Conditions on Relations

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This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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10.1 Functional and Total Relations

10.1.1 Functional Relations

Let A and B be sets.

Definition 10.1.1.1.1. A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 10.1.1.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is functional.

   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. *Item 1*, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b*: Let $(b, b') \in B \times B$. We need to show that
  $$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$
  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\implies$ *Item 1a*: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  – Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.

  – Since $R \diamond R^{\dagger} \subset \chi_B$, we have
    $$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$
    and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^{\dagger}\right](b, b') = \mathrm{true}$, and thus $\chi_B(b, b') = \mathrm{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

10.1.2 Total Relations

Let A and B be sets.

Definition 10.1.2.1.1. A relation $R \colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 10.1.2.1.2. Let $R \colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is total.

   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. *Item 1*, *Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

  $$\chi_A(a, a') \preceq_{\{\mathrm{t},\mathrm{f}\}} \left[R^\dagger \diamond R\right](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\Longrightarrow$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

  $$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. $\square$

10.2 Reflexive Relations

10.2.1 Foundations

Let A be a set.

Definition 10.2.1.1.1. A **reflexive relation** is equivalently:[1]

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A,A), \chi_A)$.

Remark 10.2.1.1.2. In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 10.2.1.1.3. Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathsf{refl}}(A,A)$ of $\mathrm{Rel}(A,A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{refl}}(A,A)$ of $\mathbf{Rel}(A,A)$ spanned by the reflexive relations.

Proposition 10.2.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

10.2.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 10.2.2.1.1. The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathsf{refl}}$[2] satisfying the following universal property:[3]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathsf{refl}} \subset \sim_S$.

Construction 10.2.2.1.2. Concretely, $\sim_R^{\mathsf{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A,A), \chi_A)$[4],

[2] *Further Notation:* Also written R^{refl}.
[3] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.
[4] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$.

being given by

$$
\begin{aligned}
R^{\mathrm{refl}} &\stackrel{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\
&= R \cup \Delta_A \\
&= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}.
\end{aligned}
$$

Proof. Clear. □

Proposition 10.2.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$
\left((-)^{\mathrm{refl}} \dashv 忘\right)\colon\quad
\begin{tikzcd}
\mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"]
\end{tikzcd}
$$

   witnessed by a bijection of sets

$$
\mathbf{Rel}^{\mathrm{refl}}\left(R^{\mathrm{refl}}, S\right) \cong \mathbf{Rel}(R,S),
$$

   natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{refl}}(A,A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

3. *Idempotency.* We have

$$
\left(R^{\mathrm{refl}}\right)^{\mathrm{refl}} = R^{\mathrm{refl}}.
$$

4. *Interaction With Inverses.* We have

$$
\left(R^{\dagger}\right)^{\mathrm{refl}} = \left(R^{\mathrm{refl}}\right)^{\dagger},
\qquad
\begin{tikzcd}
\mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\
\mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A).
\end{tikzcd}
$$

5. *Interaction With Composition.* We have

$$
(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}},
\qquad
\begin{tikzcd}
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A).
\end{tikzcd}
$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 10.2.2.1.1.
Item 2, The Reflexive Closure of a Reflexive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.2.1.1.4.
□

10.3 Symmetric Relations

10.3.1 Foundations

Let A be a set.

Definition 10.3.1.1.1. A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

Remark 10.3.1.1.2. In detail, a relation R is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

Definition 10.3.1.1.3. Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathsf{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 10.3.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Clear. □

10.3.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 10.3.2.1.1. The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5] satisfying the following universal property:[6]

($\star$) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

Construction 10.3.2.1.2. Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof. Clear. □

Proposition 10.3.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

[5] *Further Notation:* Also written R^{symm}.

[6] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

4. *Interaction With Inverses*. We have

$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition*. We have

$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 10.3.2.1.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.3.1.1.4.
□

10.4 Transitive Relations

10.4.1 Foundations

Let A be a set.

Definition 10.4.1.1.1. A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

[7] Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

Remark 10.4.1.1.2. In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 10.4.1.1.3. Let A be a set.

1. The **set of transitive relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.
2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

Proposition 10.4.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: See [MSE 2096272].[8] □

10.4.2 The Transitive Closure of a Relation

Let R be a relation on A.

[8] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
  - There is some $b \in A$ such that:
    - $a \sim_R b$;
    - $b \sim_S c$;
  - There is some $d \in A$ such that:
    - $c \sim_R d$;
    - $d \sim_S e$.

Definition 10.4.2.1.1. The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[9] satisfying the following universal property:[10]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

Construction 10.4.2.1.2. Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[11], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \,\middle|\, \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

Proof. Clear. □

Proposition 10.4.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}\left(R^{\mathrm{trans}}, S\right) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{trans}}(A, A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

[9] *Further Notation:* Also written R^{trans}.

[10] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\color{red}\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 10.4.2.1.1.
Item 2, The Transitive Closure of a Transitive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: We have

$$\begin{aligned} \left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} \\ &= \bigcup_{n=1}^{\infty} \left(R^{\diamond n}\right)^{\dagger} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} \\ &= \left(R^{\mathrm{trans}}\right)^{\dagger}, \end{aligned}$$

where we have used, respectively:

- Definition 10.4.2.1.2.
- Constructions With Relations, **??** of **??**.
- Constructions With Relations, **??** of Definition 9.2.3.1.2.

- Definition 10.4.2.1.2.

This finishes the proof.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.4.1.1.4.

□

10.5 Equivalence Relations

10.5.1 Foundations

Let A be a set.

Definition 10.5.1.1.1. A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

Example 10.5.1.1.2. The **kernel of a function** $f\colon A \longrightarrow B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

Definition 10.5.1.1.3. Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

10.5.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 10.5.2.1.1. The **equivalence closure**[14] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[15] satisfying the following universal property:[16]

[12] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Constructions With Relations, **??** of **??**.

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written R^{eq}.

[16] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

Construction 10.5.2.1.2. Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left((R^{\mathrm{symm}})^{\mathrm{trans}}\right)^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad\ \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive.
2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

Proposition 10.5.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof. *Item 1*, *Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 10.5.2.1.1.
Item 2, *The Equivalence Closure of an Equivalence Relation*: Clear.
Item 3, *Idempotency*: This follows from Item 2. □

10.6 Quotients by Equivalence Relations

10.6.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 10.6.1.1.1. The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)} \end{aligned}$$

10.6.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 10.6.2.1.1. The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 10.6.2.1.2. The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.

- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

$$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[17]

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

Proposition 10.6.2.1.3. Let $f \colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets

$$X/{\sim_R^{\mathrm{eq}}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

2. *As a Pushout.* We have an isomorphism of sets[18]

$$X/{\sim_R^{\mathrm{eq}}} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim_R^{\mathrm{eq}}} \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

[17] When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

[18] Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \cong X \times_{X/{\sim_R^{\mathrm{eq}}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/{\sim_R^{\mathrm{eq}}}. \end{tikzcd}$$

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]

$$X/{\sim_{\mathrm{Ker}(f)}} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

   $$\bar{f} \colon X/{\sim_R} \to Y$$

   making the diagram

   $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\exists"', "\bar{f}"] & \end{tikzcd}$$

   commute.

   (b) We have $R \subset \mathrm{Ker}(f)$.

   (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on

[19] *Further Terminology:* The set $X/{\sim_{\mathrm{Ker}(f)}}$ is often called the **coimage of** f, and denoted by $\mathrm{CoIm}(f)$.

[20] In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f) \colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1}) \colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

of Constructions With Relations, ?? of ??.

X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R \arrow[ur, dashed, "\overline{f}"'] \arrow[ur, phantom, "\exists!", shift left=3] & \end{tikzcd}$$

commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/\sim_R, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \,\middle|\, \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

    (a) The map $\overline{f}$ is an injection.

    (b) We have $R = \mathrm{Ker}(f)$.

    (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

    (a) The map $f \colon X \longrightarrow Y$ is surjective.

    (b) The map $\overline{f} \colon X/\sim_R \longrightarrow Y$ is surjective.

9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to R. The following conditions are equivalent:

    (a) The map f satisfies the equivalent conditions of Item 4:

- There exists a map

$$\overline{f}\colon X/{\sim_R^{\mathrm{eq}}} \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R^{\mathrm{eq}}} \arrow[ru, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. *Item 1, As a Coequaliser*: Omitted.
Item 2, As a Pushout: Omitted.
Item 3, The First Isomorphism Theorem for Sets: Clear.
Item 4, Descending Functions to Quotient Sets, I: See [Pro25c].
Item 5, Descending Functions to Quotient Sets, II: See [Pro25d].
Item 6, Descending Functions to Quotient Sets, III: This follows from Items 5 and 6.
Item 7, Descending Functions to Quotient Sets, IV: See [Pro25b].
Item 8, Descending Functions to Quotient Sets, V: See [Pro25a].
Item 9, Descending Functions to Quotient Sets, VI: The implication Item 8a $\Longrightarrow$ Item 8b is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

- The following conditions are satisfied:
  - We have $x \sim_R x_1$ or $x_1 \sim_R x$;
  - We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
  - We have $y \sim_R x_n$ or $x_n \sim_R y$;
- We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

Appendices

A Other Chapters

References

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