Constructions With Relations

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July 29, 2025

00NE This chapter contains some material about constructions with relations. Notably, we discuss and explore:

029U 1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (**??**).

029V 2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, converse relations, composition of relations, and collages (Section 9.2).

This chapter is under revision. TODO:

1. Rename range to image
2. Co/limits in **Rel**.

Contents

00NF 9.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

00NZ 9.2 More Constructions With Relations

00PM 9.2.1 The Domain and Range of a Relation

Let A and B be sets.

00PN **Definition 9.2.1.1.1 ► The Domain and Range of a Relation**

Let $R\colon A \nrightarrow B$ be a relation.[1,2]

02AV 1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\text{def}}{=} \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

02AW 2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

[1]Following ??, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b\in B}(R^b_a) && (a \in A) \\ &\cong \bigvee_{b\in B} R^b_a, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a\in A}(R^b_a) && (b \in B) \end{aligned}$$

$$\cong \bigvee_{a\in A} R^b_a,$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Constructions With Sets, Definition 3.2.2.1.3.

[2]Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned}\mathrm{dom}(R) &\cong \operatorname*{colim}_{y\in Y}(R(y)) \\ &\cong \bigcup_{y\in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x\in X}(R(x)) \\ &\cong \bigcup_{x\in X} R(x),\end{aligned}$$

00PP 9.2.2 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

00PQ **Definition 9.2.2.1.1 ▶ Binary Unions of Relations**

The **union of R and S**[1] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

  for each $a \in A$.

[1]*Further Terminology:* Also called the **binary union of R and S**, for emphasis.
[2]This is the same as the union of R and S as subsets of $A \times B$.

00PR **Proposition 9.2.2.1.2 ▶ Properties of Binary Unions of Relations**

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

00PS 1. *Interaction With Converses.* We have

$$(R \cup S)^{\dagger} = R^{\dagger} \cup S^{\dagger}.$$

00PT 2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof 9.2.2.1.3 ▶ Proof of Proposition 9.2.2.1.2

Item 1: Interaction With Converses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

    or

    - $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

    and

    - $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ.

00PU
9.2.3 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

00PV

The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \stackrel{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

[1]This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

00PW **Proposition 9.2.3.1.2 ► Properties of Unions of Families of Relations**

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

00PX 1. *Interaction With Converses.* We have

$$\Big(\bigcup_{i\in I} R_i\Big)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 9.2.3.1.3 ► Proof of Proposition 9.2.3.1.2

Item 1: Interaction With Converses

Clear.

00PY 9.2.4 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

00PZ **Definition 9.2.4.1.1 ► Binary Intersections of Relations**

The **intersection of** R **and** S[1] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[2]

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

[1] *Further Terminology:* Also called the **binary intersection of** R **and** S, for emphasis.
[2] This is the same as the intersection of R and S as subsets of $A \times B$.

00Q0 **Proposition 9.2.4.1.2 ► Properties of Binary Intersections of Relations**

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

00Q1 1. *Interaction With Converses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

00Q2 2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 9.2.4.1.3 ► Proof of Proposition 9.2.4.1.2

Item 1: Interaction With Converses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

    and

    - $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

    and

    - $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$.

00Q3 9.2.5 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

00Q4 **Definition 9.2.5.1.1 ► The Intersection of a Family of Relations**

The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[1]

$$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right](a) \overset{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each $a \in A$.

[1]This is the same as the intersection of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

00Q5 **Proposition 9.2.5.1.2 ▶ Properties of Intersections of Families of Relations**

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

00Q6 1. *Interaction With Converses.* We have

$$\Big(\bigcap_{i\in I} R_i\Big)^{\dagger} = \bigcap_{i\in I} R_i^{\dagger}.$$

Proof 9.2.5.1.3 ▶ Proof of Proposition 9.2.5.1.2

Item 1: Interaction With Converses

Clear. ∎

00Q7 9.2.6 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

00Q8 **Definition 9.2.6.1.1 ▶ Binary Products of Relations**

The **product of R and S**[1] is the relation $R\times S$ from $A\times X$ to $B\times Y$ defined as follows:

- Viewing relations from $A\times X$ to $B\times Y$ as subsets of $(A\times X)\times(B\times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[2]
- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R\times S} \mathcal{P}(B) \times \mathcal{P}(Y) \xrightarrow{\mathcal{P}^{\otimes}_{B,Y}} \mathcal{P}(B \times Y)$$

  in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

[1] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.

[2] That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R\times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

00Q9 **Proposition 9.2.6.1.2 ▶ Properties of Binary Products of Relations**

Let A, B, X, and Y be sets.

00QA 1. *Interaction With Converses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

We have

$$(R \times S)^\dagger = R^\dagger \times S^\dagger.$$

00QB 2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof 9.2.6.1.3 ▶ Proof of Proposition 9.2.6.1.2

Item 1: Interaction With Converses

Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(R\times S)^\dagger} (b, y)$ iff:
  - We have $(b, y) \sim_{R\times S} (a, x)$, i.e. iff:

        * We have $b \sim_R a$;
        * We have $y \sim_S x$;

- We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
    - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
        * We have $b \sim_R a$;
        * We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
    - We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
        * There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
        * There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
- We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
    - There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
        * We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
        * We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

00QC

9.2.7 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

00QD **Definition 9.2.7.1.1 ▶ The Product of a Family of Relations**

The **product of the family** $\{R_i\}_{i\in I}$ is the relation $\prod_{i\in I} R_i$ from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i\in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i\in I} \in \prod_{i\in I}(A_i \times B_i) \,\middle|\, \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

  for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

00R2 9.2.8 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

00R3 **Definition 9.2.8.1.1 ▶ The Collage of a Relation**

The **collage of R**[1] is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \stackrel{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)}\colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$

  on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of R**.

02AZ **Notation 9.2.8.1.2 ► Notation: $\mathsf{Pos}_{/\Delta^1}(A,B)$**

We write $\mathsf{Pos}_{/\Delta^1}(A,B)$ for the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A,B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{ev}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{ev}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{ev}_{[0]}"] \arrow[dr, "\mathrm{ev}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

02B0 **Remark 9.2.8.1.3 ► Unwinding Notation 9.2.8.1.2**

In detail, $\mathsf{Pos}_{/\Delta^1}(A,B)$ is the category where:

- *Objects.* An object of $\mathsf{Pos}_{/\Delta^1}(A,B)$ is a pair (X,ϕ_X) consisting of
  - A poset X;
  - A morphism $\phi_X\colon X \to \Delta^1$;

  such that we have

  $$\begin{aligned}\phi_X^{-1}(0) &= A,\\ \phi_X^{-1}(1) &= B.\end{aligned}$$

- *Morphisms.* A morphism of $\mathsf{Pos}_{/\Delta^1}(A,B)$ from (X,ϕ_X) to (Y,ϕ_Y)

is a morphism of posets $f\colon X \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\ & \Delta^1 & \end{tikzcd}$$

commute.

00R4 **Proposition 9.2.8.1.4 ▶ Properties of Collages of Relations**

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

00R5 1. *Functoriality.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathsf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

  – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 9.2.8.1.1.

  – The morphism $\phi_R\colon \mathbf{Coll}(R) \to \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of **Coll** at (R, S) is given by sending an inclusion

$$\iota\colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[1]

00R6 2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

[1] Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

Proof 9.2.8.1.5 ► Proof of Proposition 9.2.8.1.4

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted.

Appendices

A Other Chapters