Constructions With Monoidal Categories

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O1UF This chapter contains some material on constructions with monoidal categories.

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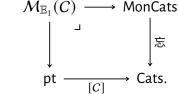
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01UH	13.1.	The Moduli Category of Monoidal Structures on a Ca	ıte-
		gory	
	Let <i>C</i> be a category.		

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DEFINITION 13.1.1.1.1 ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATE-GORY

The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\text{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats},$$



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REMARK 13.1.1.1.2 ► Unwinding Definition 13.1.1.1.1, I

In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{B}_1}(C)$ where:

- Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
- *Morphisms*. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C : C \to C$ of C.

• *Identities*. For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \stackrel{\operatorname{def}}{=} (\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes}),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of ??.

• *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,P)$$
 of $\mathcal{M}_{\mathbb{B}_{1}}(C)$ at (M,N,P) is defined by

$$\bigg(\operatorname{id}_C^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\bigg) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_1}(C)} \bigg(\operatorname{id}_C^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}\bigg) \stackrel{\mathrm{def}}{=} \bigg(\operatorname{id}_C^{\otimes,\prime} \circ \operatorname{id}_C^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime} \circ \operatorname{id}_{\mathbb{1}|C}^{\otimes}\bigg).$$

01UL REMARK 13.1.1.1.3 ► UNWINDING DEFINITION 13.1.1.1.1, II

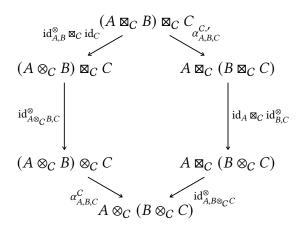
In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

1. *Naturality.* For each pair $f: A \to X$ and $g: B \to Y$ of morphisms of C, the diagram

$$\begin{array}{cccc} A \boxtimes_C B & \xrightarrow{f\boxtimes_C g} & X \boxtimes_C Y \\ \operatorname{id}_{A,B}^{\otimes} & & & & & & & | \operatorname{id}_{X,Y}^{\otimes} \\ A \otimes_C B & \xrightarrow{f\otimes_C g} & X \otimes_C Y \end{array}$$

commutes.

2. Monoidality. For each $A, B, C \in Obj(C)$, the diagram



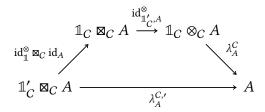
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commutes.

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3. Left Monoidal Unity. For each $A \in Obj(C)$, the diagram



commutes.

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4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} \xrightarrow{\rho_{A}^{C}} A \otimes_{C} \mathbb{1}_{C}$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A \otimes_{C} \mathbb{1}_{C}$$

commutes.

(a) The diagram

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PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUC-TURES ON A CATEGORY

Let *C* be a category.

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- 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{\mathbb{I}|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
- 01UT
- $(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\operatorname{id}_{A,B}^{\otimes} \boxtimes_{C} \operatorname{id}_{C}} (A \otimes_{C} B) \boxtimes_{C} C$ $\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \qquad \operatorname{id}_{A\otimes_{C}B,C}^{\otimes} \downarrow$ $(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} (A \otimes_{C} B) \otimes_{C} C$

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commutes.

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B \boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \qquad \downarrow \operatorname{id}_{A,B \otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

01WB

- $\rho^{C,\prime}$).
- 01WC (a) The diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C}$$

$$\downarrow^{C} \qquad \qquad \downarrow^{\rho_{\mathbb{1}_{C}}^{C,'}}$$

$$\mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes}} \mathbb{1}_{C}$$

commutes.

(b) The diagram

commutes.

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(c) The diagram

commutes.

01WF

(d) The diagram

commutes.

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3. *Mixed Associators*. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$id^{\otimes}_{-_{1},-_{2}} \colon -_{1} \boxtimes_{\mathcal{C}} -_{2} \to -_{1} \otimes_{\mathcal{C}} -_{2}$$

be a natural transformation.

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(a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes} : (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \\ \operatorname{id}_{A \otimes_C B,C}^{\otimes} \Big| & & & & \operatorname{id}_{A \otimes_C \operatorname{id}_{B,C}^{\otimes}} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \boxtimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B \boxtimes_{C} C}}$$

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$id_{A,B}^{\otimes} \otimes_{C} id_{C} \downarrow \qquad \qquad \downarrow id_{A,B \otimes_{C} C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$\begin{array}{cccc} (A\boxtimes_{C}B)\boxtimes_{C}C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_{C}(B\boxtimes_{C}C) \\ \mathrm{id}_{A\boxtimes_{C}B,C}^{\otimes} & & & & & & & \\ (A\boxtimes_{C}B)\otimes_{C}C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A\boxtimes_{C}(B\otimes_{C}C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

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making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$id_{A,B}^{\otimes} \otimes_{C} id_{C} \downarrow \qquad \qquad \downarrow id_{A} \otimes_{C} id_{B,C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \qquad \qquad \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

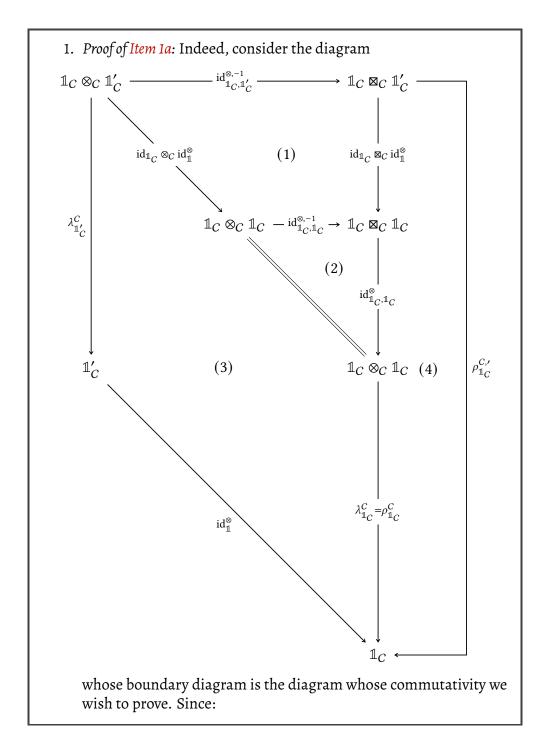
We claim that Items 1a and 1b are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_{AB}^{\otimes} and id_{C} .
- 2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and $id_{B,C}^{\otimes}$.

This finishes the proof.

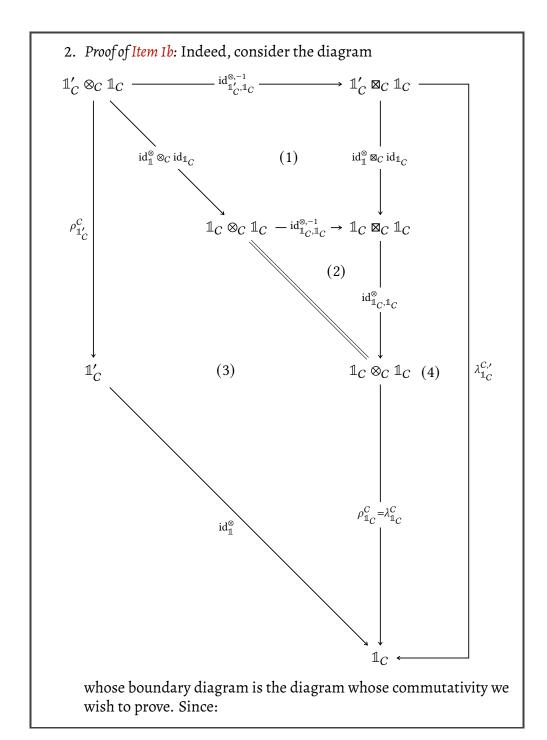
Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:



- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from $\ref{eq:composition}$;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.



- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:composition}$;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes_{C}-1}} \mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes_{C}}} \mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{C,'}_{\mathbb{1}_{C}} \qquad \qquad (\dagger) \qquad \qquad \downarrow^{\rho_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow^{C} \qquad \qquad \downarrow^{C} \qquad \qquad \downarrow^{C} \qquad \qquad \downarrow^{C} \qquad \downarrow^{C$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

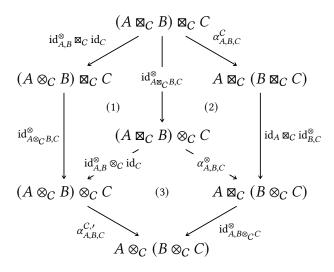
Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

1. *Proof of Item 3a*: We may partition the monoidality diagram for id^{\otimes}

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of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

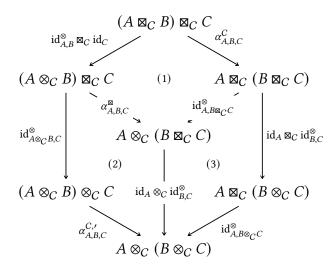
- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for id $^{\otimes}$

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of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

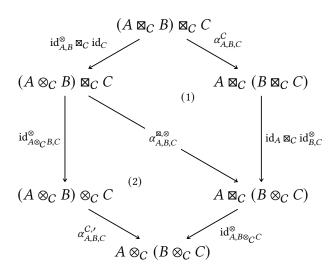
- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for id^{\otimes}

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of Item 2 of Remark 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- o1V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

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- 1. Introduction
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Monoidal Categories

13. Constructions With Monoidal gories
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14. Types of Morphisms in Bicate- 15. Notes