# Monoidal Structures on the Category of Sets

# The Clowder Project Authors

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**O1NK** This chapter contains some material on monoidal structures on Sets.

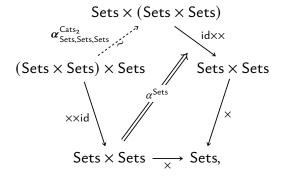
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01NP	5.1.3	<b>T</b> ]	he Monoidal Unit		
01NQ	Definition 5.1.3.1.1. The monoidal unit of the product of sets is the funct				
			$\mathbb{1}^{Sets} \colon pt \to Sets$		
	defined by $\mathbb{1}_{Sets} \stackrel{\scriptscriptstyledef}{=} pt,$				
	where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.				
01NR	5.1.4	f T	he Associator		
01NS	<b>Definition 5.1.4.1.1.</b> The <b>associator of the product of sets</b> is the natural isomorphism				

 $\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{}}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}},$ 

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \stackrel{\mathrm{def}}{=} (x,(y,z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

*Proof. Invertibility*: The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is the morphism

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \stackrel{\text{def}}{=} ((x,y),z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

• Invertibility I. We have

$$\begin{split} [\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}}]((x,y),z) &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z)) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} ((x,y),z) \\ &\stackrel{\mathsf{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}]((x,y),z) \end{split}$$

for each  $((x, y), z) \in (X \times Y) \times Z$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{(X \times Y) \times Z}$$
.

• Invertibility II. We have

$$\begin{split} \big[\alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1}\big](x,(y,z)) &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}}(\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z))) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \\ &\stackrel{\mathsf{def}}{=} (x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} \big[\mathrm{id}_{(X\times Y)\times Z}\big](x,(y,z)) \end{split}$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{X \times (Y \times Z)}$$
.

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$\begin{array}{c|c} (X\times Y)\times Z & \xrightarrow{\quad (f\times g)\times h \quad} (X'\times Y')\times Z' \\ \\ \alpha^{\mathsf{Sets}}_{X,Y,Z} & & & & & \\ x\times (Y\times Z) & \xrightarrow{\quad f\times (g\times h) \quad} X'\times (Y'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

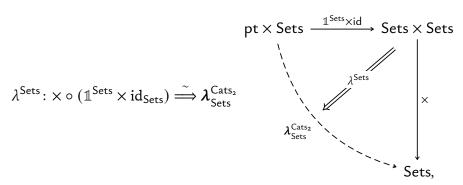
$$((x,y),z) \qquad ((x,y),z) \longmapsto ((f(x),g(y)),h(z))$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes, showing  $\alpha^{\mathsf{Sets}}$  to be a natural transformation. Being a Natural Isomorphism: Since  $\alpha^{\mathsf{Sets}}$  is natural and  $\alpha^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\alpha^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\alpha^{\mathsf{Sets},-1}$  is also natural. Thus  $\alpha^{\mathsf{Sets}}$  is a natural isomorphism.

# 01NT 5.1.5 The Left Unitor

**Definition 5.1.5.1.1.** The **left unitor of the product of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets}} \colon \mathsf{pt} \times X \xrightarrow{\sim} X$$

at  $X \in Obj(Sets)$  is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\mathrm{def}}{=} x$$

for each  $(\star, x) \in pt \times X$ .

*Proof. Invertibility*: The inverse of  $\lambda_X^{\mathsf{Sets}}$  is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathsf{pt} \times X$$

defined by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star,x)$$

for each  $x \in X$ . Indeed:

• *Invertibility I*. We have

$$\begin{split} \big[\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}\big](\mathsf{pt},x) &= \lambda_X^{\mathsf{Sets},-1}(\lambda_X^{\mathsf{Sets}}(\mathsf{pt},x)) \\ &= \lambda_X^{\mathsf{Sets},-1}(x) \\ &= (\mathsf{pt},x) \\ &= \big[\mathrm{id}_{\mathsf{pt}\times X}\big](\mathsf{pt},x) \end{split}$$

for each  $(pt, x) \in pt \times X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}} = \mathrm{id}_{\mathsf{pt} \times X} \,.$$

• Invertibility II. We have

$$[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}](x) = \lambda_X^{\mathsf{Sets}}(\lambda_X^{\mathsf{Sets},-1}(x))$$

$$= \lambda_X^{\mathsf{Sets},-1}(\mathsf{pt},x)$$

$$= x$$

$$= [\mathrm{id}_X](x)$$

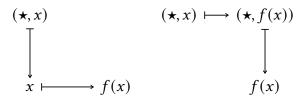
for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X \,.$$

Therefore  $\lambda_X^{\mathsf{Sets}}$  is indeed an isomorphism. Naturality: We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{c|c}
\operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\
\lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\
X & \xrightarrow{f} & Y
\end{array}$$

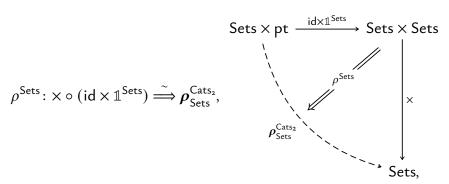
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\lambda^{\text{Sets}}$  is a natural transformation. Being a Natural Isomorphism: Since  $\lambda^{\text{Sets}}$  is natural and  $\lambda^{\text{Sets},-1}$  is a componentwise inverse to  $\lambda^{\text{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\lambda^{\text{Sets},-1}$  is also natural. Thus  $\lambda^{\text{Sets}}$  is a natural isomorphism.

# 01NV 5.1.6 The Right Unitor

**Definition 5.1.6.1.1.** The **right unitor of the product of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \mathsf{pt} \stackrel{\sim}{\dashrightarrow} X$$

at  $X \in Obj(Sets)$  is given by

$$\rho_X^{\mathsf{Sets}}(x, \star) \stackrel{\mathsf{def}}{=} x$$

for each  $(x, \star) \in X \times pt$ .

*Proof. Invertibility*: The inverse of  $\rho_X^{\rm Sets}$  is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\text{def}}{=} (x, \star)$$

for each  $x \in X$ . Indeed:

• *Invertibility I*. We have

$$\begin{split} [\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}}](x, \star) &= \rho_X^{\mathsf{Sets},-1}(\rho_X^{\mathsf{Sets}}(x, \star)) \\ &= \rho_X^{\mathsf{Sets},-1}(x) \\ &= (x, \star) \\ &= [\mathrm{id}_{X \times \mathrm{pt}}](x, \star) \end{split}$$

for each  $(x, \star) \in X \times pt$ , and therefore we have

$$\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} = \mathrm{id}_{X \times \mathrm{pt}}.$$

• Invertibility II. We have

$$\begin{split} [\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1}](x) &= \rho_X^{\mathsf{Sets}}(\rho_X^{\mathsf{Sets},-1}(x)) \\ &= \rho_X^{\mathsf{Sets},-1}(x, \bigstar) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

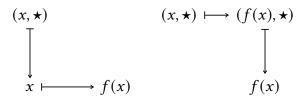
for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X.$$

Therefore  $\rho_X^{\sf Sets}$  is indeed an isomorphism. Naturality: We need to show that, given a function  $f:X\to Y$ , the diagram

$$\begin{array}{c|c} X \times \operatorname{pt} & \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} & Y \times \operatorname{pt} \\ \rho_X^{\operatorname{Sets}} & & & & \Big| \rho_Y^{\operatorname{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\rho^{\rm Sets}$  is a natural transformation. Being a Natural Isomorphism: Since  $\rho^{\rm Sets}$  is natural and  $\rho^{\rm Sets,-1}$  is a componentwise inverse to  $\rho^{\rm Sets}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\rho^{\rm Sets,-1}$  is also natural. Thus  $\rho^{\rm Sets}$  is a natural isomorphism.

# 01NX 5.1.7 The Symmetry

**Definition 5.1.7.1.1.** The **symmetry of the product of sets** is the natural isomorphism



whose component

$$\sigma_{XY}^{\mathsf{Sets}} : X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in Obj(Sets)$  is defined by

$$\sigma_{X,Y}^{\mathsf{Sets}}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

*Proof. Invertibility:* The inverse of  $\sigma_{X,Y}^{\text{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \colon Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets},-1}(y,x) \stackrel{\text{def}}{=} (x,y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

• Invertibility I. We have

$$[\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}}](x,y) \stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(\sigma_{X,Y}^{\mathsf{Sets}}(x,y))$$

$$\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(y,x)$$

$$\stackrel{\text{def}}{=} (x,y)$$

$$\stackrel{\text{def}}{=} [\mathrm{id}_{X\times Y}](x,y)$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}} = \mathrm{id}_{X \times Y}$$
.

• Invertibility II. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1}](y,x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(\sigma_{X,Y}^{\mathsf{Sets}}(y,x)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(x,y) \\ &\stackrel{\text{def}}{=} (y,x) \\ &\stackrel{\text{def}}{=} [\mathrm{id}_{Y \times X}](y,x) \end{split}$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \mathrm{id}_{Y \times X}$$
.

Therefore  $\sigma_{X,Y}^{\mathsf{Sets}}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to A$$
,  $g: Y \to B$ 

the diagram

$$\begin{array}{c|c} X \times Y & \xrightarrow{f \times g} & A \times B \\ \sigma_{X,Y}^{\mathsf{Sets}} & & & & & & \\ & & & & & & \\ & Y \times X & \xrightarrow{g \times f} & B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x,y) \longmapsto (f(x),g(y))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(y,x) \longmapsto (g(y),f(x)) \qquad \qquad (g(y),f(x))$$

and hence indeed commutes, showing  $\sigma^{\mathsf{Sets}}$  to be a natural transformation. Being a Natural Isomorphism: Since  $\sigma^{\mathsf{Sets}}$  is natural and  $\sigma^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\sigma^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\sigma^{\mathsf{Sets},-1}$  is also natural. Thus  $\sigma^{\mathsf{Sets}}$  is a natural isomorphism.

# 01NZ 5.1.8 The Diagonal

**Definition 5.1.8.1.1.** The **diagonal of the product of sets** is the natural transformation



whose component

$$\Delta_X \colon X \to X \times X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .

*Proof.* We need to show that, given a function  $f: X \to Y$ , the diagram

$$X \xrightarrow{f} Y$$

$$\Delta_X \downarrow \qquad \qquad \downarrow \Delta_Y$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
x & & x & \longrightarrow f(x) \\
\downarrow & & \downarrow \\
(x,x) & \longmapsto (f(x),f(x)) & & (f(x),f(x))
\end{array}$$

and hence indeed commutes, showing  $\Delta$  to be natural.

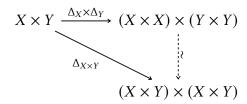
**O1P1** Proposition 5.1.8.1.2. Let X be a set.

01P2 1. Monoidality. The diagonal map

$$\Delta\colon\operatorname{id}_{\mathsf{Sets}}\Longrightarrow\times\circ\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

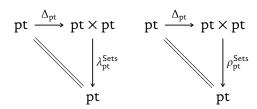
is a monoidal natural transformation:

01P3 (a) Compatibility With Strong Monoidality Constraints. For each  $X, Y \in Obj(Sets)$ , the diagram



commutes.

01P4 (b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets},-1} \\ &= \rho_{\text{pt}}^{\text{Sets},-1}, \end{split}$$

where we recall that the equalities

$$\begin{split} \lambda_{\rm pt}^{\rm Sets} &= \rho_{\rm pt}^{\rm Sets}, \\ \lambda_{\rm pt}^{\rm Sets,-1} &= \rho_{\rm pt}^{\rm Sets,-1} \end{split}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

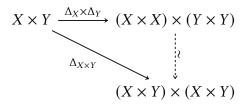
01P5 2. *The Diagonal of the Unit*. The component

$$\Delta_{pt}\colon pt\stackrel{^{\sim}}{\dashrightarrow} pt\times pt$$

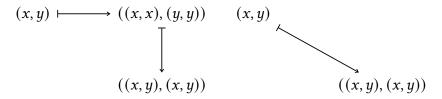
of  $\Delta$  at pt is an isomorphism.

*Proof. Item* 1, *Monoidality*: We claim that  $\Delta$  is indeed monoidal:

1. *Item 1a*: *Compatibility With Strong Monoidality Constraints*: We need to show that the diagram



commutes. Indeed, this diagram acts on elements as



and hence indeed commutes.

024T 2. *Item 1b*: *Compatibility With Strong Unitality Constraints*: As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star,x)$$

for each  $x \in X$ , so when X = pt, we have

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets},-1}(\star) \stackrel{\mathrm{def}}{=} (\star,\star),$$

and also

$$\Delta_{\mathrm{pt}}^{\mathrm{Sets}}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

so we have  $\Delta_{pt} = \lambda_{pt}^{Sets,-1}$ .

This finishes the proof.

*Item 2, The Diagonal of the Unit*: This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to ×, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor. □

# 01P6 5.1.9 The Monoidal Category of Sets and Products

- **Proposition 5.1.9.1.1.** The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:
  - The Underlying Category. The category Sets of pointed sets.
  - The Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

• The Internal Hom. The internal Hom functor

Sets: 
$$Sets^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Definition 4.3.5.1.2.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• *The Associators*. The natural isomorphism

$$\alpha^{\text{Sets}}$$
:  $\times \circ (\times \times \text{id}_{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}}_{\text{Sets},\text{Sets},\text{Sets}}$ 
of Definition 5.1.4.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} : \times \circ (\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• *The Symmetry*. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

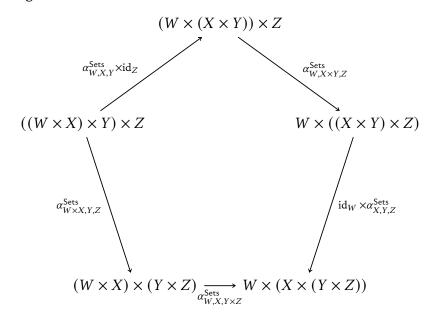
of Definition 5.1.7.1.1.

• The Diagonals. The monoidal natural transformation

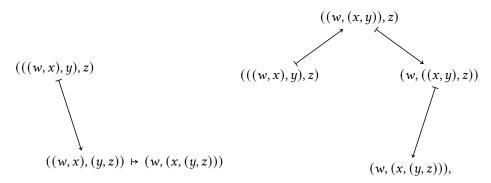
$$\Delta\colon\operatorname{id}_{\mathsf{Sets}} \Longrightarrow \mathsf{X}\circ\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

*Proof.* The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

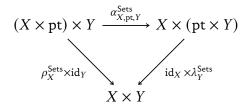


commutes. Indeed, this diagram acts on elements as

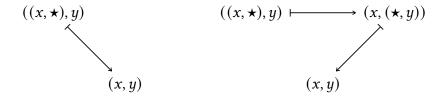


and thus the pentagon identity is satisfied.

The Triangle Identity: Let X and Y be sets. We have to show that the diagram

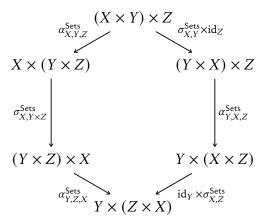


commutes. Indeed, this diagram acts on elements as

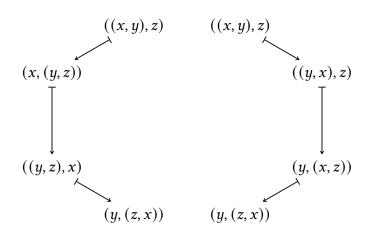


and thus the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

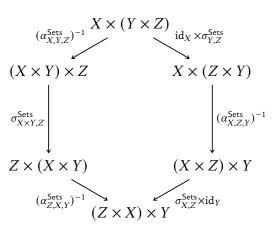


commutes. Indeed, this diagram acts on elements as

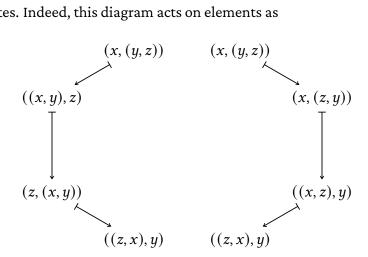


and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the right hexagon identity is satisfied.

Monoidal Closedness: This follows from Constructions With Sets, Item 2 of Definition 4.3.5.1.2

Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 5.1.8.1.2.

#### The Universal Property of (Sets, $\times$ , pt) 5.1.10

- **Theorem 5.1.10.1.1.** The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:
- 01PA 1. Existence of an Internal Hom. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom  $[-1, -2]_{Sets}$ .

01PB 2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

> More precisely, the full subcategory of the category  $\mathcal{M}^{cld}_{\mathbb{R}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:sets}$  spanned by the closed symmetric monoidal categories (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda^{\text{Sets}}$ ,  $\rho^{\text{Sets}}$ ,  $\sigma^{\text{Sets}}$ ) satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

*Proof. Unwinding the Statement*: Let (Sets,  $\otimes_{Sets}$ ,  $[-1, -2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$id_{Sets} : Sets \rightarrow Sets$$

admits a unique closed symmetric monoidal functor structure

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) to the closed symmetric monoidal category (Sets,  $\times$ , Sets $(-_1, -_2)$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda^{Sets}$ ,  $\rho^{Sets}$ ,  $\sigma^{Sets}$ ) of Definition 5.1.9.1.1.

Constructing an Isomorphism  $[-1, -2]_{Sets} \cong Sets(-1, -2)$ : By ??, we have a natural isomorphism

$$\mathsf{Sets}(\mathsf{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong \mathsf{Sets}(-_1, -_2).$$

By Constructions With Sets, Item 3 of Definition 4.3.5.1.2, we also have a natural isomorphism

Sets(pt, 
$$[-1, -2]_{Sets}$$
)  $\cong [-1, -2]_{Sets}$ .

Composing both natural isomorphisms, we obtain a natural isomorphism

$$Sets(-1, -2) \cong [-1, -2]_{Sets}.$$

Given  $A, B \in Obj(Sets)$ , we will write

$$\operatorname{id}_{A,B}^{\operatorname{Hom}} : \operatorname{Sets}(A,B) \xrightarrow{\sim} [A,B]_{\operatorname{Sets}}$$

for the component of this isomorphism at (A, B).

Constructing an Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : Since  $\otimes_{\mathsf{Sets}}$  is adjoint in each variable to  $[-_1, -_2]_{\mathsf{Sets}}$  by assumption and  $\times$  is adjoint in each variable to  $\mathsf{Sets}(-_1, -_2)$  by Constructions With Sets, Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -$$

$$-\otimes_{\mathsf{Sets}} B \cong B \times -.$$

By ??, we then have  $\otimes_{\mathsf{Sets}} \cong \times$ . We will write

$$\operatorname{id}_{\operatorname{Sets}|AB}^{\otimes} \colon A \otimes_{\operatorname{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

Alternative Construction of an Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : Alternatively, we may construct a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  as follows:

- 01PC 1. Let  $A \in Obj(Sets)$ .
- 01PD 2. Since  $\otimes_{Sets}$  is part of a closed monoidal structure, it preserves colimits in each variable by  $\ref{eq:sets}$ .
- 01PE 3. Since  $A \cong \coprod_{a \in A} pt$  and  $\otimes_{Sets}$  preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong (\coprod_{a \in A} \mathsf{pt}) \otimes_{\mathsf{Sets}} B$$

$$\cong \coprod_{a \in A} (\mathsf{pt} \otimes_{\mathsf{Sets}} B)$$

$$\cong \coprod_{a \in A} B$$

$$\cong A \times B,$$

naturally in  $B \in \text{Obj}(\mathsf{Sets})$ , where we have used that pt is the monoidal unit for  $\otimes_{\mathsf{Sets}}$ . Thus  $A \otimes_{\mathsf{Sets}} - \cong A \times -$  for each  $A \in \mathsf{Obj}(\mathsf{Sets})$ .

- **01PF** 4. Similarly,  $⊗_{Sets} B \cong × B$  for each B ∈ Obj(Sets).
- 01PG 5. By ??, we then have  $\otimes_{Sets} \cong \times$ .

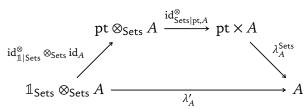
Below, we'll show that if a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \colon A \otimes_{\mathsf{Sets}} B \to A \times B$  from before.

Constructing an Isomorphism  $id_{1}^{\otimes}: 1_{Sets} \to pt$ : We define an isomorphism  $id_{1}^{\otimes}: 1_{Sets} \to pt$  as the composition

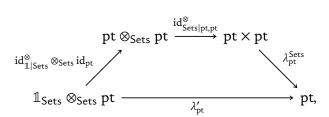
$$\mathbb{1}_{\mathsf{Sets}} \overset{\rho_{\mathbb{1}_{\mathsf{Sets}}}^{\mathsf{Sets},-1}}{\overset{\cdot \cdot \cdot}{\longrightarrow}} \mathbb{1}_{\mathsf{Sets}} \times \mathsf{pt} \overset{\mathrm{id}_{\mathsf{Sets}}^{\otimes} \mathbb{1}_{\mathsf{Sets}}}{\overset{\cdot \cdot \cdot}{\longrightarrow}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathsf{pt} \overset{\lambda_{\mathsf{pt}}'}{\overset{\cdot \cdot \cdot}{\longrightarrow}} \mathsf{pt}$$

in Sets.

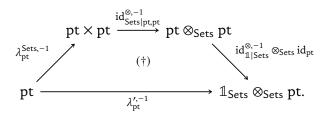
*Monoidal Left Unity of the Isomorphism*  $\otimes_{\mathsf{Sets}} \cong \mathsf{X}$ : We have to show that the diagram

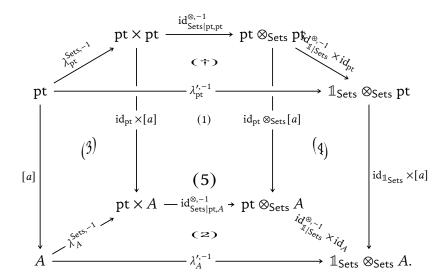


commutes. First, note that the diagram



corresponding to the case  $A = \operatorname{pt}$ , commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram



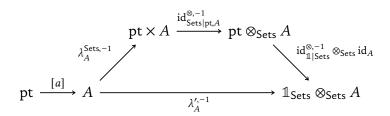


Now, let  $A \in Obj(Sets)$ , let  $a \in A$ , and consider the diagram

#### Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'^{-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{1|Sets}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\text{Sets},-1}$ .

#### it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\lambda_A^{\prime,-1}(a) = [\lambda_A^{\prime,-1} \circ [a]](\star)$$

$$\begin{split} &= \big[ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1} \circ [a] \big] (\star) \\ &= \big[ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1} \big] (a) \end{split}$$

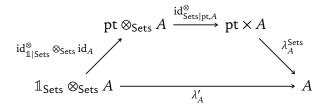
for each  $a \in A$ , and thus we have

$$\lambda_A'^{,-1} = (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \times \mathrm{id}_A),$$

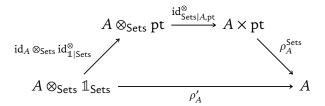
showing that the diagram



indeed commutes.

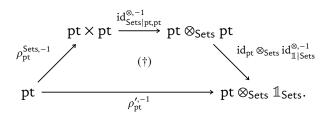
Monoidal Right Unity of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  above. For completeness, we repeat it below.

We have to show that the diagram

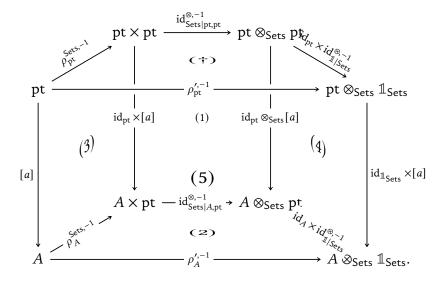


commutes. First, note that the diagram

corresponding to the case  $A = \operatorname{pt}$ , commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram



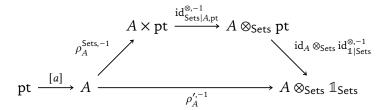
Now, let  $A \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , and consider the diagram



#### Since:

- Subdiagram (5) commutes by the naturality of  $\rho'^{-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{\mathbb{1}|Sets}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\mathrm{Sets},-1}$ .

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\rho_{A}^{\prime,-1}(a) = [\rho_{A}^{\prime,-1} \circ [a]](\star)$$

$$= [(\mathrm{id}_{A} \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_{A}^{\mathsf{Sets},-1} \circ [a]](\star)$$

$$= [(\mathrm{id}_{A} \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_{A}^{\mathsf{Sets},-1}](a)$$

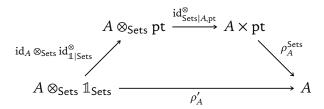
for each  $a \in A$ , and thus we have

$$\rho_A^{\prime,-1} = (\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

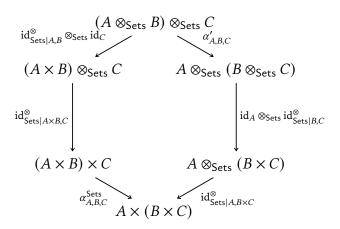
$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes}),$$

showing that the diagram

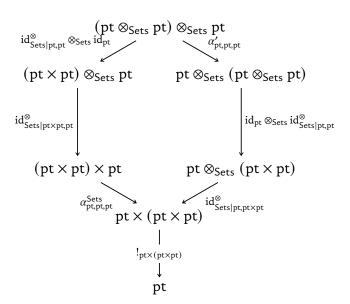


indeed commutes.

*Monoidality of the Isomorphism*  $\otimes_{\mathsf{Sets}} \cong \times$ : We have to show that the diagram

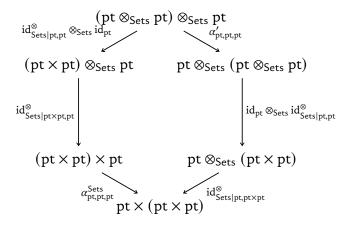


commutes. First, note that the diagram

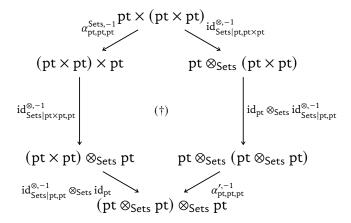


commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map  $!_{pt \times (pt \times pt)} : pt \times (pt \times pt) \rightarrow pt$  is an isomorphism (e.g. having

inverse  $\lambda_{pt}^{\mathsf{Sets},-1} \circ \lambda_{pt}^{\mathsf{Sets},-1})$  , it follows that the diagram

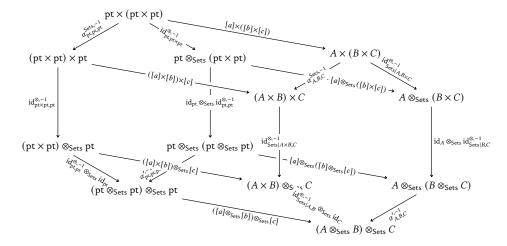


also commutes. Taking inverses, we see that the diagram

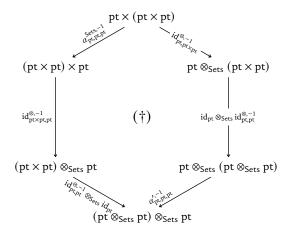


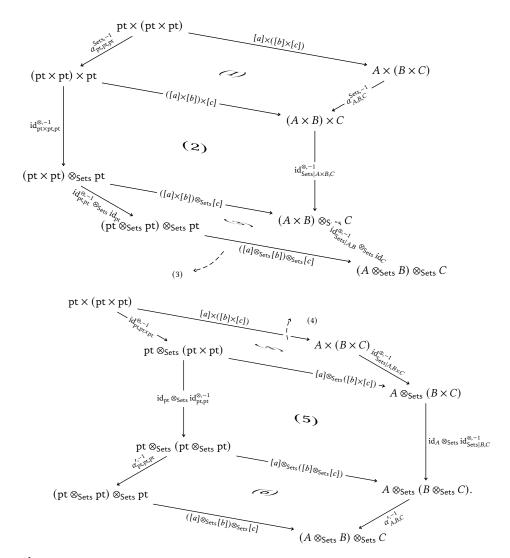
commutes as well. Now, let  $A, B, C \in Obj(Sets)$ , let  $a \in A$ , let  $b \in B$ , let  $c \in C$ ,

# and consider the diagram



# which we partition into subdiagrams as follows:



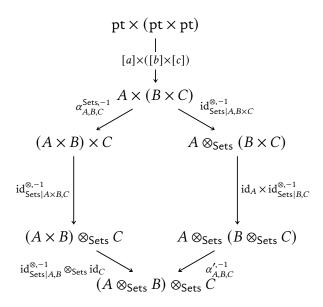


## Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\rm Sets,-1}.$
- Subdiagram (2) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (†) commutes, as proved above.

- Subdiagram (4) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (5) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .

### it follows that the diagram



also commutes. We then have

$$\begin{split} \left[ (\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ & \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) = \left[ (\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ & \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ ([a] \times ([b] \times [c])) \right] (\star,(\star,\star)) \\ & = \left[ \alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \\ & \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ ([a] \times ([b] \times [c])) \right] (\star,(\star,\star)) \\ & = \left[ \alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

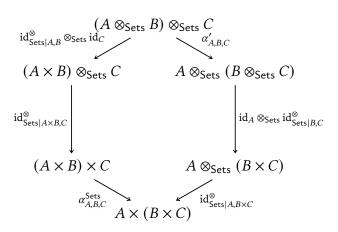
for each  $(a, (b, c)) \in A \times (B \times C)$ , and thus we have

$$(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} = \alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ (\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_{C}) = \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ (\mathrm{id}_{A} \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes}) \circ \alpha_{A,B,C}'$$

showing that the diagram



indeed commutes.

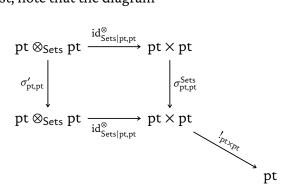
Braidedness of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : We have to show that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\sigma'_{A,B} \downarrow \qquad \qquad \qquad \downarrow \sigma^{\mathsf{Sets}}_{A,B}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

commutes. First, note that the diagram



commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map  $!_{pt \times pt} \colon pt \times pt \to pt$  is invertible (e.g. with inverse  $\lambda_{pt}^{Sets,-1}$ ), the

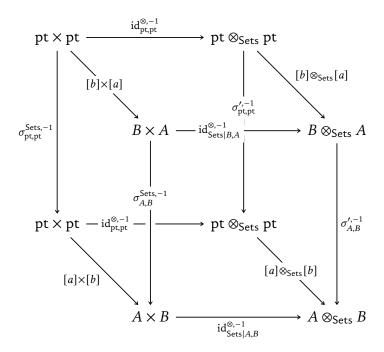
diagram

$$\begin{array}{c|c} pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes}} pt \times pt \\ \\ \sigma'_{\mathsf{pt,pt}} \downarrow & & \downarrow \sigma^{\mathsf{Sets}}_{\mathsf{pt,pt}} \\ \\ pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes}} pt \times pt \end{array}$$

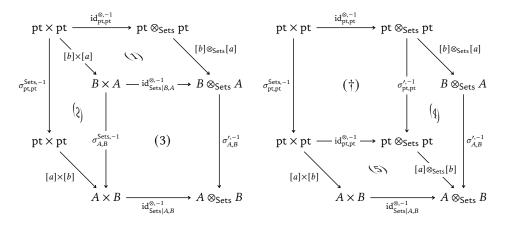
also commutes. Taking inverses, we see that the diagram

$$\begin{array}{ccc} \operatorname{pt} \times \operatorname{pt} & \xrightarrow{\operatorname{id}_{\operatorname{Sets}|\operatorname{pt,pt}}^{\otimes,-1}} & \operatorname{pt} \otimes_{\operatorname{Sets}} \operatorname{pt} \\ \sigma_{\operatorname{pt,pt}}^{\operatorname{Sets,-1}} & & (\dagger) & & & \sigma_{\operatorname{pt,pt}}^{\prime,-1} \\ \operatorname{pt} \times \operatorname{pt} & \xrightarrow{\operatorname{id}_{\operatorname{Sets}|\operatorname{pt,pt}}^{\otimes,-1}} & \operatorname{pt} \otimes_{\operatorname{Sets}} \operatorname{pt} \end{array}$$

commutes as well. Now, let  $A, B \in \mathsf{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , and consider the diagram



which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of  $\sigma^{\text{Sets},-1}$ .
- Subdiagram (5) commutes by the naturality of  $id^{\otimes,-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{-1}$ .
- Subdiagram (1) commutes by the naturality of  $id^{\otimes,-1}$ .

it follows that the diagram

$$B \times A \xrightarrow{\operatorname{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \otimes_{\mathsf{Sets}} A$$

$$\sigma_{A,B}^{\mathsf{Sets}} \downarrow \qquad \qquad \downarrow \sigma_{A,B}'$$

$$A \times B \xrightarrow{\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \otimes_{\mathsf{Sets}} B$$
Then have

commutes. We then have

$$[\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1}](b,a) = [\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a])](\star,\star)$$

$$\begin{split} &= \big[\sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a])\big](\star,\star) \\ &= \big[\sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}\big](b,a) \end{split}$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}',$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\downarrow \sigma_{A,B}^{\mathsf{Sets}} \qquad \qquad \qquad \downarrow \sigma_{A,B}^{\mathsf{Sets}}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

indeed commutes.

Uniqueness of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : Let  $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\begin{split} \lambda_B' &= \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y), \\ \lambda_B' &= \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y). \end{split}$$

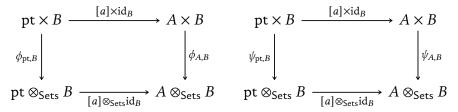
Postcomposing both sides with  $\lambda_B^{\mathsf{Sets},-1}$  gives

$$\begin{split} \lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) &= \phi_{\mathsf{pt},B}, \\ \lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) &= \psi_{\mathsf{pt},B}, \end{split}$$

and thus we have

$$\phi_{\text{pt},B} = \psi_{\text{pt},B}$$

for each  $B \in \text{Obj}(\mathsf{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams



for  $\phi$  and  $\psi$  with respect to the morphisms [a] and  $\mathrm{id}_B$ . Having shown that  $\phi_{\mathrm{pt},B}=\psi_{\mathrm{pt},B}$ , we have

$$\phi_{A,B}(a,b) = [\phi_{A,B} \circ ([a] \times id_B)](\star,b)$$

$$= [([a] \otimes_{Sets} id_B) \circ \phi_{pt,B}](\star,b)$$

$$= [([a] \otimes_{Sets} id_B) \circ \psi_{pt,B}](\star,b)$$

$$= [\psi_{A,B} \circ ([a] \times id_B)](\star,b)$$

$$= \psi_{A,B}(a,b)$$

for each  $(a, b) \in A \times B$ . Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each  $A, B \in \text{Obj}(\mathsf{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\mathsf{Sets}} \cong \mathsf{X}$  to be unique.  $\square$ 

- 01PH **Corollary 5.1.10.1.2.** The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:
- 01PJ 1. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sats}} : \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

01PK 2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:Sets}$  spanned by the symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,  $\mathbb{1}_{\mathsf{Sets}}$ ,  $\lambda^{\mathsf{Sets}}$ ,  $\rho^{\mathsf{Sets}}$ ,  $\sigma^{\mathsf{Sets}}$ ) satisfying Items 1 and 2 is contractible.

*Proof.* Since Sets is locally presentable (??), it follows from ?? that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Definition 5.1.10.1.1. 

The result then follows from Definition 5.1.10.1.1. □

# OIPL 5.2 The Monoidal Category of Sets and Coproducts

# 01PM 5.2.1 Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

# 01PN 5.2.2 The Monoidal Unit

**Definition 5.2.2.1.1.** The monoidal unit of the coproduct of sets is the functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{O}_{\mathsf{Sets}} \stackrel{\mathsf{def}}{=} \emptyset$$
,

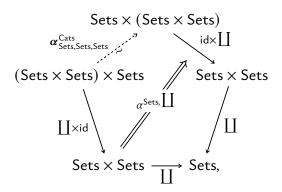
where Ø is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

## 01PQ 5.2.3 The Associator

**Definition 5.2.3.1.1.** The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} \colon \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{}}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \pmb{\alpha}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Se$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \colon (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each  $a \in (X \coprod Y) \coprod Z$ .

*Proof.* Unwinding the Definitions of  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ : Firstly, we unwind the expressions for  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ . We have

$$\begin{split} (X \coprod Y) \coprod Z &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \coprod Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ &= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ & \cup \{(1, z) \in S \mid z \in Z\}, \end{split}$$

where  $S = \{0, 1\} \times ((X \coprod Y) \cup Z)$  and

$$X \coprod (Y \coprod Z) \stackrel{\text{def}}{=} \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, a) \in S' \mid a \in Y \coprod Z \}$$
$$= \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, (0, y)) \in S' \mid y \in Y \}$$
$$\cup \{ (1, (1, z)) \in S' \mid z \in Z \},$$

where  $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$ .

*Invertibility*: The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\text{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each  $a \in X \coprod Y(\coprod Z)$ . Indeed:

• *Invertibility I*. The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  acts on elements as

$$(0,(0,x)) \mapsto (0,x) \mapsto (0,(0,x)),$$
  

$$(0,(0,y)) \mapsto (1,(0,y)) \mapsto (0,(0,y)),$$
  

$$(1,z) \mapsto (1,(1,z)) \mapsto (1,z)$$

and hence is equal to the identity map of  $(X \coprod Y) \coprod Z$ .

• *Invertibility II.* The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$  acts on elements as

$$\begin{array}{ccccc} (0,x) & \mapsto & (0,(0,x)) & \mapsto & (0,x), \\ (1,(0,y)) & \mapsto & (0,(0,y)) & \mapsto & (1,(0,y)), \\ (1,(1,z)) & \mapsto & (1,z) & \mapsto & (1,(1,z)) \end{array}$$

and hence is equal to the identity map of  $X \coprod (Y \coprod Z)$ .

Therefore  $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions

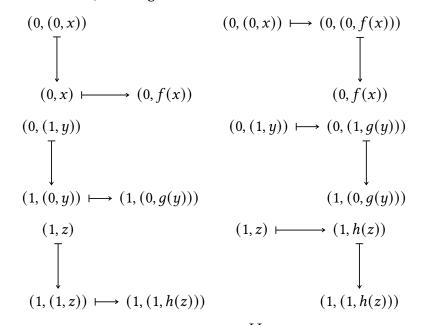
$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$(X \coprod Y) \coprod Z \xrightarrow{(f \coprod g) \coprod h} (X' \coprod Y') \coprod Z'$$

$$\begin{matrix} \xrightarrow{\alpha_{X,Y,Z}} \\ X \coprod (Y \coprod Z) \xrightarrow{f \coprod (g \coprod h)} X' \coprod (Y' \coprod Z') \end{matrix}$$

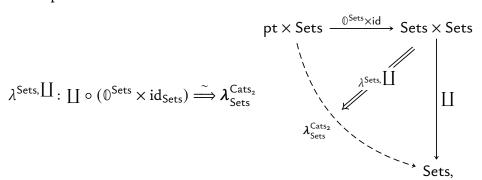
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing  $\alpha^{\mathsf{Sets}, \coprod}$  to be a natural transformation. Being a Natural Isomorphism: Since  $\alpha^{\mathsf{Sets}, \coprod}$  is natural and  $\alpha^{\mathsf{Sets}, \coprod, -1}$  is a componentwise inverse to  $\alpha^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets}, -1}$  is also natural. Thus  $\alpha^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

### 01PS 5.2.4 The Left Unitor

**Definition 5.2.4.1.1.** The **left unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} \colon \mathsf{Ø} \coprod X \stackrel{\sim}{\dashrightarrow} X$$

at X is given by

$$\lambda_X^{\mathsf{Sets},\coprod}((1,x))\stackrel{\mathrm{def}}{=} x$$

for each  $(1, x) \in \emptyset \coprod X$ .

*Proof. Unwinding the Definition of*  $\emptyset \coprod X$ : Firstly, we unwind the expressions for  $\emptyset \coprod X$ . We have

$$\emptyset \coprod X \stackrel{\text{def}}{=} \{ (0, z) \in S \mid z \in \emptyset \} \cup \{ (1, x) \in S \mid x \in X \}$$
$$= \emptyset \cup \{ (1, x) \in S \mid x \in X \}$$
$$= \{ (1, x) \in S \mid x \in X \},$$

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

*Invertibility*: The inverse of  $\lambda_X^{\mathsf{Sets}, \coprod}$  is the map

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \colon X \to \emptyset \coprod X$$

given by

$$\lambda_X^{\mathsf{Sets},\coprod,-1}(x)\stackrel{\mathrm{def}}{=} (1,x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} [\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod}](1, x) &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (\lambda_X^{\mathsf{Sets}, \coprod}(1, x)) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1}(x) \\ &= (1, x) \\ &= [\mathrm{id}_{\emptyset \coprod X}](1, x) \end{split}$$

for each  $(1, x) \in \emptyset \coprod X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$\begin{split} [\lambda_X^{\mathsf{Sets}, \coprod} \circ \lambda_X^{\mathsf{Sets}, \coprod, -1}](x) &= \lambda_X^{\mathsf{Sets}, \coprod} (\lambda_X^{\mathsf{Sets}, \coprod, -1}(x)) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1}(1, x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

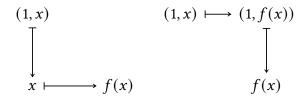
for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore  $\lambda_X^{\text{Sets},\coprod}$  is indeed an isomorphism. *Naturality*: We need to show that, given a function  $f:X\to Y$ , the diagram

$$\begin{array}{c|c}
\emptyset \coprod X & \xrightarrow{\operatorname{id}_{\emptyset} \coprod f} \emptyset \coprod Y \\
\downarrow^{\operatorname{Sets}, \coprod} & \downarrow^{\lambda^{\operatorname{Sets}, \coprod}_{Y}} \\
X & \xrightarrow{f} & Y
\end{array}$$

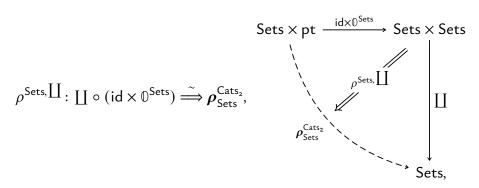
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets}, \coprod}$  is a natural transformation. Being a Natural Isomorphism: Since  $\lambda^{\mathsf{Sets}, \coprod}$  is natural and  $\lambda^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets}, -1}$  is also natural. Thus  $\lambda^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

# 01PU 5.2.5 The Right Unitor

**Definition 5.2.5.1.1.** The **right unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets},\coprod}: X \coprod \emptyset \xrightarrow{\sim} X$$

at X is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x))\stackrel{\mathrm{def}}{=} x$$

for each  $(0, x) \in X \coprod \emptyset$ .

*Proof. Unwinding the Definition of X*  $\coprod$   $\emptyset$ : Firstly, we unwind the expression for  $X \coprod \emptyset$ . We have

$$X \coprod \emptyset \stackrel{\text{def}}{=} \{ (0, x) \in S \mid x \in X \} \cup \{ (1, z) \in S \mid z \in \emptyset \}$$
$$= \{ (0, x) \in S \mid x \in X \} \cup \emptyset$$
$$= \{ (0, x) \in S \mid x \in X \},$$

where  $S=\{0,1\}\times (X\cup\emptyset)=\{0,1\}\times (\emptyset\cup X)=S.$  Invertibility: The inverse of  $\rho_X^{\mathsf{Sets},\coprod}$  is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \emptyset$$

given by

$$\rho_X^{\mathsf{Sets}, \coprod, -1}(x) \stackrel{\mathrm{def}}{=} (0, x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} [\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod}](0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} (\rho_X^{\mathsf{Sets}, \coprod}(0, x)) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1}(x) \\ &= (0, x) \\ &= [\mathrm{id}_{X \coprod \emptyset}](0, x) \end{split}$$

for each  $(0, x) \in \emptyset \coprod X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$\begin{split} [\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1}](x) &= \rho_X^{\mathsf{Sets}, \coprod} (\rho_X^{\mathsf{Sets}, \coprod, -1}(x)) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore  $\rho_X^{\text{Sets},\coprod}$  is indeed an isomorphism. Naturality: We need to show that, given a function  $f:X\to Y$ , the diagram

$$\begin{array}{c|c} X \coprod \emptyset \xrightarrow{f \coprod \mathrm{id}_{\emptyset}} Y \coprod \emptyset \\ \xrightarrow{\rho_X^{\mathsf{Sets}, \coprod}} & & \downarrow \rho_Y^{\mathsf{Sets}, \coprod} \\ X \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{bmatrix}
(0,x) & (0,x) & \longrightarrow (1,f(x)) \\
\downarrow & & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{bmatrix}$$

and hence indeed commutes. Therefore  $\rho^{\mathsf{Sets}, \coprod}$  is a natural transformation. Being a Natural Isomorphism: Since  $\rho^{\mathsf{Sets}, \coprod}$  is natural and  $\rho^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\rho^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\rho^{\mathsf{Sets}, -1}$  is also natural. Thus  $\rho^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

### 01PW 5.2.6 The Symmetry

**Definition 5.2.6.1.1.** The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}, \coprod} : \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}, \qquad \begin{array}{c} \mathsf{Sets} \times \mathsf{Sets} & \overset{\coprod}{\longleftrightarrow} \mathsf{Sets}, \\ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}} & & \downarrow & \downarrow \\ \mathsf{Sets} \times \mathsf{Sets} & & \mathsf{Sets} & \\ \mathsf{Sets} & & \\ \mathsf{Se$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}: X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at  $X, Y \in Obj(Sets)$  is defined by

$$\sigma_{X,Y}^{\text{Sets},\coprod}(x,y)\stackrel{\text{def}}{=}(y,x)$$

for each  $(x, y) \in X \times Y$ .

*Proof. Unwinding the Definitions of*  $X \coprod Y$  *and*  $Y \coprod X$ : Firstly, we unwind the expressions for  $X \coprod Y$  and  $Y \coprod X$ . We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0,x) \in S \mid x \in X\} \cup \{(1,y) \in S \mid y \in Y\},\$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},\$$

where  $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$ .

*Invertibility*: The inverse of  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is the map

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\mathrm{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = (1,x), \\ (1,y) & \text{if } z = (0,y) \end{cases}$$

for each  $z \in Y \coprod X$ . Indeed:

• Invertibility I. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}](0, x) &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (\sigma_X^{\mathsf{Sets}, \coprod}(0, x)) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (1, x) \\ &= (0, x) \\ &= [\mathrm{id}_{X \coprod Y}](0, x) \end{split}$$

for each  $(0, x) \in X \coprod Y$  and

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}](1,y) &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (\sigma_X^{\mathsf{Sets}, \coprod}(1,y)) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (0,y) \\ &= (1,y) \\ &= [\mathrm{id}_{X \coprod Y}](1,y) \end{split}$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod} = \mathrm{id}_{X\coprod Y}.$$

• Invertibility II. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}](0, y) &= \sigma_{X}^{\mathsf{Sets}, \coprod} (\sigma_{X}^{\mathsf{Sets}, \coprod, -1}(0, y)) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1}(1, y) \\ &= (0, y) \\ &= [\mathrm{id}_{Y \coprod X}](0, y) \end{split}$$

for each  $(0, y) \in Y \coprod X$  and

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}](1, x) &= \sigma_{X}^{\mathsf{Sets}, \coprod} (\sigma_{X}^{\mathsf{Sets}, \coprod, -1}(1, x)) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= (1, x) \\ &= [\mathrm{id}_{Y \coprod X}](1, x) \end{split}$$

for each  $(1, x) \in Y \coprod X$ , and therefore we have

$$\sigma_X^{\mathsf{Sets},\coprod} \circ \sigma_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_{Y\coprod X}.$$

Therefore  $\sigma_{X,Y}^{\mathsf{Sets}, \coprod}$  is indeed an isomorphism. Naturality: We need to show that, given functions  $f: A \to X$  and  $g: B \to Y$ , the diagram

$$A \coprod B \xrightarrow{f \coprod g} X \coprod Y$$

$$\downarrow_{\sigma_{A,B}}^{\mathsf{Sets}, \coprod} \qquad \qquad \downarrow_{\sigma_{X,Y}}^{\mathsf{Sets}, \coprod}$$

$$B \coprod A \xrightarrow{a \coprod f} Y \coprod X$$

commutes. Indeed, this diagram acts on elements as

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes. Therefore  $\sigma^{\mathsf{Sets},\coprod}$  is a natural transformation.

Being a Natural Isomorphism: Since  $\sigma^{Sets, \coprod}$  is natural and  $\sigma^{Sets, -1}$  is a componentwise inverse to  $\sigma^{Sets, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\sigma^{Sets, -1}$  is also natural. Thus  $\sigma^{Sets, \coprod}$  is a natural isomorphism.

# 01PY 5.2.7 The Monoidal Category of Sets and Coproducts

- **Proposition 5.2.7.1.1.** The category Sets admits a closed symmetric monoidal category structure consisting of:
  - The Underlying Category. The category Sets of pointed sets.
  - The Monoidal Product. The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.

• The Monoidal Unit. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
of Definition 5.2.3.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\text{Sets},\coprod}: \coprod \circ (\mathbb{O}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of Definition 5.2.4.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\text{Sets}, \coprod} : \coprod \circ (\text{id} \times \mathbb{O}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

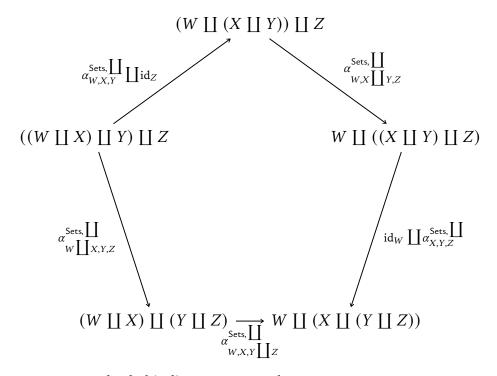
of Definition 5.2.5.1.1.

• *The Symmetry*. The natural isomorphism

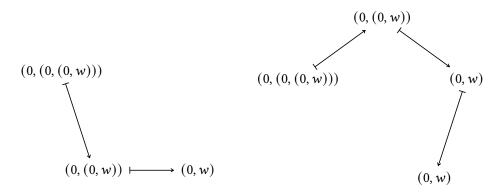
$$\sigma^{\mathsf{Sets},\coprod}: imes \stackrel{\sim}{\Longrightarrow} imes \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

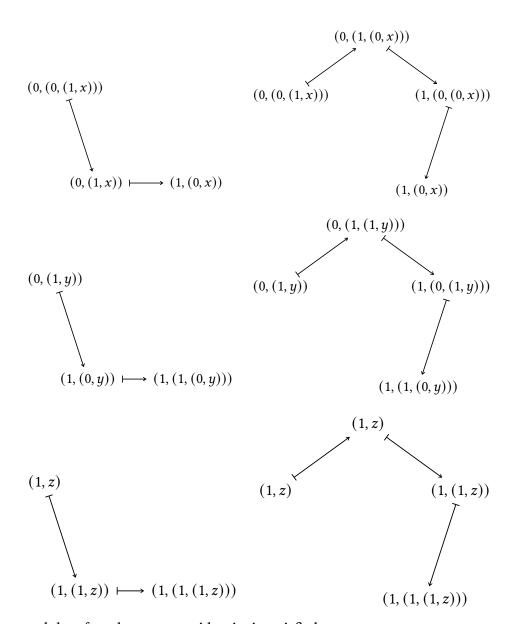
of Definition 5.2.6.1.1.

*Proof.* The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram



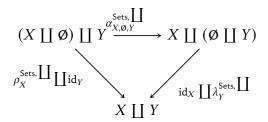
commutes. Indeed, this diagram acts on elements as



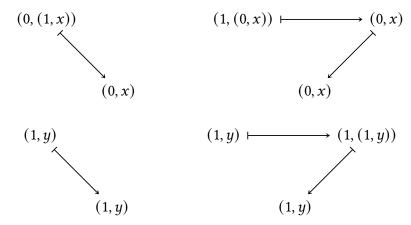


and therefore the pentagon identity is satisfied.

*The Triangle Identity*: Let *X* and *Y* be sets. We have to show that the diagram

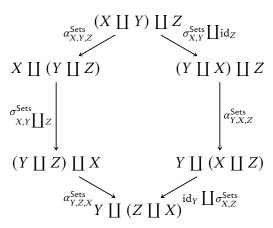


commutes. Indeed, this diagram acts on elements as

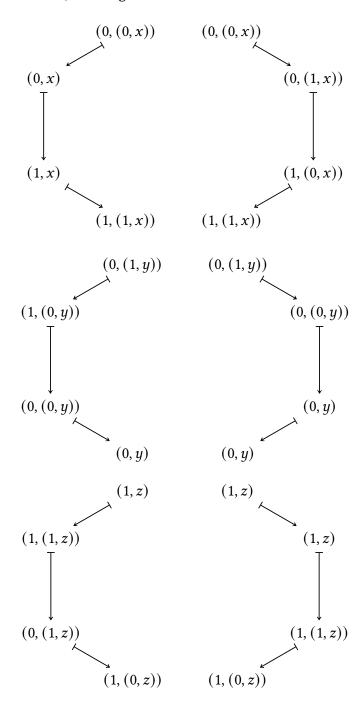


and therefore the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

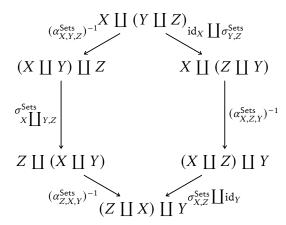


commutes. Indeed, this diagram acts on elements as

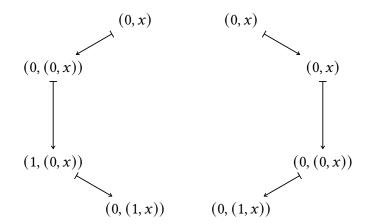


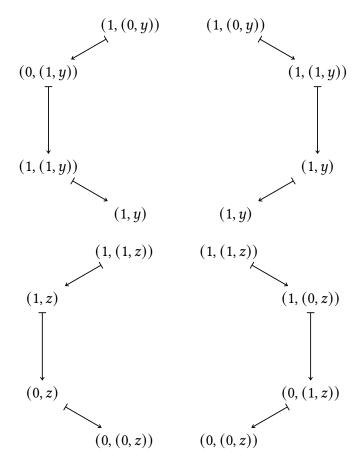
and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as





and thus the right hexagon identity is satisfied.

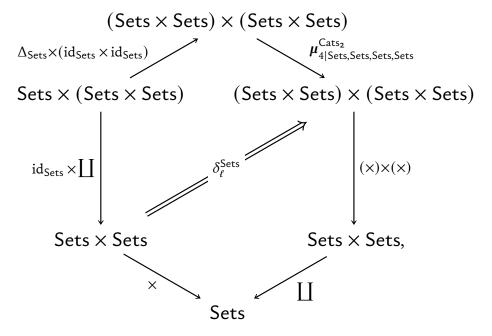
# oloo 5.3 The Bimonoidal Category of Sets, Products, and Coproducts

# 0101 5.3.1 The Left Distributor

01Q2 Definition 5.3.1.1.1. The left distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \stackrel{\widetilde{-}}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{4 \mid \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$

as in the diagram



whose component

$$\delta^{\mathsf{Sets}}_{\ell \mid X,Y,Z} \colon X \times (Y \coprod Z) \xrightarrow{\sim} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets}}(x,a) \stackrel{\text{def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each  $(x, a) \in X \times (Y \coprod Z)$ .

*Proof. Invertibility*: The inverse of  $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$  is the map

$$\delta^{\mathsf{Sets},-1}_{\ell\mid X,Y,Z} \colon (X\times Y) \ {\textstyle\coprod} \ (X\times Z) \stackrel{\widetilde{}_{-}}{\dashrightarrow} X\times (Y \ {\textstyle\coprod} \ Z)$$

given by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} (x,(0,y)) & \text{if } a = (0,(x,y)), \\ (x,(1,z)) & \text{if } a = (1,(x,z)) \end{cases}$$

for  $a \in (X \times Y) \coprod (X \times Z)$ . Indeed:

• Invertibility I. The map  $\delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z}\circ\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$  acts on elements as

$$(x, (0, y)) \mapsto (0, (x, y)) \mapsto (x, (0, y)),$$
  
 $(x, (1, z)) \mapsto (1, (x, z)) \mapsto (x, (1, z)),$ 

but these are the two possible cases for elements of  $X \times (Y \coprod Z)$ . Hence the map is equal to the identity.

• Invertibility II. The map  $\delta^{\sf Sets}_{\ell|X,Y,Z} \circ \delta^{\sf Sets,-1}_{\ell|X,Y,Z}$  acts on elements as

$$(0,(x,y)) \mapsto (x,(0,y)) \mapsto (0,(x,y)), (1,(x,z)) \mapsto (x,(1,z)) \mapsto (1,(x,z)),$$

but these are the two possible cases for elements of  $(X \times Y) \coprod (X \times Z)$ . Hence the map is equal to the identity.

Thus  $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$  is an isomorphism for all X,Y,Z. *Naturality*: We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$\begin{array}{c|c} X\times (Y\coprod Z) & \xrightarrow{f\times \left(g\coprod h\right)} & X'\times (Y'\coprod Z') \\ \delta^{\mathsf{Sets}}_{\ell\mid X,Y,Z} & & & & & \\ (X\times Y)\coprod (X\times Z) & \xrightarrow{(f\times g)\coprod (f\times h)} & (X'\times Y')\coprod (X'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x, (0, y)) \qquad (x, (0, y)) \longmapsto (f(x), (0, f(y)))$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(x, (1, z)) \qquad (x, (1, z)) \longmapsto (f(x), (1, h(z)))$$

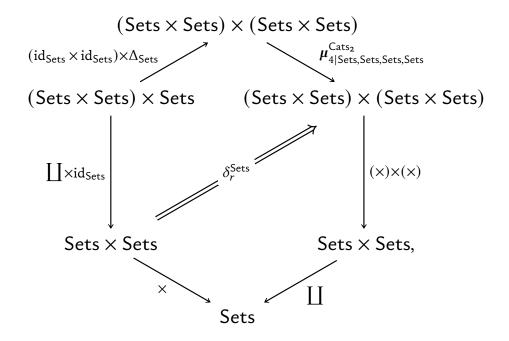
$$\downarrow \qquad \qquad \downarrow \qquad$$

so it commutes, showing  $\delta_\ell^{\mathsf{Sets}}$  to be a natural transformation.  $\mathsf{Being}\ a\ \mathsf{Natural}\ \mathsf{Isomorphism}$ : Since  $\delta_\ell^{\mathsf{Sets}}$  is natural and  $\delta_\ell^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\delta_\ell^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\delta_\ell^{\mathsf{Sets},-1}$  is also natural. Thus  $\delta_\ell^{\mathsf{Sets}}$  is a natural isomorphism.  $\square$ 

# 01Q3 5.3.2 The Right Distributor

# **Definition 5.3.2.1.1.** The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
 as in the diagram



whose component

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each  $(a, z) \in (X \mid Y) \times Z$ .

*Proof.* Invertibility: The inverse of  $\delta^{\mathsf{Sets}}_{r|X,Y,Z}$  is the map

$$\delta^{\mathsf{Sets},-1}_{r|X,Y,Z} \colon (X \times Z) \ \coprod \ (Y \times Z) \ \stackrel{\scriptstyle \sim}{\dashrightarrow} \ (X \ \coprod \ Y) \times Z$$

given by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} ((0,x),z) & \text{if } a = (0,(x,z)), \\ ((1,y),z) & \text{if } a = (1,(y,z)) \end{cases}$$

for  $a \in (X \times Z) \coprod (Y \times Z)$ . Indeed:

• *Invertibility I.* The map  $\delta_{r|X,Y,Z}^{\mathsf{Sets},-1} \circ \delta_{r|X,Y,Z}^{\mathsf{Sets}}$  acts on elements as

$$((0,x),z) \mapsto (0,(x,z)) \mapsto (0,(x,z)),$$
  
$$((1,y),z) \mapsto (1,(y,z)) \mapsto (1,(y,z)),$$

but these are the two possible cases for elements of  $(X \coprod Y) \times Z$ . Hence the map is equal to the identity.

• Invertibility II. The map  $\delta_{r|X,Y,Z}^{\mathsf{Sets}} \circ \delta_{r|X,Y,Z}^{\mathsf{Sets},-1}$  acts on elements as

$$(0,(x,z)) \mapsto ((0,x),z) \mapsto (0,(x,z)), (1,(y,z)) \mapsto ((1,y),z) \mapsto (1,(y,z)),$$

but these are the two possible cases for elements of  $(X \times Z) \coprod (Y \times Z)$ . Hence the map is equal to the identity.

So  $\delta^{\mathsf{Sets}}_{r|X,Y,Z}$  is an isomorphism for all X,Y,Z.

Naturality: We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$\begin{array}{c|c} (X \coprod Y) \times Z' & \xrightarrow{\qquad \qquad \qquad } (f \coprod g) \times h & \\ \delta^{\mathsf{Sets}}_{r \mid X, Y, Z} & & & & & \\ \delta^{\mathsf{Sets}}_{r \mid X, Y, Z'} & & & & & \\ (X \times Z) \coprod (Y \times Z) & \xrightarrow{\qquad \qquad } (f \times h) \coprod (g \times h) & & (X' \times Z') \coprod (Y' \times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

so it commutes and  $\delta_r^{\rm Sets}$  is a natural transformation.

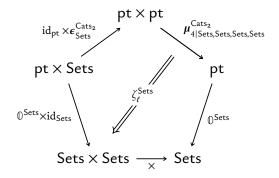
Being a Natural Isomorphism: Since  $\delta_r^{\mathsf{Sets}}$  is natural and  $\delta_r^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\delta_r^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\delta_r^{\mathsf{Sets},-1}$  is also natural. Thus  $\delta_r^{\mathsf{Sets}}$  is a natural isomorphism.

# 01Q5 5.3.3 The Left Annihilator

**Definition 5.3.3.1.1.** The **left annihilator of the product of sets** is the natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \emptyset \times A \xrightarrow{\sim} \emptyset$$

given by  $\zeta_{\ell|A}^{\mathsf{Sets}} \stackrel{\mathsf{def}}{=} \mathsf{pr}_1$ .

*Proof. Invertibility*: The inverse of  $\zeta_{\ell|A}^{\mathsf{Sets}}$  is the map

$$\zeta_{\ell|A}^{\mathsf{Sets},-1} \colon \varnothing \xrightarrow{\sim} \varnothing \times A$$

given by

$$\zeta_{\ell|A}^{\mathsf{Sets},-1} \stackrel{\mathrm{def}}{=} \iota_A,$$

where  $\iota_A$  is as defined in Constructions With Sets, Definition 4.2.1.1.2:

- *Invertibility I.* The map  $\zeta_{\ell|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \to \emptyset$  is equal to  $\mathrm{id}_\emptyset$ , as  $\emptyset$  is the initial object of Sets.
- *Invertibility II*. The map  $\iota_A \circ \zeta_{\ell|A}^{\mathsf{Sets}}$  is equal to the identity on every  $(x, a) \in \emptyset \times A$ , of which there are none.

Hence  $\zeta_{\ell|A}^{\mathsf{Sets}}$  is an isomorphism.

*Naturality*: We need to show that given a function  $f: A \rightarrow B$ , the diagram

$$\begin{array}{c|c}
\emptyset \times A & \xrightarrow{\mathrm{id}_{\emptyset} \times f} & \emptyset \times B \\
\downarrow^{\zeta_{\ell|A}} & & & \downarrow^{\zeta_{\ell|B}} \\
\emptyset & \xrightarrow{\mathrm{id}_{\sigma}} & \emptyset
\end{array}$$

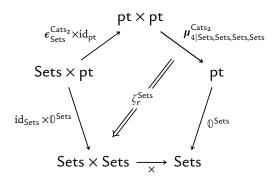
commutes. But since  $\emptyset \times A$  has no elements, this is trivially true. Being a Natural Isomorphism: Since  $\zeta_\ell^{\mathsf{Sets}}$  is natural and  $\zeta_\ell^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\zeta_\ell^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\zeta_\ell^{\mathsf{Sets},-1}$  is also natural. Thus  $\zeta_\ell^{\mathsf{Sets}}$  is a natural isomorphism.

## 0107 5.3.4 The Right Annihilator

**Definition 5.3.4.1.1.** The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}) \xrightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}})$$

as in the diagram



with components

$$\zeta_{r|A}^{\mathsf{Sets}} \colon A \times \emptyset \xrightarrow{\sim} \emptyset$$

given by  $\zeta_{r|A}^{\text{Sets}} \stackrel{\text{def}}{=} \text{pr}_2$ .

*Proof. Invertibility*: The inverse of  $\zeta_{r|A}^{\mathsf{Sets}}$  is the map

$$\zeta_{r|A}^{\mathsf{Sets},-1} \colon \emptyset \xrightarrow{\sim} A \times \emptyset$$

given by

$$\zeta_{r|A}^{\mathsf{Sets},-1} \stackrel{\mathrm{def}}{=} \iota_A,$$

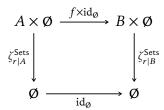
where  $\iota_A$  is as defined in Constructions With Sets, Definition 4.2.1.1.2:

• *Invertibility I.* The map  $\zeta_{r|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \to \emptyset$  is equal to  $\mathrm{id}_{\emptyset}$ , as  $\emptyset$  is the initial object of Sets.

• *Invertibility II*. The map  $\iota_A \circ \zeta_{r|A}^{\mathsf{Sets}}$  is equal to the identity on every  $(a, x) \in A \times \emptyset$ , of which there are none.

Hence  $\zeta_{r|A}^{\mathsf{Sets}}$  is an isomorphism.

*Naturality*: We need to show that given a function  $f: A \rightarrow B$ , the diagram



commutes. But since  $A \times \emptyset$  has no elements, this is trivially true. Being a Natural Isomorphism: Since  $\zeta_r^{\mathsf{Sets}}$  is natural and  $\zeta_r^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\zeta_r^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\zeta_r^{\mathsf{Sets},-1}$  is also natural. Thus  $\zeta_r^{\mathsf{Sets}}$  is a natural isomorphism.

# 01Q9 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

- **Proposition 5.3.5.1.1.** The category Sets admits a closed symmetric bimonoidal category structure consisting of:
  - The Underlying Category. The category Sets of pointed sets.
  - *The Additive Monoidal Product*. The coproduct functor

II: Sets 
$$\times$$
 Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.

• The Multiplicative Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

• *The Monoidal Unit*. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Monoidal Zero. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, ?? of ??.

• The Additive Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \boldsymbol{\alpha}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}}$$
of Definition 5.2.3.1.1.

• The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Additive Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ (\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}) \stackrel{\widetilde{\longrightarrow}}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.2.5.1.1.

• The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}:\coprod\stackrel{\widetilde{}}{\Longrightarrow}\coprod\circ\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}\,\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

• The Multiplicative Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$
 of Definition 5.1.4.1.1.

• The Multiplicative Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \mathsf{X} \circ (\mathbb{1}^{\mathsf{Sets}} \mathsf{X} \operatorname{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Multiplicative Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \mathsf{X} \circ (\mathsf{id} \mathsf{X} \, \mathbb{1}^{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• The Multiplicative Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} : \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$$
of Definition 5.3.1.1.1.

• The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.1.1.

• The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \mu_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\mathrm{id}_{\mathsf{pt}} \times \epsilon_{\mathsf{Sets}}^{\mathsf{Cats}_2}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$
of Definition 5.3.3.1.1.

• The Right Annihilator. The natural isomorphism

$$\zeta_r^{\mathsf{Sets}} : \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}) \xrightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}})$$
of Definition 5.3.4.1.1.

Proof. Omitted.

# **Appendices**

# A Other Chapters

### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

### **Relations**

- 8. Relations
- 9. Constructions With Relations

### 10. Conditions on Relations

### **Categories**

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

### **Monoidal Categories**

13. Constructions With Monoidal Categories

### **Bicategories**

14. Types of Morphisms in Bicategories

#### Extra Part

15. Notes