Categories

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This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 11.1).
2. Examples of categories (Section 11.2).
3. The quadruple adjunction $\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}$ between the category of categories and the category of sets (Section 11.3).
4. Groupoids, categories in which all morphisms admit inverses (Section 11.4).
5. Functors (Section 11.5).
6. The conditions one may impose on functors in decreasing order of importance:
   (a) Section 11.6 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.
   (b) Section 11.7 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 11.6, such as being dominant, being a monomorphism, being pseudomonic, etc.
   (c) Section 11.8 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.
7. Natural transformations (Section 11.9).

8. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 11.10).

This chapter is under active revision. TODO:

- Fix categories having an underlying set of objects by having them have an underlying setoid of objects (not necessarily by definition, as that'll likely be bothersome; at least Section 11.3 should be fixed and several remarks should be added at several points). Related: Definition 11.3.1.1.2

Contents

11.1 Categories

11.1.1 Foundations

Definition 11.1.1.1.1. A **category** $\left(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}}\right)$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**.
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

  called the **unit map of $\mathcal{C}$ at A**, determining a morphism

  $$\mathrm{id}_A \colon A \to A$$

  of $\mathcal{C}$, called the **identity morphism of A**.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$

  called the **composition map of $\mathcal{C}$ at (A, B, C)**.

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{tikzcd}
 & & \mathrm{Hom}_{\mathcal{C}}(C,D)\times(\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[drr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)}\times\circ^{\mathcal{C}}_{A,B,C}"] & & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(B,C))\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[urr, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[dr, "\circ^{\mathcal{C}}_{B,C,D}\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & & & \mathrm{Hom}_{\mathcal{C}}(C,D)\times\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[dl, "\circ^{\mathcal{C}}_{A,C,D}"] \\
 & \mathrm{Hom}_{\mathcal{C}}(B,D)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D) &
\end{tikzcd}$$

commutes, i.e. for each composable triple (f,g,h) of morphisms of $\mathcal{C}$, we have

$$(f\circ g)\circ h = f\circ(g\circ h).$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt}\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_B\times\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,B)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A\to B$ of $\mathcal{C}$, we have

$$\mathrm{id}_B\circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\times\mathbb{1}^{\mathcal{C}}_A"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A\to B$ of $\mathcal{C}$, we have

$$f\circ\mathrm{id}_A = f.$$

Notation 11.1.1.1.2. Let $\mathcal{C}$ be a category.

1. We also write $\mathcal{C}(A, B)$ for $\mathrm{Hom}_{\mathcal{C}}(A, B)$.
2. We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 11.1.1.1.3. Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A, B)/\{\text{isomorphisms}\}$$

   is a set.
3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

11.1.2 Subcategories

Let $\mathcal{C}$ be a category.

Definition 11.1.2.1.1. A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$

3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathbb{1}_A^{\mathcal{A}} = \mathbb{1}_A^{\mathcal{C}}.$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have

$$\circ_{A,B,C}^{\mathcal{A}} = \circ_{A,B,C}^{\mathcal{C}}.$$

Definition 11.1.2.1.2. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$

is surjective (and thus bijective).

Definition 11.1.2.1.3. A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[1]

Definition 11.1.2.1.4. A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[2] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$.

11.1.3 Skeletons of Categories

Definition 11.1.3.1.1. A[3] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

Definition 11.1.3.1.2. A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[4]

Proposition 11.1.3.1.3. Let $\mathcal{C}$ be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.
2. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor
$$\mathsf{Sk} \colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$
3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.
4. *Inclusions of Skeletons Are Equivalences.* The inclusion
$$\iota_{\mathcal{C}} \colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$
of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

[1] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.
[2] *Further Terminology:* Also called **lluf**.
[3] Due to Item 3 of Definition 11.1.3.1.3, which states that any two skeletons of a category are equivalent, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.
[4] That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

Proof. *Item 1, Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
Item 2, Pseudofunctoriality: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
Item 3, Uniqueness Up to Equivalence: Omitted.
Item 4, Inclusions of Skeletons Are Equivalences: Omitted. □

11.1.4 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C, X \in \text{Obj}(\mathcal{C})$.

Definition 11.1.4.1.1. Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

1. The **precomposition function associated to** f is the function

$$f^*\colon \text{Hom}_{\mathcal{C}}(B, X) \to \text{Hom}_{\mathcal{C}}(A, X)$$

defined by

$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

for each $\phi \in \text{Hom}_{\mathcal{C}}(B, X)$.

2. The **postcomposition function associated to** g is the function

$$g_*\colon \text{Hom}_{\mathcal{C}}(X, B) \to \text{Hom}_{\mathcal{C}}(X, C)$$

defined by

$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

for each $\phi \in \text{Hom}_{\mathcal{C}}(X, B)$.

Proposition 11.1.4.1.2. Let $A, B, C, D, X \in \text{Obj}(\mathcal{C})$.

1. *Interaction Between Precomposition and Postcomposition.* Let $f\colon A \to B$ and $g\colon X \to Y$ be morphisms of $\mathcal{C}$. We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \text{Hom}_{\mathcal{C}}(B, X) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \text{Hom}_{\mathcal{C}}(B, Y) \arrow[d, "f^*"] \\ \text{Hom}_{\mathcal{C}}(A, X) \arrow[r, "g_*"'] & \text{Hom}_{\mathcal{C}}(A, Y). \end{tikzcd}$$

2. *Interaction With Composition I.* Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$. We have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

3. *Interaction With Composition II.* Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$. We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* Let $f\colon X \to A$ and $g\colon C \to D$ be morphisms of $\mathcal{C}$. We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (\mathsf{id} \times f^*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathsf{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (g_* \times \mathsf{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathsf{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$\begin{aligned} \mathrm{id}_A^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}, \\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}. \end{aligned}$$

Proof. *Item 1, Interaction Between Precomposition and Postcomposition*: For each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, we have

$$\begin{aligned}[g_* \circ f^*](\phi) &= g_*(\phi \circ f) \\ &= g \circ (\phi \circ f) \\ &= (g \circ \phi) \circ f \\ &= f^*(g \circ \phi) \\ &= [f^* \circ g_*](\phi).\end{aligned}$$

Thus $g_* \circ f^* = f^* \circ g_*$.
Item 2, Interaction With Composition I: **$(g \circ f)_* = g_* \circ f_*$.** For each $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, we have

$$\begin{aligned}(g \circ f)_*(\phi) &= (g \circ f) \circ \phi \\ &= g \circ (f \circ \phi) \\ &= g \circ f_*(\phi) \\ &= g_*(f_*(\phi)) \\ &= [g_* \circ f_*](\phi).\end{aligned}$$

Thus $(g \circ f)_* = g_* \circ f_*$.
$(g \circ f)^* = g^* \circ f^*$. For each $\phi \in \mathrm{Hom}_{\mathcal{C}}(C, X)$, we have

$$\begin{aligned}(g \circ f)^*(\phi) &= \phi \circ (g \circ f) \\ &= (\phi \circ g) \circ f \\ &= (g^*(\phi)) \circ f \\ &= f^*(g^*(\phi)) \\ &= [f^* \circ g^*](\phi).\end{aligned}$$

Thus $(g \circ f)^* = g^* \circ f^*$.
Item 3, Interaction With Composition II: It suffices to show the equalities of the maps on $\star \in \mathrm{pt}$. We have

$$\begin{aligned}[g \circ f](\star) &= g \circ f \\ &= g_*(f) \\ &= g_*([f](\star)) \\ &= (g_* \circ [f])(\star),\end{aligned}$$

and

$$[g \circ f](\star) = g \circ f$$

$$
\begin{aligned}
&= f^*(g) \\
&= f^*([g](\star)) \\
&= (f^* \circ [g])(\star).
\end{aligned}
$$

Thus $[g \circ f] = g_* \circ [f]$ and $[g \circ f] = f^* \circ [g]$.

Item 4, Interaction With Composition III: $f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (\mathsf{id} \times f^*)$**.** For each $(\psi, \phi) \in \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have

$$
\begin{aligned}
\left[f^* \circ \circ^{\mathcal{C}}_{A,B,C}\right](\psi, \phi) &= f^*(\psi \circ \phi) \\
&= (\psi \circ \phi) \circ f \\
&= \psi \circ (\phi \circ f) \\
&= \circ^{\mathcal{C}}_{X,B,C}(\psi, \phi \circ f) \\
&= \circ^{\mathcal{C}}_{X,B,C}(\psi, f^*(\phi)) \\
&= \left[\circ^{\mathcal{C}}_{X,B,C} \circ (\mathsf{id} \times f^*)\right](\psi, \phi).
\end{aligned}
$$

Thus $f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (\mathsf{id} \times f^*)$.

$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (g_* \times \mathsf{id})$**.** For each $(\psi, \phi) \in \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have

$$
\begin{aligned}
\left[g_* \circ \circ^{\mathcal{C}}_{A,B,C}\right](\psi, \phi) &= g_*(\psi \circ \phi) \\
&= g \circ (\psi \circ \phi) \\
&= (g \circ \psi) \circ \phi \\
&= \circ^{\mathcal{C}}_{A,B,D}(g \circ \psi, \phi) \\
&= \circ^{\mathcal{C}}_{A,B,D}(g_*(\psi), \phi) \\
&= \left[\circ^{\mathcal{C}}_{A,B,D} \circ (g_* \times \mathsf{id})\right](\psi, \phi).
\end{aligned}
$$

Thus $g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (g_* \times \mathsf{id})$.

Item 5, Interaction With Identities: We have

$$
\begin{aligned}
\mathrm{id}^*_A(\phi) &= \phi \circ \mathrm{id}_A \\
&= \phi \\
&= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}(\phi)
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathrm{id}_B)_*(\phi) &= \mathrm{id}_B \circ \phi \\
&= \phi \\
&= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}(\phi)
\end{aligned}
$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. $\square$

11.2 Examples of Categories

11.2.1 The Empty Category

Example 11.2.1.1.1. The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset.$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset.$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

11.2.2 The Punctual Category

Example 11.2.2.1.1. The **punctual category**[5] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathsf{pt}) \overset{\text{def}}{=} \{\star\}.$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathsf{pt}}(\star,\star) \overset{\text{def}}{=} \{\mathrm{id}_\star\}.$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathsf{pt}}_\star \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{pt}}(\star,\star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathsf{pt}}_\star \overset{\text{def}}{=} \mathrm{id}_\star\,.$$
- *Composition.* The composition map
$$\circ^{\mathsf{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathsf{pt}}(\star,\star) \times \mathrm{Hom}_{\mathsf{pt}}(\star,\star) \to \mathrm{Hom}_{\mathsf{pt}}(\star,\star)$$
of pt at $(\star,\star,\star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[5] *Further Terminology:* Also called the **singleton category**.

11.2.3 Monoids as One-Object Categories

Example 11.2.3.1.1. We have an isomorphism of categories[6]

$$\mathsf{Mon} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B}\colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**, exhibiting monoids as exactly those categories having a single object.

Proof. Omitted. □

11.2.4 Ordinal Categories

Example 11.2.4.1.1. The n**th ordinal category** is the category $\mathbb{n}$ where[7]

[6] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{\mathsf{2disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{\mathsf{2disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{\mathsf{2disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{\mathsf{2disc}} \end{tikzcd}$$

between the discrete 2-category $\mathsf{Mon}_{\mathsf{2disc}}$ on Mon and the 2-category of pointed categories with one object.

[7] In other words, $\mathbb{n}$ is the category associated to the poset

$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$

The category $\mathbb{n}$ for $n \geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (Constructions With Categories, **??**): we have isomorphisms of categories

$$\begin{aligned} \mathbb{1} &\cong \mathbb{0} \star \mathbb{0}, \\ \mathbb{2} &\cong \mathbb{1} \star \mathbb{0} \\ &\cong (\mathbb{0} \star \mathbb{0}) \star \mathbb{0}, \\ \mathbb{3} &\cong \mathbb{2} \star \mathbb{0} \\ &\cong (\mathbb{1} \star \mathbb{0}) \star \mathbb{0} \\ &\cong ((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \\ \mathbb{4} &\cong \mathbb{3} \star \mathbb{0} \end{aligned}$$

- *Objects.* We have

$$\mathrm{Obj}(\mathbb{n}) \overset{\text{def}}{=} \{[0], \ldots, [n]\}.$$

- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have

$$\mathrm{Hom}_{\mathbb{n}}([i],[j]) \overset{\text{def}}{=} \begin{cases} \left\{\mathrm{id}_{[i]}\right\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$

- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map

$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$

of $\mathbb{n}$ at $[i]$ is defined by

$$\mathrm{id}^{\mathbb{n}}_{[i]} \overset{\text{def}}{=} \mathrm{id}_{[i]}\,.$$

- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\mathbb{n})$, the composition map

$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$

of $\mathbb{n}$ at $([i],[j],[k])$ is defined by

$$\begin{gathered} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} = \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) = ([i] \to [k]). \end{gathered}$$

11.2.5 The Walking Arrow

Definition 11.2.5.1.1. The **walking arrow** is the category $\mathbb{1}$ defined as the first ordinal category.

Remark 11.2.5.1.2. In detail, the walking arrow is the category $\mathbb{1}$ where:

- *Objects.* We have $\mathrm{Obj}(\mathbb{1}) = \{0, 1\}$.

$$\begin{aligned} &\cong (\mathbb{2} \star \mathbb{0}) \star \mathbb{0} \\ &\cong ((\mathbb{1} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0} \\ &\cong (((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \end{aligned}$$

and so on.

- *Morphisms.* We have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{1}}(0,0) &= \{\mathrm{id}_0\},\\ \mathrm{Hom}_{\mathbb{1}}(1,1) &= \{\mathrm{id}_1\},\\ \mathrm{Hom}_{\mathbb{1}}(0,1) &= \{f_{01}\},\\ \mathrm{Hom}_{\mathbb{1}}(1,0) &= \emptyset.\end{aligned}$$

- *Identities and Composition.* The identities and composition of $\mathbb{1}$ are completely determined by the unitality and associativity axioms for $\mathbb{1}$.

11.2.6 More Examples of Categories

Example 11.2.6.1.1. Here we list some of the other categories appearing throughout this work.

1. The category Sets_* of pointed sets of Pointed Sets, Definition 6.1.3.1.1.
2. The category Rel of sets and relations of Relations, Definition 8.3.2.1.1.
3. The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of **??**, **??**.
4. The category $\mathsf{ISets}(K)$ of K-indexed sets of Indexed Sets, **??**.
5. The category ISets of indexed sets of Indexed Sets, **??**.
6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of Fibred Sets, **??**.
7. The category $\mathsf{FibSets}$ of fibred sets of Fibred Sets, **??**.
8. Categories of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ as in Definition 11.10.1.1.1.
9. The category of categories Cats of Definition 11.10.2.1.1.
10. The category of groupoids Grpd of Definition 11.10.4.1.1.

11.2.7 Posetal Categories

Definition 11.2.7.1.1. Let $(X, \preceq_X)$ be a poset.

1. The **posetal category associated to** $(X, \preceq_X)$ is the category X_{pos} where

- *Objects.* We have

$$\mathrm{Obj}(X_{\mathsf{pos}}) \overset{\mathrm{def}}{=} X.$$

- *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathsf{pos}})$, we have

$$\mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$

- *Identities.* For each $a \in \mathrm{Obj}(X_{\mathsf{pos}})$, the unit map

$$\mathbb{1}^{X_{\mathsf{pos}}}_a \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, a)$$

  of X_{pos} at a is given by the identity map.

- *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathsf{pos}})$, the composition map

$$\circ^{X_{\mathsf{pos}}}_{a,b,c} \colon \mathrm{Hom}_{X_{\mathsf{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, c)$$

  of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.

2. A category $\mathcal{C}$ is **posetal**[8] if $\mathcal{C}$ is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

Proposition 11.2.7.1.2. Let $(X, \preceq_X)$ be a poset and let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathsf{pos}}$ defines a functor

$$(-)_{\mathsf{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$

   where:

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Pos})$, we have

$$\left[(-)_{\mathsf{pos}}\right](X) \overset{\mathrm{def}}{=} X_{\mathsf{pos}},$$

     where X_{pos} is the category of of Item 1 of Definition 11.2.7.1.1.

   - *Action on Morphisms.* For each morphism of posets $f \colon X \to Y$

[8]*Further Terminology:* Also called a **thin** category or a $(0, 1)$**-category**.

in Pos, the image

$$f_{\mathsf{pos}}\colon X_{\mathsf{pos}} \to Y_{\mathsf{pos}}$$

of f by $(-)_{\mathsf{pos}}$ is the functor defined as follows:

- *The Action of f_{pos} on Objects.* For each $x \in \mathrm{Obj}(X_{\mathsf{pos}})$, we have
$$f_{\mathsf{pos}}(x) \overset{\text{def}}{=} f(x).$$
- *The Action of f_{pos} on Morphisms.* For each $x, y \in \mathrm{Obj}(X_{\mathsf{pos}})$, the action

$$f_{\mathsf{pos}|x,y}\colon \mathrm{Hom}_{X_{\mathsf{pos}}}(x,y) \to \mathrm{Hom}_{Y_{\mathsf{pos}}}(f(x), f(y))$$

of f at (x,y) is given by

$$f_{\mathsf{pos}|x,y}\Big(\mathrm{pt}_{\mathrm{Hom}_{X_{\mathsf{pos}}}(x,y)}\Big) \overset{\text{def}}{=} \mathrm{pt}_{\mathrm{Hom}_{Y_{\mathsf{pos}}}(f(x),f(y))}$$

if $x \preceq_X y$ or, otherwise, by the inclusion of the empty set into $\mathrm{Hom}_{Y_{\mathsf{pos}}}(f(x), f(y))$.

2. *Fully Faithfulness.* The functor $(-)_{\mathsf{pos}}$ of Item 1 is fully faithful.

3. *Characterisations.* The following conditions are equivalent:

   (a) The category $\mathcal{C}$ is posetal.

   (b) For each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have $f = g$.

4. *Automatic Commutativity of Diagrams.* Every diagram in a posetal category commutes.

Proof. *Item 1*, *Functoriality*: First, note that given a morphism of posets $f\colon X \to Y$, the corresponding functor $f_{\mathsf{pos}}\colon X_{\mathsf{pos}} \to Y_{\mathsf{pos}}$ is indeed a functor: since all morphisms in the Hom-sets of Y_{pos} are equal, it preserves identities and compositions trivially.

Next, we claim that $(-)_{\mathsf{pos}}$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Pos})$. Given $x, y \in X$ with $x \preceq_X y$, we have

$$(\mathrm{id}_X)_{\mathsf{pos}}(x) = \mathrm{id}_X(x)$$

$$= \mathrm{id}_{X_{\mathsf{pos}}}(x),$$

so $(\mathrm{id}_X)_{\mathsf{pos}}$ acts like the identity functor of X_{pos} on objects, and

$$\begin{aligned}(\mathrm{id}_X)_{\mathsf{pos}}\Big(\mathrm{pt}_{\mathrm{Hom}_{X_{\mathsf{pos}}}(x,y)}\Big) &= \mathrm{pt}_{\mathrm{Hom}_{X_{\mathsf{pos}}}\left((\mathrm{id}_X)_{\mathsf{pos}}(x),(\mathrm{id}_X)_{\mathsf{pos}}(y)\right)} \\ &= \mathrm{pt}_{\mathrm{Hom}_{X_{\mathsf{pos}}}(a,b)},\end{aligned}$$

so the same holds for morphisms. Thus $(\mathrm{id}_X)_{\mathsf{pos}} = \mathrm{id}_{X_{\mathsf{pos}}}$.

- *Preservation of Composition.* Let $X, Y, Z \in \mathrm{Obj}(\mathsf{Pos})$. Given morphisms of posets $f\colon X \to Y$ and $g\colon Y \to Z$, we need to show

  $$(g \circ f)_{\mathsf{pos}} = g_{\mathsf{pos}} \circ f_{\mathsf{pos}}.$$

  Indeed, given $x \in X$, we have

  $$\begin{aligned}(g \circ f)_{\mathsf{pos}}(x) &= (g \circ f)(x) \\ &= g(f(x)) \\ &= g_{\mathsf{pos}}(f_{\mathsf{pos}}(x)) \\ &= [g_{\mathsf{pos}} \circ f_{\mathsf{pos}}](x),\end{aligned}$$

  so the identity holds on objects. Since Z_{pos} is a posetal category, the identity automatically holds on morphisms since

  $$\begin{aligned}(g \circ f)_{\mathsf{pos}}\Big(\mathrm{pt}_{\mathrm{Hom}_{X_{\mathsf{pos}}}(x,y)}\Big) &= \mathrm{pt}_{\mathrm{Hom}_{Z_{\mathsf{pos}}}(g_{\mathsf{pos}}(f_{\mathsf{pos}}(x)),g_{\mathsf{pos}}(f_{\mathsf{pos}}(y)))} \\ &= [g_{\mathsf{pos}} \circ f_{\mathsf{pos}}]\Big(\mathrm{pt}_{\mathrm{Hom}_{X_{\mathsf{pos}}}(x,y)}\Big)\end{aligned}$$

  for each $x, y \in X$ with $x \preceq_X y$.

Thus $(-)_{\mathsf{pos}}$ is indeed a functor.
Item 2, *Fully Faithfulness*: Omitted.
Item 3, *Characterisations*: Omitted.
Item 4, *Automatic Commutativity of Diagrams*: This follows from the fact that if $\mathcal{C}$ is posetal, then there is at most one morphism between any two objects, namely pt. □

11.3 The Quadruple Adjunction With Sets

11.3.1 Statement

Let $\mathcal{C}$ be a category.

Proposition 11.3.1.1.1. We have a quadruple adjunction

$$
(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon
\begin{tikzcd}
\mathsf{Sets} \arrow[r, shift left=2, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=6, "\perp"] & \mathsf{Cats}, \arrow[l, bend right=60, "\pi_0"'] \arrow[l, shift left=2, "\mathrm{Obj}" description]
\end{tikzcd}
$$

witnessed by bijections of sets

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}),\\
\mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Obj}(\mathcal{C})),\\
\mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}),
\end{aligned}
$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 11.3.2.2.1.

- The functor
$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.

- The functor
$$\mathrm{Obj}\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.

- The functor
$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

Proof. Omitted. □

Warning 11.3.1.1.2. (This is a stub, to be revised and expanded upon later.)

The discrete category functor of Definition 11.3.1.1.1 lifts to a 2-functor, but it fails to preserve 2-categorical colimits, and hence lacks a right 2-adjoint. For instance, the 2-pushout of $\mathrm{pt} \leftarrow S^0 \rightarrow \mathrm{pt}$ in $\mathsf{Sets}_{\mathsf{ldisc}}$ is pt, but in Cats_2 it is given by $\mathsf{B}\mathbb{Z}$.

11.3.2 Connected Components and Connected Categories

11.3.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 11.3.2.1.1. A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[9]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

11.3.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 11.3.2.2.1. The **set of connected components of $\mathcal{C}$** is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 11.3.2.2.2. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \to \mathsf{Sets}.$$

[9]In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/{\sim}$ with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, shift left=1.5ex, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] \arrow[r, phantom, shift left=4ex, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=4ex, "\perp"] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, shift left=1.5ex, "\mathrm{Obj}" description] \end{tikzcd}$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{gathered} \pi_0(\mathcal{C} \coprod \mathcal{D}) \cong \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}), \\ \pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) \cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}), \\ \pi_0\left(\mathrm{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) \cong \mathrm{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right), \end{gathered}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right)\colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\begin{gathered} \pi_{0|\mathcal{C},\mathcal{D}}^{\coprod}\colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) \xrightarrow{\sim} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}), \\ \pi_{0|\mathbb{1}}^{\coprod}\colon \emptyset \xrightarrow{\sim} \pi_0(\emptyset_{\mathsf{cat}}), \end{gathered}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\times}, \pi_{0|\mathbb{1}}^{\times}\right) : (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\times} : \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\sim} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\times} : \mathrm{pt} \xrightarrow{\sim} \pi_0(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: This is proved in Definition 11.3.1.1.1.
Item 3, Interaction With Groupoids: Omitted.
Item 4, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Products: Omitted. □

11.3.2.3 Connected Categories

Definition 11.3.2.3.1. A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[10,11]

11.3.3 Discrete Categories

Definition 11.3.3.1.1. Let X be a set.

1. The **discrete category on** X is the category X_{disc} where
   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\text{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
   $$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$

[10] *Further Terminology:* A category is **disconnected** if it is not connected.
[11] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
$$\mathbb{1}^{X_{\mathsf{disc}}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
of X_{disc} at A is defined by
$$\mathrm{id}^{X_{\mathsf{disc}}}_{A} \overset{\mathrm{def}}{=} \mathrm{id}_{A}\,.$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map
$$\circ^{X_{\mathsf{disc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$\mathrm{id}_{A} \circ \mathrm{id}_{A} \overset{\mathrm{def}}{=} \mathrm{id}_{A}\,.$$

2. A category $\mathcal{C}$ is **discrete** if it is equivalent to X_{disc} for some set X.

Proposition 11.3.3.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$
2. *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=50, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=50, "\pi_0"'] \arrow[l, bend left=15, "\mathrm{Obj}"] \end{tikzcd}$$
3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure
$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}),$$
being equipped with isomorphisms
$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \textstyle\coprod Y_{\mathsf{disc}} \xrightarrow{\sim} (X \textstyle\coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\sim} \emptyset_{\mathsf{disc}},$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} : X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \overset{\sim}{\dashrightarrow} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} : \mathsf{pt} \overset{\sim}{\dashrightarrow} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: This is proved in Definition 11.3.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 4, Symmetric Strong Monoidality With Respect to Products: Omitted. □

11.3.4 Indiscrete Categories

Definition 11.3.4.1.1. Let X be a set.

1. The **indiscrete category on X**[12] is the category X_{indisc} where

   - *Objects.* We have

   $$\mathrm{Obj}(X_{\mathsf{indisc}}) \overset{\text{def}}{=} X.$$

   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have

   $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\overset{\text{def}}{=} \{[A] \to [B]\}\\ &\cong \mathrm{pt}.\end{aligned}$$

   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map

   $$\mathbb{1}^{X_{\mathsf{indisc}}}_{A} : \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$

   of X_{indisc} at A is defined by

   $$\mathrm{id}^{X_{\mathsf{indisc}}}_{A} \overset{\text{def}}{=} \{[A] \to [A]\}.$$

[12] *Further Terminology:* Sometimes called the **chaotic category on X**.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map

$$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$

of X_{disc} at (A, B, C) is defined by

$$([B] \to [C]) \circ ([A] \to [B]) \stackrel{\text{def}}{=} ([A] \to [C]).$$

2. A category $\mathcal{C}$ is **indiscrete** if it is equivalent to X_{indisc} for some set X.

Proposition 11.3.4.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor

$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=50, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=50, "\pi_0"'] \arrow[l, bend left=15, "\mathrm{Obj}"] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{indisc}|X,Y} \colon X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\sim} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\times}_{\mathsf{indisc}|\mathbb{1}} \colon \mathsf{pt} \xrightarrow{\sim} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: This is proved in Definition 11.3.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Products: Omitted. □

11.4 Groupoids

11.4.1 Isomorphisms

Let $\mathcal{C}$ be a category.

Definition 11.4.1.1.1. A morphism $f\colon A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1}\colon B \to A$ of $\mathcal{C}$ such that

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A\,. \end{aligned}$$

Notation 11.4.1.1.2. We write $\mathrm{Iso}_{\mathcal{C}}(A, B)$ for the set of all isomorphisms in $\mathcal{C}$ from A to B.

11.4.2 Groupoids

Definition 11.4.2.1.1. A **groupoid** is a category in which every morphism is an isomorphism.

Example 11.4.2.1.2. The isomorphism of categories of Definition 11.2.3.1.1 restricts to an isomorphism

$$\mathsf{Grp} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Grpd}, \qquad \begin{tikzcd} \mathsf{Grp} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Grpd} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

where Grpd is the full subcategory of Cats spanned by the groupoids.

In other words, we have an identification

$$\{\text{Groups}\} \cong \{\text{One-object groupoids}\}.$$

11.4.3 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

Definition 11.4.3.1.1. The **groupoid completion of $\mathcal{C}$**[13] is the pair $(\mathrm{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

[13] *Further Terminology:* Also called the **Grothendieck groupoid of $\mathcal{C}$** or the

- A groupoid $\mathrm{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[14]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

Construction 11.4.3.1.2. Concretely, the groupoid completion of $\mathcal{C}$ is the Gabriel–Zisman localisation $\mathrm{Mor}(\mathcal{C})^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$; see Constructions With Categories, **??**.

(To be expanded upon later on.)

Proof. Omitted. □

Proposition 11.4.3.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor
$$\mathrm{K}_0 \colon \mathsf{Cats} \to \mathsf{Grpd}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor
$$\mathrm{K}_0 \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$
3. *Adjointness.* We have an adjunction
$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd} \arrow[l, bend left, "\iota"] \end{tikzcd},$$

Grothendieck groupoid completion of $\mathcal{C}$.

[14] See Item 5 of Definition 11.4.3.1.3 for an explicit construction.

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Definition 11.4.4.1.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Grpd} \arrow[l, "\iota"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathcal{G}),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G},\mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd} \arrow[l, shift left, "\iota"] \end{tikzcd},$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) \cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Definition 11.4.4.1.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, phantom, shift right=3, "\perp_2"] & \mathsf{Grpd} \arrow[l, "\iota"'] \end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),\\ \mathsf{Fun}(\mathcal{G},\mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids
$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$
natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram
$$\begin{tikzcd}
\mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] & \mathsf{Grp} \arrow[d, phantom, ""] \\
\mathsf{sSets} \arrow[r, "|-|"'] \arrow[ur, phantom, "\wr"] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"']
\end{tikzcd}$$
commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure
$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$
being equipped with isomorphisms
$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} \colon \mathrm{K}_0(\mathcal{C}) \coprod \mathrm{K}_0(\mathcal{D}) \overset{\sim}{\dashrightarrow} \mathrm{K}_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\coprod} \colon \emptyset_{\mathsf{cat}} \overset{\sim}{\dashrightarrow} \mathrm{K}_0(\emptyset_{\mathsf{cat}}),$$
natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure
$$\left(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}\right) \colon (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Grpd}, \times, \mathsf{pt})$$
being equipped with isomorphisms
$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \overset{\sim}{\dashrightarrow} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\times} \colon \mathsf{pt} \overset{\sim}{\dashrightarrow} \mathrm{K}_0(\mathsf{pt}),$$
natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. Item 1, Functoriality: Omitted.

Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Classifying Spaces: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 7, Symmetric Strong Monoidality With Respect to Products: Omitted.

□

11.4.4 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 11.4.4.1.1. The **core** of $\mathcal{C}$ is the pair $(\mathsf{Core}(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{Core}(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathsf{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathsf{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

Notation 11.4.4.1.2. We also write $\mathcal{C}^{\simeq}$ for $\mathsf{Core}(\mathcal{C})$.

Construction 11.4.4.1.3. The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[15]

1. *Objects.* We have

$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \overset{\text{def}}{=} \mathrm{Obj}(\mathcal{C}).$$

[15] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

Proof. This follows from the fact that functors preserve isomorphisms (Item 1 of Definition 11.5.1.1.6). □

Proposition 11.4.4.1.4. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor

$$\mathsf{Core}\colon \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor

$$\mathsf{Core}\colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Definition 11.4.3.1.3, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Definition 11.4.3.1.3, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}}\right)\colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\begin{aligned} \mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}}\colon \mathsf{Core}(\mathcal{C}) \times \mathsf{Core}(\mathcal{D}) &\overset{\sim}{\dashrightarrow} \mathsf{Core}(\mathcal{C} \times \mathcal{D}), \\ \mathsf{Core}^{\times}_{\mathbb{1}}\colon \mathrm{pt} &\overset{\sim}{\dashrightarrow} \mathsf{Core}(\mathrm{pt}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \coprod \mathsf{Core}(\mathcal{D}) \overset{\sim}{\dashrightarrow} \mathsf{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \overset{\sim}{\dashrightarrow} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Products: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted. □

11.5 Functors

11.5.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.5.1.1.1. A **functor** $F \colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[16] consists of:

1. *Action on Objects.* A map of sets

$$F \colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B)),$$

called the **action on morphisms of** F **at** (A, B)[17].

satisfying the following conditions:

[16] *Further Terminology:* Also called a **covariant functor**.
[17] *Further Terminology:* Also called **action on** Hom**-sets of** F **at** (A, B).

1. *Preservation of Identities.* For each $A \in \text{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \text{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_A"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \text{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \text{Hom}_{\mathcal{D}}(F(A),F(A)) \end{tikzcd}$$

   commutes, i.e. we have

$$F(\text{id}_A) = \text{id}_{F(A)}\,.$$

2. *Preservation of Composition.* For each $A, B, C \in \text{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \text{Hom}_{\mathcal{C}}(B,C) \times \text{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \text{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\ \text{Hom}_{\mathcal{D}}(F(B),F(C)) \times \text{Hom}_{\mathcal{D}}(F(A),F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \text{Hom}_{\mathcal{D}}(F(A),F(C)) \end{tikzcd}$$

   commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(g) \circ F(f).$$

Notation 11.5.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of Constructions With Categories, **??**.

1. Given a functor

$$F : \mathcal{C} \to \mathcal{D},$$

   we also write F_A for $F(A)$.

2. Given a functor

$$F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$

   we also write F^A for $F(A)$.

3. Given a functor

$$F : \mathcal{C} \times \mathcal{C} \to \mathcal{D},$$

   we also write $F_{A,B}$ for $F(A, B)$.

4. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D},$$
we also write F^A_B for $F(A,B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F\colon \mathcal{C} \to \mathcal{D}$.

Notation 11.5.1.1.3. Following the notation $[\![x \mapsto f(x)]\!]$ for a function $f\colon X \to Y$ introduced in Sets, Definition 3.1.1.1.2, we will sometimes denote a functor $F\colon \mathcal{C} \to \mathcal{D}$ by

$$F \overset{\text{def}}{=} [\![A \mapsto F(A)]\!],$$

specially when the action on morphisms of F is clear from its action on objects.

Example 11.5.1.1.4. The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A) \overset{\text{def}}{=} A.$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(\mathrm{id}_{\mathcal{C}})_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,B)}$$
of $\mathrm{id}_{\mathcal{C}}$ at (A,B) is defined by
$$(\mathrm{id}_{\mathcal{C}})_{A,B} \overset{\text{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\,.$$

Proof. Preservation of Identities: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \overset{\text{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions: For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &\overset{\text{def}}{=} g \circ f \\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof. □

Definition 11.5.1.1.5. The **composition** of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$[G \circ F](A) \stackrel{\text{def}}{=} G(F(A)).$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(G \circ F)_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$
of $G \circ F$ at (A, B) is defined by
$$[G \circ F](f) \stackrel{\text{def}}{=} G(F(f)).$$

Proof. *Preservation of Identities*: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\begin{aligned} G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F\text{)} \\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G\text{)} \end{aligned}$$
Preservation of Composition: For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have
$$\begin{aligned} G_{F_{g \circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$
This finishes the proof. □

Proposition 11.5.1.1.6. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[18]

Proof. *Item 1, Preservation of Isomorphisms*: Indeed, we have
$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\Big(f^{-1} \circ f\Big) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$
and
$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\Big(f \circ f^{-1}\Big) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$
showing $F(f)$ to be an isomorphism. □

[18] When the converse holds, we call F *conservative*, see Definition 11.6.4.1.1.

11.5.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of Constructions With Categories, **??**.

Definition 11.5.2.1.1. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

Remark 11.5.2.1.2. In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets

$$F\colon \operatorname{Obj}(\mathcal{C}) \to \operatorname{Obj}(\mathcal{D}),$$

   called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \operatorname{Obj}(\mathcal{C})$, a map

$$F_{A,B}\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{D}}(F(B),F(A)),$$

   called the **action on morphisms of** F **at** (A,B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \operatorname{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \operatorname{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \operatorname{Hom}_{\mathcal{D}}(F(A),F(A)) \end{tikzcd}$$

   commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}\,.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$
\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \arrow[dr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[ur, "F_{B,C} \times F_{A,B}"] \arrow[d, "\circ^{\mathcal{C}}_{A,B,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \arrow[d, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
\mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C), F(A))
\end{tikzcd}
$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

Remark 11.5.2.1.3. Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

11.5.3 Forgetful Functors

Definition 11.5.3.1.1. There isn't a precise definition of a **forgetful functor**.

Remark 11.5.3.1.2. Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Definitions 11.5.3.1.3 and 11.5.3.1.4).

Example 11.5.3.1.3. Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \to \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.

2. *Forgetting Topologies.* The functor $\mathsf{Top} \to \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.

3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \to \mathsf{Sets}$ sending a K-fibred set $\phi_X\colon X \to K$ to the set X, forgetting the map ϕ_X and the base set K.

Example 11.5.3.1.4. Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor $\iota\colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.

2. *Forgetting Inverses.* The inclusion functor $\iota\colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

Notation 11.5.3.1.5. Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$忘\colon \mathsf{Grp} \to \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Definition 11.5.3.1.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.

2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.

3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.

4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánzǐ*, meaning *forgetful functor*.

Remark 11.5.3.1.6. Here we collect the pronunciation of the words in Definition 11.5.3.1.5 for accuracy and completeness.

1. Pronunciation of 忘れる:
    - See here.
    - IPA broad transcription: [wäsɯɾe̞ɾɯ].
    - IPA narrow transcription: [ɰ□äsɨ□ɾe̞ɾɯ̟□].
2. Pronunciation of 忘却関手: Pronunciation:
    - See here.
    - IPA broad transcription: [bo̞ːkʲäkɯ kä̃ũ̟ɕɯ].
    - IPA narrow transcription: [bo̞ːkʲäkɯ̟□ kä̃ũ̟ɕɯ□].
3. Pronunciation of 忘记:
    - See here.
    - Broad IPA transcription: [wɑŋtɕi].
    - Sinological IPA transcription: [wɑŋ$^{51-53}$t͡ɕi^{51}].
4. Pronunciation of 遗忘函子:
    - See here.
    - Broad IPA transcription: [iwɑŋ xänt͡sz̩ɨ].
    - Sinological IPA transcription: [i^{35}wɑŋ^{51} xän^{35}t͡sz̩$^{214-21(4)}$].

11.5.4 The Natural Transformation Associated to a Functor

Definition 11.5.4.1.1. Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[19]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{F^{\dagger}}{\Longrightarrow}"] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

[19]This is the 1-categorical version of Constructions With Sets, ?? of ??.

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \operatorname{Hom}_{\mathcal{C}}(A,B)\to\operatorname{Hom}_{\mathcal{D}}(F_A,F_B)\right\}_{(A,B)\in\operatorname{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B}\stackrel{\text{def}}{=}F_{A,B}.$$

Proof. The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y)\to(A,B)$$

of $\mathcal{C}^{\mathsf{op}}\times\mathcal{C}$, the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^{*}\circ\psi_{*}=\psi_{*}\circ\phi^{*}"] \arrow[d, "F_{X,Y}"'] & \operatorname{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\
\operatorname{Hom}_{\mathcal{D}}(F_X,F_Y) \arrow[r, "F(\phi)^{*}\circ F(\psi)_{*}=F(\psi)_{*}\circ F(\phi)^{*}"'] & \operatorname{Hom}_{\mathcal{D}}(F_A,F_B),
\end{tikzcd}$$

acting on elements as

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \psi\circ f\circ\phi \arrow[d, mapsto] \\
F(f) \arrow[r, mapsto] & F(\psi)\circ F(f)\circ F(\psi)=F(\psi\circ f\circ\phi)
\end{tikzcd}$$

commutes, which follows from the functoriality of F. □

Proposition 11.5.4.1.2. Let $F\colon\mathcal{C}\to\mathcal{D}$ and $G\colon\mathcal{D}\to\mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^{\dagger}\colon\operatorname{Hom}_{\mathcal{C}}\Longrightarrow\operatorname{Hom}_{\mathcal{D}}\circ(F^{\mathsf{op}}\times F)$ associated to F is a natural isomorphism.

   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"', ""{name=L}] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}} \times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] \arrow[dl, phantom, "\Rightarrow F^{\dagger}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \arrow[l, phantom, "\Rightarrow G^{\dagger}" near end, shift right=6] \\
& \mathsf{Sets} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G \circ F)^{\mathsf{op}} \times (G \circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\Rightarrow (G \circ F)^{\dagger}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
& \mathsf{Sets} &
\end{tikzcd}
$$

in Cats_2, i.e. we have

$$(G \circ F)^{\dagger} = \left(G^{\dagger} \star \mathrm{id}_{F^{\mathsf{op}} \times F}\right) \circ F^{\dagger}.$$

3. *Interaction With Identities.* We have

$$\mathrm{id}_{\mathcal{C}}^{\dagger} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)},$$

i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

Proof. *Item 1*, *Interaction With Natural Isomorphisms*: Omitted.
Item 2, *Interaction With Composition*: Omitted.
Item 3, *Interaction With Identities*: Omitted. □

11.6 Conditions on Functors

11.6.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.1.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

Proposition 11.6.1.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Composition.* If F and G are faithful, then so is $G \circ F$.

2. *Interaction With Postcomposition.* The following conditions are equivalent:

(a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful.

(b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is faithful.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 14.1.1.1.1.

3. *Interaction With Precomposition I.* Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

(a) If F is faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be faithful.

(b) Conversely, if the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful, then F ***can fail*** to be faithful.

4. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

5. *Interaction With Precomposition III.* The following conditions are equivalent:

(a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

(b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

(c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is monadic.

(d) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 14.2.1.1.1.

(e) The components

$$\eta_G\colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all monomorphisms.

(f) The components

$$\epsilon_G\colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon\colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all epimorphisms.

(g) The functor F is dominant (Definition 11.7.1.1.1), i.e. every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A of $\mathcal{C}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \mathrm{id}_B$.

Proof. Item 1, Interaction With Composition: Since the map

$$(G \circ F)_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

defined as the composition

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{F_{A,B}} \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) \xrightarrow{G_{F(A),F(B)}} \mathrm{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

is a composition of injective functions, it follows from ?? that it is also injective. Therefore $G \circ F$ is faithful.
Item 2, Interaction With Postcomposition: Omitted.
Item 3, Interaction With Precomposition I: See [MSE 733163] for Item 3a. Item 3b follows from Item 4 and the fact that there are essentially surjective functors that are not faithful.
Item 4, Interaction With Precomposition II: Omitted, but see https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html for a formalised proof.
Item 5, Interaction With Precomposition III: We claim Items 5a to 5g are equivalent:

- *Items 5a and 5d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Types of Morphisms in Bicategories, Definition 14.2.1.1.1.
- *Items 5a to 5c and 5g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 5a and 5g.
- *Items 5a, 5e and 5f Are Equivalent:* See **??**, **??** of **??**.

This finishes the proof. □

11.6.2 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

Proposition 11.6.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Composition.* If F and G are full, then so is $G \circ F$.
2. *Interaction With Postcomposition I.* If F is full, then the postcomposition functor
$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
can fail to be full.

3. *Interaction With Postcomposition II.* If, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
is full, then F is also full.

4. *Interaction With Precomposition I.* If F is full, then the precomposition functor
$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
can fail to be full.

5. *Interaction With Precomposition II.* If, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is full, then F ***can fail*** to be full.

6. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor
$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is full (and also faithful by Item 4 of Definition 11.6.1.1.2).

7. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is full.

   (b) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably full morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 14.2.1.1.1.

   (c) The components
   $$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$
   of the unit
   $$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D}, \mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$
   of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

(d) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

(e) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following condition:

($\star$) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each pair of morphisms

$$\begin{aligned} r &\colon F(A) \to B, \\ s &\colon B \to F(A) \end{aligned}$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. Item 1, Interaction With Composition: Since the map

$$(G \circ F)_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

defined as the composition

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{F_{A,B}} \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) \xrightarrow{G_{F(A),F(B)}} \mathrm{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

is a composition of surjective functions, it follows from **??** that it is also surjective. Therefore $G \circ F$ is full.

Item 2, Interaction With Postcomposition I: We follow the proof (completely formalised in cubical Agda!) given by Naïm Camille Favier in [**favier:postcompose-not-full**]. Let $\mathcal{C}$ be the category where:

- *Objects.* We have $\mathrm{Obj}(\mathcal{C}) = \{A, B\}$.
- *Morphisms.* We have

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(A, A) &= \{e_A, \mathrm{id}_A\},\\
\mathrm{Hom}_{\mathcal{C}}(B, B) &= \{e_B, \mathrm{id}_B\},\\
\mathrm{Hom}_{\mathcal{C}}(A, B) &= \{f, g\},\\
\mathrm{Hom}_{\mathcal{C}}(B, A) &= \emptyset.
\end{aligned}
$$

- *Composition.* The nontrivial compositions in $\mathcal{C}$ are the following:

$$
\begin{aligned}
e_A \circ e_A &= \mathrm{id}_A, & f \circ e_A &= g, & e_B \circ f &= f,\\
e_B \circ e_B &= \mathrm{id}_B, & g \circ e_A &= f, & e_B \circ g &= g.
\end{aligned}
$$

We may picture $\mathcal{C}$ as follows:

$$
\begin{tikzcd}
A \arrow[loop left, "e_A"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[loop right, "e_B"]
\end{tikzcd}.
$$

Next, let $\mathcal{D}$ be the walking arrow category $\mathbb{1}$ of Definition 11.2.5.1.1 and let $F \colon \mathcal{C} \to \mathbb{1}$ be the functor given on objects by

$$
\begin{aligned}
F(A) &= 0,\\
F(B) &= 1
\end{aligned}
$$

and on non-identity morphisms by

$$
\begin{aligned}
F(f) &= f_{01}, & F(e_A) &= \mathrm{id}_0,\\
F(g) &= f_{01}, & F(e_B) &= \mathrm{id}_1 .
\end{aligned}
$$

Finally, let $\mathcal{X} = \mathsf{B}\mathbb{Z}_{/2}$ be the walking involution and let $\iota_A, \iota_B \colon \mathsf{B}\mathbb{Z}_{/2} \rightrightarrows \mathcal{C}$ be the inclusion functors from $\mathsf{B}\mathbb{Z}_{/2}$ to $\mathcal{C}$ with

$$
\begin{aligned}
\iota_A(\bullet) &= A,\\
\iota_B(\bullet) &= B.
\end{aligned}
$$

Since every morphism in $\mathbb{1}$ has a preimage in $\mathcal{C}$ by F, the functor F is full. Now, for F_* to be full, the map

$$
\begin{aligned}
F_{*|\iota_A,\iota_B} \colon \mathrm{Nat}(\iota_A, \iota_B) &\longrightarrow \mathrm{Nat}(F \circ \iota_A, F \circ \iota_B)\\
\alpha &\longmapsto \mathrm{id}_F \star \alpha
\end{aligned}
$$

would need to be surjective. However, as we will show next, we have

$$\mathrm{Nat}(\iota_A, \iota_B) = \emptyset,$$
$$\mathrm{Nat}(F \circ \iota_A, F \circ \iota_B) \cong \mathrm{pt},$$

so this is impossible:

- *Proof of* $\mathrm{Nat}(\iota_A, \iota_B) = \emptyset$*:* A natural transformation $\alpha\colon \iota_A \Rightarrow \iota_B$ consists of a morphism

  $$\alpha\colon \underbrace{\iota_A(\bullet)}_{=A} \to \underbrace{\iota_B(\bullet)}_{=B}$$

  in $\mathcal{C}$ making the diagram

  $$\begin{tikzcd} \iota_A(\bullet) \arrow[r, "\iota_A(e)"] \arrow[d, "\alpha"'] & \iota_A(\bullet) \arrow[d, "\alpha"] \\ \iota_B(\bullet) \arrow[r, "\iota_B(e)"'] & \iota_B(\bullet) \end{tikzcd}$$

  commute for each $e \in \mathrm{Hom}_{\mathsf{B}\mathbb{Z}_{/2}}(\bullet, \bullet) \cong \mathbb{Z}_{/2}$. We have two cases:

  1. If $\alpha = f$, the naturality diagram for the unique nonidentity element of $\mathbb{Z}_{/2}$ is given by

     $$\begin{tikzcd} A \arrow[r, "e_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"'] & B. \end{tikzcd}$$

     However, $e_B \circ f = f$ and $f \circ e_A = g$, so this diagram does not commute.
  2. If $\alpha = g$, the naturality diagram for the unique nonidentity element of $\mathbb{Z}_{/2}$ is given by

     $$\begin{tikzcd} A \arrow[r, "e_A"] \arrow[d, "g"'] & A \arrow[d, "g"] \\ B \arrow[r, "e_B"'] & B. \end{tikzcd}$$

However, $e_B \circ g = g$ and $g \circ e_A = f$, so this diagram does not commute.

As a result, there are no natural transformations from ι_A to ι_B.

- *Proof of* $\mathrm{Nat}(F \circ \iota_A, F \circ \iota_B) \cong \mathrm{pt}$*:* A natural transformation

$$\beta\colon F \circ \iota_A \Rightarrow F \circ \iota_B$$

consists of a morphism

$$\beta\colon \underbrace{[F \circ \iota_A](\bullet)}_{=0} \to \underbrace{[F \circ \iota_B](\bullet)}_{=1}$$

in $\mathbb{1}$ making the diagram

$$\begin{tikzcd}
{[F \circ \iota_A](\bullet)} \arrow[r, "{[F\circ\iota_A](e)}"] \arrow[d, "\beta"'] & {[F \circ \iota_A](\bullet)} \arrow[d, "\beta"] \\
{[F \circ \iota_B](\bullet)} \arrow[r, "{[F\circ\iota_B](e)}"'] & {[F \circ \iota_B](\bullet)}
\end{tikzcd}$$

commute for each $e \in \mathrm{Hom}_{\mathsf{B}\mathbb{Z}_{/2}}(\bullet, \bullet) \cong \mathbb{Z}_{/2}$. Since the only morphism from 0 to 1 in $\mathbb{1}$ is f_{01}, we must have $\beta = f_{01}$ if such a transformation were to exist, and in fact it indeed does, as in this case the naturality diagram above becomes

$$\begin{tikzcd}
0 \arrow[r, "\mathrm{id}_0"] \arrow[d, "f_{01}"'] & 0 \arrow[d, "f_{01}"] \\
1 \arrow[r, "\mathrm{id}_1"'] & 1
\end{tikzcd}$$

for each $e \in \mathbb{Z}_{/2}$, and this diagram indeed commutes, making β into a natural transformation.

This finishes the proof.

Item 3, Interaction With Postcomposition II: Taking $\mathcal{X} = \mathsf{pt}$, it follows by assumption that the functor

$$F_*\colon \mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \to \mathsf{Fun}(\mathsf{pt}, \mathcal{D})$$

is full. However, by Item 5 of Definition 11.10.1.1.2, we have isomorphisms of categories

$$\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C},$$
$$\mathsf{Fun}(\mathsf{pt}, \mathcal{D}) \cong \mathcal{D}$$

and the diagram

$$\begin{tikzcd}
\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \arrow[r, "F_*"] \arrow[d, dashed, "\wr"] & \mathsf{Fun}(\mathsf{pt}, \mathcal{D}) \arrow[d, dashed, "\wr"'] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}$$

commutes. It then follows from Item 1 that F is full.
Item 4, Interaction With Precomposition I: Omitted.
Item 5, Interaction With Precomposition II: See [BS10, p. 47].
Item 6, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 7, Interaction With Precomposition IV: We claim Items 7a to 7e are equivalent:

- *Items 7a and 7b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Types of Morphisms in Bicategories, Definition 14.2.2.1.1.

- *Items 7a, 7c and 7d Are Equivalent:* See **??**, **??** of **??**.

- *Items 7a and 7e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof. □

Question 11.6.2.1.3. Item 7 of Definition 11.6.2.1.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones?

This question also appears as [`MO 468121b`].

11.6.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

Proposition 11.6.3.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is fully faithful.
   (b) We have a pullback square

$$\mathsf{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D}\times\mathcal{D}} \mathsf{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathsf{Arr}(\mathcal{C}) \arrow[r, "\mathsf{Arr}(F)"] \arrow[d, "\mathrm{src}\times\mathrm{tgt}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src}\times\mathrm{tgt}"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D}\times\mathcal{D} \end{tikzcd}$$

   in Cats.

2. *Interaction With Composition.* If F and G are fully faithful, then so is $G \circ F$.

3. *Conservativity.* If F is fully faithful, then F is conservative.

4. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

5. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

6. *Interaction With Postcomposition.* The following conditions are equivalent:
   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably fully faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 14.1.3.1.1.

7. *Interaction With Precomposition I.* If F is fully faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be fully faithful.

8. *Interaction With Precomposition II.* If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful, then F ***can fail*** to be fully faithful (and in fact it can also fail to be either full or faithful).

9. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

10. *Interaction With Precomposition IV.* The following conditions are equivalent:

(a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

(b) The precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$$

is fully faithful.

(c) The functor

$$\mathrm{Lan}_F\colon \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D}, \mathsf{Sets})$$

is fully faithful.

(d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 14.2.3.1.1.

(e) The functor F is absolutely dense.

(f) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

(g) The components

$$\epsilon_G\colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon\colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

(h) The natural transformation

$$\alpha\colon \mathrm{Lan}_{h_F}\left(h^F\right) \Longrightarrow h$$

with components

$$\alpha_{B',B}\colon \int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi,\psi)]) = \psi \circ \phi$$

is a natural isomorphism.

(i) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of C;
- A morphism $s_B\colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B\colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have $r_B \circ s_B = \mathrm{id}_B$.

ii. For each morphism $f \colon B' \to B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. Item 1, Characterisations: Omitted.

Item 2, Interaction With Composition: Since the map

$$(G \circ F)_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

defined as the composition

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{F_{A,B}} \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) \xrightarrow{G_{F(A),F(B)}} \mathrm{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

is a composition of bijective functions, it follows from **??** that it is also bijective. Therefore $G \circ F$ is fully faithful.

Item 3, Conservativity: This is a repetition of Item 2 of Definition 11.6.4.1.2, and is proved there.

Item 4, Essential Injectivity: Omitted.

Item 5, Interaction With Co/Limits: Omitted.

Item 6, Interaction With Postcomposition: This follows from Item 2 of Definition 11.6.1.1.2 and **??** of Definition 11.6.2.1.2.

Item 7, Interaction With Precomposition I: See [`MSE 733161`] for an example of a fully faithful functor whose precomposition with which fails to be full.

Item 8, Interaction With Precomposition II: See [`MSE 749304`, Item 3].

Item 9, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.

Item 10, Interaction With Precomposition IV: We claim Items 10a to 10i are equivalent:

- *Items 10a and 10d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Types of Morphisms in Bicategories, Definition 14.2.3.1.1.

- *Items 10a, 10f and 10g Are Equivalent:* See **??**, **??** of **??**.

- *Items 10a to 10c Are Equivalent:* This follows from [Low15, Proposition A.1.5].

- *Items 10a, 10e, 10h and 10i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

11.6.4 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[20]

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 11.6.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is conservative.
   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.
2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.
3. *Interaction With Precomposition.* The following conditions are equivalent:
   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is conservative.
   (b) The equivalent conditions of Item 5 of Definition 11.6.1.1.2 are satisfied.

Proof. *Item 1, Characterisations*: This follows from Item 1 of Definition 11.5.1.1.6.

Item 2, Interaction With Fully Faithfulness: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$F(\mathrm{id}_B) = \mathrm{id}_{F(B)}$$

[20] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

$$
\begin{aligned}
&= F(f) \circ F(f)^{-1} \\
&= F\Big(f \circ f^{-1}\Big).
\end{aligned}
$$

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$
\begin{aligned}
f \circ f^{-1} &= \mathrm{id}_B, \\
f^{-1} \circ f &= \mathrm{id}_A,
\end{aligned}
$$

showing f to be an isomorphism. Thus F is conservative. □

Question 11.6.4.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

($\star$) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

11.6.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

($\star$) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

Question 11.6.5.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

11.6.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[21] if it satisfies the following condition:

($\star$) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

Question 11.6.6.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is essentially surjective?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X},\mathcal{C}) \to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

11.6.7 Equivalences of Categories

Definition 11.6.7.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$\begin{aligned} F&\colon \mathcal{C} \to \mathcal{D},\\ G&\colon \mathcal{D} \to \mathcal{C} \end{aligned}$$

together with natural isomorphisms

$$\begin{aligned} \eta&\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F,\\ \epsilon&\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}\,. \end{aligned}$$

[21] *Further Terminology:* Also called an **eso** functor, meaning *essentially surjective on*

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 11.6.7.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[22], then the following conditions are equivalent:[23]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor

   $$F|_{\mathsf{Sk}(\mathcal{C})}\colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

   is an *isomorphism* of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an equivalence of categories.

   (e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an equivalence of categories.

2. *Two-Out-of-Three.* Let

   $$\begin{tikzcd} \mathcal{C} \arrow[rr, "G \circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

   be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

---

objects.

[22] Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[23] In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

   be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[24]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection

$$\pi_0(F) \colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

$$F_{x,x} \colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\Longrightarrow$ *Item 1b:* Omitted.

2. *Item 1b* $\Longrightarrow$ *Item 1a:* Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B \colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j \colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B \colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta \colon \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon \colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

the law of excluded middle.

[24]More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint

3. *Item 1a* $\implies$ *Item 1c:* This follows from Item 4 of Definition 11.1.3.1.3.

4. *Item 1c* $\implies$ *Item 1a:* Omitted.

5. *Items 1a, 1d and 1e Are Equivalent:* This follows from **??**.

This finishes the proof of Item 1.
Item 2, Two-Out-of-Three: Omitted.
Item 3, Stability Under Composition: Omitted.
Item 4, Equivalences vs.Adjoint Equivalences: See [Rie16, Proposition 4.4.5].
Item 5, Interaction With Groupoids: See [nLa25, Proposition 4.4]. □

11.6.8 Isomorphisms of Categories

Definition 11.6.8.1.1. An **isomorphism of categories** is a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D},$$
$$G\colon \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 11.6.8.1.2. Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 11.6.8.1.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

   (a) The functor F is an isomorphism of categories.

   (b) The functor F is fully faithful and bijective on objects.

   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an isomorphism of categories.

(d) For each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an isomorphism of categories.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Definition 11.6.7.1.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

11.7 More Conditions on Functors

11.7.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.1.1.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r \colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

Proposition 11.7.1.1.2. Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I \colon \mathcal{X} \to \mathcal{C}$ be a functor.

1. *Interaction With Right Whiskering.* If I is full and dominant, then the map
$$- \star \mathrm{id}_I \colon \mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ I, G \circ I)$$
is a bijection.

2. *Interaction With Adjunctions.* Let $(F, G) \colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction.
   (a) If F is dominant, then G is faithful.
   (b) The following conditions are equivalent:
      i. The functor G is full.
      ii. The restriction
      $$G|_{\mathrm{Im}_F} \colon \mathrm{Im}(F) \to \mathcal{C}$$
      of G to $\mathrm{Im}(F)$ is full.

Proof. *Item 1, Interaction With Right Whiskering*: See [DFH75, Proposition 1.4].
Item 2, Interaction With Adjunctions: See [DFH75, Proposition 1.7]. □

Question 11.7.1.1.3. Is there a characterisation of functors $F \colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
is dominant?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
is dominant?

This question also appears as [MO 468121a].

11.7.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**, **??**).

Proposition 11.7.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is a monomorphism of categories.
   (b) The functor F is injective on objects and morphisms, i.e. F is injective on objects and the map
   $$F\colon \operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$$
   is injective.

Proof. *Item 1, Characterisations*: Omitted. □

Question 11.7.2.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \operatorname{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is a monomorphism of categories?
2. For each $\mathcal{X} \in \operatorname{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is a monomorphism of categories?

This question also appears as [MO 468121a].

11.7.3 Epimorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**, **??**).

Proposition 11.7.3.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:[25]

   (a) The functor F is a epimorphism of categories.

   (b) For each morphism $f\colon A \to B$ of $\mathcal{D}$, we have a diagram

$$\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"'] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}$$

   in $\mathcal{D}$ satisfying the following conditions:

   i. We have $f = \alpha_0 \circ \phi_1$.

   ii. We have $f = \psi_m \circ \alpha_{2m}$.

   iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.

2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

Proof. *Item 1, Characterisations*: See [Isb68].
Item 2, Surjectivity on Objects: Omitted. □

Question 11.7.3.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an epimorphism of categories?

equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

[25] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an epimorphism of categories?

This question also appears as [MO 468121a].

11.7.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Proposition 11.7.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is pseudomonic.
   (b) The functor F satisfies the following conditions:
      i. The functor F is faithful, i.e. for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
      $$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$
      of F at (A,B) is injective.
      ii. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the restriction
      $$F^{\mathrm{iso}}_{A,B}\colon \mathrm{Iso}_{\mathcal{C}}(A,B) \to \mathrm{Iso}_{\mathcal{D}}(F_A, F_B)$$
      of the action on morphisms of F at (A,B) to isomorphisms is surjective.
   (c) We have an isocomma square of the form
   $$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{D}} \mathcal{C}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$
   in Cats_2 up to equivalence.
   (d) We have an isocomma square of the form
   $$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathsf{Arr}(\mathcal{D})} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathsf{Arr}(F)"] \\ \mathcal{D} \arrow[r, hook] & \mathsf{Arr}(\mathcal{D}) \end{tikzcd}$$
   in Cats_2 up to equivalence.

(e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition[26] functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

2. *Conservativity.* If F is pseudomonic, then F is conservative.

3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Conservativity: Omitted.
Item 3, Essential Injectivity: Omitted. □

11.7.5 Pseudoepic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.5.1.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

[26] Asking the precomposition functors

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Definition 11.7.5.1.2.

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & \mathcal{X} \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & \mathcal{X} \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"{name=U}] \arrow[r, bend right, "\psi\circ F"'{name=D}] & \mathcal{X} \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F\,.$$

Proposition 11.7.5.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is pseudoepic.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   given by precomposition by F is pseudomonic.

   (c) We have an isococomma square of the form

   $$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] \arrow[ur, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

   in Cats_2 up to equivalence.

2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 11.7.1.1.1).

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Dominance: If F is pseudoepic, then

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 5g of Item 5 of Definition 11.6.1.1.2, this is equivalent to requiring F to be dominant. $\square$

Question 11.7.5.1.3. Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Definition 11.7.4.1.2?

This question also appears as [MO 321971].

Question 11.7.5.1.4. A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?

This question also appears as [MO 468334].

Question 11.7.5.1.5. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is pseudoepic?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is pseudoepic?

This question also appears as [MO 468121a].

11.8 Even More Conditions on Functors

11.8.1 Injective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.1.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **injective on objects** if the action on objects

$$F\colon\ \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

Proposition 11.8.1.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is injective on objects.

   (b) The functor F is an isocofibration in Cats_2.

Proof. *Item 1, Characterisations*: Omitted. □

11.8.2 Surjective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

11.8.3 Bijective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **bijective on objects**[27] if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

11.8.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 11.8.4.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

Question 11.8.4.1.3. Is there a characterisation of functors representably faithful on cores?

[27] *Further Terminology:* Also called a **bo** functor.

11.8.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 11.8.5.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 11.8.5.1.3. Is there a characterisation of functors representably full on cores?

This question also appears as [MO 468121a].

11.8.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 11.8.6.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Definitions 11.8.4.1.2 and 11.8.5.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

with α and β natural isomorphisms, if we have $\mathrm{id}_F \star\alpha = \mathrm{id}_F \star\beta$, then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of $\mathcal{C}$, there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star\alpha.$$

Question 11.8.6.1.3. Is there a characterisation of functors representably fully faithful on cores?

11.8.7 Functors Corepresentably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.7.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 11.8.7.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

Question 11.8.7.1.3. Is there a characterisation of functors corepresentably faithful on cores?

11.8.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.8.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 11.8.8.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F\,.$$

Question 11.8.8.1.3. Is there a characterisation of functors corepresentably full on cores?

This question also appears as [MO 468121a].

11.8.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.9.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{D}))$$

is fully faithful.

Remark 11.8.9.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Definitions 11.8.7.1.2 and 11.8.8.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"{name=U}] \arrow[r, bend right, "\psi\circ F"'{name=D}] & \mathcal{X}, \arrow[from=U, to=D, phantom, "\Downarrow\beta"] \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & \mathcal{X} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"{name=U}] \arrow[r, bend right, "F\circ\psi"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow\beta"] \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F\,.$$

Question 11.8.9.1.3. Is there a characterisation of functors corepresentably fully faithful on cores?

11.9 Natural Transformations

11.9.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 11.9.1.1.1. A **transformation**[28] $\alpha\colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

Notation 11.9.1.1.2. We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

Remark 11.9.1.1.3. We have an isomorphism

$$\mathrm{Trans}(F, G) \cong \prod_{A\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F_A, G_A).$$

Proof. Omitted. □

[28] *Further Terminology:* Also called an **unnatural transformation** for emphasis.

11.9.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 11.9.2.1.1. A **natural transformation** $\alpha\colon F \Rightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.

Remark 11.9.2.1.2. Let $\alpha\colon F \Rightarrow G$ be a natural transformation.

1. For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** α **at** A.

2. We denote natural transformations such as α in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "G"', ""{name=D}] & \mathcal{D}. \arrow[Rightarrow, from=U, to=D, "\alpha"'] \end{tikzcd}$$

Notation 11.9.2.1.3. We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Definition 11.9.2.1.4. Two natural transformations $\alpha, \beta\colon F \Rightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

11.9.3 Examples of Natural Transformations

Example 11.9.3.1.1. The **identity natural transformation** $\mathrm{id}_F\colon F \Rightarrow F$ **of** F is the natural transformation consisting of the collection

$$\{(\mathrm{id}_F)_A\colon F(A) \to F(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

defined by

$$(\mathrm{id}_F)_A \overset{\mathrm{def}}{=} \mathrm{id}_{F(A)}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof. The naturality condition for id_F is the requirement that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

commutes. This follows from unitality of the composition of $\mathcal{D}$, as we have

$$\begin{aligned}
F(f) \circ \mathrm{id}_{F(A)} &= F(f) \\
&= \mathrm{id}_{F(B)} \circ F(f),
\end{aligned}$$

where we have applied unitality twice. □

Example 11.9.3.1.2. Let A and B be monoids and let $f, g\colon A \rightrightarrows B$ be morphisms of monoids. Applying the delooping construction of **??**, we obtain functors $\mathsf{B}f, \mathsf{B}g\colon \mathsf{B}A \rightrightarrows \mathsf{B}B$. We then have

$$\mathrm{Nat}(\mathsf{B}f, \mathsf{B}g) \cong \left\{ b \in B \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } bf(a) = g(a)b\end{array} \right\}.$$

Proof. Unwinding the definitions in this case, we see that a transformation α from $\mathsf{B}f$ to $\mathsf{B}g$ consists of a collection

$$\{\alpha_\bullet\colon \bullet \to \bullet\}_{\bullet\in\mathrm{Obj}(\mathsf{B}A)}$$

of morphisms of $\mathsf{B}B$ indexed by $\mathrm{Obj}(\mathsf{B}A)$. Since $\mathrm{Obj}(\mathsf{B}A) = \mathrm{pt}$ and the morphisms of $\mathsf{B}B$ are precisely the elements of B, it follows that α corresponds precisely to the data of an element $b \in B$. Now, a transformation $[b]\colon \mathsf{B}f \Rightarrow \mathsf{B}g$ is natural precisely if, for each $a \in \mathrm{Hom}_{\mathsf{B}A}(\bullet, \bullet) \stackrel{\text{def}}{=} A$, the diagram

$$\begin{tikzcd}
\mathsf{B}f(\bullet) \arrow[r, "\mathsf{B}f(a)"] \arrow[d, "{[b]_\bullet}"'] & \mathsf{B}f(\bullet) \arrow[d, "{[b]_\bullet}"] \\
\mathsf{B}g(\bullet) \arrow[r, "\mathsf{B}g(a)"'] & \mathsf{B}g(\bullet)
\end{tikzcd}$$

commutes. Unwinding the definitions, we see that this diagram is given by

$$\begin{tikzcd}
\bullet \arrow[r, "f(a)"] \arrow[d, "b"'] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "g(a)"'] & \bullet,
\end{tikzcd}$$

and hence corresponds precisely to the condition $g(a)b = bf(a)$. □

11.9.4 Vertical Composition of Natural Transformations

Definition 11.9.4.1.1. The **vertical composition** of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D}
\arrow[from=F, to=G, Rightarrow, "\alpha"']
\arrow[from=G, to=H, Rightarrow, "\beta"']
\end{tikzcd}$$

is the natural transformation $\beta \circ \alpha\colon F \Longrightarrow H$ consisting of the collection

$$\left\{(\beta \circ \alpha)_A\colon F(A) \to H(A)\right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \stackrel{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof. The naturality condition for $\beta\circ\alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\
H(A) \arrow[r, "H(f)"'] & H(B)
\end{tikzcd}$$

commutes. Since

- Subdiagram (1) commutes by the naturality of α.
- Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta\circ\alpha$ is a natural transformation. $\square$

Proposition 11.9.4.1.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta,\alpha)\mapsto\beta\circ\alpha$ defines a function

$$\circ_{F,G,H}\colon \operatorname{Nat}(G,H)\times\operatorname{Nat}(F,G)\to\operatorname{Nat}(F,H).$$

2. *Associativity.* Let $F,G,H,K\colon\mathcal{C}\overset{\rightrightarrows}{\rightrightarrows}\mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
 & \operatorname{Nat}(H,K)\times(\operatorname{Nat}(G,H)\times\operatorname{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\operatorname{Nat}(H,K)}\times\circ_{F,G,H}"] & \\
(\operatorname{Nat}(H,K)\times\operatorname{Nat}(G,H))\times\operatorname{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\operatorname{Nat}(H,K),\operatorname{Nat}(G,H),\operatorname{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K}\times\mathrm{id}_{\operatorname{Nat}(F,G)}"'] & & \operatorname{Nat}(H,K)\times\operatorname{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\operatorname{Nat}(G,K)\times\operatorname{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \operatorname{Nat}(F,K)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F, G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G, G) \times \mathrm{Nat}(F, G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F, G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

   (b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F, G) \times \mathrm{pt} \arrow[d, "{\mathrm{id}_{\mathrm{Nat}(F,G)} \times [\mathrm{id}_F]}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F, G) \times \mathrm{Nat}(F, F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F, G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(G_1, G_2)) \times (\mathrm{Nat}(F_2, F_3) \times \mathrm{Nat}(F_1, F_2)) \arrow[rr, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(F_2, F_3)) \times (\mathrm{Nat}(G_1, G_2) \times \mathrm{Nat}(F_1, F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1, G_3) \times \mathrm{Nat}(F_1, F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=50, "F_3"'{name=D}] \arrow[from=U, to=M, phantom, "\alpha\Downarrow"] \arrow[from=M, to=D, phantom, "\alpha'\Downarrow"] & \mathcal{D} \arrow[r, bend left=50, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=50, "G_3"'{name=D2}] \arrow[from=U2, to=M2, phantom, "\beta\Downarrow"] \arrow[from=M2, to=D2, phantom, "\beta'\Downarrow"] & \mathcal{E}
\end{tikzcd}
$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Omitted.
Item 2, Associativity: Indeed, we have

$$
\begin{aligned}
((\gamma \circ \beta) \circ \alpha)_A &\stackrel{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\
&\stackrel{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\
&= \gamma_A \circ (\beta_A \circ \alpha_A) \\
&\stackrel{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\
&\stackrel{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A
\end{aligned}
$$

for each $A \in \text{Obj}(\mathcal{C})$, showing the desired equality.
Item 3, Unitality: We have

$$
\begin{aligned}
(\text{id}_G \circ \alpha)_A &= \text{id}_G \circ \alpha_A \\
&= \alpha_A, \\
(\alpha \circ \text{id}_F)_A &= \alpha_A \circ \text{id}_F \\
&= \alpha_A
\end{aligned}
$$

for each $A \in \text{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, Middle Four Exchange: This is proved in Item 4 of Definition 11.9.5.1.3.

□

11.9.5 Horizontal Composition of Natural Transformations

Definition 11.9.5.1.1. The **horizontal composition**[29,30] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\beta\star\alpha \Downarrow"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\left\{(\beta \star \alpha)_A\colon H(F(A)) \to K(G(A))\right\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned} (\beta\star\alpha)_A &\overset{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A) \\ &= K(\alpha_A) \circ \beta_{F(A)}, \end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

Proof. First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

[29] *Further Terminology:* Also called the **Godement product** of α and β.
[30] Horizontal composition forms a map

$$\star_{(F,H),(G,K)}\colon \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H\circ F, K \circ G).$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A\colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd}
H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\
H(G(A)) \arrow[r, "H(G(f))" description] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\
K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B))
\end{tikzcd}$$

commutes. Since

- Subdiagram (1) commutes by the naturality of α.
- Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[31] □

Definition 11.9.5.1.2. Let

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y}
\end{tikzcd}$$

be a diagram in Cats_2.

1. The **left whiskering of α with G** is the natural transformation[32]

$$\mathrm{id}_G \star \alpha\colon G \circ \phi \Longrightarrow G \circ \psi.$$

[31] *Reference:* [Bor94, Proposition 1.3.4].

[32] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations in this work.

2. The **right whiskering of α with F** is the natural transformation[33]

$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

Proposition 11.9.5.1.3. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\
\mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathcal{F},
\end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\
\mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F)
\end{tikzcd}$$

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}\,.$$

[33] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(G_1,G_2))\times(\mathrm{Nat}(F_2,F_3)\times\mathrm{Nat}(F_1,F_2)) \arrow[r, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3}\times\circ_{F_1,F_2,F_3}"'] & (\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(F_2,F_3))\times(\mathrm{Nat}(G_1,G_2)\times\mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3}\times\star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3)\times\mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & \mathrm{Nat}(G_2\circ F_2,G_3\circ F_3)\times\mathrm{Nat}(G_1\circ F_1,G_2\circ F_2) \arrow[d, "\circ_{G_1\circ F_1,G_2\circ F_2,G_3\circ F_3}"] \\
 & \mathrm{Nat}(G_1\circ F_1,G_3\circ F_3)
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left=50, "F_1"] \arrow[rr, "F_2" description] \arrow[rr, bend right=50, "F_3"'] & \arrow[u, phantom, "\alpha\Downarrow"] \arrow[d, phantom, "\alpha'\Downarrow"] & \mathcal{D} \arrow[rr, bend left=50, "G_1"] \arrow[rr, "G_2" description] \arrow[rr, bend right=50, "G_3"'] & \arrow[u, phantom, "\beta\Downarrow"] \arrow[d, phantom, "\beta'\Downarrow"] & \mathcal{E}
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Omitted.
Item 2, Associativity: Omitted.
Item 3, Interaction With Identities: We have

$$\begin{aligned}
(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
&\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \\
&\overset{\text{def}}{=} (\mathrm{id}_{G\circ F})_A
\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

notations in this work.

Item 4, Middle Four Exchange: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

The top composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad \begin{tikzcd} G_1(F_2(A)) \arrow[r, "G_1(\alpha'_A)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\ G_2(F_2(A)) \arrow[r, "G_2(\alpha'_A)"'] & G_2(F_3(A)) \end{tikzcd}$$

for $\beta\colon G_1 \Longrightarrow G_2$ at $\alpha'_A\colon F_2(A) \to F_3(A)$, and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof. $\square$

11.9.6 Properties of Natural Transformations

Proposition 11.9.6.1.1. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[34]

1. A natural transformation $\alpha\colon F \Longrightarrow G$.
2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$\begin{tikzcd} & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\mathbb{1}}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & \mathcal{D} \end{tikzcd}$$

[34]Taken from [MO 64365].

3. A functor $[\alpha]\colon \mathcal{C} \times \mathbb{1} \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "F"] & \\ \mathcal{C} \times \mathbb{1} \arrow[u, two heads, "\mathrm{ev}_0"] \arrow[d, two heads, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\ \mathcal{C} \arrow[ur, "G"'] & \end{tikzcd}$$

Proof. From Item 1 to Item 2 and Back: We may identify $\mathcal{D}^{\mathbb{1}}$ with $\mathsf{Arr}(\mathcal{D})$. Given a natural transformation $\alpha\colon F \Longrightarrow G$, we have a functor

$$\begin{aligned} [\alpha]\colon \mathcal{C} &\longrightarrow \mathcal{D}^{\mathbb{1}} \\ A &\longmapsto \alpha_A \\ (f\colon A \to B) &\longmapsto \left(\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}\right) \end{aligned}$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back: This follows from Item 3 of Definition 11.10.1.1.2. □

11.9.7 Natural Isomorphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 11.9.7.1.1. A natural transformation $\alpha\colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1}\colon G \Longrightarrow F$ such that

$$\begin{aligned} \alpha^{-1} \circ \alpha &= \mathrm{id}_F, \\ \alpha \circ \alpha^{-1} &= \mathrm{id}_G\,. \end{aligned}$$

Proposition 11.9.7.1.2. Let $\alpha\colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The natural transformation α is a natural isomorphism.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A\colon F_A \to G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1}\colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\begin{aligned}\alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)},\\ \alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}.\end{aligned}$$
Then α^{-1} is a natural transformation.

Proof. *Item 1, Characterisations*: The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.
Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations: The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)" description] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B), \end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. $\square$

11.10 Categories of Categories

11.10.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

Definition 11.10.1.1.1. The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[35] is the category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$[36] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$.
- *Morphisms.* For each $F, G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, we have

  $$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F, G) \overset{\mathrm{def}}{=} \mathrm{Nat}(F, G).$$

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the unit map

  $$\mathbb{1}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \colon \mathrm{pt} \to \mathrm{Nat}(F, F)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at F is given by

  $$\mathrm{id}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \overset{\mathrm{def}}{=} \mathrm{id}_F,$$

  where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Definition 11.9.3.1.1.

[35] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
[36] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C}, \mathcal{D}]$.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map

  $$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$

  of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at (F, G, H) is given by

  $$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \overset{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Definition 11.9.4.1.2.

Proposition 11.10.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors

   $$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -) \colon &\quad \mathsf{Cats} \to \mathsf{Cats}, \\ \mathsf{Fun}(-, \mathcal{D}) \colon &\quad \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats}, \\ \mathsf{Fun}(-_1, -_2) \colon &\quad \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}. \end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors

   $$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -) \colon &\quad \mathsf{Cats}_2 \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-, \mathcal{D}) \colon &\quad \mathsf{Cats}_2^{\mathsf{op}} \to \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, -_2) \colon &\quad \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \to \mathsf{Cats}_2. \end{aligned}$$

3. *Adjointness.* We have adjunctions

   $$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd},$$

   $$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats} \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd},$$

   witnessed by bijections of sets

   $$\begin{aligned} \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})), \end{aligned}$$

   natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$
(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)): \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}
$$

$$
(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)): \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}
$$

witnessed by isomorphisms of categories

$$
\begin{aligned}
\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\
\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),
\end{aligned}
$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$
\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C},
$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$
D \colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})
$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$
\begin{aligned}
\lim(D)_A &\cong \lim_{i \in \mathcal{I}}(D_i(A)),\\
\operatorname{colim}(D)_A &\cong \operatorname*{colim}_{i \in \mathcal{I}}(D_i(A)),
\end{aligned}
$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Monomorphisms and Epimorphisms.* Let $\alpha \colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$
   \alpha \colon F \Longrightarrow G
   $$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

(b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\alpha_A : F_A \to G_A$$

is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Punctual Categories: Omitted.
Item 6, Objectwise Computation of Co/Limits: Omitted.
Item 7, Interaction With Co/Completeness: This follows from **??**.
Item 8, Monomorphisms and Epimorphisms: Omitted. □

11.10.2 The Category of Categories and Functors

Definition 11.10.2.1.1. The **category of** (**small**) **categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \overset{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}} : \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

where $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Definition 11.5.1.1.4.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

$$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} : \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

$$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \overset{\text{def}}{=} G \circ F,$$

where $G \circ F : \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 11.5.1.1.5.

Proposition 11.10.2.1.2. Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.

2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathsf{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof. *Item 1, Co/Completeness*: Omitted.
Item 2, Cartesian Monoidal Structure: Omitted. □

11.10.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 11.10.3.1.1. The 2**-category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.

- Hom*-Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

$$\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

$$\mathbb{1}_{\mathcal{C}}^{\mathsf{Cats}_2} \colon \mathsf{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

  - *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}(\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F.$$

  - *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow$

(G, F) of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 11.9.5.1.1.

Proposition 11.10.3.1.2. Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof. *Item 1, Co/Completeness*: Omitted. □

11.10.4 The Category of Groupoids

Definition 11.10.4.1.1. The **category of (small) groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

11.10.5 The 2-Category of Groupoids

Definition 11.10.5.1.1. The **2-category of (small) groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

A Other Chapters

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