Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

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13.1 Moduli Categories of Monoidal Structures

13.1.1 The Moduli Category of Monoidal Structures on a Category

Let *C* be a category.

Definition 13.1.1.1.1. The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathrm{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, egin{pmatrix} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow \mathsf{MonCats} \\ & & \downarrow & & \downarrow \\ & \mathsf{pt} & \longrightarrow \mathsf{Cats}. \end{pmatrix}$$

Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
- *Morphisms*. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C : C \to C$ of C.

• *Identities.* For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\mathrm{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \stackrel{\mathrm{def}}{=} \left(\mathrm{id}_{C}^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right),$$

where $\left(id_{C}^{\otimes}, id_{1|C}^{\otimes}\right)$ is the identity monoidal functor of C of ??.

• Composition. For each $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\begin{split} \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,N) &\to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,P) \\ & \text{of } \mathcal{M}_{\mathbb{B}_1}(C) \text{ at } (M,N,P) \text{ is defined by} \\ & \left(\operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{\mathbb{A}|C}^{\otimes,\prime} \right) \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \left(\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{A}|C}^{\otimes} \right) \overset{\text{def}}{=} \left(\operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{A}|C}^{\otimes,\prime} \circ \operatorname{id}_{\mathbb{A}|C}^{\otimes} \right). \end{split}$$

Remark 13.1.1.1.3. In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

1. *Naturality*. For each pair $f:A\to X$ and $g:B\to Y$ of morphisms of C, the diagram

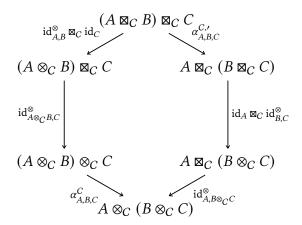
$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

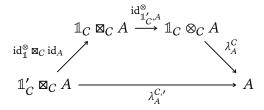
commutes.

2. *Monoidality*. For each $A, B, C \in Obj(C)$, the diagram



commutes.

3. *Left Monoidal Unity*. For each $A \in Obj(C)$, the diagram



commutes.

4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A \otimes_{C} \mathbb{1}_{C}$$

commutes.

Proposition 13.1.1.4. Let C be a category.

- 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

$$A\boxtimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\boxtimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\boxtimes_{C}(B\otimes_{C}C)$$

$$\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}\downarrow \qquad \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C}C}^{\otimes}$$

$$A\otimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\otimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\otimes_{C}(B\otimes_{C}C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

$$\mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow^{\rho_{\mathbb{1}_{C}}^{C,'}} \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes.

3. *Mixed Associators*. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} : -_1 \boxtimes_{\mathcal{C}} -_2 \longrightarrow -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{A.B.C}^{\otimes} \colon (A \otimes_{C} B) \boxtimes_{C} C \to A \otimes_{C} (B \boxtimes_{C} C)$$

making the diagrams

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A \otimes_{C} B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A\boxtimes_C B)\otimes_C C \xrightarrow{\alpha_{A,B,C}^\boxtimes} A\boxtimes_C (B\otimes_C C) \\ \operatorname{id}_{A,B}^\otimes\otimes_C \operatorname{id}_C \downarrow & \qquad & \operatorname{id}_{A,B\otimes_C^C}^\otimes \\ (A\otimes_C B)\otimes_C C \xrightarrow{\alpha_{A,B,C}^C} A\otimes_C (B\otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$id_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \downarrow id_{A}\boxtimes_{C} id_{B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C} & & & & \operatorname{id}_{A,C}^{\otimes} \\ & (A \otimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

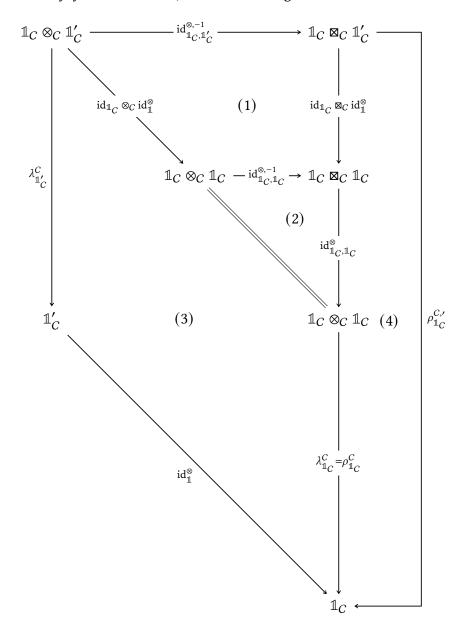
Proof. Item 1, Extra Monoidality Conditions: We claim that *Items 1a* and 1b are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of id^\otimes with respect to the morphisms $\mathrm{id}_{A,B}^\otimes$ and id_C .
- 2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and $id_{B,C}^{\otimes}$.

This finishes the proof.

Item 2, Extra Monoidal Unity Constraints: We claim that *Items 2a* and **2b** are indeed true:



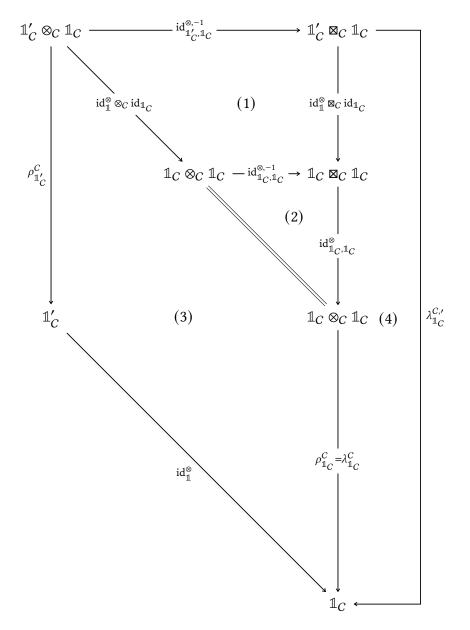


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$; so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:composition}$;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$; so does the boundary diagram, and we are done.
- 3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

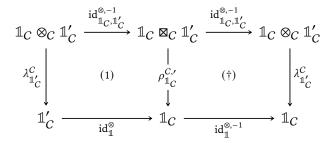
$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{\rho_{\mathbb{1}'_{C}}^{C}} \qquad \qquad \downarrow^{\rho_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d*: Indeed, consider the diagram



Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

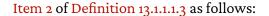
$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

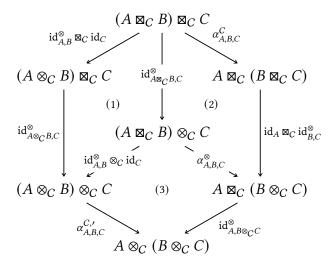
commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for id^{\otimes} of



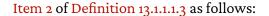


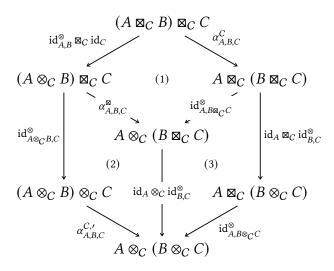
Since:

- Subdiagram (1) commutes by Item 12 of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. *Proof of Item 3b*: We may partition the monoidality diagram for id^{\otimes} of



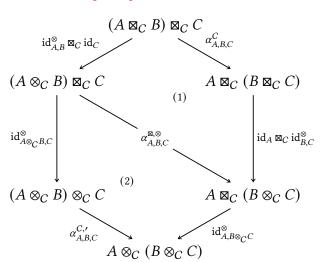


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item 3c*: We may partition the monoidality diagram for id^{\otimes} of



Item 2 of Definition 13.1.1.1.3 as follows:

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
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Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes