# Constructions With Monoidal Categories

# The Clowder Project Authors July 29, 2025

This chapter contains some material on constructions with monoidal categories.

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13.1 Moduli Categories of Monoidal Structures	
13.1.1 The Moduli Category of Monoidal Structures on a Categ	ory
Let <i>C</i> be a category.	
Definition 13.1.1.1. The moduli category of monoidal structures on	C is

the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

Remark 13.1.1.2. In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{B}_1}(C)$  where:

- *Objects*. The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is C.
- *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{I}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C: C \to C$  of C.

• *Identities*. For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{MM}^{\mathcal{M}_{\mathbb{B}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M, M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\mathrm{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\mathrm{def}}{=}(\mathrm{id}_{C}^{\otimes},\mathrm{id}_{\mathbb{1}|C}^{\otimes}),$$

where  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  is the identity monoidal functor of C of ??.

• Composition. For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M,P)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at (M, N, P) is defined by

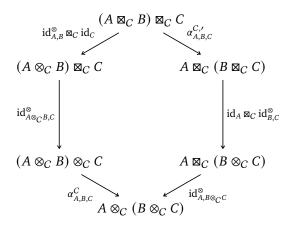
$$\left(\operatorname{id}_{C}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{H}_{1}}(C)}\left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right)\stackrel{\operatorname{def}}{=}\left(\operatorname{id}_{C}^{\otimes,\prime}\circ\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right).$$

**Remark 13.1.1.1.3.** In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

1. *Naturality*. For each pair  $f: A \to X$  and  $g: B \to Y$  of morphisms of C, the diagram

commutes.

2. Monoidality. For each  $A, B, C \in Obj(C)$ , the diagram



commutes.

3. *Left Monoidal Unity.* For each  $A \in Obj(C)$ , the diagram

$$\mathbb{1}_{C}\boxtimes_{C}A \xrightarrow{\operatorname{id}_{\mathbb{1}'_{C},A}^{\otimes}} \mathbb{1}_{C}\otimes_{C}A$$

$$\operatorname{id}_{\mathbb{1}}^{\otimes}\boxtimes_{C}\operatorname{id}_{A} \xrightarrow{\lambda_{A}^{C}} A$$

$$\mathbb{1}'_{C}\boxtimes_{C}A \xrightarrow{\lambda_{A}^{C,\prime}} A$$

commutes.

4. Right Monoidal Unity. For each  $A \in Obj(C)$ , the diagram

$$A\boxtimes_{C}\mathbb{1}_{C}\stackrel{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\vee}}{\longrightarrow}A\otimes_{C}\mathbb{1}_{C}$$

$$\operatorname{id}_{A}\boxtimes_{C}\operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A\boxtimes_{C}\mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,\prime}} A$$

commutes.

#### **Proposition 13.1.1.1.4.** Let C be a category.

- 1. Extra Monoidality Conditions. Let  $(\mathrm{id}_C^\otimes,\mathrm{id}_{\mathbb{I}|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C,\otimes_C,\mathbb{1}_C,\alpha^C,\lambda^C,\rho^C)$  to  $(C,\boxtimes_C,\mathbb{1}_C',\alpha^{C,\prime},\lambda^{C,\prime},\rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

commutes.

- 2. Extra Monoidal Unity Constraints. Let  $(\mathrm{id}_C^\otimes,\mathrm{id}_{\mathbb{1}|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C,\otimes_C,\mathbb{1}_C,\alpha^C,\lambda^C,\rho^C)$  to  $(C,\boxtimes_C,\mathbb{1}_C',\alpha^{C,\prime},\lambda^{C,\prime},\rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. *Mixed Associators*. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$id^{\otimes}_{-_1,-_2} \colon \mathrel{-_1} \boxtimes_{\mathcal{C}} \mathrel{-_2} \to \mathrel{-_1} \otimes_{\mathcal{C}} \mathrel{-_2}$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{A.B.C}^{\otimes} \colon (A \otimes_C B) \boxtimes_C C \longrightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

and

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

and

$$\begin{array}{cccc} (A\boxtimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A\boxtimes_C (B\boxtimes_C C) \\ & \mathrm{id}_{A\boxtimes_C B,C}^{\otimes} & & & & & & & \\ & (A\boxtimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A\boxtimes_C (B\otimes_C C) \end{array}$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & & & & & & & & & \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) & & & & & & \end{array}$$

and

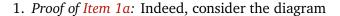
commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

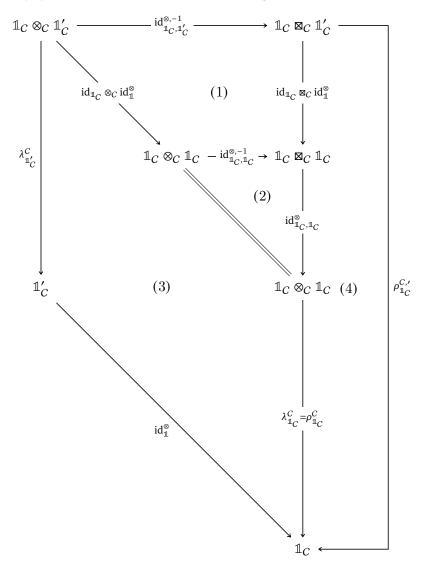
*Proof. Item 1, Extra Monoidality Conditions*: We claim that *Items 1a* and *1b* are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A,B}^{\otimes}$  and  $id_{C}$ .
- 2. *Proof of Item 1b*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

*Item 2, Extra Monoidal Unity Constraints*: We claim that Items 2a and 2b are indeed true:



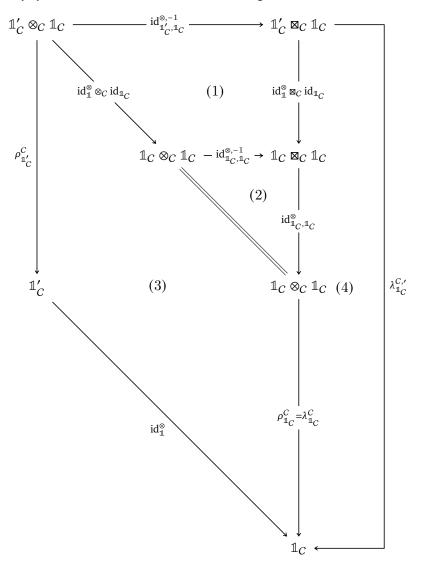


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

2. Proof of Item 1b: Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$  comes from **??**;

- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ; so does the boundary diagram, and we are done.
- 3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $id_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

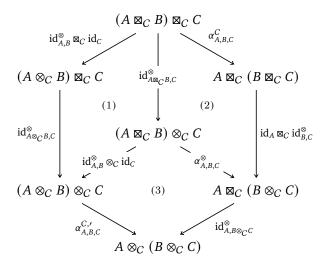
it follows that the diagram

commutes. But since  $id_{\mathbb{1}}^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

*Item 3*, *Mixed Associators*: We claim that *Items 3a* to *3c* are indeed true:

1. *Proof of Item 3a*: We may partition the monoidality diagram for id<sup>⊗</sup> of Item 2 of Definition 13.1.1.1.3 as follows:

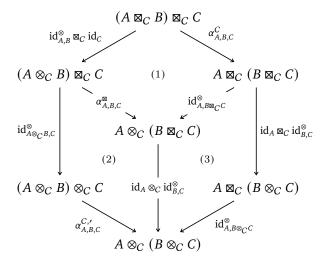


Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. *Proof of Item 3b*: We may partition the monoidality diagram for id<sup>⊗</sup> of Item 2 of Definition 13.1.1.1.3 as follows:



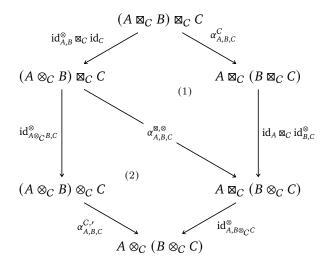
#### Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for  $id^{\otimes}$  of

#### Item 2 of Definition 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

## **Appendices**

## **A** Other Chapters

#### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

#### Relations

- 8. Relations
- 9. Constructions With Relations

#### 10. Conditions on Relations

#### Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

13. Constructions With Monoidal Categories

#### **Bicategories**

14. Types of Morphisms in Bicategories

#### Extra Part

15. Notes