Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

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13.1 Moduli Categories of Monoidal Structures

13.1.1 The Moduli Category of Monoidal Structures on a Category

Let *C* be a category.

Definition 13.1.1.1.1. The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathrm{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, egin{pmatrix} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow \mathsf{MonCats} \\ & & \downarrow & \downarrow \\ & \mathsf{pt} & \longrightarrow \mathsf{Cats}. \end{pmatrix}$$

Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
- *Morphisms.* A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{1|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C : C \to C$ of C.

• *Identities*. For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\mathrm{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\mathrm{def}}{=}\left(\mathrm{id}_{C}^{\otimes},\mathrm{id}_{1|C}^{\otimes}\right),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of \ref{C} ?

• Composition. For each $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\begin{split} \circ^{\mathcal{M}_{E_1}(C)}_{\mathcal{M},N,P} &: \ \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(\mathcal{M},N) \to \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(\mathcal{M},P) \\ & \text{of} \ \mathcal{M}_{E_1}(C) \ \operatorname{at} \ (\mathcal{M},N,P) \ \operatorname{is} \ \operatorname{defined} \ \operatorname{by} \\ & \left(\operatorname{id}_{C}^{\otimes,\prime},\operatorname{id}_{1|C}^{\otimes,\prime} \right) \circ^{\mathcal{M}_{E_1}(C)}_{\mathcal{M},N,P} \left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{1|C}^{\otimes} \right) \stackrel{\operatorname{def}}{=} \left(\operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes},\operatorname{id}_{1|C}^{\otimes,\prime} \circ \operatorname{id}_{1|C}^{\otimes} \right). \end{split}$$

Remark 13.1.1.1.3. In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

I. *Naturality.* For each pair $f:A\to X$ and $g:B\to Y$ of morphisms of C, the diagram

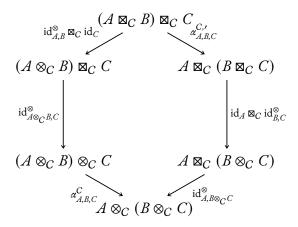
$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\operatorname{id}_{A,B}^{\otimes} \downarrow \qquad \qquad \qquad \operatorname{id}_{X,Y}^{\otimes}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

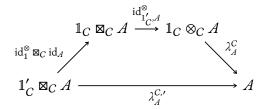
commutes.

2. *Monoidality*. For each $A, B, C \in Obj(C)$, the diagram



commutes.

3. *Left Monoidal Unity*. For each $A \in Obj(C)$, the diagram



commutes.

4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbf{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbf{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A$$

commutes.

Proposition 13.1.1.1.4. Let *C* be a category.

- 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B\boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{E_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. Mixed Associators. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^C)$ be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} : -_1 \boxtimes_C -_2 \longrightarrow -_1 \otimes_C -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes}: (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \downarrow^{\operatorname{id}_{A \otimes_C B,C}} & & \downarrow^{\operatorname{id}_A \otimes_C \operatorname{id}_{B,C}^{\otimes}} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_C (B \boxtimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ & & & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & &$$

commute, then the natural transformation id[®] satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

and

$$\begin{array}{c|c} (A\boxtimes_{C}B)\boxtimes_{C}C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_{C}(B\boxtimes_{C}C) \\ \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} & \operatorname{id}_{A\boxtimes_{C}}\operatorname{id}_{B,C}^{\otimes} \\ (A\boxtimes_{C}B)\otimes_{C}C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A\boxtimes_{C}(B\otimes_{C}C) \end{array}$$

commute, then the natural transformation id[®] satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes,\otimes}\colon (A\boxtimes_C B)\otimes_C C\to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

and

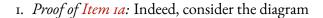
commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

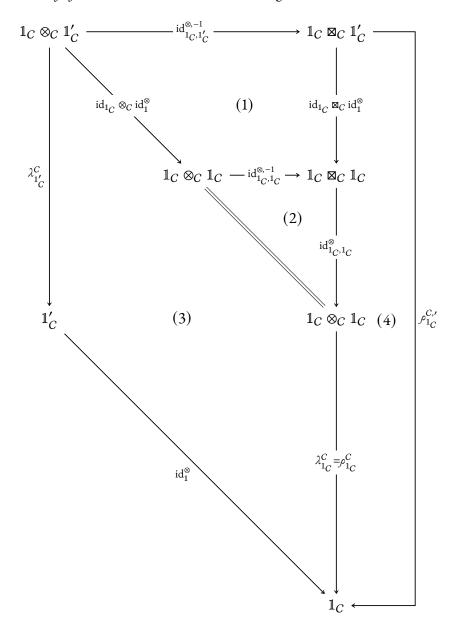
Proof. Item 1, *Extra Monoidality Conditions*: We claim that Items 1a and 1b are indeed true:

- I. *Proof of Item 1a:* This follows from the naturality of id^{\otimes} with respect to the morphisms $id_{A,B}^{\otimes}$ and id_{C} .
- 2. *Proof of Item 1b:* This follows from the naturality of id^{\otimes} with respect to the morphisms id_{A} and $id_{B,C}^{\otimes}$.

This finishes the proof.

Item 2, Extra Monoidal Unity Constraints: We claim that Items 2a and 2b are indeed true:

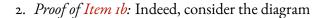


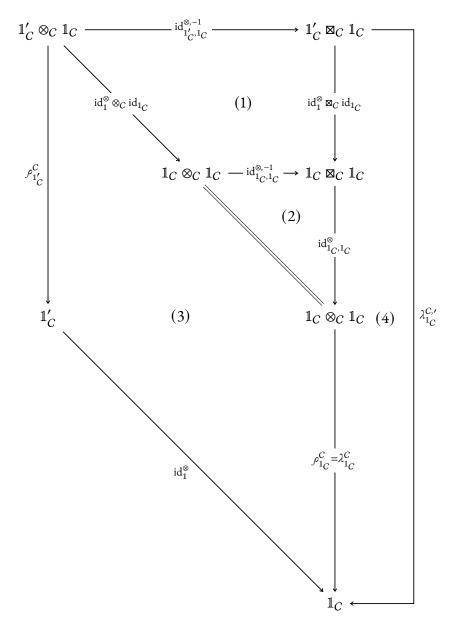


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{1_C}^C=\lambda_{1_C}^C$ comes from $\ref{eq:compare}$;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$; so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{1_C}^C = \lambda_{1_C}^C$ comes from $\ref{eq:compare}$;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$;

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

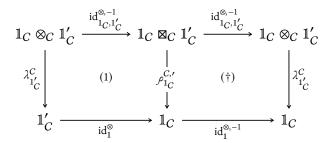
Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram



Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$1_{C} \otimes_{C} 1_{C}' \xrightarrow{\operatorname{id}_{1_{C},1_{C}'}^{\otimes,-1}} 1_{C} \boxtimes_{C} 1_{C}' \xrightarrow{\operatorname{id}_{1_{C},1_{C}'}^{\otimes,-1}} 1_{C} \otimes_{C} 1_{C}'$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{1_{C}'}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{1_{C}'}^{C}$$

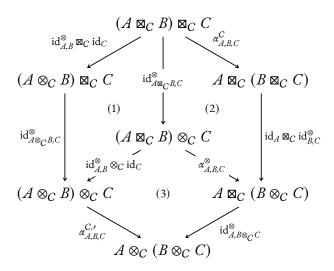
commutes. But since $id_1^{\otimes,-1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for id^{\otimes} of





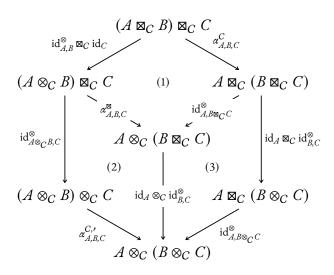
Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. *Proof of Item 3b*: We may partition the monoidality diagram for id^{\otimes} of



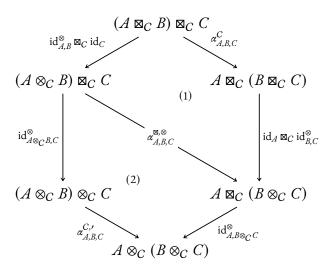


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for id^{\otimes} of



Item 2 of Definition 13.1.1.1.3 as follows:

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id[®] satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- I. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes