Constructions With Monoidal Categories

The Clowder Project Authors

July 29, 2025

O1UF This chapter contains some material on constructions with monoidal categories.

Contents

13.1	Moduli Categories of Monoidal Structures	2
	13.1.1 The Moduli Category of Monoidal Structures on a Cate-	
gory		2
	13.1.2 The Moduli Category of Braided Monoidal Structures on	
a Cat	egory	16
	13.1.3 The Moduli Category of Symmetric Monoidal Structures	
on a	Category	16
13.2	Moduli Categories of Closed Monoidal Structures	16
13.3	Moduli Categories of Refinements of Monoidal Structures	16
	13.3.1 The Moduli Category of Braided Refinements of a Monoidal	
Struc	cture	16
A	Other Chapters	17

01UG 13.1 Moduli Categories of Monoidal Structures

01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let *C* be a category.

Definition 13.1.1.1.1. The **moduli category of monoidal structures on** C is the category $\mathcal{M}_{\mathbb{B}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathrm{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, egin{pmatrix} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow \mathsf{MonCats} \\ & & \downarrow & & \downarrow \\ & \mathsf{pt} & \xrightarrow{[C]} & \mathsf{Cats}. \end{pmatrix}$$

- 01UK Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:
 - Objects. The objects of $\mathcal{M}_{\mathbb{B}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
 - *Morphisms*. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C : C \to C$ of C.

• *Identities.* For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{B}_{1}}(C)}\stackrel{\operatorname{def}}{=}(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}),$$

where $\left(id_{C}^{\otimes}, id_{1|C}^{\otimes}\right)$ is the identity monoidal functor of C of ??.

• *Composition*. For each $M, N, P \in \mathsf{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

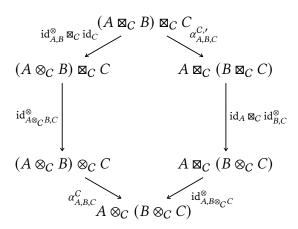
$$\begin{split} \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,N) &\to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,P) \\ & \text{of } \mathcal{M}_{\mathbb{B}_1}(C) \text{ at } (M,N,P) \text{ is defined by} \\ & \left(\operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{\mathbb{B}|C}^{\otimes,\prime} \right) \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \left(\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{B}|C}^{\otimes} \right) \overset{\text{def}}{=} \left(\operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{B}|C}^{\otimes,\prime} \circ \operatorname{id}_{\mathbb{B}|C}^{\otimes} \right). \end{split}$$

- **Remark 13.1.1.13.** In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:
- 01UM 1. Naturality. For each pair $f:A\to X$ and $g:B\to Y$ of morphisms of C, the diagram

$$\begin{array}{cccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \operatorname{id}_{A,B}^{\otimes} & & & & & \operatorname{id}_{X,Y}^{\otimes} \\ A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y \end{array}$$

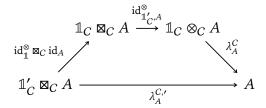
commutes.

Olum 2. *Monoidality*. For each $A, B, C \in Obj(C)$, the diagram



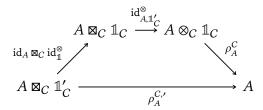
commutes.

01UP 3. Left Monoidal Unity. For each $A \in Obj(C)$, the diagram



commutes.

01UQ 4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram



commutes.

Olum Proposition 13.1.1.4. Let C be a category.

01US

1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

01UT (a) The diagram

commutes.

01UU (b) The diagram

commutes.

01WB 2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

01WC (a) The diagram

commutes.

01WD (b) The diagram

commutes.

01WE (c) The diagram

commutes.

01WF (d) The diagram

commutes.

01UV 3. Mixed Associators. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes} : (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \otimes_{C} B) \boxtimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C) \\ \operatorname{id}_{A \otimes_{C} B,C}^{\otimes} & & & & \operatorname{id}_{A,C}^{\otimes} \\ (A \otimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C) \end{array}$$

and

$$\begin{array}{c|c} (A \boxtimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_C (B \boxtimes_C C) \\ \downarrow^{\operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C} & & \downarrow^{\operatorname{id}_{A,B\boxtimes_C} C} \\ (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UX (b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_C (B \otimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & \operatorname{id}_{A,B \otimes_C C}^{\otimes} \\ & (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \qquad \operatorname{id}_{A\boxtimes_{C}} \operatorname{id}_{B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C} & & & & \operatorname{id}_{A,C}^{\otimes} \\ & (A \otimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

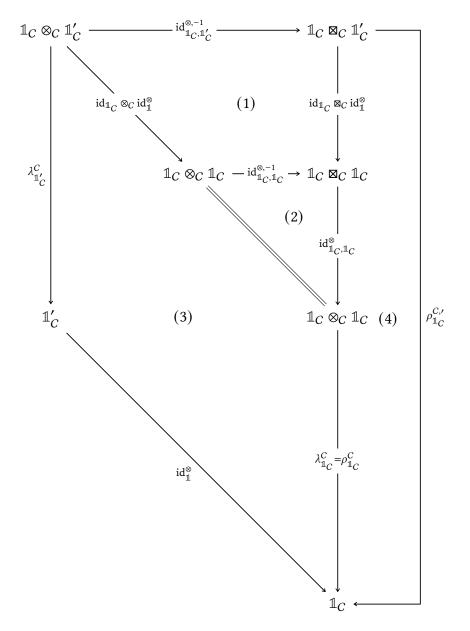
Proof. Item 1, Extra Monoidality Conditions: We claim that *Items 1a* and 1b are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of id^{\otimes} with respect to the morphisms $\mathrm{id}_{A,B}^{\otimes}$ and id_{C} .
- 2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and $id_{B,C}^{\otimes}$.

This finishes the proof.

Item 2, Extra Monoidal Unity Constraints: We claim that *Items 2a* and **2b** are indeed true:





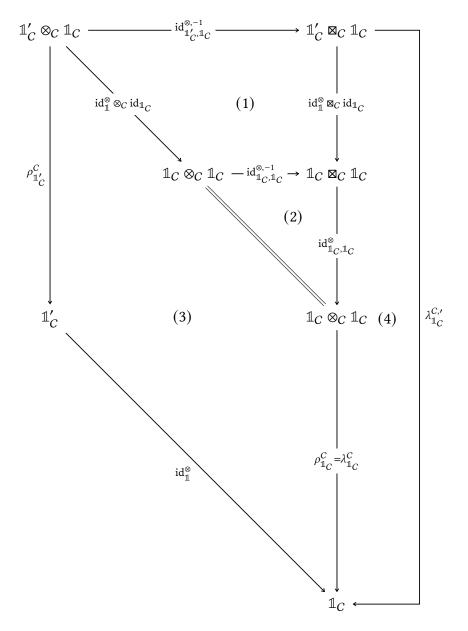
whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$;

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:composition}$;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$;

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

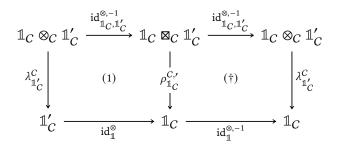
Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d*: Indeed, consider the diagram



Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

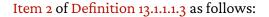
$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

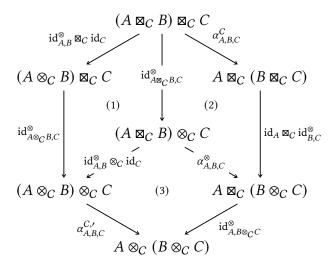
commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

01UZ 1. Proof of Item 3a: We may partition the monoidality diagram for id^{\otimes} of



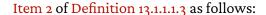


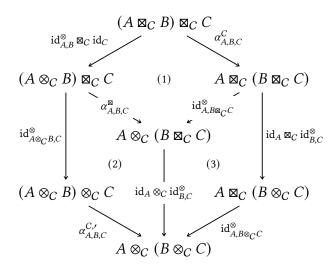
Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01V0 2. Proof of Item 3b: We may partition the monoidality diagram for id^{\otimes} of



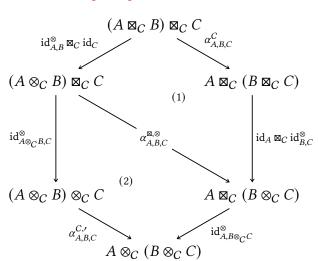


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01V1 3. Proof of Item 3c: We may partition the monoidality diagram for id^{\otimes} of



Item 2 of Definition 13.1.1.1.3 as follows:

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- 01V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes