Tensor Products of Pointed Sets

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In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

introduced in Section 7.5.1, defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f\colon X \wedge Y \longrightarrow Z$.
- Maps of sets $f\colon X \times Y \longrightarrow Z$ satisfying

  $$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ \rhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to $\wedge$, which turns out to endow Sets_* with a monoidal category structure (Definition 7.5.9.1.1), these do not admit invertible associators and unitors, but do endow Sets_* with the structure of a skew monoidal category, however (Definitions 7.3.8.1.1 and 7.4.8.1.1).

Finally, in addition to the tensor products $\lhd$, $\rhd$, and $\wedge$, we also have a ”tensor product‘ of the form

$$\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned}
\otimes_{k,\ell}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) &\longrightarrow \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets}),\\
\lhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\longrightarrow \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\\
\rhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\longrightarrow \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),
\end{aligned}$$

where $k, \ell, i \in \mathbb{N}$ with $i \leq k-1$. Together with the Cartesian product $\times$ of Sets, the tensor products studied in this chapter form the cases:

- $(k, \ell) = (-1, -1)$ for the Cartesian product of Sets;
- $(k, \ell) = (0, -1)$ and $(-1, 0)$ for the tensor of sets with pointed sets of Definition 7.2.1.1.1;
- $(i, k) = (-1, 0)$ for the left and right tensor products of pointed sets of Sections 7.3 and 7.4;
- $(k, \ell) = (-1, -1)$ for the smash product of pointed sets of Section 7.5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in **??**, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k, \ell \leq 2$, and hence *all* cases since $\mathbb{E}_k$-monoids on Sets are the same as $\mathbb{E}_2$-monoids on Sets when $k \geq 2$.

Contents

7.1 Bilinear Morphisms of Pointed Sets

7.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 7.1.1.1.1. A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[1,2]

($\star$) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

[1] *Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

Definition 7.1.1.1.2. The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathsf{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathsf{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

7.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 7.1.2.1.1. A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[3,4]

(★) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 7.1.2.1.2. The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathsf{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathsf{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

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for each $y \in Y$.

[3] *Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

for each $x \in X$.

7.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 7.1.3.1.1. A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

that is both left bilinear and right bilinear.

Remark 7.1.3.1.2. In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[5,6]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ \mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X \times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

[5] *Slogan:* The map f is bilinear if it preserves basepoints in each argument.

[6] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd} & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\ X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X\times[y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\ X\times Y \arrow[rr, "f"'] & & Z \end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 7.1.3.1.3. The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

7.2 Tensors and Cotensors of Pointed Sets by Sets

7.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 7.2.1.1.1. The **tensor of** (X, x_0) **by** A[7] is the tensor $A \odot (X, x_0)$[8] of (X, x_0) by A as in Limits and Colimits, **??**.

Remark 7.2.1.1.2. In detail, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(★) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

This universal property is in turn equivalent to the following one:

[7] *Further Terminology:* Also called the **copower of** (X, x_0) **by** A.
[8] *Further Notation:* Often written $A \odot X$ for simplicity.

(★) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } f(a, x_0) = k_0\end{array} \right\}.$$

Proof. We claim that we have a bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}(X, K)) \cong \mathsf{Sets}(A \times X, K)$$

of Constructions With Sets, Item 2 of Definition 4.1.3.1.3:

- A map
$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$
in $\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$ gets sent to the map
$$\xi^{\dagger} \colon A \times X \to K$$
defined by
$$\xi^{\dagger}(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$
for each $(a, x) \in A \times X$, which indeed lies in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$, as we have
$$\begin{aligned} \xi^{\dagger}(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &\stackrel{\text{def}}{=} k_0 \end{aligned}$$
for each $a \in A$, where we have used that $\xi_a \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map
$$\xi\colon A\times X\to K$$
in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A\times X,K)$ gets sent to the map
$$\begin{aligned}\xi^{\dagger}\colon A &\longrightarrow \mathsf{Sets}_{*}(X,K),\\ a &\longmapsto (\xi^{\dagger}_{a}\colon X\to K),\end{aligned}$$
where
$$\xi^{\dagger}_{a}\colon X\to K$$
is the map defined by
$$\xi^{\dagger}_{a}(x)\stackrel{\text{def}}{=}\xi(a,x)$$
for each $x\in X$, and indeed lies in $\mathsf{Sets}_{*}(X,K)$, as we have
$$\begin{aligned}\xi^{\dagger}_{a}(x_0)&\stackrel{\text{def}}{=}\xi(a,x_0)\\&\stackrel{\text{def}}{=}k_0.\end{aligned}$$

This finishes the proof. □

Construction 7.2.1.1.3. Concretely, the **tensor of** (X,x_0) **by** A is the pointed set $A\odot(X,x_0)$ consisting of:

- *The Underlying Set.* The set $A\odot X$ given by
$$A\odot X\cong\bigvee_{a\in A}(X,x_0),$$
where $\bigvee_{a\in A}(X,x_0)$ is the wedge product of the A-indexed family $((X,x_0))_{a\in A}$ of Pointed Sets, Definition 6.3.2.1.1.

- *The Basepoint.* The point $[(a,x_0)]=[(a',x_0)]$ of $\bigvee_{a\in A}(X,x_0)$.

Proof. (Proven below in a bit.) □

Notation 7.2.1.1.4. We write $a\odot x$ for the element $[(a,x)]$ of
$$\begin{aligned}A\odot X&\cong\bigvee_{a\in A}(X,x_0)\\&\stackrel{\text{def}}{=}\Big(\coprod_{i\in I}X_i\Big)/\sim.\end{aligned}$$

Remark 7.2.1.1.5. Taking the tensor of any element of A with the basepoint x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation $\sim$ on

$$\bigvee_{a\in A}(X, x_0) \stackrel{\text{def}}{=} \coprod_{a\in A} X/\sim$$

identifying (a, x_0) with (a', x_0), so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Map I.* We define a map

   $$\Phi_K : \mathsf{Sets}_*(A \odot X, K) \to \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

   by sending a morphism of pointed sets

   $$\xi : (A \odot X, a \odot x_0) \to (K, k_0)$$

   to the map of sets

   $$\begin{aligned} \xi^\dagger : A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a : X \to K), \end{aligned}$$

   where

   $$\xi_a : (X, x_0) \to (K, k_0)$$

   is the morphism of pointed sets defined by

   $$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

   for each $x \in X$. Note that we have

   $$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

   so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

2. *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \longrightarrow \mathsf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that $\xi(a) \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(A \odot X, K)} .$$

Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K([\![a \mapsto [\![x \mapsto \xi(a \odot x)]\!]]\!]) \\ &= \Psi_K([\![a' \mapsto [\![x' \mapsto \xi(a' \odot x')]\!]]\!]) \\ &= [\![a \odot x \mapsto \mathrm{ev}_x(\mathrm{ev}_a([\![a' \mapsto [\![x' \mapsto \xi(a' \odot x')]\!]]\!]))]\!] \\ &= [\![a \odot x \mapsto \mathrm{ev}_x([\![x' \mapsto \xi(a \odot x')]\!])]\!] \\ &= [\![a \odot x \mapsto \xi(a \odot x)]\!] \\ &= \xi. \end{aligned}$$

4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(X,K))}\,.$$

Indeed, given a morphism $\xi\colon A \to \mathsf{Sets}_*(X,K)$, we have

$$\begin{aligned}[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K([\![a \odot x \mapsto \xi_a(x)]\!]) \\ &= [\![a \mapsto [\![x \mapsto \xi_a(x)]\!]]\!] \\ &= [\![a \mapsto \xi(a)]\!] \\ &= \xi.\end{aligned}$$

5. *Naturality of* Φ. We need to show that, given a morphism of pointed sets

$$\phi\colon (K,k_0) \to (K',k'_0),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(A \odot X, K) \arrow[r, "\Phi_K"] \arrow[d, "\phi_*"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X,K)) \arrow[d, "(\phi_*)_*"] \\
\mathsf{Sets}_*(A \odot X, K') \arrow[r, "\Phi_{K'}"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X,K'))
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi\colon (A \odot X, a \odot x_0) \to (K,k_0),$$

we have

$$\begin{aligned}[\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\ &= \Phi_{K'}(\phi \circ \xi) \\ &= (\phi \circ \xi)^{\dagger} \\ &= [\![a \mapsto \phi \circ \xi(a \odot -)]\!] \\ &= [\![a \mapsto \phi_*(\xi(a \odot -))]\!] \\ &= (\phi_*)_*([\![a \mapsto \xi(a \odot -]\!])) \\ &= (\phi_*)_*(\Phi_K(\xi)) \\ &= [(\phi_*)_* \circ \Phi_K](\xi).\end{aligned}$$

6. *Naturality of Ψ.* Since Φ is natural and Φ is a componentwise inverse to Ψ, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural.

This finishes the proof. □

Proposition 7.2.1.1.6. Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$\begin{aligned} A \odot - \colon & \quad \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \odot X \colon & \quad \mathsf{Sets} \to \mathsf{Sets}_*, \\ -_1 \odot -_2 \colon & \quad \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given:

   - A map of sets $f \colon A \to B$;
   - A pointed map $\phi \colon (X, x_0) \to (Y, y_0)$;

   the induced map

$$f \odot \phi \colon A \odot X \to B \odot Y$$

   is given by

$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$

   for each $a \odot x \in A \odot X$.

2. *Adjointness I.* We have an adjunction

$$(- \odot X \dashv \mathsf{Sets}_*(X, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "- \odot X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X,-)"] \end{tikzcd}$$

   witnessed by a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A \odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A \pitchfork -"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Colimit.* We have

$$A \odot X \cong \mathrm{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Interaction With Homs.* We have a natural isomorphism

$$\mathsf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathsf{Sets}_*(X, -).$$

7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\odot}_{X,Y}\colon \mathsf{Sets}_*(X, Y) \odot X \to Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\odot}_{X,Y}(f \odot x) \stackrel{\mathrm{def}}{=} f(x)$$

for each $f \odot x \in \mathsf{Sets}_*(X, Y) \odot X$.

8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$ and each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{coev}^{\odot}_{A,X}\colon A \to \mathsf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{coev}^{\odot}_{A,X}(a) \stackrel{\mathrm{def}}{=} [\![x \mapsto a \odot x]\!]$$

for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 7.2.1.1.1.
Item 3, *: Adjointness II*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, *As a Weighted Colimit*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 5, *Iterated Tensors*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, *Interaction With Homs*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, *The Tensor Evaluation Map*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, *The Tensor Coevaluation Map*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$. □

7.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 7.2.2.1.1. The **cotensor of** (X, x_0) **by** A[9] is the cotensor $A \pitchfork (X, x_0)$[10] of (X, x_0) by A as in Limits and Colimits, **??**.

Remark 7.2.2.1.2. In detail, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(★) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

This universal property is in turn equivalent to the following one:

(★) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

[9] *Further Terminology:* Also called the **power of** (X, x_0) **by** A.
[10] *Further Notation:* Often written $A \pitchfork X$ for simplicity.

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0\end{array} \right\}.$$

Proof. This follows from the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ constructed in the proof of Definition 7.2.1.1.2. □

Construction 7.2.2.1.3. Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

  where $\bigwedge_{a \in A}(X, x_0)$ is the smash product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 7.6.1.1.1.

- *The Basepoint.* The point $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Map I.* We define a map

$$\Phi_K \colon \mathsf{Sets}_*(K, A \pitchfork X) \longrightarrow \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

   by sending a morphism of pointed sets

$$\xi \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a \in A}])$$

   to the map of sets

$$\begin{aligned} \xi^{\dagger} \colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a \colon K \to X), \end{aligned}$$

where

$$\xi_a \colon (K, k_0) \to (X, x_0)$$

is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x_a^k & \text{if } \xi(k) \neq [(x_0)_{a\in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a\in A}] \end{cases}$$

for each $k \in K$, where x_a^k is the ath component of $\xi(k) = [(x_a^k)_{a\in A}]$. Note that:

(a) The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned} \xi(k) &= [(x_a^k)_{a\in A}] \\ &= [(y_a^k)_{a\in A}] \end{aligned}$$

with $x_a^k \neq y_a^k$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x_{a_x}^k = y_{a_y}^k = x_0$. The equivalence relation $\sim$ on $\prod_{a\in A} X$ then forces

$$\begin{aligned} [(x_a^k)_{a\in A}] &= [(x_0)_{a\in A}], \\ [(y_a^k)_{a\in A}] &= [(x_0)_{a\in A}], \end{aligned}$$

however, and $\xi_a(k)$ is defined to be x_0 in this case.

(b) The map ξ_a is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since $\xi(k_0) = [(x_0)_{a\in A}]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the ath component of $[(x_0)_{a\in A}]$, is equal to x_0.

2. *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \to \mathsf{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a \colon K \to X), \end{aligned}$$

to the morphism of pointed sets

$$\xi^{\dagger}\colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

defined by

$$\xi^{\dagger}(k) \stackrel{\text{def}}{=} [(\xi_a(k))_{a\in A}]$$

for each $k \in K$. Note that $\xi^{\dagger}$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\xi^{\dagger}(k_0) &\stackrel{\text{def}}{=} [(\xi_a(k_0))_{a\in A}]\\ &= x_0,\end{aligned}$$

where we have used that $\xi_a \in \mathsf{Sets}_*(K, X)$ is a morphism of pointed sets for each $a \in A$.

3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi\colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi))\\ &= \Psi_K([\![a \mapsto \xi_a]\!])\\ &= \Psi_K([\![a' \mapsto \xi_{a'}]\!])\\ &= [\![k \mapsto [(\mathrm{ev}_a([\![a' \mapsto \xi_{a'}(k)]\!]))_{a\in A}]]\!]\\ &= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!].\end{aligned}$$

Now, we have two cases:

(a) If $\xi(k) = [(x_0)_{a\in A}]$, we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!]\\ &= [\![k \mapsto [(x_0)_{a\in A}]]\!]\\ &= [\![k \mapsto \xi(k)]\!]\\ &= \xi.\end{aligned}$$

(b) If $\xi(k) \neq [(x_0)_{a\in A}]$ and $\xi(k) = [(x^k_a)_{a\in A}]$ instead, we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!]\\ &= [\![k \mapsto [(x^k_a)_{a\in A}]]\!]\\ &= [\![k \mapsto \xi(k)]\!]\\ &= \xi.\end{aligned}$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(K,X))}.$$

Indeed, given a morphism $\xi\colon A \to \mathsf{Sets}_*(K,X)$, we have

$$\begin{aligned}[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi))\\ &= \Phi_K([\![k \mapsto [(\xi_a(k))_{a\in A}]]\!])\\ &= [\![a \mapsto [\![k \mapsto \xi_a(k)]\!]]\!]\\ &= \xi\end{aligned}$$

5. *Naturality of Ψ.* We need to show that, given a morphism of pointed sets

$$\phi\colon (K,k_0) \to (K',k'_0),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A,\mathsf{Sets}_*(K',X)) \arrow[r, "\Psi_{K'}"] \arrow[d, "(\phi^*)_*"'] & \mathsf{Sets}_*(K', A \pitchfork X) \arrow[d, "\phi^*"] \\
\mathsf{Sets}(A,\mathsf{Sets}_*(K,X)) \arrow[r, "\Psi_K"'] & \mathsf{Sets}_*(K, A \pitchfork X)
\end{tikzcd}$$

commutes. Indeed, given a map of sets

$$\begin{aligned}\xi\colon A &\longrightarrow \mathsf{Sets}_*(K',X),\\ a &\longmapsto (\xi_a\colon K' \to X),\end{aligned}$$

we have

$$[\Psi_K \circ (\phi^*)_*](\xi) = \Psi_K((\phi^*)_*(\xi))$$

$$
\begin{aligned}
&= \Psi_K((\phi^*)_*([\![a \mapsto \xi_a]\!])) \\
&= \Psi_K(([\![a \mapsto \phi^*(\xi_a)]\!])) \\
&= \Psi_K(([\![a \mapsto [\![k \mapsto \xi_a(\phi(k))]\!]]\!])) \\
&= [\![k \mapsto [(\xi_a(\phi(k)))_{a\in A}]]\!] \\
&= \phi^*([\![k' \mapsto [(\xi_a(k'))_{a\in A}]]\!]) \\
&= \phi^*(\Psi_{K'}(\xi)) \\
&= [\phi^* \circ \Psi_{K'}](\xi).
\end{aligned}
$$

6. *Naturality of* Φ. Since Ψ is natural and Ψ is a componentwise inverse to Φ, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Φ is also natural.

This finishes the proof. □

Proposition 7.2.2.1.4. Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors

$$
\begin{aligned}
A \pitchfork - \colon &\quad \mathsf{Sets}_* &&\to \mathsf{Sets}_*, \\
- \pitchfork X \colon &\quad \mathsf{Sets}^{\mathsf{op}} &&\to \mathsf{Sets}_*, \\
-_1 \pitchfork -_2 \colon &\quad \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* &&\to \mathsf{Sets}_*.
\end{aligned}
$$

   In particular, given:

   - A map of sets $f\colon A \to B$;
   - A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

   the induced map

$$f \odot \phi\colon A \pitchfork X \to B \pitchfork Y$$

   is given by

$$[f \odot \phi]([(x_a)_{a\in A}]) \stackrel{\text{def}}{=} [(\phi(x_{f(a)}))_{a\in A}]$$

   for each $[(x_a)_{a\in A}] \in A \pitchfork X$.

2. *Adjointness I.* We have an adjunction

$$
(- \pitchfork X \dashv \mathsf{Sets}_*(-, X))\colon \quad
\begin{tikzcd}
\mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "-\pitchfork X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(-,X)"]
\end{tikzcd}
$$

witnessed by a bijection

$$\mathsf{Sets}_*^{\mathsf{op}}(A \pitchfork X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

i.e. by a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork-"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \longrightarrow \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \mathsf{Sets}_*(X, -) &\cong \mathsf{Sets}_*(A \odot X, -), \\ A \pitchfork \mathsf{Sets}_*(-, Y) &\cong \mathsf{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

7. *The Cotensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\pitchfork}_{X,Y} \colon X \longrightarrow \mathsf{Sets}_*(X,Y) \pitchfork Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\pitchfork}_{X,Y}(x) \stackrel{\mathrm{def}}{=} [(f(x))_{f \in \mathsf{Sets}_*(X,Y)}]$$

for each $x \in X$.

8. *The Cotensor Coevaluation Map.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have a map

$$\mathrm{coev}^{\pitchfork}_{A,X} \colon A \longrightarrow \mathsf{Sets}_*(A \pitchfork X, X),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $A \in \mathrm{Obj}(\mathsf{Sets})$, and given by

$$\mathrm{coev}^{\pitchfork}_{A,X}(a) \stackrel{\mathrm{def}}{=} [\![[(x_b)_{b \in A}] \mapsto x_a]\!]$$

for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 7.2.2.1.1.
Item 3, *: Adjointness II*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, *As a Weighted Limit*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 5, *Iterated Cotensors*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, *Commutativity With Homs*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, *The Cotensor Evaluation Map*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, *The Cotensor Coevaluation Map*: This is the special case of Limits and Colimits, **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$. □

7.3 The Left Tensor Product of Pointed Sets

7.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.3.1.1.1. The **left tensor product of pointed sets** is the functor[11]

$$\lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id}\times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\boldsymbol{\beta}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\boldsymbol{\beta}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}\colon \mathsf{Sets}_* \times \mathsf{Sets} \overset{\sim}{\dashrightarrow} \mathsf{Sets} \times \mathsf{Sets}_*$ is the braiding of Cats_2, i.e. the functor witnessing the isomorphism

$$\mathsf{Sets}_* \times \mathsf{Sets} \cong \mathsf{Sets} \times \mathsf{Sets}_*.$$

- $\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$ is the tensor functor of Item 1 of Definition 7.2.1.1.6.

Remark 7.3.1.1.2. The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f\colon X \lhd Y \longrightarrow Z$.
2. Maps of sets $f\colon X \times Y \longrightarrow Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

Remark 7.3.1.1.3. The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \lhd Y, x_0 \lhd y_0), \iota)$ consisting of
  - A pointed set $(X \lhd Y, x_0 \lhd y_0)$;
  - A left bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \longrightarrow X \lhd Y$;

  satisfying the following universal property:

[11] *Further Notation:* Also written $\lhd_{\mathsf{Sets}_*}$.

($\star$) Given another such pair $((Z, z_0), f)$ consisting of

* A pointed set (Z, z_0);
* A left bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \longrightarrow X \lhd Y$;

there exists a unique morphism of pointed sets $X \lhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \lhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

Construction 7.3.1.1.4. In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd Y, [x_0])$ consisting of:

- *The Underlying Set.* The set $X \lhd Y$ defined by

$$\begin{aligned} X \lhd Y &\overset{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0), \end{aligned}$$

  where $|Y|$ denotes the underlying set of (Y, y_0).

- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y \in Y}(X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

Proof. Since $\bigvee_{y \in Y}(X, x_0)$ is defined as the quotient of $\coprod_{y \in Y} X$ by the equivalence relation R generated by declaring $(y, x) \sim (y', x')$ if $x = x' = x_0$, we have, by Conditions on Relations, **??**, a natural bijection

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(\coprod_{y \in Y} X, Z),$$

where $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^R_{\mathsf{Sets}}(\coprod_{y \in Y} X, Z) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(\coprod_{y \in Y} X, Z) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (y, x) \sim_R (y', x'), \text{ then} \\ f(y, x) = f(y', x') \end{array} \right\}.$$

However, the condition $(y, x) \sim_R (y', x')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. We have $x = x' = x_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(\coprod_{y\in Y} X, Z)$ with a corresponding $\overline{f}\colon X \lhd Y \to Z$, the latter case above implies

$$\begin{aligned} f([(y,x_0)]) &= f([(y',x_0)]) \\ &= f([(y_0,x_0)]), \end{aligned}$$

and since $\overline{f}\colon X \lhd Y \to Z$ is a pointed map, we have

$$\begin{aligned} f([(y_0,x_0)]) &= \overline{f}([(y_0,x_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X\times Y, Z)$ are precisely those functions $f\colon X\times Y \to Z$ satisfying the equality

$$f(x_0, y) = z_0$$

for each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes,\mathsf{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. ☐

Notation 7.3.1.1.5. We write[12] $x \lhd y$ for the element $[(y,x)]$ of

$$X \lhd Y \cong |Y| \odot X.$$

Remark 7.3.1.1.6. Employing the notation introduced in Definition 7.3.1.1.5, we have

$$x_0 \lhd y_0 = x_0 \lhd y$$

for each $y \in Y$, and

$$x_0 \lhd y = x_0 \lhd y'$$

for each $y, y' \in Y$.

[12] *Further Notation:* Also written $x \lhd_{\mathsf{Sets}_*} y$.

Proposition 7.3.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd Y$ define functors

$$\begin{aligned} X \lhd - \colon & \quad \mathsf{Sets}_* \quad \to \mathsf{Sets}_*, \\ - \lhd Y \colon & \quad \mathsf{Sets}_* \quad \to \mathsf{Sets}_*, \\ -_1 \lhd -_2 \colon & \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f \colon (X, x_0) &\to (A, a_0), \\ g \colon (Y, y_0) &\to (B, b_0), \end{aligned}$$

   the induced map

$$f \lhd g \colon X \lhd Y \to A \lhd B$$

   is given by

$$[f \lhd g](x \lhd y) \stackrel{\text{def}}{=} f(x) \lhd g(y)$$

   for each $x \lhd y \in X \lhd Y$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*})$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 7.3.2.1.1.

3. *Adjointness II.* The functor

$$X \lhd - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   does not admit a right adjoint.

4. *Adjointness III*. We have a 忘-relative adjunction

$$(X \lhd - \dashv \mathsf{Sets}_*(X,-)) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \lhd -"] \arrow[r, phantom, "\perp_{忘}"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X,-)"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1*, *Functoriality*: This follows from the definition of $\lhd$ as a composition of functors (Definition 7.3.1.1.1).
Item 2, *Adjointness I*: This follows from Item 3 of Definition 7.2.1.1.6.
Item 3, *Adjointness II*: For $X \lhd -$ to admit a right adjoint would require it to preserve colimits by **??**, **??** of **??**. However, we have

$$\begin{aligned} X \lhd \mathrm{pt} &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $X \lhd -$ does not have a right adjoint.
Item 4, *Adjointness III*: This follows from Item 2 of Definition 7.2.1.1.6. □

Remark 7.3.1.1.8. Here is some intuition on why $X \lhd -$ fails to be a left adjoint. Item 4 of Definition 7.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give $X \lhd - \dashv \mathbf{Sets}_*(X, -)$. However, such a bijection would require every map

$$f \colon X \lhd Y \to Z$$

to satisfy

$$f(x \lhd y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \lhd y$. Thus $\mathbf{Sets}_*(X, -)$ can,t be a right adjoint for $X \lhd -$, and as shown by Item 3 of Definition 7.3.1.1.7, no functor can.[13]

[13]The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed sets

7.3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.3.2.1.1. The **left internal Hom**[14] **of pointed sets** is the functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Definition 7.2.2.1.4.

Remark 7.3.2.1.2. The left internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathsf{Sets}_*(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*})$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \lhd Y \to Z$.
2. Pointed maps $f\colon X \to [Y, Z]^{\lhd}_{\mathsf{Sets}_*}$.

Remark 7.3.2.1.3. In detail, the **left internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $([X, Y]^{\lhd}_{\mathsf{Sets}_*}, [(y_0)_{x\in X}])$ consisting of:

- *The Underlying Set.* The set $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X, Y]^{\lhd}_{\mathsf{Sets}_*} &\overset{\mathrm{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X} (Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

of Definition 7.5.1.1.1. See Item 2 of Definition 7.5.1.1.10.

[14] For a proof that $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ is indeed the left internal Hom of Sets_* with respect to the left

- *The Underlying Basepoint.* The point $[(y_0)_{x\in X}]$ of $\bigwedge_{x\in X}(Y, y_0)$.

Proposition 7.3.2.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\lhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}
[X, -]^{\lhd}_{\mathsf{Sets}_*} &\colon \quad \mathsf{Sets}_* \quad \to \mathsf{Sets}_*,\\
[-, Y]^{\lhd}_{\mathsf{Sets}_*} &\colon \quad \mathsf{Sets}_*^{\mathsf{op}} \quad \to \mathsf{Sets}_*,\\
[-_1, -_2]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned}
f &\colon (X, x_0) \to (A, a_0),\\
g &\colon (Y, y_0) \to (B, b_0),
\end{aligned}$$

   the induced map

$$[f, g]^{\lhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\lhd}_{\mathsf{Sets}_*} \to [X, B]^{\lhd}_{\mathsf{Sets}_*}$$

   is given by

$$[f, g]^{\lhd}_{\mathsf{Sets}_*}([(y_a)_{a\in A}]) \overset{\text{def}}{=} [(g(y_{f(x)}))_{x\in X}]$$

   for each $[(y_a)_{a\in A}] \in [A, Y]^{\lhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*})$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$

3. *Adjointness II.* The functor

$$X \lhd - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

   does not admit a right adjoint.

Proof. *Item 1*, *Functoriality*: This follows from the definition of $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ as a composition of functors (Definition 7.3.2.1.1).
Item 2, *Adjointness I*: This is a repetition of Item 2 of Definition 7.3.1.1.7, and is proved there.
Item 3, *Adjointness II*: This is a repetition of Item 3 of Definition 7.3.1.1.7, and is proved there. □

7.3.3 The Left Skew Unit

Definition 7.3.3.1.1. The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd}\colon \mathrm{pt} \to \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\lhd}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

7.3.4 The Left Skew Associator

Definition 7.3.4.1.1. The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd \circ (\lhd \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\lhd"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\lhd\times\mathrm{id}"'] & & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dl, "\lhd"] \\
 & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\lhd"'] \arrow[urr, phantom, "\overset{\alpha^{\mathsf{Sets}_*,\lhd}}{\Nearrow}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}\colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

tensor product of pointed sets, see Item 2 of Definition 7.3.1.1.7.

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
(X \lhd Y) \lhd Z &\overset{\text{def}}{=} |Z| \odot (X \lhd Y) \\
&\overset{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
&\cong \bigvee_{z \in Z} |Y| \odot X \\
&\cong \bigvee_{z \in Z} \Big(\bigvee_{y \in Y} X \Big) \\
&\to \bigvee_{[(z,y)] \in \bigvee_{z \in Z} Y} X \\
&\cong \bigvee_{[(z,y)] \in |Z| \odot Y} X \\
&\cong ||Z| \odot Y| \odot X \\
&\overset{\text{def}}{=} |Y \lhd Z| \odot X \\
&\overset{\text{def}}{=} X \lhd (Y \lhd Z),
\end{aligned}$$

where the map

$$\bigvee_{z \in Z} \Big(\bigvee_{y \in Y} X \Big) \to \bigvee_{(z,y) \in \bigvee_{z \in Z} Y} X$$

is given by $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$.

Proof. (Proven below in a bit.) □

Remark 7.3.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}
[(z, [(y, x)])] &\overset{\text{def}}{=} [(z, x \lhd y)] \\
&\overset{\text{def}}{=} (x \lhd y) \lhd z
\end{aligned}$$

and

$$\begin{aligned}
[([(z, y)], x)] &\overset{\text{def}}{=} [(y \lhd z, x)] \\
&\overset{\text{def}}{=} x \lhd (y \lhd z).
\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*, \lhd}_{X,Y,Z}((x \lhd y) \lhd z) \overset{\text{def}}{=} x \lhd (y \lhd z)$$

for each $(x \lhd y) \lhd z \in (X \lhd Y) \lhd Z$.

Remark 7.3.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y_0) \lhd z) \stackrel{\text{def}}{=} x \lhd (y_0 \lhd z).$$

However, by the definition of $\lhd$, we have $y_0 \lhd z = y_0 \lhd z'$ for all $z, z' \in Z$, preventing $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x_0 \lhd y_0) \lhd z_0) = x_0 \lhd (y_0 \lhd z_0).$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0), \\ g &\colon (Y, y_0) \to (Y', y'_0), \\ h &\colon (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \lhd Y) \lhd Z \arrow[r, "(f \lhd g) \lhd h"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"'] & (X' \lhd Y') \lhd Z' \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X',Y',Z'}"] \\ X \lhd (Y \lhd Z) \arrow[r, "f \lhd (g \lhd h)"'] & X' \lhd (Y' \lhd Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x \lhd y) \lhd z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \lhd g(y)) \lhd h(z) \arrow[d, mapsto] \\ x \lhd (y \lhd z) \arrow[r, mapsto] & f(x) \lhd (g(y) \lhd h(z)) \end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

7.3.5 The Left Skew Left Unitor

Definition 7.3.5.1.1. The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

$$\begin{tikzcd} \mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*,\lhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X\colon S^0 \lhd X \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} S^0 \lhd X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\to X, \end{aligned}$$

where $\bigvee_{x\in X} S^0 \to X$ is the map given by

$$\begin{aligned} [(x,0)] &\mapsto x_0, \\ [(x,1)] &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. (Proven below in a bit.) □

Remark 7.3.5.1.2. In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned} \lambda^{\mathsf{Sets}_*,\lhd}_X(0 \lhd x) &\overset{\text{def}}{=} x_0, \\ \lambda^{\mathsf{Sets}_*,\lhd}_X(1 \lhd x) &\overset{\text{def}}{=} x \end{aligned}$$

for each $1 \lhd x \in S^0 \lhd X$.

Remark 7.3.5.1.3. The morphism $\lambda_X^{\mathsf{Sets}_*,\lhd}$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to S^0 \lhd X$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} 1 \lhd x$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}[\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi](x) &= \lambda_X^{\mathsf{Sets}_*,\lhd}(\phi(x))\\ &= \lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}](1 \lhd x) &= \phi(\lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x))\\ &= \phi(x)\\ &= 1 \lhd x\\ &= [\mathrm{id}_{S^0 \lhd X}](1 \lhd x),\end{aligned}$$

but

$$\begin{aligned}[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}](0 \lhd x) &= \phi(\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x))\\ &= \phi(x_0)\\ &= 1 \lhd x_0,\end{aligned}$$

where $0 \lhd x \neq 1 \lhd x_0$. Thus

$$\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd} \stackrel{?}{=} \mathrm{id}_{S^0 \lhd X}$$

holds for all elements in $S^0 \lhd X$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \colon S^0 \lhd X \to X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x_0) = x_0.$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} S^0 \lhd X \arrow[r, "\mathrm{id}_{S^0} \lhd f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\lhd}"'] & S^0 \lhd Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\lhd}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \lhd x \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} 0 \lhd x \arrow[r, mapsto] & 0 \lhd f(x) \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \lhd x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \lhd f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. $\square$

7.3.6 The Left Skew Right Unitor

Definition 7.3.6.1.1. The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}), \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \\ & \mathsf{Sets}_*, \arrow[from=2-2, to=1-2, phantom, "\Nearrow \rho^{\mathsf{Sets}_*,\lhd}"] \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd S^0$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \lhd S^0, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 7.3.6.1.2. In other words, $\rho^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho^{\mathsf{Sets}_*,\lhd}_X(x) \overset{\text{def}}{=} x \lhd 1$$

for each $x \in X$.

Remark 7.3.6.1.3. The morphism $\rho^{\mathsf{Sets}_*,\lhd}_X$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \lhd 0$ of $X \lhd S^0$ with $x \neq x_0$ are outside the image of $\rho^{\mathsf{Sets}_*,\lhd}_X$, which sends x to $x \lhd 1$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned}\rho_X^{\mathsf{Sets}_*,\lhd}(x_0) &= x_0 \lhd 1\\ &= x_0 \lhd 0.\end{aligned}$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\lhd}"'] & Y \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\lhd}"] \\ X \lhd S^0 \arrow[r, "f \lhd \mathrm{id}_{S^0}"'] & Y \lhd S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \lhd 0 \arrow[r, mapsto] & f(x) \lhd 0 \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

7.3.7 The Diagonal

Definition 7.3.7.1.1. The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\lhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\lhd}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\lhd"'] & \end{tikzcd}$$

whose component

$$\Delta^{\lhd}_X \colon (X, x_0) \to (X \lhd X, x_0 \lhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\lhd}_X(x) \stackrel{\text{def}}{=} x \lhd x$$

for each $x \in X$.

Proof. *Being a Morphism of Pointed Sets*: We have

$$\Delta^{\lhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \lhd x_0,$$

and thus $\Delta^{\lhd}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\lhd}_X"'] & Y \arrow[d, "\Delta^{\lhd}_Y"] \\ X \lhd X \arrow[r, "f \lhd f"'] & Y \lhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \lhd x \arrow[r, mapsto] & f(x) \lhd f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\lhd}$ to be natural. □

7.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$

Proposition 7.3.8.1.1. The category Sets_* admits a left-closed left skew monoidal category structure consisting of:

- *The Underlying Category.* The category Sets_* of pointed sets.
- *The Left Skew Monoidal Product.* The left tensor product functor

$$\lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

 of Definition 7.3.1.1.1.
- *The Left Internal Skew Hom.* The left internal Hom functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*}\colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

 of Definition 7.3.2.1.1.
- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

 of Definition 7.3.3.1.1.
- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd \circ (\lhd \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

 of Definition 7.3.4.1.1.
- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

 of Definition 7.3.5.1.1.
- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*})$$

 of Definition 7.3.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \lhd (X \lhd Y)) \lhd Z \arrow[dr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X\lhd Y,Z}"] & \\
((W \lhd X) \lhd Y) \lhd Z \arrow[ur, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y} \lhd \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{W\lhd X,Y,Z}"'] & & W \lhd ((X \lhd Y) \lhd Z) \arrow[d, "\mathrm{id}_W \lhd \alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"] \\
(W \lhd X) \lhd (Y \lhd Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y\lhd Z}"'] & & W \lhd (X \lhd (Y \lhd Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \lhd (x \lhd y)) \lhd z \arrow[dr, mapsto] & \\
((w \lhd x) \lhd y) \lhd z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \lhd ((x \lhd y) \lhd z) \arrow[d, mapsto] \\
(w \lhd x) \lhd (y \lhd z) \arrow[rr, mapsto] & & w \lhd (x \lhd (y \lhd z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Left Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
(S^0 \lhd X) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{S^0,X,Y}"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & S^0 \lhd (X \lhd Y) \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"] \\
 & X \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x_0 \lhd y = x_0 \lhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
(1 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x \lhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Left Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"'] \arrow[dr, "\mathrm{id}_X \lhd \rho^{\mathsf{Sets}_*,\lhd}_Y"] & \\
(X \lhd Y) \lhd S^0 \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,S^0}"'] & X \lhd (Y \lhd S^0)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(x \lhd y) \lhd 1 \arrow[r, mapsto] & x \lhd (y \lhd 1)
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Left Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[rr, equal] \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & & X \lhd Y \\
(X \lhd S^0) \lhd Y \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,S^0,Y}"'] & & X \lhd (S^0 \lhd Y) \arrow[u, "\mathrm{id}_A \lhd \lambda^{\mathsf{Sets}_*,\lhd}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \lhd y \\
(x \lhd 1) \lhd y \arrow[r, mapsto] & x \lhd (1 \lhd y) \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,\lhd}_{S^0}"] \arrow[dr, equal] & S^0 \lhd S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{S^0}"] \\
 & S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd 1 \arrow[d, mapsto] \\
 & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd 1 \arrow[d, mapsto] \\
 & 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Left Skew Monoidal Left-Closedness: This follows from Item 2 of Definition 7.3.1.1.7.

□

7.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

Proposition 7.3.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ is isomorphic to the category of "monoids with left zero'[15] and morphisms between them.

Proof. *Monoids on* $(\mathsf{Sets}_*, \lhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \lhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \lhd A \to A,$$

  determining a left bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

[15] A monoid with left zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

1. *Associativity.* The diagram

$$
\begin{tikzcd}
 & A \lhd (A \lhd A) \arrow[dr, "\mathrm{id}_A \lhd \mu_A"] & \\
(A \lhd A) \lhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*,\lhd}_{A,A,A}"] \arrow[d, "\mu_A \lhd \mathrm{id}_A"'] & & A \lhd A \arrow[d, "\mu_A"] \\
A \lhd A \arrow[rr, "\mu_A"'] & & A
\end{tikzcd}
$$

2. *Left Unitality.* The diagram

$$
\begin{tikzcd}
S^0 \lhd A \arrow[r, "\eta_A \times \mathrm{id}_A"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_A"'] & A \lhd A \arrow[d, "\mu_A"] \\
 & A
\end{tikzcd}
$$

commutes.

3. *Right Unitality.* The diagram

$$
\begin{tikzcd}
A \arrow[r, "\rho^{\mathsf{Sets}_*,\lhd}_A"] \arrow[d, equal] & A \lhd S^0 \arrow[d, "\mathrm{id}_A \times \eta_A"] \\
A & A \lhd A \arrow[l, "\mu_A"]
\end{tikzcd}
$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$
\begin{tikzcd}
(a \lhd b) \lhd c \arrow[d, mapsto] & \\
ab \lhd c \arrow[r, mapsto] & (ab)c
\end{tikzcd}
\qquad
\begin{tikzcd}
& a \lhd (b \lhd c) \arrow[dr, mapsto] & \\
(a \lhd b) \lhd c \arrow[ur, mapsto] & & a \lhd bc \arrow[d, mapsto] \\
& & a(bc)
\end{tikzcd}
$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts:

(a) On $0 \lhd a$ as

$$
\begin{tikzcd}
0 \lhd a \arrow[dr, mapsto] & \\
& 0_A
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 0_A \lhd a \arrow[d, mapsto] \\
& 0_A a.
\end{tikzcd}
$$

(b) On $1 \lhd a$ as

$$
\begin{tikzcd}
1 \lhd a \arrow[dr, mapsto] & \\
& a
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 1_A \lhd a \arrow[d, mapsto] \\
& 1_A a.
\end{tikzcd}
$$

This gives

$$
\begin{aligned}
1_A a &= a, \\
0_A a &= 0_A
\end{aligned}
$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] & a \arrow[r, mapsto] & a \lhd 1 \arrow[d, mapsto] \\ a & a1_A & a \lhd 1_A \arrow[l, mapsto] \end{tikzcd}$$

   This gives

$$a1_A = a$$

   for each $a \in A$.

Thus we see that monoids with respect to $\lhd$ are exactly monoids with left zero.
Morphisms of Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \lhd A \arrow[r, "f \lhd f"] \arrow[d, "\mu_A"'] & B \lhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

   commutes.

These act on elements as

$$
\begin{tikzcd}
a \lhd b \arrow[d, mapsto] & & a \lhd b \arrow[r, mapsto] & f(a) \lhd f(b) \arrow[d, mapsto] \\
ab \arrow[r, mapsto] & f(ab) & & f(a)f(b)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\
& 0_B & & f(0_A)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\
& 1_B & & f(1_A)
\end{tikzcd}
$$

giving

$$
\begin{aligned}
f(ab) &= f(a)f(b), \\
f(0_A) &= 0_B, \\
f(1_A) &= 1_B,
\end{aligned}
$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \lhd, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. □

7.4 The Right Tensor Product of Pointed Sets

7.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.4.1.1.1. The **right tensor product of pointed sets** is the functor[16]

$$\rhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot\colon \mathsf{Sets}\times\mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$ is the tensor functor of Item 1 of Definition 7.2.1.1.6.

Remark 7.4.1.1.2. The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f\colon X \rhd Y \longrightarrow Z$.
2. Maps of sets $f\colon X \times Y \longrightarrow Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

Remark 7.4.1.1.3. The right tensor product of pointed sets may be described as follows:

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \rhd Y, x_0 \rhd y_0), \iota)$ consisting of
  - A pointed set $(X \rhd Y, x_0 \rhd y_0)$;
  - A right bilinear morphism of pointed sets $\iota\colon (X\times Y, (x_0, y_0)) \longrightarrow X \rhd Y$;

  satisfying the following universal property:

  (★) Given another such pair $((Z, z_0), f)$ consisting of
  - ⋆ A pointed set (Z, z_0);
  - ⋆ A right bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \longrightarrow X \rhd Y$;

[16] *Further Notation:* Also written $\rhd_{\mathsf{Sets}_*}$.

there exists a unique morphism of pointed sets $X \rhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \rhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

Construction 7.4.1.1.4. In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \rhd Y$ defined by

$$\begin{aligned} X \rhd Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x \in X}(Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

Proof. Since $\bigvee_{y \in Y}(X, x_0)$ is defined as the quotient of $\coprod_{x \in X} Y$ by the equivalence relation R generated by declaring $(x, y) \sim (x', y')$ if $y = y' = y_0$, we have, by Conditions on Relations, **??**, a natural bijection

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(\coprod_{X \in X} Y, Z),$$

where $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^R_{\mathsf{Sets}}(\coprod_{x \in X} Y, Z) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(\coprod_{x \in X} Y, Z) \,\middle|\, \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. We have $y = y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(\coprod_{x\in X} Y, Z)$ with a corresponding $\overline{f}\colon X \rhd Y \longrightarrow Z$, the latter case above implies

$$\begin{aligned} f([(x, y_0)]) &= f([(x', y_0)]) \\ &= f([(x_0, y_0)]), \end{aligned}$$

and since $\overline{f}\colon X \rhd Y \longrightarrow Z$ is a pointed map, we have

$$\begin{aligned} f([(x_0, y_0)]) &= \overline{f}([(x_0, y_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X\times Y, Z)$ are precisely those functions $f\colon X\times Y \longrightarrow Z$ satisfying the equality

$$f(x, y_0) = z_0$$

for each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. □

Notation 7.4.1.1.5. We write[17] $x \rhd y$ for the element $[(x, y)]$ of

$$X \rhd Y \cong |X| \odot Y.$$

Remark 7.4.1.1.6. Employing the notation introduced in Definition 7.4.1.1.5, we have

$$x_0 \rhd y_0 = x \rhd y_0$$

for each $x \in X$, and

$$x \rhd y_0 = x' \rhd y_0$$

for each $x, x' \in X$.

Proposition 7.4.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

[17] *Further Notation:* Also written $x \rhd_{\mathsf{Sets}_*} y$.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd Y$ define functors

$$\begin{aligned} X \rhd -\colon & \quad \mathsf{Sets}_* && \longrightarrow \mathsf{Sets}_*, \\ - \rhd Y\colon & \quad \mathsf{Sets}_* && \longrightarrow \mathsf{Sets}_*, \\ -_1 \rhd -_2\colon & \quad \mathsf{Sets}_* \times \mathsf{Sets}_* && \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\longrightarrow (A, a_0), \\ g\colon (Y, y_0) &\longrightarrow (B, b_0), \end{aligned}$$

   the induced map

$$f \rhd g\colon X \rhd Y \longrightarrow A \rhd B$$

   is given by

$$[f \rhd g](x \rhd y) \overset{\text{def}}{=} f(x) \rhd g(y)$$

   for each $x \rhd y \in X \rhd Y$.

2. *Adjointness I.* We have an adjunction

$$\Big(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\Big)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*})$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 7.4.2.1.1.

3. *Adjointness II.* The functor

$$- \rhd Y\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

   does not admit a right adjoint.

4. *Adjointness III.* We have a 忘-relative adjunction

$$(- \rhd Y \dashv \mathsf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \rhd Y"] \arrow[r, phantom, "\perp_{忘}"] & \mathsf{Sets}_*, \arrow[l, bend left, "{\mathsf{Sets}_*(Y,-)}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: This follows from the definition of $\triangleright$ as a composition of functors (Definition 7.4.1.1.1).
Item 2, Adjointness I: This follows from Item 3 of Definition 7.2.1.1.6.
Item 3, Adjointness II: For $- \triangleright Y$ to admit a right adjoint would require it to preserve colimits by **??**, **??** of **??**. However, we have

$$\begin{aligned} \mathrm{pt} \triangleright X &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \triangleright Y$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Definition 7.2.1.1.6. □

Remark 7.4.1.1.8. Here is some intuition on why $- \triangleright Y$ fails to be a left adjoint. Item 4 of Definition 7.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give $- \triangleright Y \dashv \mathbf{Sets}_*(Y, -)$. However, such a bijection would require every map

$$f : X \triangleright Y \to Z$$

to satisfy

$$f(x_0 \triangleright y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \triangleright y_0$. Thus $\mathbf{Sets}_*(Y, -)$ can,t be a right adjoint for $- \triangleright Y$, and as shown by Item 3 of Definition 7.4.1.1.7, no functor can.[18]

[18] The functor $\mathbf{Sets}_*(Y, -)$ is instead right adjoint to $- \wedge Y$, the smash product of pointed sets of Definition 7.5.1.1.1. See Item 2 of Definition 7.5.1.1.10.

7.4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.4.2.1.1. The **right internal Hom**[19] **of pointed sets** is the functor

$$[-,-]^{\rhd}_{\mathsf{Sets}_*}\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- 忘: $\mathsf{Sets}_* \longrightarrow \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Definition 7.2.2.1.4.

Remark 7.4.2.1.2. We have

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} = [-,-]^{\rhd}_{\mathsf{Sets}_*}.$$

Remark 7.4.2.1.3. The right internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathsf{Sets}_*(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*})$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \rhd Y \longrightarrow Z$.
2. Pointed maps $f\colon Y \longrightarrow [X, Z]^{\rhd}_{\mathsf{Sets}_*}$.

Remark 7.4.2.1.4. In detail, the **right internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $([X, Y]^{\rhd}_{\mathsf{Sets}_*}, [(y_0)_{x\in X}])$ consisting of:

- *The Underlying Set.* The set $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X, Y]^{\rhd}_{\mathsf{Sets}_*} &\overset{\mathrm{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X}(Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

[19] For a proof that $[-,-]^{\rhd}_{\mathsf{Sets}_*}$ is indeed the right internal Hom of Sets_* with respect to the right

- *The Underlying Basepoint.* The point $[(y_0)_{x\in X}]$ of $\bigwedge_{x\in X}(Y,y_0)$.

Proposition 7.4.2.1.5. Let (X,x_0) and (Y,y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X,Y) \mapsto [X,Y]^{\rhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}
[X,-]^{\rhd}_{\mathsf{Sets}_*}&\colon \quad \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*,\\
[-,Y]^{\rhd}_{\mathsf{Sets}_*}&\colon \quad \mathsf{Sets}_*^{\mathsf{op}} \longrightarrow \mathsf{Sets}_*,\\
[-_1,-_2]^{\rhd}_{\mathsf{Sets}_*}&\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*.
\end{aligned}$$

In particular, given pointed maps

$$\begin{aligned}
f&\colon (X,x_0) \longrightarrow (A,a_0),\\
g&\colon (Y,y_0) \longrightarrow (B,b_0),
\end{aligned}$$

the induced map

$$[f,g]^{\rhd}_{\mathsf{Sets}_*}\colon [A,Y]^{\rhd}_{\mathsf{Sets}_*} \longrightarrow [X,B]^{\rhd}_{\mathsf{Sets}_*}$$

is given by

$$[f,g]^{\rhd}_{\mathsf{Sets}_*}([(y_a)_{a\in A}]) \overset{\text{def}}{=} [(g(y_{f(x)}))_{x\in X}]$$

for each $[(y_a)_{a\in A}] \in [A,Y]^{\rhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\Big(X \rhd - \dashv [X,-]^{\rhd}_{\mathsf{Sets}_*}\Big)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*})$$

natural in $(X,x_0), (Y,y_0), (Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X,Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 7.4.2.1.1.

3. *Adjointness II.* The functor

$$- \rhd Y\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1*, *Functoriality*: This follows from the definition of $[-,-]^{\triangleright}_{\mathsf{Sets}_*}$ as a composition of functors (Definition 7.4.2.1.1).
Item 2, *Adjointness I*: This is a repetition of Item 2 of Definition 7.4.1.1.7, and is proved there.
Item 3, *Adjointness II*: This is a repetition of Item 3 of Definition 7.4.1.1.7, and is proved there. □

7.4.3 The Right Skew Unit

Definition 7.4.3.1.1. The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\triangleright}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

7.4.4 The Right Skew Associator

Definition 7.4.4.1.1. The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright}\colon \triangleright\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\triangleright) \Longrightarrow \triangleright\circ(\triangleright\times\mathrm{id}_{\mathsf{Sets}_*})\circ\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & (\mathsf{Sets}_*\times\mathsf{Sets}_*)\times\mathsf{Sets}_* \arrow[dr, "\triangleright\times\mathrm{id}"] & \\
\mathsf{Sets}_*\times(\mathsf{Sets}_*\times\mathsf{Sets}_*) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\mathrm{id}\times\triangleright"'] & \arrow[d, phantom, "\Uparrow\alpha^{\mathsf{Sets}_*,\triangleright}"] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\triangleright"] \\
\mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[rr, "\triangleright"'] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}\colon X\triangleright(Y\triangleright Z)\longrightarrow(X\triangleright Y)\triangleright Z$$

tensor product of pointed sets, see Item 2 of Definition 7.4.1.1.7.

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
X \rhd (Y \rhd Z) &\stackrel{\text{def}}{=} |X| \odot (Y \rhd Z) \\
&\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
&\cong \bigvee_{x \in X} (|Y| \odot Z) \\
&\cong \bigvee_{x \in X} \Big(\bigvee_{y \in Y} Z\Big) \\
&\to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\
&\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\
&\cong ||X| \odot Y| \odot Z \\
&\stackrel{\text{def}}{=} |X \rhd Y| \odot Z \\
&\stackrel{\text{def}}{=} (X \rhd Y) \rhd Z,
\end{aligned}$$

where the map

$$\bigvee_{x \in X} \Big(\bigvee_{y \in Y} Z\Big) \to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z$$

is given by $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$.

Proof. (Proven below in a bit.) □

Remark 7.4.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}
[(x, [(y, z)])] &\stackrel{\text{def}}{=} [(x, y \rhd z)] \\
&\stackrel{\text{def}}{=} x \rhd (y \rhd z)
\end{aligned}$$

and

$$\begin{aligned}
[([(x, y)], z)] &\stackrel{\text{def}}{=} [(x \rhd y, z)] \\
&\stackrel{\text{def}}{=} (x \rhd y) \rhd z.
\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}(x \rhd (y \rhd z)) \stackrel{\text{def}}{=} (x \rhd y) \rhd z$$

for each $x \rhd (y \rhd z) \in X \rhd (Y \rhd Z)$.

Remark 7.4.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}(x \triangleright (y_0 \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y_0) \triangleright z.$$

However, by the definition of $\triangleright$, we have $x \triangleright y_0 = x' \triangleright y_0$ for all $x, x' \in X$, preventing $\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z} : X \triangleright (Y \triangleright Z) \to (X \triangleright Y) \triangleright Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}(x_0 \triangleright (y_0 \triangleright z_0)) = (x_0 \triangleright y_0) \triangleright z_0.$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0), \\ g &\colon (Y, y_0) \to (Y', y'_0), \\ h &\colon (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} X \triangleright (Y \triangleright Z) \arrow[r, "f \triangleright (g \triangleright h)"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"'] & X' \triangleright (Y' \triangleright Z') \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{X',Y',Z'}"] \\ (X \triangleright Y) \triangleright Z \arrow[r, "(f \triangleright g) \triangleright h"'] & (X' \triangleright Y') \triangleright Z' \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \triangleright (y \triangleright z) \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \triangleright (g(y) \triangleright h(z)) \arrow[d, mapsto] \\ (x \triangleright y) \triangleright z \arrow[r, mapsto] & (f(x) \triangleright g(y)) \triangleright h(z) \end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. $\square$

7.4.5 The Right Skew Left Unitor

Definition 7.4.5.1.1. The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \qquad \begin{tikzcd} \mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \arrow[dl, phantom, "\Uparrow \lambda^{\mathsf{Sets}_*,\triangleright}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X \colon X \to S^0 \triangleright X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \triangleright X, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 7.4.5.1.2. In other words, $\lambda^{\mathsf{Sets}_*,\triangleright}_X$ acts on elements as

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X(x) \overset{\mathrm{def}}{=} [(1, x)]$$

i.e. by

$$\lambda^{\mathsf{Sets}_*,\triangleright}_X(x) \overset{\mathrm{def}}{=} 1 \triangleright x$$

for each $x \in X$.

Remark 7.4.5.1.3. The morphism $\lambda^{\mathsf{Sets}_*,\triangleright}_X$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0 \triangleright x$ of $S^0 \triangleright X$ with $x \neq x_0$ are outside the image of $\lambda^{\mathsf{Sets}_*,\triangleright}_X$, which sends x to $1 \triangleright x$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\triangleright} : X \to S^0 \triangleright X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\mathsf{Sets}_*,\triangleright}(x_0) &= 1 \triangleright x_0 \\ &= 0 \triangleright x_0. \end{aligned}$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\triangleright}"'] & Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\triangleright}"] \\ S^0 \triangleright X \arrow[r, "\mathrm{id}_{S^0} \triangleright f"'] & S^0 \triangleright Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ 1 \triangleright x \arrow[r, mapsto] & 1 \triangleright f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. $\square$

7.4.6 The Right Skew Right Unitor

Definition 7.4.6.1.1. The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*,\triangleright}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\triangleright}_X \colon X \triangleright S^0 \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X \triangleright S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\to X, \end{aligned}$$

where $\bigvee_{x \in X} S^0 \to X$ is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. (Proven below in a bit.) □

Remark 7.4.6.1.2. In other words, $\rho^{\mathsf{Sets}_*,\triangleright}_X$ acts on elements as

$$\begin{aligned} \rho^{\mathsf{Sets}_*,\triangleright}_X(x \triangleright 0) &\overset{\text{def}}{=} x_0, \\ \rho^{\mathsf{Sets}_*,\triangleright}_X(x \triangleright 1) &\overset{\text{def}}{=} x \end{aligned}$$

for each $x \triangleright 1 \in X \triangleright S^0$.

Remark 7.4.6.1.3. The morphism $\rho_X^{\mathsf{Sets}_*,\rhd}$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to X \rhd S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \rhd 1$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}
[\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi](x) &= \rho_X^{\mathsf{Sets}_*,\rhd}(\phi(x)) \\
&= \rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1) \\
&= x \\
&= [\mathrm{id}_X](x)
\end{aligned}$$

so that

$$\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}
[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}](x \rhd 1) &= \phi(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1)) \\
&= \phi(x) \\
&= x \rhd 1 \\
&= [\mathrm{id}_{X \rhd S^0}](x \rhd 1),
\end{aligned}$$

but

$$\begin{aligned}
[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}](x \rhd 0) &= \phi(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 0)) \\
&= \phi(x_0) \\
&= 1 \rhd x_0,
\end{aligned}$$

where $x \rhd 0 \neq 1 \rhd x_0$. Thus

$$\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd} \stackrel{?}{=} \mathrm{id}_{X \rhd S^0}$$

holds for all elements in $X \rhd S^0$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\rhd} \colon X \rhd S^0 \to X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\mathsf{Sets}_*,\triangleright}(x_0 \triangleright 0) = x_0.$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \triangleright S^0 \arrow[r, "f \triangleright \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\triangleright}"'] & Y \triangleright S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\triangleright}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \triangleright 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \triangleright 0 \arrow[r, mapsto] & f(x) \triangleright 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \triangleright 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \triangleright 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. □

7.4.7 The Diagonal

Definition 7.4.7.1.1. The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\rhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, swap, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"] & \arrow[d, phantom, "\Downarrow \Delta^{\rhd}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, swap, "\rhd"] & \end{tikzcd}$$

whose component

$$\Delta^{\rhd}_X \colon (X, x_0) \to (X \rhd X, x_0 \rhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\rhd}_X(x) \stackrel{\text{def}}{=} x \rhd x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\rhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \rhd x_0,$$

and thus $\Delta^{\rhd}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, swap, "\Delta^{\rhd}_X"] & Y \arrow[d, "\Delta^{\rhd}_Y"] \\ X \rhd X \arrow[r, swap, "f \rhd f"] & Y \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \rhd x \arrow[r, mapsto] & f(x) \rhd f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\rhd}$ to be natural. $\square$

7.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

Proposition 7.4.8.1.1. The category Sets_* admits a right-closed right skew monoidal category structure consisting of:

- *The Underlying Category.* The category Sets_* of pointed sets.
- *The Right Skew Monoidal Product.* The right tensor product functor

  $$\triangleright \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.4.1.1.1.
- *The Right Internal Skew Hom.* The right internal Hom functor

  $$[-,-]^{\triangleright}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.4.2.1.1.
- *The Right Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*,\triangleright} \colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.4.3.1.1.
- *The Right Skew Associators.* The natural transformation

  $$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 7.4.4.1.1.
- *The Right Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*})$$

  of Definition 7.4.5.1.1.
- *The Right Skew Right Unitors.* The natural transformation

  $$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.4.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & W \rhd ((X \rhd Y) \rhd Z) \arrow[dr, "\alpha^{\mathsf{Sets}_*,\rhd}_{W,X\rhd Y,Z}"] & \\
W \rhd (X \rhd (Y \rhd Z)) \arrow[ur, "\alpha^{\mathsf{Sets}_*,\rhd}_{W,X,Y} \rhd \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{W\rhd X,Y,Z}"'] & & (W \rhd (X \rhd Y)) \rhd Z \arrow[d, "\mathrm{id}_W \rhd \alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}"] \\
(W \rhd X) \rhd (Y \rhd Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\rhd}_{W,X,Y\rhd Z}"'] & & ((W \rhd X) \rhd Y) \rhd Z
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & w \rhd ((x \rhd y) \rhd z) \arrow[dr, mapsto] & \\
w \rhd (x \rhd (y \rhd z)) \arrow[ur, mapsto] \arrow[d, mapsto] & & (w \rhd (x \rhd y)) \rhd z \arrow[d, mapsto] \\
(w \rhd x) \rhd (y \rhd z) \arrow[rr, mapsto] & & ((w \rhd x) \rhd y) \rhd z
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Right Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"'] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\rhd}_{X} \rhd \mathrm{id}_Y"] & \\
S^0 \rhd (X \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (S^0 \rhd X) \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
1 \rhd (x \rhd y) \arrow[r, mapsto] & (1 \rhd x) \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Right Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd (Y \rhd S^0) \arrow[r, "\mathrm{id}_X \rhd \rho^{\mathsf{Sets}_*,\rhd}_{Y}"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (X \rhd Y) \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{X \rhd Y}"] \\
 & X \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd (y \rhd 0) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 0 \arrow[d, mapsto] \\
 & x \rhd y_0 = x_0 \rhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \rhd (y \rhd 1) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 1 \arrow[d, mapsto] \\
 & x \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Right Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \triangleright Y \arrow[r, equal] \arrow[d, "\mathrm{id}_X \triangleright \lambda_Y^{\mathsf{Sets}_*, \triangleright}"'] & X \triangleright Y \\
X \triangleright (S^0 \triangleright Y) \arrow[r, "\alpha^{\mathsf{Sets}_*, \triangleright}_{X, S^0, Y}"'] & (X \triangleright S^0) \triangleright Y \arrow[u, "\rho_X^{\mathsf{Sets}_*, \triangleright} \triangleright \mathrm{id}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \triangleright y \arrow[r, mapsto] \arrow[d, mapsto] & x \triangleright y \\
x \triangleright (1 \triangleright y) \arrow[r, mapsto] & (x \triangleright 1) \triangleright y \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\lambda_{S^0}^{\mathsf{Sets}_*, \triangleright}"] \arrow[dr, equal] & S^0 \triangleright S^0 \arrow[d, "\rho_{S^0}^{\mathsf{Sets}_*, \triangleright}"] \\
& S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \triangleright 0 \arrow[d, mapsto] \\
& 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \triangleright 1 \arrow[d, mapsto] \\
& 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Right Skew Monoidal Right-Closedness: This follows from Item 2 of Definition 7.4.1.1.7.

□

7.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

Proposition 7.4.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ is isomorphic to the category of "monoids with right zero'[20] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \rhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \rhd A \to A,$$

  determining a right bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

[20] A monoid with right zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

1. *Associativity.* The diagram

$$
\begin{tikzcd}
 & A \triangleright (A \triangleright A) \arrow[dr, "\mathrm{id}_A \triangleright \mu_A"] & \\
(A \triangleright A) \triangleright A \arrow[ur, "\alpha^{\mathsf{Sets}_{*,\triangleright}}_{A,A,A}"] \arrow[d, "\mu_A \triangleright \mathrm{id}_A"'] & & A \triangleright A \arrow[d, "\mu_A"] \\
A \triangleright A \arrow[rr, "\mu_A"'] & & A
\end{tikzcd}
$$

2. *Left Unitality.* The diagram

$$
\begin{tikzcd}
A \arrow[r, "\lambda^{\mathsf{Sets}_{*,\triangleright}}_{A}"] \arrow[d, equal] & S^0 \triangleright A \arrow[d, "\eta_A \times \mathrm{id}_A"] \\
A & A \triangleright A \arrow[l, "\mu_A"]
\end{tikzcd}
$$

commutes.

3. *Right Unitality.* The diagram

$$
\begin{tikzcd}
A \triangleright S^0 \arrow[r, "\mathrm{id}_A \times \eta_A"] \arrow[dr, "\rho^{\mathsf{Sets}_{*,\triangleright}}_{A}"'] & A \triangleright A \arrow[d, "\mu_A"] \\
 & A
\end{tikzcd}
$$

commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$
\begin{tikzcd}
(a \triangleright b) \triangleright c \arrow[d, mapsto] & \\
ab \triangleright c \arrow[r, mapsto] & (ab)c
\end{tikzcd}
\qquad
\begin{tikzcd}
 & a \triangleright (b \triangleright c) \arrow[dr, mapsto] & \\
(a \triangleright b) \triangleright c \arrow[ur, mapsto] & & a \triangleright bc \arrow[d, mapsto] \\
 & & a(bc)
\end{tikzcd}
$$

   This gives

$$(ab)c = a(bc)$$

   for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts as

$$
\begin{tikzcd}
a \arrow[d, mapsto] \\
a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] & 1 \triangleright a \arrow[d, mapsto] \\
1_A a & 1_A \triangleright a \arrow[l, mapsto]
\end{tikzcd}
$$

   This gives

$$1_A a = a$$

   for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts:

   (a) On $1 \triangleright 0$ as

$$
\begin{tikzcd}
1 \triangleright 0 \arrow[dr, mapsto] & \\
 & 0_A
\end{tikzcd}
\qquad
\begin{tikzcd}
a \triangleright 0 \arrow[r, mapsto] \arrow[dr, mapsto] & a \triangleright 0_A \arrow[d, mapsto] \\
 & a 0_A.
\end{tikzcd}
$$

   (b) On $a \triangleright 1$ as

$$
\begin{tikzcd}
a \triangleright 1 \arrow[dr, mapsto] & \\
 & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \triangleright 1 \arrow[r, mapsto] \arrow[dr, mapsto] & a \triangleright 1_A \arrow[d, mapsto] \\
 & a 1_A.
\end{tikzcd}
$$

This gives

$$\begin{aligned} a1_A &= a, \\ a0_A &= 0_A \end{aligned}$$

for each $a \in A$.

Thus we see that monoids with respect to $\rhd$ are exactly monoids with right zero.
Morphisms of Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \rhd A \arrow[r, "f \rhd f"] \arrow[d, "\mu_A"'] & B \rhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

   commutes.

These act on elements as

$$\begin{tikzcd} a \rhd b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \rhd b \arrow[r, mapsto] & f(a) \rhd f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & 0_B & & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & 1_B & & f(1_A) \end{tikzcd}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \rhd, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. □

7.5 The Smash Product of Pointed Sets

7.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.5.1.1.1. The **smash product of** (X, x_0) **and** (Y, y_0)[21] is the pointed set $X \wedge Y$[22] satisfying the bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathsf{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[21] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also called the **tensor product of $\mathbb{F}_1$-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[22] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash

Remark 7.5.1.1.2. That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f\colon X \wedge Y \to Z$.
- Maps of sets $f\colon X \times Y \to Z$ satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Remark 7.5.1.1.3. The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of
  - A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
  - A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  satisfying the following universal property:

  (★) Given another such pair $((Z, z_0), f)$ consisting of
  - * A pointed set (Z, z_0);
  - * A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

Construction 7.5.1.1.4. Concretely, the smash product of (X, x_0) and (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of:

product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$X \wedge Y \cong (X \times Y)/{\sim_R},$$

where $\sim_R$ is the equivalence relation on $X \times Y$ obtained by declaring

$$\begin{aligned}(x_0, y) &\sim_R (x_0, y'),\\ (x, y_0) &\sim_R (x', y_0)\end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

Proof. By Conditions on Relations, **??**, we have a natural bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z),$$

where $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; \begin{array}{l}\text{for each } x, y \in X, \text{ if}\\ (x, y) \sim_R (x', y'), \text{ then}\\ f(x, y) = f(x', y')\end{array} \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. The following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ with a corresponding $\overline{f}\colon X \wedge Y \to Z$, the latter case above implies

$$\begin{aligned} f(x_0, y) &= f(x, y_0)\\ &= f(x_0, y_0), \end{aligned}$$

and since $\overline{f}\colon X \wedge Y \to Z$ is a pointed map, we have

$$f(x_0, y_0) = \overline{f}(x_0, y_0)$$

$$= z_0.$$

Thus the elements f in $\mathsf{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f\colon X \times Y \to Z$ satisfying the equalities

$$\begin{aligned}f(x_0, y) &= z_0,\\ f(x, y_0) &= z_0\end{aligned}$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathsf{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) = \mathsf{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathsf{Hom}^R_{\mathsf{Sets}}(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. □

Remark 7.5.1.1.5. It is also somewhat common to write

$$X \wedge Y \overset{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

Construction 7.5.1.1.6. Alternatively, the smash product of (X, x_0) and (Y, y_0) may be constructed as the pointed set $X \wedge Y$ given by

$$\begin{aligned}X \wedge Y &\cong \bigvee_{x \in X^-} Y\\ &\cong \bigvee_{y \in Y^-} X.\end{aligned}$$

Proof. Indeed, since $X \cong \bigvee_{x \in X^-} S^0$, we have

$$\begin{aligned}X \wedge Y &\cong \Big(\bigvee_{x \in X^-} S^0\Big) \wedge Y\\ &\cong \bigvee_{x \in X^-} S^0 \wedge Y\end{aligned}$$

$$\cong \bigvee_{x \in X^-} Y,$$

where we have used that $\wedge$ preserves colimits in both variables via **??** for the second isomorphism above, since it has right adjoints in both variables by Item 2.

A similar proof applies to the isomorphism $X \wedge Y \cong \bigvee_{y \in Y^-} X$. □

Notation 7.5.1.1.7. We write $x \wedge y$ for the element $[(x, y)]$ of $X \wedge Y \cong X \times Y/\sim$.

Remark 7.5.1.1.8. Employing the notation introduced in Definition 7.5.1.1.7, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Example 7.5.1.1.9. Here are some examples of smash products of pointed sets.

1. *Smashing With pt.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} \mathrm{pt} \wedge X &\cong \mathrm{pt}, \\ X \wedge \mathrm{pt} &\cong \mathrm{pt}. \end{aligned}$$

2. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 7.5.1.1.10. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors

$$
\begin{aligned}
X \wedge -\colon &\quad \mathsf{Sets}_* \phantom{{}\times \mathsf{Sets}_*} \to \mathsf{Sets}_*,\\
- \wedge Y\colon &\quad \mathsf{Sets}_* \phantom{{}\times \mathsf{Sets}_*} \to \mathsf{Sets}_*,\\
-_1 \wedge -_2\colon &\quad \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}
$$

   In particular, given pointed maps

$$
\begin{aligned}
f\colon (X, x_0) &\to (A, a_0),\\
g\colon (Y, y_0) &\to (B, b_0),
\end{aligned}
$$

   the induced map

$$f \wedge g\colon X \wedge Y \to A \wedge B$$

   is given by

$$[f \wedge g](x \wedge y) \overset{\text{def}}{=} f(x) \wedge g(y)$$

   for each $x \wedge y \in X \wedge Y$.

2. *Adjointness.* We have adjunctions

$$
(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad
\begin{tikzcd}
\mathsf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"]
\end{tikzcd}
$$

$$
(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad
\begin{tikzcd}
\mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"]
\end{tikzcd}
$$

   witnessed by bijections

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\
\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),
\end{aligned}
$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$
(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad
\begin{tikzcd}
\mathbf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"]
\end{tikzcd}
$$

$$
(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad
\begin{tikzcd}
\mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"]
\end{tikzcd}
$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

natural in $X, Y \in \mathrm{Obj}(\mathrm{Sets}_*)$, where the pushout is taken in Sets, and the embedding $\iota\colon X \vee Y \hookrightarrow X \times Y$ is defined following Definition 7.5.1.1.5.

5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned}X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z),\\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathrm{Sets}_*)$.

Proof. *Item 1, Functoriality*: The map $f \wedge g$ comes from Conditions on Relations, Item 4 of Definition 10.6.2.1.3 via the map

$$f \wedge g\colon X \times Y \to A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where $\sim_R$ is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y)/\sim_R$$

in Definition 7.5.1.1.4. The condition defining $\sim$ is that at least one of the following conditions is satisfied:

1. We have $x = x'$ and $y = y'$;

2. Both of the following conditions are satisfied:

   (a) We have $x = x_0$ or $y = y_0$.

   (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

1. In the first case, we clearly have

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

   since $x = x'$ and $y = y'$.

2. If $x = x_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y)\\ &= a_0 \wedge g(y)\\ &= a_0 \wedge g(y')\\ &= f(x_0) \wedge g(y')\\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

3. If $x = x_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y)\\ &= a_0 \wedge g(y)\\ &= a_0 \wedge b_0\\ &= f(x') \wedge b_0\\ &= f(x') \wedge g(y_0)\\ &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

4. If $y = y_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0)\\ &= f(x) \wedge b_0\\ &= a_0 \wedge b_0\\ &= a_0 \wedge g(y')\\ &= f(x_0) \wedge g(y')\\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

5. If $y = y_0$ and $y' = y_0$, we have

$$
\begin{aligned}
[f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\
&= f(x) \wedge b_0 \\
&= f(x') \wedge b_0 \\
&= f(x) \wedge g(y_0) \\
&\stackrel{\text{def}}{=} [f \wedge g](x', y_0).
\end{aligned}
$$

Thus $f \wedge g$ is well-defined. Next, we claim that $\wedge$ preserves identities and composition:

- *Preservation of Identities.* We have

$$
\begin{aligned}
[\mathrm{id}_X \wedge \mathrm{id}_Y](x \wedge y) &\stackrel{\text{def}}{=} \mathrm{id}_X(x) \wedge \mathrm{id}_Y(y) \\
&= x \wedge y \\
&= [\mathrm{id}_{X \wedge Y}](x \wedge y)
\end{aligned}
$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$
\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y} .
$$

- *Preservation of Composition.* Given pointed maps

$$
\begin{aligned}
f &\colon (X, x_0) \to (X', x_0'), \\
h &\colon (X', x_0') \to (X'', x_0''), \\
g &\colon (Y, y_0) \to (Y', y_0'), \\
k &\colon (Y', y_0') \to (Y'', y_0''),
\end{aligned}
$$

  we have

$$
\begin{aligned}
[(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\
&\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\
&\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\
&\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y)
\end{aligned}
$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$
(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).
$$

This finishes the proof.
Item 2, *Adjointness*: We prove only the adjunction $- \wedge Y \dashv \mathbf{Sets}_*(Y,-)$, witnessed by a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y,Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathbf{Sets}_*(X,-)$ is similar. We claim we have a bijection

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y,Z))$$

natural in $(X,x_0), (Y,y_0), (Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, impliying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(X \times Y, Z) \cong \mathsf{Sets}(X, \mathsf{Sets}(Y,Z))$$

of Constructions With Sets, Item 2 of Definition 4.1.3.1.3:

- A map
  $$\xi \colon X \times Y \to Z$$
  in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ gets sent to the pointed map
  $$\begin{aligned} \xi^{\dagger} \colon (X,x_0) &\longrightarrow (\mathbf{Sets}_*(Y,Z), \Delta_{z_0}), \\ x &\longmapsto (\xi^{\dagger}_x \colon Y \to Z), \end{aligned}$$
  where $\xi^{\dagger}_x \colon Y \to Z$ is the map defined by
  $$\xi^{\dagger}_x(y) \overset{\mathrm{def}}{=} \xi(x,y)$$
  for each $y \in Y$, where:
  - The map $\xi^{\dagger}$ is indeed pointed, as we have
    $$\begin{aligned} \xi^{\dagger}_{x_0}(y) &\overset{\mathrm{def}}{=} \xi(x_0,y) \\ &\overset{\mathrm{def}}{=} z_0 \end{aligned}$$
    for each $y \in Y$. Thus $\xi^{\dagger}_{x_0} = \Delta_{z_0}$ and $\xi^{\dagger}$ is pointed.
  - The map $\xi^{\dagger}_x$ indeed lies in $\mathbf{Sets}_*(Y,Z)$, as we have
    $$\begin{aligned} \xi^{\dagger}_x(y_0) &\overset{\mathrm{def}}{=} \xi(x,y_0) \\ &\overset{\mathrm{def}}{=} z_0. \end{aligned}$$

- Conversely, a map

$$\begin{aligned} \xi\colon (X,x_0) &\to (\mathsf{Sets}_*(Y,Z),\Delta_{z_0}),\\ x &\longmapsto (\xi_x\colon Y\to Z), \end{aligned}$$

 in $\mathrm{Hom}_{\mathsf{Sets}_*}(X,\mathsf{Sets}_*(Y,Z))$ gets sent to the map

$$\xi^\dagger\colon X\times Y\to Z$$

defined by

$$\xi^\dagger(x,y)\overset{\text{def}}{=}\xi_x(y)$$

for each $(x,y)\in X\times Y$, which indeed lies in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X\times Y,Z)$, as:

  - *Left Bilinearity*. We have

$$\begin{aligned} \xi^\dagger(x_0,y) &\overset{\text{def}}{=} \xi_{x_0}(y)\\ &\overset{\text{def}}{=} \Delta_{z_0}(y)\\ &\overset{\text{def}}{=} z_0 \end{aligned}$$

    for each $y\in Y$, since $\xi_{x_0}=\Delta_{z_0}$ as ξ is assumed to be a pointed map.

  - *Right Bilinearity*. We have

$$\begin{aligned} \xi^\dagger(x,y_0) &\overset{\text{def}}{=} \xi_x(y_0)\\ &\overset{\text{def}}{=} z_0 \end{aligned}$$

    for each $x\in X$, since $\xi_x\in\mathsf{Sets}_*(Y,Z)$ is a morphism of pointed sets.

This finishes the proof.

Item 3, *Enriched Adjointness*: This follows from Item 2 and Monoidal Categories, **??** of **??**.

Item 4, *As a Pushout*: Following the description of Constructions With Sets, Definition 4.2.4.1.3, we have

$$\mathrm{pt}\coprod_{X\vee Y}(X\times Y)\cong(\mathrm{pt}\times(X\times Y))/\sim,$$

where $\sim$ identifies the elemenet $\star$ in pt with all elements of the form (x_0,y) and (x,y_0) in $X\times Y$. Thus Conditions on Relations, Item 4 of Definition 10.6.2.1.3 coupled with Definition 7.5.1.1.8 then gives us a well-defined map

$$\mathrm{pt}\coprod_{X\vee Y}(X\times Y)\to X\wedge Y$$

via $[(\star,(x,y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \to \mathrm{pt} \textstyle\coprod_{X \vee Y} (X \times Y)$$

given by $x \wedge y \mapsto [(\star,(x,y))]$.
Item 5, Distributivity Over Wedge Sums: This follows from Definition 7.5.9.1.1, Monoidal Categories, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_* (Pointed Sets, Definition 6.3.3.1.1). □

7.5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.5.2.1.1. The **internal Hom**[23] **of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$[24] consisting of:

- *The Underlying Set.* The set $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0).
- *The Basepoint.* The element

  $$\Delta_{y_0} \colon (X, x_0) \to (Y, y_0)$$

  of $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ given by

  $$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

  for each $x \in X$.

Proposition 7.5.2.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$ define functors

$$\begin{aligned} \mathbf{Sets}_*(X, -) \colon & \quad \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ \mathbf{Sets}_*(-, Y) \colon & \quad \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*, \\ \mathbf{Sets}_*(-_1, -_2) \colon & \quad \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$f \colon (X, x_0) \to (A, a_0),$$

[23] For a proof that $\mathbf{Sets}_*$ is indeed the internal Hom of Sets_* with respect to the smash product of pointed sets, see Item 2 of Definition 7.5.1.1.10.
[24] *Further Notation:* Also written $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$.

$$g\colon (Y, y_0) \longrightarrow (B, b_0),$$

the induced map

$$\mathbf{Sets}_*(f, g)\colon \mathbf{Sets}_*(A, Y) \longrightarrow \mathbf{Sets}_*(X, B)$$

is given by

$$[\mathbf{Sets}_*(f, g)](\phi) \stackrel{\text{def}}{=} g \circ \phi \circ f$$

for each $\phi \in \mathbf{Sets}_*(A, Y)$.

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof. *Item 1, Functoriality*: This follows from Constructions With Sets, Item 1 of Definition 4.3.5.1.2 and from the equalities

$$\begin{aligned} g \circ \Delta_{y_0} &= \Delta_{z_0}, \\ \Delta_{y_0} \circ f &= \Delta_{y_0} \end{aligned}$$

for morphisms $f\colon (K, k_0) \to (X, x_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2, Adjointness: This is a repetition of Item 2 of Definition 7.5.1.1.10, and is proved there.

Item 3, Enriched Adjointness: This is a repetition of Item 3 of Definition 7.5.1.1.10, and is proved there. □

7.5.3 The Monoidal Unit

Definition 7.5.3.1.1. The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \to \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

7.5.4 The Associator

Definition 7.5.4.1.1. The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\wedge"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\wedge\times\mathrm{id}"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"'] \arrow[urr, phantom, "\Uparrow \alpha^{\mathsf{Sets}_*}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\colon (X \wedge Y) \wedge Z \overset{\sim}{\dashrightarrow} X \wedge (Y \wedge Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x \wedge y) \wedge z) \overset{\text{def}}{=} x \wedge (y \wedge z)$$

for each $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$.

Proof. *Well-Definedness*: Let $[((x, y), z)] = [((x', y'), z')]$ be an element in $(X \wedge Y) \wedge Z$. Then either:

1. We have $x = x'$, $y = y'$, and $z = z'$.

2. Both of the following conditions are satisfied:

   (a) We have $x = x_0$ or $y = y_0$ or $z = z_0$.

   (b) We have $x' = x_0$ or $y' = y_0$ or $z' = z_0$.

In the first case, $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ clearly sends both elements to the same element in $X \wedge (Y \wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0 \wedge y_0) \wedge z_0$ of $(X \wedge Y) \wedge Z$, which gets sent to the basepoint $x_0 \wedge (y_0 \wedge z_0)$ of $X \wedge (Y \wedge Z)$.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x_0 \wedge y_0) \wedge z_0) \overset{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is a morphism of pointed sets.

Invertibility: The inverse of $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is given by the morphism

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}\colon X \wedge (Y \wedge Z) \overset{\sim}{\dashrightarrow} (X \wedge Y) \wedge Z$$

defined by

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}(x \wedge (y \wedge z)) \overset{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.

Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f\colon (X, x_0) &\to (X', x'_0), \\ g\colon (Y, y_0) &\to (Y', y'_0), \end{aligned}$$

$$h\colon (Z, z_0) \to (Z', z'_0)$$

the diagram

$$\begin{tikzcd}
(X \wedge Y) \wedge Z \arrow[r, "(f\wedge g)\wedge h"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X',Y',Z'}"] \\
X \wedge (Y \wedge Z) \arrow[r, "f\wedge(g\wedge h)"'] & X' \wedge (Y' \wedge Z')
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \wedge y) \wedge z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \wedge g(y)) \wedge h(z) \arrow[d, mapsto] \\
x \wedge (y \wedge z) \arrow[r, mapsto] & f(x) \wedge (g(y) \wedge h(z))
\end{tikzcd}$$

and hence indeed commutes, showing α^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since α^{Sets_*} is natural and $\alpha^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to α^{Sets_*}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\alpha^{\mathsf{Sets}_*,-1}$ is also natural. Thus α^{Sets_*} is a natural isomorphism. □

7.5.5 The Left Unitor

Definition 7.5.5.1.1. The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}_*}\colon \wedge \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \qquad \begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}"] \arrow[rd, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[ld, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*}_X\colon S^0 \wedge X \overset{\sim}{\dashrightarrow} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}0 \wedge x &\mapsto x_0,\\ 1 \wedge x &\mapsto x\end{aligned}$$

for each $x \in X$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $S^0 \wedge X$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = 0$ or $y = x_0$.
   (b) We have $x' = 0$ or $y' = x_0$.

In the first case, $\lambda^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $0 \wedge x_0$ of $S^0 \wedge X$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\lambda^{\mathsf{Sets}_*}_X(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus $\lambda^{\mathsf{Sets}_*}_X$ is a morphism of pointed sets.
Invertibility: The inverse of $\lambda^{\mathsf{Sets}_*}_X$ is the morphism

$$\lambda^{\mathsf{Sets}_*,-1}_X : X \overset{\sim}{\dashrightarrow} S^0 \wedge X$$

defined by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$. Indeed:

1. *Invertibility I.* We have

$$\begin{aligned}[\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X](0 \wedge x) &= \lambda^{\mathsf{Sets}_*,-1}_X(\lambda^{\mathsf{Sets}_*}_X(0 \wedge x))\\ &= \lambda^{\mathsf{Sets}_*,-1}_X(x_0)\\ &= 1 \wedge x_0\\ &= 0 \wedge x,\end{aligned}$$

and

$$\begin{aligned}
[\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X](1 \wedge x) &= \lambda^{\mathsf{Sets}_*,-1}_X(\lambda^{\mathsf{Sets}_*}_X(1 \wedge x)) \\
&= \lambda^{\mathsf{Sets}_*,-1}_X(x) \\
&= 1 \wedge x
\end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X = \mathrm{id}_{S^0 \wedge X} .$$

2. *Invertibility II*. We have

$$\begin{aligned}
[\lambda^{\mathsf{Sets}_*}_X \circ \lambda^{\mathsf{Sets}_*,-1}_X](x) &= \lambda^{\mathsf{Sets}_*}_X(\lambda^{\mathsf{Sets}_*,-1}_X(x)) \\
&= \lambda^{\mathsf{Sets}_*,-1}_X(1 \wedge x) \\
&= x
\end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda^{\mathsf{Sets}_*}_X \circ \lambda^{\mathsf{Sets}_*,-1}_X = \mathrm{id}_X .$$

This shows $\lambda^{\mathsf{Sets}_*}_X$ to be invertible.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \longrightarrow (Y, y_0),$$

the diagram

$$\begin{tikzcd}
S^0 \wedge X \arrow[r, "\mathrm{id}_{S^0} \wedge f"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_X"'] & S^0 \wedge Y \arrow[d, "\lambda^{\mathsf{Sets}_*}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \wedge x \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
0 \wedge x \arrow[r, mapsto] & 0 \wedge f(x) \arrow[d, mapsto] \\
& y_0
\end{tikzcd}$$

and

$$\begin{tikzcd} 1 \wedge x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \wedge f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*}$ to be a natural transformation.
Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets}_*}$ is natural and $\lambda^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}_*}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets}_*,-1}$ is also natural. Thus $\lambda^{\mathsf{Sets}_*}$ is a natural isomorphism. □

7.5.6 The Right Unitor

Definition 7.5.6.1.1. The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}_*} \colon \wedge \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[rd, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[ld, phantom, "\Swarrow \rho^{\mathsf{Sets}_*}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*}_X \colon X \wedge S^0 \overset{\sim}{\dashrightarrow} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge S^0$. Then either:

1. We have $x = x'$ and $y = y'$.

2. Both of the following conditions are satisfied:

    (a) We have $x = x_0$ or $y = 0$.

    (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge 0$ of $X \wedge S^0$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\rho_X^{\mathsf{Sets}_*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus $\rho_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: The inverse of $\rho_X^{\mathsf{Sets}_*}$ is the morphism

$$\rho_X^{\mathsf{Sets}_*,-1} \colon X \overset{\sim}{\dashrightarrow} X \wedge S^0$$

defined by

$$\rho_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

1. *Invertibility I*. We have

$$\begin{aligned}
[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}](x \wedge 0) &= \rho_X^{\mathsf{Sets}_*,-1}(\rho_X^{\mathsf{Sets}_*}(x \wedge 0)) \\
&= \rho_X^{\mathsf{Sets}_*,-1}(x_0) \\
&= x_0 \wedge 1 \\
&= x \wedge 0,
\end{aligned}$$

   and

$$\begin{aligned}
[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}](x \wedge 1) &= \rho_X^{\mathsf{Sets}_*,-1}(\rho_X^{\mathsf{Sets}_*}(x \wedge 1)) \\
&= \rho_X^{\mathsf{Sets}_*,-1}(x) \\
&= x \wedge 1
\end{aligned}$$

   for each $x \in X$, and thus we have

$$\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*} = \mathrm{id}_{X \wedge S^0}.$$

2. *Invertibility II.* We have

$$\begin{aligned}[\rho^{\mathsf{Sets}_*}_X \circ \rho^{\mathsf{Sets}_*,-1}_X](x) &= \rho^{\mathsf{Sets}_*}_X(\rho^{\mathsf{Sets}_*,-1}_X(x)) \\ &= \rho^{\mathsf{Sets}_*,-1}_X(x \wedge 1) \\ &= x\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho^{\mathsf{Sets}_*}_X \circ \rho^{\mathsf{Sets}_*,-1}_X = \mathrm{id}_X .$$

This shows $\rho^{\mathsf{Sets}_*}_X$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \wedge S^0 \arrow[r, "f \wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*}_X"'] & Y \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \wedge 0 \arrow[r, mapsto] & f(x) \wedge 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \wedge 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing ρ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since ρ^{Sets_*} is natural and $\rho^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to ρ^{Sets_*}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\rho^{\mathsf{Sets}_*,-1}$ is also natural. Thus ρ^{Sets_*} is a natural isomorphism. □

7.5.7 The Symmetry

Definition 7.5.7.1.1. The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}"'] \arrow[dr, phantom, shift left=4, "\overset{\|}{\underset{\Downarrow}{\sigma^{\mathsf{Sets}_*}}}"] & & \mathsf{Sets}_*, \\ & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}_*}_{X,Y} \colon X \wedge Y \overset{\sim}{\dashrightarrow} Y \wedge X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ is defined by

$$\sigma^{\mathsf{Sets}_*}_{X,Y}(x \wedge y) \overset{\mathrm{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge Y$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\sigma^{\mathsf{Sets}_*}_X(x_0 \wedge y_0) \overset{\mathrm{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma^{\mathsf{Sets}_*}_X$ is a morphism of pointed sets.
Invertibility: The inverse of $\sigma^{\mathsf{Sets}_*}_{X,Y}$ is given by the morphism

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y} \colon Y \wedge X \overset{\sim}{\dashrightarrow} X \wedge Y$$

defined by

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y}(y \wedge x) \stackrel{\text{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.

Naturality: We need to show that, given morphisms of pointed sets

$$f\colon (X, x_0) \to (A, a_0),$$
$$g\colon (Y, y_0) \to (B, b_0)$$

the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "f \wedge g"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"'] & A \wedge B \arrow[d, "\sigma^{\mathsf{Sets}_*}_{A,B}"] \\ Y \wedge X \arrow[r, "g \wedge f"'] & B \wedge A \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge g(y) \arrow[d, mapsto] \\ y \wedge x \arrow[r, mapsto] & g(y) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since σ^{Sets_*} is natural and $\sigma^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to σ^{Sets_*}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\sigma^{\mathsf{Sets}_*,-1}$ is also natural. Thus σ^{Sets_*} is a natural isomorphism. □

7.5.8 The Diagonal

Definition 7.5.8.1.1. The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \Big\| \ \Downarrow \Delta^{\wedge} & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \wedge \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta^{\wedge}_X \colon (X, x_0) \to (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}(X, x_0) &\xrightarrow{\Delta^{\wedge}_X} (X \times X, (x_0, x_0))\\ &\twoheadrightarrow ((X \times X)/\sim, [(x_0, x_0)])\\ &\overset{\text{def}}{=\!=} (X \wedge X, x_0 \wedge x_0)\end{aligned}$$

in Sets_*, and thus by

$$\Delta^{\wedge}_X(x) \overset{\text{def}}{=} x \wedge x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\wedge}_X(x_0) \overset{\text{def}}{=} x_0 \wedge x_0,$$

and thus $\Delta^{\wedge}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\wedge}_X"'] & Y \arrow[d, "\Delta^{\wedge}_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \wedge x \arrow[r, mapsto] & f(x) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\wedge}$ to be natural. □

Proposition 7.5.8.1.2. Let $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Monoidality.* The diagonal
$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},$$
of the smash product of pointed sets is a monoidal natural transformation:
   (a) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the diagram
$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$
   commutes.
   (b) *Compatibility With Strong Unitality Constraints.* The diagrams
$$\begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd} \qquad \begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd}$$
   commute, i.e. we have
$$\begin{aligned} \Delta^{\wedge}_{S^0} &= \lambda^{\mathsf{Sets}_*, -1}_{S^0} \\ &= \rho^{\mathsf{Sets}_*, -1}_{S^0}, \end{aligned}$$
   where we recall that the equalities
$$\begin{aligned} \lambda^{\mathsf{Sets}_*}_{S^0} &= \rho^{\mathsf{Sets}_*}_{S^0}, \\ \lambda^{\mathsf{Sets}_*, -1}_{S^0} &= \rho^{\mathsf{Sets}_*, -1}_{S^0} \end{aligned}$$
   are always true in any monoidal category by Monoidal Categories, **??** of **??**.

2. *The Diagonal of the Unit.* The component

$$\Delta^{\wedge}_{S^0}\colon S^0 \overset{\sim}{\dashrightarrow} S^0 \wedge S^0$$

of $\Delta^{\wedge}$ at S^0 is an isomorphism.

Proof. *Item 1*, *Monoidality*: We claim that $\Delta^{\wedge}$ is indeed monoidal:

1. *Item 1a*: *Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X\wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\
& (X \wedge Y) \wedge (X \wedge Y)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \wedge y \arrow[r, mapsto] \arrow[dr, mapsto] & (x \wedge x) \wedge (y \wedge y) \arrow[d, mapsto] \\
& (x \wedge y) \wedge (x \wedge y)
\end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b*: *Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 7.5.5.1.1, the inverse of the left unitor of Sets_* with respect to to the smash product of pointed sets at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x) \overset{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\begin{aligned}
\lambda^{\mathsf{Sets}_*,-1}_{S^0}(0) &\overset{\text{def}}{=} 1 \wedge 0,\\
\lambda^{\mathsf{Sets}_*,-1}_{S^0}(1) &\overset{\text{def}}{=} 1 \wedge 1.
\end{aligned}$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\begin{aligned}
\Delta^{\wedge}_{S^0}(0) &\overset{\text{def}}{=} 0 \wedge 0,\\
\Delta^{\wedge}_{S^0}(1) &\overset{\text{def}}{=} 1 \wedge 1,
\end{aligned}$$

it follows that we indeed have $\Delta^{\wedge}_{S^0} = \lambda^{\mathsf{Sets}_*,-1}_{S^0}$.

This finishes the proof.
Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets_* with respect to $\wedge$, proved in the proof of Definition 7.5.5.1.1 for the left unitor or the proof of Definition 7.5.6.1.1 for the right unitor. □

7.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

Proposition 7.5.9.1.1. The category Sets_* admits a closed monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category Sets_* of pointed sets.
- *The Monoidal Product.* The smash product functor

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Definition 7.5.1.1.10.
- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}_*\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Definition 7.5.2.1.2.
- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 7.5.3.1.1.
- *The Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 7.5.4.1.1.
- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}_*}\colon \wedge \circ (\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.5.5.1.1.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}_*} \colon \wedge \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 7.5.6.1.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}$$

of Definition 7.5.7.1.1.

- *The Diagonals.* The monoidal natural transformation

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 7.5.8.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \wedge (X \wedge Y)) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{W,X\wedge Y,Z}"] & \\
((W \wedge X) \wedge Y) \wedge Z \arrow[ur, "\alpha^{\mathsf{Sets}_*}_{W,X,Y} \wedge \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{W\wedge X,Y,Z}"'] & & W \wedge ((X \wedge Y) \wedge Z) \arrow[d, "\mathrm{id}_W \wedge \alpha^{\mathsf{Sets}_*}_{X,Y,Z}"] \\
(W \wedge X) \wedge (Y \wedge Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{W,X,Y\wedge Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \wedge (x \wedge y)) \wedge z \arrow[dr, mapsto] & \\
((w \wedge x) \wedge y) \wedge z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \wedge ((x \wedge y) \wedge z) \arrow[d, mapsto] \\
(w \wedge x) \wedge (y \wedge z) \arrow[rr, mapsto] & & w \wedge (x \wedge (y \wedge z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.
The Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
(X \wedge S^0) \wedge Y \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{X,S^0,Y}"] \arrow[dr, "\rho^{\mathsf{Sets}_*}_X \wedge \mathrm{id}_Y"'] & & X \wedge (S^0 \wedge Y) \arrow[dl, "\mathrm{id}_X \wedge \lambda^{\mathsf{Sets}_*}_Y"] \\
 & X \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[dr, mapsto] & \\
 & x_0 \wedge y
\end{tikzcd}
\qquad
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[r, mapsto] & x \wedge (0 \wedge y) \arrow[dl, mapsto] \\
x \wedge y_0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(x \wedge 1) \wedge y \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \wedge (1 \wedge y) \arrow[dl, mapsto] \\
 & x \wedge y, &
\end{tikzcd}$$

and thus we see that the triangle identity is satisfied.
The Left Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}[column sep=small] & (X \wedge Y) \wedge Z \arrow[dl, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] \arrow[dr, "\beta^{\mathsf{Sets}_*}_{X,Y} \wedge \mathrm{id}_Z"] & \\ X \wedge (Y \wedge Z) \arrow[d, "\beta^{\mathsf{Sets}_*}_{X,Y \wedge Z}"'] & & (Y \wedge X) \wedge Z \arrow[d, "\alpha^{\mathsf{Sets}_*}_{Y,X,Z}"] \\ (Y \wedge Z) \wedge X \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{Y,Z,X}"'] & & Y \wedge (X \wedge Z) \arrow[dl, "\mathrm{id}_Y \wedge \beta^{\mathsf{Sets}_*}_{X,Z}"] \\ & Y \wedge (Z \wedge X) & \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}[column sep=small] & (x \wedge y) \wedge z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x \wedge (y \wedge z) \arrow[d, mapsto] & & (y \wedge x) \wedge z \arrow[d, mapsto] \\ (y \wedge z) \wedge x \arrow[dr, mapsto] & & y \wedge (x \wedge z) \arrow[dl, mapsto] \\ & y \wedge (z \wedge x) & \end{tikzcd}$$

and thus we see that the left hexagon identity is satisfied.
The Right Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have

to show that the diagram

$$\begin{tikzcd}
 & X \wedge (Y \wedge Z) \arrow[dl, "(\alpha^{\mathsf{Sets}_*}_{X,Y,Z})^{-1}"'] \arrow[dr, "\mathrm{id}_X \wedge \beta^{\mathsf{Sets}_*}_{Y,Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\beta^{\mathsf{Sets}_*}_{X \wedge Y, Z}"'] & & X \wedge (Z \wedge Y) \arrow[d, "(\alpha^{\mathsf{Sets}_*}_{X,Z,Y})^{-1}"] \\
Z \wedge (X \wedge Y) \arrow[dr, "(\alpha^{\mathsf{Sets}_*}_{Z,X,Y})^{-1}"'] & & (X \wedge Z) \wedge Y \arrow[dl, "\beta^{\mathsf{Sets}_*}_{X,Z} \wedge \mathrm{id}_Y"] \\
 & (Z \wedge X) \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & x \wedge (y \wedge z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(x \wedge y) \wedge z \arrow[d, mapsto] & & x \wedge (z \wedge y) \arrow[d, mapsto] \\
z \wedge (x \wedge y) \arrow[dr, mapsto] & & (x \wedge z) \wedge y \arrow[dl, mapsto] \\
 & (z \wedge x) \wedge y &
\end{tikzcd}$$

and thus we see that the right hexagon identity is satisfied.

Monoidal Closedness: This follows from Item 2 of Definition 7.5.1.1.10.

Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 7.5.8.1.2.

□

7.5.10 The Universal Property of $(\mathsf{Sets}_*, \wedge, S^0)$

Theorem 7.5.10.1.1. The symmetric monoidal structure on the category Sets_* of Definition 7.5.9.1.1 is uniquely determined by the following requirements:

1. *Existence of an Internal Hom*. The tensor product

$$\otimes_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

of Sets_* admits an internal Hom $[-_1, -_2]_{\mathsf{Sets}_*}$.

2. *The Unit Object Is S^0*. We have $\mathbb{1}_{\mathsf{Sets}_*} \cong S^0$.

More precisely, the full subcategory of the category $\mathcal{M}^{\mathsf{cld}}_{\mathbb{E}_\infty}(\mathsf{Sets}_*)$ of **??** spanned by the closed symmetric monoidal categories $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, [-_1, -_2]_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

Proof. *Unwinding the Statement*: Let $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, [-_1, -_2]_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda', \rho', \sigma')$ be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$\mathrm{id}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*} &\colon & X \otimes_{\mathsf{Sets}_*} Y &\overset{\sim}{\dashrightarrow} X \wedge Y, \\ \mathrm{id}^{\mathsf{Hom}}_{\mathsf{Sets}_*} &\colon & [X, Y]_{\mathsf{Sets}_*} &\overset{\sim}{\dashrightarrow} \mathsf{Sets}_*(X, Y), \\ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} &\colon & \mathbb{1}_{\mathsf{Sets}_*} &\overset{\sim}{\dashrightarrow} S^0, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, [-_1, -_2]_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda', \rho', \sigma')$ to the closed symmetric monoidal category $(\mathsf{Sets}_*, \times, \mathsf{Sets}_*(-_1, -_2), \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ of Definition 7.5.9.1.1.

Constructing an Isomorphism $[-_1, -_2]_{\mathsf{Sets}_*} \cong \mathsf{Sets}_*(-_1, -_2)$: By **??**, we have a natural isomorphism

$$\mathsf{Sets}_*(S^0, [-_1, -_2]_{\mathsf{Sets}_*}) \cong \mathsf{Sets}_*(-_1, -_2).$$

By Pointed Sets, Item 4 of Definition 6.1.4.1.1, we also have a natural isomorphism

$$\mathsf{Sets}_*(S^0, [-_1, -_2]_{\mathsf{Sets}_*}) \cong [-_1, -_2]_{\mathsf{Sets}_*}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}_*(-_1, -_2) \cong [-_1, -_2]_{\mathsf{Sets}_*}.$$

Given $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we will write

$$\mathrm{id}^{\mathsf{Hom}}_{X,Y} \colon \mathsf{Sets}_*(X, Y) \overset{\sim}{\dashrightarrow} [X, Y]_{\mathsf{Sets}_*}$$

for the component of this isomorphism at (X, Y).

Constructing an Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: Since $\otimes_{\mathsf{Sets}_*}$ is adjoint in each variable to $[-_1, -_2]_{\mathsf{Sets}_*}$ by assumption and $\wedge$ is adjoint in each variable to $\mathsf{Sets}_*(-_1, -_2)$ by Constructions With Sets, Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (**??**) gives us natural isomorphisms

$$\begin{aligned} X \otimes_{\mathsf{Sets}_*} - &\cong X \wedge -, \\ - \otimes_{\mathsf{Sets}_*} Y &\cong Y \wedge -. \end{aligned}$$

By **??**, we then have $\otimes_{\mathsf{Sets}_*} \cong \wedge$. We will write

$$\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \colon X \otimes_{\mathsf{Sets}_*} Y \overset{\sim}{\dashrightarrow} X \wedge Y$$

for the component of this isomorphism at (X, Y).

Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: Alternatively, we may construct a natural isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$ as follows:

1. Let $X \in \mathrm{Obj}(\mathsf{Sets}_*)$.

2. Since $\otimes_{\mathsf{Sets}_*}$ is part of a closed monoidal structure, it preserves colimits in each variable by **??**.

3. Since $X \cong \bigvee_{x \in X^-} S^0$ and $\otimes_{\mathsf{Sets}_*}$ preserves colimits in each variable, we have

$$\begin{aligned} X \otimes_{\mathsf{Sets}_*} Y &\cong \Big(\bigvee_{x \in X^-} S^0\Big) \otimes_{\mathsf{Sets}_*} Y \\ &\cong \bigvee_{x \in X^-} (S^0 \otimes_{\mathsf{Sets}_*} Y) \\ &\cong \bigvee_{x \in X^-} Y \\ &\cong \bigvee_{x \in X^-} S^0 \wedge Y \\ &\cong \Big(\bigvee_{x \in X^-} S^0\Big) \wedge Y \\ &\cong X \wedge Y, \end{aligned}$$

   naturally in $Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, where we have used that S^0 is the monoidal unit for $\otimes_{\mathsf{Sets}_*}$. Thus $X \otimes_{\mathsf{Sets}_*} - \cong X \wedge -$ for each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. Similarly, $- \otimes_{\mathsf{Sets}_*} Y \cong - \wedge Y$ for each $Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. By **??**, we then have $\otimes_{\mathsf{Sets}_*} \cong \wedge$.

Below, we,ll show that if a natural isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \colon X \otimes_{\mathsf{Sets}_*} Y \to X \wedge Y$ from before.

Constructing an Isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}_*} \to S^0$: We define an isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}_*} \to S^0$ as the composition

$$\mathbb{1}_{\mathsf{Sets}_*} \underset{\sim}{\overset{\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}}{\dashrightarrow}} \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \underset{\sim}{\overset{\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*}}}{\dashrightarrow}} \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \underset{\sim}{\overset{\lambda'_{S^0}}{\dashrightarrow}} S^0$$

in Sets_*.

Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$\begin{tikzcd}
 & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X}"] & S^0 \wedge X \arrow[dr, "\lambda^{\mathsf{Sets}_*}_X"] & \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_X"] \arrow[rrr, "\lambda'_X"'] & & & X
\end{tikzcd}$$

commutes. To this end, we will first show that the diagram

$$\begin{tikzcd}
 & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, phantom, "(\dagger)"] & S^0 \wedge S^0 \arrow[dr, "\lambda^{\mathsf{Sets}_*}_{S^0}"] & \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[rrr, "\lambda'_{S^0}"'] & {} & & S^0,
\end{tikzcd}$$

corresponding to the case $X = S^0$, commutes. Indeed, consider the diagram

$$
\begin{tikzcd}
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,-1}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*}, S^0}"'] \arrow[dd, equals, bend right=60] \arrow[dr, phantom, "(1)"] & (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*}, S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0, S^0}"'] \arrow[dr, phantom, "(2)"] & (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\lambda'_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0, S^0}"'] \arrow[dr, phantom, "(3)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0, S^0}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[r, bend left, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}} \wedge \mathrm{id}_{S^0}"] \arrow[r, bend right, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"'] \arrow[r, phantom, "(4)"] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*}, S^0}"'] \arrow[dr, phantom, "(5)"] & (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*}, S^0} \wedge \mathrm{id}_{S^0}"] \arrow[dr, phantom, "(6)"] & (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \wedge S^0 \arrow[r, "\lambda'_{S^0} \wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"'] \arrow[dr, phantom, "(7)"] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0} = \lambda^{\mathsf{Sets}_*}_{S^0}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, equals] \arrow[urr, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}" description] & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\lambda'_{S^0}"'] & S^0,
\end{tikzcd}
$$

whose boundary diagram corresponds to the diagram (†) above. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$.
- Subdiagram (4) commutes by **??**.
- Subdiagram (5) commutes by the naturality of $\rho^{\mathsf{Sets}_*,-1}$.
- Subdiagram (6) commutes trivially.
- Subdiagram (7) commutes by the naturality of ρ^{Sets_*}, where the equality $\rho^{\mathsf{Sets}_*}_{S^0} = \lambda^{\mathsf{Sets}_*}_{S^0}$ comes from **??**.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram (†) above. As a result, the diagram

$$
\begin{tikzcd}
 & S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0, S^0}"] \arrow[d, phantom, "(\ddagger)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] & \\
S^0 \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\lambda'^{,-1}_{S^0}"'] & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0
\end{tikzcd}
$$

also commutes. Now, let $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, and consider the diagram

$$\begin{tikzcd}
 & S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[r, phantom, "(\ddagger)"', shift right=4] \arrow[dd, "\mathrm{id}_{S^0} \wedge [x]"'] \arrow[ddr, phantom, "(1)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0}"] \arrow[dd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} [x]"'] \arrow[ddr, phantom, "(4)"] & \\
S^0 \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\lambda'^{,-1}_{S^0}"] \arrow[dd, "{[x]}"'] \arrow[ddr, phantom, "(3)"] & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}_*}} \wedge [x]"] \\
 & S^0 \wedge X \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X}"] \arrow[r, phantom, "(5)", shift left=6] \arrow[r, phantom, "(2)"', shift right=4] & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X"] & \\
X \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\lambda'^{,-1}_X"] & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X.
\end{tikzcd}$$

Since:

- Subdiagram (5) commutes by the naturality of $\lambda'^{,-1}$.
- Subdiagram $(\ddagger)$ commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathsf{Sets}_*,-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & & S^0 \wedge X \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X}"] & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_X"] & \\
S^0 \arrow[r, "{[x]}"] & X \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\lambda'^{,-1}_X"'] & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X
\end{tikzcd}$$

Here,s a step-by-step showcase of this argument: [Link]. We then have

$$\lambda'^{,-1}_X(x) = [\lambda'^{,-1}_X \circ [x]](1)$$

$$\begin{aligned}
&= [(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \lambda^{\mathsf{Sets}_*,-1}_X \circ [x]](1)\\
&= [(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \lambda^{\mathsf{Sets}_*,-1}_X](x)
\end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda'^{,-1}_X = (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \lambda^{\mathsf{Sets}_*,-1}_X.$$

Taking inverses then gives

$$\lambda'_X = \lambda^{\mathsf{Sets}_*}_X \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X} \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X),$$

showing that the diagram

$$\begin{tikzcd}
 & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X}"] & S^0 \wedge X \arrow[dr, "\lambda^{\mathsf{Sets}_*}_X"] & \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_X"] \arrow[rrr, "\lambda'_X"'] & & & X
\end{tikzcd}$$

indeed commutes.

Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$\begin{tikzcd}
X \otimes_{\mathsf{Sets}_*} Y \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y}"] \arrow[d, "\sigma'_{X,Y}"'] & X \wedge Y \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"] \\
Y \otimes_{\mathsf{Sets}_*} X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,X}"'] & Y \wedge X
\end{tikzcd}$$

commutes. To this end, we will first show that the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, "\sigma'_{S^0,S^0}"'] \arrow[dr, phantom, "(\dagger)"] & S^0 \wedge S^0 \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,S^0}"] \\
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"'] & S^0 \wedge S^0
\end{tikzcd}$$

commutes. To that end, we will first show that the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(\ddagger)"] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0
\end{tikzcd}$$

commutes, and, to this end, we will first show that the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[dr, phantom, "(\S)"] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[u, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[r, "\rho'_{S^0}"'] & S^0
\end{tikzcd}$$

commutes. Indeed, consider the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrrr, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[dddd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"'] \arrow[ddrrr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] & & & & S^0 \wedge S^0 \arrow[dddd, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\
& & & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0}"] & \\
& \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[uul, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[rr, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[dr, "\rho'_{\mathbb{1}_{\mathsf{Sets}_*}} = \lambda'_{\mathbb{1}_{\mathsf{Sets}_*}}"] & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[u, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[ddr, "\lambda'_{S^0}"] & \\
& & \mathbb{1}_{\mathsf{Sets}_*} \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] & & \\
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[uur, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[rrrr, "\rho'_{S^0}"'] & & & & S^0
\arrow[from=2-3, to=2-4, phantom, "(1)"]
\arrow[from=3-1, to=3-2, phantom, "(2)"]
\arrow[from=2-2, to=3-3, phantom, "(3)"]
\arrow[from=3-4, to=3-5, phantom, "(4)"]
\arrow[from=4-3, to=4-4, phantom, "(5)"]
\arrow[from=4-2, to=5-2, phantom, "(6)"]
\end{tikzcd}
$$

whose boundary diagram corresponds to diagram (§) above. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagrams (2) and (3) commute by the functoriality of $\otimes$;
- Subdiagram (4) commutes by the left monoidal unity of $(\mathrm{id}^{\otimes}, \mathrm{id}^{\otimes}_{\mathbb{1}})$, which we proved above;
- Subdiagram (5) commutes by the naturality of λ';
- Subdiagram (6) commutes by the naturality of ρ', where the equality $\rho'_{\mathbb{1}_{\mathsf{Sets}_*}} = \lambda'_{\mathbb{1}_{\mathsf{Sets}_*}}$ comes from **??**;

it follows that the boundary diagram, i.e. diagram (§), also commutes. Next, consider the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[rrr, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dr, "\mathrm{id}_{S^0}\otimes_{\mathsf{Sets}_*}\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[ddd, "\rho'_{S^0}"] \arrow[dddd, bend right=60, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dddd, phantom, "(1)"', shift right=8] & & & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[ddd, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dddd, bend left=60, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dddd, phantom, "(6)", shift left=8] \\
& S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[urr, phantom, "(2)"] \arrow[ddl, phantom, "(\S)"] & S^0 \wedge S^0 \arrow[ur, "\mathrm{id}_{S^0}\otimes_{\mathsf{Sets}_*}\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}}"'] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \arrow[ddr, phantom, "(3)"] & \\
& & S^0 \arrow[dll, equal] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}}"] & \\
S^0 \arrow[rrr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[d, "\lambda'^{,-1}_{S^0}"] \arrow[urr, phantom, "(4)"', shift right=4] \arrow[drrr, phantom, "(5)"] & & & \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & & & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0
\end{tikzcd}
$$

whose boundary diagram corresponds to the diagram $(\ddagger)$ above. Since:

- Subdiagrams (1) and (6) commute by **??**;
- Subdiagram (2) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagram $(\S)$ commutes, as was shown above;
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathsf{Sets}_*}$;
- Subdiagram (4) commutes trivially;
- Subdiagram (5) commutes by Constructions With Monoidal Categories, Item 2c of Item 2 of Definition 13.1.1.1.4, whose proof uses only the left monoidal unity of $(\mathrm{id}^{\otimes}, \mathrm{id}^{\otimes}_{\mathbb{1}})$, which has been proven above;

it follows that the boundary diagram, i.e. diagram $(\ddagger)$, also commutes. Next, consider the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\mathrm{id}^{\otimes}_{S^0,S^0}"] \arrow[ddd, "\sigma'_{S^0,S^0}"'] \arrow[dr, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}}"] & & & S^0 \wedge S^0 \arrow[ddd, "\sigma^{\mathsf{Sets}_*}_{S^0,S^0}"] \arrow[dl, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] \\
& S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}^{\otimes}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] & \\
& \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[dr, "\mathrm{id}^{\otimes}_{\mathbb{1}} \wedge \mathrm{id}_{S^0}"] & \\
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\mathrm{id}^{\otimes}_{S^0,S^0}"'] & & & S^0 \wedge S^0,
\end{tikzcd}$$

(1) (2) $(\ddagger)$ (3) (4)

whose boundary diagram corresponds to the diagram $(\dagger)$. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagram (2) commutes by the naturality of σ' and the fact that $\mathrm{id}^{\otimes}_{\mathbb{1}}$ is invertible;
- Subdiagram $(\ddagger)$ commutes as proved above;
- Subdiagram (3) commutes by the naturality of σ^{Sets_*} and the fact that $\mathrm{id}^{\otimes}_{\mathbb{1}}$ is invertible;
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;

it follows that the boundary diagram, i.e. diagram (†) also commutes. Taking inverses for the diagram (†), we see that the diagram

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] \arrow[dr, phantom, "(\P)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\sigma'^{,-1}_{S^0,S^0}"] \\
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"'] & S^0 \otimes_{\mathsf{Sets}_*} S^0
\end{tikzcd}$$

commutes as well. Now, let $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, let $y \in Y$, and consider the diagram

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dr, "{[y]\wedge[x]}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[y]\otimes_{\mathsf{Sets}_*}[x]}"] \arrow[dd, "\sigma'^{,-1}_{S^0,S^0}" near start] & \\
& Y \wedge X \arrow[rr, crossing over, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,A}" near start] & & Y \otimes_{\mathsf{Sets}_*} X \arrow[dd, "\sigma'^{,-1}_{A,Y}"] \\
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}" near start] \arrow[dr, "{[x]\wedge[y]}"'] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[x]\otimes_{\mathsf{Sets}_*}[y]}"] & \\
& X \wedge Y \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|A,Y}"'] \arrow[from=uu, crossing over, "\sigma^{\mathsf{Sets}_*,-1}_{A,Y}" near start] & & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}$$

which we partition into subdiagrams as follows:

$$
\begin{tikzcd}
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] \arrow[dr, "{[y]\wedge[x]}"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[y]\otimes_{\mathsf{Sets}_*}[x]}"] \arrow[d, phantom, "(1)"] & \\
 & Y \wedge X \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{X,Y}"'] \arrow[ddr, phantom, "(3)"] & Y \otimes_{\mathsf{Sets}_*} X \arrow[dd, "\sigma'^{,-1}_{X,Y}"] \\
S^0 \wedge S^0 \arrow[dr, "{[x]\wedge[y]}"'] \arrow[ur, phantom, "(2)"] & & \\
 & X \wedge Y \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y}"'] & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}
\qquad
\begin{tikzcd}
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] \arrow[ddr, phantom, "(\P)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[y]\otimes_{\mathsf{Sets}_*}[x]}"] \arrow[dd, "\sigma'^{,-1}_{S^0,S^0}"] & \\
 & & Y \otimes_{\mathsf{Sets}_*} X \arrow[dd, "\sigma'^{,-1}_{X,Y}"] \arrow[dl, phantom, "(4)"] \\
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dr, "{[x]\wedge[y]}"'] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[x]\otimes_{\mathsf{Sets}_*}[y]}"] \arrow[d, phantom, "(5)"] & \\
 & X \wedge Y \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y}"'] & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}
$$

Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\mathsf{Sets}_*,-1}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.
- Subdiagram (¶) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\sigma'^{,-1}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.

it follows that the diagram

$$
\begin{tikzcd}
S^0 \wedge S^0 \arrow[dr, "{[y]\wedge[x]}"] & & \\
 & Y \wedge X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,X}"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"'] & Y \otimes_{\mathsf{Sets}_*} X \arrow[d, "\sigma'_{X,Y}"] \\
 & X \wedge Y \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y}"'] & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}
$$

commutes. We then have

$$
\begin{aligned}
[\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \circ \sigma^{\mathsf{Sets}_*,-1}_{X,Y}](y,x) &= [\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \circ \sigma^{\mathsf{Sets}_*,-1}_{X,Y} \circ ([y] \wedge [x])](1,1) \\
&= [\sigma'^{,-1}_{X,Y} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X} \circ ([y] \wedge [x])](1,1)
\end{aligned}
$$

$$= [\sigma'^{,-1}_{X,Y} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X}](y,x)$$

for each $(y,x) \in Y \wedge X$, and thus we have

$$\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \circ \sigma^{\mathsf{Sets}_*,-1}_{X,Y} = \sigma'^{,-1}_{X,Y} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X}.$$

Taking inverses then gives

$$\sigma^{\mathsf{Sets}_*}_{X,Y} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} = \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,X} \circ \sigma'_{X,Y},$$

showing that the diagram

$$\begin{tikzcd}
A \otimes_{\mathsf{Sets}_*} B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|A,B}"] \arrow[d, "\sigma'_{A,B}"'] & A \wedge B \arrow[d, "\sigma^{\mathsf{Sets}_*}_{A,B}"] \\
B \otimes_{\mathsf{Sets}_*} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|B,A}"'] & B \wedge A
\end{tikzcd}$$

indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$\begin{tikzcd}
& X \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,S^0}"] & X \wedge S^0 \arrow[dr, "\rho^{\mathsf{Sets}_*}_X"] & \\
X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[ur, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[rrr, "\rho'_X"'] & & & X
\end{tikzcd}$$

commutes. To this end, we will first show that the diagram

$$\begin{tikzcd}
& S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, phantom, "(\dagger)"] & S^0 \wedge S^0 \arrow[dr, "\rho^{\mathsf{Sets}_*}_{S^0}"] & \\
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[rrr, "\rho'_{S^0}"'] & {} & & S^0,
\end{tikzcd}$$

corresponding to the case $X = S^0$, commutes. First, notice that we may write

$$\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}} \colon S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \longrightarrow \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0$$

as the composition

$$
\begin{aligned}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} &\xrightarrow{\mathrm{id}^{\otimes}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}} S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \\
&\xrightarrow{\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}} \mathbb{1}_{\mathsf{Sets}_*} \\
&\xrightarrow{\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}} \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \\
&\xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}} \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0.
\end{aligned}
$$

Indeed, we may write this composition as part of the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}^{\otimes}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(1)"] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dr, phantom, "(2)"] & \mathbb{1}_{\mathsf{Sets}_*} \arrow[dl, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0,
\end{tikzcd}
$$

which commutes since:

- Subdiagram (1) commutes by the braidedness of $\mathrm{id}^{\otimes}$, as proved above.
- Subdiagram (2) commutes by **??**.

Next, consider the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(1)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \arrow[dr, phantom, "(2)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \lambda'_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"] \arrow[dr, phantom, "(3)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \\
S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}_{S^0} \wedge \rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(4)"] & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[r, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \arrow[dr, phantom, "(5)"] & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[r, "\mathrm{id}_{S^0} \wedge \lambda'_{S^0}"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"] \arrow[dr, phantom, "(6)"] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0} = \rho^{\mathsf{Sets}_*}_{S^0}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\lambda'_{S^0}"] & S^0,
\end{tikzcd}
$$

whose boundary diagram corresponds to the diagram (†) above, since the composition in red is equal to $\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}$ as proved above, and then the composition in red composed with λ'_{S^0} is equal to ρ'_{S^0} by **??**. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$.
- Subdiagrams (4), (5), and (6) commute by the naturality of $\lambda^{\mathsf{Sets}_*}$, where the equality $\lambda^{\mathsf{Sets}_*}_{S^0} = \rho^{\mathsf{Sets}_*}_{S^0}$ comes from **??**.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram (†) above. As a result, the diagram

$$
\begin{tikzcd}
& S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] & \\
S^0 \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\rho'^{,-1}_{S^0}"'] & & \arrow[u, phantom, "(\ddagger)"] & S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}.
\end{tikzcd}
$$

also commutes. Now, let $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, and consider the diagram

$$
\begin{tikzcd}
 & S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[r, phantom, shift right=4ex, "(\dagger)"] \arrow[dd, "\mathrm{id}_{S^0} \wedge [x]"'] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[dd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} [x]"] & \\
S^0 \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\rho'^{,-1}_{S^0}"] \arrow[dd, "{[x]}"'] \arrow[dr, phantom, "(3)"] & & & S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}_*}} \wedge [x]"] \arrow[dl, phantom, "(4)"] \\
 & X \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,S^0}"] \arrow[r, phantom, shift left=5ex, "(5)"] \arrow[r, phantom, shift right=4ex, "(2)"] & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] & \\
X \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\rho'^{,-1}_X"] & & & X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}.
\end{tikzcd}
$$

Since:

- Subdiagram (5) commutes by the naturality of $\rho'^{,-1}$.
- Subdiagram $(\dagger)$ commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\mathsf{Sets}_*,-1}$.

it follows that the diagram

$$
\begin{tikzcd}
 & & X \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,S^0}"] & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] & \\
S^0 \arrow[r, "{[x]}"] & X \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\rho'^{,-1}_X"'] & & & X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}
\end{tikzcd}
$$

Here,s a step-by-step showcase of this argument: [Link]. We then have

$$\rho'^{,-1}_X(a) = [\rho'^{,-1}_X \circ [x]](1)$$

$$
\begin{aligned}
&= [(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \rho^{\mathsf{Sets}_*,-1}_X \circ [x]](1)\\
&= [(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \rho^{\mathsf{Sets}_*,-1}_X](a)
\end{aligned}
$$

for each $a \in X$, and thus we have

$$\rho'^{,-1}_X = (\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \rho^{\mathsf{Sets}_*,-1}_X.$$

Taking inverses then gives

$$\rho'_X = \rho^{\mathsf{Sets}_*}_X \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X} \circ (\mathrm{id}_X \wedge \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}),$$

showing that the diagram

$$
\begin{tikzcd}
 & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,S^0}"] & X \wedge S^0 \arrow[dr, "\rho^{\mathsf{Sets}_*}_X"] & \\
X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[ur, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[rrr, "\rho'_X"'] & & & X
\end{tikzcd}
$$

indeed commutes.

Monoidality of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$
\begin{tikzcd}
 & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] \arrow[dr, "\alpha'_{X,Y,Z}"] & \\
(X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X\wedge Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,Z}"] \\
(X \wedge Y) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y\wedge Z}"] \\
 & X \wedge (Y \wedge Z) &
\end{tikzcd}
$$

commutes. To this end, we will first prove that the diagram

$$
\begin{tikzcd}
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[dr, "\alpha_{S^0,S^0,S^0}"] & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge S^0,S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \\
(S^0 \wedge S^0) \wedge S^0 \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,S^0,S^0}"'] & (\dagger) & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0 \wedge S^0}"] \\
 & S^0 \wedge (S^0 \wedge S^0) &
\end{tikzcd}
$$

commutes, and, to that end, we will first show that the diagram

$$
\begin{tikzcd}
 & (S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[dr, "\alpha_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \wedge S^0 \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & (\ddagger) & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \\
 & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) &
\end{tikzcd}
$$

commutes. Indeed, consider the diagram

$$
\begin{tikzcd}
 & & (S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[d, "(\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[ddrr, "\rho'_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[ddrrr, "\alpha'_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & & & \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[ddd, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*},S^0}"'] \arrow[rr, phantom, "(1)"] & & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[dr, phantom, "(2)"] \arrow[rrr, phantom, "(3)"] & & & \\
 & (S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\rho^{\mathsf{Sets}_*}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge S^0,S^0}"] \arrow[drr, phantom, "(5)"] & & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \lambda'^{,-1}_{S^0}"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"'] \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[dr, phantom, "(9)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \\
 & (S^0 \wedge S^0) \wedge S^0 \arrow[rr, bend left, "\rho^{\mathsf{Sets}_*}_{S^0} \wedge \mathrm{id}_{S^0}"] \arrow[rr, bend right, "\rho^{\mathsf{Sets}_*}_{S^0 \wedge S^0}"'] \arrow[rr, phantom, "(6)"] \arrow[dl, "(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}) \wedge \mathrm{id}_{S^0}"] \arrow[l, phantom, "(4)"] \arrow[drr, phantom, "(7)"] & & S^0 \wedge S^0 \arrow[d, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"'] \arrow[r, phantom, "(8)"] & S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, phantom, "(11)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[ddll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \wedge S^0 \arrow[rrr, "\rho^{\mathsf{Sets}_*}_{S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[drrr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] \arrow[drr, phantom, "(10)"] & & & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\mathrm{id}_{S^0} \wedge \rho^{\mathsf{Sets}_*}_{S^0}"] & & \\
 & & & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) & &
\end{tikzcd}
$$

whose boundary diagram corresponds to diagram (‡) above. Since:

- Subdiagrams (1), (4), (5), (8), and (11) commute by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagram (2) commutes by the right monoidal unity of $(\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*})$;
- Subdiagram (3) commutes by the triangle identity for $(\alpha', \lambda', \rho')$;
- Subdiagram (6) commutes by **??**;
- Subdiagram (7) commutes by the naturality of ρ^{Sets_*};
- Subdiagram (9) commutes by **??**;

- Subdiagram (10) commutes by **??**;

it follows that the boundary diagram, i.e. diagram $(\ddagger)$, also commutes. Consider now the diagram

$$
\begin{tikzcd}[column sep=tiny]
 & & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[d, "(\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}" description] \arrow[drr, "\alpha'_{S^0,S^0,S^0}"] & & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, phantom, "(1)"] \arrow[dr, "(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}" description] \arrow[dddd, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge S^0,S^0}"'] \arrow[ddr, phantom, "(3)"] & & (S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, phantom, "(2)"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}" description] \arrow[dr, "\alpha'_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}" description] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dl, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0})" description] \arrow[dddd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[ddl, phantom, "(4)"] \\
 & (S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dd, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*},S^0}" description] \arrow[ddrr, phantom, "(\ddagger)"] & & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[dd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}" description] & \\
 & & & & \\
 & (S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \wedge S^0 \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}" description] & & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}" description] & \\
(S^0 \wedge S^0) \wedge S^0 \arrow[ur, "(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}) \wedge \mathrm{id}_{S^0}" description] \arrow[rr, phantom, "(5)"] \arrow[drr, "\alpha^{\mathsf{Sets}_*}_{S^0,S^0,S^0}"'] & & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[rr, phantom, "(6)"] \arrow[d, "\mathrm{id}_{S^0} \wedge (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0})" description] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[ul, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0})" description] \arrow[dll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0 \wedge S^0}"] \\
 & & S^0 \wedge (S^0 \wedge S^0) & &
\end{tikzcd}
$$

whose boundary corresponds to diagram $(\dagger)$ above. Since:

- Subdiagrams (1), (3), (4), and (6) commute by the naturality of $\mathrm{id}^{\otimes}_{Sets_*}$;
- Subdiagram $(\ddagger)$ commutes, as proved above;
- Subdiagram (2) commutes by the naturality of α';
- Subdiagram (5) commutes by the naturality of α^{Sets_*};

it follows that the boundary diagram, i.e. diagram $(\dagger)$, also commutes. Taking inverses on the diagram (†), we see that the diagram

$$
\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0\wedge S^0}"] & \\
(S^0 \wedge S^0) \wedge S^0 \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0\wedge S^0,S^0}"'] & (\dagger) & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"] \\
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 &
\end{tikzcd}
$$

commutes as well. Now, let $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, let $y \in Y$, let $z \in Z$, and consider the diagram

$$
\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0\wedge S^0}"] \arrow[rrr, "{[x]\wedge([y]\wedge[z])}"] & & & X \wedge (Y \wedge Z) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}"] & \\
(S^0 \wedge S^0) \wedge S^0 \arrow[d, "\mathrm{id}^{\otimes,-1}_{S^0\wedge S^0,S^0}"'] \arrow[rrr, "{([x]\wedge[y])\wedge[z]}"] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[rrr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\wedge[z])}"] & (X \wedge Y) \wedge Z \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z}"] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}"] \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[rrr, "{([x]\wedge[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"] \arrow[rrr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\otimes_{\mathsf{Sets}_*}[z])}"] & (X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z) \arrow[dl, "\alpha'^{,-1}_{X,Y,Z}"] \\
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "{([x]\otimes_{\mathsf{Sets}_*}[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z &
\end{tikzcd}
$$

which we partition into subdiagrams as follows:

$$
\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[dr, dashed, "\mathrm{id}^{\otimes,-1}_{S^0,S^0 \wedge S^0}"] & \\
(S^0 \wedge S^0) \wedge S^0 \arrow[d, "\mathrm{id}^{\otimes,-1}_{S^0 \wedge S^0,S^0}"'] & (\dagger) & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"] \\
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[rrr, "{[x]\wedge([y]\wedge[z])}"] & & & X \wedge (Y \wedge Z) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}"] \\
(S^0 \wedge S^0) \wedge S^0 \arrow[d, "\mathrm{id}^{\otimes,-1}_{S^0 \wedge S^0,S^0}"'] \arrow[rrr, "{([x]\wedge[y])\wedge[z]}"] & & (1) & (X \wedge Y) \wedge Z \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X \wedge Y,Z}"] & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[rrr, "{([x]\wedge[y])\otimes_{\mathsf{Sets}_*}[z]}"] & (2) & & (X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"] & \\
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "{([x]\otimes_{\mathsf{Sets}_*}[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z \\
 & & (3) & &
\end{tikzcd}
$$

$$\begin{tikzcd}
S^0 \wedge (S^0 \wedge S^0) \arrow[rr, "{[x]\wedge([y]\wedge[z])}"] \arrow[dr, "{\mathrm{id}^{\otimes,-1}_{S^0,S^0\wedge S^0}}"'] \arrow[drrr, phantom, "(4)"] & & X \wedge (Y \wedge Z) \arrow[dr, "{\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}}"] & \\
& S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[rr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\wedge[z])}"] \arrow[d, "{\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{S^0,S^0}}"] \arrow[drr, phantom, "(5)"] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[d, "{\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}}"] \\
& S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[rr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\otimes_{\mathsf{Sets}_*}[z])}"] \arrow[dl, "{\alpha'^{,-1}_{S^0,S^0,S^0}}"'] \arrow[dr, phantom, "(6)"] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z). \arrow[dl, "{\alpha'^{,-1}_{X,Y,Z}}"] \\
(S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, "{([x]\otimes_{\mathsf{Sets}_*}[y])\otimes_{\mathsf{Sets}_*}[z]}"'] & & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z &
\end{tikzcd}$$

Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{\mathsf{Sets}_*,-1}$.
- Subdiagram (2) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (3) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram $(\dagger)$ commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (6) commutes by the naturality of $\alpha'^{,-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[d, "{[x] \wedge ([y] \wedge [z])}"] & \\
 & X \wedge (Y \wedge Z) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[d, "\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}"] \\
(X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z) \arrow[dl, "\alpha'^{,-1}_{X,Y,Z}"] \\
 & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z &
\end{tikzcd}$$

also commutes. We then have

$$\begin{aligned}
\Big[(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z} & \\
\circ\, \alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}\Big](x,(y,z)) &= \Big[(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z} \\
&\quad \circ\, \alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \circ ([x] \wedge ([y] \wedge [z]))\Big](1,(1,1)) \\
&= \Big[\alpha'^{,-1}_{X,Y,Z} \circ (\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}) \\
&\quad \circ\, \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z} \circ ([x] \wedge ([y] \wedge [z]))\Big](1,(1,1)) \\
&= [\alpha'^{,-1}_{X,Y,Z} \circ (\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}](x,(y,z))
\end{aligned}$$

for each $(x,(y,z)) \in X \wedge (Y \wedge Z)$, and thus we have

$$(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z} \circ \alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} = \alpha'^{,-1}_{X,Y,Z} \circ (\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}.$$

Taking inverses then gives

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X\wedge Y,Z} \circ (\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z) = \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y\wedge Z} \circ (\mathrm{id}_X \wedge \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,Z}) \circ \alpha'_{X,Y,Z},$$

showing that the diagram

$$
\begin{tikzcd}[column sep=tiny]
 & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] \arrow[dr, "\alpha'_{X,Y,Z}"] & \\
(X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X\wedge Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,Z}"] \\
(X \wedge Y) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y\wedge Z}"] \\
 & X \wedge (Y \wedge Z) &
\end{tikzcd}
$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: Let $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}_*} -_2 \Rightarrow -_1 \wedge -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets_* with respect to $\wedge$ and $\otimes$ (as shown above), we have

$$
\begin{aligned}
\lambda'_Y &= \lambda^{\mathsf{Sets}_*}_Y \circ \phi_{S^0,Y} \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y),\\
\lambda'_Y &= \lambda^{\mathsf{Sets}_*}_Y \circ \psi_{S^0,Y} \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y).
\end{aligned}
$$

Postcomposing both sides with $\lambda^{\mathsf{Sets}_*,-1}_Y$ and then precomposing both sides with $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y$ gives

$$
\begin{aligned}
\lambda^{\mathsf{Sets}_*,-1}_Y \circ \lambda'_Y \circ (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) &= \phi_{S^0,Y},\\
\lambda^{\mathsf{Sets}_*,-1}_Y \circ \lambda'_Y \circ (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) &= \psi_{S^0,Y},
\end{aligned}
$$

and thus we have

$$\phi_{S^0,Y} = \psi_{S^0,Y}$$

for each $Y \in \mathrm{Obj}(\mathsf{Sets}_*)$. Now, let $x \in X$ and consider the naturality diagrams

$$
\begin{tikzcd}
S^0 \wedge Y \arrow[r, "{[x] \wedge \mathrm{id}_Y}"] \arrow[d, "\phi_{S^0,Y}"'] & X \wedge Y \arrow[d, "\phi_{X,Y}"] \\
S^0 \otimes_{\mathsf{Sets}_*} Y \arrow[r, "{[x] \otimes_{\mathsf{Sets}_*} \mathrm{id}_Y}"'] & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}
\qquad
\begin{tikzcd}
S^0 \wedge Y \arrow[r, "{[x] \wedge \mathrm{id}_Y}"] \arrow[d, "\psi_{S^0,Y}"'] & X \wedge Y \arrow[d, "\psi_{X,Y}"] \\
S^0 \otimes_{\mathsf{Sets}_*} Y \arrow[r, "{[x] \otimes_{\mathsf{Sets}_*} \mathrm{id}_Y}"'] & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}
$$

for ϕ and ψ with respect to the morphisms $[x]$ and id_Y. Having shown that $\phi_{S^0,Y} = \psi_{S^0,Y}$, we have

$$\begin{aligned}\phi_{X,Y}(x,y) &= [\phi_{X,Y} \circ ([x] \wedge \mathrm{id}_Y)](1,y) \\ &= [([x] \otimes_{\mathsf{Sets}_*} \mathrm{id}_Y) \circ \phi_{S^0,Y}](1,y) \\ &= [([x] \otimes_{\mathsf{Sets}_*} \mathrm{id}_Y) \circ \psi_{S^0,Y}](1,y) \\ &= [\psi_{X,Y} \circ ([x] \wedge \mathrm{id}_Y)](1,y) \\ &= \psi_{X,Y}(x,y)\end{aligned}$$

for each $(x,y) \in X \wedge Y$. Therefore we have

$$\phi_{X,Y} = \psi_{X,Y}$$

for each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathsf{Sets}_*} \cong \times$ to be unique. □

Corollary 7.5.10.1.2. The symmetric monoidal structure on the category Sets_* of Definition 7.5.9.1.1 is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\mathsf{Sets}_*} : \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

   of Sets_* preserves colimits separately in each variable.

2. *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} \cong S^0$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathsf{Sets}_*)$ of **??** spanned by the symmetric monoidal categories $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ satisfying Items 1 and 2 is contractible.

Proof. Since Sets_* is locally presentable (**??**), it follows from **??** that Definition 7.5.10.1.2 is equivalent to the existence of an internal Hom as in Item 1 of Definition 7.5.10.1.1. The result then follows from Definition 7.5.10.1.1. □

Corollary 7.5.10.1.3. The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ : \mathsf{Sets} \longrightarrow \mathsf{Sets}_*$$

admits a symmetric monoidal structure, i.e. the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathsf{Sets}_*)$ of **??** spanned by the symmetric monoidal categories $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ with respect to which $(-)^+$ admits a symmetric monoidal structure is contractible.

Proof. Let $(\otimes_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ be a symmetric monoidal structure on Sets_* such that $(-)^+$ admits a symmetric monoidal structure with respect to $\otimes_{\mathsf{Sets}_*}$ and $\wedge$. We have isomorphisms

$$\begin{aligned} X \otimes_{\mathsf{Sets}_*} Y &\cong (X^-)^+ \otimes_{\mathsf{Sets}_*} (Y^-)^+ \\ &\cong (X^- \times Y^-)^+ \\ &\cong (X^-)^+ \wedge (Y^-)^+ \\ &\cong X \wedge Y, \end{aligned}$$

all natural in X and Y. Now, since $\wedge$ preserves colimits in both variables and $\otimes_{\mathsf{Sets}_*} \cong \wedge$, it follows that $\otimes_{\mathsf{Sets}_*}$ also preserves colimits in both variables, so the result then follows from Definition 7.5.10.1.2. □

7.5.11 Monoids With Respect to the Smash Product of Pointed Sets

Proposition 7.5.11.1.1. The category of monoids on $(\mathsf{Sets}_*, \wedge, S^0)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof. See **??**, in particular **??**, **??**, **??**, and **??**. □

7.5.12 Comonoids With Respect to the Smash Product of Pointed Sets

Proposition 7.5.12.1.1. The symmetric monoidal functor

$$((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

of Pointed Sets, Item 4 of Definition 6.4.1.1.2 lifts to an equivalence of categories

$$\begin{aligned} \mathsf{CoMon}(\mathsf{Sets}_*, \wedge, S^0) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt}) \\ &\cong \mathsf{Sets}. \end{aligned}$$

Proof. We follow [PS19, Lemma 2.4].
Faithfulness: Given morphisms $f, g \colon X \to Y$, if $f^+ = g^+$, then we have

$$\begin{aligned} f(x) &\overset{\text{def}}{=} f^+(x) \\ &= g^+(x) \\ &\overset{\text{def}}{=} g(x) \end{aligned}$$

for each $x \in X^+$, and thus $f = g$, showing $(-)^+$ to be faithful.
Fullness: Let $f\colon X^+ \to Y^+$ be a morphism of comonoids in Sets_*. By counitality, the diagram

$$\begin{tikzcd} X^+ \arrow[rr, "f"] \arrow[dr, "\epsilon^+_X"'] & & Y^+ \arrow[dl, "\epsilon^+_Y"] \\ & S^0 & \end{tikzcd}$$

commutes. If $f(x) = \star_Y$ for $x \neq \star_X$, the commutativity of this diagram then gives

$$\begin{aligned} 1 &= \epsilon^+_X(x) \\ &= \epsilon^+_Y(f(x)) \\ &= \epsilon^+_Y(\star_Y) \\ &= 0, \end{aligned}$$

which is a contradiction. Thus f is an active morphism of pointed sets, so there exists a map f^- such that $(f^-)^+ = f$ (Pointed Sets, Item 1 of Definition 6.4.2.1.2).
Essential Surjectivity: Let $(X, \Delta_X, \epsilon_X)$ be a comonoid in Sets_*. We claim that

$$\Delta_X(x) = x \wedge x$$

for each $x \in X$ with $x \neq \star_X$. Indeed:

- Suppose that $x \neq \star_X$ and write $\Delta_X(x) = x_1 \wedge x_2$.
- Since $\mathrm{id}_X \wedge \epsilon_X$ is pointed, we have

  $$[\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) = \star_{X \wedge S^0}.$$

- The counitality condition for Δ_X, corresponding to the commutativity of the diagram

  $$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[dr, "\rho^{\mathsf{Sets}_*,-1}_X"'] & X \wedge X \arrow[d, "\mathrm{id}_X \wedge \epsilon_X"] \\ & X \wedge S^0 \end{tikzcd}$$

  gives

  $$x \wedge 1 = \rho^{\mathsf{Sets}_*,-1}_X(x)$$

$$
\begin{aligned}
&= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x)\\
&= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x))\\
&= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2)\\
&= \star_{X \wedge S^0},
\end{aligned}
$$

which is a contradiction. Thus $x_1 \neq \star_X$.

- Similarly, if $x \neq \star_X$, then $x_2 \neq \star_X$.
- Next, we claim that $\epsilon_X(x_2) = 1$, as otherwise we would have

$$
\begin{aligned}
\star_{X \wedge S^0} &= x_1 \wedge 0\\
&= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2)\\
&= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x))\\
&= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x)\\
&= \rho_X^{\mathsf{Sets}_*,-1}(x)\\
&= x \wedge 1,
\end{aligned}
$$

a contradiction. Thus $\epsilon_X(x_2) = 1$.

- Similarly, if $x \neq \star_X$, then $\epsilon_X(x_1) = 1$.
- Now, since Δ_X is counital, we have

$$
\begin{aligned}
x \wedge 1 &= \rho_X^{\mathsf{Sets}_*,-1}(x)\\
&= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x)\\
&= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x))\\
&= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2)\\
&= x_1 \wedge 1,
\end{aligned}
$$

so $x = x_1$.

- Similarly, $x = x_2$, and we are done.

Next, notice that $X \cong \epsilon_X^{-1}(0) \coprod \epsilon_X^{-1}(1)$, and let $x \in \epsilon_X^{-1}(0)$. We then have

$$
\begin{aligned}
[(\mathrm{id}_X \wedge \epsilon_X) \circ \Delta_X](x) &= [\mathrm{id}_X \wedge \epsilon_X](x \wedge x)\\
&= x \wedge 0
\end{aligned}
$$

$$= \star_{X \wedge S^0}.$$

The counitality condition for Δ_X then gives $x = \star_X$, so $\epsilon_X^{-1}(0) = \{\star_X\}$. Thus we have $(\epsilon_X^{-1}(1))^+ \cong X$, and this isomorphism is compatible with the comonoid structures when equipping $\epsilon_X^{-1}(1)$ with its unique comonoid structure. This shows that $(-)^+$ is essentially surjective.
Equivalence: Since $(-)^+$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1b of Item 1 of Definition 11.6.7.1.2. □

7.6 Miscellany

7.6.1 The Smash Product of a Family of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 7.6.1.1.1. The **smash product of the family** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pointed set $\bigwedge_{i\in I} X_i$ consisting of:

- *The Underlying Set*. The set $\bigwedge_{i\in I} X_i$ defined by

  $$\bigwedge_{i\in I} X_i \overset{\text{def}}{=} (\prod_{i\in I} X_i)/\sim,$$

  where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ obtained by declaring

  $$(x_i)_{i\in I} \sim (y_i)_{i\in I}$$

  if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$.

- *The Basepoint*. The element $[(x_0)_{i\in I}]$ of $\bigwedge_{i\in I} X_i$.

Appendices

A Other Chapters

References

[PS19] Maximilien Péroux and Brooke Shipley. "Coalgebras in Symmetric Monoidal Categories of Spectra'. In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. ISSN: 1532-0073. DOI: 10.4310/HHA.2019.v21.n1.a1. URL: https://doi.org/10.4310/HHA.2019.v21.n1.a1 (cit. on p. 129).