

# Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

## Contents

<b>13.1</b>	<b>Moduli Categories of Monoidal Structures .....</b>	<b>2</b>
13.1.1	The Moduli Category of Monoidal Structures on a Category .....	2
13.1.2	The Moduli Category of Braided Monoidal Structures on a Category .....	18
13.1.3	The Moduli Category of Symmetric Monoidal Structures on a Category .....	18
<b>13.2</b>	<b>Moduli Categories of Closed Monoidal Structures .....</b>	<b>18</b>
<b>13.3</b>	<b>Moduli Categories of Refinements of Monoidal Structures ....</b>	<b>18</b>
13.3.1	The Moduli Category of Braided Refinements of a Monoidal Structure .....	18
<b>A</b>	<b>Other Chapters .....</b>	<b>18</b>

## 13.1 Moduli Categories of Monoidal Structures

### 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let  $C$  be a category.

#### DEFINITION 13.1.1.1.1 ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

The **moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow \text{forget} \\ \text{pt} & \xrightarrow{[C]} & \text{Cats.} \end{array}$$

#### REMARK 13.1.1.1.2 ► UNWINDING DEFINITION 13.1.1.1.1, I

In detail, **the moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- *Objects.* The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is  $C$ .
- *Morphisms.* A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  is a strong monoidal functor structure

$$\text{id}_C^{\otimes}: A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$

$$\text{id}_{\mathbb{1}_C}^{\otimes}: \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor  $\text{id}_C: C \rightarrow C$  of  $C$ .

- *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{pt} \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $M$  is defined by

$$\mathrm{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \stackrel{\mathrm{def}}{=} (\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}),$$

where  $(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes})$  is the identity monoidal functor of  $C$  of ??.

- *Composition.* For each  $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} : \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N, P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, N) \rightarrow \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, P)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $(M, N, P)$  is defined by

$$(\mathrm{id}_C^{\otimes, \prime}, \mathrm{id}_{\mathbb{1}|C}^{\otimes, \prime}) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} (\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}) \stackrel{\mathrm{def}}{=} (\mathrm{id}_C^{\otimes, \prime} \circ \mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes, \prime} \circ \mathrm{id}_{\mathbb{1}|C}^{\otimes}).$$

#### REMARK 13.1.1.1.3 ► UNWINDING DEFINITION 13.1.1.1, II

In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  satisfies the following conditions:

1. *Naturality.* For each pair  $f: A \rightarrow X$  and  $g: B \rightarrow Y$  of morphisms of  $C$ , the diagram

$$\begin{array}{ccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \mathrm{id}_{A,B}^{\otimes} \downarrow & & \downarrow \mathrm{id}_{X,Y}^{\otimes} \\ A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y \end{array}$$

commutes.

2. *Monoidality.* For each  $A, B, C \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^{C'} \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^C & & \swarrow \text{id}_{A,B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}$$

commutes.

3. *Left Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & \mathbb{1}_C \boxtimes_C A & \xrightarrow{\text{id}_{\mathbb{1}_C, A}^\otimes} \mathbb{1}_C \otimes_C A \\
 \text{id}_{\mathbb{1}}^\otimes \boxtimes \text{id}_A \nearrow & & \searrow \lambda_A^C \\
 \mathbb{1}_C' \boxtimes_C A & \xrightarrow{\lambda_A^{C'}} & A
 \end{array}$$

commutes.

4. *Right Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & A \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{A, \mathbb{1}_C}^\otimes} A \otimes_C \mathbb{1}_C \\
 \text{id}_A \boxtimes \text{id}_{\mathbb{1}}^\otimes \nearrow & & \searrow \rho_A^C \\
 A \boxtimes_C \mathbb{1}_C' & \xrightarrow{\rho_A^{C'}} & A
 \end{array}$$

commutes.

**PROPOSITION 13.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY**

Let  $\mathcal{C}$  be a category.

1. *Extra Monoidality Conditions.* Let  $(\text{id}_{\mathcal{C}}^{\otimes}, \text{id}_{\mathbb{1}|\mathcal{C}}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{B}_1}(\mathcal{C})$  from  $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$  to  $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$ .

(a) The diagram

$$\begin{array}{ccc}
 (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow{\text{id}_{A,B}^{\otimes} \boxtimes_{\mathcal{C}} \text{id}_C} & (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \\
 \text{id}_{A \boxtimes_{\mathcal{C}} B, C}^{\otimes} \downarrow & & \downarrow \text{id}_{A \otimes_{\mathcal{C}} B, C}^{\otimes} \\
 (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow{\text{id}_{A,B}^{\otimes} \otimes_{\mathcal{C}} \text{id}_C} & (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C
 \end{array}$$

commutes.

(b) The diagram

$$\begin{array}{ccc}
 A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) & \xrightarrow{\text{id}_A \boxtimes_{\mathcal{C}} \text{id}_{B,C}^{\otimes}} & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \\
 \text{id}_{A, B \boxtimes_{\mathcal{C}} C}^{\otimes} \downarrow & & \downarrow \text{id}_{A, B \otimes_{\mathcal{C}} C}^{\otimes} \\
 A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) & \xrightarrow{\text{id}_A \otimes_{\mathcal{C}} \text{id}_{B,C}^{\otimes}} & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
 \end{array}$$

commutes.

2. *Extra Monoidal Unity Constraints.* Let  $(\text{id}_{\mathcal{C}}^{\otimes}, \text{id}_{\mathbb{1}|\mathcal{C}}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{B}_1}(\mathcal{C})$  from  $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$  to  $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$ .

(a) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & & \downarrow \rho_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

(b) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\
 \rho_{\mathbb{1}'_C}^C \downarrow & & \downarrow \lambda_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

(c) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \lambda_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

(d) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \rho_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

3. *Mixed Associators.* Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  be monoidal structures on  $C$  and let

$$\mathrm{id}_{-1, -2}^{\otimes} : -_1 \boxtimes_C -_2 \rightarrow -_1 \otimes_C -_2$$

be a natural transformation.

- (a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} : (A \otimes_C B) \boxtimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \mathrm{id}_{A \otimes_C B, C}^{\otimes} \downarrow & & \downarrow \mathrm{id}_A \otimes_C \mathrm{id}_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C'}} & A \boxtimes_C (B \boxtimes_C C) \\ \mathrm{id}_{A,B}^{\otimes} \boxtimes_C \mathrm{id}_C \downarrow & & \downarrow \mathrm{id}_{A,B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation  $\mathrm{id}^{\otimes}$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.

- (b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \\ \mathrm{id}_{A,B}^{\otimes} \otimes_C \mathrm{id}_C \downarrow & & \downarrow \mathrm{id}_{A,B \otimes_C C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,'}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \boxtimes_C \text{id}_{B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} & A \boxtimes_C (B \otimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes, \otimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C) \\
 \text{id}_{A, B}^\otimes \otimes \text{id}_C \downarrow & & \downarrow \text{id}_A \otimes_C \text{id}_{B, C}^\otimes \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,'}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A, B \boxtimes_C C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.



**PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4****Item 1: Extra Monoidality Conditions**

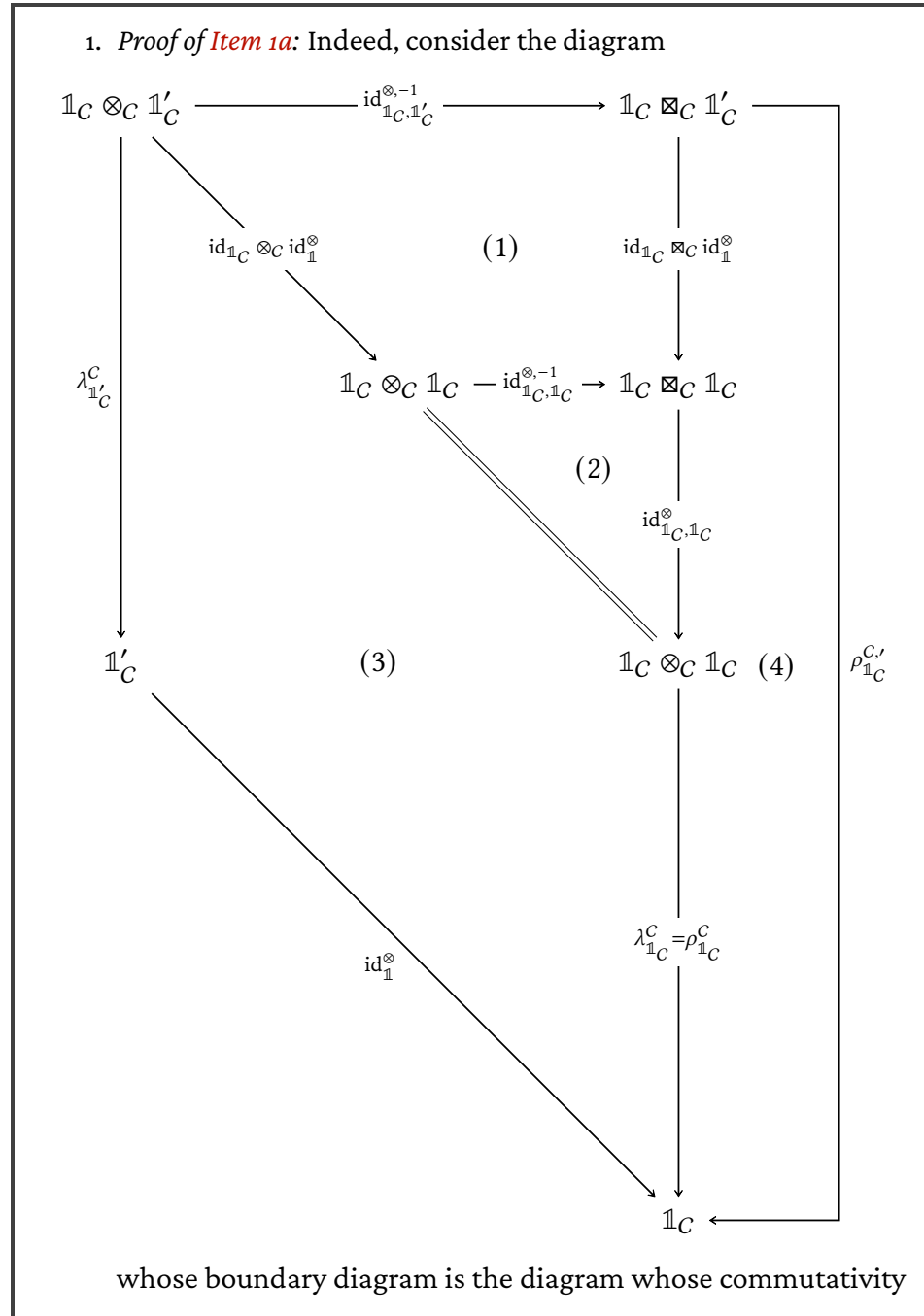
We claim that **Items 1a** and **1b** are indeed true:

1. *Proof of **Item 1a***: This follows from the naturality of  $\mathrm{id}^{\otimes}$  with respect to the morphisms  $\mathrm{id}_{A,B}^{\otimes}$  and  $\mathrm{id}_C$ .
2. *Proof of **Item 1b***: This follows from the naturality of  $\mathrm{id}^{\otimes}$  with respect to the morphisms  $\mathrm{id}_A$  and  $\mathrm{id}_{B,C}^{\otimes}$ .

This finishes the proof.

**Item 2: Extra Monoidal Unity Constraints**

We claim that **Items 2a** and **2b** are indeed true:

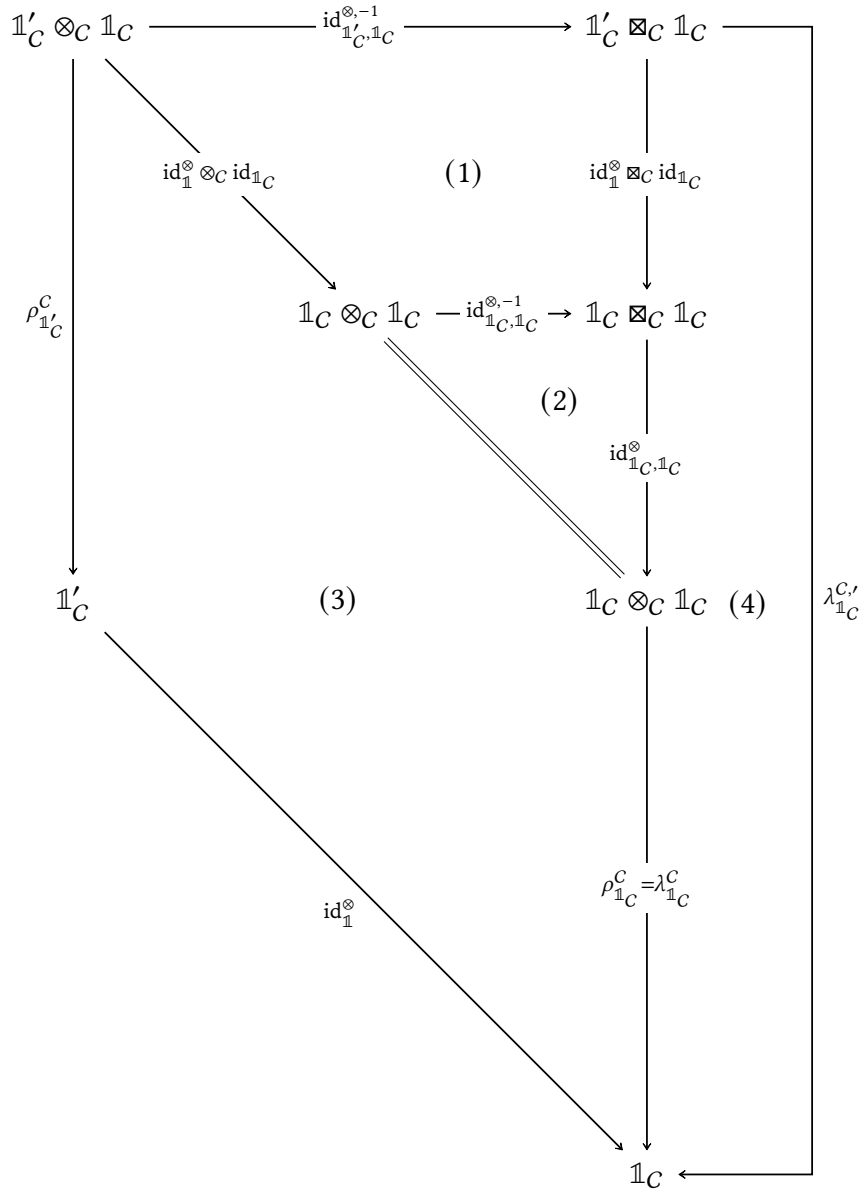


we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of  $(\text{id}_C, \text{id}_C^{\otimes}, \text{id}_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram



we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of  $(\text{id}_C, \text{id}_C^{\otimes}, \text{id}_{C|\mathbb{1}}^{\otimes})$ ;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \rho_{\mathbb{1}'_C}^C \downarrow & (1) & \lambda_{\mathbb{1}_C}^{C, ' } \downarrow & (\dagger) & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C.
 \end{array}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1b**;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 & & \lambda_{\mathbb{1}_C}^{C, ' } \downarrow & (\dagger) & \downarrow \rho_{\mathbb{1}'_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes. But since  $\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \downarrow \lambda_{\mathbb{1}'_C}^C & & \downarrow \rho_{\mathbb{1}_C}^{C, \prime} & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}
 \quad (1) \qquad (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1a**;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 & & \downarrow \rho_{\mathbb{1}_C}^{C, \prime} & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}
 \quad (\dagger)$$

commutes. But since  $\text{id}_{\mathbb{1}}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

#### Item 3: Mixed Associators

We claim that **Items 3a to 3c** are indeed true:

1. *Proof of Item 3a:* We may partition the monoidality diagram for  $\text{id}^\otimes$  of Item 2 of Remark 13.1.1.1.3 as follows:

$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C & \downarrow & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) & \downarrow \text{id}_{A \boxtimes_C B, C}^\otimes & (2) & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 & \swarrow \text{id}_{A,B}^\otimes \otimes_C \text{id}_C & \searrow \alpha_{A,B,C}^\otimes & & \\
 & (A \otimes_C B) \otimes_C C & (3) & & A \boxtimes_C (B \otimes_C C) \\
 & \searrow \alpha_{A,B,C}^{C'} & & \swarrow \text{id}_{A,B \otimes_C C}^\otimes & \\
 & A \otimes_C (B \otimes_C C) & & & 
 \end{array}$$

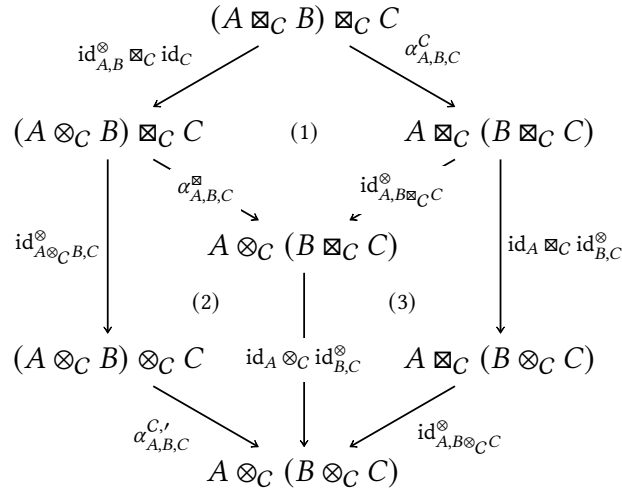
Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. *Proof of Item 3b:* We may partition the monoidality diagram for  $\text{id}^\otimes$

of **Item 2** of **Remark 13.1.1.3** as follows:



Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by **Item 1b** of **Item 1**.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.


3. *Proof of **Item 3c***: We may partition the monoidality diagram for  $\text{id}^\otimes$



of **Item 2** of **Remark 13.1.1.1.3** as follows:

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^C \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \searrow \alpha_{A,B,C}^{\boxtimes, \otimes} & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^{C, \prime} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}
 \quad (1) \quad (2)$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

This finishes the proof. 

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category**
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category**
- 13.2 Moduli Categories of Closed Monoidal Structures**
- 13.3 Moduli Categories of Refinements of Monoidal Structures**
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure**

## Appendices

### A Other Chapters

#### Preliminaries

- 1. [Introduction](#)
- 2. [A Guide to the Literature](#)

#### Sets

- 3. [Sets](#)
- 4. [Constructions With Sets](#)
- 5. [Monoidal Structures on the Category of Sets](#)
- 6. [Pointed Sets](#)

- 7. [Tensor Products of Pointed Sets](#)

#### Relations

- 8. [Relations](#)
- 9. [Constructions With Relations](#)
- 10. [Conditions on Relations](#)

#### Categories

- 11. [Categories](#)
- 12. [Presheaves and the Yoneda Lemma](#)

#### Monoidal Categories

13. **Constructions With Monoidal Categories**

**Bicategories**

**Extra Part**

14. **Types of Morphisms in Bicate-**      15. **Notes**