Constructions With Relations

The Clowder Project Authors

July 29, 2025

This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (**??**).
2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, converse relations, composition of relations, and collages (Section 9.2).

This chapter is under revision. TODO:

1. Rename range to image
2. Co/limits in **Rel**.

Contents

9.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

9.2 More Constructions With Relations

9.2.1 The Domain and Range of a Relation

Let A and B be sets.

Definition 9.2.1.1.1 ▶ The Domain and Range of a Relation

Let $R\colon A \nrightarrow B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by
$$\mathrm{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists some } b \in B\\ \text{such that } a \sim_R b\end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by
$$\mathrm{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in A\\ \text{such that } a \sim_R b\end{array} \right\}.$$

[1] Following **??**, **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:
$$\begin{aligned}
\chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b\in B}(R^b_a) \qquad (a \in A)\\
&\cong \bigvee_{b\in B} R^b_a,\\
\chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a\in A}(R^b_a) \qquad (b \in B)\\
&\cong \bigvee_{a\in A} R^b_a,
\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$ of Constructions With Sets, Definition 3.2.2.1.3.

[2]Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned}\mathrm{dom}(R) &\cong \operatorname*{colim}_{y\in Y}(R(y))\\ &\cong \bigcup_{y\in Y} R(y),\\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x\in X}(R(x))\\ &\cong \bigcup_{x\in X} R(x),\end{aligned}$$

9.2.2 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 9.2.2.1.1 ▶ Binary Unions of Relations

The **union of R and S**[1] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cup S \overset{\mathrm{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \overset{\mathrm{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

[1]*Further Terminology:* Also called the **binary union of R and S**, for emphasis.

[2]This is the same as the union of R and S as subsets of $A \times B$.

Proposition 9.2.2.1.2 ▶ Properties of Binary Unions of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Converses.* We have

$$(R \cup S)^{\dagger} = R^{\dagger} \cup S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

PROOF 9.2.2.1.3 ► PROOF OF PROPOSITION 9.2.2.1.2

Item 1: Interaction With Converses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

    or
    - $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

    and
    - $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ.

9.2.3 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 9.2.3.1.1 ► The Union of a Family of Relations

The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \,\middle|\, \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \stackrel{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

[1]This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

Proposition 9.2.3.1.2 ► Properties of Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Converses.* We have

$$\Big(\bigcup_{i\in I} R_i\Big)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 9.2.3.1.3 ► Proof of Proposition 9.2.3.1.2

Item 1: Interaction With Converses

Clear. ∎

9.2.4 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 9.2.4.1.1 ▶ Binary Intersections of Relations

The **intersection of** R **and** S[1] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary intersection of** R **and** S, for emphasis.
[2] This is the same as the intersection of R and S as subsets of $A \times B$.

Proposition 9.2.4.1.2 ▶ Properties of Binary Intersections of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Converses.* We have

$$(R \cap S)^{\dagger} = R^{\dagger} \cap S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 9.2.4.1.3 ▶ Proof of Proposition 9.2.4.1.2

Item 1: Interaction With Converses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

    and
    - $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

    and
    - $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$.

9.2.5 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 9.2.5.1.1 ▶ The Intersection of a Family of Relations

The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right](a) \overset{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each $a \in A$.

[1] This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

Proposition 9.2.5.1.2 ▶ Properties of Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

1. *Interaction With Converses.* We have

$$\Big(\bigcap_{i \in I} R_i\Big)^{\dagger} = \bigcap_{i \in I} R_i^{\dagger}.$$

Proof 9.2.5.1.3 ▶ Proof of Proposition 9.2.5.1.2

Item 1: Interaction With Converses

Clear. □

9.2.6 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

Definition 9.2.6.1.1 ▶ Binary Products of Relations

The **product of** R **and** S[1] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[2]

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

[1] *Further Terminology:* Also called the **binary product of** R **and** S, for emphasis.

[2] That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R\times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

Proposition 9.2.6.1.2 ▶ Properties of Binary Products of Relations

Let A, B, X, and Y be sets.

1. *Interaction With Converses.* Let

$$R \colon A \nrightarrow A,$$
$$S \colon X \nrightarrow X$$

We have
$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$R_1 \colon A \nrightarrow B,$$
$$S_1 \colon B \nrightarrow C,$$
$$R_2 \colon X \nrightarrow Y,$$
$$S_2 \colon Y \nrightarrow Z$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof 9.2.6.1.3 ▶ Proof of Proposition 9.2.6.1.2

Item 1: Interaction With Converses

Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(R\times S)^{\dagger}} (b, y)$ iff:
  - We have $(b, y) \sim_{R\times S} (a, x)$, i.e. iff:

- * We have $b \sim_R a$;
  * We have $y \sim_S x$;
- We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
  - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
    * We have $b \sim_R a$;
    * We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
  - We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
    * There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
    * There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
- We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
  - There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
    * We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
    * We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal.

9.2.7 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i \colon A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 9.2.7.1.1 ▶ The Product of a Family of Relations

The **product of the family** $\{R_i\}_{i\in I}$ is the relation $\prod_{i\in I} R_i$ from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i\in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i\in I} \in \prod_{i\in I}(A_i \times B_i) \,\middle|\, \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

  for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

9.2.8 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 9.2.8.1.1 ▶ The Collage of a Relation

The **collage of** R[1] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)}\colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$

  on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of** R.

Notation 9.2.8.1.2 ▶ Notation: $\mathsf{Pos}_{/\Delta^1}(A, B)$

We write $\mathsf{Pos}_{/\Delta^1}(A, B)$ for the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \overset{\mathrm{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{ev}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{ev}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{ev}_{[0]}"] \arrow[dr, "\mathrm{ev}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Remark 9.2.8.1.3 ▶ Unwinding Notation 9.2.8.1.2

In detail, $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category where:

- *Objects.* An object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of
  - A poset X;
  - A morphism $\phi_X : X \longrightarrow \Delta^1$;

  such that we have

$$\begin{aligned}\phi_X^{-1}(0) &= A,\\ \phi_X^{-1}(1) &= B.\end{aligned}$$

- *Morphisms.* A morphism of $\mathsf{Pos}_{/\Delta^1}(A, B)$ from (X, ϕ_X) to (Y, ϕ_Y)

is a morphism of posets $f\colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & \Delta^1 & \end{tikzcd}$$

commute.

Proposition 9.2.8.1.4 ► Properties of Collages of Relations

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathsf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where
  - The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 9.2.8.1.1.
  - The morphism $\phi_R\colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

  for each $x \in \mathbf{Coll}(R)$.
- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

$$\iota\colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \longrightarrow \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[1]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

[1]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

PROOF 9.2.8.1.5 ▶ PROOF OF PROPOSITION 9.2.8.1.4

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted. □

Appendices

A Other Chapters