# Constructions With Monoidal Categories

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**O1UF** This chapter contains some material on constructions with monoidal categories.

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#### Moduli Categories of Monoidal Structures 01UG **I3.I**

#### The Moduli Category of Monoidal Structures on a 01UH 13.I.I Category

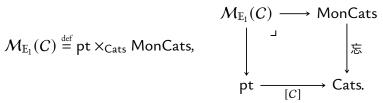
Let *C* be a category.

01UJ

**DEFINITION 13.1.1.1.1** ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATE-

The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

$$\mathcal{M}_{\mathbb{E}_1}(C)\stackrel{\scriptscriptstyle
m def}{=}\operatorname{pt} imes_{\mathsf{Cats}}\operatorname{\mathsf{MonCats}},$$



01UK

#### REMARK 13.1.1.1.2 ► UNWINDING DEFINITION 13.1.1.1.1, I

In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- Objects. The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{I}_C,$  $\alpha^{C}$ ,  $\lambda^{C}$ ,  $\rho^{C}$ ) whose underlying category is C.
- *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \alpha^C, \alpha^C, \lambda^C, \rho^C)$  $\mathbb{1}'_{C}, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{1|C}^{\otimes} \colon \mathbb{1}_{C}' \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C: C \to C$  of C.

• *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C)),$ the unit map

$$\mathbb{1}_{MM}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\mathrm{id}_{\mathcal{M}}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\mathrm{def}}{=} \left(\mathrm{id}_{C}^{\otimes},\mathrm{id}_{1|C}^{\otimes}\right),$$

where  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  is the identity monoidal functor of C of  $\ref{C}$ ?

• *Composition*. For each M, N,  $P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{E_1}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(M,P)$$
 of  $\mathcal{M}_{E_1}(C)$  at  $(M,N,P)$  is defined by

$$\left( \operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{1|C}^{\otimes,\prime} \right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \left( \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes} \right) \stackrel{\scriptscriptstyle \mathsf{def}}{=} \left( \operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes,\prime} \circ \operatorname{id}_{1|C}^{\otimes} \right).$$

#### 01UL REMAR

#### REMARK 13.1.1.1.3 ► Unwinding Definition 13.1.1.1.1, II

In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

01UM

I. *Naturality*. For each pair  $f:A\to X$  and  $g:B\to Y$  of morphisms of C, the diagram

$$A \boxtimes_C B \xrightarrow{f \boxtimes_C g} X \boxtimes_C Y$$

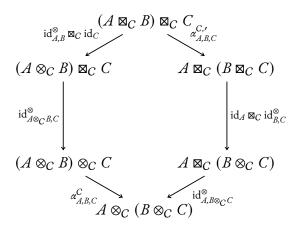
$$\downarrow_{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow_{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_C B \xrightarrow{f \otimes_C g} X \otimes_C Y$$

commutes.

01UN

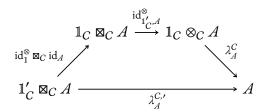
2. *Monoidality*. For each A, B,  $C \in Obj(C)$ , the diagram



commutes.

01UP

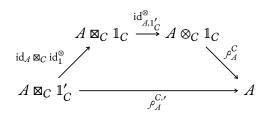
3. Left Monoidal Unity. For each  $A \in \text{Obj}(C)$ , the diagram



commutes.

01UQ

4. Right Monoidal Unity. For each  $A \in Obj(C)$ , the diagram



commutes.

**01UR** 

# PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

Let *C* be a category.

01US

- 1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
- 01UT (a) The diagram

commutes.

01UU

(b) The diagram

commutes.

01WB

2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{I}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{I}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .

01WC

(a) The diagram

commutes.

01WD (b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

01WE

01WF

01UV

3. *Mixed Associators.* Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_C -_2 \to -_1 \otimes_C -_2$$

be a natural transformation.

01UW

(a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} \colon (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \operatorname{id}_{A \otimes_C B,C}^{\otimes} \downarrow & & \operatorname{id}_{A \otimes_C \operatorname{id}_{B,C}^{\otimes}} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{c|c} (A\boxtimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_C (B\boxtimes_C C) \\ \mathrm{id}_{A,B}^\otimes\boxtimes_C \mathrm{id}_C & & \downarrow \mathrm{id}_{A,B\boxtimes_C C} \\ (A\otimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^\otimes} A\otimes_C (B\boxtimes_C C) \end{array}$$

commute, then the natural transformation id<sup>⊗</sup> satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UX

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} \colon (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

making the diagrams

and

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

and

$$\begin{array}{c|c} (A\boxtimes_{C}B)\boxtimes_{C}C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_{C}(B\boxtimes_{C}C) \\ \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} & & \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes} \\ (A\boxtimes_{C}B)\otimes_{C}C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A\otimes_{C}(B\boxtimes_{C}C) \end{array}$$

commute, then the natural transformation id  $^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

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#### PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

### Item 1: Extra Monoidality Conditions

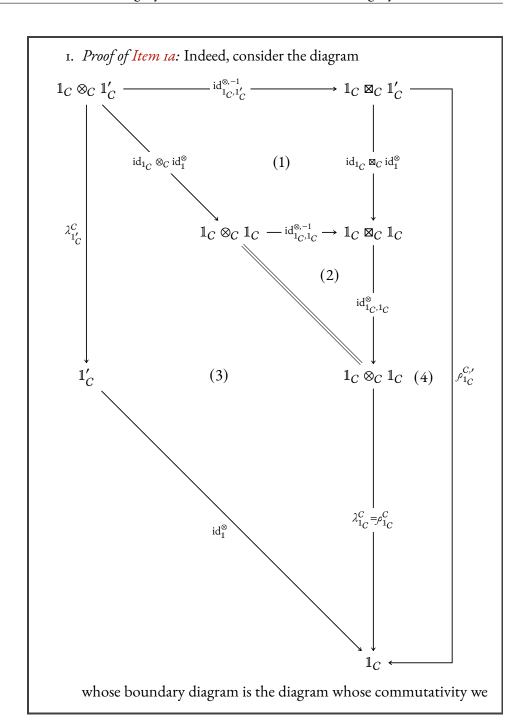
We claim that Items 1a and 1b are indeed true:

- I. *Proof of Item 1a:* This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A,B}^{\otimes}$  and  $id_{C}$ .
- 2. *Proof of Item 1b*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A}$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

### Item 2: Extra Monoidal Unity Constraints

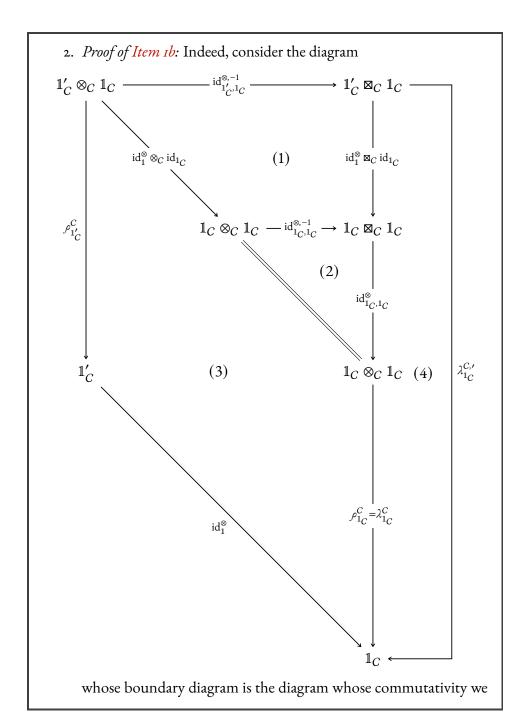
We claim that Items 2a and 2b are indeed true:



wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{C}^{\otimes,-1};$
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from  $\ref{eq:comparison}$ ;
- Subdiagram (4) commutes by the right monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.



wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $id_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from  $\ref{eq:composition}$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $\mathrm{id}_{1_C,1_C'}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

commutes. But since  $id_1^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

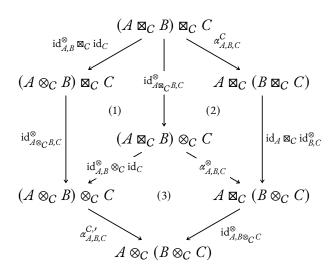
This finishes the proof.

#### Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

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I. *Proof of Item 3a:* We may partition the monoidality diagram for id<sup>⊗</sup> of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

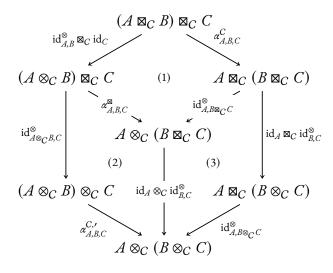
- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id<sup>⊗</sup> satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

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2. *Proof of Item 3b*: We may partition the monoidality diagram for  $id^{\otimes}$ 

#### of Item 2 of Remark 13.1.1.1.3 as follows:



#### Since:

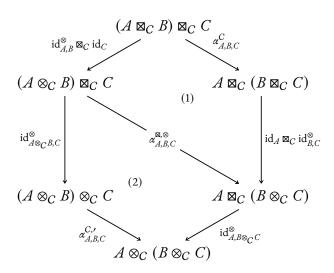
- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id<sup>⊗</sup> satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. *Proof of Item 3c:* We may partition the monoidality diagram for  $id^{\otimes}$ 

01V1

### of Item 2 of Remark 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- one of Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# Appendices

# A Other Chapters

#### **Preliminaries**

- I. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets

7. Tensor Products of Pointed Sets

#### Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

#### Categories

- II. Categories
- 12. Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

13. Constructions With Monoidal gories
Categories

## **Bicategories**

## Extra Part

14. Types of Morphisms in Bicate- 15. Notes