

# Constructions With Monoidal Categories

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01UF This chapter contains some material on constructions with monoidal categories.

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## 01UG 13.1 Moduli Categories of Monoidal Structures

### 01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let  $C$  be a category.

01UJ

**DEFINITION 13.1.1.1 ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY**

The **moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow \text{忘} \\ \text{pt} & \xrightarrow{[C]} & \text{Cats}. \end{array}$$

01UK

**REMARK 13.1.1.1.2 ► UNWINDING DEFINITION 13.1.1.1, I**

In detail, **the moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- *Objects.* The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is  $C$ .
- *Morphisms.* A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  is a strong monoidal functor structure

$$\text{id}_C^\otimes: A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$

$$\text{id}_{\mathbb{1}|C}^\otimes: \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor  $\text{id}_C: C \rightarrow C$  of  $C$ .

- *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{pt} \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $M$  is defined by

$$\text{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \stackrel{\text{def}}{=} (\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes),$$

where  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  is the identity monoidal functor of  $C$  of??.

- *Composition.* For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} : \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N, P) \times \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, N) \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, P)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $(M, N, P)$  is defined by

$$\left( \text{id}_C^{\otimes, \prime}, \text{id}_{\mathbb{1}|C}^{\otimes, \prime} \right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \left( \text{id}_C^{\otimes}, \text{id}_{\mathbb{1}|C}^{\otimes} \right) \stackrel{\text{def}}{=} \left( \text{id}_C^{\otimes, \prime} \circ \text{id}_C^{\otimes}, \text{id}_{\mathbb{1}|C}^{\otimes, \prime} \circ \text{id}_{\mathbb{1}|C}^{\otimes} \right).$$

01UL

REMARK 13.1.1.1.3 ► UNWINDING DEFINITION 13.1.1.1.1, II

In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, \prime}, \lambda^{C, \prime}, \rho^{C, \prime})$  satisfies the following conditions:

01UM

1. *Naturality.* For each pair  $f: A \rightarrow X$  and  $g: B \rightarrow Y$  of morphisms of  $C$ , the diagram

$$\begin{array}{ccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \text{id}_{A,B}^{\otimes} \downarrow & & \downarrow \text{id}_{X,Y}^{\otimes} \\ A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y \end{array}$$

commutes.

01UN

2. *Monoidality.* For each  $A, B, C \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc} & (A \boxtimes_C B) \boxtimes_C C & \\ \text{id}_{A,B}^{\otimes} \boxtimes \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^{C, \prime} \\ (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \otimes_C B, C}^{\otimes} \downarrow & & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\ \alpha_{A,B,C}^C \swarrow & & \swarrow \text{id}_{A,B \otimes_C C}^{\otimes} \\ & A \otimes_C (B \otimes_C C) & \end{array}$$

commutes.

01UP

3. *Left Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & \mathbb{1}_C \boxtimes_C A & \xrightarrow{\text{id}_{\mathbb{1}'_C, A}^\otimes} \mathbb{1}_C \otimes_C A \\
 \text{id}_{\mathbb{1}}^\otimes \boxtimes_C \text{id}_A \nearrow & & \searrow \lambda_A^C \\
 \mathbb{1}'_C \boxtimes_C A & \xrightarrow{\lambda_A^{C, '}} & A
 \end{array}$$

commutes.

01UQ

4. *Right Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & A \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{A, \mathbb{1}'_C}^\otimes} A \otimes_C \mathbb{1}_C \\
 \text{id}_A \boxtimes_C \text{id}_{\mathbb{1}}^\otimes \nearrow & & \searrow \rho_A^C \\
 A \boxtimes_C \mathbb{1}'_C & \xrightarrow{\rho_A^{C, '}} & A
 \end{array}$$

commutes.

**PROPOSITION 13.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY**

01UR

Let  $C$  be a category.

01US

1. *Extra Monoidality Conditions.* Let  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, '}, \lambda^{C, '}, \rho^{C, '})$ .

01UT

(a) The diagram

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\text{id}_{A, B}^\otimes \boxtimes_C \text{id}_C} & (A \otimes_C B) \boxtimes_C C \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A \otimes_C B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\text{id}_{A, B}^\otimes \otimes_C \text{id}_C} & (A \otimes_C B) \otimes_C C
 \end{array}$$

commutes.

01UU

(b) The diagram

$$\begin{array}{ccc}
 A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \boxtimes \text{id}_{B,C}^\otimes} & A \boxtimes_C (B \otimes_C C) \\
 \downarrow \text{id}_{A,B \boxtimes_C C}^\otimes & & \downarrow \text{id}_{A,B \otimes_C C}^\otimes \\
 A \otimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \otimes \text{id}_{B,C}^\otimes} & A \otimes_C (B \otimes_C C)
 \end{array}$$

commutes.

01WB

2. *Extra Monoidal Unity Constraints.* Let  $(\text{id}_C^\otimes, \text{id}_{1|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, 1_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, 1'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$ .

01WC

(a) The diagram

$$\begin{array}{ccc}
 1_C \otimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \boxtimes_C 1'_C \\
 \downarrow \lambda_{1'_C}^C & & \downarrow \rho_{1'_C}^{C'} \\
 1'_C & \xrightarrow{\text{id}_1^\otimes} & 1_C
 \end{array}$$

commutes.

01WD

(b) The diagram

$$\begin{array}{ccc}
 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1'_C, 1_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C \\
 \downarrow \rho_{1'_C}^C & & \downarrow \lambda_{1'_C}^{C'} \\
 1'_C & \xrightarrow{\text{id}_1^\otimes} & 1_C
 \end{array}$$

commutes.

01WE

(c) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \lambda_{\mathbb{1}_C}^{C, \prime} \downarrow & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

01WF

(d) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \rho_{\mathbb{1}_C}^{C, \prime} \downarrow & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

01UV

3. *Mixed Associators.* Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, \prime}, \lambda^{C, \prime}, \rho^{C, \prime})$  be monoidal structures on  $C$  and let

$$\text{id}_{-1, -2}^\otimes: -_1 \boxtimes_C -_2 \rightarrow -_1 \otimes_C -_2$$

be a natural transformation.

01UW

(a) If there exists a natural transformation

$$\alpha_{A, B, C}^\otimes: (A \otimes_C B) \boxtimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A, B, C}^\otimes} & A \otimes_C (B \boxtimes_C C) \\
 \text{id}_{A \otimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \otimes_C \text{id}_{B, C}^\otimes \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A, B, C}^C} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C'}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \downarrow & & \downarrow \text{id}_{A,B \boxtimes_C C} \\
 (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^\otimes} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

01UX

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^\boxtimes: (A \boxtimes_C B) \otimes_C C \rightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} & A \boxtimes_C (B \otimes_C C) \\
 \text{id}_{A,B}^\otimes \otimes_C \text{id}_C \downarrow & & \downarrow \text{id}_{A,B \otimes_C C}^\otimes \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C'}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} & A \boxtimes_C (B \otimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

01UY

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes, \otimes}: (A \boxtimes_C B) \otimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C) \\
 \text{id}_{A,B}^{\otimes} \otimes \text{id}_C \downarrow & & \downarrow \text{id}_A \otimes \text{id}_{B,C}^{\otimes} \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes, \otimes}} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C, \boxtimes}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^{\boxtimes} \downarrow & & \downarrow \text{id}_{A, B \boxtimes_C C}^{\boxtimes} \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^{\otimes}$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

#### PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.4

##### Item 1: Extra Monoidality Conditions

We claim that **Items 1a** and **1b** are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of  $\text{id}^{\otimes}$  with respect to the morphisms  $\text{id}_{A,B}^{\otimes}$  and  $\text{id}_C$ .
2. *Proof of Item 1b:* This follows from the naturality of  $\text{id}^{\otimes}$  with respect to the morphisms  $\text{id}_A$  and  $\text{id}_{B,C}^{\otimes}$ .

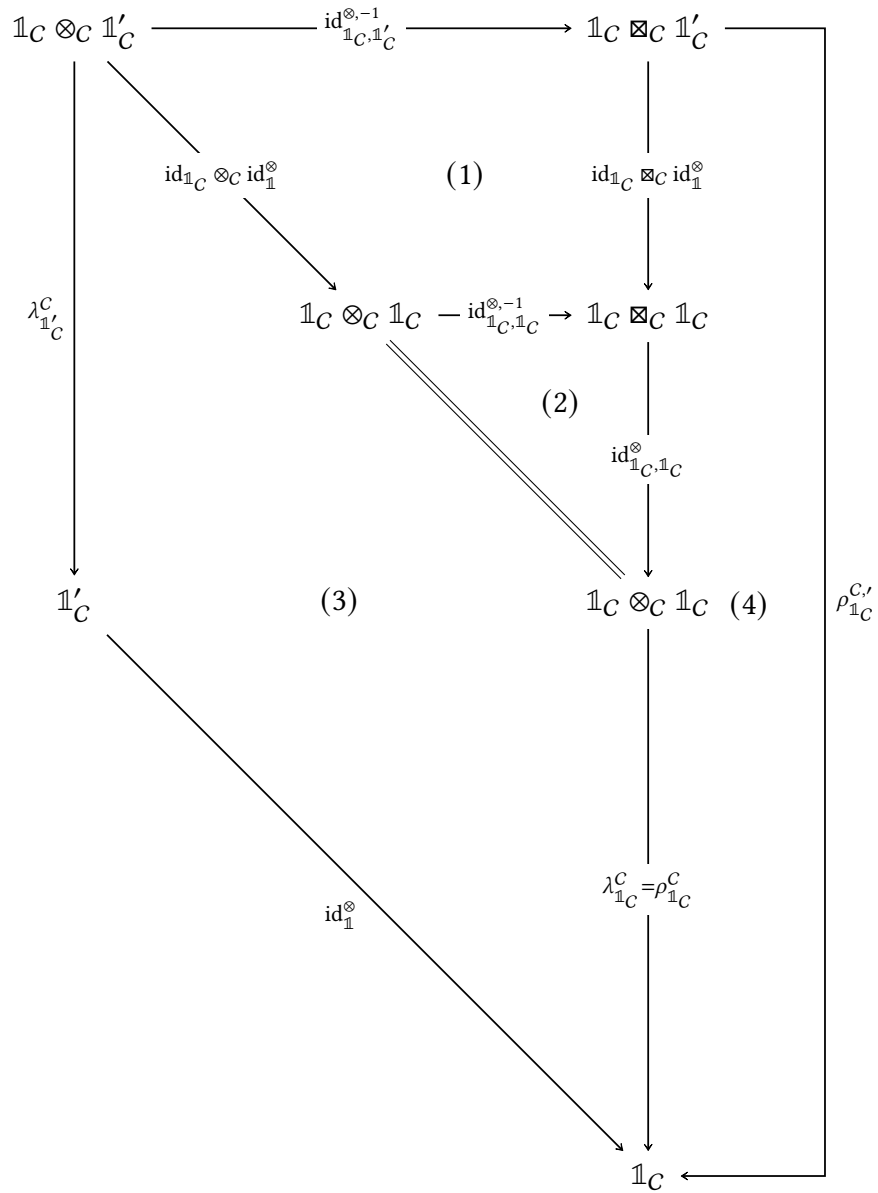
This finishes the proof.

##### Item 2: Extra Monoidal Unity Constraints

We claim that **Items 2a** and **2b** are indeed true:



1. *Proof of Item 1a:* Indeed, consider the diagram

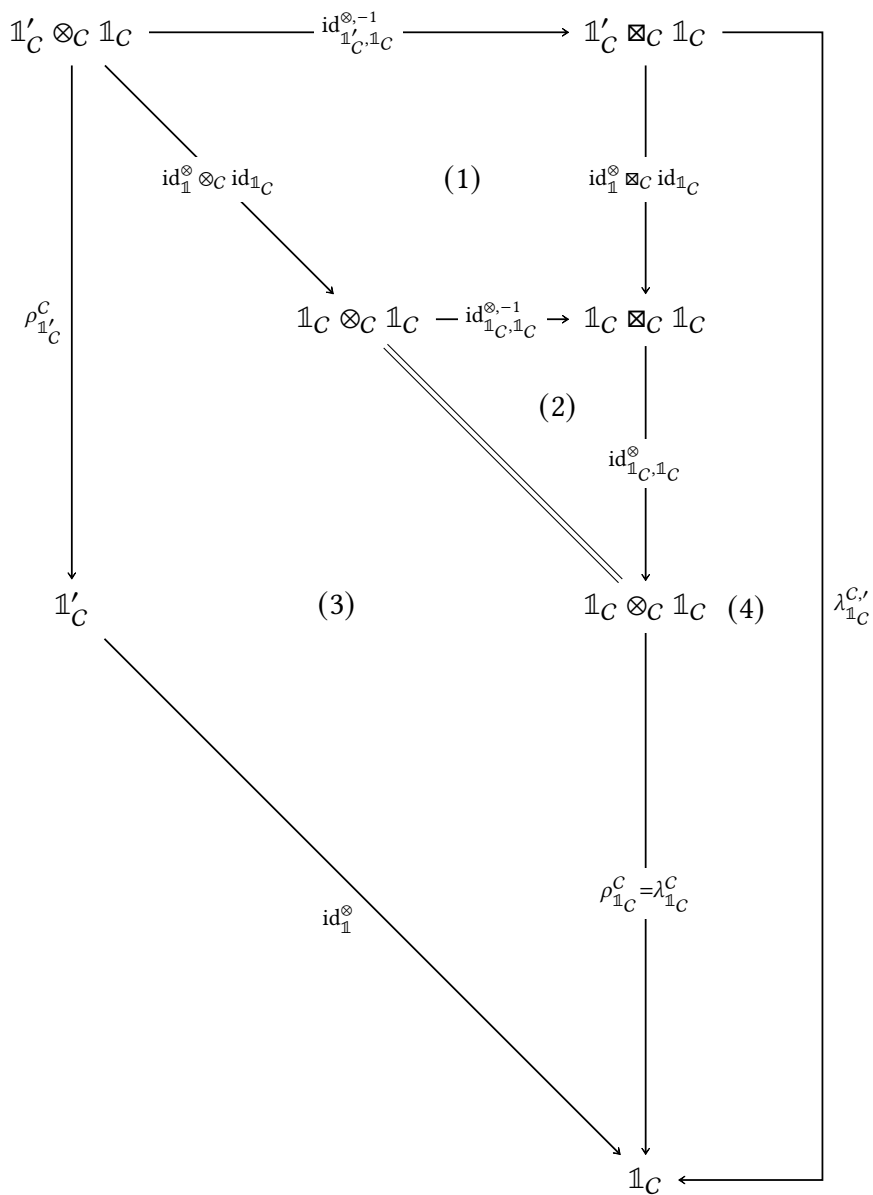


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes, -1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of  $(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b*: Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of  $(\text{id}_C, \text{id}_C^{\otimes}, \text{id}_{C|\mathbb{1}})$ ;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \downarrow \rho_{\mathbb{1}'_C}^C & & \downarrow \lambda_{\mathbb{1}_C}^{C, ' } & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C.
 \end{array}
 \quad \begin{array}{c} (1) \end{array} \quad \begin{array}{c} (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by *Item 1b*;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 & & \downarrow \lambda_{\mathbb{1}_C}^{C, ' } & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}
 \quad \begin{array}{c} (\dagger)$$

commutes. But since  $\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & & \downarrow \rho_{\mathbb{1}_C}^{C, \dagger} & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}
 \quad (1) \quad (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by *Item 1a*;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 & & \downarrow \rho_{\mathbb{1}_C}^{C, \dagger} & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}
 \quad (\dagger)$$

commutes. But since  $\text{id}_{\mathbb{1}_C}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

### Item 3: Mixed Associators

We claim that *Items 3a to 3c* are indeed true:

01UZ

1. *Proof of Item 3a:* We may partition the monoidality diagram for  $\text{id}^{\otimes}$

of **Item 2** of **Remark 13.1.1.3** as follows:

$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C & \downarrow & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) \downarrow & \downarrow & (2) \downarrow & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \swarrow \alpha_{A,B,C}^{C'} & \downarrow \text{id}_{A,B}^\otimes \otimes_C \text{id}_C & \downarrow \alpha_{A,B,C}^\otimes & \searrow \alpha_{A,B,C}^\otimes & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 & (3) & & & \\
 & A \otimes_C (B \otimes_C C) & & & 
 \end{array}$$

Since:

- Subdiagram (1) commutes by **Item 1a** of **Item 1**.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.

01V0

2. *Proof of **Item 3b***: We may partition the monoidality diagram for  $\text{id}^\otimes$

of **Item 2** of **Remark 13.1.1.3** as follows:

$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C & & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (1) & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \swarrow \alpha_{A,B,C}^\boxtimes & & \nwarrow \text{id}_{A, B \boxtimes_C C}^\otimes & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 & A \otimes_C (B \boxtimes_C C) & & & \\
 (2) & & (3) & & \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\text{id}_A \otimes_C \text{id}_{B,C}^\otimes} & A \boxtimes_C (B \otimes_C C) & & \\
 \swarrow \alpha_{A,B,C}^{C'} & & \nwarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & & 
 \end{array}$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by **Item 1b** of **Item 1**.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.3**.


01V1

3. *Proof of Item 3c:* We may partition the monoidality diagram for  $\text{id}^\otimes$

of **Item 2** of **Remark 13.1.1.1.3** as follows:

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^C \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \searrow \alpha_{A,B,C}^{\boxtimes, \otimes} & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^{C, \otimes} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}
 \quad (1) \quad (2)$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Remark 13.1.1.1.3**.

This finishes the proof. 



- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- 01V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# Appendices

## A Other Chapters

### Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets

### 7. Tensor Products of Pointed Sets

### Relations

- 8. Relations
- 9. Constructions With Relations
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### Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

### Monoidal Categories

13. Constructions With Monoidal Categories

**Bicategories**

**Extra Part**

14. Types of Morphisms in Bicate- 15. Notes