Types of Morphisms in Bicategories

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In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2).*
   There is a large number of different notions capturing the idea of a
   "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseu-domonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

Contents

14.1   Monomorphisms in Bicategories

14.1.1   Representably Faithful Morphisms

Let C be a bicategory.

Definition 14.1.1.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

Remark 14.1.1.1.2. In detail, f is representably faithful if, for all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha\Downarrow\Downarrow\beta}{\rightrightarrows}}} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 14.1.1.1.3. Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Categories, Item 2 of Definition 11.6.1.1.2.

2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, **??** of **??**.

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[1]*Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of

14.1.2   Representably Full Morphisms

Let C be a bicategory.

Definition 14.1.2.1.1. A 1-morphism $f\colon A \to B$ of C is **representably full**[2] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is full.

Remark 14.1.2.1.2. In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rrightarrow}} B \quad \beta\Downarrow$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\Rrightarrow}} A \quad \alpha\Downarrow$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Rrightarrow}} A \xrightarrow{\ f\ } B \quad\alpha\Downarrow \qquad = \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rrightarrow}} B \quad\beta\Downarrow$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 14.1.2.1.3. Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Categories, **??** of Definition 11.6.2.1.2.

2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Relations, **??** of **??**.

Definition 14.1.1.1.3.
  [2]*Further Terminology:*   Also called simply a **full morphism**, based on Item 1 of

14.1.3   Representably Fully Faithful Morphisms

Let C be a bicategory.

Definition 14.1.3.1.1. A 1-morphism $f\colon A \to B$ of C is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1) and representably full (Definition 14.1.2.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

   $$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

   given by postcomposition by f is fully faithful.

Remark 14.1.3.1.2. In detail, f is representably fully faithful if the conditions in Definition 14.1.1.1.2 and Definition 14.1.2.1.2 hold:

1. For all diagrams in C of the form

   $$X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\rightrightarrows}}} {}^{\alpha\Vert\Vert\beta} A \xrightarrow{f} B,$$

   if we have

   $$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

   $$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f\circ\psi}{\overset{f\circ\phi}{\rightrightarrows}} {}^{\beta\Downarrow} B$$

   of C, there exists a 2-morphism

   $$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\rightrightarrows}} {}^{\alpha\Downarrow} A$$

Definition 14.1.2.1.3.

   [3]*Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Definition 14.1.3.1.3.

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Downarrow \alpha}} A \overset{f}{\longrightarrow} B \quad = \quad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Downarrow \beta}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 14.1.3.1.3. Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 6 of Definition 11.6.3.1.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, **??** of **??**) with the representably full morphisms in **Rel**, which are characterised in Relations, **??** of **??**.

14.1.4   Morphisms Representably Faithful on Cores

Let C be a bicategory.

Definition 14.1.4.1.1. A 1-morphism $f \colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

Remark 14.1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$X \overset{\phi}{\underset{\psi}{\Downarrow \alpha \Downarrow \beta}} A \overset{f}{\longrightarrow} B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

14.1.5   Morphisms Representably Full on Cores

Let C be a bicategory.

Definition 14.1.5.1.1. A 1-morphism $f \colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathsf{Core}(\mathrm{Hom}_C(X, A)) \to \mathsf{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is full.

Remark 14.1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.6   Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

Definition 14.1.6.1.1. A 1-morphism $f \colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).

2. For each $X \in \mathsf{Obj}(C)$, the functor

$$f_* \colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is fully faithful.

Remark 14.1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in Definition 14.1.4.1.2 and Definition 14.1.5.1.2 hold:

1. For all diagrams in C of the form

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha \Vert \Vert \beta} A \xrightarrow{f} B,$$

   if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathsf{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Longrightarrow}}_{\beta \Downarrow} B$$

   of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha \Downarrow} A$$

   of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha \Downarrow} A \xrightarrow{f} B \quad = \quad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Longrightarrow}}_{\beta \Downarrow} B$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.7   Representably Essentially Injective Morphisms

Let C be a bicategory.

Definition 14.1.7.1.1. A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

Remark 14.1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

14.1.8   Representably Conservative Morphisms

Let C be a bicategory.

Definition 14.1.8.1.1. A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

Remark 14.1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\Downarrow\alpha}{\rightrightarrows}}} A$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f\circ\psi}{\overset{f\circ\phi}{\underset{\mathrm{id}_f \star\alpha}{\rightrightarrows}}} B$$

is a 2-isomorphism, then so is α.

14.1.9   Strict Monomorphisms

Let C be a bicategory.

Definition 14.1.9.1.1. A 1-morphism $f\colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$

is injective.

Remark 14.1.9.1.2. In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 14.1.9.1.3. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Definition 11.7.2.1.2.

2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, **??**.

14.1.10   Pseudomonic Morphisms

Let C be a bicategory.

Definition 14.1.10.1.1. A 1-morphism $f\colon A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 14.1.10.1.2. In detail, a 1-morphism $f\colon A \to B$ of C is pseudomonic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$X \; \overset{\phi}{\underset{\psi}{\alpha \| \| \beta}} \; A \xrightarrow{\; f \;} B,$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \; \overset{f \circ \phi}{\underset{f \circ \psi}{\beta \|}} \; B$$

   of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \; \overset{\phi}{\underset{\psi}{\alpha \|}} \; A$$

   of C such that we have an equality

$$X \; \overset{\phi}{\underset{\psi}{\alpha \|}} \; A \xrightarrow{\; f \;} B \quad = \quad X \; \overset{f \circ \phi}{\underset{f \circ \psi}{\beta \|}} \; B$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 14.1.10.1.3. Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudomonic.
   (b) The morphism f is representably full on cores and representably faithful.

(c) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \stackrel{\leftrightarrow}{\times}_B A,$$

$$
\begin{array}{ccc}
A & \xrightarrow{\text{id}_A} & A \\
{\scriptstyle\text{id}_A}\downarrow & \nearrow & \downarrow{\scriptstyle F} \\
A & \xrightarrow{F} & B
\end{array}
$$

in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \stackrel{\leftrightarrow}{\times}_{\mathbb{1}\pitchfork F} B,$$

$$
\begin{array}{ccc}
A & \longrightarrow & \mathbb{1}\pitchfork A \\
{\scriptstyle F}\downarrow & \nearrow & \downarrow{\scriptstyle \mathbb{1}\pitchfork F} \\
B & \longrightarrow & \mathbb{1}\pitchfork B
\end{array}
$$

in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Interaction With Cotensors: Omitted. □

14.2 Epimorphisms in Bicategories

14.2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

Definition 14.2.1.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*\colon \text{Hom}_C(B,X) \to \text{Hom}_C(A,X)$$

given by precomposition by f is faithful.

Remark 14.2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\underset{\Downarrow}{\Rrightarrow}}} X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 14.2.1.1.3. Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 5 of Definition 11.6.1.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Relations, **??** of **??**.

14.2.2   Corepresentably Full Morphisms

Let C be a bicategory.

Definition 14.2.2.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is full.

Remark 14.2.2.1.2. In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \overset{\phi \circ f}{\underset{\psi \circ f}{\Rightarrow}} X \quad \beta \Downarrow$$

of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad B \overset{\phi}{\underset{\psi}{\Rightarrow}} X \quad \alpha \Downarrow$$

of C such that we have an equality

$$A \overset{f}{\longrightarrow} B \overset{\phi}{\underset{\psi}{\Rightarrow}} X \quad \alpha \Downarrow \qquad = \qquad A \overset{\phi \circ f}{\underset{\psi \circ f}{\Rightarrow}} X \quad \beta \Downarrow$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f\,.$$

Example 14.2.2.1.3. Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats$_2$. The corepresentably full morphisms in Cats$_2$ are characterised in Categories, Item 7 of Definition 11.6.2.1.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in Relations, **??** of **??**.

14.2.3   Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 14.2.3.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is fully faithful.

Remark 14.2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Definition 14.2.1.1.2 and Definition 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

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[4]*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

2. For each $X \in \text{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \quad A \overset{\phi \circ f}{\underset{\psi \circ f}{\beta \Downarrow}} X$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \quad B \overset{\phi}{\underset{\psi}{\alpha \Downarrow}} X$$

of C such that we have an equality

$$A \overset{f}{\longrightarrow} B \overset{\phi}{\underset{\psi}{\alpha \Downarrow}} X \quad = \quad A \overset{\phi \circ f}{\underset{\psi \circ f}{\beta \Downarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

Example 14.2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 10 of Definition 11.6.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, **??** of **??**) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, **??** of **??**.

14.2.4   Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

Definition 14.2.4.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \text{Obj}(C)$, the functor

$$f^*\colon \text{Core}(\text{Hom}_C(B, X)) \to \text{Core}(\text{Hom}_C(A, X))$$

given by precomposition by f is faithful.

Remark 14.2.4.1.2. In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

14.2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

Definition 14.2.5.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

given by precomposition by f is full.

Remark 14.2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

14.2.6   Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 14.2.6.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

2. For each $X \in \mathsf{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

   given by precomposition by f is fully faithful.

Remark 14.2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in Definition 14.2.4.1.2 and Definition 14.2.5.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

   if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathsf{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\rightrightarrows}} X$$

   of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X$$

   of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\rightrightarrows}} X$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

14.2.7   Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

Definition 14.2.7.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is essentially injective.

Remark 14.2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of C, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

14.2.8   Corepresentably Conservative Morphisms

Let C be a bicategory.

Definition 14.2.8.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is conservative.

Remark 14.2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

is a 2-isomorphism, then so is α.

14.2.9   Strict Epimorphisms

Let C be a bicategory.

Definition 14.2.9.1.1. A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathrm{Hom}_C(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_C(A, X))$$

is injective.

Remark 14.2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 14.2.9.1.3. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Definition 11.7.3.1.2.

2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, **??**.

14.2.10   Pseudoepic Morphisms

Let C be a bicategory.

Definition 14.2.10.1.1. A 1-morphism $f \colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

Remark 14.2.10.1.2. In detail, a 1-morphism $f \colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

   if we have

   $$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

   of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

   of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

   of pasting diagrams in C, i.e. such that we have

   $$\beta = \alpha \star \mathrm{id}_f \,.$$

Proposition 14.2.10.1.3. Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.
   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

(c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B,$$

$$
\begin{array}{ccc}
B & \xleftarrow{\;\text{id}_B\;} & B \\
{\scriptstyle\text{id}_B}\big\uparrow & & \big\uparrow{\scriptstyle F} \\
B & \xleftarrow{\;F\;} & A
\end{array}
$$

in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted. □

Appendices

A Other Chapters

References

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 13).