Constructions With Monoidal Categories

The Clowder Project Authors

July 29, 2025

O1UF This chapter contains some material on constructions with monoidal categories.

Contents

13.1 N	Moduli Categories of Monoidal Structures 2	,
	3.1.1 The Moduli Category of Monoidal Structures on a Cate-	
gory)
1	3.1.2 The Moduli Category of Braided Monoidal Structures	
on a Ca	ategory	7
1	3.1.3 The Moduli Category of Symmetric Monoidal Structures	
on a Ca	ategory	7
_		
13.2 N	Moduli Categories of Closed Monoidal Structures 17	•
133 N	Moduli Categories of Refinements of Monoidal Struc-	
		,
	3.3.1 The Moduli Category of Braided Refinements of a Monoidal	
	re	7
Duracuc	11	
A (Other Chapters 18	3

Moduli Categories of Monoidal Struc-01UG 13.1tures

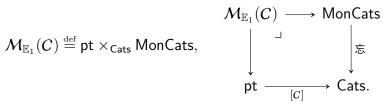
The Moduli Category of Monoidal Structures 01UH 13.1.1 on a Category

Let C be a category.

01UJ DEFINITION 13.1.1.1.1 ▶ THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{ ext{def}}{=} \mathsf{pt} imes_{\mathsf{Cats}} \mathsf{MonCats},$$



01UK REMARK 13.1.1.1.2 ➤ Unwinding Definition 13.1.1.1.1, I

In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ are monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}},$ $\mathbb{1}_C$, α^C , λ^C , ρ^C) whose underlying category is C.
- Morphisms. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ $\boxtimes_{\mathcal{C}}$, $\mathbb{1}'_{\mathcal{C}}$, $\alpha^{\mathcal{C},\prime}$, $\lambda^{\mathcal{C},\prime}$, $\rho^{\mathcal{C},\prime}$) is a strong monoidal functor structure

$$\operatorname{id}_{\mathcal{C}}^{\otimes} \colon A \boxtimes_{\mathcal{C}} B \xrightarrow{\sim} A \otimes_{\mathcal{C}} B,$$
$$\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes} \colon \mathbb{1}'_{\mathcal{C}} \xrightarrow{\sim} \mathbb{1}_{\mathcal{C}}$$

on the identity functor $id_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of \mathcal{C} .

For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in$ • Identities. $\mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at M is defined by

$$\mathrm{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(\mathcal{C})} \stackrel{\mathrm{def}}{=} \left(\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of \ref{c} ?

• Composition. For each $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} : \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M,P)
\text{of } \mathcal{M}_{\mathbb{E}_{1}}(C) \text{ at } (M,N,P) \text{ is defined by}
\left(\operatorname{id}_{C}^{\otimes,\prime},\operatorname{id}_{\mathbb{I}|C}^{\otimes,\prime}\right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{I}|C}^{\otimes}\right) \stackrel{\text{def}}{=} \left(\operatorname{id}_{C}^{\otimes,\prime}\circ\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{I}|C}^{\otimes,\prime}\circ\operatorname{id}_{C}^{\otimes}\right).$$

01UL REMARK 13.1.1.1.3 ➤ UNWINDING DEFINITION 13.1.1.1.1, II

In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

1. Naturality. For each pair $f: A \to X$ and $g: B \to Y$ of morphisms of C, the diagram

$$A \boxtimes_{\mathcal{C}} B \xrightarrow{f \boxtimes_{\mathcal{C}} g} X \boxtimes_{\mathcal{C}} Y$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{\mathcal{C}} B \xrightarrow{f \otimes_{\mathcal{C}} g} X \otimes_{\mathcal{C}} Y$$

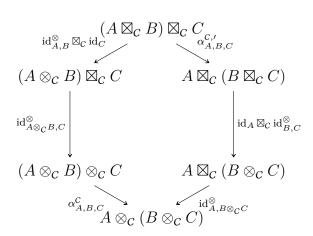
commutes.

O I UL

01UM

01UN

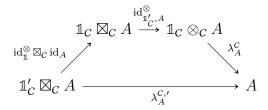
2. Monoidality. For each $A, B, C \in \text{Obj}(C)$, the diagram



commutes.

01UP

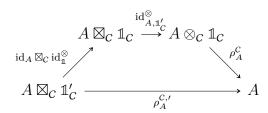
3. Left Monoidal Unity. For each $A \in \text{Obj}(\mathcal{C})$, the diagram



commutes.

01U0

4. Right Monoidal Unity. For each $A \in \text{Obj}(C)$, the diagram



commutes.

PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

Let C be a category.

- 1. Extra Monoidality Conditions. Let $(id_{\mathcal{C}}^{\otimes}, id_{\mathbb{1}|\mathcal{C}}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}', \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$.
- (a) The diagram

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\operatorname{id}_{A,B}^{\otimes} \boxtimes_{C} \operatorname{id}_{C}} (A \otimes_{C} B) \boxtimes_{C} C$$

$$\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A\otimes_{C}B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} (A \otimes_{C} B) \otimes_{C} C$$

commutes.

(b) The diagram

$$A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{\mathcal{C}} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

$$\downarrow^{\operatorname{id}_{A,B\boxtimes_{\mathcal{C}} C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B\otimes_{\mathcal{C}} C}^{\otimes}}$$

$$A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \xrightarrow[\operatorname{id}_{A} \otimes_{\mathcal{C}} \operatorname{id}_{B,C}^{\otimes}]} A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C}$$

$$\lambda_{\mathbb{1}'_{C}}^{c} \downarrow \qquad \qquad \downarrow^{\rho_{\mathbb{1}_{C}}^{c,\prime}}$$

$$\mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{A}}^{\otimes}} \mathbb{1}_{C}$$

01US

01UR

01UT

01UU

01WB

01WC

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. Mixed Associators. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} : -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

01WE

01WD

01WF

01UV

01UW

(a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} \colon (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A} \otimes_{C} B,C} \qquad \qquad \downarrow^{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B} \boxtimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B} \boxtimes_{C} C}$$

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UX

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes}$$
: $(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$ making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} C}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \xrightarrow{\alpha_{A,B,C}^{\mathcal{C},\prime}} A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{\mathcal{C}} B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A}\boxtimes_{\mathcal{C}} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_{C} B) \otimes_{C} C \to A \otimes_{C} (B\boxtimes_{C} C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

PROOF 13.1.1.1.5 ▶ PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

We claim that Items 1a and 1b are indeed true:

- 1. Proof of Item 1a: This follows from the naturality of id^{\otimes} with respect to the morphisms $id_{A,B}^{\otimes}$ and id_C .
- 2. Proof of Item 1b: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and $id_{B,C}^{\otimes}$.

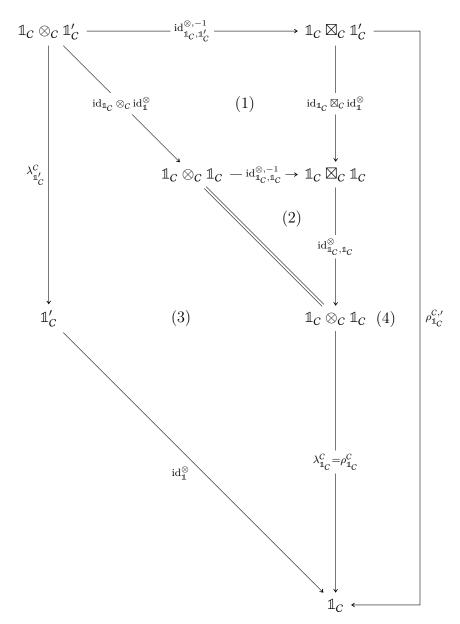
01UY

This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:

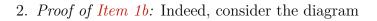
1. Proof of Item 1a: Indeed, consider the diagram

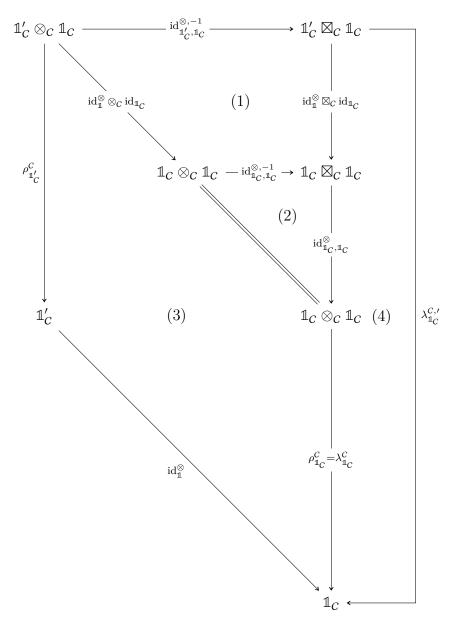


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_{\mathcal{C}}^{\otimes,-1};$
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathcal{C}}$, where the equality $\rho_{\mathbb{1}_{\mathcal{C}}}^{\mathcal{C}} = \lambda_{\mathbb{1}_{\mathcal{C}}}^{\mathcal{C}}$ comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $id_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{C,'}_{\mathbb{1}_{C}} \qquad \qquad (\dagger) \qquad \qquad \downarrow^{\rho_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes. But since $\mathrm{id}_{\mathbb{1}_{\mathcal{C}},\mathbb{1}'_{\mathcal{C}}}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \\
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}} \\
\downarrow \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}} \\
\downarrow \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}_{C}$$

commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

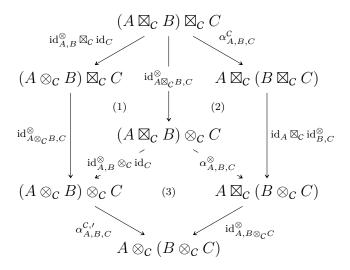
Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for

01UZ

 id^{\otimes} of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

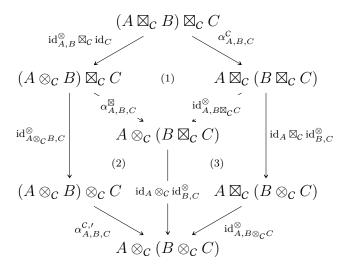
- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for

01V0

 id^{\otimes} of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

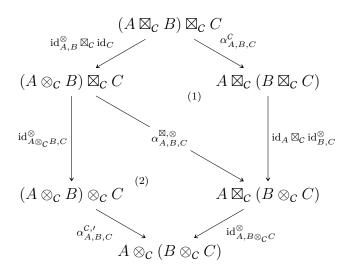
- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for

01V1

 id^{\otimes} of Item 2 of Remark 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- olva 13.2 Moduli Categories of Closed Monoidal Structures
- olv 13.3 Moduli Categories of Refinements of Monoidal Structures
- on 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

T	•	•	•
Pre	$\mathbf{l}\mathbf{m}$	una	ries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes