Types of Morphisms in Bicategories

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019H   In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

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019J **14.1   Monomorphisms in Bicategories**

019K **14.1.1   Representably Faithful Morphisms**

Let C be a bicategory.

019L

> **DEFINITION 14.1.1.1.1 ▶ REPRESENTABLY FAITHFUL MORPHISMS**
>
> A 1-morphism $f\colon A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$
>
> given by postcomposition by f is faithful.
>
> ---
> [1]*Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 14.1.1.1.3.

019M

> **REMARK 14.1.1.1.2 ▶ UNWINDING DEFINITION 14.1.1.1.1**
>
> In detail, f is representably faithful if, for all diagrams in C of the form
>
> $$X \underset{\psi}{\overset{\phi}{\underset{\alpha\Downarrow\Downarrow\beta}{\rightrightarrows}}} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

019N   **EXAMPLE 14.1.1.1.3 ▶ EXAMPLES OF REPRESENTABLY FAITHFUL MORPHISMS**

Here are some examples of representably faithful morphisms.

019P   1. *Representably Faithful Morphisms in* Cats$_2$. The representably faithful morphisms in Cats$_2$ are precisely the faithful functors; see Categories, Item 2 of Proposition 11.6.1.1.2.

019Q   2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Proposition 8.5.11.1.1.

019R   ## 14.1.2   Representably Full Morphisms

Let C be a bicategory.

019S   **DEFINITION 14.1.2.1.1 ▶ REPRESENTABLY FULL MORPHISMS**

A 1-morphism $f\colon A \to B$ of C is **representably full**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is full.

---

[1]*Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Example 14.1.2.1.3.

019T   **REMARK 14.1.2.1.2 ▶ UNWINDING DEFINITION 14.1.2.1.1**

In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Longrightarrow}} B$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \overset{\phi}{\underset{\psi}{\Downarrow \alpha}} A$$

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Downarrow \alpha}} A \xrightarrow{f} B \quad = \quad X \overset{f\circ\phi}{\underset{f\circ\psi}{\Downarrow \beta}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

019U   **EXAMPLE 14.1.2.1.3 ▶ EXAMPLES OF REPRESENTABLY FULL MORPHISMS**

Here are some examples of representably full morphisms.

019V   1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Categories, **??** of Proposition 11.6.2.1.2.

019W   2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 8.5.11.1.1.

019X   ## 14.1.3   Representably Fully Faithful Morphisms

Let C be a bicategory.

019Y   **DEFINITION 14.1.3.1.1 ▶ REPRESENTABLY FULLY FAITHFUL MORPHISMS**

A 1-morphism $f\colon A \to B$ of C is **representably fully faithful**[1] if the following equivalent conditions are satisfied:

019Z

1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1) and representably full (Definition 14.1.2.1.1).

01A0

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

---

[1]*Further Terminology*: Also called simply a **fully faithful morphism**, based on Item 1 of Example 14.1.3.1.3.

01A1

REMARK 14.1.3.1.2 ▶ UNWINDING REPRESENTABLY FULLY FAITHFUL MORPHISMS

In detail, f is representably fully faithful if the conditions in Remark 14.1.1.1.2 and Remark 14.1.2.1.2 hold:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} {}_{\alpha \Downarrow \Downarrow \beta} A \xrightarrow{f} B,$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rightarrow}} {}_{\beta \Downarrow} B$$

   of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\Rightarrow}} {}_{\alpha \Downarrow} A$$

of C such that we have an equality

$$
X \overset{\phi}{\underset{\psi}{\overset{\alpha\Downarrow}{\rightrightarrows}}} A \xrightarrow{f} B \quad = \quad X \overset{f\circ\phi}{\underset{f\circ\psi}{\overset{\beta\Downarrow}{\rightrightarrows}}} B
$$

of pasting diagrams in C, i.e. such that we have

$$
\beta = \mathrm{id}_f \star \alpha.
$$

01A2   **EXAMPLE 14.1.3.1.3 ▶ EXAMPLES OF REPRESENTABLY FULLY FAITHFUL MORPHISMS**

Here are some examples of representably fully faithful morphisms.

01A3   1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 6 of Proposition 11.6.3.1.2.

01A4   2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 8.5.11.1.1) with the representably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 8.5.11.1.1.

01A5   ## 14.1.4   Morphisms Representably Faithful on Cores

Let C be a bicategory.

01A6   **DEFINITION 14.1.4.1.1 ▶ MORPHISMS REPRESENTABLY FAITHFUL ON CORES**

A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$
f_*\colon \mathsf{Core}(\mathrm{Hom}_C(X, A)) \to \mathsf{Core}(\mathrm{Hom}_C(X, B))
$$

given by postcomposition by f is faithful.

01A7
> **REMARK 14.1.4.1.2 ▸ UNWINDING DEFINITION 14.1.4.1.1**
>
> In detail, f is representably faithful on cores if, for all diagrams in C of the form
>
> $$X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \| \| \beta}}} A \xrightarrow{f} B,$$
>
> if α and β are 2-isomorphisms and we have
>
> $$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
>
> then $\alpha = \beta$.

01A8
14.1.5   Morphisms Representably Full on Cores

Let C be a bicategory.

01A9
> **DEFINITION 14.1.5.1.1 ▸ MORPHISMS REPRESENTABLY FULL ON CORES**
>
> A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f_*\colon \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$
>
> given by postcomposition by f is full.

01AA
> **REMARK 14.1.5.1.2 ▸ UNWINDING DEFINITION 14.1.5.1.1**
>
> In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism
>
> $$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\rightrightarrows}} B$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha}}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha}}} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f\circ\psi}{\overset{f\circ\phi}{\underset{\Longrightarrow}{\beta}}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AB   14.1.6   Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

01AC   **DEFINITION 14.1.6.1.1 ▶ MORPHISMS REPRESENTABLY FULLY FAITHFUL ON CORES**

A 1-morphism $f\colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

01AD   1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).

01AE   2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is fully faithful.

01AF   **REMARK 14.1.6.1.2 ▶ UNWINDING DEFINITION 14.1.6.1.1**

In detail, f is representably fully faithful on cores if the conditions in Remark 14.1.4.1.2 and Remark 14.1.5.1.2 hold:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\Downarrow\beta}{\overset{\alpha\Downarrow}{\rightrightarrows}}}} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f\circ\psi}{\overset{f\circ\phi}{\underset{\beta\Downarrow}{\rightrightarrows}}} B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\alpha\Downarrow}{\rightrightarrows}}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha\Downarrow}{\rightrightarrows}}} A \xrightarrow{f} B \quad = \quad X \underset{f\circ\psi}{\overset{f\circ\phi}{\underset{\beta\Downarrow}{\rightrightarrows}}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AG   14.1.7   Representably Essentially Injective Morphisms

Let C be a bicategory.

01AH

DEFINITION 14.1.7.1.1 ▶ REPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

01AJ

REMARK 14.1.7.1.2 ▶ UNWINDING DEFINITION 14.1.7.1.1

In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

(★) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

01AK 14.1.8 Representably Conservative Morphisms

Let C be a bicategory.

01AL

DEFINITION 14.1.8.1.1 ▶ REPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

01AM

REMARK 14.1.8.1.2 ▶ UNWINDING DEFINITION 14.1.8.1.1

In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\Downarrow \alpha}{\rightrightarrows}}} A$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha \colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \begin{array}{c} f \circ \phi \\ \overset{\parallel}{\underset{\mathrm{id}_f \star \alpha}{\Downarrow}} \\ f \circ \psi \end{array} B$$

is a 2-isomorphism, then so is α.

01AN ## 14.1.9   Strict Monomorphisms

Let C be a bicategory.

01AP **DEFINITION 14.1.9.1.1 ▶ STRICT MONOMORPHISMS**

A 1-morphism $f \colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$

is injective.

01AQ **REMARK 14.1.9.1.2 ▶ UNWINDING DEFINITION 14.1.9.1.1**

In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \overset{f}{\longrightarrow} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

01AR

EXAMPLE 14.1.9.1.3 ▶ EXAMPLES OF STRICT MONOMORPHISMS

Here are some examples of strict monomorphisms.

01AS

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Proposition 11.7.2.1.2.

01AT

2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, Proposition 8.5.10.1.1.

01AU   ## 14.1.10   Pseudomonic Morphisms

Let C be a bicategory.

01AV

DEFINITION 14.1.10.1.1 ▶ PSEUDOMONIC MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

01AW

REMARK 14.1.10.1.2 ▶ UNWINDING DEFINITION 14.1.10.1.1

In detail, a 1-morphism $f\colon A \to B$ of C is pseudomonic if it satisfies the following conditions:

01AX

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha\Downarrow\Downarrow\beta}{\longrightarrow}}} A \xrightarrow{\;f\;} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01AY

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\underset{\beta \Downarrow}{\longrightarrow}}} B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow}{\longrightarrow}}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow}{\longrightarrow}}} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\underset{\beta \Downarrow}{\longrightarrow}}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AZ

PROPOSITION 14.1.10.1.3 ▸ PROPERTIES OF PSEUDOMONIC MORPHISMS

Let $f \colon A \to B$ be a 1-morphism of C.

01B0

1. *Characterisations.* The following conditions are equivalent:

01B1

    (a) The morphism f is pseudomonic.

01B2

    (b) The morphism f is representably full on cores and representably faithful.

01B3   (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A,$$

in C up to equivalence.

01B4   2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B,$$

in C up to equivalence.

PROOF 14.1.10.1.4 ▶ PROOF OF PROPOSITION 14.1.10.1.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted.

01B5 ## 14.2   Epimorphisms in Bicategories

01B6 ### 14.2.1   Corepresentably Faithful Morphisms

Let C be a bicategory.

01B7

A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is faithful.

01B8   **REMARK 14.2.1.1.2 ▶ UNWINDING DEFINITION 14.2.1.1.1**

In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \ \underset{\psi}{\overset{\phi}{\underset{\alpha\Downarrow\ \Downarrow\beta}{\rightrightarrows}}}\ X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01B9   **EXAMPLE 14.2.1.1.3 ▶ EXAMPLES OF COREPRESENTABLY FAITHFUL MORPHISMS**

Here are some examples of corepresentably faithful morphisms.

01BA   1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 5 of Proposition 11.6.1.1.2.

01BB   2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Relations, Item 1 of Proposition 8.5.13.1.1.

01BC   ## 14.2.2   Corepresentably Full Morphisms

Let C be a bicategory.

01BD

DEFINITION 14.2.2.1.1 ▶ COREPRESENTABLY FULL MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is full.

01BE

REMARK 14.2.2.1.2 ▶ UNWINDING DEFINITION 14.2.2.1.1

In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \quad \beta\Downarrow$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad \alpha\Downarrow$$

of C such that we have an equality

$$A \overset{f}{\longrightarrow} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad \alpha\Downarrow \qquad = \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \quad \beta\Downarrow$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01BF

EXAMPLE 14.2.2.1.3 ▶ EXAMPLES OF COREPRESENTABLY FULL MORPHISMS

Here are some examples of corepresentably full morphisms.

01BG

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 7 of Proposi-

01BH   tion 11.6.2.1.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 8.5.13.1.1.

01BJ   ## 14.2.3   Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

01BK   **DEFINITION 14.2.3.1.1 ▶ COREPRESENTABLY FULLY FAITHFUL MORPHISMS**

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[1] if the following equivalent conditions are satisfied:

01BL   1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).

01BM   2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is fully faithful.

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[1]*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

01BN   **REMARK 14.2.3.1.2 ▶ UNWINDING DEFINITION 14.2.3.1.1**

In detail, f is corepresentably fully faithful if the conditions in Remark 14.2.1.1.2 and Remark 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}}[\alpha \Downarrow \Downarrow \beta] X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}}_{\beta \Downarrow} X$$

of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}}_{\beta \Downarrow} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f \,.$$

01BP   **EXAMPLE 14.2.3.1.3 ▶ EXAMPLES OF COREPRESENTABLY FULLY FAITHFUL MORPHISMS**

Here are some examples of corepresentably fully faithful morphisms.

01BQ   1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 10 of Proposition 11.6.3.1.2.

01BR   2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 8.5.13.1.1) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 8.5.13.1.1.

01BS   ## 14.2.4   Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

01BT   **DEFINITION 14.2.4.1.1 ▶ MORPHISMS COREPRESENTABLY FAITHFUL ON CORES**

A 1-morphism $f : A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is faithful.

01BU   **REMARK 14.2.4.1.2 ▶ UNWINDING DEFINITION 14.2.4.1.1**

In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow \Downarrow \beta}{\longrightarrow}}} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01BV   ## 14.2.5   Morphisms Corepresentably Full on Cores

Let C be a bicategory.

01BW   **DEFINITION 14.2.5.1.1 ▶ MORPHISMS COREPRESENTABLY FULL ON CORES**

A 1-morphism $f : A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is full.

01BX   **REMARK 14.2.5.1.2 ▶ UNWINDING DEFINITION 14.2.5.1.1**

In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \quad \beta\!\Downarrow$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad \alpha\!\Downarrow$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad \alpha\!\Downarrow \qquad = \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \quad \beta\!\Downarrow$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

01BY   **14.2.6   Morphisms Corepresentably Fully Faithful on Cores**

Let C be a bicategory.

01BZ   **DEFINITION 14.2.6.1.1 ▶ MORPHISMS COREPRESENTABLY FULLY FAITHFUL ON CORES**

A 1-morphism $f \colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

01C0   1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

01C1

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \longrightarrow \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is fully faithful.

01C2

REMARK 14.2.6.1.2 ▶ UNWINDING DEFINITION 14.2.6.1.1

In detail, f is corepresentably fully faithful on cores if the conditions in
Remark 14.2.4.1.2 and Remark 14.2.5.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

with $\alpha \Downarrow \Downarrow \beta$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

with $\beta \Downarrow$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

with $\alpha \Downarrow$

of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

with $\alpha \Downarrow$ and $\beta \Downarrow$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

01C3 ## 14.2.7   Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

01C4 **DEFINITION 14.2.7.1.1 ▶ COREPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS**

A 1-morphism $f\colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

01C5 **REMARK 14.2.7.1.2 ▶ UNWINDING DEFINITION 14.2.7.1.1**

In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of C, the following condition is satisfied:

(★) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

01C6 ## 14.2.8   Corepresentably Conservative Morphisms

Let C be a bicategory.

01C7 **DEFINITION 14.2.8.1.1 ▶ COREPRESENTABLY CONSERVATIVE MORPHISMS**

A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is conservative.

01C8 **REMARK 14.2.8.1.2 ▶ UNWINDING DEFINITION 14.2.8.1.1**

In detail, f is **corepresentably conservative** if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ and each 2-morphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\underset{\Downarrow \alpha}{\rightrightarrows}}} X$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\underset{\Downarrow}{\overset{\|}{\rightrightarrows}}}} X \quad {}_{\alpha \star \mathrm{id}_f}$$

is a 2-isomorphism, then so is α.

01C9   14.2.9   Strict Epimorphisms

Let C be a bicategory.

01CA   **DEFINITION 14.2.9.1.1 ▶ STRICT EPIMORPHISMS**

A 1-morphism $f \colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects
$$f_* \colon \mathrm{Obj}(\mathrm{Hom}_C(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_C(A, X))$$
is injective.

01CB   **REMARK 14.2.9.1.2 ▶ UNWINDING DEFINITION 14.2.9.1.1**

In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \overset{\phi}{\underset{\psi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

01CC **EXAMPLE 14.2.9.1.3 ▶ EXAMPLES OF STRICT EPIMORPHISMS**

Here are some examples of strict epimorphisms.

01CD 1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Proposition 11.7.3.1.2.

01CE 2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, Proposition 8.5.12.1.1.

01CF ## 14.2.10 Pseudoepic Morphisms

Let C be a bicategory.

01CG **DEFINITION 14.2.10.1.1 ▶ PSEUDOEPIC MORPHISMS**

A 1-morphism $f : A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

01CH **REMARK 14.2.10.1.2 ▶ UNWINDING DEFINITION 14.2.10.1.1**

In detail, a 1-morphism $f : A \to B$ of C is pseudoepic if it satisfies the following conditions:

01CJ 1. For all diagrams in C of the form

$$A \xrightarrow{f} B \overset{\phi}{\underset{\psi}{\underset{\alpha \Downarrow\; \Downarrow \beta}{\rightrightarrows}}} X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \xRightarrow{\sim} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow_{\beta}}} X$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \xRightarrow{\sim} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow_{\alpha}}} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow_{\alpha}}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow_{\beta}}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

PROPOSITION 14.2.10.1.3 ▶ PROPERTIES OF PSEUDOEPIC MORPHISMS

Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.

   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

01CQ

(c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B,$$

$$
\begin{array}{ccc}
B & \xleftarrow{\ \text{id}_B\ } & B \\
{\scriptstyle \text{id}_B}\big\uparrow & \nearrow\!\!\!\!\!\dashrightarrow & \big\uparrow{\scriptstyle F} \\
B & \xleftarrow{\ F\ } & A
\end{array}
$$

in C up to equivalence.

PROOF 14.2.10.1.4 ► PROOF OF PROPOSITION 14.2.10.1.3

Item 1: Characterisations

Omitted.

Appendices

A Other Chapters

References

[Adá+01]   Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 18).