Types of Morphisms in Bicategories

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019H In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

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019J 14.1 Monomorphisms in Bicategories

019K 14.1.1 Representably Faithful Morphisms

Let C be a bicategory.

019L **Definition 14.1.1.1.1.** A 1-morphism $f: A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

019M **Remark 14.1.1.1.2.** In detail, f is representably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of

019N **Example 14.1.1.1.3.** Here are some examples of representably faithful morphisms.

019P 1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Categories, Item 2 of Definition 11.6.1.1.2.

019Q 2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Definition 8.5.11.1.1.

019R 14.1.2 Representably Full Morphisms

Let C be a bicategory.

019S **Definition 14.1.2.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably full**[2] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is full.

019T **Remark 14.1.2.1.2.** In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[r, "f"] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

Definition 14.1.1.1.3.

[2] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

019U **Example 14.1.2.1.3.** Here are some examples of representably full morphisms.

019V 1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Categories, **??** of Definition 11.6.2.1.2.

019W 2. *Representably Full Morphisms in* Rel. The representably full morphisms in Rel are characterised in Relations, Item 2 of Definition 8.5.11.1.1.

019X 14.1.3 Representably Fully Faithful Morphisms

Let C be a bicategory.

019Y **Definition 14.1.3.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

019Z 1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1) and representably full (Definition 14.1.2.1.1).

01A0 2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

01A1 **Remark 14.1.3.1.2.** In detail, f is representably fully faithful if the conditions in Definition 14.1.1.1.2 and Definition 14.1.2.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Definition 14.1.2.1.3.

[3] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta \colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01A2 **Example 14.1.3.1.3.** Here are some examples of representably fully faithful morphisms.

01A3 1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 6 of Definition 11.6.3.1.2.

01A4 2. *Representably Fully Faithful Morphisms in* $\mathbf{Rel}$. The representably fully faithful morphisms of $\mathbf{Rel}$ coincide (Relations, Item 3 of Definition 8.5.11.1.1) with the representably full morphisms in $\mathbf{Rel}$, which are characterised in Relations, Item 2 of Definition 8.5.11.1.1.

01A5 14.1.4 Morphisms Representably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

01A6 **Definition 14.1.4.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X,A)) \to \mathsf{Core}(\mathsf{Hom}_C(X,B))$$

given by postcomposition by f is faithful.

01A7 **Remark 14.1.4.1.2.** In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01A8 14.1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

01A9 **Definition 14.1.5.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X,A)) \to \mathsf{Core}(\mathsf{Hom}_C(X,B))$$

given by postcomposition by f is full.

01AA **Remark 14.1.5.1.2.** In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

Definition 14.1.3.1.3.

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\beta"'] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AB 14.1.6 Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

01AC **Definition 14.1.6.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

01AD 1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).

01AE 2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is fully faithful.

01AF **Remark 14.1.6.1.2.** In detail, f is representably fully faithful on cores if the conditions in Definition 14.1.4.1.2 and Definition 14.1.5.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, Rightarrow, shift right=2, "\alpha"'] \arrow[from=U, to=D, Rightarrow, shift left=2, "\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[r, "f"] \arrow[from=U, to=D, Rightarrow, "\alpha"'] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AG 14.1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

01AH **Definition 14.1.7.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

01AJ **Remark 14.1.7.1.2.** In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

01AK 14.1.8 Representably Conservative Morphisms

Let C be a bicategory.

01AL **Definition 14.1.8.1.1.** A 1-morphism $f: A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

01AM **Remark 14.1.8.1.2.** In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ and each 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow"] & A \end{tikzcd}$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha: f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[rr, bend left, "f \circ \phi"] \arrow[rr, bend right, "f \circ \psi"'] \arrow[rr, phantom, "\Downarrow \mathrm{id}_f \star \alpha"] & & B \end{tikzcd}$$

is a 2-isomorphism, then so is α.

01AN 14.1.9 Strict Monomorphisms

Let C be a bicategory.

01AP **Definition 14.1.9.1.1.** A 1-morphism $f: A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*: \mathrm{Obj}(\mathsf{Hom}_C(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_C(X, B))$$

is injective.

01AQ **Remark 14.1.9.1.2.** In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

01AR **Example 14.1.9.1.3.** Here are some examples of strict monomorphisms.

01AS 1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Definition 11.7.2.1.2.

01AT 2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, Definition 8.5.10.1.1.

01AU 14.1.10 Pseudomonic Morphisms

Let C be a bicategory.

01AV **Definition 14.1.10.1.1.** A 1-morphism $f : A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

01AW **Remark 14.1.10.1.2.** In detail, a 1-morphism $f : A \to B$ of C is pseudomonic if it satisfies the following conditions:

01AX 1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01AY 2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AZ **Proposition 14.1.10.1.3.** Let $f : A \to B$ be a 1-morphism of $\mathcal{C}$.

01B0 1. *Characterisations.* The following conditions are equivalent:

01B1 (a) The morphism f is pseudomonic.

01B2 (b) The morphism f is representably full on cores and representably faithful.

01B3 (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \\ A \arrow[r, "F"'] \arrow[ur, Leftrightarrow, dashed] & B \end{tikzcd}$$

in $\mathcal{C}$ up to equivalence.

01B4 2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1}\pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1}\pitchfork A \arrow[d, "\mathbb{1}\pitchfork F"] \arrow[dl, phantom, "\Leftrightarrow"] \\ B \arrow[r, hook] & \mathbb{1}\pitchfork B \end{tikzcd}$$

   in C up to equivalence.

Proof. *Item 1*, *Characterisations*: Omitted.
Item 2, *Interaction With Cotensors*: Omitted. □

01B5 14.2 Epimorphisms in Bicategories

01B6 14.2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

01B7 **Definition 14.2.1.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is faithful.

01B8 **Remark 14.2.1.1.2.** In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01B9 **Example 14.2.1.1.3.** Here are some examples of corepresentably faithful morphisms.

01BA 1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 5 of Definition 11.6.1.1.2.

01BB 2. *Corepresentably Faithful Morphisms in* $\mathbf{Rel}$. Every morphism of $\mathbf{Rel}$ is corepresentably faithful; see Relations, Item 1 of Definition 8.5.13.1.1.

01BC 14.2.2 Corepresentably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

01BD **Definition 14.2.2.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

01BE **Remark 14.2.2.1.2.** In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"{name=U}] \arrow[r, bend right, "\psi\circ f"'{name=D}] \arrow[from=U, to=D, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01BF **Example 14.2.2.1.3.** Here are some examples of corepresentably full morphisms.

01BG 1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 7 of Definition 11.6.2.1.2.

01BH 2. *Corepresentably Full Morphisms in* $\mathbf{Rel}$. The corepresentably full morphisms in $\mathbf{Rel}$ are characterised in Relations, Item 2 of Definition 8.5.13.1.1.

01BJ 14.2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

01BK **Definition 14.2.3.1.1.** A 1-morphism $f\colon A\to B$ of C is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

01BL 1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).

01BM 2. For each $X\in\mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B,X)\to\mathsf{Hom}_C(A,X)$$

given by precomposition by f is fully faithful.

01BN **Remark 14.2.3.1.2.** In detail, f is corepresentably fully faithful if the conditions in Definition 14.2.1.1.2 and Definition 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha\star\mathrm{id}_f=\beta\star\mathrm{id}_f,$$

then $\alpha=\beta$.

[4] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

01BP **Example 14.2.3.1.3.** Here are some examples of corepresentably fully faithful morphisms.

01BQ 1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 10 of Definition 11.6.3.1.2.

01BR 2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Definition 8.5.13.1.1) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Definition 8.5.13.1.1.

01BS 14.2.4 Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

01BT **Definition 14.2.4.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is faithful.

01BU **Remark 14.2.4.1.2.** In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01BV 14.2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

01BW **Definition 14.2.5.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is full.

01BX **Remark 14.2.5.1.2.** In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

01BY 14.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

01BZ **Definition 14.2.6.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

01C0 1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

01C1 2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is fully faithful.

01C2 **Remark 14.2.6.1.2.** In detail, f is corepresentably fully faithful on cores if the conditions in Definition 14.2.4.1.2 and Definition 14.2.5.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

01C3 14.2.7 Corepresentably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

01C4 **Definition 14.2.7.1.1.** A 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is essentially injective.

01C5 **Remark 14.2.7.1.2.** In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ of $\mathcal{C}$, the following condition is satisfied:

(★) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

01C6 14.2.8 Corepresentably Conservative Morphisms

Let C be a bicategory.

01C7 **Definition 14.2.8.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B,X) \to \mathsf{Hom}_C(A,X)$$

given by precomposition by f is conservative.

01C8 **Remark 14.2.8.1.2.** In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\Downarrow \alpha\star\mathrm{id}_f"] & X \end{tikzcd}$$

is a 2-isomorphism, then so is α.

01C9 14.2.9 Strict Epimorphisms

Let C be a bicategory.

01CA **Definition 14.2.9.1.1.** A 1-morphism $f\colon A \to B$ is a **strict epimorphism in C** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B,X) \to \mathsf{Hom}_C(A,X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathsf{Obj}(\mathsf{Hom}_C(B,X)) \to \mathsf{Obj}(\mathsf{Hom}_C(A,X))$$

is injective.

01CB **Remark 14.2.9.1.2.** In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

01CC **Example 14.2.9.1.3.** Here are some examples of strict epimorphisms.

01CD 1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Definition 11.7.3.1.2.

01CE 2. *Strict Epimorphisms in* $\mathbf{Rel}$. The strict epimorphisms in $\mathbf{Rel}$ are characterised in Relations, Definition 8.5.12.1.1.

01CF 14.2.10 Pseudoepic Morphisms

Let C be a bicategory.

01CG **Definition 14.2.10.1.1.** A 1-morphism $f : A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

01CH **Remark 14.2.10.1.2.** In detail, a 1-morphism $f : A \to B$ of C is pseudoepic if it satisfies the following conditions:

01CJ 1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01CK 2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01CL **Proposition 14.2.10.1.3.** Let $f\colon A \to B$ be a 1-morphism of $\mathcal{C}$.

01CM 1. *Characterisations.* The following conditions are equivalent:

01CN (a) The morphism f is pseudoepic.

01CP (b) The morphism f is corepresentably full on cores and corepresentably faithful.

01CQ (c) We have an isocococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \\ B \arrow[u, "\mathrm{id}_B"] & A \arrow[l, "F"] \arrow[u, "F"'] \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

in $\mathcal{C}$ up to equivalence.

Proof. *Item 1, Characterisations*: Omitted. □

Appendices

A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

References

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 14).