

Constructions With Monoidal Categories

The Clowder Project Authors

July 21, 2025

This chapter contains some material on constructions with monoidal categories.

Contents

13.1	Moduli Categories of Monoidal Structures.....	2
13.1.1	The Moduli Category of Monoidal Structures on a Category.....	2
13.1.2	The Moduli Category of Braided Monoidal Structures on a Category.....	17
13.1.3	The Moduli Category of Symmetric Monoidal Structures on a Category.....	17
13.2	Moduli Categories of Closed Monoidal Structures.....	17
13.3	Moduli Categories of Refinements of Monoidal Structures.....	17
13.3.1	The Moduli Category of Braided Refinements of a Monoidal Structure.....	17
A	Other Chapters.....	18

13.1 Moduli Categories of Monoidal Structures

13.1.1 The Moduli Category of Monoidal Structures on a Category

Let \mathcal{C} be a category.

DEFINITION 13.1.1.1.1 ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

The **moduli category of monoidal structures on \mathcal{C}** is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) \stackrel{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow \text{forget} \\ \text{pt} & \xrightarrow{[\mathcal{C}]} & \text{Cats}. \end{array}$$

REMARK 13.1.1.1.2 ► UNWINDING DEFINITION 13.1.1.1.1, I

In detail, the **moduli category of monoidal structures on \mathcal{C}** is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ are monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ whose underlying category is \mathcal{C} .
- *Morphisms.* A morphism from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$ is a strong monoidal functor structure

$$\begin{aligned} \text{id}_{\mathcal{C}}^{\otimes} : A \boxtimes_{\mathcal{C}} B &\xrightarrow{\sim} A \otimes_{\mathcal{C}} B, \\ \text{id}_{\mathbb{1}_{\mathcal{C}}}^{\otimes} : \mathbb{1}'_{\mathcal{C}} &\xrightarrow{\sim} \mathbb{1}_{\mathcal{C}} \end{aligned}$$

on the identity functor $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ of \mathcal{C} .

- *Identities.* For each $M \stackrel{\text{def}}{=} (\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} : \text{pt} \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, M)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at M is defined by

$$\mathrm{id}_M^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} \stackrel{\mathrm{def}}{=} (\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{1|\mathcal{C}}^{\otimes}),$$

where $(\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{1|\mathcal{C}}^{\otimes})$ is the identity monoidal functor of \mathcal{C} of ??.

- *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} : \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(N, P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, N) \rightarrow \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, P)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at (M, N, P) is defined by

$$(\mathrm{id}_{\mathcal{C}}^{\otimes, \prime}, \mathrm{id}_{1|\mathcal{C}}^{\otimes, \prime}) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} (\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{1|\mathcal{C}}^{\otimes}) \stackrel{\mathrm{def}}{=} (\mathrm{id}_{\mathcal{C}}^{\otimes, \prime} \circ \mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{1|\mathcal{C}}^{\otimes, \prime} \circ \mathrm{id}_{1|\mathcal{C}}^{\otimes}).$$

REMARK 13.1.1.1.3 ► UNWINDING DEFINITION 13.1.1.1, II

In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, 1'_{\mathcal{C}}, \alpha^{C, \prime}, \lambda^{C, \prime}, \rho^{C, \prime})$ satisfies the following conditions:

1. *Naturality.* For each pair $f: A \rightarrow X$ and $g: B \rightarrow Y$ of morphisms of \mathcal{C} , the diagram

$$\begin{array}{ccc} A \boxtimes_{\mathcal{C}} B & \xrightarrow{f \boxtimes_{\mathcal{C}} g} & X \boxtimes_{\mathcal{C}} Y \\ \mathrm{id}_{A,B}^{\otimes} \downarrow & & \downarrow \mathrm{id}_{X,Y}^{\otimes} \\ A \otimes_{\mathcal{C}} B & \xrightarrow{f \otimes_{\mathcal{C}} g} & X \otimes_{\mathcal{C}} Y \end{array}$$

commutes.

2. *Monoidality.* For each $A, B, C \in \text{Obj}(\mathcal{C})$, the diagram

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^{C,\prime} \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^C & & \swarrow \text{id}_{A,B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}$$

commutes.

3. *Left Monoidal Unity.* For each $A \in \text{Obj}(\mathcal{C})$, the diagram

$$\begin{array}{ccc}
 & \mathbb{1}_C \boxtimes_C A & \xrightarrow{\text{id}_{\mathbb{1}_C, A}^\otimes} \mathbb{1}_C \otimes_C A \\
 \text{id}_{\mathbb{1}}^\otimes \boxtimes \text{id}_A \nearrow & & \searrow \lambda_A^C \\
 \mathbb{1}'_C \boxtimes_C A & \xrightarrow{\lambda_A^{C,\prime}} & A
 \end{array}$$

commutes.

4. *Right Monoidal Unity.* For each $A \in \text{Obj}(\mathcal{C})$, the diagram

$$\begin{array}{ccc}
 & A \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{A, \mathbb{1}_C}^\otimes} A \otimes_C \mathbb{1}_C \\
 \text{id}_A \boxtimes \text{id}_{\mathbb{1}}^\otimes \nearrow & & \searrow \rho_A^C \\
 A \boxtimes_C \mathbb{1}'_C & \xrightarrow{\rho_A^{C,\prime}} & A
 \end{array}$$

commutes.

PROPOSITION 13.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

Let \mathcal{C} be a category.

1. *Extra Monoidality Conditions.* Let $(\text{id}_{\mathcal{C}}^{\otimes}, \text{id}_{\mathbb{1}|\mathcal{C}}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$.

(a) The diagram

$$\begin{array}{ccc} (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow{\text{id}_{A,B}^{\otimes} \boxtimes_{\mathcal{C}} \text{id}_C} & (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \\ \text{id}_{A \boxtimes_{\mathcal{C}} B, C}^{\otimes} \downarrow & & \downarrow \text{id}_{A \otimes_{\mathcal{C}} B, C}^{\otimes} \\ (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow{\text{id}_{A,B}^{\otimes} \otimes_{\mathcal{C}} \text{id}_C} & (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \end{array}$$

commutes.

(b) The diagram

$$\begin{array}{ccc} A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) & \xrightarrow{\text{id}_A \boxtimes_{\mathcal{C}} \text{id}_{B,C}^{\otimes}} & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \\ \text{id}_{A, B \boxtimes_{\mathcal{C}} C}^{\otimes} \downarrow & & \downarrow \text{id}_{A, B \otimes_{\mathcal{C}} C}^{\otimes} \\ A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) & \xrightarrow{\text{id}_A \otimes_{\mathcal{C}} \text{id}_{B,C}^{\otimes}} & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \end{array}$$

commutes.

2. *Extra Monoidal Unity Constraints.* Let $(\text{id}_{\mathcal{C}}^{\otimes}, \text{id}_{\mathbb{1}|\mathcal{C}}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$.

(a) The diagram

$$\begin{array}{ccc} \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} & \xrightarrow{\text{id}_{\mathbb{1}_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}}^{\otimes, -1}} & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \\ \lambda_{\mathbb{1}'_{\mathcal{C}}}^{\mathcal{C}} \downarrow & & \downarrow \rho_{\mathbb{1}_{\mathcal{C}}}^{C,'} \\ \mathbb{1}'_{\mathcal{C}} & \xrightarrow{\text{id}_{\mathbb{1}}} & \mathbb{1}_{\mathcal{C}} \end{array}$$

commutes.

(b) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\
 \rho_{\mathbb{1}'_C}^C \downarrow & & \downarrow \lambda_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

(c) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \lambda_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

(d) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C}^{\otimes}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \rho_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

3. *Mixed Associators.* Let $(\mathcal{C}, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(\mathcal{C}, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, ' }, \lambda^{C, ' }, \rho^{C, ' })$ be monoidal structures on \mathcal{C} and let

$$\text{id}_{-1, -2}^{\otimes}: -_1 \boxtimes_C -_2 \rightarrow -_1 \otimes_C -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes}: (A \otimes_C B) \boxtimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \text{id}_{A \otimes_C B, C}^{\otimes} \downarrow & & \downarrow \text{id}_A \otimes_C \text{id}_{B, C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C, \prime}} & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A, B}^{\otimes} \boxtimes_C \text{id}_C \downarrow & & \downarrow \text{id}_{A, B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \rightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \\ \text{id}_{A, B}^{\otimes} \otimes_C \text{id}_C \downarrow & & \downarrow \text{id}_{A, B \otimes_C C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C, \prime}} & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \boxtimes_C B, C}^{\otimes} \downarrow & & \downarrow \text{id}_A \boxtimes_C \text{id}_{B, C}^{\otimes} \\ (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \end{array}$$

commute, then the natural transformation id^\otimes satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes,\otimes}: (A \boxtimes_C B) \otimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \text{id}_{A,B}^\otimes \otimes \text{id}_C \downarrow & & \downarrow \text{id}_A \otimes \text{id}_{B,C}^\otimes \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \boxtimes_C B,C}^\otimes \downarrow & & \downarrow \text{id}_{A,B \boxtimes_C C}^\otimes \\ (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^\otimes satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

We claim that **Items 1a** and **1b** are indeed true:

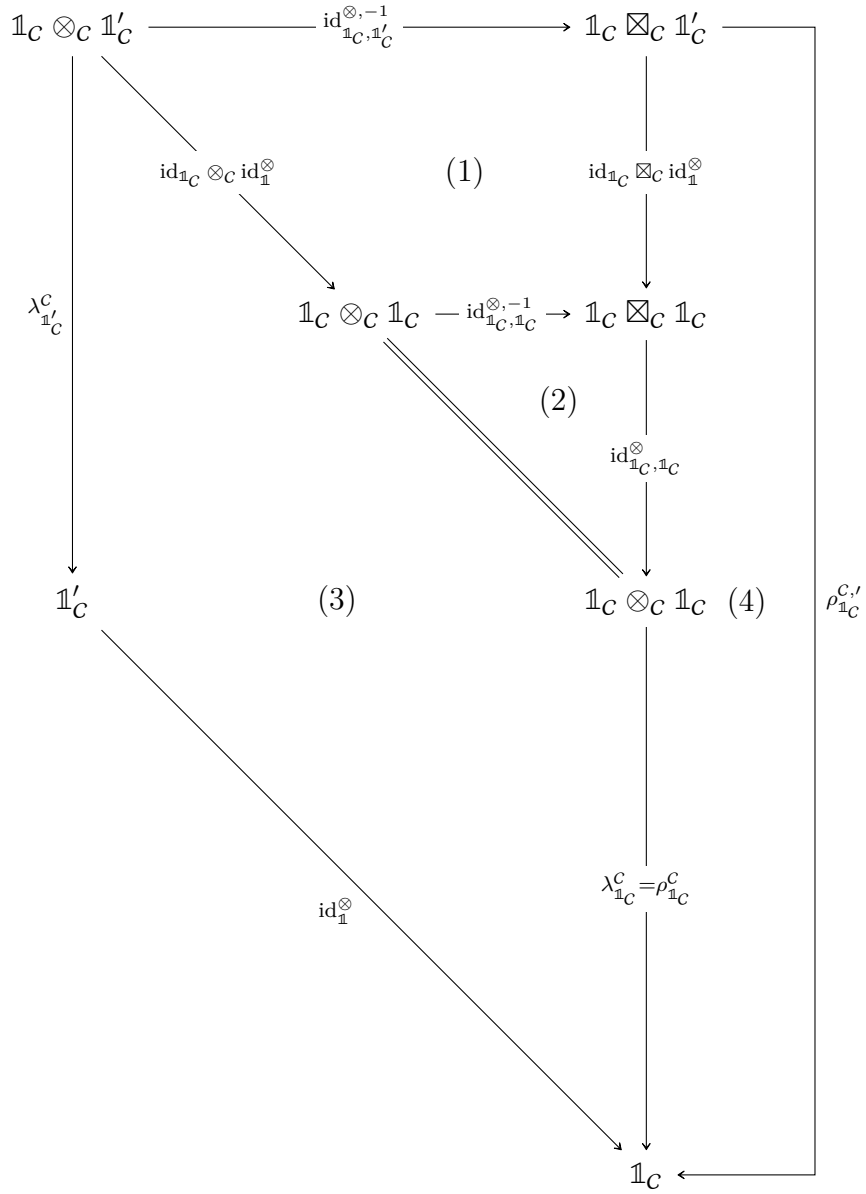
1. *Proof of Item 1a:* This follows from the naturality of id^\otimes with respect to the morphisms $\text{id}_{A,B}^\otimes$ and id_C .
2. *Proof of Item 1b:* This follows from the naturality of id^\otimes with respect to the morphisms id_A and $\text{id}_{B,C}^\otimes$.

This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

We claim that **Items 2a** and **2b** are indeed true:

1. *Proof of Item 1a:* Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes, -1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{1_C}^C = \lambda_{1_C}^C$ comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of $(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}_{C|1}^{\otimes})$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1'_C, 1_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C & & \\
 \downarrow \rho_{1'_C}^C & \searrow \text{id}_{1_1}^{\otimes} \otimes \text{id}_{1_C} & \downarrow \text{id}_{1_1}^{\otimes} \boxtimes \text{id}_{1_C} & & \\
 & & 1_C \otimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1_C}^{\otimes, -1}} & 1_C \boxtimes_C 1_C \\
 & & \parallel & & \downarrow \text{id}_{1_C, 1_C}^{\otimes} \\
 & & & & 1_C \otimes_C 1_C \quad (4) \\
 & & & & \downarrow \rho_{1_C}^C = \lambda_{1_C}^C \\
 & & & & 1_C \\
 & \searrow \text{id}_{1_1}^{\otimes} & & & \uparrow \lambda_{1_C}^{C, ' } \\
 1'_C & & & &
 \end{array}$$

(1) (2) (3)

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\text{id}_C^{\otimes, -1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{1_C}^C = \lambda_{1_C}^C$ comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of $(\text{id}_C, \text{id}_C^{\otimes}, \text{id}_{C|1})$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes}} & 1'_C \otimes_C 1_C \\
 \downarrow \rho_{1'_C}^C & & \downarrow \lambda_{1_C}^{C, '}& & \downarrow \rho_{1'_C}^C \\
 1'_C & \xrightarrow{\text{id}_{1'}^{\otimes}} & 1_C & \xrightarrow{\text{id}_{1'}^{\otimes, -1}} & 1'_C.
 \end{array}
 \quad \begin{array}{c} (1) \end{array} \quad \begin{array}{c} (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1b**;

it follows that the diagram

$$\begin{array}{ccccc}
 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes}} & 1'_C \otimes_C 1_C \\
 & & \downarrow \lambda_{1_C}^{C, '}& & \downarrow \rho_{1'_C}^C \\
 & & 1_C & \xrightarrow{\text{id}_{1'}^{\otimes, -1}} & 1'_C
 \end{array}
 \quad \begin{array}{c} (\dagger)$$

commutes. But since $\text{id}_{1_C, 1'_C}^{\otimes, -1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 1_C \otimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \boxtimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \otimes_C 1'_C \\
 \lambda_{1'_C}^C \downarrow & (1) & \rho_{1_C}^{C, ' } \downarrow & (\dagger) & \lambda_{1'_C}^C \downarrow \\
 1'_C & \xrightarrow{\text{id}_{1'}^{\otimes}} & 1_C & \xrightarrow{\text{id}_{1'}^{\otimes, -1}} & 1_C
 \end{array}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1a**;

it follows that the diagram

$$\begin{array}{ccccc}
 1_C \otimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \boxtimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \otimes_C 1'_C \\
 & & \rho_{1_C}^{C, ' } \downarrow & (\dagger) & \lambda_{1'_C}^C \downarrow \\
 & & 1_C & \xrightarrow{\text{id}_{1'}^{\otimes, -1}} & 1_C
 \end{array}$$

commutes. But since $\text{id}_{1'}^{\otimes, -1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

This finishes the proof.

Item 3: Mixed Associators

We claim that **Items 3a** to **3c** are indeed true:

1. *Proof of Item 3a:* We may partition the monoidality diagram for

id^\otimes of **Item 2** of **Remark 13.1.1.3** as follows:

$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes \text{id}_C & \downarrow & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (A \boxtimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) & \downarrow \text{id}_{A \boxtimes_C B, C}^\otimes & (2) & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \swarrow \text{id}_{A,B}^\otimes \otimes \text{id}_C & & \swarrow \alpha_{A,B,C}^\otimes & & \downarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 (A \otimes_C B) \otimes_C C & (3) & A \boxtimes_C (B \otimes_C C) & & \\
 \searrow \alpha_{A,B,C}^{C, \prime} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & &
 \end{array}$$

Since:

- Subdiagram (1) commutes by **Item 1a** of **Item 1**.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^\otimes satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.

2. *Proof of **Item 3b***: We may partition the monoidality diagram for

id^\otimes of **Item 2** of **Remark 13.1.1.3** as follows:

$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes \text{id}_C & & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (1) & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \swarrow \alpha_{A,B,C}^\boxtimes & & \nwarrow \text{id}_{A, B \boxtimes_C C}^\otimes & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\
 & A \otimes_C (B \boxtimes_C C) & & & \\
 & (2) & & (3) & \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\text{id}_A \otimes \text{id}_{B,C}^\otimes} & A \boxtimes_C (B \otimes_C C) & & \\
 \searrow \alpha_{A,B,C}^{C,\prime} & \downarrow & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & &
 \end{array}$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by **Item 1b** of **Item 1**.

it follows that the boundary diagram also commutes, i.e. id^\otimes satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.

3. *Proof of **Item 3c***: We may partition the monoidality diagram for

id^\otimes of **Item 2** of **Remark 13.1.1.3** as follows:

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^C \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \searrow \alpha_{A,B,C}^{\boxtimes, \otimes} & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^{C, \otimes} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}$$

(1) (2)

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^\otimes satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.

This finishes the proof. 

13.1.2 The Moduli Category of Braided Monoidal Structures on a Category

13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category

13.2 Moduli Categories of Closed Monoidal Structures

13.3 Moduli Categories of Refinements of Monoidal Structures

13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

Relations

8. [Relations](#)
9. [Constructions With Relations](#)

10. [Conditions on Relations](#)

Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

Monoidal Categories

13. [Constructions With Monoidal Categories](#)

Bicategories

14. [Types of Morphisms in Bicategories](#)

Extra Part

15. [Notes](#)