SETS**

The **category of pointed sets** is the category Sets_* defined equivalently as:

- The homotopy category of the ∞-category $\mathrm{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**.

- The category Sets_* of Constructions With Categories, **??**.

009N   **REMARK 6.1.3.1.2 ▶ UNWINDING DEFINITION 6.1.3.1.1**

In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.

- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.

- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} : \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[1]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X .$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} : \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

---

[1]Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
g(f(x_0)) &= g(y_0) \\
&= z_0,
\end{aligned}
$$

$$
\begin{array}{ccc}
& & \mathrm{pt} \\
& [x_0] \swarrow & \big\downarrow {\scriptstyle [y_0]} \quad \searrow {\scriptstyle [z_0]} \\
X \xrightarrow{\ f\ } & Y \xrightarrow{\ g\ } & Z
\end{array}
$$

009P   ## 6.1.4   Elementary Properties of Pointed Sets

009Q   **PROPOSITION 6.1.4.1.1 ▶ ELEMENTARY PROPERTIES OF POINTED SETS**

Let (X, x_0) be a pointed set.

009R   1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

009S   (a) Products, described as in Definition 6.2.3.1.1.

009T   (b) Pullbacks, described as in Definition 6.2.4.1.1.

009U   (c) Equalisers, described as in Definition 6.2.5.1.1.

009V   2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

009W   (a) Coproducts, described as in Definition 6.3.3.1.1.

009X   (b) Pushouts, described as in Definition 6.3.4.1.1;

009Y   (c) Coequalisers, described as in Definition 6.3.5.1.1.

009Z   3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[1]

00A0   4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[2]

$$
\mathsf{Sets}_*\left(S^0, X\right) \cong X,
$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$
\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0),
$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00A1 5. *Relation to Partial Functions.* We have an equivalence of categories[3]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

024U (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \overset{\cong}{\to} \mathsf{Sets}^{\text{part.}}$$

sends:

024V   i. A pointed set (X, x_0) to X.

024W   ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

024X (b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\text{part.}} \overset{\cong}{\to} \mathsf{Sets}_*$$

sends:

024Y   i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

024Z   ii. A partial function

$$f \colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

---

¹The category Sets$_*$ does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

²In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

³ ⚠ *Warning:* This is not an isomorphism of categories, only an equivalence.

PROOF 6.1.4.1.2 ▶ PROOF OF PROPOSITION 6.1.4.1.1

Item 1: Completeness

This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.

Item 2: Cocompleteness

This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathsf{Sets}_*\left(S^0, X\right) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}\colon S^0 \to X$, the basepoint of $\mathsf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].

01QC   **6.1.5   Active and Inert Morphisms of Pointed Sets**

01QD   **DEFINITION 6.1.5.1.1 ▶ ACTIVE AND INERT MORPHISMS OF POINTED SETS**

Let $f\colon (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

01QE   1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.

01QF   2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.
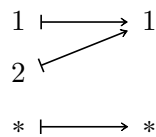
01QG   **NOTATION 6.1.5.1.2 ▶ THE CATEGORY OF POINTED SETS AND ACTIVE MORPHISMS**

We write $\mathsf{Sets}_*^{\mathrm{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

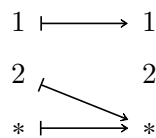01QH   **EXAMPLE 6.1.5.1.3 ▶ EXAMPLES OF ACTIVE AND INERT MAPS OF POINTED SETS**

Here are some examples of active and inert maps of pointed sets.

01QJ   1. The map $\mu\colon \langle 2 \rangle \to \langle 1 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & \nearrow & \\
& & \\
* & \longmapsto & *
\end{array}
$$

is active but not inert.

01QK   2. The map $f\colon \langle 2 \rangle \to \langle 2 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & & 2 \\
* & \longmapsto & *
\end{array}
$$

is inert but not active.

01QL   3. The map $f\colon \langle 3 \rangle \to \langle 1 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & \nearrow & \\
3 & & \\
* & \longmapsto & *
\end{array}
$$
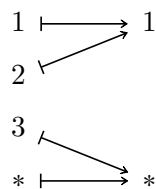
is neither inert nor active. However, it factors as $f = a \circ i$, where

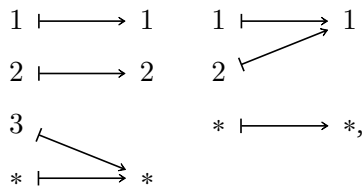$$i\colon \langle 3 \rangle \longrightarrow \langle 2 \rangle,$$
$$a\colon \langle 2 \rangle \longrightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by

$$
\begin{array}{ccc}
1 \longmapsto 1 & \quad & 1 \longmapsto 1 \\
2 \longmapsto 2 & \quad & 2 \nearrow \\
3 \searrow & \quad & * \longmapsto *, \\
* \longmapsto * & &
\end{array}
$$

with i being inert and a being active.

01QM   PROPOSITION 6.1.5.1.4 ▶ PROPERTIES OF ACTIVE AND INERT MAPS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

01QN   1. *Active-Inert Factorisation.* Every morphism of pointed sets $f\colon (X, x_0) \rightarrow (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

01QP   (a) The map $i\colon (X, x_0) \rightarrow (K, k_0)$ is an inert morphism of pointed sets

01QQ   (b) The map $a\colon (K, k_0) \rightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_*.

PROOF 6.1.5.1.5 ▶ PROOF OF PROPOSITION 6.1.5.1.4

Item 1: Active-Inert Factorisation

Let $f\colon X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$
$$= \left(X \setminus f^{-1}(y_0)\right) \cup \{x_0\};$$

- $i\colon X \to K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

  for each $x \in X$;

- $a\colon K \to Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

  for each $x \in K$.

Next, let

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & Y \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
A & \xrightarrow{\ a\ } & B
\end{array}
$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y \to A$ given by

$$\phi(y) = f\left(i^{-1}(y)\right)$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & Y \\
{\scriptstyle f}\big\downarrow & {\scriptstyle \exists!}\ \nearrow{\scriptstyle \phi} & \big\downarrow{\scriptstyle g} \\
A & \xrightarrow{\ a\ } & B.
\end{array}
$$

Indeed, this diagram commutes, as we have

$$[\phi \circ i](x) \overset{\text{def}}{=} \phi(i(x))$$
$$\overset{\text{def}}{=} f\left(i^{-1}(i(x))\right)$$
$$= f(x)$$

for each $x \in X$ and

$$[a \circ \phi](y) \overset{\text{def}}{=} a(\phi(y))$$
$$\overset{\text{def}}{=} a\left(f\left(i^{-1}(y)\right)\right)$$
$$\overset{\text{def}}{=} [a \circ f]\left(i^{-1}(y)\right)$$
$$= [g \circ i]\left(i^{-1}(y)\right)$$
$$\overset{\text{def}}{=} g\left(i\left(i^{-1}(y)\right)\right)$$
$$\overset{\text{def}}{=} g(y)$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$
\begin{array}{ccc}
X & \overset{i}{\longrightarrow} & Y \\
f \downarrow & \overset{\psi}{\nearrow} & \downarrow g \\
A & \underset{a}{\longrightarrow} & B
\end{array}
$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\psi(y) = \psi(i(x))$$
$$= f(x)$$
$$= f\left(i^{-1}(y)\right)$$
$$\overset{\text{def}}{=} \phi(y).$$

This finishes the proof.

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

00A2

00A3

00A4

> **DEFINITION 6.2.1.1.1 ▶ THE TERMINAL POINTED SET**
>
> The **terminal pointed set** is the terminal object of Sets$_*$ as in Limits and Colimits, **??**.

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> **CONSTRUCTION 6.2.1.1.2 ▶ CONSTRUCTION OF THE TERMINAL POINTED SET**
>
> Concretely, the **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:
>
> - *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
>
> - *The Cone.* The collection of morphisms of pointed sets
>
> $$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$
>
>   defined by
>
> $$!_X(x) \overset{\mathrm{def}}{=} \star$$
>
>   for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

> **PROOF 6.2.1.1.3 ▶ PROOF OF CONSTRUCTION 6.2.1.1.2**
>
> We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets$_*$. Indeed, suppose we have a diagram of the form
>
> $$(X, x_0) \qquad (\mathrm{pt}, \star)$$
>
> in Sets$_*$. Then there exists a unique morphism of pointed sets
>
> $$\phi \colon (X, x_0) \to (\mathrm{pt}, \star)$$
>
> making the diagram
>
> $$(X, x_0) \underset{\exists!}{\overset{\phi}{-\!-\!\to}} (\mathrm{pt}, \star)$$
>
> commute, namely $!_X$.

00A5   ## 6.2.2   Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

00A6 | DEFINITION 6.2.2.1.1 ▸ THE PRODUCT OF A FAMILY OF POINTED SETS |
       | --- |

The **product of** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the product of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets$_*$ as in Limits and Colimits, **??**.

0251 | CONSTRUCTION 6.2.2.1.2 ▸ CONSTRUCTION OF THE PRODUCT OF A FAMILY OF POINTED SETS |
       | --- |

Concretely, the **product of** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right), \left\{\mathrm{pr}_i\right\}_{i \in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$.

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i \colon \left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i\left(\left(x_j\right)_{j \in I}\right) \overset{\mathrm{def}}{=} x_i$$

for each $\left(x_j\right)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

PROOF 6.2.2.1.3 ▸ PROOF OF CONSTRUCTION 6.2.2.1.2

We claim that $\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right)$ is the categorical product of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets$_*$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$(P, *)$$
$$\searrow {\scriptstyle p_i}$$
$$\left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \xrightarrow[\mathrm{pr}_i]{} (X_i, x_0^i)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left(\prod_{i \in I} X_i, \left(x_0^i \right)_{i \in I} \right)$$

making the diagram

$$
\begin{array}{c}
(P, *) \\[1.2em]
\phi \Big\downarrow \exists! \qquad \searrow^{p_i} \\[1.2em]
\left(\prod_{i \in I} X_i, \left(x_0^i \right)_{i \in I} \right) \xrightarrow[\mathrm{pr}_i]{} (X_i, x_0^i)
\end{array}
$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= \left(x_0^i \right)_{i \in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. �

00A7   **PROPOSITION 6.2.2.1.4 ▶ PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS**

Let $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ be a family of pointed sets.

00A8   1. *Functoriality.* The assignment $\left\{ (X_i, x_0^i) \right\}_{i \in I} \mapsto \left(\prod_{i \in I} X_i, \left(x_0^i \right)_{i \in I} \right)$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

> **PROOF 6.2.2.1.5 ▶ PROOF OF PROPOSITION 6.2.2.1.4**
>
> **Item 1: Functoriality**
>
> This follows from Limits and Colimits, **??** of **??**.

00A9 **6.2.3   Products**

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA **DEFINITION 6.2.3.1.1 ▶ PRODUCTS OF POINTED SETS**

The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

0252 **CONSTRUCTION 6.2.3.1.2 ▶ CONSTRUCTION OF PRODUCTS OF POINTED SETS**

Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$\mathrm{pr}_1 \colon (X \times Y, (x_0, y_0)) \to (X, x_0),$$
$$\mathrm{pr}_2 \colon (X \times Y, (x_0, y_0)) \to (Y, y_0)$$

  defined by

$$\mathrm{pr}_1(x, y) \stackrel{\mathrm{def}}{=} x,$$
$$\mathrm{pr}_2(x, y) \stackrel{\mathrm{def}}{=} y$$

  for each $(x, y) \in X \times Y$.

PROOF 6.2.3.1.3 ▶ PROOF OF CONSTRUCTION 6.2.3.1.2

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets$_*$. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccc}
 & (P, *) & \\
p_1 \swarrow & & \searrow p_2 \\
(X, x_0) \xleftarrow{\mathrm{pr}_1} (X \times Y, (x_0, y_0)) & & \xrightarrow{\mathrm{pr}_2} (Y, y_0)
\end{array}
$$

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$
\begin{array}{ccc}
 & (P, *) & \\
p_1 \swarrow & \phi \downarrow \exists! & \searrow p_2 \\
(X, x_0) \xleftarrow{\mathrm{pr}_1} (X \times Y, (x_0, y_0)) & & \xrightarrow{\mathrm{pr}_2} (Y, y_0)
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\mathrm{pr}_1 \circ \phi = p_1, \\
\mathrm{pr}_2 \circ \phi = p_2
$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(*) &= (p_1(*), p_2(*)) \\
&= (x_0, y_0),
\end{aligned}
$$

where we have used that p_1 and p_2 are morphisms of pointed sets. ▨

00AB   **PROPOSITION 6.2.3.1.4 ▸ PROPERTIES OF PRODUCTS OF POINTED SETS**

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00AC   1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$
\begin{aligned}
A \times - &: & \mathsf{Sets}_* && \to \mathsf{Sets}_*, \\
- \times B &: & \mathsf{Sets}_* && \to \mathsf{Sets}_*, \\
-_1 \times -_2 &: & \mathsf{Sets}_* \times \mathsf{Sets}_* && \to \mathsf{Sets}_*,
\end{aligned}
$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

01QR   2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

00AD   3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AE   4. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{aligned}
(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\
(X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),
\end{aligned}
$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AF   5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AG   6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

PROOF 6.2.3.1.5 ▶ PROOF OF PROPOSITION 6.2.3.1.4

Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.

Item 2: Lack of Adjointness

See [MSE 2855868].

Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

Item 4: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.

Item 5: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.

🏳️‍⚧️

00AH **6.2.4 Pullbacks**

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

00AJ **DEFINITION 6.2.4.1.1 ▶ PULLBACKS OF POINTED SETS**

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets$_*$ as in Limits and Colimits, **??**.

0253 **CONSTRUCTION 6.2.4.1.2 ▶ CONSTRUCTION OF PULLBACKS OF POINTED SETS**

Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$\mathrm{pr}_1\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$

$$\mathrm{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$$

defined by

$$\mathrm{pr}_1(x, y) \overset{\mathrm{def}}{=} x,$$
$$\mathrm{pr}_2(x, y) \overset{\mathrm{def}}{=} y$$

for each $(x, y) \in X \times_Z Y$.

PROOF 6.2.4.1.3 ▶ PROOF OF CONSTRUCTION 6.2.4.1.2

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,$$

$$
\begin{CD}
(X \times_Z Y, (x_0, y_0)) @>{\mathrm{pr}_2}>> (Y, y_0) \\
@V{\mathrm{pr}_1}VV @VV{g}V \\
(X, x_0) @>>{f}> (Z, z_0).
\end{CD}
$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$
\begin{aligned}
\left[f \circ \mathrm{pr}_1 \right](x, y) &= f\left(\mathrm{pr}_1(x, y) \right) \\
&= f(x) \\
&= g(y) \\
&= g\left(\mathrm{pr}_2(x, y) \right) \\
&= \left[g \circ \mathrm{pr}_2 \right](x, y),
\end{aligned}
$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the
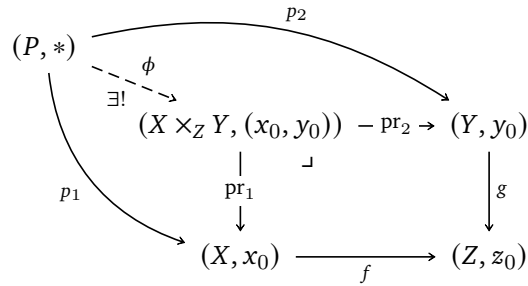
form

$$(P, *) \xrightarrow{\quad p_2 \quad}$$

$$(X \times_Z Y, (x_0, y_0)) \;-\mathrm{pr}_2\rightarrow\; (Y, y_0)$$

$$\Big\downarrow \mathrm{pr}_1 \qquad \lrcorner \qquad \Big\downarrow g$$

$$p_1 \qquad (X, x_0) \xrightarrow{\quad f \quad} (Z, z_0)$$
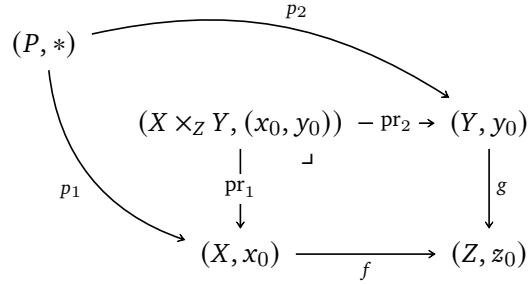
in Sets$_*$. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$(P, *) \xrightarrow{\quad p_2 \quad}$$

$$\phi$$

$$\exists!$$

$$(X \times_Z Y, (x_0, y_0)) \;-\mathrm{pr}_2\rightarrow\; (Y, y_0)$$

$$\Big\downarrow \mathrm{pr}_1 \qquad \lrcorner \qquad \Big\downarrow g$$

$$p_1 \qquad (X, x_0) \xrightarrow{\quad f \quad} (Z, z_0)$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
$$= (x_0, y_0),$$

where we have used that p_1 and p_2 are morphisms of pointed sets.

00AK

PROPOSITION 6.2.4.1.4 ▶ PROPERTIES OF PULLBACKS OF POINTED SETS

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

00AL

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor
$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:

$$
\begin{array}{ccc}
 & & \bullet \\
 & & \downarrow \\
\bullet & \longrightarrow & \bullet.
\end{array}
$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism
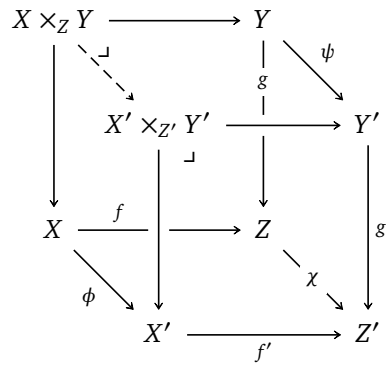

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets
$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \left(X' \times_{Z'} Y', (x_0', y_0')\right)$$
given by
$$\xi(x, y) \overset{\mathrm{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
\begin{array}{ccc}
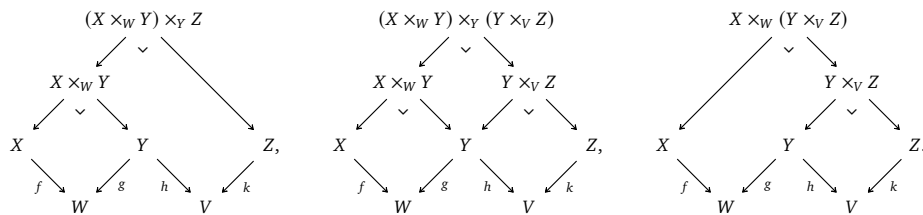X \times_Z Y & \longrightarrow & Y \\
\Big\downarrow & \searrow & \Big\downarrow{\scriptstyle g} \quad \searrow{\scriptstyle \psi} \\
 & X' \times_{Z'} Y' \longrightarrow Y' \\
 & \Big\downarrow & \Big\downarrow \\
X \xrightarrow{\ f\ } & Z & g' \\
 & \searrow{\scriptstyle \phi} \quad \searrow{\scriptstyle \chi} & \\
 & X' \xrightarrow{\ f'\ } Z' &
\end{array}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{array}{ccccc}
X & & Y & & Z \\
 & \searrow{\scriptstyle f} \quad \swarrow{\scriptstyle g} & & \searrow{\scriptstyle h} \quad \swarrow{\scriptstyle k} & \\
 & W & & V &
\end{array}
$$

in Sets_*, we have isomorphisms of pointed sets

$$
(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),
$$
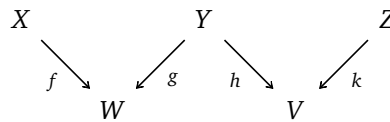
where these pullbacks are built as in the diagrams

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{array}{ccc}
A & =\!=\!= & A \\
{\scriptstyle f}\Big\downarrow & \lrcorner & \Big\downarrow{\scriptstyle f} \\
X & =\!=\!= & X
\end{array}
\qquad
\begin{array}{l}
X \times_X A \cong A, \\
A \times_X X \cong A,
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
\Big\| & \lrcorner & \Big\| \\
X & \xrightarrow{\ f\ } & X.
\end{array}
$$

00AP

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{array}{ccc}
A \times_X B & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow g \\
A & \xrightarrow{\quad f \quad} & X,
\end{array}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{array}{ccc}
B \times_X A & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow f \\
B & \xrightarrow{\quad g \quad} & X.
\end{array}
$$

00AQ

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow & \lrcorner & \downarrow {!_Y} \\
X & \xrightarrow{\quad !_X \quad} & \mathrm{pt.}
\end{array}
$$

00AR

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

PROOF 6.2.4.1.5 ▶ PROOF OF PROPOSITION 6.2.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.4.1.7.

Item 3: Unitality

This follows from Constructions With Sets, Item 6 of Proposition 4.1.4.1.7.

Item 4: Commutativity

This follows from Constructions With Sets, Item 7 of Proposition 4.1.4.1.7.

Item 5: Interaction With Products

This follows from Constructions With Sets, Item 10 of Proposition 4.1.4.1.7.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 11 of Proposition 4.1.4.1.7.

00AS **6.2.5   Equalisers**

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT   **DEFINITION 6.2.5.1.1 ▶ EQUALISERS OF POINTED SETS**

The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in Limits and Colimits, **??**.

0254   **CONSTRUCTION 6.2.5.1.2 ▶ CONSTRUCTION OF EQUALISERS OF POINTED SETS**

Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.

- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g)\colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

PROOF 6.2.5.1.3 ▶ PROOF OF CONSTRUCTION 6.2.5.1.2

We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we

have a diagram of the form

$$(\mathrm{Eq}(f,g),x_0) \xrightarrow{\mathrm{eq}(f,g)} (X,x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y,y_0)$$

$$\nearrow \, e$$

$$(E, *)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathrm{Eq}(f,g),x_0)$$

making the diagram

$$(\mathrm{Eq}(f,g),x_0) \xrightarrow{\mathrm{eq}(f,g)} (X,x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y,y_0)$$

$$\phi \uparrow \, \exists! \qquad \nearrow \, e$$

$$(E, *)$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

00AU   PROPOSITION 6.2.5.1.4 ▶ PROPERTIES OF EQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00AV   1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \; \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{\; g \;} \\ \xrightarrow{h} \end{array} \; (Y, y_0)$$

in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW   2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

00AX   3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

PROOF 6.2.5.1.5 ▶ PROOF OF PROPOSITION 6.2.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.1.5.1.4.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.1.5.1.4.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.1.5.1.4.

00AY 6.3   Colimits of Pointed Sets

00AZ 6.3.1   The Initial Pointed Set

00B0 **DEFINITION 6.3.1.1.1 ▶ THE INITIAL POINTED SET**

The **initial pointed set** is the initial object of Sets_* as in Limits and Colimits, **??**.

0255 **CONSTRUCTION 6.3.1.1.2 ▶ CONSTRUCTION OF THE INITIAL POINTED SET**

Concretely, the **initial pointed set** is the pair $\left((\mathrm{pt},\star),\{\iota_X\}_{(X,x_0)\in\mathsf{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt},\star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X\colon(\mathrm{pt},\star)\to(X,x_0)\}_{(X,x_0)\in\mathsf{Obj}(\mathsf{Sets})}$$

  defined by

$$\iota_X(\star)\overset{\mathrm{def}}{=}x_0.$$

PROOF 6.3.1.1.3 ▶ PROOF OF CONSTRUCTION 6.3.1.1.2

We claim that $(\mathrm{pt},\star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt},\star)\qquad(X,x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon(\mathrm{pt},\star)\to(X,x_0)$$

making the diagram

$$(\mathrm{pt},\star)\ \underset{\exists!}{-\overset{\phi}{-}\to}\ (X,x_0)$$

commute, namely ι_X.

00B1 6.3.2   Coproducts of Families of Pointed Sets

Let $\left\{(X_i,x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

00B2

The **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$[1] is the coproduct of $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$ in Sets_* as in Limits and Colimits, **??**.

---

[1]*Further Terminology:* Also called the **wedge sum of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$.

0256   **CONSTRUCTION 6.3.2.1.2 ▶ CONSTRUCTION OF COPRODUCTS OF FAMILIES OF POINTED SETS**

Concretely, the **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$ is the pair $\left(\left(\bigvee_{i \in I} X_i, p_0\right), \{\mathrm{inj}_i\}_{i \in I}\right)$ consisting of:

- *The Colimit.* The pointed set $\left(\bigvee_{i \in I} X_i, p_0\right)$ consisting of:

  - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

  $$\bigvee_{i \in I} X_i \overset{\mathrm{def}}{=} \left(\coprod_{i \in I} X_i\right)/\sim,$$

  where $\sim$ is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

  $$\left(i, x_0^i\right) \sim \left(j, x_0^j\right)$$

  for each $i, j \in I$.

  - *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

  $$\begin{aligned} p_0 &\overset{\mathrm{def}}{=} \left[\left(i, x_0^i\right)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$

  for any $i, j \in I$.

- *The Cocone.* The collection

  $$\left\{\mathrm{inj}_i \colon \left(X_i, x_0^i\right) \to \left(\bigvee_{i \in I} X_i, p_0\right)\right\}_{i \in I}$$

  of morphism of pointed sets given by

  $$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i, x)$$

  for each $x \in X_i$ and each $i \in I$.

PROOF 6.3.2.1.3 ▶ PROOF OF CONSTRUCTION 6.3.2.1.2

We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets$_*$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{ccc}
 & & (C, *) \\
 & \overset{\iota_i}{\nearrow} & \\
(X_i, x_0^i) & \xrightarrow[\text{inj}_i]{} & \left(\bigvee_{i \in I} X_i, p_0\right)
\end{array}
$$

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$
\phi : \left(\bigvee_{i \in I} X_i, p_0\right) \to (C, *)
$$

making the diagram

$$
\begin{array}{ccc}
 & & (C, *) \\
 & \overset{\iota_i}{\nearrow} & \phi \uparrow \, \exists! \\
(X_i, x_0^i) & \xrightarrow[\text{inj}_i]{} & \left(\bigvee_{i \in I} X_i, p_0\right)
\end{array}
$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$
\phi([(i, x)]) = \iota_i(x)
$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$
\phi(p_0) = \iota_i\left(\left[\left(i, x_0^i\right)\right]\right)
$$
$$
= *,
$$

as ι_i is a morphism of pointed sets.

00B3

PROPOSITION 6.3.2.1.4 ▶ PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS

Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

00B4

1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor
$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

PROOF 6.3.2.1.5 ▶ PROOF OF PROPOSITION 6.3.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

00B5 ## 6.3.3   Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6

DEFINITION 6.3.3.1.1 ▶ COPRODUCTS OF POINTED SETS

The **coproduct of** (X, x_0) **and** (Y, y_0)[1] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

---
[1]*Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

0257

CONSTRUCTION 6.3.3.1.2 ▶ CONSTRUCTION OF COPRODUCTS OF POINTED SETS

Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

  - *The Underlying Set.* The set $X \vee Y$ defined by

$$(X \vee Y, p_0) \overset{\mathrm{def}}{=} (X, x_0) \coprod (Y, y_0)$$
$$\cong \left(X \coprod_{\mathsf{pt}} Y, p_0\right)$$
$$\cong (X \coprod Y/\sim, p_0),$$

$$
\begin{array}{ccc}
X \vee Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow {\scriptstyle [y_0]} \\
X & \underset{[x_0]}{\longleftarrow} & \mathsf{pt},
\end{array}
$$

where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

– *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$p_0 \overset{\text{def}}{=} [(0, x_0)]$$
$$= [(1, y_0)].$$

- *The Cocone.* The morphisms of pointed sets

$$\text{inj}_1 \colon (X, x_0) \longrightarrow (X \vee Y, p_0),$$
$$\text{inj}_2 \colon (Y, y_0) \longrightarrow (X \vee Y, p_0),$$

given by

$$\text{inj}_1(x) \overset{\text{def}}{=} [(0, x)],$$
$$\text{inj}_2(y) \overset{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.3.1.3 ▶ PROOF OF CONSTRUCTION 6.3.3.1.2

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \longrightarrow (C, *)$$

making the diagram

$$
\begin{array}{ccc}
& (C, *) & \\
\iota_1 \nearrow & \uparrow \phi \;\, \exists! & \nwarrow \iota_2 \\
(X, x_0) \underset{\mathrm{inj}_1}{\longrightarrow} & (X \vee Y, p_0) & \underset{\mathrm{inj}_2}{\longleftarrow} (Y, y_0)
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\phi \circ \mathrm{inj}_X = \iota_X,
$$
$$
\phi \circ \mathrm{inj}_Y = \iota_Y
$$

via

$$
\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}
$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(p_0) &= \iota_X([(0, x_0)]) \\
&= \iota_Y([(1, y_0)]) \\
&= *,
\end{aligned}
$$

as ι_X and ι_Y are morphisms of pointed sets.

00B7

PROPOSITION 6.3.3.1.4 ▶ PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

00B8

1. *Functoriality.* The assignments

$$
(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)
$$

define functors

$$
\begin{aligned}
X \vee - &: \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
- \vee Y &: \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
-_1 \vee -_2 &: \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}
$$

00B9   2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

00BA   3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, *) \vee (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \vee (\mathrm{pt}, *) \cong (X, x_0),$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

00BB   4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

00BC   5. *Symmetric Monoidality.*   The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

00BD   6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

$$
\begin{array}{ccc}
 & \mathsf{Sets}_* \times \mathsf{Sets}_* & \\
\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \nearrow & \| \quad \Downarrow \nabla & \searrow \vee \\
\mathsf{Sets}_* & \xrightarrow[\mathrm{id}_{\mathsf{Sets}_*}]{\Downarrow} & \mathsf{Sets}_*,
\end{array}
$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

PROOF 6.3.3.1.5 ▶ PROOF OF PROPOSITION 6.3.3.1.4

> **Item 1: Functoriality**
>
> This follows from Limits and Colimits, **??** of **??**.
>
> **Item 2: Associativity**
>
> Omitted.
>
> **Item 3: Unitality**
>
> Omitted.
>
> **Item 4: Commutativity**
>
> Omitted.
>
> **Item 5: Symmetric Monoidality**
>
> Omitted.
>
> **Item 6: The Fold Map**
>
> Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$, we have
>
> $$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad
> \begin{array}{ccc}
> X \vee X & \xrightarrow{\nabla_X} & X \\
> {\scriptstyle f \vee f}\downarrow & & \downarrow{\scriptstyle f} \\
> Y \vee Y & \xrightarrow{\nabla_Y} & Y.
> \end{array}$$
>
> Indeed, we have
>
> $$\begin{aligned}
> [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\
> &= f(x) \\
> &= f(\nabla_X([(i, x)])) \\
> &= [f \circ \nabla_X]([(i, x)])
> \end{aligned}$$
>
> for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. ▨

00BE **6.3.4 Pushouts**

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BF   **DEFINITION 6.3.4.1.1 ▶ PUSHOUTS OF POINTED SETS**

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

0258   **CONSTRUCTION 6.3.4.1.2 ▶ CONSTRUCTION OF PUSHOUTS OF POINTED SETS**

Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0 \right)$, where:

    - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
    - We have $p_0 = [x_0] = [y_0]$.

- *The Cocone.* The morphisms of pointed sets

$$\mathrm{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$
$$\mathrm{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$$

    given by

$$\mathrm{inj}_1(x) \overset{\mathrm{def}}{=} [(0, x)]$$
$$\mathrm{inj}_2(y) \overset{\mathrm{def}}{=} [(1, y)]$$

    for each $x \in X$ and each $y \in Y$.

PROOF 6.3.4.1.3 ▶ PROOF OF ??

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$
$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check

that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g,$$

$$
\begin{array}{ccc}
(X \coprod_Z Y, p_0) & \xleftarrow{\ \mathrm{inj}_2\ } & (Y, y_0) \\
{\scriptstyle \mathrm{inj}_1} \uparrow & & \uparrow {\scriptstyle g} \\
(X, x_0) & \xleftarrow{\quad f \quad} & (Z, z_0).
\end{array}
$$

Indeed, given $z \in Z$, we have

$$
\begin{aligned}
\big[\mathrm{inj}_1 \circ f\big](z) &= \mathrm{inj}_1(f(z)) \\
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \mathrm{inj}_2(g(z)) \\
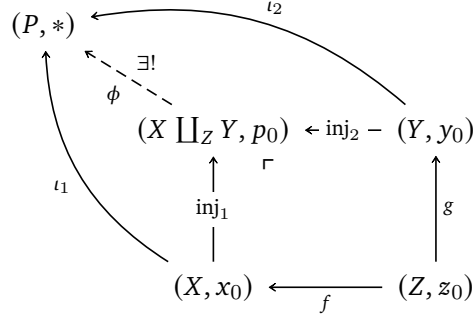&= \big[\mathrm{inj}_2 \circ g\big](z),
\end{aligned}
$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$
\begin{array}{ccc}
(P, *) & \xleftarrow{\ \ \iota_2\ \ } & \\
{\scriptstyle \iota_1} & & \\
& (X \coprod_Z Y, p_0) & \xleftarrow{\ \mathrm{inj}_2\ } (Y, y_0) \\
& {\scriptstyle \mathrm{inj}_1}\uparrow\ \ \ulcorner & \uparrow {\scriptstyle g} \\
& (X, x_0) \xleftarrow{\quad f \quad} & (Z, z_0)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$
\begin{array}{ccc}
(P, *) & \xleftarrow{\ \iota_2\ } & \\
\uparrow \nwarrow_{\exists!} & & \\
\phi & & \\
(X \amalg_Z Y, p_0) & \xleftarrow{\ \mathrm{inj}_2\ } (Y, y_0) \\
\iota_1 \quad \uparrow \mathrm{inj}_1 & \ulcorner & \uparrow g \\
(X, x_0) & \xleftarrow{\ f\ } & (Z, z_0)
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\phi \circ \mathrm{inj}_1 = \iota_1,
$$
$$
\phi \circ \mathrm{inj}_2 = \iota_2
$$

via

$$
\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}
$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)]) \\
&= \iota_1(x_0) \\
&= *,
\end{aligned}
$$

or alternatively

$$
\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)]) \\
&= \iota_2(y_0) \\
&= *,
\end{aligned}
$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

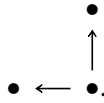**PROPOSITION 6.3.4.1.4 ▶ PROPERTIES OF PUSHOUTS OF POINTED SETS**

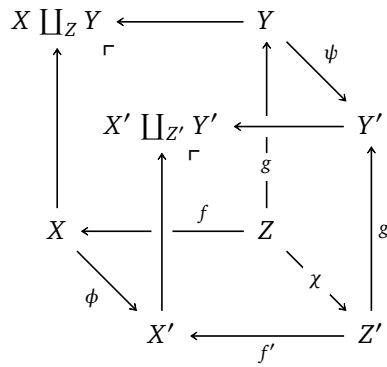Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:

$$
\begin{array}{ccc}
 & & \bullet \\
 & & \uparrow \\
\bullet & \leftarrow & \bullet.
\end{array}
$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets
$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} \left(X' \coprod_{Z'} Y', p_0' \right)$$
given by
$$\xi(p) \overset{\mathrm{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$
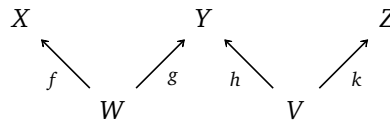
for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
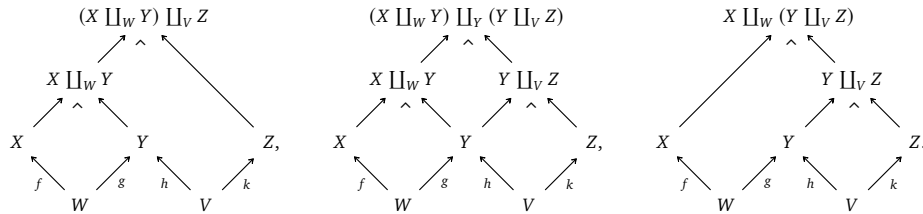\begin{array}{ccc}
X \coprod_Z Y & \longleftarrow & Y \\
\end{array}
$$
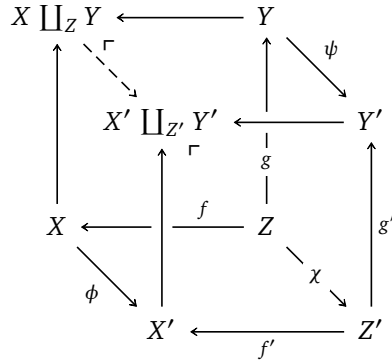
commute.

00BJ   2. *Associativity*. Given a diagram

$$
\begin{array}{ccccc}
X & & Y & & Z \\
& \nwarrow{\scriptstyle f} \quad \nearrow{\scriptstyle g} & & \nwarrow{\scriptstyle h} \quad \nearrow{\scriptstyle k} & \\
& W & & V &
\end{array}
$$

in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

00BK   3. *Unitality*. We have isomorphisms of sets

$$
\begin{array}{ccc}
A & =\!\!=\!\!= & A \\
\uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f} \\
X & =\!\!=\!\!= & X
\end{array}
\qquad
\begin{array}{l}
X \coprod_X A \cong A, \\[4pt]
A \coprod_X X \cong A,
\end{array}
\qquad
\begin{array}{ccc}
A & \xleftarrow{\;f\;} & X \\
\| & & \| \\
X & \xleftarrow{\;f\;} & X.
\end{array}
$$

00BL 4. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
X \coprod_Z Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow {\scriptstyle g} \\
X & \longleftarrow_{f} & Z,
\end{array}
\qquad
X \coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{array}{ccc}
Y \coprod_Z X & \longleftarrow & X \\
\uparrow & \ulcorner & \uparrow {\scriptstyle f} \\
Y & \longleftarrow_{g} & Z.
\end{array}
$$

00BM 5. *Interaction With Coproducts.* We have

$$
X \coprod_{\mathrm{pt}} Y \cong X \vee Y,
\qquad
\begin{array}{ccc}
X \vee Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow {\scriptstyle [y_0]} \\
X & \xrightarrow{[x_0]} & \mathrm{pt}.
\end{array}
$$

00BN 6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

PROOF 6.3.4.1.5 ▶ PROOF OF PROPOSITION 6.3.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.

Item 3: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.

Item 4: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.

Item 5: Interaction With Coproducts

Omitted.

Item 6: Symmetric Monoidality

Omitted.

00BP **6.3.5 Coequalisers**

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ **DEFINITION 6.3.5.1.1 ▶ COEQUALISERS OF POINTED SETS**

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

0259 **CONSTRUCTION 6.3.5.1.2 ▶ CONSTRUCTION OF COEQUALISERS OF POINTED SETS**

The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.

- *The Cocone.* The map

$$\mathrm{coeq}(f, g) \colon Y \twoheadrightarrow (\mathrm{CoEq}(f, g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Construction 4.2.5.1.2.

PROOF 6.3.5.1.3 ▶ PROOF OF CONSTRUCTION 6.3.5.1.2

We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$[\mathrm{coeq}(f, g) \circ f](x) \overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](f(x))$$
$$\overset{\mathrm{def}}{=} [f(x)]$$

$$= [g(x)]$$
$$\overset{\text{def}}{=} [\text{coeq}(f,g)](g(x))$$
$$\overset{\text{def}}{=} [\text{coeq}(f,g) \circ g](x)$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0) \xrightarrow{\text{coeq}(f,g)} (\text{CoEq}(f,g), [y_0])$$
$$\overset{c}{\searrow}$$
$$(C, *)$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Proposition 10.6.2.1.3 that there exists a unique map $\phi\colon \text{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$(X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0) \xrightarrow{\text{coeq}(f,g)} (\text{CoEq}(f,g), [y_0])$$
$$\overset{c}{\searrow} \qquad \phi \downarrow \exists!$$
$$(C, *)$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \text{coeq}(f,g)]([y_0])$$
$$= c([y_0])$$
$$= *,$$

where we have used that c is a morphism of pointed sets.

PROPOSITION 6.3.5.1.4 ▶ PROPERTIES OF COEQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BS   1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f,g) \circ f, \text{coeq}(f,g) \circ h)}_{=\text{CoEq}(\text{coeq}(f,g)\circ g, \text{coeq}(f,g)\circ h)} \cong \text{CoEq}(f,g,h) \cong \underbrace{\text{CoEq}(\text{coeq}(g,h) \circ f, \text{coeq}(g,h) \circ g)}_{=\text{CoEq}(\text{coeq}(g,h)\circ f, \text{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \; \overset{f}{\underset{h}{\underset{-g\rightarrow}{\longrightarrow}}} \; (Y, y_0)$$

in Sets_*.

00BT 2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, f) \cong B.$$

00BU 3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

PROOF 6.3.5.1.5 ▶ PROOF OF PROPOSITION 6.3.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.2.5.1.7.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.2.5.1.7.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.5.1.7.

00BV # 6.4 Constructions With Pointed Sets

00BW ## 6.4.1 Free Pointed Sets

Let X be a set.

00BX **DEFINITION 6.4.1.1.1 ▶ FREE POINTED SETS**

The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[1]

$$X^+ \overset{\mathrm{def}}{=} X \coprod \mathrm{pt}$$
$$\overset{\mathrm{def}}{=} X \coprod \{\star\}.$$

- *The Basepoint.* The element $\star$ of X^+.

---

[1]*Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

00BY **PROPOSITION 6.4.1.1.2 ▶ PROPERTIES OF FREE POINTED SETS**

Let X be a set.

00BZ 1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \stackrel{\mathrm{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.
- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^+ \colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

00C0 2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv \text{忘}\right) \colon \quad \mathsf{Sets} \underset{\text{忘}}{\overset{(-)^+}{\rightleftarrows}} {}^{\perp} \mathsf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*\big((X^+, \star_X), (Y, y_0)\big) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00C1

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+,$$

$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\sim} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

00C2

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, (-)^+_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^+_{X,Y} \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$

$$(-)^+_{\mathbb{1}} \colon S^0 \xrightarrow{\sim} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.4.1.1.3 ▶ PROOF OF PROPOSITION 6.4.1.1.2

Item 1: Functoriality

We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}_X^+(x) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\mathrm{id}_X^+ = \mathrm{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$f\colon X \to Y,$$
$$g\colon Y \to Z,$$

we have

$$
\begin{aligned}
\left[g^+ \circ f^+\right](x) &\overset{\mathrm{def}}{=} g^+\big(f^+(x)\big)\\
&\overset{\mathrm{def}}{=} g^+(f(x))\\
&\overset{\mathrm{def}}{=} g(f(x))\\
&\overset{\mathrm{def}}{=} [g \circ f]^+(x)
\end{aligned}
$$

for each $x \in X$ and

$$
\begin{aligned}
\left[g^+ \circ f^+\right](\star_X) &\overset{\mathrm{def}}{=} g^+\big(f^+(\star_X)\big)\\
&\overset{\mathrm{def}}{=} g^+(\star_Y)\\
&\overset{\mathrm{def}}{=} \star_Z\\
&\overset{\mathrm{def}}{=} [g \circ f]^+(\star_X),
\end{aligned}
$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2: Adjointness

We proceed in a few steps:

- *Map I.* We define a map

$$\Phi_{X,Y}\colon \mathsf{Sets}_*\big(X^+, Y\big) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi\colon \big(X^+, \star_X\big) \to (Y, y_0)$$

to the function

$$\xi^\dagger\colon X \to Y$$

given by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \xi(x)$$

for each $x \in X$.

025A

- *Map II.* We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

  given by sending a function $\xi\colon X \to Y$ to the morphism of pointed sets

$$\xi^\dagger\colon \left(X^+,\star_X\right) \to (Y,y_0)$$

  defined by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

  for each $x \in X^+$.

- *Invertibility I.* Given a morphism of pointed sets

$$\xi\colon \left(X^+,\star_X\right) \to (Y,y_0),$$

  we have

$$\begin{aligned} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\overset{\mathrm{def}}{=} \Psi_{X,Y}\big(\Phi_{X,Y}(\xi)\big) \\ &= \Psi_{X,Y}\big(\xi^\dagger\big) \\ &\overset{\mathrm{def}}{=} \left[\!\left[x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right] \\ &= \left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right] \\ &= \xi \\ &\overset{\mathrm{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}\right](\xi). \end{aligned}$$

  Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)} \ .$$

- *Invertibility II.* Given a map of sets $\xi\colon X \to Y$, we have

$$\begin{aligned} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\overset{\mathrm{def}}{=} \Phi_{X,Y}\big(\Psi_{X,Y}(\xi)\big) \\ &= \Phi_{X,Y}\big(\xi^\dagger\big) \end{aligned}$$

025B

025C

025D

$$= \Phi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\right]\!\right]\right)$$

$$= \left[\!\left[x \mapsto \xi(x)\right]\!\right]$$

$$= \xi$$

$$\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,Y)}\right](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)} .$$

025E
- *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (X', x_0'),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*(X'^{,+}, Y) & \xrightarrow{\ \Phi_{X',Y}\ } & \mathsf{Sets}(X', Y) \\
{\scriptstyle f^*}\Big\downarrow & & \Big\downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*(X^+, Y) & \xrightarrow[\ \Phi_{X,Y}\ ]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, given a morphism of pointed sets $\xi \colon X'^{,+} \to Y$, we have

$$\left[\Phi_{X,Y} \circ f^*\right](\xi) = \Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \xi \circ f$$

$$= \Phi_{X',Y}(\xi) \circ f$$

$$= f^*\left(\Phi_{X',Y}(\xi)\right)$$

$$= f^*\left(\Phi_{X',Y}(\xi)\right)$$

$$= \left[f^* \circ \Phi_{X',Y}\right](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

025F
- *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g \colon (Y, y_0) \to (Y', y_0'),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X, Y) \\
{\scriptstyle g_*} \downarrow & & \downarrow {\scriptstyle g_*} \\
\mathsf{Sets}_*(X^+, Y'), & \xrightarrow[\Phi_{X,Y'}]{} & \mathsf{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger \colon X^+ \to Y,$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*\left(\Phi_{X,Y'}(\xi)\right) \\
&= \left[g_* \circ \Phi_{X,Y'}\right](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

025G
- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^+$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{+,\coprod} : X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \coprod Y)^+$$

  is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

  for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+,\coprod,-1} : (X \coprod Y)^+ \overset{\sim}{\dashrightarrow} X^+ \vee Y^+$$

  given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

  for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} : \mathrm{pt} \overset{\sim}{\dashrightarrow} \varnothing^+$$

  is given by sending $\star_X$ to $\star_\varnothing$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

We construct the strong monoidal structure on $(-)^+$ with respect to $\times$ and $\wedge$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+ : X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+$$

is given by

$$(-)^+_{X,Y}(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)^{+,-1}_{X,Y} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)^{+,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\mathbb{1}}_{X,Y} \colon S^0 \xrightarrow{\sim} \text{pt}^+$$

is given by sending 0 to $\star_{\text{pt}}$ and 1 to $\star$, where $\text{pt}^+ = \{\star, \star_{\text{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. ⬚

6.4.2   Deleting Basepoints

Let (X, x_0) be a pointed set.

DEFINITION 6.4.2.1.1 ▶ SETS WITH DELETED BASEPOINTS

The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\text{def}}{=} X \setminus \{x_0\}.$$

PROPOSITION 6.4.2.1.2 ▶ PROPERTIES OF SETS WITH DELETED BASEPOINTS

Let (X, x_0) be a pointed set.

01QV

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- \colon \mathsf{Sets}_*^{\mathrm{actv}} \to \mathsf{Sets},$$

where:

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*^{\mathrm{actv}})$, we have

$$[(-)^-](X) \overset{\mathrm{def}}{=} X^-,$$

where X^- is the set of Definition 6.4.2.1.1.

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of $\mathsf{Sets}_*^{\mathrm{actv}}$, the image

$$f^- \colon X^- \to Y^-$$

of f by $(-)^-$ is the map defined by

$$f^-(x) \overset{\mathrm{def}}{=} f(x)$$

for each $x \in X^-$.

01QW

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$\big((-)^- \dashv (-)^+\big) \colon \quad \mathsf{Sets}_*^{\mathrm{actv}} \underset{(-)^+}{\overset{(-)^-}{\underset{\perp_{\mathrm{eq}}}{\rightleftarrows}}} \mathsf{Sets},$$

witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*(X, Y^+),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^-)^+ \cong X,$$
$$(Y^+)^- \cong Y,$$

once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

01QX

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure

$$\big((-)^-, (-)^{-,\vee}, (-)_{\mathbb{1}}^{-,\vee}\big) \colon \big(\mathsf{Sets}_*^{\mathrm{actv}}, \vee, \mathrm{pt}\big), \to \big(\mathsf{Sets}, \coprod, \varnothing\big),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^- \coprod Y^- \overset{\sim}{\dashrightarrow} (X \vee Y)^-,$$
$$(-)_{\mathbb{1}}^{-,\vee} \colon \varnothing \overset{\sim}{\dashrightarrow} \mathrm{pt}^-,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

01QY 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^-, (-)^{-,\times}, (-)_{\mathbb{1}}^{-,\times}\right) \colon \left(\mathsf{Sets}_*^{\mathrm{actv}}, \wedge, S^0\right), \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^- \colon X^- \times Y^- \overset{\sim}{\dashrightarrow} (X \wedge Y)^-,$$
$$(-)_{\mathbb{1}}^- \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^0\right)^-,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.4.2.1.3 ▶ PROOF OF PROPOSITION 6.4.2.1.2

Item 1: Functoriality

We claim that $(-)^-$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}_X^-(x) \overset{\mathrm{def}}{=} x$$

  for each $x \in X^-$, so $\mathrm{id}_X^- = \mathrm{id}_{X^-}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$
$$g \colon (Y, y_0) \to (Z, z_0),$$

  we have

$$[g^- \circ f^-](x) \overset{\mathrm{def}}{=} g^-(f^-(x))$$

$$\overset{\text{def}}{=} g^-(f(x))$$
$$\overset{\text{def}}{=} g(f(x))$$
$$\overset{\text{def}}{=} [g \circ f]^-(x)$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2: Adjoint Equivalence

We proceed in a few steps:

025H

1. *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-, Y) \longrightarrow \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)$$

by sending a map $\xi \colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^\dagger \colon X \to Y^+$$

given by

$$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

025J

2. *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+) \longrightarrow \mathsf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi \colon X \to Y^+$ to the map

$$\xi^\dagger \colon X^- \to Y$$

defined by

$$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

025K   3. *Invertibility I.* Given a map of sets $\xi\colon X^- \to Y$, we have

$$\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) \overset{\text{def}}{=} \Psi_{X,Y}\left(\Phi_{X,Y}(\xi)\right)$$

$$\overset{\text{def}}{=} \Psi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]\right)$$

$$= \left[\!\left[x \mapsto \xi(x)\right]\!\right]$$

$$= \xi$$

$$= \left[\mathrm{id}_{\mathsf{Sets}(X^-,Y)}\right](\xi).$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}\,.$$

025L   4. *Invertibility II.* Given a morphism of pointed sets

$$\xi\colon (X,x_0) \to \left(Y^+,\star_Y\right),$$

we have

$$\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) \overset{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right)$$

$$= \Phi_{X,Y}\left(\left[\!\left[x \mapsto \xi(x)\right]\!\right]\right)$$

$$= \left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]$$

$$= \xi$$

$$= \left[\mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\right](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\,.$$

025M   5. *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0) \to \left(X',x_0'\right),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}\!\left(X'^{,-},Y\right) & \xrightarrow{\Phi_{X',Y}} & \mathsf{Sets}^{\mathrm{actv}}_*(X',Y^+) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*(X^-,Y) & \xrightarrow[\Phi_{X,Y}]{} & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+)
\end{array}
$$

commutes. Indeed, given a map of sets $\xi\colon X'\to Y$, we have

$$
\begin{aligned}
\left[\Phi_{X,Y}\circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi\circ f) \\
&= \left[\!\!\left[x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x)\in X'^{,-} \\ \star_Y & \text{if } f(x)=x'_0 \end{cases}\right]\!\!\right] \\
&= f^*\!\left(\left[\!\!\left[x' \mapsto \begin{cases} \xi(x') & \text{if } x'\in X'^{,-} \\ \star_Y & \text{if } x'=x'_0 \end{cases}\right]\!\!\right]\right) \\
&= f^*\!\left(\Phi_{X',Y}(\xi)\right) \\
&= \left[f^*\circ\Phi_{X',Y}\right](\xi).
\end{aligned}
$$

Therefore we have

$$
\Phi_{X,Y}\circ f^* = f^*\circ\Phi_{X',Y},
$$

and the naturality diagram for Φ above indeed commutes.

025N   6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$
g\colon (Y,y_0)\to\left(Y',y'_0\right),
$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(X^-,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+) \\
{\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\
\mathsf{Sets}(X^-,Y') & \xrightarrow[\Phi_{X,Y'}]{} & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y'^{,+})
\end{array}
$$

commutes. Indeed, given a map of sets $\xi \colon X^- \to Y$, we have

$$
\begin{aligned}
\left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= \left[\!\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \right]\!\!\right] \\
&= g_*\left(\left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \right]\!\!\right]\right) \\
&= g_*\left(\Phi_{X,Y'}(\xi)\right) \\
&= \left[g_* \circ \Phi_{X,Y'}\right](\xi).
\end{aligned}
$$

Therefore we have

$$
\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},
$$

and the naturality diagram for Φ above indeed commutes.

025P

7. *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

025Q

8. *Fully Faithfulness of* $(-)^-$. We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$
(-)^-_{X,Y} \colon \mathsf{Sets}^{\mathrm{actv}}_*(X,Y) \overset{\sim}{\dashrightarrow} \mathsf{Sets}(X^-,Y^-).
$$

Indeed, the inverse map

$$
(-)^{-,-1}_{X,Y} \colon \mathsf{Sets}(X^-,Y^-) \overset{\sim}{\dashrightarrow} \mathsf{Sets}^{\mathrm{actv}}_*(X,Y)
$$

is given by sending a map of sets $f \colon X^- \to Y^-$ to the active morphism of pointed sets $f^\dagger \colon X \to Y$ defined by

$$
f^\dagger(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}
$$

for each $x \in X$.

025R
9. *Essential Surjectivity of* $(-)^-$. We need to show that, given an object $X \in \mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}(\mathsf{Sets}_*^{\mathrm{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$
\begin{aligned}
\left(X^+\right)^- &\overset{\mathrm{def}}{=} \left(X \cup \{\star_X\}\right)^- \\
&\overset{\mathrm{def}}{=} \left(X \cup \{\star_X\}\right) \setminus \{\star_X\} \\
&= X,
\end{aligned}
$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

025S
10. *The Functor* $(-)^-$ *Is an Equivalence.* Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Proposition 11.6.7.1.2.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^-$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$
(-)^{-,\vee}_{X,Y} : X^- \coprod Y^- \overset{\sim}{\dashrightarrow} (X \vee Y)^-
$$

is given by

$$
(-)^{-,\vee}_{X,Y}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}
$$

for each $z \in X^- \coprod Y^-$, with inverse

$$
(-)^{-,\vee,-1}_{X,Y} : (X \vee Y)^- \overset{\sim}{\dashrightarrow} X^- \coprod Y^-
$$

given by

$$
(-)^{-,\vee,-1}_{X,Y}(z) \overset{\mathrm{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}
$$

for each $z \in (X \vee Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\vee,\mathbb{1}} \colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^-$$

  is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Products

We construct the strong monoidal structure on $(-)^+$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^- \colon X^- \times Y^- \overset{\sim}{\dashrightarrow} (X \wedge Y)^-$$

  is given by

$$(-)_{X,Y}^-(x,y) = x \wedge y$$

  for each $(x,y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1} \colon (X \wedge Y)^- \overset{\sim}{\dashrightarrow} X^- \times Y^-$$

  given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \overset{\mathrm{def}}{=} (x,y)$$

  for each $x \wedge y \in (X \wedge Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{-,\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^0\right)^-$$

  is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Appendices

A   Other Chapters

References

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[MSE 884460]   Martin Brandenburg. *Why are the category of pointed sets and the category of sets and partial functions "essentially the same"?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/884460 (cit. on p. 7).