

Constructions With Sets

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see [Definitions 4.2.4.1.1](#), [4.2.4.1.3](#), [4.2.5.1.1](#) and [4.2.5.1.3](#)).
2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:
 - (a) A discussion of the internal Hom of a powerset ([Section 4.4.7](#)).
 - (b) A o-categorical version of the Yoneda lemma ([Presheaves and the Yoneda Lemma](#), [Definition 12.1.5.1.1](#)), which we term the *Yoneda lemma for sets* ([Definition 4.5.5.1.1](#)).
 - (c) A characterisation of powersets as free cocompletions ([Section 4.4.5](#)), mimicking the corresponding statement for categories of presheaves (??).
 - (d) A characterisation of powersets as free completions ([Section 4.4.6](#)), mimicking the corresponding statement for categories of copresheaves (??).
 - (e) A (-1) -categorical version of un/straightening ([Item 2](#) of [Definition 4.5.1.1.4](#) and [Definition 4.5.1.1.5](#)).
 - (f) A o-categorical form of Isbell duality internal to powersets ([Section 4.4.8](#)).

3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_*: \mathcal{P}(A) \xrightarrow{\cong} \mathcal{P}(B)$$

of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f: A \rightarrow B$, including in particular:

- (a) How f^{-1} can be described as a precomposition while $f_!$ and f_* can be described as Kan extensions ([Definitions 4.6.1.1.4](#), [4.6.2.1.2](#) and [4.6.3.1.4](#)).
- (b) An extensive list of the properties of $f_!$, f^{-1} , and f_* ([Definitions 4.6.1.1.5](#), [4.6.1.1.6](#), [4.6.2.1.3](#), [4.6.2.1.4](#), [4.6.3.1.7](#) and [4.6.3.1.8](#)).
- (c) How the functors $f_!$, f^{-1} , f_* , along with the functors

$$\begin{aligned} -_1 \cap -_2: \mathcal{P}(X) \times \mathcal{P}(X) &\rightarrow \mathcal{P}(X), \\ [-_1, -_2]_X: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \end{aligned}$$

may be viewed as a six-functor formalism with the empty set \emptyset as the dualising object ([Section 4.6.4](#)).

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4.1 Limits of Sets

4.1.1 The Terminal Set

Definition 4.1.1.1. The **terminal set** is the terminal object of **Sets** as in Limits and Colimits, ??.

Construction 4.1.1.2. Concretely, the terminal set is the pair $(\text{pt}, \{!_A\}_{A \in \text{Obj}(\text{Sets})})$ consisting of:

1. *The Limit.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
2. *The Cone.* The collection of maps

$$\{!_A: A \rightarrow \text{pt}\}_{A \in \text{Obj}(\text{Sets})}$$

defined by

$$!_A(a) \stackrel{\text{def}}{=} \star$$

for each $a \in A$ and each $A \in \text{Obj}(\text{Sets})$.

Proof. We claim that pt is the terminal object of **Sets**. Indeed, suppose we have a diagram of the form

$$A \quad \text{pt}$$

in **Sets**. Then there exists a unique map $\phi: A \rightarrow \text{pt}$ making the diagram

$$A \xrightarrow[\exists!]{\phi} \text{pt}$$

commute, namely $!_A$. □

4.1.2 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 4.1.2.1.1. The **product**¹ of $\{A_i\}_{i \in I}$ is the product of $\{A_i\}_{i \in I}$ in **Sets** as in Limits and Colimits, ??.

Construction 4.1.2.1.2. Concretely, the product of $\{A_i\}_{i \in I}$ is the pair $(\prod_{i \in I} A_i, \{\text{pr}_i\}_{i \in I})$ consisting of:

¹*Further Terminology:* Also called the **Cartesian product** of $\{A_i\}_{i \in I}$.

1. *The Limit.* The set $\prod_{i \in I} A_i$ defined by

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \text{Sets} \left(I, \bigcup_{i \in I} A_i \right) \left| \begin{array}{l} \text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i \end{array} \right. \right\}.$$

2. *The Cone.* The collection

$$\left\{ \text{pr}_i: \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i \in I} A_i$ is the categorical product of $\{A_i\}_{i \in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} P & & \\ & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

in Sets. Then there exists a unique map $\phi: P \rightarrow \prod_{i \in I} A_i$ making the diagram

$$\begin{array}{ccc} P & & \\ \downarrow \phi \quad \exists! & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. □

Remark 4.1.2.1.3. Less formally, we may think of Cartesian products and projection maps as follows:

1. We think of $\prod_{i \in I} A_i$ as the set whose elements are I -indexed collections $(a_i)_{i \in I}$ with $a_i \in A_i$ for each $i \in I$.
2. We view the projection maps

$$\left\{ \text{pr}_i : \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

as being given by

$$\text{pr}_i \left((a_j)_{j \in I} \right) \stackrel{\text{def}}{=} a_i$$

for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

Proposition 4.1.2.1.4. Let $\{A_i\}_{i \in I}$ be a family of sets.

- I. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}) \rightarrow \text{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, we have

$$\left[\prod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \text{Sets} \left(\prod_{i \in I} A_i, \prod_{i \in I} B_i \right)$$

of $\prod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i : A_i \rightarrow B_i\}_{i \in I}$$

in $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\prod_{i \in I} f_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$$

defined by

$$\left[\prod_{i \in I} f_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i \in I}$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} A_i$.

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??. \square

4.1.3 Binary Products of Sets

Let A and B be sets.

Definition 4.1.3.1.1. The **product of A and B** ² is the product of A and B in Sets as in Limits and Colimits, ??.

Construction 4.1.3.1.2. Concretely, the product of A and B is the pair $(A \times B, \{\text{pr}_1, \text{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \text{Sets}(\{0, 1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\ &\cong \left\{ \begin{array}{l} \text{ordered pairs } (a, b) \text{ with} \\ a \in A \text{ and } b \in B \end{array} \right\}. \end{aligned}$$

2. *The Cone.* The maps

$$\begin{aligned} \text{pr}_1 : A \times B &\rightarrow A, \\ \text{pr}_2 : A \times B &\rightarrow B \end{aligned}$$

²*Further Terminology:* Also called the **Cartesian product of A and B** .

defined by

$$\begin{aligned} \text{pr}_1(a, b) &\stackrel{\text{def}}{=} a, \\ \text{pr}_2(a, b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each $(a, b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in the category of sets. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & P & \\ p_1 \swarrow & & \searrow p_2 \\ A & \xleftarrow{\text{pr}_1} A \times B \xrightarrow{\text{pr}_2} & B \end{array}$$

in **Sets**. Then there exists a unique map $\phi: P \rightarrow A \times B$ making the diagram

$$\begin{array}{ccc} & P & \\ p_1 \swarrow & \downarrow \phi \mid \exists! & \searrow p_2 \\ A & \xleftarrow{\text{pr}_1} A \times B \xrightarrow{\text{pr}_2} & B \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

Proposition 4.1.3.1.3. Let A, B, C , and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -: \quad \text{Sets} &\rightarrow \text{Sets}, \\ - \times B: \quad \text{Sets} &\rightarrow \text{Sets}, \\ -_1 \times -_2: \text{Sets} \times \text{Sets} &\rightarrow \text{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} : \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \times B, X \times Y)$$

of \times at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g : A \times B \rightarrow X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \text{Obj}(\text{Sets})$.

2. *Adjointness I.* We have adjunctions

$$\begin{aligned} (A \times - \dashv \text{Sets}(A, -)) : \quad & \text{Sets} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets}, \\ (- \times B \dashv \text{Sets}(B, -)) : \quad & \text{Sets} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets}, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\text{Sets}} \dashv -_1 \times -_2): \text{Sets} \begin{array}{c} \xrightarrow{\Delta_{\text{Sets}}} \\ \perp \\ \xleftarrow{-_1 \times -_2} \end{array} \text{Sets} \times \text{Sets},$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets} \times \text{Sets}}((A, A), (B, C)) \cong \text{Sets}(A, B \times C),$$

natural in $A \in \text{Obj}(\text{Sets})$ and in $(B, C) \in \text{Obj}(\text{Sets} \times \text{Sets})$.

4. *Associativity.* We have an isomorphism of sets

$$\alpha_{A,B,C}^{\text{Sets}}: (A \times B) \times C \xrightarrow{\sim} A \times (B \times C),$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

5. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \lambda_A^{\text{Sets}}: \text{pt} \times A &\xrightarrow{\sim} A, \\ \rho_A^{\text{Sets}}: A \times \text{pt} &\xrightarrow{\sim} A, \end{aligned}$$

natural in $A \in \text{Obj}(\text{Sets})$.

6. *Commutativity.* We have an isomorphism of sets

$$\sigma_{A,B}^{\text{Sets}}: A \times B \xrightarrow{\sim} B \times A,$$

natural in $A, B \in \text{Obj}(\text{Sets})$.

7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\begin{aligned} \partial_\ell^{\text{Sets}}: A \times (B \amalg C) &\xrightarrow{\sim} (A \times B) \amalg (A \times C), \\ \partial_r^{\text{Sets}}: (A \amalg B) \times C &\xrightarrow{\sim} (A \times C) \amalg (B \times C), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned}\zeta_\ell^{\text{Sets}} : \emptyset \times A &\xrightarrow{\sim} \emptyset, \\ \zeta_r^{\text{Sets}} : A \times \emptyset &\xrightarrow{\sim} \emptyset,\end{aligned}$$

natural in $A \in \text{Obj}(\text{Sets})$.

9. *Distributivity Over Unions.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \cup W) &= (U \times V) \cup (U \times W), \\ (U \cup V) \times W &= (U \times W) \cup (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

10. *Distributivity Over Intersections.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \cap W) &= (U \times V) \cap (U \times W), \\ (U \cap V) \times W &= (U \times W) \cap (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

11. *Distributivity Over Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \setminus W) &= (U \times V) \setminus (U \times W), \\ (U \setminus V) \times W &= (U \times W) \setminus (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

12. *Distributivity Over Symmetric Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \triangle W) &= (U \times V) \triangle (U \times W), \\ (U \triangle V) \times W &= (U \times W) \triangle (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$\begin{array}{ccc}
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \xrightarrow{\cap \times \cap} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \downarrow \mathcal{P}_{X,X}^\times \times \mathcal{P}_{X,X}^\times & & \downarrow \mathcal{P}_{X,X}^\times \\
 \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) & \xrightarrow{\cap} & \mathcal{P}(X \times X)
 \end{array}$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).$$

for each $U, V, W, T \in \mathcal{P}(X)$.

14. *Symmetric Monoidality.* The 8-tuple $(\mathbf{Sets}, \times, \text{pt}, \mathbf{Sets}(-1, -2), \alpha^{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$ is a closed symmetric monoidal category.

15. *Symmetric Bimonoidality.* The 18-tuple

$$\begin{aligned}
 & \left(\mathbf{Sets}, \coprod, \times, \emptyset, \text{pt}, \mathbf{Sets}(-1, -2), \alpha^{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}}, \right. \\
 & \left. \alpha^{\mathbf{Sets}, \coprod}, \lambda^{\mathbf{Sets}, \coprod}, \rho^{\mathbf{Sets}, \coprod}, \sigma^{\mathbf{Sets}, \coprod}, \delta_\ell^{\mathbf{Sets}}, \delta_r^{\mathbf{Sets}}, \zeta_\ell^{\mathbf{Sets}}, \zeta_r^{\mathbf{Sets}} \right),
 \end{aligned}$$

is a symmetric closed bimonoidal category, where $\alpha^{\mathbf{Sets}, \coprod}, \lambda^{\mathbf{Sets}, \coprod}, \rho^{\mathbf{Sets}, \coprod}$, and $\sigma^{\mathbf{Sets}, \coprod}$ are the natural transformations from **Items 3** to **5** of **Definition 4.2.3.1.3**.

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??.

Item 2, Adjointness: We prove only that there's an adjunction $- \times B \dashv \mathbf{Sets}(B, -)$, witnessed by a bijection

$$\mathbf{Sets}(A \times B, C) \cong \mathbf{Sets}(A, \mathbf{Sets}(B, C)),$$

natural in $B, C \in \text{Obj}(\mathbf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathbf{Sets}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

$$\Phi_{B,C}: \mathbf{Sets}(A \times B, C) \rightarrow \mathbf{Sets}(A, \mathbf{Sets}(B, C)),$$

by sending a function

$$\xi: A \times B \rightarrow C$$

to the function

$$\begin{aligned} \xi^\dagger: A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a^\dagger: B \rightarrow C), \end{aligned}$$

where we define

$$\xi_a^\dagger(b) \stackrel{\text{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $\llbracket a \mapsto f(a) \rrbracket$ notation of [Sets, Definition 3.1.1.2](#), we have

$$\xi^\dagger \stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket.$$

- *Map II.* We define a map

$$\Psi_{B,C}: \text{Sets}(A, \text{Sets}(B, C)) \rightarrow \text{Sets}(A \times B, C)$$

given by sending a function

$$\begin{aligned} \xi: A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a: B \rightarrow C), \end{aligned}$$

to the function

$$\xi^\dagger: A \times B \rightarrow C$$

defined by

$$\begin{aligned} \xi^\dagger(a, b) &\stackrel{\text{def}}{=} \text{ev}_b(\text{ev}_a(\xi)) \\ &\stackrel{\text{def}}{=} \text{ev}_b(\xi_a) \\ &\stackrel{\text{def}}{=} \xi_a(b) \end{aligned}$$

for each $(a, b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \text{id}_{\text{Sets}(A \times B, C)}.$$

Indeed, given a function $\xi: A \times B \rightarrow C$, we have

$$\begin{aligned} [\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\ &= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\ &= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\ &= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket) \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\text{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket) \rrbracket) \rrbracket \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\ &= \llbracket (a, b) \mapsto \xi(a, b) \rrbracket \\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \text{id}_{\text{Sets}(A, \text{Sets}(B, C))}.$$

Indeed, given a function

$$\begin{aligned} \xi: A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a: B \rightarrow C), \end{aligned}$$

we have

$$\begin{aligned} [\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\stackrel{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a, b) \mapsto \xi_a(b) \rrbracket) \\ &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \text{ev}_{(a,b)}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\ &\stackrel{\text{def}}{=} \xi. \end{aligned}$$

- *Naturality for Φ , Part I.* We need to show that, given a function $g: B \rightarrow$

B' , the diagram

$$\begin{array}{ccc} \text{Sets}(A \times B', C) & \xrightarrow{\Phi_{B',C}} & \text{Sets}(A, \text{Sets}(B', C)), \\ \text{id}_A \times g^* \downarrow & & \downarrow (g^*)_! \\ \text{Sets}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Sets}(A, \text{Sets}(B, C)) \end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B' \rightarrow C,$$

we have

$$\begin{aligned} [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\ &= \Phi_{B,C}(\xi(-_1, g(-_2))) \\ &= [\xi(-_1, g(-_2))]^\dagger \\ &= \xi_{-1}^\dagger(g(-_2)) \\ &= (g^*)_!(\xi^\dagger) \\ &= (g^*)_!(\Phi_{B',C}(\xi)) \\ &= [(g^*)_! \circ \Phi_{B',C}](\xi). \end{aligned}$$

Alternatively, using the $\llbracket a \mapsto f(a) \rrbracket$ notation of [Sets, Definition 3.1.1.1.2](#), we have

$$\begin{aligned} [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\ &= \Phi_{B,C}([\text{id}_A \times g^*](\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\ &= \Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, g(b)) \rrbracket) \\ &= \llbracket a \mapsto \llbracket b \mapsto \xi(a, g(b)) \rrbracket \rrbracket \\ &= \llbracket a \mapsto g^*(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\ &= (g^*)_!(\llbracket a \mapsto \llbracket b' \mapsto \xi(a, b') \rrbracket \rrbracket) \\ &= (g^*)_!(\Phi_{B',C}(\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\ &= (g^*)_!(\Phi_{B',C}(\xi)) \\ &= [(g^*)_! \circ \Phi_{B',C}](\xi). \end{aligned}$$

- *Naturality for Φ , Part II.* We need to show that, given a function $b: C \rightarrow C'$, the diagram

$$\begin{array}{ccc} \text{Sets}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Sets}(A, \text{Sets}(B, C)), \\ \downarrow b_! & & \downarrow (b_!)_! \\ \text{Sets}(A \times B, C') & \xrightarrow{\Phi_{B,C'}} & \text{Sets}(A, \text{Sets}(B, C')) \end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B \rightarrow C,$$

we have

$$\begin{aligned} [\Phi_{B,C} \circ b_!](\xi) &= \Phi_{B,C}(b_!(\xi)) \\ &= \Phi_{B,C}(b_!(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\ &= \Phi_{B,C}(\llbracket (a, b) \mapsto b(\xi(a, b)) \rrbracket) \\ &= \llbracket a \mapsto \llbracket b \mapsto b(\xi(a, b)) \rrbracket \rrbracket \\ &= \llbracket a \mapsto b_!(\llbracket b \mapsto \xi(a, b) \rrbracket) \rrbracket \\ &= (b_!)_!(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\ &= (b_!)_!(\Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\ &= (b_!)_!(\Phi_{B,C}(\xi)) \\ &= [(b_!)_! \circ \Phi_{B,C}](\xi). \end{aligned}$$

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2](#) of [Definition 11.9.7.1.2](#) that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Adjointness II: This follows from the universal property of the product.

Item 4, Associativity: This is proved in the proof of [Monoidal Structures on the Category of Sets, Definition 5.1.4.1.1](#).

Item 5, Unitality: This is proved in the proof of [Monoidal Structures on the Category of Sets, Definitions 5.1.5.1.1 and 5.1.6.1.1](#).

Item 6, Commutativity: This is proved in the proof of [Monoidal Structures on the Category of Sets, Definition 5.1.7.1.1](#).

Item 7, Distributivity Over Coproducts: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.3.1.1.1 and 5.3.2.1.1.

Item 8, Annihilation With the Empty Set: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.3.3.1.1 and 5.3.4.1.1.

Item 9, Distributivity Over Unions: See [Pro25c].

Item 10, Distributivity Over Intersections: See [Pro25d, Corollary 1].

Item 11, Distributivity Over Differences: See [Pro25a].

Item 12, Distributivity Over Symmetric Differences: See [Pro25b].

Item 13, Middle-Four Exchange With Respect to Intersections: See [Pro25d, Corollary 1].

Item 14, Symmetric Monoidality: This is a repetition of **Monoidal Structures on the Category of Sets**, Definition 5.1.9.1.1, and is proved there.

Item 15, Symmetric Bimonoidality: This is a repetition of **Monoidal Structures on the Category of Sets**, Definition 5.3.5.1.1, and is proved there. \square

Remark 4.1.3.1.4. As shown in **Item 1** of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

$$-_1 \times -_2 : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}.$$

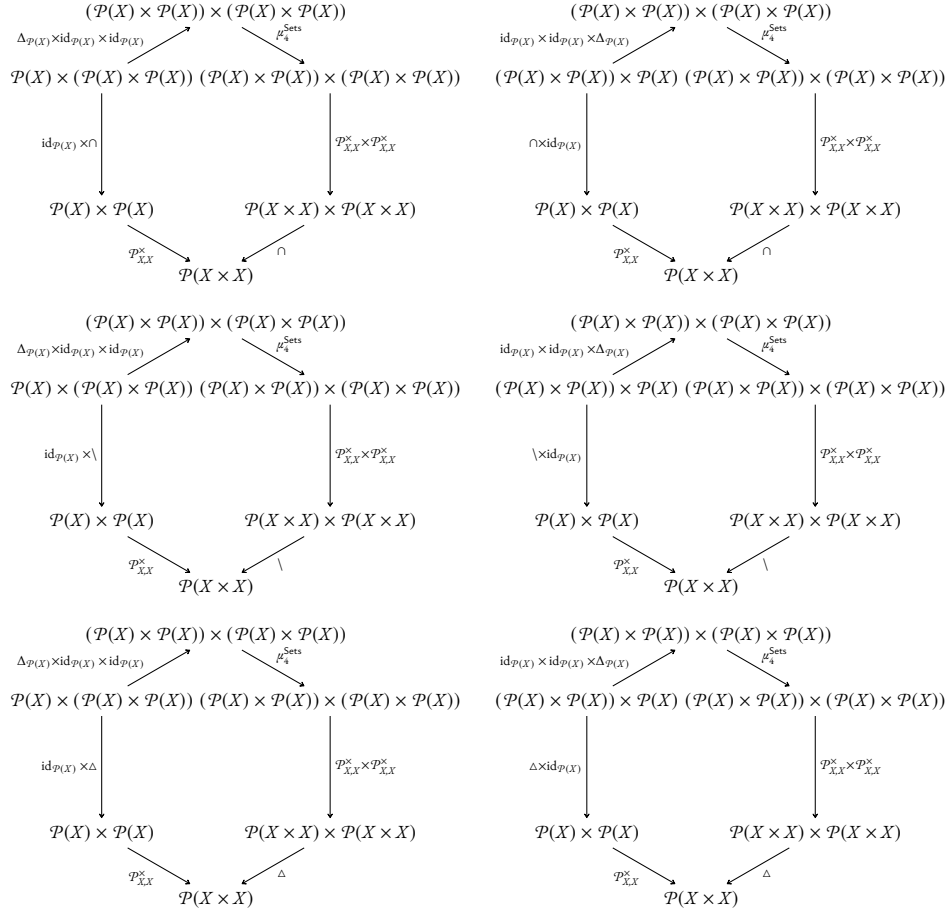
This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k, \ell} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_\ell}(\mathbf{Sets}) \rightarrow \mathbf{Mon}_{\mathbb{E}_{k+\ell}}(\mathbf{Sets})$$

of tensor products of \mathbb{E}_k -monoid objects on **Sets** with \mathbb{E}_ℓ -monoid objects on **Sets**; see ??.

Remark 4.1.3.1.5. We may state the equalities in **Items 9** to **12** of Definition 4.1.3.1.3 as the commutativity of the following diagrams:

$$\begin{array}{ccc}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \Delta_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)} \nearrow & & \searrow \mu_{\mathbf{Sets}}^{\mathcal{P}} \\
 \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \downarrow \text{id}_{\mathcal{P}(X)} \times \cup & & \downarrow \mathcal{P}_{XX}^\times \times \mathcal{P}_{XX}^\times \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) & \\
 \searrow \mathcal{P}_{XX}^\times & \swarrow \cup & \\
 & \mathcal{P}(X \times X) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \text{id}_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)} \times \Delta_{\mathcal{P}(X)} \nearrow & & \searrow \mu_{\mathbf{Sets}}^{\mathcal{P}} \\
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \downarrow \cup \times \text{id}_{\mathcal{P}(X)} & & \downarrow \mathcal{P}_{XX}^\times \times \mathcal{P}_{XX}^\times \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) & \\
 \searrow \mathcal{P}_{XX}^\times & \swarrow \cup & \\
 & \mathcal{P}(X \times X) &
 \end{array}$$



4.1.4 Pullbacks

Let A , B , and C be sets and let $f: A \rightarrow C$ and $g: B \rightarrow C$ be functions.

Definition 4.1.4.1.1. The **pullback of A and B over C along f and g** ³ is the pullback of A and B over C along f and g in **Sets** as in Limits and Colimits, ??.

Construction 4.1.4.1.2. Concretely, the pullback of A and B over C along f and g is the pair $(A \times_C B, \{\text{pr}_1, \text{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

³*Further Terminology:* Also called the **fibre product of A and B over C along f and g** .

2. *The Cone.* The maps⁴

$$\begin{aligned}\mathrm{pr}_1 &: A \times_C B \rightarrow A, \\ \mathrm{pr}_2 &: A \times_C B \rightarrow B\end{aligned}$$

defined by

$$\begin{aligned}\mathrm{pr}_1(a, b) &\stackrel{\mathrm{def}}{=} a, \\ \mathrm{pr}_2(a, b) &\stackrel{\mathrm{def}}{=} b\end{aligned}$$

for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in **Sets**. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,$$

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\mathrm{pr}_2} & B \\ \mathrm{pr}_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C. \end{array}$$

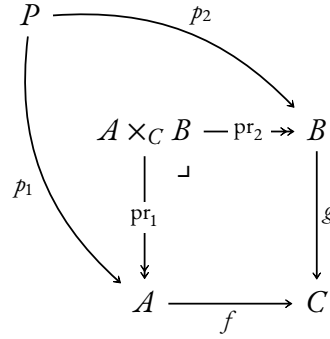
Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\mathrm{pr}_2(a, b)) \\ &= [g \circ \mathrm{pr}_2](a, b),\end{aligned}$$

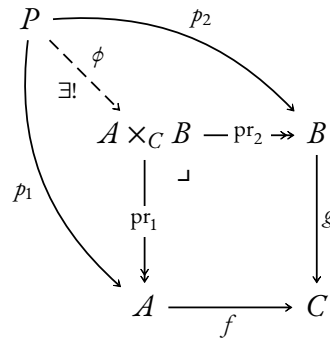
where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies

⁴*Further Notation:* Also written $\mathrm{pr}_1^{A \times_C B}$ and $\mathrm{pr}_2^{A \times_C B}$.

the universal property of the pullback. Suppose we have a diagram of the form



in Sets. Then there exists a unique map $\phi: P \rightarrow A \times_C B$ making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

Remark 4.1.4.1.3. It is common practice to write $A \times_C B$ for the pullback of A and B over C along f and g , omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \times_C B$ depends very much on the maps f and g , and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

Example 4.1.4.1.4. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let X be a set. We have

$$A \cap B \cong A \times_{A \cup B} B,$$

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \iota_B \\ A & \xrightarrow{\iota_A} & A \cup B \end{array}$$

for each $A, B \in \mathcal{P}(X)$.

Proof. **Item 1, Unions via Intersections:** Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

Proposition 4.1.4.1.5. Let A, B, C , and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

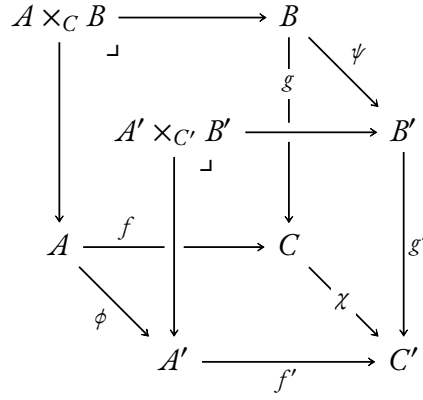
$$-_1 \times_{-3} -_1: \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a

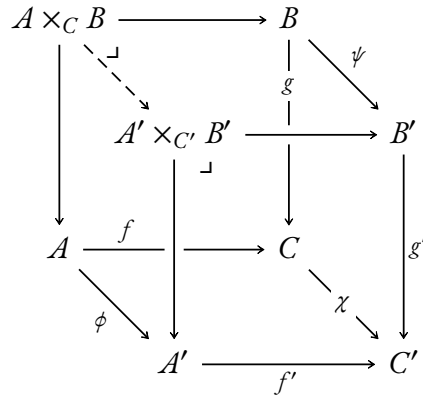
morphism



in $\text{Fun}(\mathcal{P}, \text{Sets})$ to the map $\xi: A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram



commute.

2. *Adjointness I.* We have adjunctions

$$\begin{aligned} (A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)) : \quad \mathbf{Sets}_{/X} &\overset{A \times_X -}{\underset{\mathbf{Sets}_{/X}(A, -)}{\rightleftarrows}} \mathbf{Sets}_{/X}, \\ (- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)) : \quad \mathbf{Sets}_{/X} &\overset{- \times_X B}{\underset{\mathbf{Sets}_{/X}(B, -)}{\rightleftarrows}} \mathbf{Sets}_{/X}, \end{aligned}$$

witnessed by bijections

$$\mathbf{Sets}_{/X}(A \times_X B, C) \cong \mathbf{Sets}_{/X}(A, \mathbf{Sets}_{/X}(B, C)),$$

$$\mathbf{Sets}_{/X}(A \times_X B, C) \cong \mathbf{Sets}_{/X}(B, \mathbf{Sets}_{/X}(A, C)),$$

natural in $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \mathbf{Obj}(\mathbf{Sets}_{/X})$, where $\mathbf{Sets}_{/X}(A, B)$ is the object of $\mathbf{Sets}_{/X}$ consisting of (see Fibred Sets, ??):

- *The Set.* The set $\mathbf{Sets}_{/X}(A, B)$ defined by

$$\mathbf{Sets}_{/X}(A, B) \stackrel{\text{def}}{=} \coprod_{x \in X} \mathbf{Sets}(\phi_A^{-1}(x), \phi_Y^{-1}(x))$$

- *The Map to X.* The map

$$\phi_{\mathbf{Sets}_{/X}(A, B)} : \mathbf{Sets}_{/X}(A, B) \rightarrow X$$

defined by

$$\phi_{\mathbf{Sets}_{/X}(A, B)}(x, f) \stackrel{\text{def}}{=} x$$

for each $(x, f) \in \mathbf{Sets}_{/X}(A, B)$.

3. *Adjointness II.* We have an adjunction

$$\left(\Delta_{\mathbf{Sets}_{/X}} \dashv -_1 \times -_2 \right) : \mathbf{Sets}_{/X} \begin{array}{c} \xrightarrow{\Delta_{\mathbf{Sets}_{/X}}} \\ \perp \\ \xleftarrow{-_1 \times -_2} \end{array} \mathbf{Sets}_{/X} \times \mathbf{Sets}_{/X},$$

witnessed by a bijection

$$\mathbf{Hom}_{\mathbf{Sets}_{/X} \times \mathbf{Sets}_{/X}}((A, A), (B, C)) \cong \mathbf{Sets}_{/X}(A, B \times_X C),$$

natural in $A \in \mathbf{Obj}(\mathbf{Sets}_{/X})$ and in $(B, C) \in \mathbf{Obj}(\mathbf{Sets}_{/X} \times \mathbf{Sets}_{/X})$.

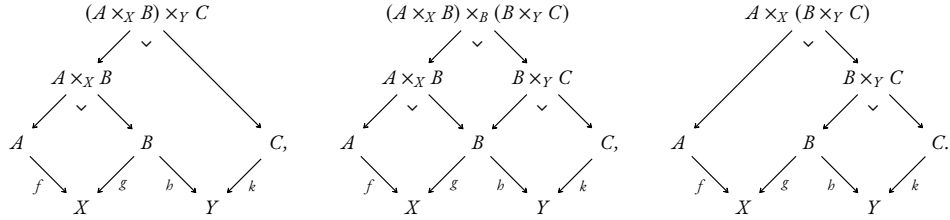
4. *Associativity.* Given a diagram

$$\begin{array}{ccccc} A & & B & & C \\ & \searrow f & \swarrow g & \searrow b & \swarrow k \\ & X & & Y & \end{array}$$

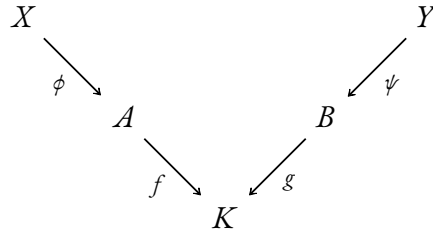
in **Sets**, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams



5. *Interaction With Composition.* Given a diagram



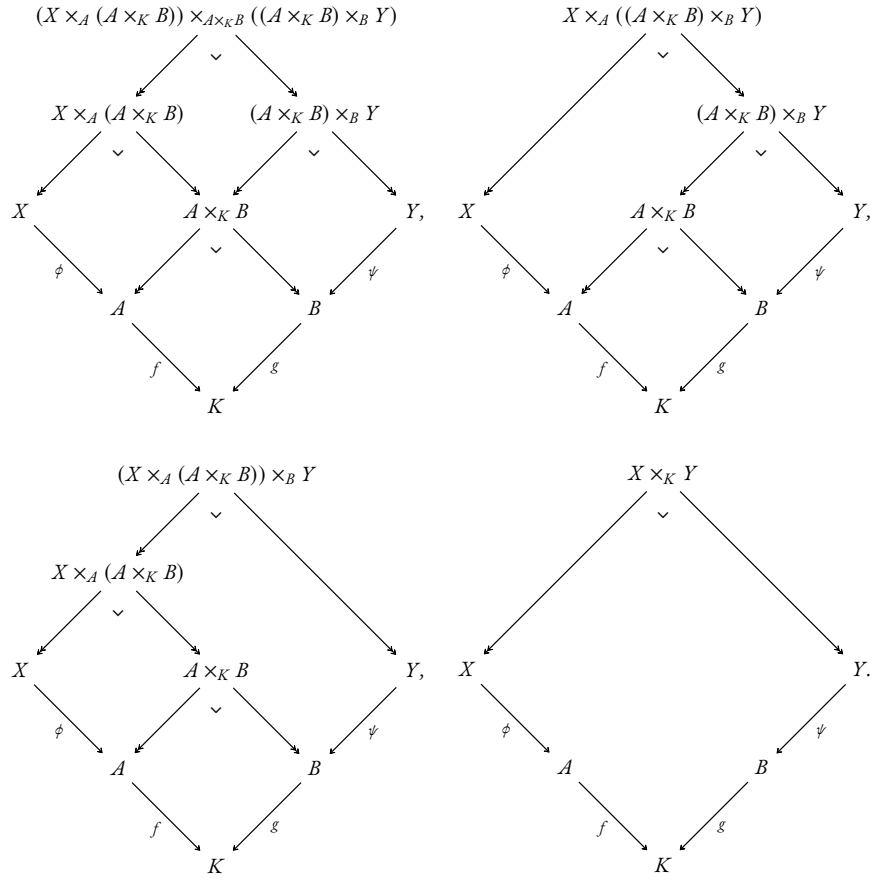
in **Sets**, we have isomorphisms of sets

$$\begin{aligned} X \times_K^{f \circ \phi, g \circ \psi} Y &\cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f, g} B \right) \right) \times_{A \times_K^{f, g} B}^{p_2, p_1} \left(\left(A \times_K^{f, g} B \right) \times_B^{q_2, \psi} Y \right) \\ &\cong X \times_A^{\phi, p} \left(\left(A \times_K^{f, g} B \right) \times_B^{q_2, \psi} Y \right) \\ &\cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f, g} B \right) \right) \times_B^{q, \psi} Y \end{aligned}$$

where

$$\begin{aligned} q_1 &= \text{pr}_1^{A \times_K^{f, g} B}, & q_2 &= \text{pr}_2^{A \times_K^{f, g} B}, \\ p_1 &= \text{pr}_1^{(A \times_K^{f, g} B) \times_Y^{q_2, \psi}}, & p_2 &= \text{pr}_2^{X \times_{A \times_K^{f, g} B}^{\phi, q_1} (A \times_K^{f, g} B)}, \\ p &= q_1 \circ \text{pr}_1^{(A \times_K^{f, g} B) \times_B^{q_2, \psi} Y}, & q &= q_2 \circ \text{pr}_2^{X \times_A^{\phi, q_1} (A \times_K^{f, g} B)}, \end{aligned}$$

and where these pullbacks are built as in the following diagrams:



6. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \downarrow f & \lrcorner & \downarrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 \lambda_A^{\mathbf{Sets}/X} : X \times_X A \xrightarrow{\sim} A, \\
 \rho_A^{\mathbf{Sets}/X} : A \times_X X \xrightarrow{\sim} A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xrightarrow{f} & X,
 \end{array}$$

natural in $(A, f) \in \mathbf{Obj}(\mathbf{Sets}/X)$.

7. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 \sigma_{A,B}^{\text{Sets}/X} : A \times_X B \xrightarrow{\sim} B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X,
 \end{array}$$

natural in $(A, f), (B, g) \in \text{Obj}(\text{Sets}/X)$.

8. *Distributivity Over Coproducts.* Let A, B , and C be sets and let $\phi_A: A \rightarrow X$, $\phi_B: B \rightarrow X$, and $\phi_C: C \rightarrow X$ be morphisms of sets. We have isomorphisms of sets

$$\begin{aligned}
 \delta_\ell^{\text{Sets}/X} : A \times_X (B \amalg C) &\xrightarrow{\sim} (A \times_X B) \amalg (A \times_X C), \\
 \delta_r^{\text{Sets}/X} : (A \amalg B) \times_X C &\xrightarrow{\sim} (A \times_X C) \amalg (B \times_X C),
 \end{aligned}$$

as in the diagrams

$$\begin{array}{ccc}
 (A \times_X B) \amalg (A \times_X C) & \longrightarrow & B \amalg C \\
 \downarrow \lrcorner & & \downarrow \phi_B \amalg \phi_C \\
 A & \xrightarrow{\phi_A} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 (A \times_X C) \amalg (B \times_X C) & \longrightarrow & C \\
 \downarrow \lrcorner & & \downarrow \phi_C \\
 A \amalg B & \xrightarrow{\phi_A \amalg \phi_B} & X
 \end{array}$$

natural in $A, B, C \in \text{Obj}(\text{Sets}/X)$.

9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & \emptyset \\
 \downarrow \lrcorner & & \downarrow \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 \begin{aligned}
 \zeta_\ell^{\text{Sets}/X} : A \times_X \emptyset &\xrightarrow{\sim} \emptyset, \\
 \zeta_r^{\text{Sets}/X} : \emptyset \times_X A &\xrightarrow{\sim} \emptyset,
 \end{aligned}
 \quad
 \begin{array}{ccc}
 \emptyset & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 \emptyset & \longrightarrow & X,
 \end{array}$$

natural in $(A, f) \in \text{Obj}(\text{Sets}/X)$.

10. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\text{pt}} B \cong A \times B,$$

$$\begin{array}{ccc} A \times B & \longrightarrow & B \\ \downarrow \lrcorner & & \downarrow !_B \\ A & \xrightarrow{!_A} & \text{pt.} \end{array}$$

11. *Symmetric Monoidality.* The 8-tuple $(\mathbf{Sets}/_X, \times_X, X, \mathbf{Sets}/_X, \alpha^{\mathbf{Sets}/_X}, \lambda^{\mathbf{Sets}/_X}, \rho^{\mathbf{Sets}/_X}, \sigma^{\mathbf{Sets}/_X})$ is a symmetric closed monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Adjointness I: This is a repetition of Fibred Sets, ?? of ??, and is proved there.

Item 3, Adjointness II: This follows from the universal property of the product (pullbacks are products in $\mathbf{Sets}/_X$).

Item 4, Associativity: We have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a, b), c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{((a, b), c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a, (b, c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a, (b, c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned} (A \times_X B) \times_B (B \times_Y C) &\cong \{((a, b), (b', c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\ &\cong \left\{ ((a, b), (b', c)) \in (A \times B) \times (B \times C) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\ &\cong \left\{ (a, (b, (b', c))) \in A \times (B \times (B \times C)) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\ &\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times B) \times C) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\ &\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times_B B) \times C) \mid \begin{array}{l} f(a) = g(b) \text{ and } \\ h(b') = k(c) \end{array} \right\} \\ &\cong \{(a, (b, c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \end{aligned}$$

$$\cong A \times_X (B \times_Y C),$$

where we have used **Item 6** for the isomorphism $B \times_B B \cong B$.

Item 5, Interaction With Composition: By **Item 4**, it suffices to construct only the isomorphism

$$X \times_K^{f \circ \phi, g \circ \psi} Y \cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f, g} B \right) \right) \times_{A \times_K^{f, g} B}^{p_2, p_1} \left(\left(A \times_K^{f, g} B \right) \times_B^{q_2, \psi} Y \right).$$

We have

$$\begin{aligned} \left(X \times_A^{\phi, q_1} \left(A \times_K^{f, g} B \right) \right) &\stackrel{\text{def}}{=} \left\{ (x, (a, b)) \in X \times \left(A \times_K^{f, g} B \right) \mid \phi(x) = q_1(a, b) \right\} \\ &\stackrel{\text{def}}{=} \left\{ (x, (a, b)) \in X \times \left(A \times_K^{f, g} B \right) \mid \phi(x) = a \right\} \\ &\cong \left\{ (x, (a, b)) \in X \times (A \times B) \mid \phi(x) = a \text{ and } f(a) = g(b) \right\}, \\ \left(\left(A \times_K^{f, g} B \right) \times_B^{q_2, \psi} Y \right) &\stackrel{\text{def}}{=} \left\{ ((a, b), y) \in \left(A \times_K^{f, g} B \right) \times Y \mid q_2(a, b) = \psi(y) \right\} \\ &\stackrel{\text{def}}{=} \left\{ ((a, b), y) \in \left(A \times_K^{f, g} B \right) \times Y \mid b = \psi(y) \right\} \\ &\cong \left\{ ((a, b), y) \in (A \times B) \times Y \mid b = \psi(y) \text{ and } f(a) = g(b) \right\}, \end{aligned}$$

so writing

$$\begin{aligned} S &= \left(X \times_A^{\phi, q_1} \left(A \times_K^{f, g} B \right) \right) \\ S' &= \left(\left(A \times_K^{f, g} B \right) \times_B^{q_2, \psi} Y \right), \end{aligned}$$

we have

$$\begin{aligned} S \times_{A \times_K^{f, g} B}^{p_2, p_1} S' &\stackrel{\text{def}}{=} \left\{ ((x, (a, b)), ((a', b'), y)) \in S \times S' \mid p_1(x, (a, b)) = p_2((a', b'), y) \right\} \\ &\stackrel{\text{def}}{=} \left\{ ((x, (a, b)), ((a', b'), y)) \in S \times S' \mid (a, b) = (a', b') \right\} \\ &\cong \left\{ ((x, a, b, y)) \in X \times A \times B \times Y \mid \phi(x) = a, \psi(y) = b, \text{ and } f(a) = g(b) \right\} \\ &\cong \left\{ ((x, a, b, y)) \in X \times A \times B \times Y \mid f(\phi(x)) = g(\psi(y)) \right\} \\ &\stackrel{\text{def}}{=} X \times_K Y. \end{aligned}$$

This finishes the proof.

Item 6, Unitality: We have

$$\begin{aligned} X \times_X A &\cong \{ (x, a) \in X \times A \mid f(a) = x \}, \\ A \times_X X &\cong \{ (a, x) \in X \times A \mid f(a) = x \}, \end{aligned}$$

which are isomorphic to A via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$. The proof of the naturality of $\lambda^{\mathbf{Sets}/X}$ and $\rho^{\mathbf{Sets}/X}$ is omitted.

Item 7, Commutativity: We have

$$\begin{aligned} A \times_C B &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\} \\ &= \{(a, b) \in A \times B \mid g(b) = f(a)\} \\ &\cong \{(b, a) \in B \times A \mid g(b) = f(a)\} \\ &\stackrel{\text{def}}{=} B \times_C A. \end{aligned}$$

The proof of the naturality of $\sigma^{\mathbf{Sets}/X}$ is omitted.

Item 8, Distributivity Over Coproducts: We have

$$\begin{aligned} A \times_X (B \amalg C) &\stackrel{\text{def}}{=} \{(a, z) \in A \times (B \amalg C) \mid \phi_A(a) = \phi_{B \amalg C}(z)\} \\ &= \{(a, z) \in A \times (B \amalg C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, z) \in A \times (B \amalg C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c)\} \\ &= \{(a, z) \in A \times (B \amalg C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, z) \in A \times (B \amalg C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c)\} \\ &\cong \{(a, b) \in A \times B \mid \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, c) \in A \times C \mid \phi_A(a) = \phi_C(c)\} \\ &\stackrel{\text{def}}{=} (A \times_X B) \cup (A \times_X C) \\ &\cong (A \times_X B) \amalg (A \times_X C), \end{aligned}$$

with the construction of the isomorphism

$$\delta_r^{\mathbf{Sets}/X} : (A \amalg B) \times_X C \xrightarrow{\sim} (A \times_X C) \amalg (B \times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\mathbf{Sets}/X}$ and $\delta_r^{\mathbf{Sets}/X}$ is omitted.

Item 9, Annihilation With the Empty Set: We have

$$\begin{aligned} A \times_X \emptyset &\stackrel{\text{def}}{=} \{(a, b) \in A \times \emptyset \mid f(a) = g(b)\} \\ &= \{k \in \emptyset \mid f(a) = g(b)\} \\ &= \emptyset, \end{aligned}$$

and similarly for $\emptyset \times_X A$, where we have used *Item 8* of *Definition 4.1.3.1.3*. The proof of the naturality of $\zeta_\ell^{\mathbf{Sets}/X}$ and $\zeta_r^{\mathbf{Sets}/X}$ is omitted.

Item 10, Interaction With Products: We have

$$\begin{aligned}
 A \times_{\text{pt}} B &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid !_A(a) = !_B(b)\} \\
 &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid \star = \star\} \\
 &= \{(a, b) \in A \times B\} \\
 &= A \times B.
 \end{aligned}$$

Item 11, Symmetric Monoidality: Omitted. □

4.1.5 Equalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

Definition 4.1.5.1.1. The **equaliser of f and g** is the equaliser of f and g in **Sets** as in Limits and Colimits, ??.

Construction 4.1.5.1.2. Concretely, the equaliser of f and g is the pair $(\text{Eq}(f, g), \text{eq}(f, g))$ consisting of:

1. *The Limit.* The set $\text{Eq}(f, g)$ defined by

$$\text{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

2. *The Cone.* The inclusion map

$$\text{eq}(f, g): \text{Eq}(f, g) \rightarrow A.$$

Proof. We claim that $\text{Eq}(f, g)$ is the categorical equaliser of f and g in **Sets**. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc}
 \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A \xrightleftharpoons[g]{f} B \\
 & \nearrow e & \\
 E & &
 \end{array}$$

in Sets. Then there exists a unique map $\phi: E \rightarrow \text{Eq}(f, g)$ making the diagram

$$\begin{array}{ccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A \xrightleftharpoons[g]{f} B \\ \uparrow \phi \quad \exists! & \nearrow e & \\ E & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. □

Proposition 4.1.5.1.3. Let A, B , and C be sets.

1. *Associativity.* We have isomorphisms of sets⁵

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

⁵That is, the following three ways of forming “the” equaliser of (f, g, h) agree:

1. Take the equaliser of (f, g, h) , i.e. the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

in Sets.

2. First take the equaliser of f and g , forming a diagram

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) = \text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))$$

of $\text{Eq}(f, g)$.

3. First take the equaliser of g and h , forming a diagram

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) = \text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))$$

of $\text{Eq}(g, h)$.

where $\text{Eq}(f, g, b)$ is the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{} \\ \xrightarrow{b} \end{array} B$$

in **Sets**, being explicitly given by

$$\text{Eq}(f, g, b) \cong \{a \in A \mid f(a) = g(a) = b(a)\}.$$

4. *Unitality*. We have an isomorphism of sets

$$\text{Eq}(f, f) \cong A.$$

5. *Commutativity*. We have an isomorphism of sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

6. *Interaction With Composition*. Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{} \end{array} B \begin{array}{c} \xrightarrow{b} \\ \xrightarrow[k]{} \end{array} C$$

be functions. We have an inclusion of sets

$$\text{Eq}(b \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) \subset \text{Eq}(b \circ f, k \circ g),$$

where $\text{Eq}(b \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ is the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{} \end{array} B \begin{array}{c} \xrightarrow{b} \\ \xrightarrow[k]{} \end{array} C.$$

Proof. **Item 1, Associativity**: We first prove that $\text{Eq}(f, g, b)$ is indeed given by

$$\text{Eq}(f, g, b) \cong \{a \in A \mid f(a) = g(a) = b(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & \text{Eq}(f, g, b) & \xrightarrow{\text{eq}(f, g, b)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{} \\ \xrightarrow{b} \end{array} B \\ & \nearrow e & \\ E & & \end{array}$$

in Sets. Then there exists a unique map $\phi: E \rightarrow \text{Eq}(f, g, h)$, uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g, h)$.

We now check the equalities

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) \cong \text{Eq}(f, g, h) \cong \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)).$$

Indeed, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) &\cong \{x \in \text{Eq}(g, h) \mid [f \circ \text{eq}(g, h)](a) = [g \circ \text{eq}(g, h)](a)\} \\ &\cong \{x \in \text{Eq}(g, h) \mid f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) &\cong \{x \in \text{Eq}(f, g) \mid [f \circ \text{eq}(f, g)](a) = [h \circ \text{eq}(f, g)](a)\} \\ &\cong \{x \in \text{Eq}(f, g) \mid f(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

Item 4, Unitality: Indeed, we have

$$\begin{aligned} \text{Eq}(f, f) &\stackrel{\text{def}}{=} \{a \in A \mid f(a) = f(a)\} \\ &= A. \end{aligned}$$

Item 5, Commutativity: Indeed, we have

$$\text{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}$$

$$= \{a \in A \mid g(a) = f(a)\} \\ \stackrel{\text{def}}{=} \text{Eq}(g, f).$$

Item 6, Interaction With Composition: Indeed, we have

$$\begin{aligned} \text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) &\cong \{a \in \text{Eq}(f, g) \mid h(f(a)) = k(g(a))\} \\ &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}. \end{aligned}$$

and

$$\text{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ to $\text{Eq}(h \circ f, k \circ g)$. \square

4.1.6 Inverse Limits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I} : (I, \preceq) \rightarrow \text{Sets}$ be an inverse system of sets.

Definition 4.1.6.1.1. The **inverse limit** of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ in Sets as in Limits and Colimits, ??.

Construction 4.1.6.1.2. Concretely, the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the pair $(\varprojlim_{\alpha \in I} (X_\alpha), \{\text{pr}_\alpha\}_{\alpha \in I})$ consisting of:

1. *The Limit.* The set $\varprojlim_{\alpha \in I} (X_\alpha)$ defined by

$$\varprojlim_{\alpha \in I} (X_\alpha) \stackrel{\text{def}}{=} \left\{ (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \left| \begin{array}{l} \text{for each } \alpha, \beta \in I, \text{ if } \alpha \preceq \beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta) \end{array} \right. \right\}.$$

2. *The Cone.* The collection

$$\left\{ \text{pr}_\gamma : \varprojlim_{\alpha \in I} (X_\alpha) \rightarrow X_\gamma \right\}_{\gamma \in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{ \text{pr}_\gamma: \prod_{\alpha \in I} X_\alpha \rightarrow X_\gamma \right\}_{\gamma \in I}$$

of **Item 2** of **Definition 4.1.2.1.2** to $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ and hence given by

$$\text{pr}_\gamma((x_\alpha)_{\alpha \in I}) \stackrel{\text{def}}{=} x_\gamma$$

for each $\gamma \in I$ and each $(x_\alpha)_{\alpha \in I} \in \lim_{\leftarrow \alpha \in I} (X_\alpha)$.

Proof. We claim that $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ is the limit of the inverse system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$.

First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$\begin{array}{ccc} & \lim_{\leftarrow \alpha \in I} (X_\alpha) & \\ & \uparrow \text{pr}_\alpha \quad \downarrow \text{pr}_\beta & \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

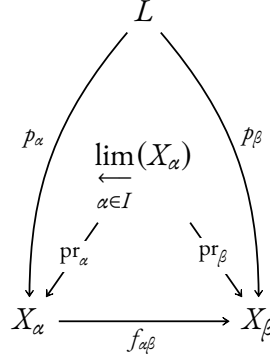
$f_{\alpha\beta} \circ \text{pr}_\alpha = \text{pr}_\beta$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma \in I} \in \lim_{\leftarrow \gamma \in I} (X_\gamma)$, we have

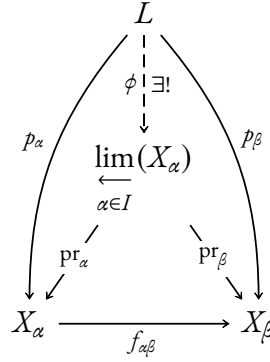
$$\begin{aligned} [f_{\alpha\beta} \circ \text{pr}_\alpha]((x_\gamma)_{\gamma \in I}) &\stackrel{\text{def}}{=} f_{\alpha\beta}(\text{pr}_\alpha((x_\gamma)_{\gamma \in I})) \\ &\stackrel{\text{def}}{=} f_{\alpha\beta}(x_\alpha) \\ &= x_\beta \\ &\stackrel{\text{def}}{=} \text{pr}_\beta((x_\gamma)_{\gamma \in I}), \end{aligned}$$

where the third equality comes from the definition of $\lim_{\leftarrow \alpha \in I} (X_\alpha)$. Next, we prove that $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that

we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form



in Sets. Then there indeed exists a unique map $\phi: L \xrightarrow{\exists!} \varprojlim_{\alpha \in I} (X_\alpha)$ making the diagram



commute, being uniquely determined by the family of conditions

$$\{p_\alpha = \text{pr}_\alpha \circ \phi\}_{\alpha \in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha \in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$ indeed lies in $\varprojlim_{\alpha \in I} (X_\alpha)$, as we have

$$\begin{aligned}
 f_{\alpha\beta}(p_\alpha(\ell)) &\stackrel{\text{def}}{=} [f_{\alpha\beta} \circ p_\alpha](\ell) \\
 &\stackrel{\text{def}}{=} p_\beta(\ell)
 \end{aligned}$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $(L, \{p_\alpha\}_{\alpha \in I})$. \square

Example 4.1.6.1.3. Here are some examples of inverse limits of sets.

1. *The p -Adic Integers.* The ring of p -adic integers \mathbb{Z}_p of ?? is the inverse limit

$$\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N}} (\mathbb{Z}/p^n);$$

see ??.

2. *Rings of Formal Power Series.* The ring $R[[t]]$ of formal power series in a variable t is the inverse limit

$$R[[t]] \cong \varprojlim_{n \in \mathbb{N}} (R[t]/t^n R[t]);$$

see ??.

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see ??.

4.2 Colimits of Sets

4.2.1 The Initial Set

Definition 4.2.1.1.1. The **initial set** is the initial object of **Sets** as in Limits and Colimits, ??.

Construction 4.2.1.1.2. Concretely, the initial set is the pair $(\emptyset, \{\iota_A\}_{A \in \text{Obj}(\text{Sets})})$ consisting of:

1. *The Colimit.* The empty set \emptyset of **Definition 4.3.1.1.1.**
2. *The Cocone.* The collection of maps

$$\{\iota_A: \emptyset \rightarrow A\}_{A \in \text{Obj}(\text{Sets})}$$

given by the inclusion maps from \emptyset to A .

Proof. We claim that \emptyset is the initial object of **Sets**. Indeed, suppose we have a

diagram of the form

$$\emptyset \quad A$$

in **Sets**. Then there exists a unique map $\phi: \emptyset \rightarrow A$ making the diagram

$$\emptyset \xrightarrow[\exists!]{\phi} A$$

commute, namely the inclusion map ι_A . □

4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 4.2.2.1.1. The **coproduct** of $\{A_i\}_{i \in I}$ ⁶ is the coproduct of $\{A_i\}_{i \in I}$ in **Sets** as in Limits and Colimits, ??.

Construction 4.2.2.1.2. Concretely, the disjoint union of $\{A_i\}_{i \in I}$ is the pair $\left(\coprod_{i \in I} A_i, \{\text{inj}_i\}_{i \in I} \right)$ consisting of:

1. *The Colimit.* The set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ (i, x) \in I \times \left(\bigcup_{i \in I} A_i \right) \mid x \in A_i \right\}.$$

2. *The Cocone.* The collection

$$\left\{ \text{inj}_i: A_i \rightarrow \coprod_{i \in I} A_i \right\}_{i \in I}$$

of maps given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i \in I} A_i$ is the categorical coproduct of $\{A_i\}_{i \in I}$ in **Sets**.

⁶*Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i \in I}$.

Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

in **Sets**. Then there exists a unique map $\phi: \coprod_{i \in I} A_i \rightarrow C$ making the diagram

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \uparrow \phi \exists! \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i, x)) = \iota_i(x)$$

for each $(i, x) \in \coprod_{i \in I} A_i$. □

Proposition 4.2.2.1.3. Let $\{A_i\}_{i \in I}$ be a family of sets.

I. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \coprod_{i \in I} A_i$ defines a functor

$$\coprod_{i \in I}: \text{Fun}(I_{\text{disc}}, \mathbf{Sets}) \rightarrow \mathbf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \mathbf{Sets}))$, we have

$$\left[\coprod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \coprod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \mathbf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \mathbf{Sets} \left(\coprod_{i \in I} A_i, \coprod_{i \in I} B_i \right)$$

of $\coprod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i: A_i \rightarrow B_i\}_{i \in I}$$

in $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\coprod_{i \in I} f_i: \coprod_{i \in I} A_i \rightarrow \coprod_{i \in I} B_i$$

defined by

$$\left[\coprod_{i \in I} f_i \right] (i, a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i \in I} A_i$.

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??. \square

4.2.3 Binary Coproducts

Let A and B be sets.

Definition 4.2.3.1.1. The **coproduct of A and B** ⁷ is the coproduct of A and B in Sets as in Limits and Colimits, ??.

Construction 4.2.3.1.2. Concretely, the coproduct of A and B is the pair $(A \coprod B, \{\text{inj}_1, \text{inj}_2\})$ consisting of:

1. *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in A\} \cup \{(1, b) \in S \mid b \in B\}, \end{aligned}$$

where $S = \{0, 1\} \times (A \cup B)$.

2. *The Cocone.* The maps

$$\begin{aligned} \text{inj}_1: A &\rightarrow A \coprod B, \\ \text{inj}_2: B &\rightarrow A \coprod B, \end{aligned}$$

⁷ *Further Terminology:* Also called the **disjoint union of A and B** .

given by

$$\begin{aligned}\text{inj}_1(a) &\stackrel{\text{def}}{=} (0, a), \\ \text{inj}_2(b) &\stackrel{\text{def}}{=} (1, b),\end{aligned}$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \amalg B$ is the categorical coproduct of A and B in **Sets**. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & C & \\ \iota_1 \nearrow & & \nwarrow \iota_2 \\ A & \xrightarrow{\text{inj}_1} & A \amalg B \xleftarrow{\text{inj}_2} B \end{array}$$

in **Sets**. Then there exists a unique map $\phi: A \amalg B \rightarrow C$ making the diagram

$$\begin{array}{ccc} & C & \\ \iota_1 \nearrow & \uparrow \phi \mid \exists! & \nwarrow \iota_2 \\ A & \xrightarrow{\text{inj}_1} & A \amalg B \xleftarrow{\text{inj}_2} B \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_A &= \iota_A, \\ \phi \circ \text{inj}_B &= \iota_B\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \amalg B$. □

Proposition 4.2.3.1.3. Let A, B, C , and X be sets.

I. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \amalg B$ defines functors

$$\begin{aligned}A \amalg -: \quad \text{Sets} &\rightarrow \text{Sets}, \\ - \amalg B: \quad \text{Sets} &\rightarrow \text{Sets}, \\ -_1 \amalg -_2: \text{Sets} \times \text{Sets} &\rightarrow \text{Sets},\end{aligned}$$

where $-_1 \amalg -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$, we have

$$[-_1 \amalg -_2](A, B) \stackrel{\text{def}}{=} A \amalg B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\amalg_{(A,B),(X,Y)} : \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \amalg B, X \amalg Y)$$

of \amalg at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \amalg g : A \amalg B \rightarrow X \amalg Y$$

defined by

$$[f \amalg g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \amalg B$.

and where $A \amalg -$ and $- \amalg B$ are the partial functors of $-_1 \amalg -_2$ at $A, B \in \text{Obj}(\text{Sets})$.

2. *Adjointness.* We have an adjunction

$$(-_1 \amalg -_2 \dashv \Delta_{\text{Sets}}) : \text{Sets} \times \text{Sets} \begin{array}{c} \xrightarrow{-_1 \amalg -_2} \\ \perp \\ \xleftarrow{\Delta_{\text{Sets}}} \end{array} \text{Sets},$$

witnessed by a bijection

$$\text{Sets}(A \amalg B, C), \cong \text{Hom}_{\text{Sets} \times \text{Sets}}((A, B), (C, C))$$

natural in $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$ and in $C \in \text{Obj}(\text{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg} : (X \amalg Y) \amalg Z \xrightarrow{\sim} X \amalg (Y \amalg Z),$$

natural in $X, Y, Z \in \text{Obj}(\text{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned}\lambda_X^{\text{Sets}, \amalg} : \emptyset \amalg X &\xrightarrow{\sim} X, \\ \rho_X^{\text{Sets}, \amalg} : X \amalg \emptyset &\xrightarrow{\sim} X,\end{aligned}$$

natural in $X \in \text{Obj}(\text{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$\sigma_{X,Y}^{\text{Sets}, \amalg} : X \amalg Y \xrightarrow{\sim} Y \amalg X,$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

6. *Symmetric Monoidality.* The 7-tuple $(\text{Sets}, \amalg, \emptyset, \alpha_{\amalg}^{\text{Sets}}, \lambda_{\amalg}^{\text{Sets}}, \rho_{\amalg}^{\text{Sets}}, \sigma^{\text{Sets}})$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality:* This follows from Limits and Colimits, ?? of ??.

Item 2, Adjointness: This follows from the universal property of the coproduct.

Item 3, Associativity: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.2.3.1.1.

Item 4, Unitality: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.2.4.1.1 and 5.2.5.1.1.

Item 5, Commutativity: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.2.6.1.1.

Item 6, Symmetric Monoidality: This is a repetition of **Monoidal Structures on the Category of Sets**, Definition 5.2.7.1.1, and is proved there. \square

4.2.4 Pushouts

Let A, B , and C be sets and let $f : C \rightarrow A$ and $g : C \rightarrow B$ be functions.

Definition 4.2.4.1.1. The **pushout of A and B over C along f and g** ⁸ is the pushout of A and B over C along f and g in **Sets** as in Limits and Colimits, ??.

Construction 4.2.4.1.2. Concretely, the pushout of A and B over C along f and g is the pair $(A \amalg_C B, \{\text{inj}_1, \text{inj}_2\})$ consisting of:

⁸*Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g** .

1. *The Colimit.* The set $A \amalg_C B$ defined by

$$A \amalg_C B \stackrel{\text{def}}{=} A \amalg B / \sim_C,$$

where \sim_C is the equivalence relation on $A \amalg B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

2. *The Cocone.* The maps

$$\text{inj}_1: A \rightarrow A \amalg_C B,$$

$$\text{inj}_2: B \rightarrow A \amalg_C B$$

given by

$$\text{inj}_1(a) \stackrel{\text{def}}{=} [(0, a)]$$

$$\text{inj}_2(b) \stackrel{\text{def}}{=} [(1, b)]$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \amalg_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

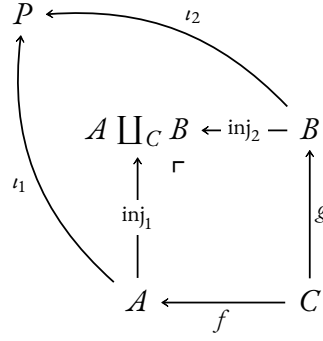
$$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$$

$$\begin{array}{ccc} A \amalg_C B & \xleftarrow{\text{inj}_2} & B \\ \text{inj}_1 \uparrow & & \uparrow g \\ A & \xleftarrow{f} & C. \end{array}$$

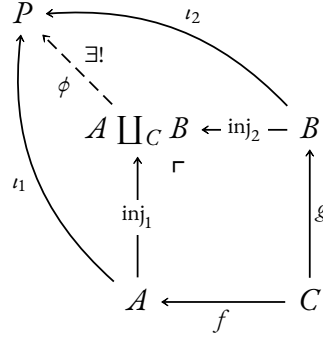
Indeed, given $c \in C$, we have

$$\begin{aligned} [\text{inj}_1 \circ f](c) &= \text{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \text{inj}_2(g(c)) \\ &= [\text{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation \sim on $A \amalg B$. Next, we prove that $A \amalg_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets. Then there exists a unique map $\phi: A \amalg_C B \rightarrow P$ making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_1 &= \iota_1, \\ \phi \circ \text{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \amalg_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation \sim on $A \amalg B$ as follows:

- i. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by [Definition 4.2.4.1.3](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by [Definition 4.2.4.1.3](#), we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by [Definition 4.2.4.1.3](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned} \phi([x]) &= \phi([(0, f(c))]) \\ &\stackrel{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]), \end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned} (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b) \end{aligned}$$

gives

$$\begin{aligned} \phi([(0, a)]) &= \phi([(0, a')]), \\ \phi([(1, b)]) &= \phi([(1, b')]), \\ \phi([(0, a)]) &= \phi([(1, b)]), \end{aligned}$$

showing ϕ to be well-defined. □

Remark 4.2.4.1.3. In detail, by [Conditions on Relations](#), [Definition 10.5.2.1.2](#), the relation \sim of [Definition 4.2.4.1.1](#) is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a, b \in A$ and $a = b$.
2. We have $a, b \in B$ and $a = b$.
3. There exist $x_1, \dots, x_n \in A \amalg B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
 - (a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
 - (b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

In other words, there exist $x_1, \dots, x_n \in A \amalg B$ satisfying the following conditions:

- (c) There exists $c_0 \in C$ satisfying one of the following conditions:
 - i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.
 - ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.
- (d) For each $1 \leq i \leq n - 1$, there exists $c_i \in C$ satisfying one of the following conditions:
 - i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
 - ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
- (e) There exists $c_n \in C$ satisfying one of the following conditions:
 - i. We have $x_n = f(c_n)$ and $b = g(c_n)$.
 - ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

Remark 4.2.4.1.4. It is common practice to write $A \amalg_C B$ for the pushout of A and B over C along f and g , omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \amalg_C B$ depends very much on the maps f and g , and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \amalg_{f,C,g} B$ or $A \amalg_C^{f,g} B$ for $A \amalg_C B$.

Example 4.2.4.1.5. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **Pointed Sets, Definition 6.3.3.1.1** is an example of a pushout of sets.

2. *Intersections via Unions.* Let X be a set. We have

$$A \cup B \cong A \amalg_{A \cap B} B,$$

$$\begin{array}{ccc} A \cup B & \longleftarrow & B \\ \uparrow \ulcorner & & \updownarrow \\ A & \longleftarrow & A \cap B \end{array}$$

for each $A, B \in \mathcal{P}(X)$.

Proof. **Item 1, Wedge Sums of Pointed Sets:** This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.

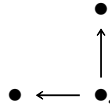
Item 2, Intersections via Unions: Indeed, $A \amalg_{A \cap B} B$ is the quotient of $A \amalg B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. \square

Proposition 4.2.4.1.6. Let A, B, C , and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \amalg_{f, C, g} B$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-_1 \amalg_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} A \amalg_C B & \longleftarrow & B & & \\ \uparrow \ulcorner & & \updownarrow & \searrow \psi & \\ A' \amalg_{C'} B' & \longleftarrow & B' & & \\ \uparrow \ulcorner & & \downarrow g & & \updownarrow g' \\ A & \xleftarrow{f} & C & \searrow \chi & \\ \downarrow \phi & & & & \\ A' & \xleftarrow{f'} & C' & & \end{array}$$

in $\text{Fun}(\mathcal{P}, \text{Sets})$ to the map $\xi: A \amalg_C B \xrightarrow{\exists!} A' \amalg_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \amalg_C B$, which is the unique map making the diagram

$$\begin{array}{ccccc} A \amalg_C B & \xleftarrow{\quad} & B & & \\ & \searrow \ulcorner & \uparrow & \searrow \psi & \\ & & A' \amalg_{C'} B' & \xleftarrow{\quad} & B' \\ & & \uparrow \ulcorner & \downarrow g & \uparrow \\ A & \xleftarrow{\quad} & C & \xrightarrow{f} & C' \\ & \searrow \phi & \downarrow & \searrow \chi & \uparrow g' \\ & & A' & \xleftarrow{f'} & C' \end{array}$$

commute.

2. *Adjointness.* We have an adjunction

$$\left(-1 \amalg_{X^{-2}} \dashv \Delta_{\text{Sets}_{X/}} \right): \text{Sets}_{X/} \times \text{Sets}_{X/} \begin{array}{c} \xrightarrow{-1 \amalg_{X^{-2}}} \\ \perp \\ \xleftarrow{\Delta_{\text{Sets}_{X/}}} \end{array} \text{Sets}_{X/},$$

witnessed by a bijection

$$\text{Sets}_{X/}(A \amalg_X B, C), \cong \text{Hom}_{\text{Sets}_{X/} \times \text{Sets}_{X/}}((A, B), (C, C))$$

natural in $(A, B) \in \text{Obj}(\text{Sets}_{X/} \times \text{Sets}_{X/})$ and in $C \in \text{Obj}(\text{Sets}_{X/})$.

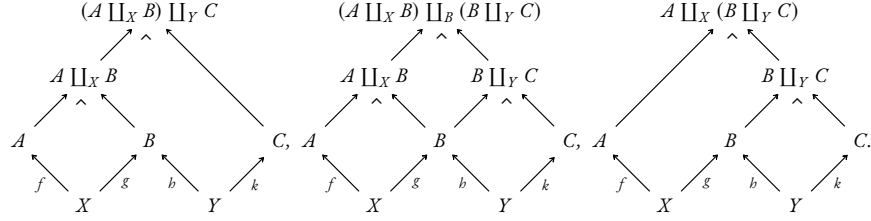
3. *Associativity.* Given a diagram

$$\begin{array}{ccccc} & A & & B & & C \\ & \swarrow f & & \swarrow g & & \swarrow h \\ & X & & Y & & \\ & \searrow & & \searrow k & & \end{array}$$

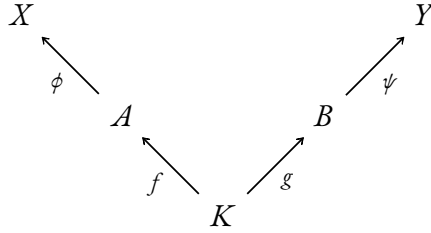
in **Sets**, we have isomorphisms of sets

$$(A \amalg_X B) \amalg_Y C \cong (A \amalg_X B) \amalg_B (B \amalg_Y C) \cong A \amalg_X (B \amalg_Y C)$$

where these pullbacks are built as in the diagrams



4. *Interaction With Composition.* Given a diagram



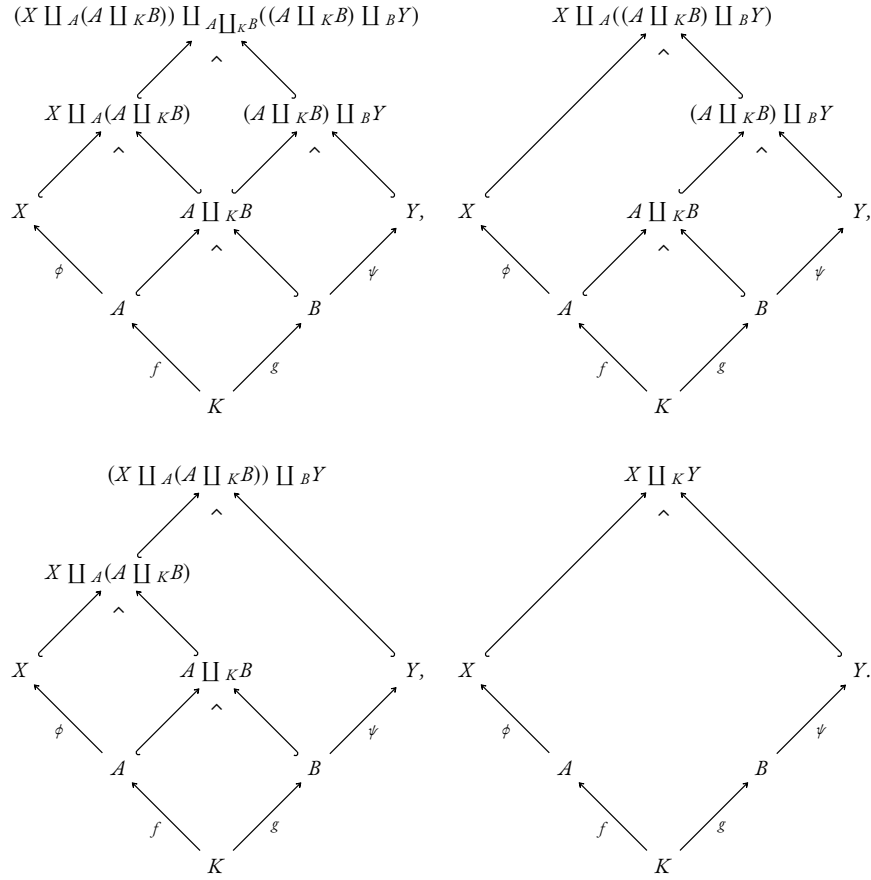
in **Sets**, we have isomorphisms of sets

$$\begin{aligned} X \amalg_K^{\phi \circ f, \psi \circ g} Y &\cong \left(X \amalg_A^{\phi, j_1} (A \amalg_K^{f, g} B) \right) \amalg_{A \amalg_K^{f, g} B}^{i_2, i_1} \left((A \amalg_K^{f, g} B) \amalg_B^{j_2, \psi} Y \right) \\ &\cong X \amalg_A^{\phi, i} \left((A \amalg_K^{f, g} B) \amalg_B^{j_2, \psi} Y \right) \\ &\cong \left(X \amalg_A^{\phi, i_1} (A \amalg_K^{f, g} B) \right) \amalg_B^{j, \psi} Y \end{aligned}$$

where

$$\begin{aligned} j_1 &= \text{inj}_1^{A \times_K^{f, g} B}, & j_2 &= \text{inj}_2^{A \times_K^{f, g} B}, \\ i_1 &= \text{inj}_1^{(A \times_K^{f, g} B) \times_Y^{q_2, \psi}}, & i_2 &= \text{inj}_2^{X \times_{A \times_K^{f, g} B}^{\phi, q_1} (A \times_K^{f, g} B)}, \\ i &= j_1 \circ \text{inj}_1^{(A \times_K^{f, g} B) \times_B^{q_2, \psi} Y}, & j &= j_2 \circ \text{inj}_2^{X \times_A^{\phi, q_1} (A \times_K^{f, g} B)}, \end{aligned}$$

and where these pullbacks are built as in the diagrams



5. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc} \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \uparrow f & \lrcorner & \uparrow f \\ X & \xlongequal{\quad} & X \end{array} & \begin{array}{l} \lambda_A^{\mathbf{Sets}_{X/}} : X \amalg_X A \xrightarrow{\sim} A, \\ \rho_A^{\mathbf{Sets}_{X/}} : A \amalg_X X \xrightarrow{\sim} A, \end{array} & \begin{array}{ccc} A & \xleftarrow{f} & X \\ \parallel & \lrcorner & \parallel \\ X & \xleftarrow{f} & X \end{array} \end{array}$$

natural in $(A, f) \in \mathbf{Obj}(\mathbf{Sets}_{X/})$.

6. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc} A \amalg_X B & \longleftarrow & B \\ \uparrow \ulcorner & & \uparrow g \\ A & \xleftarrow{f} & X, \end{array} \quad \sigma_A^{\mathbf{Sets}_{X/}} : A \amalg_X B \xrightarrow{\sim} B \amalg_X A \quad \begin{array}{ccc} B \amalg_X A & \longleftarrow & A \\ \uparrow \ulcorner & & \uparrow f \\ B & \xleftarrow{g} & X. \end{array}$$

natural in $(A, f), (B, g) \in \mathbf{Obj}(\mathbf{Sets}_{X/})$.

7. *Interaction With Coproducts.* We have

$$A \amalg_{\emptyset} B \cong A \amalg B, \quad \begin{array}{ccc} A \amalg B & \longleftarrow & B \\ \uparrow \ulcorner & & \uparrow \iota_B \\ A & \xleftarrow{\iota_A} & \emptyset. \end{array}$$

8. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_{X/}, \amalg_X, X)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, : Adjointness: This follows from the universal property of the coproduct (pushouts are coproducts in $\mathbf{Sets}_{X/}$).

Item 3, Associativity: Omitted.

Item 4, Interaction With Composition: Omitted.

Item 5, Unitality: Omitted.

Item 6, Commutativity: Omitted.

Item 7, Interaction With Coproducts: Omitted.

Item 8, Symmetric Monoidality: Omitted. □

4.2.5 Coequalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

Definition 4.2.5.1.1. The **coequaliser of f and g** is the coequaliser of f and g in \mathbf{Sets} as in Limits and Colimits, ??.

Construction 4.2.5.1.2. Concretely, the coequaliser of f and g is the pair $(\text{CoEq}(f, g), \text{coeq}(f, g))$ consisting of:

1. *The Colimit.* The set $\text{CoEq}(f, g)$ defined by

$$\text{CoEq}(f, g) \stackrel{\text{def}}{=} B/\sim,$$

where \sim is the equivalence relation on B generated by $f(a) \sim g(a)$.

2. *The Cocone.* The map

$$\text{coeq}(f, g) : B \twoheadrightarrow \text{CoEq}(f, g)$$

given by the quotient map $\pi : B \twoheadrightarrow B/\sim$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\text{CoEq}(f, g)$ is the categorical coequaliser of f and g in **Sets**. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](a) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(a)) \\ &\stackrel{\text{def}}{=} [f(a)] \\ &= [g(a)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(a)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](a) \end{aligned}$$

for each $a \in A$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\ & \searrow c & \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & C \end{array}$$

in **Sets**. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **Conditions on Relations, Items 4 and 5** of **Definition 10.6.2.1.3** that there exists a unique

map $\text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc}
 A & \xrightleftharpoons[g]{f} & B & \xrightarrow{\text{coeq}(f, g)} & \text{CoEq}(f, g) \\
 & & \searrow c & & \downarrow \exists! \\
 & & & & C
 \end{array}$$

commute. □

Remark 4.2.5.1.3. In detail, by [Conditions on Relations](#), [Definition 10.5.2.1.2](#), the relation \sim of [Definition 4.2.5.1.1](#) is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a = b$;
2. There exist $x_1, \dots, x_n \in B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
 - (a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
 - (b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

In other words, there exist $x_1, \dots, x_n \in B$ satisfying the following conditions:

- (a) There exists $z_0 \in A$ satisfying one of the following conditions:
 - i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.
 - ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.
- (b) For each $1 \leq i \leq n - 1$, there exists $z_i \in A$ satisfying one of the following conditions:
 - i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
 - ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
- (c) There exists $z_n \in A$ satisfying one of the following conditions:
 - i. We have $x_n = f(z_n)$ and $b = g(z_n)$.
 - ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 4.2.5.1.4. Here are some examples of coequalisers of sets.

- I. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X . We have a bijection of sets

$$X/\sim_R \cong \text{CoEq}\left(R \rightarrow X \times X \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} X\right).$$

Proof. *Item 1, Quotients by Equivalence Relations:* See [Pro25z].

□

Proposition 4.2.5.1.5. Let A , B , and C be sets.

1. *Associativity.* We have isomorphisms of sets⁹

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ b)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ b)} \cong \text{CoEq}(f, g, b) \cong \underbrace{\text{CoEq}(\text{coeq}(g, b) \circ f, \text{coeq}(g, b) \circ g)}_{=\text{CoEq}(\text{coeq}(g, b) \circ f, \text{coeq}(g, b) \circ b)},$$

⁹That is, the following three ways of forming “the” coequaliser of (f, g, b) agree:

1. Take the coequaliser of (f, g, b) , i.e. the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{b} \end{array} B$$

in Sets.

2. First take the coequaliser of f and g , forming a diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{} \end{array} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[b]{} \end{array} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ b) = \text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ b)$$

of $\text{CoEq}(f, g)$

3. First take the coequaliser of g and b , forming a diagram

$$A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow[b]{} \end{array} B \xrightarrow{\text{coeq}(g, b)} \text{CoEq}(g, b)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{} \end{array} B \xrightarrow{\text{coeq}(g, b)} \text{CoEq}(g, b),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(g, b) \circ f, \text{coeq}(g, b) \circ g) = \text{CoEq}(\text{coeq}(g, b) \circ f, \text{coeq}(g, b) \circ b)$$

of $\text{CoEq}(g, b)$.

where $\text{CoEq}(f, g, b)$ is the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \\ \xrightarrow{b} \end{array} B$$

in Sets.

4. *Unitality*. We have an isomorphism of sets

$$\text{CoEq}(f, f) \cong B.$$

5. *Commutativity*. We have an isomorphism of sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

6. *Interaction With Composition*. Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow[k]{\quad} \end{array} C$$

be functions. We have a surjection

$$\text{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$$

exhibiting $\text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$ as a quotient of $\text{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. **Item 1, Associativity**: Omitted.

Item 4, Unitality: Omitted.

Item 5, Commutativity: Omitted.

Item 6, Interaction With Composition: Omitted. □

4.2.6 Direct Colimits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I} : (I, \preceq) \rightarrow \Pi$ be a direct system of sets.

Definition 4.2.6.1.1. The **direct colimit** of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ in Sets as in Limits and Colimits, ??.

Construction 4.2.6.1.2. Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the pair $\left(\varinjlim (X_\alpha), \{\text{inj}_\alpha\}_{\alpha \in I} \right)$ consisting of:

1. *The Colimit.* The set $\varinjlim_{\alpha \in I} (X_\alpha)$ defined by

$$\varinjlim_{\alpha \in I} (X_\alpha) \stackrel{\text{def}}{=} \left(\coprod_{\alpha \in I} X_\alpha \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{\alpha \in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

- (a) We have $\alpha \preceq \gamma$.
- (b) We have $\beta \preceq \gamma$.
- (c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

2. *The Cocone.* The collection

$$\left\{ \text{inj}_\gamma : X_\gamma \rightarrow \varinjlim_{\alpha \in I} (X_\alpha) \right\}_{\gamma \in I}$$

of maps of sets defined by

$$\text{inj}_\gamma(x) \stackrel{\text{def}}{=} [(\gamma, x)]$$

for each $\gamma \in I$ and each $x \in X_\gamma$.

Proof. We will prove **Definition 4.2.6.1.2** below in a bit, but first we need a lemma (which is interesting in its own right). \square

Lemma 4.2.6.1.3. For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have

$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in $\varinjlim_{\alpha \in I} (X_\alpha)$.

Proof. Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \stackrel{\text{def}}{=} \text{id}_{X_\beta}$, and we have

$$\begin{aligned} f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} \text{id}_{X_\beta}(f_{\alpha\beta}(x)), \\ &= f_{\alpha\beta}(x). \end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, **Items ia to ic** of **Definition 4.2.6.1.2** are met. Thus we have $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$. \square

We can now prove **Definition 4.2.6.1.2**:

Proof. We claim that $\text{colim}_{\alpha \in I} (X_\alpha)$ is the colimit of the direct system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$.

Commutativity of the Colimit Diagram: First, we need to check that the colimit diagram defined by $\text{colim}_{\alpha \in I} (X_\alpha)$ commutes, i.e. that we have

$$\begin{array}{ccc} & \text{colim}(X_\alpha) & \\ & \xrightarrow{\alpha \in I} & \\ \text{inj}_\alpha = \text{inj}_\beta \circ f_{\alpha\beta} & \nearrow \text{inj}_\alpha & \nwarrow \text{inj}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

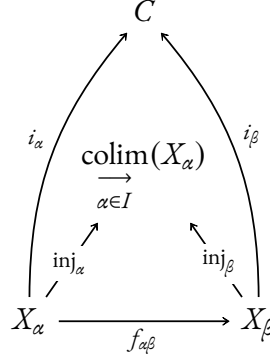
for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$\begin{aligned} [\text{inj}_\beta \circ f_{\alpha\beta}](x) &\stackrel{\text{def}}{=} \text{inj}_\beta(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\ &= [(\alpha, x)] \\ &\stackrel{\text{def}}{=} \text{inj}_\alpha(x), \end{aligned}$$

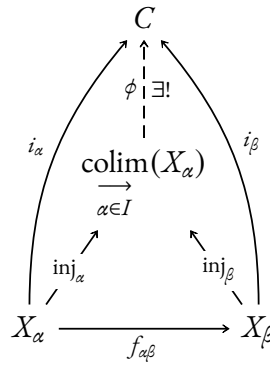
where we have used **Definition 4.2.6.1.3** for the third equality.

Proof of the Universal Property of the Colimit: Next, we prove that $\text{colim}_{\alpha \in I} (X_\alpha)$ as constructed in **Definition 4.2.6.1.2** satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the

form



in Sets. We claim that there exists a unique map $\phi: \text{colim}(X_\alpha) \xrightarrow[\alpha \in I]{\exists!} C$ making the diagram



commute. To this end, first consider the diagram

$$\begin{array}{ccc} \coprod_{\alpha \in I} X_\alpha & \xrightarrow{\text{pr}} & \text{colim}(X_\alpha) \\ & \searrow & \downarrow \\ & \coprod_{\alpha \in I} i_\alpha & C. \end{array}$$

Lemma. If $(\alpha, x) \sim (\beta, y)$, then we have

$$\left[\coprod_{\alpha \in I} i_\alpha \right] (x) = \left[\coprod_{\alpha \in I} i_\alpha \right] (y).$$

Proof. Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

1. We have $\alpha \preceq \gamma$.
2. We have $\beta \preceq \gamma$.
3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$\begin{aligned}
 \left[\coprod_{\alpha \in I} i_\alpha \right] (x) &\stackrel{\text{def}}{=} i_\alpha(x) \\
 &\stackrel{\text{def}}{=} [i_\gamma \circ f_{\alpha\gamma}](x) \\
 &\stackrel{\text{def}}{=} i_\gamma(f_{\alpha\gamma}(x)) \\
 &= i_\gamma(f_{\beta\gamma}(y)) \\
 &\stackrel{\text{def}}{=} [i_\gamma \circ f_{\beta\gamma}](y) \\
 &= i_\beta(y) \\
 &\stackrel{\text{def}}{=} \left[\coprod_{\alpha \in I} i_\alpha \right] (y).
 \end{aligned}$$

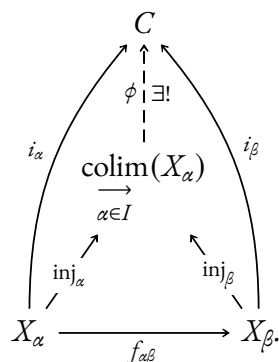
This finishes the proof of the lemma. Continuing, by **Conditions on Relations**, ?? of **Definition 10.6.2.1.3**, there then exists a map $\phi: \underset{\alpha \in I}{\text{colim}}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc}
 \coprod_{\alpha \in I} X_\alpha & \xrightarrow{\text{pr}} & \underset{\alpha \in I}{\text{colim}}(X_\alpha) \\
 & \searrow & \downarrow \phi \\
 & \coprod_{\alpha \in I} i_\alpha & C
 \end{array}$$

commute. In particular, this implies that the diagram

$$\begin{array}{ccc}
 X_\alpha & \xrightarrow{\text{inj}_\alpha} & \underset{\alpha \in I}{\text{colim}}(X_\alpha) \\
 & \searrow i_\alpha & \downarrow \phi \\
 & & C
 \end{array}$$

also commutes, and thus so does the diagram



This finishes the proof.¹⁰

□

Example 4.2.6.1.4. Here are some examples of direct colimits of sets.

1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \stackrel{\text{def}}{=} \text{colim}_{n \in \mathbb{N}} (\mathbb{Z}/p^n);$$

see ??.

4.3 Operations With Sets

4.3.1 The Empty Set

Definition 4.3.1.1.1. The **empty set** is the set \emptyset defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where X is the set in the set existence axiom, ?? of ??.

¹⁰Incidentally, the conditions

$$\{i_\alpha = \phi \circ \text{inj}_\alpha\}_{\alpha \in I}$$

show that ϕ must be given by

$$\phi([(\alpha, x)]) = (i_\alpha(x))_{\alpha \in I}$$

for each $[(\alpha, x)] \in \text{colim}_{\alpha \in I} (X_\alpha)$, although we would need to show that this assignment is

4.3.2 Singleton Sets

Let X be a set.

Definition 4.3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself of [Definition 4.3.3.1.1](#).

4.3.3 Pairings of Sets

Let X and Y be sets.

Definition 4.3.3.1.1. The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in \mathcal{A} \mid x = X \text{ or } x = Y\},$$

where \mathcal{A} is the set in the axiom of pairing, ?? of ??.

4.3.4 Ordered Pairs

Let A and B be sets.

Definition 4.3.4.1.1. The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 4.3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C , and D be sets. The following conditions are equivalent:

- (a) We have $(A, B) = (C, D)$.
- (b) We have $A = C$ and $B = D$.

Proof. [Item 1](#), *Uniqueness*: See [[Cie97](#), Theorem 1.2.3]. □

well-defined were we to prove [Definition 4.2.6.1.2](#) in this way. Instead, invoking [Conditions on Relations](#), ?? of [Definition 10.6.2.1.3](#) gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

4.3.5 Sets of Maps

Let A and B be sets.

Definition 4.3.5.1.1. The **set of maps from A to B** ¹¹ is the set $\text{Sets}(A, B)$ ¹² whose elements are the functions from A to B .

Proposition 4.3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \text{Hom}_{\text{Sets}}(X, Y)$ define functors

$$\begin{aligned} \text{Sets}(X, -) &: \text{Sets} \rightarrow \text{Sets}, \\ \text{Sets}(-, Y) &: \text{Sets}^{\text{op}} \rightarrow \text{Sets}, \\ \text{Sets}(-_1, -_2) &: \text{Sets}^{\text{op}} \times \text{Sets} \rightarrow \text{Sets}. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (A \times - \dashv \text{Sets}(A, -)) &: \text{Sets} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets}, \\ (- \times B \dashv \text{Sets}(B, -)) &: \text{Sets} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets}, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

3. *Maps From the Punctual Set.* We have a bijection

$$\text{Sets}(\text{pt}, A) \cong A,$$

natural in $A \in \text{Obj}(\text{Sets})$.

¹¹*Further Terminology:* Also called the **Hom set from A to B** .

¹²*Further Notation:* Also written $\text{Hom}_{\text{Sets}}(A, B)$.

4. *Maps to the Punctual Set.* We have a bijection

$$\mathbf{Sets}(A, \mathbf{pt}) \cong \mathbf{pt},$$

natural in $A \in \mathbf{Obj}(\mathbf{Sets})$.

Proof. Item 1, Functoriality: This follows from **Categories**, **Items 2** and **5** of **Definition II.1.4.1.2**.

Item 2, Adjointness: This is a repetition of **Item 2** of **Definition 4.1.3.1.3** and is proved there.

Item 3, Maps From the Punctual Set: The bijection

$$\Phi_A: \mathbf{Sets}(\mathbf{pt}, A) \xrightarrow{\sim} A$$

is given by

$$\Phi_A(f) \stackrel{\text{def}}{=} f(\star)$$

for each $f \in \mathbf{Sets}(\mathbf{pt}, A)$, admitting an inverse

$$\Phi_A^{-1}: A \xrightarrow{\sim} \mathbf{Sets}(\mathbf{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \stackrel{\text{def}}{=} \llbracket \star \mapsto a \rrbracket$$

for each $a \in A$. Indeed, we have

$$\begin{aligned} [\Phi_A^{-1} \circ \Phi_A](f) &\stackrel{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\ &\stackrel{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\ &\stackrel{\text{def}}{=} \llbracket \star \mapsto f(\star) \rrbracket \\ &\stackrel{\text{def}}{=} f \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathbf{Sets}(\mathbf{pt}, A)}](f) \end{aligned}$$

for each $f \in \mathbf{Sets}(\mathbf{pt}, A)$ and

$$\begin{aligned} [\Phi_A \circ \Phi_A^{-1}](a) &\stackrel{\text{def}}{=} \Phi_A(\Phi_A^{-1}(a)) \\ &\stackrel{\text{def}}{=} \Phi_A(\llbracket \star \mapsto a \rrbracket) \\ &\stackrel{\text{def}}{=} \text{ev}_{\star}(\llbracket \star \mapsto a \rrbracket) \\ &\stackrel{\text{def}}{=} a \end{aligned}$$

$$\stackrel{\text{def}}{=} [\text{id}_A](a)$$

for each $a \in A$, and thus we have

$$\begin{aligned}\Phi_A^{-1} \circ \Phi_A &= \text{id}_{\text{Sets}(\text{pt}, A)} \\ \Phi_A \circ \Phi_A^{-1} &= \text{id}_A.\end{aligned}$$

To prove naturality, we need to show that the diagram

$$\begin{array}{ccc}\text{Sets}(\text{pt}, A) & \xrightarrow{f_!} & \text{Sets}(\text{pt}, B) \\ \Phi_A \downarrow & & \downarrow \Phi_B \\ A & \xrightarrow{f} & B\end{array}$$

commutes. Indeed, we have

$$\begin{aligned}[f \circ \Phi_A](\phi) &\stackrel{\text{def}}{=} f(\Phi_A(\phi)) \\ &\stackrel{\text{def}}{=} f(\phi(\star)) \\ &\stackrel{\text{def}}{=} [f \circ \phi](\star) \\ &\stackrel{\text{def}}{=} \Phi_B(f \circ \phi) \\ &\stackrel{\text{def}}{=} \Phi_B(f_!(\phi)) \\ &\stackrel{\text{def}}{=} [\Phi_B \circ f_!](\phi)\end{aligned}$$

for each $\phi \in \text{Sets}(\text{pt}, A)$. This finishes the proof.

Item 4, Maps to the Punctual Set: This follows from the universal property of pt as the terminal set, [Definition 4.1.1.1.1](#). \square

4.3.6 Unions of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.6.1.1. The **union of** \mathcal{U} is the set $\bigcup_{U \in \mathcal{U}} U$ defined by

$$\bigcup_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}.$$

Proposition 4.3.6.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$ defines a functor

$$\bigcup: (\mathcal{P}(\mathcal{P}(X)), \subset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

$$(\star) \text{ If } \mathcal{U} \subset \mathcal{V}, \text{ then } \bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

2. *Associativity.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \bigcup} & \mathcal{P}(\mathcal{P}(X)) \\ \bigcup \star \text{id}_{\mathcal{P}(X)} \downarrow & & \downarrow \bigcup \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{A \in \mathcal{A}} \bigcup_{U \in A} U = \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \chi_{\mathcal{P}(X)} \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \{U\}} V = U$$

for each $U \in \mathcal{P}(X)$.

4. *Right Unitality.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \mathcal{P}(\chi_X) \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{\{u\} \in \chi_X(U)} \{u\} = U$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Unions I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \times \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{U \cup -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{- \cup V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$U \cup \left(\bigcup_{V \in \mathcal{V}} V \right) = \bigcup_{V \in \mathcal{V}} (U \cup V),$$

$$\left(\bigcup_{U \in \mathcal{U}} U \right) \cup V = \bigcup_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \times \cup \downarrow & \wr & \downarrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X), \end{array}$$

with components

$$\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \subset \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \cup \left(\bigcup_{V \in \mathcal{V}} V \right) &= \bigcup_{V \in \mathcal{V}} (U \cup V), \\ \left(\bigcup_{U \in \mathcal{U}} U \right) \cup V &= \bigcup_{U \in \mathcal{U}} (U \cup V) \end{aligned}$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\setminus} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup \times \cup & \text{X} & \downarrow \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\setminus} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup^{\text{op}} & \text{X} & \downarrow \cup \\
 \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{U \in \mathcal{U}^c} U \neq \bigcup_{U \in \mathcal{U}} U^c$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$\begin{array}{ccccc}
 & & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \text{---} \sim \text{---} \searrow & & \\
 \mathcal{P}(\mathcal{P}(X)) & & & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cup & & & & \downarrow \cap^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}}, & &
 \end{array}$$

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U \right)^c = \bigcap_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & \\ \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \dashrightarrow \sim \\ \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\ \cap \searrow & & \searrow \cup^{\text{op}} \\ \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}}, \end{array}$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U \right)^c = \bigcup_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \times \cup \downarrow & \text{X} & \downarrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U \right) \Delta \left(\bigcup_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup^{\text{op}} \times \cup^{\text{op}} & \text{X} & \downarrow \cup \\
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\
 \swarrow \sim & & \searrow \cup^{\text{op}} \\
 \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & \mathcal{P}(X)^{\text{op}} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & \downarrow [-, V]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f!(U) = \bigcup_{V \in f!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f!(\mathcal{U}) \stackrel{\text{def}}{=} (f!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{V})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a map of sets. The

diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Intersections of Families I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(x)) \\ \downarrow \cup \star \text{id}_{\mathcal{P}(X)} & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{\cap} & X \end{array}$$

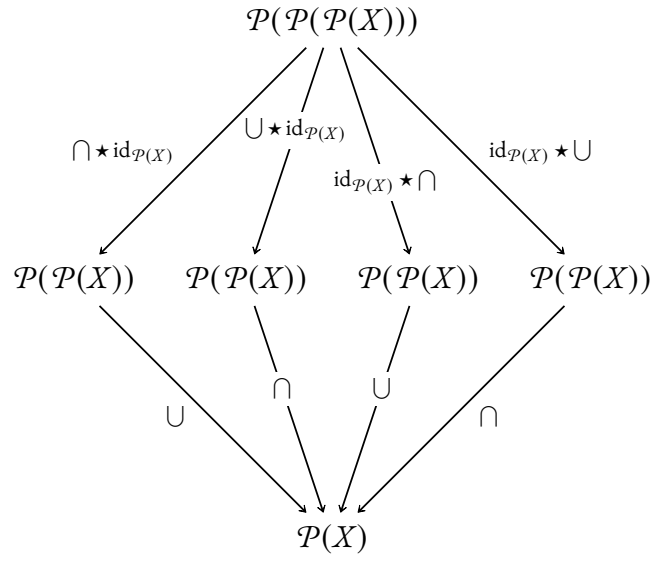
commutes, i.e. we have

$$\bigcap_{\substack{U \in \\ A \in \mathcal{A}}} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in \mathcal{A}} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Intersections of Families II.* Let X be a set and consider

the compositions



given by

$$\begin{aligned}
 \mathcal{A} &\mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, & \mathcal{A} &\mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U, \\
 \mathcal{A} &\mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), & \mathcal{A} &\mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)
 \end{aligned}$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\begin{array}{ccc}
 & \bigcup_{U \in \bigcap_{A \in \mathcal{A}} U} U & \\
 & \uparrow & \searrow \\
 \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) & & \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \\
 & \downarrow & \swarrow \\
 & \bigcap_{U \in \bigcup_{A \in \mathcal{A}} U} U &
 \end{array}$$

All other possible inclusions fail to hold in general.

Proof. Item 1, Functoriality: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition (\star) . So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

Item 2, Associativity: We have

$$\begin{aligned}
 \bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{there exists some } U \in \bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right. \right\} \\
 &= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right. \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right).
\end{aligned}$$

This finishes the proof.

Item 3, Left Unitality: We have

$$\begin{aligned}
\bigcup_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in V \end{array} \right. \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

Item 4, Right Unitality: We have

$$\begin{aligned}
\bigcup_{\{u\} \in \chi_X(U)} \{u\} &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right. \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

Item 5, Interaction With Unions I: We have

$$\begin{aligned}
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some } \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right. \right\}
\end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{there exists some } W \in \mathcal{U} \\ \text{such that we have } x \in W \end{array} \right. \right\} \\
&\cup \left\{ x \in X \left| \begin{array}{l} \text{there exists some } W \in \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left(\bigcup_{W \in \mathcal{U}} W \right) \cup \left(\bigcup_{W \in \mathcal{V}} W \right) \\
&= \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$\begin{aligned}
\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cap \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right. \right\} \\
&\subset \left\{ x \in X \left| \begin{array}{l} \text{there exists some } U \in \mathcal{U} \text{ and some } V \in \mathcal{V} \\ \text{such that we have } x \in U \text{ and } x \in V \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right. \right\} \\
&\cup \left\{ x \in X \left| \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that we have } x \in V \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}$$

This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned}
\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W &= \bigcup_{W \in \{\{0, 1\}\}} W \\
&= \{0, 1\},
\end{aligned}$$

whereas

$$\begin{aligned} \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right) &= \{0, 1\} \setminus \{0\} \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \{1\} = \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0, 1\}$ and let $\mathcal{U} = \{0\}$. We have

$$\begin{aligned} \bigcup_{U \in \mathcal{U}^c} U &= \bigcup_{U \in \{\emptyset, \{1\}, \{0, 1\}\}} U \\ &= \{0, 1\}, \end{aligned}$$

whereas

$$\begin{aligned} \bigcup_{U \in \mathcal{U}} U^c &= \{0\}^c \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcup_{U \in \mathcal{U}^c} U = \{0, 1\} \neq \{1\} = \bigcup_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

Item 11, Interaction With Complements II: Omitted.

Item 12, Interaction With Complements III: Omitted.

Item 13, Interaction With Symmetric Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned} \bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W &= \bigcup_{W \in \{\{0\}\}} W \\ &= \{0\}, \end{aligned}$$

whereas

$$\begin{aligned} \left(\bigcup_{U \in \mathcal{U}} U \right) \Delta \left(\bigcup_{V \in \mathcal{V}} V \right) &= \{0, 1\} \Delta \{0, 1\} \\ &= \emptyset, \end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U \in \mathcal{U}} U \right) \Delta \left(\bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of **Item 7** of **Definition 4.4.7.1.3** and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of **Item 8** of **Definition 4.4.7.1.3** and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of **Item 9** of **Definition 4.4.7.1.3** and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of **Item 3** of **Definition 4.6.1.1.5** and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of **Item 3** of **Definition 4.6.2.1.3** and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of **Item 3** of **Definition 4.6.3.1.7** and is proved there.

Item 20, Interaction With Intersections of Families I: We have

$$\begin{aligned} \bigcap_{A \in \mathcal{A}} A &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right. \right\} \\ &= \left\{ x \in X \left| \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right. \right\} \\ &\stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right). \end{aligned}$$

This finishes the proof.

Item 21, Interaction With Intersections of Families II: Omitted. □

4.3.7 Intersections of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.7.1.1. The **intersection** of \mathcal{U} is the set $\bigcap_{U \in \mathcal{U}} U$ defined by

$$\bigcap_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right. \right\}.$$

Proposition 4.3.7.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcap_{U \in \mathcal{U}} U$ defines a functor

$$\bigcap: (\mathcal{P}(\mathcal{P}(X)), \supset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

$$(\star) \text{ If } \mathcal{U} \subset \mathcal{V}, \text{ then } \bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

2. *Op-lax Associativity.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \bigcap} & \mathcal{P}(\mathcal{P}(X)) \\ \bigcap \star \text{id}_{\mathcal{P}(X)} \downarrow & \wr & \downarrow \bigcap \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcap} & \mathcal{P}(X) \end{array}$$

with components

$$\bigcap_{A \in {}^c\mathcal{A}} \left(\bigcap_{U \in A} U \right) \subset \bigcap_{U \in \bigcap_{A \in {}^c\mathcal{A}} A} U$$

for each ${}^c\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \downarrow \chi_{\mathcal{P}(X)} & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \{U\}} V = U.$$

for each $U \in \mathcal{P}(X)$.

4. *Oplax Right Unitality*. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \mathcal{P}(\chi_X) \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

(An orange 'X' is placed in the center of the diagram, indicating it does not commute.)

does not commute in general, i.e. we may have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \neq U$$

in general, where $U \in \mathcal{P}(X)$. However, when U is nonempty, we have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \subset U.$$

5. *Interaction With Unions I*. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap \downarrow & & \downarrow \cap \\
 \mathcal{P}(X) & \xrightarrow{U \cup -} & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap \downarrow & & \downarrow \cap \\
 \mathcal{P}(X) & \xrightarrow{- \cup V} & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have

$$\begin{aligned}
 U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) &= \bigcap_{V \in \mathcal{V}} (U \cup V), \\
 \left(\bigcap_{U \in \mathcal{U}} U \right) \cup V &= \bigcap_{U \in \mathcal{U}} (U \cup V)
 \end{aligned}$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap \times \cap \downarrow & \supset & \downarrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X),
 \end{array}$$

with components

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) \subset \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap \downarrow & & \downarrow \cap \\
 \mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap \downarrow & & \downarrow \cap \\
 \mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) = \bigcap_{V \in \mathcal{V}} (U \cup V),$$

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cup V = \bigcap_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\setminus} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\setminus} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(\mathcal{P}(X)) \\ \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U}^c} W \neq \bigcap_{U \in \mathcal{U}} U^c$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

II. *Interaction With Complements II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \dashrightarrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cap & & \downarrow \cup^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U \right)^c = \bigcup_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \dashrightarrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cup & & \downarrow \cap^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U \right)^c = \bigcap_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \Delta \left(\bigcap_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-, -]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap^{\text{op}} \times \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-, -]_X} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\ & \nearrow \text{dashed} & & \searrow \cap^{\text{op}} & \\ \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & & & \mathcal{P}(X)^{\text{op}} \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [-, \cdot]_X & & & & \downarrow [-, \cdot]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) & & \end{array}$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \\ \text{id}_{\mathcal{P}(X)} \star [U, -]_X \downarrow & & \downarrow [U, -]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{U}} V \right]_X = \bigcap_{V \in \mathcal{U}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f!(U) = \bigcap_{V \in f!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f!(\mathcal{U}) \stackrel{\text{def}}{=} (f!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a map of sets. The

diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{U}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(Y))$, where $f^{-1}(\mathcal{U}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{U})$.

19. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Unions of Families I.* The diagram

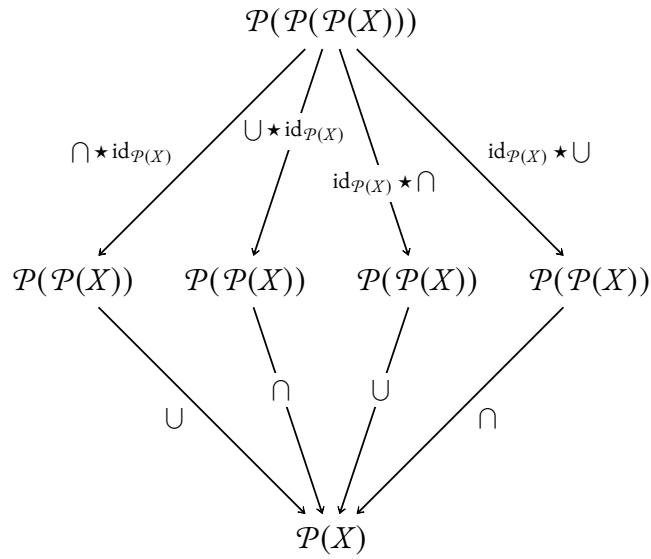
$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(x)) \\ \cup \star \text{id}_{\mathcal{P}(X)} \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{\cap} & X \end{array}$$

commutes, i.e. we have

$$\bigcap_{\substack{U \in \\ A \in \mathcal{A}}} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Unions of Families II.* Let X be a set and consider the compositions



given by

$$\begin{aligned} \mathcal{A} &\mapsto \bigcup_{\substack{U \in \\ A \in \mathcal{A}}} U, & \mathcal{A} &\mapsto \bigcap_{\substack{U \in \\ A \in \mathcal{A}}} U, \\ \mathcal{A} &\mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), & \mathcal{A} &\mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \end{aligned}$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\begin{array}{ccc}
 & \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U & \\
 & \uparrow & \searrow \\
 \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) & & \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \\
 & \downarrow & \swarrow \\
 & \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U &
 \end{array}$$

All other possible inclusions fail to hold in general.

Proof. Item 1, Functoriality: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition (\star) . So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion.

Item 2, Oplax Associativity: We have

$$\begin{aligned}
 \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \bigcap_{U \in A} U \end{array} \right. \right\} \\
 &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right. \right\} \\
 &= \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right. \right\}
 \end{aligned}$$

$$\begin{aligned}
& \subset \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right. \right\} \\
& \stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.
\end{aligned}$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic ([Categories, Item 4](#) of [Definition 11.2.7.1.2](#)). This finishes the proof.

Item 3, Left Unitality: We have

$$\begin{aligned}
\bigcap_{V \in \{U\}} V & \stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in U \end{array} \right. \right\} \\
& = \{x \in X \mid x \in U\} \\
& = U.
\end{aligned}$$

This finishes the proof.

Item 4, Oplax Right Unitality: If $U = \emptyset$, then we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} & = \bigcap_{\{u\} \in \emptyset} \{u\} \\
& = X,
\end{aligned}$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$. When U is nonempty, we have two cases:

1. If U is a singleton, say $U = \{u\}$, we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} & = \{u\} \\
& \stackrel{\text{def}}{=} U.
\end{aligned}$$

2. If U contains at least two elements, we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} & = \emptyset \\
& \subset U.
\end{aligned}$$

This finishes the proof.

Item 5, Interaction With Unions I: We have

$$\begin{aligned}
 \bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right. \right\} \\
 &= \left\{ x \in X \left| \begin{array}{l} \text{for each } W \in \mathcal{U} \text{ and each} \\ W \in \mathcal{V}, \text{ we have } x \in W \end{array} \right. \right\} \\
 &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } W \in \mathcal{U}, \\ \text{we have } x \in W \end{array} \right. \right\} \\
 &\quad \cap \left\{ x \in X \left| \begin{array}{l} \text{for each } W \in \mathcal{V}, \\ \text{we have } x \in W \end{array} \right. \right\} \\
 &\stackrel{\text{def}}{=} \left(\bigcap_{W \in \mathcal{U}} W \right) \cap \left(\bigcap_{W \in \mathcal{V}} W \right) \\
 &= \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right).
 \end{aligned}$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$\begin{aligned}
 \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right. \right\} \\
 &\quad \cup \left\{ x \in X \left| \begin{array}{l} \text{for each } V \in \mathcal{V}, \\ \text{we have } x \in V \end{array} \right. \right\} \\
 &= \left\{ x \in X \left| \begin{array}{l} \text{for each } W \in \mathcal{U} \cap \mathcal{V}, \\ \text{we have } x \in W \end{array} \right. \right\} \\
 &\subset \left\{ x \in X \left| \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right. \right\} \\
 &\stackrel{\text{def}}{=} \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W.
 \end{aligned}$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Definition 11.2.7.1.2](#)). This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0\}, \{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} U &= \bigcap_{W \in \{\{0, 1\}\}} W \\ &= \{0, 1\}, \end{aligned}$$

whereas

$$\begin{aligned} \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right) &= \{0\} \setminus \{0\} \\ &= \emptyset. \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \emptyset = \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0, 1\}$ and let $\mathcal{U} = \{\{0\}\}$.

We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U}^c} U &= \bigcap_{W \in \{\emptyset, \{1\}, \{0, 1\}\}} W \\ &= \emptyset, \end{aligned}$$

whereas

$$\begin{aligned} \bigcap_{U \in \mathcal{U}} U^c &= \{0\}^c \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U}^c} U = \emptyset \neq \{1\} = \bigcap_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

Item 11, Interaction With Complements II: This is a repetition of **Item 12** of **Definition 4.3.6.1.2** and is proved there.

Item 12, Interaction With Complements III: This is a repetition of **Item 11** of **Definition 4.3.6.1.2** and is proved there.

Item 13, Interaction With Symmetric Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W &= \bigcap_{W \in \{\{0\}\}} W \\ &= \{0\}, \end{aligned}$$

whereas

$$\begin{aligned} \left(\bigcap_{U \in \mathcal{U}} U \right) \Delta \left(\bigcap_{V \in \mathcal{V}} V \right) &= \{0, 1\} \Delta \{0\} \\ &= \emptyset, \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcap_{U \in \mathcal{U}} U \right) \Delta \left(\bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of **Item 10** of **Definition 4.4.7.1.3** and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of **Item 11** of **Definition 4.4.7.1.3** and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of **Item 12** of **Definition 4.4.7.1.3** and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of **Item 4** of **Definition 4.6.1.1.5** and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of **Item 4** of **Definition 4.6.2.1.3** and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of **Item 4** of **Definition 4.6.3.1.7** and is proved there.

Item 20, Interaction With Unions of Families I: This is a repetition of **Item 20** of **Definition 4.3.6.1.2** and is proved there.

Item 21, Interaction With Unions of Families II: This is a repetition of **Item 21** of **Definition 4.3.6.1.2** and is proved there. \square

4.3.8 Binary Unions

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.8.1.1. The **union of U and V** is the set $U \cup V$ defined by

$$\begin{aligned} U \cup V &\stackrel{\text{def}}{=} \bigcup_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

Proposition 4.3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \cup V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cup -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

- (a) If $U \subset A$, then $U \cup V \subset A \cup V$.
- (b) If $V \subset B$, then $U \cup V \subset U \cup B$.
- (c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

2. *Associativity.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\ \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \nearrow & & \searrow \text{id}_{\mathcal{P}(X)} \times \cup \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cup \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X), \end{array}$$

commutes, i.e. we have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

3. *Unitality*. The diagrams

$$\begin{array}{ccc}
 \text{pt} \times \mathcal{P}(X) & \xrightarrow{[\emptyset] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \lambda_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [\emptyset]} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \rho_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$\emptyset \cup U = U,$$

$$U \cup \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

4. *Commutativity*. The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \cup & & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup V = V \cup U$$

for each $U, V \in \mathcal{P}(X)$.

5. *Annihilation With X*. The diagrams

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \\
 \text{id}_{\text{pt}} \times \varepsilon_{\mathcal{P}(X)}^{\text{Sets}} \nearrow & & \nwarrow \mu_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \text{pt} \times \mathcal{P}(X) & & \text{pt} \\
 \downarrow [X] \times \text{id}_{\mathcal{P}(X)} & & \downarrow [X] \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 & \text{pt} \times \text{pt} & \\
 \varepsilon_{\mathcal{P}(X)}^{\text{Sets}} \times \text{id}_{\text{pt}} \nearrow & & \nwarrow \mu_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \mathcal{P}(X) \times \text{pt} & & \text{pt} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \times [X] & & \downarrow [X] \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

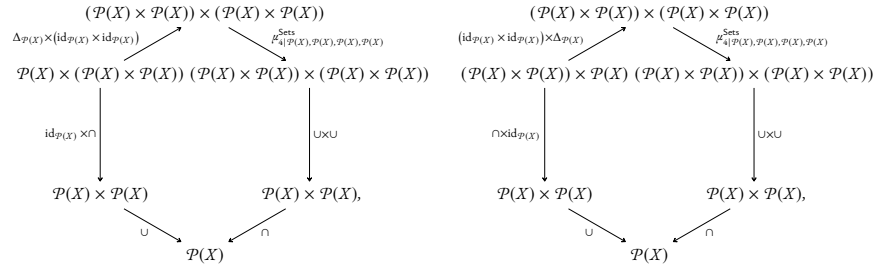
commute, i.e. we have equalities of sets

$$U \cup X = X,$$

$$X \cup V = X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Distributivity of Unions Over Intersections.* The diagrams

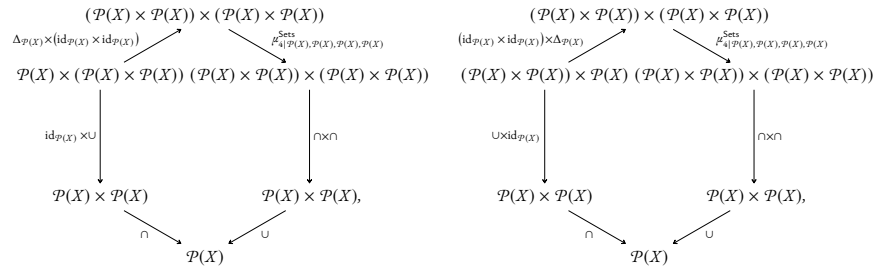


commute, i.e. we have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Distributivity of Intersections Over Unions.* The diagrams



commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Idempotency.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) & \xrightarrow{\Delta_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 & \searrow \text{id}_{\mathcal{P}(X)} & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each $U \in \mathcal{P}(X)$.

9. *Via Intersections and Symmetric Differences.* The diagram

$$\begin{array}{ccc}
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \xrightarrow{\Delta \times \cap} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \nearrow \Delta_{\mathcal{P}(X) \times \mathcal{P}(X)} & & \searrow \Delta \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup V = (U \Delta V) \Delta (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cup & \subset & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality:* See [Pro25an].

Item 2, Associativity: See [Pro25ba].

Item 3, Unitality: This follows from [Pro25bd] and *Item 4*.

Item 4, Commutativity: See [Pro25bb].

Item 5, Annihilation With X : We have

$$\begin{aligned} U \cup X &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\ &= \{x \in X \mid x \in X\}, \\ &= X \end{aligned}$$

and

$$\begin{aligned} X \cup V &\stackrel{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\ &= \{x \in X \mid x \in X\} \\ &= X. \end{aligned}$$

This finishes the proof.

Item 6, Distributivity of Unions Over Intersections: See [Pro25az].

Item 7, Distributivity of Intersections Over Unions: See [Pro25aj].

Item 8, Idempotency: See [Pro25am].

Item 9, Via Intersections and Symmetric Differences: See [Pro25ay].

Item 10, Interaction With Characteristic Functions I: See [Pro25h].

Item 11, Interaction With Characteristic Functions II: See [Pro25h].

Item 12, Interaction With Direct Images: See [Pro25p].

Item 13, Interaction With Inverse Images: See [Pro25y].

Item 14, Interaction With Codirect Images: This is a repetition of *Item 5* of [Definition 4.6.3.1.7](#) and is proved there.

Item 15, Interaction With Powersets and Semirings: This follows from *Items 2* to *4* and *8* of this proposition and *Items 3* to *6* and *8* of [Definition 4.3.9.1.2](#). \square

4.3.9 Binary Intersections

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.9.1.1. The **intersection of U and V** is the set $U \cap V$ defined by

$$\begin{aligned} U \cap V &\stackrel{\text{def}}{=} \bigcap_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

Proposition 4.3.9.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \cap V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cap -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

- (a) If $U \subset A$, then $U \cap V \subset A \cap V$.
- (b) If $V \subset B$, then $U \cap V \subset U \cap B$.
- (c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (U \cap - \dashv [U, -]_X): \mathcal{P}(X) &\overset{U \cap -}{\underset{[U, -]_X}{\rightleftarrows}} \mathcal{P}(X), \\ (- \cap V \dashv [V, -]_X): \mathcal{P}(X) &\overset{- \cap V}{\underset{[V, -]_X}{\rightleftarrows}} \mathcal{P}(X), \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(V, [U, W]_X), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is the bifunctor of [Section 4.4.7](#). In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

- i. We have $U \cap V \subset W$.
- ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

- i. We have $U \cap V \subset W$.
- ii. We have $V \subset [U, W]_X$.

3. *Associativity.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \nearrow & & \searrow \text{id}_{\mathcal{P}(X)} \times \cap \\
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \cap \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X),
 \end{array}$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* The diagrams

$$\begin{array}{ccc}
 \text{pt} \times \mathcal{P}(X) & \xrightarrow{[X] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \downarrow \alpha_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [X]} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \downarrow \beta_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$X \cap U = U,$$

$$U \cap X = U$$

for each $U \in \mathcal{P}(X)$.

5. *Commutativity.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 & \searrow \cap & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each $U, V \in \mathcal{P}(X)$.

6. *Annihilation With the Empty Set.* The diagrams

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \\
 \text{id}_{\text{pt}} \times \varepsilon_{\mathcal{P}(X)}^{\text{Sets}} \nearrow & & \nwarrow \mu_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \text{pt} \times \mathcal{P}(X) & & \text{pt} \\
 [\emptyset] \times \text{id}_{\mathcal{P}(X)} \searrow & & \nearrow [\emptyset] \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 & \text{pt} \times \text{pt} & \\
 \varepsilon_{\mathcal{P}(X)}^{\text{Sets}} \times \text{id}_{\text{pt}} \nearrow & & \nwarrow \mu_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \mathcal{P}(X) \times \text{pt} & & \text{pt} \\
 \text{id}_{\mathcal{P}(X)} \times [\emptyset] \searrow & & \nearrow [\emptyset] \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
 \emptyset \cap X &= \emptyset, \\
 X \cap \emptyset &= \emptyset
 \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

7. *Distributivity of Unions Over Intersections.* The diagrams

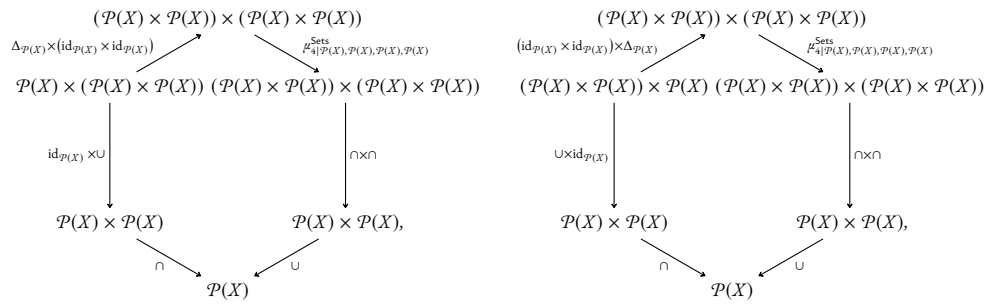
$$\begin{array}{ccc}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \Delta_{\mathcal{P}(X)} \times (\text{id}_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)}) \nearrow & & \nwarrow \mu_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
 \text{id}_{\mathcal{P}(X)} \times \cap \searrow & & \nearrow \cup \times \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X), \\
 \cup \nearrow & & \nwarrow \cap \\
 & \mathcal{P}(X) &
 \end{array}
 \quad
 \begin{array}{ccc}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 (\text{id}_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)}) \times \Delta_{\mathcal{P}(X)} \nearrow & & \nwarrow \mu_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
 \cap \times \text{id}_{\mathcal{P}(X)} \searrow & & \nearrow \cup \times \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X), \\
 \cup \nearrow & & \nwarrow \cap \\
 & \mathcal{P}(X) &
 \end{array}$$

commute, i.e. we have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Distributivity of Intersections Over Unions.* The diagrams

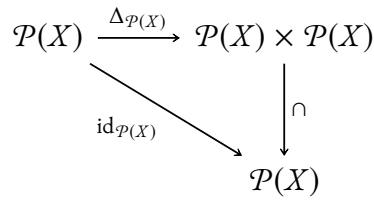


commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Idempotency.* The diagram



commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f! \times f!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & \subset & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

with components

$$f!(U \cap V) \subset f!(U) \cap f!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cap & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.
16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. **Item 1, Functoriality:** See [Pro25al].

Item 2, Adjointness: See [MSE 267469].

Item 3, Associativity: See [Pro25r].

Item 4, Unitality: This follows from [Pro25v] and **Item 5**.

Item 5, Commutativity: See [Pro25s].

Item 6, Annihilation With the Empty Set: This follows from [Pro25t] and **Item 5**.

Item 7, Distributivity of Unions Over Intersections: See [Pro25az].

Item 8, Distributivity of Intersections Over Unions: See [Pro25aj].

Item 9, Idempotency: See [Pro25ak].

Item 10, Interaction With Characteristic Functions I: See [Pro25e].

Item 11, Interaction With Characteristic Functions II: See [Pro25e].

Item 12, Interaction With Direct Images: See [Pro25n].

Item 13, Interaction With Inverse Images: See [Pro25w].

Item 14, Interaction With Codirect Images: This is a repetition of **Item 6** of **Definition 4.6.3.1.7** and is proved there.

Item 15, Interaction With Powersets and Monoids With Zero: This follows from **Items 3** to **6**.

Item 16, Interaction With Powersets and Semirings: This follows from **Items 2** to **4** and **8** and **Items 3** to **6** and **8** of **Definition 4.3.9.1.2**. \square

4.3.10 Differences

Let X and Y be sets.

Definition 4.3.10.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 4.3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus - &: (\mathcal{P}(X), \supset) && \rightarrow (\mathcal{P}(X), \subset), \\ - \setminus V &: (\mathcal{P}(X), \subset) && \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

- (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.
- (b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.
- (c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^c$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \Delta (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

13. *Right Annihilation.* We have

$$U \setminus X = U$$

for each $U \in \mathcal{P}(X)$.

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

15. *Interaction With Containment.* The following conditions are equivalent:

(a) We have $V \setminus U \subset W$.

(b) We have $V \setminus W \subset U$.

16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Direct Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \setminus & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

18. *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\
 \downarrow \setminus & & \downarrow \setminus \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

19. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{array}{ccc}
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\
 \downarrow \setminus & \supset & \downarrow \setminus \\
 \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
 \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. **Item 1, Functoriality:** See [Pro25ad] and [Pro25ah].

Item 2, De Morgan's Laws: See [Pro25k].

Item 3, Interaction With Unions I: See [Pro25l].

Item 4, Interaction With Unions II: Omitted.

Item 5, Interaction With Unions III: See [Pro25ai].

Item 6, Interaction With Unions IV: See [Pro25ac].

Item 7, Interaction With Intersections: See [Pro25u].

Item 8, Interaction With Complements: See [Pro25aa].

Item 9, Interaction With Symmetric Differences: See [Pro25ab].

Item 10, Triple Differences: See [Pro25ag].

Item 11, Left Annihilation: Omitted.

Item 12, Right Unitality: See [Pro25ae].

Item 13, Right Annihilation: Omitted.

Item 14, Invertibility: See [Pro25af].

Item 15, Interaction With Containment: Omitted.

Item 16, Interaction With Characteristic Functions: See [Pro25f].

Item 17, Interaction With Direct Images: See [Pro25o].

Item 18, Interaction With Inverse Images: See [Pro25x]. □

4.3.II Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.3.II.1.1. The **complement of U** is the set U^c defined by

$$\begin{aligned} U^c &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 4.3.II.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^c$ defines a functor

$$(-)^c: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X).$$

In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

(★) If $U \subset V$, then $V^c \subset U^c$.

2. *De Morgan's Laws.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cup^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ \downarrow (-)^c \times (-)^c & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ \downarrow (-)^c \times (-)^c & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have equalities of sets

$$\begin{aligned} (U \cup V)^c &= U^c \cap V^c, \\ (U \cap V)^c &= U^c \cup V^c \end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)^{\text{op}}} & \downarrow (-)^{c, \text{op}} \\ & & \mathcal{P}(X)^{\text{op}} \end{array}$$

commutes, i.e. we have

$$(U^c)^c = U$$

for each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^c} \equiv 1 - \chi_U \pmod{2}$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U^c) = f_*(U)^c$$

for each $U \in \mathcal{P}(X)$.

6. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U^c) = f^{-1}(U)^c$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U^c) = f_!(U)^c$$

for each $U \in \mathcal{P}(X)$.

Proof. **Item 1, Functoriality:** This follows from **Item 1** of **Definition 4.3.10.1.2**.

Item 2, De Morgan's Laws: See [Pro25k].

Item 3, Involutority: See [Pro25j].

Item 4, Interaction With Characteristic Functions: Omitted.

Item 5, Interaction With Direct Images: This is a repetition of **Item 8** of **Definition 4.6.1.1.5** and is proved there.

Item 6, Interaction With Inverse Images: This is a repetition of **Item 8** of **Definition 4.6.2.1.3** and is proved there.

Item 7, Interaction With Codirect Images: This is a repetition of **Item 7** of **Definition 4.6.3.1.7** and is proved there. \square

4.3.12 Symmetric Differences

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.12.1.1. The **symmetric difference of U and V** is the set $U \triangle V$

defined by¹³

$$U \Delta V \stackrel{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

Proposition 4.3.12.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \Delta V$ **does not** in general define functors

$$\begin{aligned} U \Delta -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \Delta V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \Delta -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have

$$U \Delta V = (U \cup V) \setminus (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram



3. *Symmetric Differences of Disjoint Sets.* If U and V are disjoint, then we have

$$U \Delta V = U \cup V.$$

4. *Associativity.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\ \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \nearrow & & \searrow \text{id}_{\mathcal{P}(X)} \times \Delta \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\ \Delta \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \Delta \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X), \end{array}$$

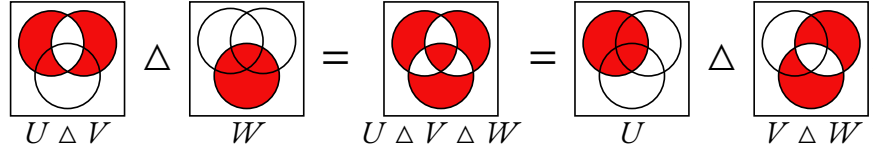
¹³Illustration:



commutes, i.e. we have

$$(U \Delta V) \Delta W = U \Delta (V \Delta W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram



5. *Unitality.* The diagrams

$$\begin{array}{ccc} \text{pt} \times \mathcal{P}(X) & \xrightarrow{[\emptyset] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ & \searrow \lambda_{\mathcal{P}(X)}^{\text{Sets}} & \downarrow \Delta \\ & & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [\emptyset]} & \mathcal{P}(X) \times \mathcal{P}(X) \\ & \searrow \rho_{\mathcal{P}(X)}^{\text{Sets}} & \downarrow \Delta \\ & & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \Delta \emptyset &= U, \\ \emptyset \Delta U &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

6. *Commutativity.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ & \searrow \Delta & \downarrow \Delta \\ & & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$U \Delta V = V \Delta U$$

for each $U, V \in \mathcal{P}(X)$.

7. *Invertibility.* We have

$$U \Delta U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Unions.* We have

$$(U \Delta V) \cup (V \Delta T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Interaction With Complements I.* We have

$$U \Delta U^c = X$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements II.* We have

$$\begin{aligned} U \Delta X &= U^c, \\ X \Delta U &= U^c \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

11. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X) \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$U^c \Delta V^c = U \Delta V$$

for each $U, V \in \mathcal{P}(X)$.

12. “*Transitivity*”. We have

$$(U \Delta V) \Delta (V \Delta W) = U \Delta W$$

for each $U, V, W \in \mathcal{P}(X)$.

13. *The Triangle Inequality for Symmetric Differences*. We have

$$U \Delta W \subset U \Delta V \cup V \Delta W$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Distributivity Over Intersections*. We have

$$\begin{aligned} U \cap (V \Delta W) &= (U \cap V) \Delta (U \cap W), \\ (U \Delta V) \cap W &= (U \cap W) \Delta (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Characteristic Functions*. We have

$$\chi_{U \Delta V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \Delta V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

16. *Bijectivity*. Given $U, V \in \mathcal{P}(X)$, the maps

$$\begin{aligned} U \Delta -: \mathcal{P}(X) &\rightarrow \mathcal{P}(X), \\ - \Delta V: \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (U \Delta -)^{-1} &= - \cup (U \cap -), \\ (- \Delta V)^{-1} &= - \cup (V \cap -). \end{aligned}$$

Moreover, the map

$$\begin{aligned} \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ C &\longmapsto C \Delta (U \Delta V) \end{aligned}$$

is a bijection of $\mathcal{P}(X)$ onto itself sending U to V and V to U .

17. *Interaction With Powersets and Groups.* Let X be a set.

- (a) The quadruple $(\mathcal{P}(X), \Delta, \emptyset, \text{id}_{\mathcal{P}(X)})$ is an abelian group.¹⁴
- (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to Δ , and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

- The group $\mathcal{P}(X)$ of **Item 17**;
- The map $\alpha_{\mathcal{P}(X)}: \mathbb{F}_2 \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

is an \mathbb{F}_2 -vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

- (a) The set of singletons sets on the elements of X forms a basis for the \mathbb{F}_2 -vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of **Item 4**.
- (b) We have

$$\dim(\mathcal{P}(X)) = \#X.$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \Delta, \cap, \emptyset, X)$

¹⁴Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \Delta, \emptyset, \text{id}_{\mathcal{P}(\emptyset)}) \cong \text{pt}.$$

2. When $X = \text{pt}$, we have an isomorphism of groups between $\mathcal{P}(\text{pt})$ and \mathbb{Z}_2 :

$$(\mathcal{P}(\text{pt}), \Delta, \emptyset, \text{id}_{\mathcal{P}(\text{pt})}) \cong \mathbb{Z}_2.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$(\mathcal{P}(\{0, 1\}), \Delta, \emptyset, \text{id}_{\mathcal{P}(\{0, 1\})}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

is a commutative ring.¹⁵

7. *Interaction With Direct Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \wr & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \Delta f_!(V) \subset f_!(U \Delta V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \Delta \downarrow & & \downarrow \Delta \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$


i.e. we have

$$f^{-1}(U) \Delta f^{-1}(V) = f^{-1}(U \Delta V)$$

for each $U, V \in \mathcal{P}(Y)$.

9. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \wr & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

¹⁵  *Warning:* The analogous statement replacing intersections by unions (i.e. that the

with components

$$f_*(U \Delta V) \subset f_*(U) \Delta f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. **Item 1, Lack of Functoriality:** Omitted.

Item 2, Via Unions and Intersections: See [Pro25m].

Item 3, Symmetric Differences of Disjoint Sets: Since U and V are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$\begin{aligned} U \Delta V &= (U \cup V) \setminus (U \cap V) \\ &= (U \cup V) \setminus \emptyset \\ &= U \cup V, \end{aligned}$$

where we've used **Item 2** and **Item 12** of **Definition 4.3.10.1.2**.

Item 4, Associativity: See [Pro25ao].

Item 5, Unitality: This follows from **Item 6** and [Pro25at].

Item 6, Commutativity: See [Pro25ap].

Item 7, Invertibility: See [Pro25av].

Item 8, Interaction With Unions: See [Pro25bc].

Item 9, Interaction With Complements I: See [Pro25as].

Item 10, Interaction With Complements II: This follows from **Item 6** and [Pro25ax].

Item 11, Interaction With Complements III: See [Pro25aq].

Item 12, "Transitivity": We have

$$\begin{aligned} (U \Delta V) \Delta (V \Delta W) &= U \Delta (V \Delta (V \Delta W)) && \text{(by Item 4)} \\ &= U \Delta ((V \Delta V) \Delta W) && \text{(by Item 4)} \\ &= U \Delta (\emptyset \Delta W) && \text{(by Item 7)} \\ &= U \Delta W. && \text{(by Item 5)} \end{aligned}$$

This finishes the proof.

Item 13, The Triangle Inequality for Symmetric Differences: This follows from **Items 2** and **12**.

Item 14, Distributivity Over Intersections: See [Pro25q].

Item 15, Interaction With Characteristic Functions: See [Pro25g].

Item 16, Bijectivity: Omitted.

quintuple $(\mathcal{P}(X), \Delta, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25aw] for a proof.

Item 17, Interaction With Powersets and Groups: **Item 17a** follows from **Items 4** to **7**, while **Item 3b** follows from **Item 7**.¹⁶

Item 4, Interaction With Powersets and Vector Spaces I: See [MSE 2719059].

Item 5, Interaction With Powersets and Vector Spaces II: See [MSE 2719059].

Item 6, Interaction With Powersets and Rings: This follows from **Items 6** and **15** of **Definition 4.3.9.1.2** and **Items 14** and **17**.¹⁷

Item 7, Interaction With Direct Images: This is a repetition of **Item 9** of **Definition 4.6.1.1.5** and is proved there.

Item 8, Interaction With Inverse Images: This is a repetition of **Item 9** of **Definition 4.6.2.1.3** and is proved there.

Item 9, Interaction With Codirect Images: This is a repetition of **Item 8** of **Definition 4.6.3.1.7** and is proved there. \square

4.4 Powersets

4.4.1 Foundations

Let X be a set.

Definition 4.4.1.1.1. The **powerset of X** is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, ?? of ??.

Remark 4.4.1.1.2. Under the analogy that $\{\mathbf{t}, \mathbf{f}\}$ should be the (-1) -categorical analogue of **Sets**, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set X is equivalently (**Item 2** of **Definition 4.5.1.1.4**) the set

$$\mathbf{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$$

of functions from X to the set $\{\mathbf{t}, \mathbf{f}\}$ of classical truth values.

- The category of presheaves on a category C is the category

$$\mathbf{Fun}(C^{\text{op}}, \mathbf{Sets})$$

of functors from C^{op} to the category **Sets** of sets.

¹⁶Reference: [Pro25ar].

¹⁷Reference: [Pro25au].

Notation 4.4.I.I.3. Let X be a set.

1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of X .
2. We write $\mathcal{P}_{\text{fin}}(X)$ for the set of finite subsets of X .

Proposition 4.4.I.I.4. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:
 - (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.
 - (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.
 - (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.
2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.
3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \text{Rel}(\text{pt}, X), \\ \mathcal{P}(X) &\cong \text{Rel}(X, \text{pt}),\end{aligned}$$

natural in $X \in \text{Obj}(\text{Sets})$.

4. *Interaction With Products I.* The map

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \amalg Y) \\ (U, V) &\longmapsto U \cup V\end{aligned}$$

is an isomorphism of sets, natural in $X, Y \in \text{Obj}(\text{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* on \mathcal{P} of [Definition 4.4.2.I.I.](#) Moreover, this makes each of $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* into a symmetric monoidal functor.

5. *Interaction With Products II.* The map

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \amalg Y) \\ (U, V) &\longmapsto U \boxtimes_{X \times Y} V,\end{aligned}$$

where¹⁸

$$U \boxtimes_{X \times Y} V \stackrel{\text{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

is an inclusion of sets, natural in $X, Y \in \text{Obj}(\text{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* on \mathcal{P} of [Definition 4.4.2.1.1](#). Moreover, this makes each of $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* into a symmetric monoidal functor.

6. *Interaction With Products III*. We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in $X, Y \in \text{Obj}(\text{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* on \mathcal{P} of [Definition 4.4.2.1.1](#), where \otimes denotes the tensor product of suplattices of [??](#). Moreover, this makes each of $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* into a symmetric monoidal functor.

Proof. [Item 1](#), *Co/Completeness*: Omitted.

[Item 2](#), *Cartesian Closedness*: See [Section 4.4.7](#).

[Item 3](#), *Powersets as Sets of Relations*: Indeed, we have

$$\begin{aligned} \text{Rel}(\text{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\text{pt} \times X) \\ &\cong \mathcal{P}(X) \end{aligned}$$

and

$$\begin{aligned} \text{Rel}(X, \text{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \text{pt}) \\ &\cong \mathcal{P}(X), \end{aligned}$$

where we have used [Item 5](#) of [Definition 4.1.3.1.3](#).

[Item 4](#), *Interaction With Products I*: The inverse of the map in the statement is the map

$$\Phi: \mathcal{P}(X \amalg Y) \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \stackrel{\text{def}}{=} (S_X, S_Y)$$

¹⁸The set $U \boxtimes_{X \times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this somewhat

for each $S \in \mathcal{P}(X \amalg Y)$, where

$$\begin{aligned} S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\} \\ S_Y &\stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}. \end{aligned}$$

The rest of the proof is omitted.

Item 5, Interaction With Products II: Omitted.

Item 6, Interaction With Products III: Omitted. □

4.4.2 Functoriality of Powersets

Proposition 4.4.2.1.1. Let X be a set.

- I. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! : \mathbf{Sets} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathbf{Obj}(\mathbf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} : \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by sending a map of sets $f : A \rightarrow B$ to the map

$$\mathcal{P}_!(f) : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in **Definition 4.6.1.1.1.**

weird way to highlight the similarity to external tensor products in six-functor formalisms (see also **Section 4.6.4**).

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1}: \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathbf{Sets})$, the action on morphisms

$$\mathcal{P}_{A,B}^{-1}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of \mathcal{P}^{-1} at (A, B) is the map defined by sending a map of sets $f: A \rightarrow B$ to the map

$$\mathcal{P}^{-1}(f): \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in [Definition 4.6.2.1.1.](#)

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_*: \mathbf{Sets} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathbf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\mathbf{Sets})$, the action on morphisms

$$\mathcal{P}_{|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of \mathcal{P}_* at (A, B) is the map defined by sending a map of sets $f: A \rightarrow B$ to the map

$$\mathcal{P}_*(f): \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in **Definition 4.6.3.1.1**.

Proof. **Item 1, Functoriality I**: This follows from **Items 3** and **4** of **Definition 4.6.1.1.6**.

Item 2, Functoriality II: This follows from **Items 3** and **4** of **Definition 4.6.2.1.4**.

Item 3, Functoriality III: This follows from **Items 3** and **4** of **Definition 4.6.3.1.8**. \square

4.4.3 Adjointness of Powersets I

Proposition 4.4.3.1.1. We have an adjunction

$$(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1, \text{op}}): \text{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{P}^{-1}} \\ \perp \\ \xleftarrow{\mathcal{P}^{-1, \text{op}}} \end{array} \text{Sets},$$

witnessed by a bijection

$$\underbrace{\text{Sets}^{\text{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \text{Sets}(Y, \mathcal{P}(X))} \cong \text{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $Y \in \text{Obj}(\text{Sets}^{\text{op}})$.

Proof. We have

$$\begin{aligned} \text{Sets}^{\text{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \text{Sets}(B, \mathcal{P}(A)) \\ &\cong \text{Sets}(B, \text{Sets}(A, \{\text{t}, \text{f}\})) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\ &\cong \text{Sets}(A \times B, \{\text{t}, \text{f}\}) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \text{Sets}(A, \text{Sets}(B, \{\text{t}, \text{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \text{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)} \end{aligned}$$

where all bijections are natural in A and B .¹⁹ \square

¹⁹Here we are using **Item 3** of **Definition 4.5.1.1.4**.

4.4.4 Adjointness of Powersets II

Proposition 4.4.4.I.I. We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_!): \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_!} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(X), Y) \cong \text{Sets}(X, \mathcal{P}(Y))$$

natural in $X \in \text{Obj}(\text{Sets})$ and $Y \in \text{Obj}(\text{Rel})$, where Gr is the graph functor of [Relations, Item 1 of Definition 8.2.2.I.2](#) and $\mathcal{P}_!$ is the functor of [Relations, Definition 8.7.5.I.I.](#)

Proof. We have

$$\begin{aligned} \text{Rel}(\text{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\ &\cong \text{Sets}(A \times B, \{\text{t}, \text{f}\}) && \text{(by Item 2 of Definition 4.5.I.I.4)} \\ &\cong \text{Sets}(A, \text{Sets}(B, \{\text{t}, \text{f}\})) && \text{(by Item 2 of Definition 4.I.3.I.3)} \\ &\cong \text{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.I.I.4)} \end{aligned}$$

where all bijections are natural in A , (where we are using [Item 3 of Definition 4.5.I.I.4](#)). Explicitly, this isomorphism is given by sending a relation $R: \text{Gr}(A) \rightarrow B$ to the map $R^\dagger: A \rightarrow \mathcal{P}(B)$ sending a to the subset $R(a)$ of B , as in [Relations, Definition 8.1.1.I.1](#).

Naturality in B is then the statement that given a relation $R: B \rightarrow B'$, the diagram

$$\begin{array}{ccc} \text{Rel}(\text{Gr}(A), B) & \xrightarrow{R \diamond -} & \text{Rel}(\text{Gr}(A), B') \\ \downarrow \wr & & \downarrow \wr \\ \text{Sets}(A, \mathcal{P}(B)) & \xrightarrow{R_!} & \text{Sets}(A, \mathcal{P}(B')) \end{array}$$

commutes, which follows from [Relations, Definition 8.7.I.I.3](#). \square

4.4.5 Powersets as Free Cocompletions

Let X be a set.

Proposition 4.4.5.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of X of **Definition 4.4.1.1.1**;
- The characteristic embedding $\chi_{(-)}: X \rightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of **Definition 4.5.4.1.1**;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

- A suplattice (Y, \preceq) ;
- A function $f: X \rightarrow Y$;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc} & & \mathcal{P}(X) \\ & \nearrow \chi_X & \downarrow \exists! \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

Proof. This is a rephrasing of **Definition 4.4.5.1.2**, which we prove below.²⁰ \square

Proposition 4.4.5.1.2. We have an adjunction

$$(\mathcal{P} \dashv \overline{\omega}): \text{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\overline{\omega}} \end{array} \text{SupLat},$$

²⁰Here we only remark that the unique morphism of suplattices in the statement is given by

witnessed by a bijection

$$\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, \preceq) \in \text{Obj}(\text{SupLat})$, where:

- The category SupLat is the category of suplattices of ??.
- The map

$$\chi_X^*: \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of suplattices $f: \mathcal{P}(X) \rightarrow Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

$$\text{Lan}_{\chi_X}: \text{Sets}(X, Y) \rightarrow \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function $f: X \rightarrow Y$ to its left Kan extension along χ_X ,

$$\text{Lan}_{\chi_X}(f): \mathcal{P}(X) \rightarrow Y,$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$ of **Item 2** of **Definition 4.5.1.1.4**, $\text{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$[\text{Lan}_{\chi_X}(f)](U) = \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)$$

the left Kan extension $\text{Lan}_{\chi_X}(f)$ of f along χ_X .

$$\begin{aligned}
&= \int^{x \in X} \chi_U(x) \odot f(x) \\
&= \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \\
&= \left(\bigvee_{x \in U} (\chi_U(x) \odot f(x)) \right) \vee \left(\bigvee_{x \in U^c} (\chi_U(x) \odot f(x)) \right) \\
&= \left(\bigvee_{x \in U} f(x) \right) \vee \left(\bigvee_{x \in U^c} \emptyset_Y \right) \\
&= \bigvee_{x \in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- We have used ?? for the first equality.
- We have used [Definition 4.5.5.1.1](#) for the second equality.
- We have used ?? for the third equality.
- The symbol \vee denotes the join in (Y, \preceq) .
- The symbol \odot denotes the tensor of an element of Y by a truth value as in ??. In particular, we have

$$\begin{aligned}
\text{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\
\text{false} \odot f(x) &\stackrel{\text{def}}{=} \emptyset_Y,
\end{aligned}$$

where \emptyset_Y is the bottom element of (Y, \preceq) .

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B , the Kan extension $\text{Lan}_{\chi_X}(f)$ is given by

$$\begin{aligned}
[\text{Lan}_{\chi_X}(f)](U) &= \bigvee_{x \in U} f(x) \\
&= \bigcup_{x \in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y} : \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, i.e. by

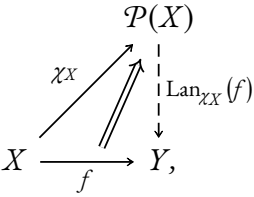
$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Map II: We define a map

$$\Psi_{X,Y} : \text{Sets}(X, Y) \rightarrow \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f),$$


for each $f \in \text{Sets}(X, Y)$.

Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))}.$$

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. We now claim that

$$\text{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. Indeed, we have

$$[\text{Lan}_{\chi_X}(f \circ \chi_X)](U) = \bigvee_{x \in U} f(\chi_X(x))$$

$$\begin{aligned}
&= f\left(\bigvee_{x \in U} \chi_X(x)\right) \\
&= f\left(\bigcup_{x \in U} \{x\}\right) \\
&= f(U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))}$ of $\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)}.$$

We have

$$\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Lan}_{\chi_X}(f)) \\
&\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f) \circ \chi_X
\end{aligned}$$

for each $f \in \text{Sets}(X, Y)$. We now claim that

$$\text{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \text{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned}
[\text{Lan}_{\chi_X}(f) \circ \chi_X](x) &= \bigvee_{y \in \{x\}} f(y) \\
&= f(x)
\end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \text{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{Sets}(X,Y)}$ of $\text{Sets}(X, Y)$.

Naturality for Φ , Part I: We need to show that, given a function $f: X \rightarrow X'$, the diagram

$$\begin{array}{ccc} \text{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ \mathcal{P}_!(f)^* \downarrow & & \downarrow f^* \\ \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, we have

$$\begin{aligned} [\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f!) \\ &\stackrel{\text{def}}{=} (\xi \circ f!) \circ \chi_X \\ &= \xi \circ (f! \circ \chi_X) \\ &\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\ &= (\xi \circ \chi_{X'}) \circ f \\ &\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\ &\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\ &\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi), \end{aligned}$$

for each $\xi \in \text{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$, where we have used [Item 1 of Definition 4.5.4.1.3](#) for the fifth equality above.

Naturality for Φ , Part II: We need to show that, given a morphism of suplattices

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc} \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\ g! \downarrow & & \downarrow g! \\ \text{SupLat}((\mathcal{P}(X), \subset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y') \end{array}$$

commutes. Indeed, we have

$$[\Phi_{X,Y'} \circ g!](\xi) \stackrel{\text{def}}{=} \Phi_{X,Y'}(g!(\xi))$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} g_!(\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

for each $\xi \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Naturality for Ψ : Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2 of Definition 11.9.7.1.2](#) that Ψ is also natural in each argument. \square

Warning 4.4.5.1.3. Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion of X* , it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.

4.4.6 Powersets as Free Completions

Let X be a set.

Proposition 4.4.6.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset of X together with reverse inclusion $\mathcal{P}(X)^{\text{op}} = (\mathcal{P}(X), \supset)$ of [Definition 4.4.1.1.1](#);
- The characteristic embedding $\chi_{(-)}: X \rightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of [Definition 4.5.4.1.1](#);

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

- An inflattice (Y, \preceq) ;
- A function $f: X \rightarrow Y$;

there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X)^{\text{op}} & \\
 \nearrow \chi_X & \downarrow \exists! & \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commute.

Proof. This is a rephrasing of [Definition 4.4.6.1.2](#), which we prove below.²¹ \square

Proposition 4.4.6.1.2. We have an adjunction

$$(\mathcal{P} \dashv \overline{\text{忘}}): \text{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\overline{\text{忘}}} \end{array} \text{InfLat},$$

witnessed by a bijection

$$\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, \preceq) \in \text{Obj}(\text{InfLat})$, where:

- The category InfLat is the category of inflattices of ??.
- The map

$$\chi_X^*: \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of inflattices $f: \mathcal{P}(X)^{\text{op}} \rightarrow Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X)^{\text{op}} \xrightarrow{f} Y.$$

²¹Here we only remark that the unique morphism of inflattices in the statement is given by the right Kan extension $\text{Ran}_{\chi_X}(f)$ of f along χ_X .

- The map

$$\text{Ran}_{\chi_X} : \text{Sets}(X, Y) \rightarrow \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function $f : X \rightarrow Y$ to its right Kan extension along χ_X ,

$$\text{Ran}_{\chi_X}(f) : \mathcal{P}(X)^{\text{op}} \rightarrow Y,$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$ of **Item 2** of **Definition 4.5.I.1.4**, $\text{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned} [\text{Ran}_{\chi_X}(f)](U) &= \int_{x \in X} \chi_{\mathcal{P}(X)^{\text{op}}}(\chi_x, U) \multimap f(x) \\ &= \int_{x \in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \multimap f(x) \\ &= \int_{x \in X} \chi_U(x) \multimap f(x) \\ &= \bigwedge_{x \in X} \chi_U(x) \multimap f(x) \\ &= \left(\bigwedge_{x \in U} \chi_U(x) \multimap f(x) \right) \wedge \left(\bigwedge_{x \in U^c} \chi_U(x) \multimap f(x) \right) \\ &= \left(\bigwedge_{x \in U} f(x) \right) \wedge \left(\bigwedge_{x \in U^c} \infty_Y \right) \\ &= \left(\bigwedge_{x \in U} f(x) \right) \wedge \infty_Y \\ &= \bigwedge_{x \in U} f(x) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- We have used ?? for the first equality.

- We have used [Definition 4.5.5.I.1](#) for the second equality.
- We have used ?? for the third equality.
- The symbol \wedge denotes the meet in (Y, \preceq) .
- The symbol \pitchfork denotes the cotensor of an element of Y by a truth value as in ?. In particular, we have

$$\begin{aligned}\text{true} \pitchfork f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \pitchfork f(x) &\stackrel{\text{def}}{=} \infty_Y,\end{aligned}$$

where ∞_Y is the top element of (Y, \preceq) .

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B , the Kan extension $\text{Ran}_{\chi_X}(f)$ is given by

$$\begin{aligned}[\text{Ran}_{\chi_X}(f)](U) &= \bigwedge_{x \in U} f(x) \\ &= \bigcap_{x \in U} f(x)\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y}: \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Map II: We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f),$$

for each $f \in \text{Sets}(X, Y)$.

Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}.$$

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\text{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned} [\text{Ran}_{\chi_X}(f \circ \chi_X)](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\ &= f\left(\bigwedge_{x \in U} \chi_X(x)\right) \\ &= f\left(\bigcup_{x \in U} \{x\}\right) \\ &= f(U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}$ of $\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)}.$$

We have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Ran}_{\chi_X}(f)) \\ &\stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f) \circ \chi_X \end{aligned}$$

for each $f \in \text{Sets}(X, Y)$. We now claim that

$$\text{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \text{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned} [\text{Ran}_{\chi_X}(f) \circ \chi_X](x) &= \bigwedge_{y \in \{x\}} f(y) \\ &= f(x) \end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \text{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{Sets}(X,Y)}$ of $\text{Sets}(X, Y)$.

Naturality for Φ , Part I: We need to show that, given a function $f: X \rightarrow X'$, the diagram

$$\begin{array}{ccc} \text{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ \mathcal{P}_!(f)^* \downarrow & & \downarrow f^* \\ \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, we have

$$\begin{aligned} [\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f!) \\ &\stackrel{\text{def}}{=} (\xi \circ f!) \circ \chi_X \\ &= \xi \circ (f! \circ \chi_X) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

for each $\xi \in \text{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used **Item 1** of **Definition 4.5.4.1.3** for the fifth equality above.

Naturality for Φ , Part II: We need to show that, given a cocontinuous morphism of posets

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc}
\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\
\downarrow g! & & \downarrow g! \\
\text{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y')
\end{array}$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g!](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g!(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} g!(\Phi_{X,Y}(\xi)) \\
&\stackrel{\text{def}}{=} [g! \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

for each $\xi \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Naturality for Ψ : Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that Ψ is also natural in each argument. \square

Warning 4.4.6.1.3. Although the assignment $X \mapsto \mathcal{P}(X)^{\text{op}}$ is called the *free completion of X* , it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\text{op}})^{\text{op}} \neq \mathcal{P}(X)^{\text{op}}$.

4.4.7 The Internal Hom of a Powerset

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Proposition 4.4.7.1.1. The **internal Hom of $\mathcal{P}(X)$ from U to V** is the subset $[U, V]_X$ ²² of X given by

$$\begin{aligned} [U, V]_X &= U^c \cup V \\ &= (U \setminus V)^c \end{aligned}$$

where U^c is the complement of U of **Definition 4.3.11.1.1**.

Proof. Proof of the Equality $U^c \cup V = (U \setminus V)^c$: We have

$$\begin{aligned} (U \setminus V)^c &\stackrel{\text{def}}{=} X \setminus (U \setminus V) \\ &= (X \cap V) \cup (X \setminus U) \\ &= V \cup (X \setminus U) \\ &\stackrel{\text{def}}{=} V \cup U^c \\ &= U^c \cup V, \end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.10.1.2** for the second equality.
2. **Item 4** of **Definition 4.3.9.1.2** for the third equality.
3. **Item 4** of **Definition 4.3.8.1.2** for the last equality.

This finishes the proof.

Proof that $U^c \cup V$ Is Indeed the Internal Hom: This follows from **Item 2** of **Definition 4.3.9.1.2**. \square

Remark 4.4.7.1.2. Henning Makhholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from U to V ([MSE 267365]):

1. Since products in $\mathcal{P}(X)$ are given by binary intersections (**Item 1** of **Definition 4.4.1.1.4**), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.

²²*Further Notation:* Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$.

2. Under the Curry–Howard correspondence (??), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.
3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.
4. The expression $\neg U \vee V$ then corresponds to the set $U^c \cup V$ in $\mathcal{P}(X)$.
5. The set $U^c \cup V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

Proposition 4.4.7.1.3. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$\begin{aligned} [U, -]_X &: (\mathcal{P}(X), \supset) && \rightarrow (\mathcal{P}(X), \subset), \\ [-, V]_X &: (\mathcal{P}(X), \subset) && \rightarrow (\mathcal{P}(X), \subset), \\ [-, -]_X &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

- (a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.
- (b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.
- (c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (U \cap - \dashv [U, -]_X) &: \mathcal{P}(X) \begin{array}{c} \xrightarrow{U \cap -} \\ \perp \\ \xleftarrow{[U, -]_X} \end{array} \mathcal{P}(X), \\ (- \cap V \dashv [V, -]_X) &: \mathcal{P}(X) \begin{array}{c} \xrightarrow{- \cap V} \\ \perp \\ \xleftarrow{[V, -]_X} \end{array} \mathcal{P}(X), \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \mathbf{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathbf{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \mathbf{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathbf{Hom}_{\mathcal{P}(X)}(V, [U, W]_X). \end{aligned}$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

- i. We have $U \cap V \subset W$.
- ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

- i. We have $U \cap V \subset W$.
- ii. We have $V \subset [U, W]_X$.

3. *Interaction With the Empty Set I.* We have

$$\begin{aligned} [U, \emptyset]_X &= U^c, \\ [\emptyset, V]_X &= X, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Interaction With X .* We have

$$\begin{aligned} [U, X]_X &= X, \\ [X, V]_X &= V, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Interaction With the Empty Set II.* The functor

$$D_X: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X)$$

defined by

$$\begin{aligned} D_X &\stackrel{\text{def}}{=} [-, \emptyset]_X \\ &= (-)^c \end{aligned}$$

is an involutory isomorphism of categories, making \emptyset into a dualising object for $(\mathcal{P}(X), \cap, X, [-, -]_X)$ in the sense of ???. In particular:

(a) The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)^{\text{op}}} & \downarrow D_X \\ & & \mathcal{P}(X)^{\text{op}} \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X} = U$$

for each $U \in \mathcal{P}(X)$.

(b) The diagram

$$\begin{array}{ccc} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} \xrightarrow{\cap^{\text{op}}} \mathcal{P}(X)^{\text{op}} & \\ \text{id}_{\mathcal{P}(X)^{\text{op}}} \times D_X \nearrow & & \searrow D_X \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-, -]_X} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With the Empty Set III.* Let $f: X \rightarrow Y$ be a function.

(a) *Interaction With Direct Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(b) *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ D_Y \downarrow & & \downarrow D_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

(c) *Interaction With Codirect Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\ \cup^{\text{op}} \times \cup^{\text{op}} \downarrow & \text{X} & \downarrow \cup \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\
 \nearrow \sim & & \searrow \cup^{\text{op}} \\
 \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & \mathcal{P}(X)^{\text{op}} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & \downarrow [-, V]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{U}} V \right]_X = \bigcup_{V \in \mathcal{U}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cap^{\text{op}} \times \cap^{\text{op}} & \text{X} & \downarrow \cap \\
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\ \nearrow \sim & & \searrow \cap^{\text{op}} \\ \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & \mathcal{P}(X)^{\text{op}} \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & \downarrow [-, V]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Binary Unions.* We have equalities of sets

$$\begin{aligned}[U \cap V, W]_X &= [U, W]_X \cup [V, W]_X, \\ [U, V \cap W]_X &= [U, V]_X \cap [U, W]_X\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Binary Intersections.* We have equalities of sets

$$\begin{aligned}[U \cup V, W]_X &= [U, W]_X \cap [V, W]_X, \\ [U, V \cup W]_X &= [U, V]_X \cup [U, W]_X\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Differences.* We have equalities of sets

$$\begin{aligned}[U \setminus V, W]_X &= [U, W]_X \cup [V^c, W]_X \\ &= [U, W]_X \cup [U, V]_X, \\ [U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

16. *Interaction With Complements.* We have equalities of sets

$$\begin{aligned}[U^c, V]_X &= U \cup V, \\ [U, V^c]_X &= U \cap V, \\ [U, V]_X^c &= U \setminus V\end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U, V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod{2}, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

18. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

19. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

20. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & \wr & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. **Item 1, Functoriality:** Since $\mathcal{P}(X)$ is posetal, it suffices to prove **Items 1a** to **1c**.

1. *Proof of Item 1a:* We have

$$\begin{aligned} [A, V]_X &\stackrel{\text{def}}{=} A^c \cup V \\ &\subset U^c \cup V \\ &\stackrel{\text{def}}{=} [U, V]_X, \end{aligned}$$

where we have used:

- (a) **Item 1** of **Definition 4.3.11.1.2**, which states that if $U \subset A$, then $A^c \subset U^c$.
- (b) **Item 1a** of **Item 1** of **Definition 4.3.11.1.2**, which states that if $A^c \subset U^c$, then $A^c \cup K \subset U^c \cup K$ for any $K \in \mathcal{P}(X)$.

2. *Proof of Item 1b:* We have

$$\begin{aligned} [U, V]_X &\stackrel{\text{def}}{=} U^c \cup V \\ &\subset U^c \cup B \\ &\stackrel{\text{def}}{=} [U, B]_X, \end{aligned}$$

where we have used **Item 1b** of **Item 1** of **Definition 4.3.11.1.2**, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

3. *Proof of Item 1c:* We have

$$\begin{aligned} [A, V]_X &\subset [U, V]_X \\ &\subset [U, B]_X, \end{aligned}$$

where we have used **Items 1a** and **1b**.

This finishes the proof.

Item 2, Adjointness: This is a repetition of **Item 2** of **Definition 4.3.9.1.2** and is proved there.

Item 3, Interaction With the Empty Set I: We have

$$\begin{aligned} [U, \emptyset]_X &\stackrel{\text{def}}{=} U^c \cup \emptyset \\ &= U^c, \end{aligned}$$

where we have used **Item 3** of **Definition 4.3.8.1.2**, and we have

$$[\emptyset, V]_X \stackrel{\text{def}}{=} \emptyset^c \cup V$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} (X \setminus \emptyset) \cup V \\
&= X \cup V \\
&= X,
\end{aligned}$$

where we have used:

1. **Item 12** of **Definition 4.3.10.1.2** for the first equality.
2. **Item 5** of **Definition 4.3.8.1.2** for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**).

Item 4, Interaction With X : We have

$$\begin{aligned}
[U, X]_X &\stackrel{\text{def}}{=} U^c \cup X \\
&= X,
\end{aligned}$$

where we have used **Item 5** of **Definition 4.3.8.1.2**, and we have

$$\begin{aligned}
[X, V]_X &\stackrel{\text{def}}{=} X^c \cup V \\
&\stackrel{\text{def}}{=} (X \setminus X) \cup V \\
&= \emptyset \cup V \\
&= V,
\end{aligned}$$

where we have used **Item 3** of **Definition 4.3.8.1.2** for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**).

Item 5, Interaction With the Empty Set II: We have

$$\begin{aligned}
D_X(D_X(U)) &\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X \\
&= [U^c, \emptyset]_X \\
&= (U^c)^c \\
&= U,
\end{aligned}$$

where we have used:

1. **Item 3** for the second and third equalities.
2. **Item 3** of **Definition 4.3.11.1.2** for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Definition 11.2.7.1.2](#)), and thus we have

$$[[-, \emptyset]_X, \emptyset]_X \cong \text{id}_{\mathcal{P}(X)}$$

This finishes the proof.

Item 6, Interaction With the Empty Set III: Since $D_X = (-)^c$, this is essentially a repetition of the corresponding results for $(-)^c$, namely [Items 5 to 7](#) of [Definition 4.3.11.1.2](#).

Item 7, Interaction With Unions of Families of Subsets I: By [Item 3](#) of [Definition 4.4.7.1.3](#), we have

$$\begin{aligned} [\mathcal{U}, \emptyset]_{\mathcal{P}(X)} &= \mathcal{U}^c, \\ [U, \emptyset]_X &= U^c. \end{aligned}$$

With this, the counterexample given in the proof of [Item 10](#) of [Definition 4.3.6.1.2](#) then applies.

Item 8, Interaction With Unions of Families of Subsets II: We have

$$\begin{aligned} \left[\bigcup_{U \in \mathcal{U}} U, V \right]_X &\stackrel{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right)^c \cup V \\ &= \left(\bigcap_{U \in \mathcal{U}} U^c \right) \cup V \\ &= \bigcap_{U \in \mathcal{U}} (U^c \cup V) \\ &\stackrel{\text{def}}{=} \bigcap_{U \in \mathcal{U}} [U, V]_X, \end{aligned}$$

where we have used:

1. [Item 11](#) of [Definition 4.3.6.1.2](#) for the second equality.
2. [Item 6](#) of [Definition 4.3.7.1.2](#) for the third equality.

This finishes the proof.

Item 9, Interaction With Unions of Families of Subsets III: We have

$$\bigcup_{V \in \mathcal{V}} [U, V]_X \stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} (U^c \cup V)$$

$$\begin{aligned}
&= U^c \cup \left(\bigcup_{V \in \mathcal{V}} V \right) \\
&\stackrel{\text{def}}{=} \left[U, \bigcup_{V \in \mathcal{V}} V \right]_X.
\end{aligned}$$

where we have used **Item 6**. This finishes the proof.

Item 10, Interaction With Intersections of Families of Subsets I: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned}
\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W &= \bigcap_{W \in \mathcal{P}(X)} W \\
&= \{0, 1\},
\end{aligned}$$

whereas

$$\begin{aligned}
\left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X &= [\{0, 1\}, \{0\}] \\
&= \{0\},
\end{aligned}$$

Thus we have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X.$$

This finishes the proof.

Item 11, Interaction With Intersections of Families of Subsets II: We have

$$\begin{aligned}
\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X &\stackrel{\text{def}}{=} \left(\bigcap_{U \in \mathcal{U}} U \right)^c \cup V \\
&= \left(\bigcup_{U \in \mathcal{U}} U^c \right) \cup V \\
&= \bigcup_{U \in \mathcal{U}} (U^c \cup V) \\
&\stackrel{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X,
\end{aligned}$$

where we have used:

1. **Item 12** of **Definition 4.3.6.1.2** for the second equality.
2. **Item 6** of **Definition 4.3.7.1.2** for the third equality.

This finishes the proof.

Item 12, Interaction With Intersections of Families of Subsets III: We have

$$\begin{aligned}
 \bigcap_{V \in \mathcal{U}} [U, V]_X &\stackrel{\text{def}}{=} \bigcap_{V \in \mathcal{U}} (U^c \cup V) \\
 &= U^c \cup \left(\bigcap_{V \in \mathcal{U}} V \right) \\
 &\stackrel{\text{def}}{=} \left[U, \bigcap_{V \in \mathcal{U}} V \right]_X.
 \end{aligned}$$

where we have used **Item 6**. This finishes the proof.

Item 13, Interaction With Binary Unions: We have

$$\begin{aligned}
 [U \cap V, W]_X &\stackrel{\text{def}}{=} (U \cap V)^c \cup W \\
 &= (U^c \cup V^c) \cup W \\
 &= (U^c \cup V^c) \cup (W \cup W) \\
 &= (U^c \cup W) \cup (V^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cup [V, W]_X,
 \end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.3.11.1.2** for the second equality.
2. **Item 8** of **Definition 4.3.8.1.2** for the third equality.
3. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the fourth equality.

For the second equality in the statement, we have

$$\begin{aligned}
 [U, V \cap W]_X &\stackrel{\text{def}}{=} U^c \cup (V \cap W) \\
 &= (U^c \cup V) \cap (U^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, V]_X \cap [U, W]_X,
 \end{aligned}$$

where we have used **Item 6** of **Definition 4.3.8.1.2** for the second equality.

Item 14, Interaction With Binary Intersections: We have

$$\begin{aligned} [U \cup V, W]_X &\stackrel{\text{def}}{=} (U \cup V)^c \cup W \\ &= (U^c \cap V^c) \cup W \\ &= (U^c \cup W) \cap (V^c \cup W) \\ &\stackrel{\text{def}}{=} [U, W]_X \cap [V, W]_X, \end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.3.11.1.2** for the second equality.
2. **Item 6** of **Definition 4.3.8.1.2** for the third equality.

Now, for the second equality in the statement, we have

$$\begin{aligned} [U, V \cup W]_X &\stackrel{\text{def}}{=} U^c \cup (V \cup W) \\ &= (U^c \cup U^c) \cup (V \cup W) \\ &= (U^c \cup V) \cup (U^c \cup W) \\ &\stackrel{\text{def}}{=} [U, V]_X \cup [U, W]_X, \end{aligned}$$

where we have used:

1. **Item 8** of **Definition 4.3.8.1.2** for the second equality.
2. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the third equality.

This finishes the proof.

Item 15, Interaction With Differences: We have

$$\begin{aligned} [U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^c \cup W \\ &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\ &= ((X \cap V) \cup (X \setminus U)) \cup W \\ &= (V \cup (X \setminus U)) \cup W \\ &\stackrel{\text{def}}{=} (V \cup U^c) \cup W \\ &= (V \cup (U^c \cup U^c)) \cup W \\ &= (U^c \cup W) \cup (U^c \cup V) \\ &\stackrel{\text{def}}{=} [U, W]_X \cup [U, V]_X, \end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.10.1.2** for the third equality.
2. **Item 4** of **Definition 4.3.9.1.2** for the fourth equality.
3. **Item 8** of **Definition 4.3.8.1.2** for the sixth equality.
4. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the seventh equality.

We also have

$$\begin{aligned}
 [U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^c \cup W \\
 &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
 &= ((X \cap V) \cup (X \setminus U)) \cup W \\
 &= (V \cup (X \setminus U)) \cup W \\
 &\stackrel{\text{def}}{=} (V \cup U^c) \cup W \\
 &= (V \cup U^c) \cup (W \cup W) \\
 &= (U^c \cup W) \cup (V \cup W) \\
 &= (U^c \cup W) \cup ((V^c)^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cup [V^c, W]_X,
 \end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.10.1.2** for the third equality.
2. **Item 4** of **Definition 4.3.9.1.2** for the fourth equality.
3. **Item 8** of **Definition 4.3.8.1.2** for the sixth equality.
4. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the seventh equality.
5. **Item 3** of **Definition 4.3.11.1.2** for the eighth equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
 [U, V \setminus W]_X &\stackrel{\text{def}}{=} U^c \cup (V \setminus W) \\
 &= (V \setminus W) \cup U^c \\
 &= (V \cup U^c) \setminus (W \setminus U^c)
 \end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} (V \cup U^c) \setminus (W \setminus (X \setminus U)) \\
&= (V \cup U^c) \setminus ((W \cap U) \cup (W \setminus X)) \\
&= (V \cup U^c) \setminus ((W \cap U) \cup \emptyset) \\
&= (V \cup U^c) \setminus (W \cap U) \\
&= (V \cup U^c) \setminus (U \cap W) \\
&\stackrel{\text{def}}{=} [U, V]_X \setminus (U \cap W)
\end{aligned}$$

where we have used:

1. **Item 4** of **Definition 4.3.8.I.2** for the second equality.
2. **Item 4** of **Definition 4.3.10.I.2** for the third equality.
3. **Item 10** of **Definition 4.3.10.I.2** for the fifth equality.
4. **Item 13** of **Definition 4.3.10.I.2** for the sixth equality.
5. **Item 3** of **Definition 4.3.8.I.2** for the seventh equality.
6. **Item 5** of **Definition 4.3.9.I.2** for the eighth equality.

This finishes the proof.

Item 16, Interaction With Complements: We have

$$\begin{aligned}
[U^c, V]_X &\stackrel{\text{def}}{=} (U^c)^c \cup V, \\
&= U \cup V,
\end{aligned}$$

where we have used **Item 3** of **Definition 4.3.11.I.2**. We also have

$$\begin{aligned}
[U, V^c]_X &\stackrel{\text{def}}{=} U^c \cup V^c \\
&= U \cap V
\end{aligned}$$

where we have used **Item 2** of **Definition 4.3.11.I.2**. Finally, we have

$$\begin{aligned}
[U, V]_X^c &= ((U \setminus V)^c)^c \\
&= U \setminus V,
\end{aligned}$$

where we have used **Item 2** of **Definition 4.3.11.I.2**.

Item 17, Interaction With Characteristic Functions: We have

$$\begin{aligned}\chi_{[U,V]_{\mathcal{P}(X)}}(x) &\stackrel{\text{def}}{=} \chi_{U^c \cup V}(x) \\ &= \max(\chi_{U^c}, \chi_V) \\ &= \max(1 - \chi_U \pmod{2}, \chi_V),\end{aligned}$$

where we have used:

1. *Item 10* of *Definition 4.3.8.1.2* for the second equality.
2. *Item 4* of *Definition 4.3.11.1.2* for the third equality.

This finishes the proof.

Item 18, Interaction With Direct Images: This is a repetition of *Item 10* of *Definition 4.6.1.1.5* and is proved there.

Item 19, Interaction With Inverse Images: This is a repetition of *Item 10* of *Definition 4.6.2.1.3* and is proved there.

Item 20, Interaction With Codirect Images: This is a repetition of *Item 9* of *Definition 4.6.3.1.7* and is proved there. \square

4.4.8 Isbell Duality for Sets

Let X be a set.

Definition 4.4.8.1.1. The **Isbell function** of X is the map

$$I: \mathcal{P}(X) \rightarrow \text{Sets}(X, \mathcal{P}(X))$$

defined by

$$I(U) \stackrel{\text{def}}{=} \llbracket x \mapsto [U, \{x\}]_X \rrbracket$$

for each $U \in \mathcal{P}(X)$.

Remark 4.4.8.1.2. Recall from *Definition 4.4.1.1.2* that we may view the power-set $\mathcal{P}(X)$ of a set X as the decategorification of the category of presheaves $\text{PSh}(C)$ of a category C . Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\text{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \text{Nat}(\mathcal{F}, h_{(-)})$$

for each $\mathcal{F} \in \text{Obj}(\text{PSh}(C))$. To this end, we could define

$$I(U) \stackrel{\text{def}}{=} [U, \chi_{(-)}]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of C into $\mathbf{PSh}(C)$ with the characteristic embedding $x \mapsto \chi_x$ of X into $\mathcal{P}(X)$ of [Definition 4.5.4.1.1](#).
- The internal Hom Nat of $\mathbf{PSh}(C)$ with the internal Hom $[-, -]_X$ of $\mathcal{P}(X)$ of [Definition 4.4.7.1.1](#).

However, since $[U, \chi_x]_X$ is a subset of U instead of a truth value, we get a function

$$! : \mathcal{P}(X) \rightarrow \mathbf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$! : \mathcal{P}(X) \rightarrow \mathcal{P}(X).$$

This makes some of the properties involving $!$ a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $! \circ !$ evaluates to $\text{id}_{\mathcal{P}(X)}$ in the sense of [Definition 4.4.8.1.3](#).

Proposition 4.4.8.1.3. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{!} & \mathbf{Sets}(X, \mathcal{P}(X)) \\ & \searrow \Delta_{\Delta_{\text{id}_{\mathcal{P}(X)}}} & \downarrow ! \\ & & \mathbf{Sets}(X, \mathbf{Sets}(X, \mathcal{P}(X))) \end{array}$$

commutes, i.e. we have

$$!(!(U)) = \llbracket x \mapsto \llbracket y \mapsto U \rrbracket \rrbracket$$

for each $U \in \mathcal{P}(X)$.

Proof. We have

$$\begin{aligned} !(!(U)) &\stackrel{\text{def}}{=} !(\llbracket x \mapsto U^c \cup \{x\} \rrbracket) \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto !(U^c \cup \{x\}) \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \llbracket y \mapsto (U^c \cup \{x\})^c \cup \{x\} \rrbracket \rrbracket \\ &= \llbracket x \mapsto \llbracket y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\} \rrbracket \rrbracket \\ &= \llbracket x \mapsto \llbracket y \mapsto (U \setminus \{x\}) \cup \{x\} \rrbracket \rrbracket \\ &= \llbracket x \mapsto \llbracket y \mapsto U \rrbracket \rrbracket, \end{aligned}$$

where we have used [Item 2](#) of [Definition 4.3.11.1.2](#) for the fourth equality above.

□

4.5 Characteristic Functions

4.5.1 The Characteristic Function of a Subset

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.5.1.1.1. The **characteristic function of U** ²³ is the function $\chi_U: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$ ²⁴ defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

Remark 4.5.1.1.2. Under the analogy that $\{\mathbf{t}, \mathbf{f}\}$ should be the (-1) -categorical analogue of **Sets**, we may view a function

$$f: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

as a decategorification of presheaves and copresheaves

$$\begin{aligned} \mathcal{F}: C^{\text{op}} &\rightarrow \mathbf{Sets}, \\ F: C &\rightarrow \mathbf{Sets}. \end{aligned}$$

The characteristic functions χ_U of the subsets of X are then the primordial examples of such functions (and, in fact, all of them).

Notation 4.5.1.1.3. We will often employ the bijection $\{\mathbf{t}, \mathbf{f}\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when discussing characteristic functions.

Examples of this include **Items 4 to 11** of **Definition 4.5.1.1.4** below.

Proposition 4.5.1.1.4. Let X be a set.

1. *Functionality.* The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)}: \mathcal{P}(X) \rightarrow \mathbf{Sets}(X, \{\mathbf{t}, \mathbf{f}\}).$$

2. *Bijectivity.* The function $\chi_{(-)}$ from **Item 1** is bijective.

²³*Further Terminology:* Also called the **indicator function of U** .

²⁴*Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

3. *Naturality.* The collection

$$\{\chi_{(-)}: \mathcal{P}(X) \rightarrow \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})\}_{X \in \text{Obj}(\text{Sets})}$$

defines a natural isomorphism between \mathcal{P}^{-1} and $\text{Sets}(-, \{\mathbf{t}, \mathbf{f}\})$. In particular, given a function $f: X \rightarrow Y$, the diagram

$$\begin{array}{ccc} \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \\ \chi_{(-)} \downarrow & & \downarrow \chi_{(-)} \\ \text{Sets}(Y, \{\mathbf{t}, \mathbf{f}\}) & \xrightarrow{f^*} & \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\}) \end{array}$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Complements.* We have

$$\chi_{U^c} \equiv 1 - \chi_U \pmod{2}$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \Delta V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \Delta V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Internal Homs.* We have

$$\chi_{[U, V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod{2}, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

Proof. **Item 1, Functionality:** There is nothing to prove.

Item 2, Bijectivity: We proceed in three steps:

I. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi: \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\}) \xrightarrow{\sim} \mathcal{P}(X),$$

defined by

$$\begin{aligned} \Phi(f) &\stackrel{\text{def}}{=} U_f \\ &\stackrel{\text{def}}{=} f^{-1}(\mathbf{true}) \\ &\stackrel{\text{def}}{=} \{x \in X \mid f(x) = \mathbf{true}\} \end{aligned}$$

for each $f \in \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$.

2. *Invertibility I.* We have

$$\begin{aligned}
 [\Phi \circ \chi_{(-)}](U) &\stackrel{\text{def}}{=} \Phi(\chi_U) \\
 &\stackrel{\text{def}}{=} \chi_U^{-1}(\text{true}) \\
 &\stackrel{\text{def}}{=} \{x \in X \mid \chi_U(x) = \text{true}\} \\
 &\stackrel{\text{def}}{=} \{x \in X \mid x \in U\} \\
 &= U \\
 &\stackrel{\text{def}}{=} [\text{id}_{\mathcal{P}(X)}](U)
 \end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus, we have

$$\Phi \circ \chi_{(-)} = \text{id}_{\mathcal{P}(X)}.$$

3. *Invertibility II.* We have

$$\begin{aligned}
 [\chi_{(-)} \circ \Phi](U) &\stackrel{\text{def}}{=} \chi_{\Phi(f)} \\
 &\stackrel{\text{def}}{=} \chi_{f^{-1}(\text{true})} \\
 &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \text{true} & \text{if } x \in f^{-1}(\text{true}) \\ \text{false} & \text{otherwise} \end{cases} \rrbracket \\
 &= \llbracket x \mapsto f(x) \rrbracket \\
 &= f \\
 &\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(X, \{\text{t}, \text{f}\})}](f)
 \end{aligned}$$

for each $f \in \text{Sets}(X, \{\text{t}, \text{f}\})$. Thus, we have

$$\chi_{(-)} \circ \Phi = \text{id}_{\text{Sets}(X, \{\text{t}, \text{f}\})}.$$

This finishes the proof.

Item 3, Naturality: We proceed in two steps:

1. *Naturality of $\chi_{(-)}$.* We have

$$\begin{aligned}
 [\chi_V \circ f](v) &\stackrel{\text{def}}{=} \chi_V(f(v)) \\
 &= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases}
 \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases} \\
&\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v)
\end{aligned}$$

for each $v \in V$.

2. *Naturality of Φ* . Since $\chi_{(-)}$ is natural and a componentwise inverse to Φ , it follows from [Categories](#), [Item 2](#) of [Definition 11.9.7.1.2](#) that Φ is also natural in each argument.

This finishes the proof.

Item 4, Interaction With Unions I: This is a repetition of [Item 10](#) of [Definition 4.3.8.1.2](#) and is proved there.

Item 5, Interaction With Unions II: This is a repetition of [Item 11](#) of [Definition 4.3.8.1.2](#) and is proved there.

Item 6, Interaction With Intersections I: This is a repetition of [Item 10](#) of [Definition 4.3.9.1.2](#) and is proved there.

Item 7, Interaction With Intersections II: This is a repetition of [Item 11](#) of [Definition 4.3.9.1.2](#) and is proved there.

Item 8, Interaction With Differences: This is a repetition of [Item 16](#) of [Definition 4.3.10.1.2](#) and is proved there.

Item 9, Interaction With Complements: This is a repetition of [Item 4](#) of [Definition 4.3.11.1.2](#) and is proved there.

Item 10, Interaction With Symmetric Differences: This is a repetition of [Item 15](#) of [Definition 4.3.12.1.2](#) and is proved there.

Item 11, Interaction With Internal Homs: This is a repetition of [Item 17](#) of [Definition 4.4.7.1.3](#) and is proved there. \square

Remark 4.5.1.1.5. The bijection

$$\mathcal{P}(X) \cong \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$$

of [Item 2](#) of [Definition 4.5.1.1.4](#), which

- Takes a subset $U \rightarrow X$ of X and *straightens* it to a function $\chi_U: X \rightarrow \{\text{true}, \text{false}\}$;
- Takes a function $f: X \rightarrow \{\text{true}, \text{false}\}$ and *unstraightens* it to a subset $f^{-1}(\text{true}) \rightarrow X$ of X ;

may be viewed as the (-1) -categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\text{FibSets}_X}_{\stackrel{\text{def}}{=} \text{Sets}/X} \cong \underbrace{\text{ISets}_X}_{\stackrel{\text{def}}{=} \text{Fun}(X_{\text{disc}}, \text{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, ???. Here we view:

- Subsets $U \rightarrow X$ as being analogous to X -fibred sets $\phi_X: A \rightarrow X$.
- Functions $f: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$ as being analogous to X -indexed sets $A: X_{\text{disc}} \rightarrow \text{Sets}$.

4.5.2 The Characteristic Function of a Point

Let X be a set and let $x \in X$.

Definition 4.5.2.1.1. The **characteristic function of x** is the function²⁵

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathbf{true} & \text{if } x = y, \\ \mathbf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

Remark 4.5.2.1.2. Expanding upon [Definition 4.5.1.1.2](#), we may think of the characteristic function

$$\chi_x: X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of an *element* x of X as a decategorification of the representable presheaf and of the representable copresheaf

$$\begin{aligned} h_X: C^{\text{op}} &\rightarrow \text{Sets}, \\ h^X: C &\rightarrow \text{Sets} \end{aligned}$$

associated of an *object* X of a category C .

²⁵*Further Notation:* Also written χ^x , $\chi_X(x, -)$, or $\chi_X(-, x)$.

4.5.3 The Characteristic Relation of a Set

Let X be a set.

Definition 4.5.3.1.1. The **characteristic relation on X** ²⁶ is the relation²⁷

$$\chi_X(-_1, -_2): X \times X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

on X defined by²⁸

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

Remark 4.5.3.1.2. Expanding upon [Definitions 4.5.1.1.2](#) and [4.5.2.1.2](#), we may view the characteristic relation

$$\chi_X(-_1, -_2): X \times X \rightarrow \{\mathbf{t}, \mathbf{f}\}$$

of X as a decategorification of the Hom profunctor

$$\text{Hom}_C(-_1, -_2): C^{\text{op}} \times C \rightarrow \text{Sets}$$

of a category C .

Proposition 4.5.3.1.3. Let $f: X \rightarrow Y$ be a function.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f: A \rightarrow B$ be a function. We have an inclusion²⁹

$$\chi_B \circ (f \times f) \subset \chi_A, \quad \begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ & \searrow \chi_A \quad \supset \quad \swarrow \chi_B & \\ & \{\mathbf{t}, \mathbf{f}\}. & \end{array}$$

Proof. **Item 1, The Inclusion of Characteristic Relations Associated to a Function:** The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement “if $a = b$, then $f(a) = f(b)$ ”, which is true. \square

²⁶*Further Terminology:* Also called the **identity relation on X** .

²⁷*Further Notation:* Also written χ_X^{-1} , or \sim_{id} in the context of relations.

²⁸Under the bijection $\text{Sets}(X \times X, \{\mathbf{t}, \mathbf{f}\}) \cong \mathcal{P}(X \times X)$ of [Item 2](#) of [Definition 4.5.1.1.4](#), the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X .

²⁹*Note:* This is the 0-categorical version of [Categories, Definition 11.5.4.1.1](#).

4.5.4 The Characteristic Embedding of a Set

Let X be a set.

Definition 4.5.4.1.1. The **characteristic embedding**³⁰ of X into $\mathcal{P}(X)$ is the function

$$\chi(-) : X \rightarrow \mathcal{P}(X)$$

defined by³¹

$$\begin{aligned} \chi(-)(x) &\stackrel{\text{def}}{=} \chi_x \\ &= \{x\} \end{aligned}$$

for each $x \in X$.

Remark 4.5.4.1.2. Expanding upon [Definitions 4.5.1.1.2](#), [4.5.2.1.2](#) and [4.5.3.1.2](#), we may view the characteristic embedding

$$\chi(-) : X \rightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$\mathfrak{Y} : C^{\text{op}} \rightarrow \text{PSh}(C)$$

of a category C into $\text{PSh}(C)$.

Proposition 4.5.4.1.3. Let $f : X \rightarrow Y$ be a map of sets.

1. *Interaction With Functions.* We have

$$f! \circ \chi_X = \chi_Y \circ f,$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \chi_X \downarrow & & \downarrow \chi_Y \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y). \end{array}$$

³⁰The name “characteristic *embedding*” is justified by [Definition 4.5.5.1.2](#), which gives an analogue of fully faithfulness for $\chi(-)$.

³¹Here we are identifying $\mathcal{P}(X)$ with $\text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$ as per [Item 2](#) of [Definition 4.5.1.1.4](#).

Proof. **Item 1, Interaction With Functions:** Indeed, we have

$$\begin{aligned}
 [f \circ \chi_X](x) &\stackrel{\text{def}}{=} f(\chi_X(x)) \\
 &\stackrel{\text{def}}{=} f(\{x\}) \\
 &= \{f(x)\} \\
 &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\
 &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x),
 \end{aligned}$$

for each $x \in X$, showing the desired equality. \square

4.5.5 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X .

Proposition 4.5.5.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } U \subset V, \\ \text{false} & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned}
 \chi_{\mathcal{P}(X)}(\chi_x, \chi_U) &\stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } \{x\} \subset U, \\ \text{false} & \text{otherwise} \end{cases} \\
 &= \begin{cases} \text{true} & \text{if } x \in U \\ \text{false} & \text{otherwise} \end{cases} \\
 &\stackrel{\text{def}}{=} \chi_U(x).
 \end{aligned}$$

This finishes the proof. \square

Corollary 4.5.5.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x \chi_y) \cong \chi_X(x, y)$$

for each $x, y \in X$.

Proof. We have

$$\begin{aligned} \chi_{\mathcal{P}(X)}(\chi_x \chi_y) &= \chi_y(x) \\ &\stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in \{y\} \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } x = y \\ \text{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_X(x, y). \end{aligned}$$

where we have used [Definition 4.5.5.1.1](#) for the first equality. \square

4.6 The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

4.6.1 Direct Images

Let $f: X \rightarrow Y$ be a function.

Definition 4.6.1.1.1. The **direct image function associated to f** is the function³²

$$f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by³³

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ y \in Y \mid \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\} \\ &= \{f(x) \in Y \mid x \in U\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

³²*Further Notation:* Also written simply $f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$.

³³*Further Terminology:* The set $f(U)$ is called the **direct image of U by f** .

Notation 4.6.1.1.2. Sometimes one finds the notation

$$\exists_f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \exists_f(U)$.
- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.1.1.3. Notation for direct images between powersets is tricky:

1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:

- (a) *Powersets.* Given a function $f: X \rightarrow Y$, we have an inverse image functor

$$f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X).$$

The *left* adjoint of this functor is the usual direct image, defined above in [Definition 4.6.1.1.1](#).

- (b) *Presheaves.* Given a morphism of topological spaces $f: X \rightarrow Y$, we have an inverse image functor

$$f^{-1}: \mathbf{PSh}(Y) \rightarrow \mathbf{PSh}(X).$$

The *right* adjoint of this functor is the direct image functor of presheaves, defined in ??.

2. The presheaf direct image functor is denoted f_* , but the direct image functor for powersets is denoted $f_!$ (as it's a left adjoint).
3. Adding to the confusion, it's somewhat common for $f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ to be denoted f_* .

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

SITUATION	ADJOINT STRING
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*) : \mathcal{P}(X) \xrightarrow{\cong} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*) : \mathbf{PSh}(X) \xrightarrow{\cong} \mathbf{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*) : \mathcal{C}_{/X} \xrightarrow{\cong} \mathcal{C}_{/Y}$
Kan Extensions	$(F_! \dashv F^* \dashv F_*) : \mathbf{Fun}(\mathcal{C}, \mathcal{E}) \xrightarrow{\cong} \mathbf{Fun}(\mathcal{D}, \mathcal{E})$

Remark 4.6.I.I.4. Identifying $\mathcal{P}(X)$ with $\mathbf{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$ via [Item 2 of Definition 4.5.I.I.4](#), we see that the direct image function associated to f is equivalently the function

$$f_! : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by

$$\begin{aligned}
 f_!(\chi_U) &\stackrel{\text{def}}{=} \mathbf{Lan}_f(\chi_U) \\
 &= \text{colim} \left(\left(f \times \underline{(-)} \right) \xrightarrow{\text{pr}} \mathcal{A} \xrightarrow{\chi_U} \{\mathbf{t}, \mathbf{f}\} \right) \\
 &= \text{colim}_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)) \\
 &= \bigvee_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),
 \end{aligned}$$

where we have used ?? for the second equality. In other words, we have

$$\begin{aligned}
 [f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\
 &= \begin{cases} \text{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
 &= \begin{cases} \text{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \text{false} & \text{otherwise} \end{cases}
 \end{aligned}$$

for each $y \in Y$.

Proposition 4.6.1.1.5. Let $f: X \rightarrow Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f_!(U) \subset f_!(V).$$

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \mathcal{P}(X) \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{array} \mathcal{P}(Y),$$

witnessed by:

- (a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\rightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\rightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\rightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\rightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

- (b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

- i. The following conditions are equivalent:
A. We have $f_!(U) \subset V$.

- B. We have $U \subset f^{-1}(V)$.
- ii. The following conditions are equivalent:
- A. We have $f^{-1}(U) \subset V$.
- B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f!(U) = \bigcup_{V \in f!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f!(\mathcal{U}) \stackrel{\text{def}}{=} (f!)_!(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f!(U) = \bigcap_{V \in f!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f!(\mathcal{U}) \stackrel{\text{def}}{=} (f!)_!(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f! \times f!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & \subset & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

7. *Interaction With Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \setminus \downarrow & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U^c) = f_*(U)^c$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \wr & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \Delta f_!(V) \subset f_!(U \Delta V)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ [-1, -2]_X \downarrow & & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_!(U) \cup f_!(V) &= f_!(U \cup V), \\ f_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U \cap V) &\subset f_!(U) \cap f_!(V), \\ f_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_!, f_!^\otimes, f_{!|1}^\otimes): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^\otimes: f_!(U) \cup f_!(V) &\xrightarrow{=} f_!(U \cup V), \\ f_{!|1}^\otimes: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(f_!, f_!^\otimes, f_{!|1}^\otimes): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^\otimes: f_!(U \cap V) &\rightarrow f_!(U) \cap f_!(V), \\ f_{!|1}^\otimes: f_!(X) &\rightarrow Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

15. *Interaction With Coproducts.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \amalg g)_!(U \amalg V) = f_!(U) \amalg g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Interaction With Products.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

17. *Relation to Codirect Images.* We have

$$\begin{aligned} f_!(U) &= f_*(U^c)^c \\ &\stackrel{\text{def}}{=} Y \setminus f_*(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. **Item 1, Functoriality:** Omitted.

Item 2, Triple Adjointness: This follows from **Definition 4.6.1.1.4**, **Definition 4.6.2.1.2**, **Definition 4.6.3.1.4**, and Kan Extensions, ?? of ??.

Item 3, Interaction With Unions of Families of Subsets: We have

$$\begin{aligned} \bigcup_{V \in f_!(\mathcal{U})} V &= \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.

Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned} \bigcap_{V \in f_!(\mathcal{U})} V &= \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.

Item 5, Interaction With Binary Unions: See [Pro25p].

Item 6, Interaction With Binary Intersections: See [Pro25n].

Item 7, Interaction With Differences: See [Pro25o].

Item 8, Interaction With Complements: Applying *Item 17* to $X \setminus U$, we have

$$\begin{aligned} f_!(U^c) &= f_!(X \setminus U) \\ &= Y \setminus f_*(X \setminus (X \setminus U)) \\ &= Y \setminus f_*(U) \\ &= f_*(U)^c. \end{aligned}$$

This finishes the proof.

Item 9, Interaction With Symmetric Differences: We have

$$\begin{aligned} f_!(U) \triangle f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\ &\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\ &= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\ &\subset f_!((U \cup V) \setminus (U \cap V)) \\ &= f_!(U \triangle V), \end{aligned}$$

where we have used:

1. *Item 2* of *Definition 4.3.12.1.2* for the first equality.
2. *Item 6* of this proposition together with *Item 1* of *Definition 4.3.10.1.2* for the first inclusion.
3. *Item 5* for the second equality.
4. *Item 7* for the second inclusion.
5. *Item 2* of *Definition 4.3.12.1.2* for the tchird equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (*Categories*, *Item 4* of *Definition 11.2.7.1.2*). This finishes the proof.

Item 10, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned} f_!([U, V]_X) &\stackrel{\text{def}}{=} f_!(U^c \cup V) \\ &= f_!(U^c) \cup f_!(V) \end{aligned}$$

$$\begin{aligned}
&= f_*(U)^c \cup f_!(V) \\
&\stackrel{\text{def}}{=} [f_*(U), f_!(V)]_Y,
\end{aligned}$$

where we have used:

1. **Item 5** for the second equality.
2. **Item 17** for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Definition II.2.7.1.2**). This finishes the proof.

Item 11, *Preservation of Colimits*: This follows from **Item 2** and ??, ?? of ??.³⁴

Item 12, *Oplax Preservation of Limits*: The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25n] for the other inclusions.

Item 13, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **Item 11**.

Item 14, *Symmetric Oplax Monoidality With Respect to Intersections*: The inclusions in the statement follow from **Item 12**. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (**Categories**, **Item 4** of **Definition II.2.7.1.2**).

Item 15, *Interaction With Coproducts*: Omitted.

Item 16, *Interaction With Products*: Omitted.

Item 17, *Relation to Codirect Images*: Applying **Item 16** of **Definition 4.6.3.1.7** to $X \setminus U$, we have

$$\begin{aligned}
f_*(X \setminus U) &= B \setminus f_!(X \setminus (X \setminus U)) \\
&= B \setminus f_!(U).
\end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned}
f_!(U) &= B \setminus (B \setminus f_*(X \setminus U)), \\
&= B \setminus f_*(X \setminus U),
\end{aligned}$$

which finishes the proof. □

Proposition 4.6.1.1.6. Let $f: X \rightarrow Y$ be a function.

³⁴Reference: [Pro25p].

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$(\mathrm{id}_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \\ & \searrow (g \circ f)_! & \downarrow g_! \\ & & \mathcal{P}(Z). \end{array}$$

Proof. **Item 1, Functionality I:** There is nothing to prove.

Item 2, Functionality II: This follows from **Item 1** of **Definition 4.6.1.1.5**.

Item 3, Interaction With Identities: This follows from **Definition 4.6.1.1.4** and Kan Extensions, ?? of ??.

Item 4, Interaction With Composition: This follows from **Definition 4.6.1.1.4** and Kan Extensions, ?? of ??. \square

4.6.2 Inverse Images

Let $f: X \rightarrow Y$ be a function.

Definition 4.6.2.1.1. The **inverse image function associated to f** is the function³⁵

$$f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

³⁵*Further Notation:* Also written $f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$.

defined by³⁶

$$f^{-1}(V) \stackrel{\text{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 4.6.2.1.2. Identifying $\mathcal{P}(Y)$ with $\text{Sets}(Y, \{\mathbf{t}, \mathbf{f}\})$ via **Item 2** of **Definition 4.5.1.1.4**, we see that the inverse image function associated to f is equivalently the function

$$f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\mathbf{true}, \mathbf{false}\}$$

in Sets .

Proposition 4.6.2.1.3. Let $f: X \rightarrow Y$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}: (\mathcal{P}(Y), \subset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f^{-1}(U) \subset f^{-1}(V).$$

2. *Triple Adjointness.* We have a triple adjunction

$$(f! \dashv f^{-1} \dashv f_*): \quad \begin{array}{ccc} & \overset{f!}{\curvearrowright} & \\ & \perp & \\ \mathcal{P}(X) & \xleftarrow{f^{-1}} & \mathcal{P}(Y), \\ & \underset{f_*}{\curvearrowright} & \\ & \perp & \end{array}$$

witnessed by:

³⁶*Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of V by f** .

(a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\rightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\rightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\rightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\rightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

- i. The following conditions are equivalent:
 - A. We have $f_!(U) \subset V$.
 - B. We have $U \subset f^{-1}(V)$.
- ii. The following conditions are equivalent:
 - A. We have $f^{-1}(U) \subset V$.
 - B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{V})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{U}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(Y))$, where $f^{-1}(\mathcal{U}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

6. *Interaction With Binary Intersections.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow \setminus & & \downarrow \setminus \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U^c) = f^{-1}(U)^c$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow \Delta & & \downarrow \Delta \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

i.e. we have

$$f^{-1}(U) \Delta f^{-1}(V) = f^{-1}(U \Delta V)$$

for each $U, V \in \mathcal{P}(Y)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\
 \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned}
 f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\
 f^{-1}(\emptyset) &= \emptyset,
 \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned}
 f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\
 f^{-1}(Y) &= X,
 \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_1^{-1, \otimes}): (\mathcal{P}(Y), \cup, \emptyset) \rightarrow (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1, \otimes}: f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f_1^{-1, \otimes}: \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_1^{-1, \otimes}): (\mathcal{P}(Y), \cap, Y) \rightarrow (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1, \otimes}: f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cap V), \\ f_1^{-1, \otimes}: X &\xrightarrow{=} f^{-1}(Y), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

15. *Interaction With Coproducts.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \amalg g)^{-1}(U' \amalg V') = f^{-1}(U') \amalg g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

16. *Interaction With Products.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

Proof. **Item 1, Functoriality:** Omitted.

Item 2, Triple Adjointness: This follows from **Definition 4.6.1.1.4**, **Definition 4.6.2.1.2**, **Definition 4.6.3.1.4**, and Kan Extensions, ?? of ??.

Item 3, Interaction With Unions of Families of Subsets: We have

$$\begin{aligned} \bigcup_{U \in f^{-1}(\mathcal{V})} U &= \bigcup_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcup_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.

Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned} \bigcap_{U \in f^{-1}(\mathcal{V})} U &= \bigcap_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcap_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.

Item 5, Interaction With Binary Unions: See [Pro25y].

Item 6, Interaction With Binary Intersections: See [Pro25w].

Item 7, Interaction With Differences: See [Pro25x].

Item 8, Interaction With Complements: See [Pro25j].

Item 9, Interaction With Symmetric Differences: We have

$$\begin{aligned} f^{-1}(U \triangle V) &= f^{-1}((U \cup V) \setminus (U \cap V)) \\ &= f^{-1}(U \cup V) \setminus f^{-1}(U \cap V) \\ &= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U \cap V) \\ &= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U) \cap f^{-1}(V) \\ &= f^{-1}(U) \triangle f^{-1}(V), \end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.3.12.1.2** for the first equality.
2. **Item 7** for the second equality.
3. **Item 5** for the third equality.

4. **Item 6** for the fourth equality.
5. **Item 2** of **Definition 4.3.12.1.2** for the fifth equality.

This finishes the proof.

Item 10, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned}
 f^{-1}([U, V]_Y) &\stackrel{\text{def}}{=} f^{-1}(U^c \cup V) \\
 &= f^{-1}(U^c) \cup f^{-1}(V) \\
 &= f^{-1}(U)^c \cup f^{-1}(V) \\
 &\stackrel{\text{def}}{=} [f^{-1}(U), f^{-1}(V)]_X,
 \end{aligned}$$

where we have used:

1. **Item 8** for the second equality.
2. **Item 5** for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**Categories, Item 4** of **Definition II.2.7.1.2**). This finishes the proof.

Item 11, Preservation of Colimits: This follows from **Item 2** and ??, ?? of ??.³⁷

Item 12, Preservation of Limits: This follows from **Item 2** and ??, ?? of ??.³⁸

Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from **Item 11**.

Item 14, Symmetric Strict Monoidality With Respect to Intersections: This follows from **Item 12**.

Item 15, Interaction With Coproducts: Omitted.

Item 16, Interaction With Products: Omitted. □

Proposition 4.6.2.1.4. Let $f : X \rightarrow Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{X,Y}^{-1} : \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{X,Y}^{-1} : \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

³⁷Reference: [Pro25y].

³⁸Reference: [Pro25w].

3. *Interaction With Identities.* For each $X \in \text{Obj}(\text{Sets})$, we have

$$\text{id}_X^{-1} = \text{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y) \\ & \searrow (g \circ f)^{-1} & \downarrow f^{-1} \\ & & \mathcal{P}(X). \end{array}$$

Proof. **Item 1, Functionality I:** There is nothing to prove.

Item 2, Functionality II: This follows from **Item 1** of **Definition 4.6.2.1.3**.

Item 3, Interaction With Identities: This follows from **Definition 4.6.2.1.2** and **Categories, Item 5** of **Definition 11.1.4.1.2**.

Item 4, Interaction With Composition: This follows from **Definition 4.6.2.1.2** and **Categories, Item 2** of **Definition 11.1.4.1.2**. \square

4.6.3 Codirect Images

Let $f: X \rightarrow Y$ be a function.

Definition 4.6.3.1.1. The **codirect image function associated to f** is the function

$$f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by^{39,40}

$$f_*(U) \stackrel{\text{def}}{=} \left\{ y \in Y \left| \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right. \right\}$$

³⁹*Further Terminology:* The set $f_*(U)$ is called the **codirect image of U by f** .

⁴⁰We also have

$$\begin{aligned} f_*(U) &= f_!(U^c)^c \\ &\stackrel{\text{def}}{=} Y \setminus f_!(X \setminus U); \end{aligned}$$

see **Item 16** of **Definition 4.6.3.1.7**.

$$= \{y \in Y \mid \text{we have } f^{-1}(y) \subset U\}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.3.1.2. Sometimes one finds the notation

$$\forall_f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.
- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.3.1.3. See [Definition 4.6.1.1.3](#).

Remark 4.6.3.1.4. Identifying $\mathcal{P}(X)$ with $\text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$ via [Item 2](#) of [Definition 4.5.1.1.4](#), we see that the codirect image function associated to f is equivalently the function

$$f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \text{Ran}_f(\chi_U) \\ &= \lim \left(\left(\underline{(-1)} \xrightarrow{\quad} f \right) \xrightarrow{\text{pr}} X \xrightarrow{\chi_U} \{\text{true}, \text{false}\} \right) \\ &= \lim_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)) \\ &= \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)). \end{aligned}$$

where we have used ?? for the second equality. In other words, we have

$$[f_*(\chi_U)](y) = \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x))$$

$$\begin{aligned}
&= \begin{cases} \text{true} & \text{if, for each } x \in X \text{ such that} \\ & f(x) = y, \text{ we have } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } f^{-1}(y) \subset U \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $y \in Y$.

Definition 4.6.3.1.5. Let U be a subset of X .^{41,42}

1. The **image part of the codirect image** $f_*(U)$ of U is the set $f_{*,\text{im}}(U)$ defined by

$$\begin{aligned}
f_{*,\text{im}}(U) &\stackrel{\text{def}}{=} f_*(U) \cap \text{Im}(f) \\
&= \left\{ y \in Y \left| \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{array} \right. \right\}.
\end{aligned}$$

2. The **complement part of the codirect image** $f_*(U)$ of U is the set $f_{*,\text{cp}}(U)$ defined by

$$f_{*,\text{cp}}(U) \stackrel{\text{def}}{=} f_*(U) \cap (Y \setminus \text{Im}(f))$$

⁴¹Note that we have

$$f_*(U) = f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U),$$

as

$$\begin{aligned}
f_*(U) &= f_*(U) \cap Y \\
&= f_*(U) \cap (\text{Im}(f) \cup (Y \setminus \text{Im}(f))) \\
&= (f_*(U) \cap \text{Im}(f)) \cup (f_*(U) \cap (Y \setminus \text{Im}(f))) \\
&\stackrel{\text{def}}{=} f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U).
\end{aligned}$$

⁴²In terms of the meet computation of $f_*(U)$ of **Definition 4.6.3.1.4**, namely

$$f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),$$

we see that $f_{*,\text{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\text{cp}}$ corresponds to meets indexed over the empty set.

$$\begin{aligned}
&= Y \setminus \text{Im}(f) \\
&= \left\{ y \in Y \left| \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{array} \right. \right\} \\
&= \{y \in Y \mid f^{-1}(y) = \emptyset\}.
\end{aligned}$$

Example 4.6.3.1.6. Here are some examples of codirect images.

1. *Multiplication by Two.* Consider the function $f: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(n) \stackrel{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned}
f_{*,\text{im}}(U) &= f_!(U) \\
f_{*,\text{cp}}(U) &= \{\text{odd natural numbers}\}
\end{aligned}$$

for any $U \subset \mathbb{N}$. In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

2. *Parabolas.* Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) \stackrel{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{*,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned}
f_{*,\text{im}}([0, 1]) &= \{0\}, \\
f_{*,\text{im}}([-1, 1]) &= [0, 1], \\
f_{*,\text{im}}([1, 2]) &= \emptyset, \\
f_{*,\text{im}}([-2, -1] \cup [1, 2]) &= [1, 4].
\end{aligned}$$

3. *Circles.* Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x, y) \stackrel{\text{def}}{=} x^2 + y^2$$

for each $(x, y) \in \mathbb{R}^2$. We have

$$f_{*,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0, 0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{*,\text{im}}([-1, 1] \times [-1, 1]) &= [0, 1], \\ f_{*,\text{im}}(([-1, 1] \times [-1, 1]) \setminus [-1, 1] \times \{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.6.3.1.7. Let $f: X \rightarrow Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f_*(U) \subset f_*(V).$$

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \quad \mathcal{P}(X) \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{array} \mathcal{P}(Y),$$

witnessed by:

- (a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\rightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\rightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\rightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\rightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),\end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

- i. The following conditions are equivalent:
 - A. We have $f_!(U) \subset V$.
 - B. We have $U \subset f^{-1}(V)$.
- ii. The following conditions are equivalent:
 - A. We have $f^{-1}(U) \subset V$.
 - B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc}\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)\end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc}\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)\end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

5. *Interaction With Binary Unions.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cup & \subset & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cap & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U^c) = f_!(U)^c$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \supset & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U \Delta V) \subset f_*(U) \Delta f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ [-1, -2]_X \downarrow & \supset & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U) \cup f_*(V) &\rightarrow f_*(U \cup V), \\ \emptyset &\rightarrow f_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

II. *Preservation of Limits.* We have an equality of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_*(U) \cap f^{-1}(V), \\ f_*(X) &= Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of **Item 1** has a symmetric lax monoidal structure

$$(f_*, f_*^\otimes, f_{*|1}^\otimes): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U) \cup f_*(V) &\rightarrow f_*(U \cup V), \\ f_{*|1}^\otimes: \emptyset &\rightarrow f_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_*, f_*^\otimes, f_{*|1}^\otimes): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U \cap V) &\xrightarrow{=} f_*(U) \cap f_*(V), \\ f_{*|1}^\otimes: f_*(X) &\xrightarrow{=} Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Interaction With Coproducts.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \amalg g)_*(U \amalg V) = f_*(U) \amalg g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

15. *Interaction With Products.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_*(U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Relation to Direct Images.* We have

$$\begin{aligned} f_*(U) &= f_!(U^c)^c \\ &= Y \setminus f_!(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

17. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{*,\text{im}}(U) &= f_!(U), \\ f_{*,\text{cp}}(U) &= Y \setminus \text{Im}(f), \end{aligned}$$

and so

$$\begin{aligned} f_*(U) &= f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U) \\ &= f_!(U) \cup (Y \setminus \text{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

18. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{*,\text{im}}(U) &\subset f_!(U), \\ f_{*,\text{cp}}(U) &= \emptyset, \end{aligned}$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

Proof. **Item 1, Functoriality:** Omitted.

Item 2, Triple Adjointness: This follows from **Definition 4.6.1.1.4**, **Definition 4.6.2.1.2**, **Definition 4.6.3.1.4**, and Kan Extensions, ?? of ??.

Item 3, Interaction With Unions of Families of Subsets: We have

$$\begin{aligned} \bigcup_{V \in f_*(\mathcal{U})} V &= \bigcup_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.

Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned} \bigcap_{V \in f_*(\mathcal{U})} V &= \bigcap_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.

Item 5, Interaction With Binary Unions: We have

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_!(U^c)^c \cup f_!(V^c)^c \\ &= (f_!(U^c) \cap f_!(V^c))^c \\ &\subset (f_!(U^c \cap V^c))^c \\ &= f_!((U \cup V)^c)^c \\ &= f_*(U \cup V), \end{aligned}$$

where:

1. We have used **Item 16** for the first equality.
2. We have used **Item 2** of **Definition 4.3.11.1.2** for the second equality.
3. We have used **Item 6** of **Definition 4.6.1.1.5** for the third equality.
4. We have used **Item 2** of **Definition 4.3.11.1.2** for the fourth equality.
5. We have used **Item 16** for the last equality.

This finishes the proof.

Item 6, Interaction With Binary Intersections: This follows from **Item 11**.

Item 7, Interaction With Complements: Omitted.

Item 8, Interaction With Symmetric Differences: Omitted.

Item 9, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned} [f_!(U), f^!(V)]_X &\stackrel{\text{def}}{=} f_!(U)^c \cup f_*(V) \\ &= f_*(U^c) \cup f_*(V) \\ &\subset f_*(U^c \cup V) \\ &\stackrel{\text{def}}{=} f_*([U, V]_X), \end{aligned}$$

where we have used:

1. **Item 7** of **Definition 4.6.3.1.7** for the second equality.
2. **Item 5** of **Definition 4.6.3.1.7** for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**). This finishes the proof.

Item 10, Lax Preservation of Colimits: Omitted.

Item 11, Preservation of Limits: This follows from **Item 2** and ??, ?? of ??.

Item 12, Symmetric Lax Monoidality With Respect to Unions: This follows from **Item 10**.

Item 13, Symmetric Strict Monoidality With Respect to Intersections: This follows from **Item 11**.

Item 14, Interaction With Coproducts: Omitted.

Item 15, Interaction With Products: Omitted.

Item 16, Relation to Direct Images: We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication.* We claim that

$$f_*(U) \subset Y \setminus f_!(X \setminus U).$$

Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.

This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).

Thus $y \in Y \setminus f_!(X \setminus U)$.

- *The Second Implication.* We claim that

$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$

Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.

Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.

Thus $y \in f_*(U)$.

This finishes the proof of **Item 16**.

Item 17, *Interaction With Injections*: Omitted.

Item 18, *Interaction With Surjections*: Omitted. \square

Proposition 4.6.3.1.8. Let $f: X \rightarrow B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_X)_* = \text{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$(g \circ f)_* = g_* \circ f_*$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \\ & \searrow (g \circ f)_* & \downarrow g_* \\ & & \mathcal{P}(Z). \end{array}$$

Proof. **Item 1**, *Functionality I*: There is nothing to prove.

Item 2, *Functionality II*: This follows from **Item 1** of **Definition 4.6.3.1.7**.

Item 3, *Interaction With Identities*: This follows from **Definition 4.6.3.1.4** and Kan Extensions, ?? of ??.

Item 4, *Interaction With Composition*: This follows from **Definition 4.6.3.1.4** and Kan Extensions, ?? of ?? \square

4.6.4 A Six-Functor Formalism for Sets

Remark 4.6.4.1.1. The assignment $X \mapsto \mathcal{P}(X)$ together with the functors f_* , f^{-1} , and $f_!$ of [Item 1 of Definition 4.6.1.1.5](#), [Item 1 of Definition 4.6.2.1.3](#), and [Item 1 of Definition 4.6.3.1.7](#), and the functors

$$\begin{aligned} -_1 \cap -_2 &: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X), \\ [-_1, -_2]_X &: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X) \end{aligned}$$

of [Item 1 of Definition 4.3.9.1.2](#) and [Item 1 of Definition 4.4.7.1.3](#) satisfy several properties reminiscent of a six functor formalism in the sense of ??.

We collect these properties in [Definition 4.6.4.1.2](#) below.⁴³

Proposition 4.6.4.1.2. Let X be a set.

1. *The Beck–Chevalley Condition.* Let

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\text{pr}_2} & Y \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

be a pullback diagram in Sets. We have

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{pr}_1^{-1}} & \mathcal{P}(X \times_Z Y) \\ f_! \downarrow & & \downarrow (\text{pr}_2)_! \\ \mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y), \end{array} \quad \begin{array}{l} g^{-1} \circ f_! = (\text{pr}_2)_! \circ \text{pr}_1^{-1}, \\ f^{-1} \circ g_! = (\text{pr}_1)_! \circ \text{pr}_2^{-1}, \end{array} \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{pr}_2^{-1}} & \mathcal{P}(X \times_Z Y) \\ g_! \downarrow & & \downarrow (\text{pr}_1)_! \\ \mathcal{P}(Z) & \xrightarrow{f^{-1}} & \mathcal{P}(Y). \end{array}$$

⁴³See also [nLaz5].

2. *The Projection Formula I.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times \mathcal{P}(X) & \\
 \text{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
 f_! \times \text{id}_{\mathcal{P}(Y)} \searrow & & \searrow f_! \\
 \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\cap} & \mathcal{P}(Y),
 \end{array}$$

commutes, i.e. we have

$$f_!(U \cap f^{-1}(V)) = f_!(U) \cap V$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

3. *The Projection Formula II.* We have a natural transformation

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times \mathcal{P}(X) & \\
 \text{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
 f_* \times \text{id}_{\mathcal{P}(Y)} \searrow & \cup & \searrow f_* \\
 \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\cap} & \mathcal{P}(Y),
 \end{array}$$

with components

$$f_*(U) \cap V \subset f_*(U \cap f^{-1}(V))$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

4. *Strong Closed Monoidality.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\
 \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *The External Tensor Product.* We have an external tensor product

$$-_1 \boxtimes_{X \times Y} -_2: \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathcal{P}(X \times Y)$$

given by

$$\begin{aligned} U \boxtimes_{X \times Y} V &\stackrel{\text{def}}{=} \text{pr}_1^{-1}(U) \cap \text{pr}_2^{-1}(V) \\ &= \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}. \end{aligned}$$

This is the same map as the one in [Item 5](#) of [Definition 4.4.I.1.4](#). Moreover, the following conditions are satisfied:

(a) *Interaction With Direct Images.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f! \times g!} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \boxtimes_{X \times Y} \downarrow & & \downarrow \boxtimes_{X' \times Y'} \\ \mathcal{P}(X \times Y) & \xrightarrow{f! \times g!} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, i.e. we have

$$[f! \times g!](U \boxtimes_{X \times Y} V) = f!(U) \boxtimes_{X' \times Y'} g!(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(b) *Interaction With Inverse Images.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X') \times \mathcal{P}(Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X) \times \mathcal{P}(Y) \\ \boxtimes_{X' \times Y'} \downarrow & & \downarrow \boxtimes_{X \times Y} \\ \mathcal{P}(X' \times Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X \times Y) \end{array}$$

commutes, i.e. we have

$$[f^{-1} \times g^{-1}](U \boxtimes_{X' \times Y'} V) = f^{-1}(U) \boxtimes_{X \times Y} g^{-1}(V)$$

for each $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$.

- (c) *Interaction With Codirect Images.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \boxtimes_{X \times Y} \downarrow & & \downarrow \boxtimes_{X' \times Y'} \\ \mathcal{P}(X \times Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, i.e. we have

$$[f_* \times g_*](U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

- (d) *Interaction With Diagonals.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\boxtimes_{X \times X}} & \mathcal{P}(X \times X) \\ & \searrow \cap & \downarrow \Delta_X^{-1} \\ & & \mathcal{P}(X), \end{array}$$

i.e. we have

$$U \cap V = \Delta_X^{-1}(U \boxtimes_{X \times X} V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *The Dualisation Functor.* We have a functor

$$D_X: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X)$$

given by

$$D_X(U) \stackrel{\text{def}}{=} [U, \emptyset]_X \\ \stackrel{\text{def}}{=} U^c$$

for each $U \in \mathcal{P}(X)$, as in [Item 5](#) of [Definition 4.4.7.1.3](#), satisfying the following conditions:

(a) *Duality*. We have

$$D_X(D_X(U)) = U,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)} & \downarrow D_X \\ & & \mathcal{P}(X). \end{array}$$

(b) *Duality*. The diagram

$$\begin{array}{ccc} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} \xrightarrow{\cap^{\text{op}}} \mathcal{P}(X)^{\text{op}} & \\ \text{id}_{\mathcal{P}(X)^{\text{op}}} \times D_X \nearrow & & \searrow D_X \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-, -]_X} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

(c) *Interaction With Direct Images*. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(d) *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ D_Y \downarrow & & \downarrow D_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

(e) *Interaction With Codirect Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

Proof. **Item 1, The Beck–Chevalley Condition:** We have

$$\begin{aligned} [g^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} g^{-1}(f_!(U)) \\ &\stackrel{\text{def}}{=} \{y \in Y \mid g(y) \in f_!(U)\} \\ &= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right. \right\} \\ &= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \end{array} \right. \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } y = y \end{array} \right. \right\} \\
&= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } \text{pr}_2(x, y) = y \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} (\text{pr}_2)_!(\{(x, y) \in X \times_Z Y \mid x \in U\}) \\
&= (\text{pr}_2)_!(\{(x, y) \in X \times_Z Y \mid \text{pr}_1(x, y) \in U\}) \\
&\stackrel{\text{def}}{=} (\text{pr}_2)_!(\text{pr}_1^{-1}(U)) \\
&\stackrel{\text{def}}{=} [(\text{pr}_2)_! \circ \text{pr}_1^{-1}](U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$g^{-1} \circ f_! = (\text{pr}_2)_! \circ \text{pr}_1^{-1}.$$

For the second equality, we have

$$\begin{aligned}
[f^{-1} \circ g_!](U) &\stackrel{\text{def}}{=} f^{-1}(g_!(U)) \\
&\stackrel{\text{def}}{=} \{x \in X \mid f(x) \in g_!(V)\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } x = x \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } \text{pr}_1(x, y) = x \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} (\text{pr}_1)_!(\{(x, y) \in X \times_Z Y \mid y \in V\}) \\
&= (\text{pr}_1)_!(\{(x, y) \in X \times_Z Y \mid \text{pr}_2(x, y) \in V\}) \\
&\stackrel{\text{def}}{=} (\text{pr}_1)_!(\text{pr}_2^{-1}(V))
\end{aligned}$$

$$\stackrel{\text{def}}{=} [(\text{pr}_1)_! \circ \text{pr}_2^{-1}](V)$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$f^{-1} \circ g_! = (\text{pr}_1)_! \circ \text{pr}_2^{-1}.$$

This finishes the proof.

Item 2, The Projection Formula I: We claim that

$$f_!(U) \cap V \subset f_!(U \cap f^{-1}(V)).$$

Indeed, we have

$$\begin{aligned} f_!(U) \cap V &\subset f_!(U) \cap f_!(f^{-1}(V)) \\ &= f_!(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

1. *Item 2* of **Definition 4.6.1.1.5** for the inclusion.
2. *Item 6* of **Definition 4.6.1.1.5** for the equality.

Conversely, we claim that

$$f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V.$$

Indeed:

1. Let $y \in f_!(U \cap f^{-1}(V))$.
2. Since $y \in f_!(U \cap f^{-1}(V))$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.
3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.
4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.
5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.
6. But $y = f(x)$, so $y \in f_!(U) \cap V$.
7. Thus $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$.

This finishes the proof.

Item 3, The Projection Formula II: We have

$$\begin{aligned} f_*(U) \cap V &\subset f_*(U) \cap f_*(f^{-1}(V)) \\ &= f_*(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

1. *Item 2* of *Definition 4.6.3.1.7* for the inclusion.
2. *Item 6* of *Definition 4.6.3.1.7* for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (*Categories*, *Item 4* of *Definition 11.2.7.1.2*).

Item 4, Strong Closed Monoidality: This is a repetition of *Item 19* of *Definition 4.4.7.1.3* and is proved there.

Item 5, The External Tensor Product: We have

$$\begin{aligned} U \boxtimes_{X \times Y} V &\stackrel{\text{def}}{=} \text{pr}_1^{-1}(U) \cap \text{pr}_2^{-1}(V) \\ &\stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid \text{pr}_1(x, y) \in U\} \\ &\quad \cup \{(x, y) \in X \times Y \mid \text{pr}_2(x, y) \in V\} \\ &= \{(x, y) \in X \times Y \mid x \in U\} \\ &\quad \cup \{(x, y) \in X \times Y \mid y \in V\} \\ &= \{(x, y) \in X \times Y \mid x \in U \text{ and } y \in V\} \\ &\stackrel{\text{def}}{=} U \times V. \end{aligned}$$

Next, we claim that *Items 5a* to *5d* are indeed true:

1. *Proof of Item 5a:* This is a repetition of *Item 16* of *Definition 4.6.1.1.5* and is proved there.
2. *Proof of Item 5b:* This is a repetition of *Item 16* of *Definition 4.6.2.1.3* and is proved there.
3. *Proof of Item 5c:* This is a repetition of *Item 15* of *Definition 4.6.3.1.7* and is proved there.

4. *Proof of Item 5d:* We have

$$\begin{aligned}\Delta_X^{-1}(U \boxtimes_{X \times X} V) &\stackrel{\text{def}}{=} \{x \in X \mid (x, x) \in U \boxtimes_{X \times X} V\} \\ &= \{x \in X \mid (x, x) \in \{(u, v) \in X \times X \mid u \in U \text{ and } v \in V\}\} \\ &= U \cap V.\end{aligned}$$

This finishes the proof.

Item 6, The Dualisation Functor: This is a repetition of *Items 5* and *6* of *Definition 4.4.7.1.3* and is proved there. \square

Appendices

A Other Chapters

Preliminaries

1. *Introduction*
2. *A Guide to the Literature*

Sets

3. *Sets*
4. *Constructions With Sets*
5. *Monoidal Structures on the Category of Sets*
6. *Pointed Sets*
7. *Tensor Products of Pointed Sets*

Relations

8. *Relations*
9. *Constructions With Relations*

10. *Conditions on Relations*

Categories

11. *Categories*
12. *Presheaves and the Yoneda Lemma*

Monoidal Categories

13. *Constructions With Monoidal Categories*

Bicategories

14. *Types of Morphisms in Bicategories*

Extra Part

15. *Notes*

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