

# Types of Morphisms in Bicategories

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**019H** In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories* (*Sections 14.1 and 14.2*). There is a large number of different notions capturing the idea of a “monomorphism” or of an “epimorphism” in a bicategory.

Arguably, the notion that best captures these concepts is that of a *pseudomononic morphism* (*Definition 14.1.10.1.1*) and of a *pseudoepic morphism* (*Definition 14.2.10.1.1*), although the other notions introduced in *Sections 14.1* and *14.2* are also interesting on their own.

## Contents

<b>14.1</b>	<b>Monomorphisms in Bicategories .....</b>	<b>2</b>
14.1.1	Representably Faithful Morphisms .....	2
14.1.2	Representably Full Morphisms.....	4
14.1.3	Representably Fully Faithful Morphisms.....	5
14.1.4	Morphisms Representably Faithful on Cores.....	8
14.1.5	Morphisms Representably Full on Cores .....	8
14.1.6	Morphisms Representably Fully Faithful on Cores .....	9
14.1.7	Representably Essentially Injective Morphisms.....	11
14.1.8	Representably Conservative Morphisms.....	11
14.1.9	Strict Monomorphisms.....	12
14.1.10	Pseudomononic Morphisms.....	13
<b>14.2</b>	<b>Epimorphisms in Bicategories.....</b>	<b>16</b>
14.2.1	Corepresentably Faithful Morphisms.....	16
14.2.2	Corepresentably Full Morphisms.....	17

14.2.3	Corepresentably Fully Faithful Morphisms .....	19
14.2.4	Morphisms Corepresentably Faithful on Cores .....	20
14.2.5	Morphisms Corepresentably Full on Cores.....	21
14.2.6	Morphisms Corepresentably Fully Faithful on Cores.....	22
14.2.7	Corepresentably Essentially Injective Morphisms.....	24
14.2.8	Corepresentably Conservative Morphisms .....	24
14.2.9	Strict Epimorphisms .....	25
14.2.10	Pseudoepic Morphisms .....	26

## A Other Chapters..... 28

### 019J 14.1 Monomorphisms in Bicategories

#### 019K 14.1.1 Representably Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

##### 019L DEFINITION 14.1.1.1.1 ► REPRESENTABLY FAITHFUL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably faithful**<sup>1</sup> if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is faithful.

<sup>1</sup>*Further Terminology:* Also called simply a **faithful morphism**, based on [Item 1](#) of [Example 14.1.1.1.3](#).

##### 019M REMARK 14.1.1.1.2 ► UNWINDING DEFINITION 14.1.1.1

In detail,  $f$  is representably faithful if, for all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .



Here are some examples of representably faithful morphisms.

019P

1. *Representably Faithful Morphisms in  $\mathbf{Cats}_2$* . The representably faithful morphisms in  $\mathbf{Cats}_2$  are precisely the faithful functors; see [Categories](#), [Item 2](#) of [Proposition 11.6.1.1.2](#).

019Q

2. *Representably Faithful Morphisms in  $\mathbf{Rel}$* . Every morphism of  $\mathbf{Rel}$  is representably faithful; see [Relations](#), ?? of ??.

## 019R 14.1.2 Representably Full Morphisms

Let  $\mathcal{C}$  be a bicategory.

019S

### DEFINITION 14.1.2.1.1 ► REPRESENTABLY FULL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably full**<sup>1</sup> if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is full.

<sup>1</sup>*Further Terminology:* Also called simply a **full morphism**, based on [Item 1](#) of [Example 14.1.2.1.3](#).

019T

### REMARK 14.1.2.1.2 ► UNWINDING DEFINITION 14.1.2.1.1

In detail,  $f$  is representably full if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: f \circ \phi \Longrightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

019U

#### EXAMPLE 14.1.2.1.3 ► EXAMPLES OF REPRESENTABLY FULL MORPHISMS

Here are some examples of representably full morphisms.

019V

1. *Representably Full Morphisms in  $\mathbf{Cats}_2$ .* The representably full morphisms in  $\mathbf{Cats}_2$  are precisely the full functors; see [Categories](#), ?? of [Proposition 11.6.2.1.2](#).

019W

2. *Representably Full Morphisms in  $\mathbf{Rel}$ .* The representably full morphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), ?? of ??.

### 019X 14.1.3 Representably Fully Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

019Y

#### DEFINITION 14.1.3.1.1 ► REPRESENTABLY FULLY FAITHFUL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably fully faithful**<sup>1</sup> if the following equivalent conditions are satisfied:

019Z

1. The 1-morphism  $f$  is representably faithful ([Definition 14.1.1.1.1](#)) and representably full ([Definition 14.1.2.1.1](#)).

01A0

2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is fully faithful.

<sup>1</sup>*Further Terminology:* Also called simply a **fully faithful morphism**, based on [Item 1](#) of [Example 14.1.3.1.3](#).

---

01A1

In detail,  $f$  is representably fully faithful if the conditions in [Remark 14.1.1.1.2](#) and [Remark 14.1.2.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $X \in \mathrm{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01A2

#### EXAMPLE 14.1.3.1.3 ► EXAMPLES OF REPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of representably fully faithful morphisms.

01A3

1. *Representably Fully Faithful Morphisms in  $\mathbf{Cats}_2$ .* The representably fully faithful morphisms in  $\mathbf{Cats}_2$  are precisely the fully faithful functors; see [Categories](#), [Item 6](#) of [Proposition 11.6.3.1.2](#).

01A4

2. *Representably Fully Faithful Morphisms in  $\mathbf{Rel}$ .* The representably fully faithful morphisms of  $\mathbf{Rel}$  coincide ([Relations](#), ?? of ??) with the representably full morphisms in  $\mathbf{Rel}$ , which are characterised in [Relations](#), ?? of ??.

### 01A5 14.1.4 Morphisms Representably Faithful on Cores

Let  $\mathcal{C}$  be a bicategory.

01A6

#### DEFINITION 14.1.4.1.1 ► MORPHISMS REPRESENTABLY FAITHFUL ON CORES

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably faithful on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is faithful.

01A7

#### REMARK 14.1.4.1.2 ► UNWINDING DEFINITION 14.1.4.1.1

In detail,  $f$  is representably faithful on cores if, for all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

### 01A8 14.1.5 Morphisms Representably Full on Cores

Let  $\mathcal{C}$  be a bicategory.



01A9

**DEFINITION 14.1.5.1.1 ► MORPHISMS REPRESENTABLY FULL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably full on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is full.

01AA

**REMARK 14.1.5.1.2 ► UNWINDING DEFINITION 14.1.5.1.1**

In detail,  $f$  is representably full on cores if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

**01AB 14.1.6 Morphisms Representably Fully Faithful on Cores**

Let  $\mathcal{C}$  be a bicategory.

01AC

**DEFINITION 14.1.6.1.1 ► MORPHISMS REPRESENTABLY FULLY FAITHFUL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

01AD

1. The 1-morphism  $f$  is representably faithful on cores ([Definition 14.1.5.1.1](#)) and representably full on cores ([Definition 14.1.4.1.1](#)).

01AE

2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is fully faithful.

01AF

**REMARK 14.1.6.1.2 ► UNWINDING DEFINITION 14.1.6.1.1**

In detail,  $f$  is representably fully faithful on cores if the conditions in [Remark 14.1.4.1.2](#) and [Remark 14.1.5.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

### 01AG 14.1.7 Representably Essentially Injective Morphisms

Let  $\mathcal{C}$  be a bicategory.

01AH

#### DEFINITION 14.1.7.1.1 ► REPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably essentially injective** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is essentially injective.

01AJ

#### REMARK 14.1.7.1.2 ► UNWINDING DEFINITION 14.1.7.1.1

In detail,  $f$  is representably essentially injective if, for each pair of morphisms  $\phi, \psi: X \rightrightarrows A$  of  $\mathcal{C}$ , the following condition is satisfied:

( $\star$ ) If  $f \circ \phi \cong f \circ \psi$ , then  $\phi \cong \psi$ .

### 01AK 14.1.8 Representably Conservative Morphisms

Let  $\mathcal{C}$  be a bicategory.

01AL

#### DEFINITION 14.1.8.1.1 ► REPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably conservative** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is conservative.

01AM

**REMARK 14.1.8.1.2 ► UNWINDING DEFINITION 14.1.8.1.1**

In detail,  $f$  is representably conservative if, for each pair of morphisms  $\phi, \psi: X \rightrightarrows A$  and each 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$ , if the 2-morphism

$$\mathrm{id}_f \star \alpha: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \parallel \\ \mathrm{id}_f \star \alpha \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

is a 2-isomorphism, then so is  $\alpha$ .

## 01AN 14.1.9 Strict Monomorphisms

Let  $\mathcal{C}$  be a bicategory.

01AP

**DEFINITION 14.1.9.1.1 ► STRICT MONOMORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is a **strict monomorphism** if, for each  $X \in \mathrm{Obj}(\mathcal{C})$ , the functor

$$f_*: \mathrm{Hom}_{\mathcal{C}}(X, A) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is injective on objects, i.e. its action on objects

$$f_*: \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \rightarrow \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

is injective.

01AQ

**REMARK 14.1.9.1.2 ► UNWINDING DEFINITION 14.1.9.1.1**

In detail,  $f$  is a strict monomorphism in  $\mathcal{C}$  if, for each diagram in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $f \circ \phi = f \circ \psi$ , then  $\phi = \psi$ .

01AR

**EXAMPLE 14.1.9.1.3 ► EXAMPLES OF STRICT MONOMORPHISMS**

Here are some examples of strict monomorphisms.

01AS

1. *Strict Monomorphisms in  $\mathbf{Cats}_2$* . The strict monomorphisms in  $\mathbf{Cats}_2$  are precisely the functors which are injective on objects and injective on morphisms; see [Categories](#), [Item 1](#) of [Proposition 11.7.2.1.2](#).

01AT

2. *Strict Monomorphisms in  $\mathbf{Rel}$* . The strict monomorphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), ??.

**01AU 14.1.10 Pseudomonic Morphisms**

Let  $\mathcal{C}$  be a bicategory.

01AV

**DEFINITION 14.1.10.1.1 ► PSEUDOMONIC MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **pseudomonic** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is pseudomonic.

01AW

**REMARK 14.1.10.1.2 ► UNWINDING DEFINITION 14.1.10.1.1**

In detail, a 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is pseudomonic if it satisfies the following conditions:

01AX

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then  $\alpha = \beta$ .

01AY

2. For each  $X \in \mathrm{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AZ

**PROPOSITION 14.1.10.1.3 ► PROPERTIES OF PSEUDOMONIC MORPHISMS**

Let  $f: A \rightarrow B$  be a 1-morphism of  $\mathcal{C}$ .

01B0

1. *Characterisations.* The following conditions are equivalent:

01B1

- (a) The morphism  $f$  is pseudomononic.

01B2

(b) The morphism  $f$  is representably full on cores and representably faithful.

01B3

(c) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_B A, \quad \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & \nearrow \text{dashed} & \downarrow F \\ A & \xrightarrow{F} & B \end{array}$$

in  $\mathcal{C}$  up to equivalence.

01B4

2. *Interaction With Cotensors.* If  $\mathcal{C}$  has cotensors with  $\mathbb{1}$ , then the following conditions are equivalent:

(a) The morphism  $f$  is pseudomononic.

(b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_{\mathbb{1} \pitchfork F} B, \quad \begin{array}{ccc} A & \hookrightarrow & \mathbb{1} \pitchfork A \\ F \downarrow & \nearrow \text{dashed} & \downarrow \mathbb{1} \pitchfork F \\ B & \hookrightarrow & \mathbb{1} \pitchfork B \end{array}$$

in  $\mathcal{C}$  up to equivalence.

#### PROOF 14.1.10.1.4 ► PROOF OF PROPOSITION 14.1.10.1.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted.



## 01B5 14.2 Epimorphisms in Bicategories

### 01B6 14.2.1 Corepresentably Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.



01B7

**DEFINITION 14.2.1.1.1 ► COREPRESENTABLY FAITHFUL MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably faithful** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is faithful.

01B8

**REMARK 14.2.1.1.2 ► UNWINDING DEFINITION 14.2.1.1.1**

In detail,  $f$  is corepresentably faithful if, for all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \parallel \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

01B9

**EXAMPLE 14.2.1.1.3 ► EXAMPLES OF COREPRESENTABLY FAITHFUL MORPHISMS**

Here are some examples of corepresentably faithful morphisms.

01BA

1. *Corepresentably Faithful Morphisms in  $\mathbf{Cats}_2$* . The corepresentably faithful morphisms in  $\mathbf{Cats}_2$  are characterised in [Categories](#), [Item 5](#) of [Proposition 11.6.1.1.2](#).

01BB

2. *Corepresentably Faithful Morphisms in  $\mathbf{Rel}$* . Every morphism of  $\mathbf{Rel}$  is corepresentably faithful; see [Relations](#), ?? of ??.

**01BC 14.2.2 Corepresentably Full Morphisms**

Let  $\mathcal{C}$  be a bicategory.

01BD

**DEFINITION 14.2.2.1.1 ► COREPRESENTABLY FULL MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably full** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is full.

01BE

**REMARK 14.2.2.1.2** ► UNWINDING DEFINITION 14.2.2.1.1

In detail,  $f$  is corepresentably full if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: \phi \circ f \Longrightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01BF

**EXAMPLE 14.2.2.1.3** ► EXAMPLES OF COREPRESENTABLY FULL MORPHISMS

Here are some examples of corepresentably full morphisms.

01BG

1. *Corepresentably Full Morphisms in  $\mathbf{Cats}_2$ .* The corepresentably full morphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 7 of Proposition 11.6.2.1.2](#).

01BH

2. *Corepresentably Full Morphisms in **Rel***. The corepresentably full morphisms in **Rel** are characterised in **Relations**, ?? of ??.

### 01BJ 14.2.3 Corepresentably Fully Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

01BK

#### DEFINITION 14.2.3.1.1 ► COREPRESENTABLY FULLY FAITHFUL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably fully faithful**<sup>1</sup> if the following equivalent conditions are satisfied:

01BL

1. The 1-morphism  $f$  is corepresentably full (**Definition 14.2.2.1.1**) and corepresentably faithful (**Definition 14.2.1.1.1**).

01BM

2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is fully faithful.

<sup>1</sup>*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name “corepresentably fully faithful morphism” instead in this work.

01BN

#### REMARK 14.2.3.1.2 ► UNWINDING DEFINITION 14.2.3.1.1

In detail,  $f$  is corepresentably fully faithful if the conditions in **Remark 14.2.1.1.2** and **Remark 14.2.2.1.2** hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \parallel \downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X \quad = \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01BP

#### EXAMPLE 14.2.3.1.3 ► EXAMPLES OF COREPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of corepresentably fully faithful morphisms.

01BQ

1. *Corepresentably Fully Faithful Morphisms in  $\mathbf{Cats}_2$* . The fully faithful epimorphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 10](#) of [Proposition 11.6.3.1.2](#).

01BR

2. *Corepresentably Fully Faithful Morphisms in  $\mathbf{Rel}$* . The corepresentably fully faithful morphisms of  $\mathbf{Rel}$  coincide ([Relations](#), ?? of ??) with the corepresentably full morphisms in  $\mathbf{Rel}$ , which are characterised in [Relations](#), ?? of ??.

### 01BS 14.2.4 Morphisms Corepresentably Faithful on Cores

Let  $\mathcal{C}$  be a bicategory.

01BT

**DEFINITION 14.2.4.1.1 ► MORPHISMS COREPRESENTABLY FAITHFUL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably faithful on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is faithful.

01BU

**REMARK 14.2.4.1.2 ► UNWINDING DEFINITION 14.2.4.1.1**

In detail,  $f$  is corepresentably faithful on cores if, for all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

**01BV 14.2.5 Morphisms Corepresentably Full on Cores**

Let  $\mathcal{C}$  be a bicategory.

01BW

**DEFINITION 14.2.5.1.1 ► MORPHISMS COREPRESENTABLY FULL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably full on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is full.

01BX

**REMARK 14.2.5.1.2 ► UNWINDING DEFINITION 14.2.5.1.1**

In detail,  $f$  is corepresentably full on cores if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

## 01BY 14.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let  $\mathcal{C}$  be a bicategory.

### 01BZ DEFINITION 14.2.6.1.1 ► MORPHISMS COREPRESENTABLY FULLY FAITHFUL ON CORES

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

- 01C0 1. The 1-morphism  $f$  is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).
- 01C1 2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is fully faithful.

01C2

**REMARK 14.2.6.1.2 ► UNWINDING DEFINITION 14.2.6.1.1**

In detail,  $f$  is corepresentably fully faithful on cores if the conditions in [Remark 14.2.4.1.2](#) and [Remark 14.2.5.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

### 01C3 14.2.7 Corepresentably Essentially Injective Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### 01C4 DEFINITION 14.2.7.1.1 ► COREPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably essentially injective** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is essentially injective.

#### 01C5 REMARK 14.2.7.1.2 ► UNWINDING DEFINITION 14.2.7.1.1

In detail,  $f$  is corepresentably essentially injective if, for each pair of morphisms  $\phi, \psi: B \rightrightarrows X$  of  $\mathcal{C}$ , the following condition is satisfied:

( $\star$ ) If  $\phi \circ f \cong \psi \circ f$ , then  $\phi \cong \psi$ .

### 01C6 14.2.8 Corepresentably Conservative Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### 01C7 DEFINITION 14.2.8.1.1 ► COREPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably conservative** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is conservative.

#### 01C8 REMARK 14.2.8.1.2 ► UNWINDING DEFINITION 14.2.8.1.1

In detail,  $f$  is corepresentably conservative if, for each pair of morphisms  $\phi, \psi: B \rightrightarrows X$  and each 2-morphism

$$\alpha: \phi \rightrightarrows \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \\ \xrightarrow{\psi} \end{array} X$$



of  $\mathcal{C}$ , if the 2-morphism

$$\alpha \star \text{id}_f: \phi \circ f \Longrightarrow \psi \circ f, \quad \begin{array}{ccc} & \phi \circ f & \\ \curvearrowright & \parallel & \curvearrowright \\ A & \alpha \star \text{id}_f & X \\ \curvearrowleft & \downarrow & \curvearrowleft \\ & \psi \circ f & \end{array}$$

is a 2-isomorphism, then so is  $\alpha$ .

### 01C9 14.2.9 Strict Epimorphisms

Let  $\mathcal{C}$  be a bicategory.

#### 01CA DEFINITION 14.2.9.1.1 ► STRICT EPIMORPHISMS

A 1-morphism  $f: A \rightarrow B$  is a **strict epimorphism in  $\mathcal{C}$**  if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is injective on objects, i.e. its action on objects

$$f_*: \text{Obj}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Obj}(\text{Hom}_{\mathcal{C}}(A, X))$$

is injective.

#### 01CB REMARK 14.2.9.1.2 ► UNWINDING DEFINITION 14.2.9.1.1

In detail,  $f$  is a strict epimorphism if, for each diagram in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \xrightleftharpoons[\psi]{\phi} X,$$

if  $\phi \circ f = \psi \circ f$ , then  $\phi = \psi$ .

#### 01CC EXAMPLE 14.2.9.1.3 ► EXAMPLES OF STRICT EPIMORPHISMS

Here are some examples of strict epimorphisms.

01CD

1. *Strict Epimorphisms in  $\mathbf{Cats}_2$* . The strict epimorphisms in  $\mathbf{Cats}_2$  are characterised in **Categories**, **Item 1** of **Proposition 11.7.3.1.2**.

01CE

2. *Strict Epimorphisms in  $\mathbf{Rel}$* . The strict epimorphisms in  $\mathbf{Rel}$  are characterised in **Relations**, ??.

01CF **14.2.10 Pseudoepic Morphisms**

Let  $\mathcal{C}$  be a bicategory.

01CG

**DEFINITION 14.2.10.1.1 ► PSEUDOEPIC MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **pseudoepic** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is pseudomonic.

01CH

**REMARK 14.2.10.1.2 ► UNWINDING DEFINITION 14.2.10.1.1**

In detail, a 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is pseudoepic if it satisfies the following conditions:

01CJ

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

01CK

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xrightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \downarrow \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X \quad = \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01CL

### PROPOSITION 14.2.10.1.3 ► PROPERTIES OF PSEUDOEPIC MORPHISMS

Let  $f: A \rightarrow B$  be a 1-morphism of  $\mathcal{C}$ .

01CM

1. *Characterisations.* The following conditions are equivalent:

01CN

(a) The morphism  $f$  is pseudoepic.

01CP

(b) The morphism  $f$  is corepresentably full on cores and corepresentably faithful.

01CQ

(c) We have an isococcomma square of the form

$$B \xrightarrow{\text{eq.}} B \coprod_A B, \quad \begin{array}{ccc} B & \xleftarrow{\text{id}_B} & B \\ \text{id}_B \uparrow & \nearrow \text{dashed} & \uparrow F \\ B & \xleftarrow{F} & A \end{array}$$

in  $\mathcal{C}$  up to equivalence.

PROOF 14.2.10.1.4 ► PROOF OF PROPOSITION 14.2.10.1.3

Item 1: Characterisations

Omitted.



# Appendices

## A Other Chapters

### Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

### Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

### Relations

8. [Relations](#)
9. [Constructions With Relations](#)

### 10. [Conditions on Relations](#)

### Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

### Monoidal Categories

13. [Constructions With Monoidal Categories](#)

### Bicategories

14. [Types of Morphisms in Bicategories](#)

### Extra Part

15. [Notes](#)

## References

- [Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. “On Functors Which Are Lax Epimorphisms”. In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. [19](#)).