Conditions on Relations

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00TJ   This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

Contents

02D0   # 10.1   Functional and Total Relations

00JC   ## 10.1.1   Functional Relations

Let A and B be sets.

00JD   **Definition 10.1.1.1.1.** A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

00JE   **Proposition 10.1.1.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JF   1. *Characterisations.* The following conditions are equivalent:

00JG   (a) The relation R is functional.

00JH   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. Item 1, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* Let $(b, b') \in B \times B$. We need to show that
$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{t,f\}} \chi_B(b, b'),$$
i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b $\implies$ Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:
  - Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  - Since $R \diamond R^{\dagger} \subset \chi_B$, we have
  $$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{t,f\}} \chi_B(b, b'),$$
  and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^{\dagger}\right](b, b') = \text{true}$, and thus $\chi_B(b, b') = \text{true}$ as well, i.e. $b = b'$.

This finishes the proof. $\qquad\square$

00JJ   **10.1.2   Total Relations**

Let A and B be sets.

00JK   **Definition 10.1.2.1.1.** A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

00JL   **Proposition 10.1.2.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JM   1. *Characterisations.* The following conditions are equivalent:

00JN   (a) The relation R is total.

00JP   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. Item 1, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b*: We have to show that, for each $(a, a') \in A$, we have

$$\chi_A(a, a') \preceq_{\{t,f\}} \left[R^\dagger \diamond R\right](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b $\implies$ Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. $\square$

00TK   **10.2   Reflexive Relations**

00TL   **10.2.1   Foundations**

Let A be a set.

00TM   **Definition 10.2.1.1.1.** A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

00TN   **Remark 10.2.1.1.2.** In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

00TP   **Definition 10.2.1.1.3.** Let A be a set.

00TQ   1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.

00TR   2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

00TS   **Proposition 10.2.1.1.4.** Let R and S be relations on A.

00TT   1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.

00TU   2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. Item 1, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear.   □

00TV   10.2.2   The Reflexive Closure of a Relation

Let R be a relation on A.

00TW   **Definition 10.2.2.1.1.** The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[2] satisfying the following universal property:[3]

(★)   Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

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[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

[2] *Further Notation:* Also written R^{refl}.

[3] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

00TX **Construction 10.2.2.1.2.** Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $\left(\mathbf{Rel}(A,A), \chi_A\right)^4$, being given by

$$R^{\mathrm{refl}} \stackrel{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A$$
$$= R \cup \Delta_A$$
$$= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}.$$

Proof. Clear.   □

00TY **Proposition 10.2.2.1.3.** Let R be a relation on A.

00TZ 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{refl}} \dashv \text{忘}\right): \quad \mathbf{Rel}(A,A) \overset{(-)^{\mathrm{refl}}}{\underset{\text{忘}}{\overset{\perp}{\rightleftarrows}}} \mathbf{Rel}^{\mathrm{refl}}(A,A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\left(R^{\mathrm{refl}}, S\right) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{refl}}(A,A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00U0 2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

00U1 3. *Idempotency.* We have

$$\left(R^{\mathrm{refl}}\right)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

00U2 4. *Interaction With Inverses.* We have

$$\left(R^\dagger\right)^{\mathrm{refl}} = \left(R^{\mathrm{refl}}\right)^\dagger,$$

$$\begin{array}{ccc}
\mathrm{Rel}(A,A) & \xrightarrow{(-)^{\mathrm{refl}}} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^\dagger}\downarrow & & \downarrow{\scriptstyle (-)^\dagger} \\
\mathrm{Rel}(A,A) & \xrightarrow[(-)^{\mathrm{refl}}]{} & \mathrm{Rel}(A,A).
\end{array}$$

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[4]Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$.

00U3   5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}},$$

$$\begin{array}{ccc}
\mathrm{Rel}(A, A) \times \mathrm{Rel}(A, A) & \xrightarrow{\diamond} & \mathrm{Rel}(A, A) \\
{\scriptstyle (-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathrm{refl}}} \\
\mathrm{Rel}(A, A) \times \mathrm{Rel}(A, A) & \xrightarrow[\diamond]{} & \mathrm{Rel}(A, A).
\end{array}$$

Proof. Item 1, *Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 10.2.2.1.1.
Item 2, *The Reflexive Closure of a Reflexive Relation*: Clear.
Item 3, *Idempotency*: This follows from Item 2.
Item 4, *Interaction With Inverses*: Clear.
Item 5, *Interaction With Composition*: This follows from Item 2 of Definition 10.2.1.1.4.

$\square$

00U4   10.3   Symmetric Relations

00U5   10.3.1   Foundations

Let A be a set.

00U6   **Definition 10.3.1.1.1.** A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

00U7   **Remark 10.3.1.1.2.** In detail, a relation R is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

00U8   **Definition 10.3.1.1.3.** Let A be a set.

00U9   1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

00UA   2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UB   **Proposition 10.3.1.1.4.** Let R and S be relations on A.

00UC   1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.

00UD   2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. Item 1, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear.

$\square$

00UE   ## 10.3.2   The Symmetric Closure of a Relation

Let R be a relation on A.

00UF   **Definition 10.3.2.1.1.** The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5] satisfying the following universal property:[6]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

00UG   **Construction 10.3.2.1.2.** Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$R^{\mathrm{symm}} \overset{\mathrm{def}}{=} R \cup R^{\dagger}$$
$$= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}.$$

Proof. Clear.   □

00UH   **Proposition 10.3.2.1.3.** Let R be a relation on A.

00UJ   1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{symm}} \dashv 忘\right): \quad \mathbf{Rel}(A, A) \overset{(-)^{\mathrm{symm}}}{\underset{忘}{\rightleftarrows}} \bot \quad \mathbf{Rel}^{\mathrm{symm}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{symm}}(A, A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, A))$.

00UK   2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

00UL   3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

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[5] *Further Notation:* Also written R^{symm}.
[6] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

00UM   4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger},$$

$$
\begin{array}{ccc}
\mathrm{Rel}(A,A) & \xrightarrow{(-)^{\mathrm{symm}}} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^{\dagger}}\big\downarrow & & \big\downarrow{\scriptstyle (-)^{\dagger}} \\
\mathrm{Rel}(A,A) & \xrightarrow[(-)^{\mathrm{symm}}]{} & \mathrm{Rel}(A,A).
\end{array}
$$

00UN   5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}},$$

$$
\begin{array}{ccc}
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) & \xrightarrow{\diamond} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}}\big\downarrow & & \big\downarrow{\scriptstyle (-)^{\mathrm{symm}}} \\
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) & \xrightarrow[\diamond]{} & \mathrm{Rel}(A,A).
\end{array}
$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 10.3.2.1.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.3.1.1.4.
□

00UP 10.4 Transitive Relations

00UQ 10.4.1 Foundations

Let A be a set.

00UR **Definition 10.4.1.1.1.** A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A,A)), \diamond)$.

- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

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[7]Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

00US   **Remark 10.4.1.1.2.** In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00UT   **Definition 10.4.1.1.3.** Let A be a set.

00UU   1. The **set of transitive relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.

00UV   2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UW   **Proposition 10.4.1.1.4.** Let R and S be relations on A.

00UX   1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.

00UY   2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ **may fail to be transitive**.

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: See [MSE 2096272].[8]   □

00UZ   10.4.2   The Transitive Closure of a Relation

Let R be a relation on A.

---

[8]*Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
    - There is some $b \in A$ such that:
        * $a \sim_R b$;
        * $b \sim_S c$;
    - There is some $d \in A$ such that:
        * $c \sim_R d$;
        * $d \sim_S e$.

00V0 **Definition 10.4.2.1.1.** The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[9] satisfying the following universal property:[10]

(⋆) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

00V1 **Construction 10.4.2.1.2.** Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[11], being given by

$$
\begin{aligned}
R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\
&\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\
&\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \,\middle|\,
\begin{array}{l}
\text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\
\text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b
\end{array}
\right\}.
\end{aligned}
$$

Proof. Clear. □

00V2 **Proposition 10.4.2.1.3.** Let R be a relation on A.

00V3 1. *Adjointness.* We have an adjunction

$$
\left((-)^{\mathrm{trans}} \dashv \text{忘} \right): \quad \mathbf{Rel}(A, A) \underset{\text{忘}}{\overset{(-)^{\mathrm{trans}}}{\rightleftarrows}} \bot \; \mathbf{Rel}^{\mathrm{trans}}(A, A),
$$

witnessed by a bijection of sets

$$
\mathbf{Rel}^{\mathrm{trans}}\left(R^{\mathrm{trans}}, S \right) \cong \mathbf{Rel}(R, S),
$$

natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{trans}}(A, A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

00V4 2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

00V5 3. *Idempotency.* We have

$$
\left(R^{\mathrm{trans}} \right)^{\mathrm{trans}} = R^{\mathrm{trans}}.
$$

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[9] *Further Notation:* Also written R^{trans}.

[10] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

00V6   4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\text{trans}} = \left(R^{\text{trans}}\right)^{\dagger},$$

$$
\begin{array}{ccc}
\mathrm{Rel}(A,A) & \xrightarrow{(-)^{\text{trans}}} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^{\dagger}}\downarrow & & \downarrow{\scriptstyle (-)^{\dagger}} \\
\mathrm{Rel}(A,A) & \xrightarrow[(-)^{\text{trans}}]{} & \mathrm{Rel}(A,A).
\end{array}
$$

00V7   5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{trans}} \overset{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}},$$

$$
\begin{array}{ccc}
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) & \xrightarrow{\diamond} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^{\text{trans}} \times (-)^{\text{trans}}}\downarrow & \textcolor{red}{\times} & \downarrow{\scriptstyle (-)^{\text{trans}}} \\
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) & \xrightarrow[\diamond]{} & \mathrm{Rel}(A,A).
\end{array}
$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 10.4.2.1.1.
Item 2, The Transitive Closure of a Transitive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: We have

$$
\begin{aligned}
\left(R^{\dagger}\right)^{\text{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} \\
&= \bigcup_{n=1}^{\infty} \left(R^{\diamond n}\right)^{\dagger} \\
&= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} \\
&= \left(R^{\text{trans}}\right)^{\dagger},
\end{aligned}
$$

where we have used, respectively:

- Definition 10.4.2.1.2.

- Constructions With Relations, **??** of **??**.

- Constructions With Relations, **??** of Definition 9.2.3.1.2.

- Definition 10.4.2.1.2.

This finishes the proof.

Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.4.1.1.4.

<div style="text-align: right;">□</div>

10.5 Equivalence Relations
00V8

10.5.1 Foundations
00V9

Let A be a set.

00VA **Definition 10.5.1.1.1.** A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

00VB **Example 10.5.1.1.2.** The **kernel of a function** $f\colon A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

00VC **Definition 10.5.1.1.3.** Let A and B be sets.

00VD 1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

00VE 2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

10.5.2 The Equivalence Closure of a Relation
00VF

Let R be a relation on A.

00VG **Definition 10.5.2.1.1.** The **equivalence closure**[14] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[15] satisfying the following universal property:[16]

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[12] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Constructions With Relations, **??** of **??**.

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written R^{eq}.

[16] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

$(\star)$ Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

00VH **Construction 10.5.2.1.2.** Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$R^{\mathrm{eq}} \overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}} \right)^{\mathrm{symm}} \right)^{\mathrm{trans}}$$

$$= \left((R^{\mathrm{symm}})^{\mathrm{trans}} \right)^{\mathrm{refl}}$$

$$= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\[4pt] \quad 1. \text{ The following conditions are satisfied:} \\[4pt] \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\qquad \text{for each } 1 \le i \le n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\[4pt] \quad 2. \text{ We have } a = b. \end{array} \right\}.$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

00VJ   1. The symmetric closure of a reflexive relation is still reflexive.

00VK   2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. $\qquad\square$

00VL **Proposition 10.5.2.1.3.** Let R be a relation on A.

00VM   1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv \text{忘} \right): \quad \mathbf{Rel}(A,B) \underset{\text{忘}}{\overset{(-)^{\mathrm{eq}}}{\rightleftarrows}} \bot \; \mathbf{Rel}^{\mathrm{eq}}(A,B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

00VN   2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

00VP   3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof. *Item 1*, *Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 10.5.2.1.1.
Item 2, *The Equivalence Closure of an Equivalence Relation*: Clear.
Item 3, *Idempotency*: This follows from Item 2.   □

00VQ 10.6   Quotients by Equivalence Relations

00VR 10.6.1   Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

02B1   **Definition 10.6.1.1.1.** The **equivalence class associated to** a is the set $[a]$ defined by

$$
\begin{aligned}
[a] &\overset{\mathrm{def}}{=} \{x \in X \mid x \sim_R a\} \\
&= \{x \in X \mid a \sim_R x\}. \qquad \text{(since R is symmetric)}
\end{aligned}
$$

02B2 10.6.2   Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

02B3   **Definition 10.6.2.1.1.** The **quotient of** X **by** R is the set $X/\!\sim_R$ defined by

$$X/\!\sim_R \overset{\mathrm{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00VV   **Remark 10.6.2.1.2.** The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.

- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

$$[a] \stackrel{\mathrm{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \stackrel{\mathrm{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[17]

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \not\sim_R b$, and equal otherwise.

00VW   **Proposition 10.6.2.1.3.** Let $f : X \to Y$ be a function and let R be a relation on X.

02B4   1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\mathrm{eq}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X \right),$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

02B5   2. *As a Pushout.* We have an isomorphism of sets[18]

$$X/\sim_R^{\mathrm{eq}} \cong X \coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X,$$

$$
\begin{array}{ccc}
X/\sim_R^{\mathrm{eq}} & \longleftarrow & X \\
\uparrow & \ulcorner & \uparrow \\
X & \longleftarrow & \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2).
\end{array}
$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

---

[17]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

[18]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/\sim_R^{\mathrm{eq}}} X,$$

$$
\begin{array}{ccc}
\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & X/\sim_R^{\mathrm{eq}}.
\end{array}
$$

02B6   3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]
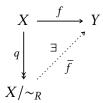
$$X/{\sim}_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

00W0   4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

02B7   (a) There exists a map

$$\bar{f}\colon X/{\sim}_R \to Y$$

making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & Y \\
{\scriptstyle q}\downarrow & {\scriptstyle \exists}\ \ {\nearrow}\ {\scriptstyle \bar{f}} & \\
X/{\sim}_R & &
\end{array}
$$

commute.

02B8   (b) We have $R \subset \mathrm{Ker}(f)$.

02B9   (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

---

[19]*Further Terminology:* The set $X/{\sim}_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{CoIm}(f)$.

[20]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image
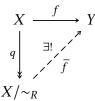
$$
\begin{aligned}
\mathrm{Ker}(f)\colon X &\nrightarrow X, \\
\mathrm{Im}(f) &\subset Y
\end{aligned}
$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$
(\mathrm{Gr}(f) \dashv f^{-1})\colon \quad A \underset{f^{-1}}{\overset{\mathrm{Gr}(f)}{\rightleftarrows}} B
$$

of Constructions With Relations, **??** of **??**.

00W1   5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\bar{f}$ is the *unique* map making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle q}\big\downarrow & {\scriptstyle \exists!} \nearrow & \\
X/\!\sim_R & {}^{\bar{f}} &
\end{array}
$$

commute.

00W2   6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$
\mathrm{Hom}_{\mathsf{Sets}}(X/\!\sim_R, Y) \cong \mathrm{Hom}_{\mathsf{Sets}}^{R}(X, Y),
$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \bar{f}$ of Items 4 and 5, where $\mathrm{Hom}_{\mathsf{Sets}}^{R}(X, Y)$ is the set defined by

$$
\mathrm{Hom}_{\mathsf{Sets}}^{R}(X, Y) \stackrel{\mathrm{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \,\middle|\, \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.
$$

02BA   7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BB   (a) The map $\bar{f}$ is an injection.

02BC   (b) We have $R = \mathrm{Ker}(f)$.

02BD   (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

02BE   8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BF   (a) The map $f\colon X \to Y$ is surjective.

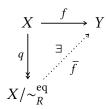02BG   (b) The map $\bar{f}\colon X/\!\sim_R \to Y$ is surjective.

02BH   9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

02BJ   (a) The map f satisfies the equivalent conditions of Item 4:

  • There exists a map

$$\overline{f} \colon X/\!\sim_R^{\mathrm{eq}} \to Y$$

  making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & Y \\
{\scriptstyle q}\big\downarrow & {\scriptstyle \exists}\;\;\raisebox{-0.5em}{$\overline{f}$} & \\
X/\!\sim_R^{\mathrm{eq}} & &
\end{array}
$$

  commute.
  • For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

02BK   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. Item 1, *As a Coequaliser*: Omitted.
Item 2, *As a Pushout*: Omitted.
Item 3, *The First Isomorphism Theorem for Sets*: Clear.
Item 4, *Descending Functions to Quotient Sets, I*: See [Pro25c].
Item 5, *Descending Functions to Quotient Sets, II*: See [Pro25d].
Item 6, *Descending Functions to Quotient Sets, III*: This follows from Items 5 and 6.
Item 7, *Descending Functions to Quotient Sets, IV*: See [Pro25b].
Item 8, *Descending Functions to Quotient Sets, V*: See [Pro25a].
Item 9, *Descending Functions to Quotient Sets, VI*: The implication Item 8a $\implies$ Item 8b is clear.
   Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

(★) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

  – The following conditions are satisfied:

* We have $x \sim_R x_1$ or $x_1 \sim_R x$;
* We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
* We have $y \sim_R x_n$ or $x_n \sim_R y$;
 – We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$f(x) = f(x_1),$$
$$f(x_1) = f(x_2),$$
$$\vdots$$
$$f(x_{n-1}) = f(x_n),$$
$$f(x_n) = f(y),$$

and $f(x) = f(y)$, as we wanted to show. □

Appendices

A   Other Chapters

References

[MSE 2096272] Akiva Weinberger. *Is composition of two transitive relations transitive? If not, can you give me a counterexample?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2096272 (cit. on p. 9).

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