

# Constructions With Sets

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see [Definitions 4.2.4.1.1](#), [4.2.4.1.3](#), [4.2.5.1.1](#) and [4.2.5.1.3](#)).
2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:
  - (a) A discussion of the internal Hom of a powerset ([Section 4.4.7](#)).
  - (b) A 0-categorical version of the Yoneda lemma ([Presheaves and the Yoneda Lemma](#), [Definition 12.1.5.1.1](#)), which we term the *Yoneda lemma for sets* ([Definition 4.5.5.1.1](#)).
  - (c) A characterisation of powersets as free cocompletions ([Section 4.4.5](#)), mimicking the corresponding statement for categories of presheaves (??).
  - (d) A characterisation of powersets as free completions ([Section 4.4.6](#)), mimicking the corresponding statement for categories of copresheaves (??).
  - (e) A  $(-1)$ -categorical version of un/straightening ([Item 2](#) of [Definition 4.5.1.1.4](#) and [Definition 4.5.1.1.5](#)).
  - (f) A 0-categorical form of Isbell duality internal to powersets ([Section 4.4.8](#)).
3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* : \mathcal{P}(A) \xrightarrow{\cong} \mathcal{P}(B)$$

of functors (i.e. morphisms of posets) between  $\mathcal{P}(A)$  and  $\mathcal{P}(B)$  induced by a map of sets  $f : A \rightarrow B$ , including in particular:

- (a) How  $f^{-1}$  can be described as a precomposition while  $f_!$  and  $f_*$  can be described as Kan extensions (Definitions 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).
- (b) An extensive list of the properties of  $f_!$ ,  $f^{-1}$ , and  $f_*$  (Definitions 4.6.1.1.5, 4.6.1.1.6, 4.6.2.1.3, 4.6.2.1.4, 4.6.3.1.7 and 4.6.3.1.8).
- (c) How the functors  $f_!$ ,  $f^{-1}$ ,  $f_*$ , along with the functors

$$\begin{aligned} -_1 \cap -_2 &: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X), \\ [-_1, -_2]_X &: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X) \end{aligned}$$

may be viewed as a six-functor formalism with the empty set  $\emptyset$  as the dualising object (Section 4.6.4).

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## 4.1 Limits of Sets

### 4.1.1 The Terminal Set

**Definition 4.1.1.1.1.** The **terminal set** is the terminal object of Sets as in Limits and Colimits, ??.

**Construction 4.1.1.1.2.** Concretely, the terminal set is the pair  $(\text{pt}, \{!_A\}_{A \in \text{Obj}(\text{Sets})})$  consisting of:

1. *The Limit.* The punctual set  $\text{pt} \stackrel{\text{def}}{=} \{\star\}$ .
2. *The Cone.* The collection of maps

$$\{!_A : A \rightarrow \text{pt}\}_{A \in \text{Obj}(\text{Sets})}$$

defined by

$$!_A(a) \stackrel{\text{def}}{=} \star$$

for each  $a \in A$  and each  $A \in \text{Obj}(\text{Sets})$ .

*Proof.* We claim that  $\text{pt}$  is the terminal object of  $\text{Sets}$ . Indeed, suppose we have a diagram of the form

$$A \quad \text{pt}$$

in  $\text{Sets}$ . Then there exists a unique map  $\phi : A \rightarrow \text{pt}$  making the diagram

$$A \xrightarrow[\exists!]{\phi} \text{pt}$$

commute, namely  $!_A$ . □

## 4.1.2 Products of Families of Sets

Let  $\{A_i\}_{i \in I}$  be a family of sets.

**Definition 4.1.2.1.1.** The **product**<sup>1</sup> of  $\{A_i\}_{i \in I}$  is the product of  $\{A_i\}_{i \in I}$  in  $\text{Sets}$  as in Limits and Colimits, ??.

**Construction 4.1.2.1.2.** Concretely, the product of  $\{A_i\}_{i \in I}$  is the pair  $(\prod_{i \in I} A_i, \{\text{pr}_i\}_{i \in I})$  consisting of:

1. *The Limit.* The set  $\prod_{i \in I} A_i$  defined by

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \text{Sets} \left( I, \bigcup_{i \in I} A_i \right) \mid \begin{array}{l} \text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \text{pr}_i : \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

---

<sup>1</sup>*Further Terminology:* Also called the **Cartesian product** of  $\{A_i\}_{i \in I}$ .

of maps given by

$$\text{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each  $f \in \prod_{i \in I} A_i$  and each  $i \in I$ .

*Proof.* We claim that  $\prod_{i \in I} A_i$  is the categorical product of  $\{A_i\}_{i \in I}$  in Sets. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$\begin{array}{ccc} P & & \\ & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

in Sets. Then there exists a unique map  $\phi: P \rightarrow \prod_{i \in I} A_i$  making the diagram

$$\begin{array}{ccc} P & & \\ \downarrow \phi \quad \exists! & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

commute, being uniquely determined by the condition  $\text{pr}_i \circ \phi = p_i$  for each  $i \in I$  via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each  $x \in P$ . □

**Remark 4.1.2.1.3.** Less formally, we may think of Cartesian products and projection maps as follows:

1. We think of  $\prod_{i \in I} A_i$  as the set whose elements are  $I$ -indexed collections  $(a_i)_{i \in I}$  with  $a_i \in A_i$  for each  $i \in I$ .
2. We view the projection maps

$$\left\{ \text{pr}_i: \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

as being given by

$$\text{pr}_i((a_j)_{j \in I}) \stackrel{\text{def}}{=} a_i$$

for each  $(a_j)_{j \in I} \in \prod_{i \in I} A_i$  and each  $i \in I$ .

**Proposition 4.1.2.1.4.** Let  $\{A_i\}_{i \in I}$  be a family of sets.

1. *Functoriality.* The assignment  $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$  defines a functor

$$\prod_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}) \rightarrow \text{Sets}$$

where

- *Action on Objects.* For each  $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$ , we have

$$\left[ \prod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each  $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$ , the action on Hom-sets

$$\left( \prod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \text{Sets} \left( \prod_{i \in I} A_i, \prod_{i \in I} B_i \right)$$

of  $\prod_{i \in I}$  at  $((A_i)_{i \in I}, (B_i)_{i \in I})$  is defined by sending a map

$$\{f_i : A_i \rightarrow B_i\}_{i \in I}$$

in  $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$  to the map of sets

$$\prod_{i \in I} f_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$$

defined by

$$\left[ \prod_{i \in I} f_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i \in I}$$

for each  $(a_i)_{i \in I} \in \prod_{i \in I} A_i$ .

*Proof.* **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??.  $\square$

### 4.1.3 Binary Products of Sets

Let  $A$  and  $B$  be sets.

**Definition 4.1.3.1.1.** The **product of  $A$  and  $B$** <sup>2</sup> is the product of  $A$  and  $B$  in Sets as in Limits and Colimits, ??.

**Construction 4.1.3.1.2.** Concretely, the product of  $A$  and  $B$  is the pair  $(A \times B, \{\text{pr}_1, \text{pr}_2\})$  consisting of:

1. *The Limit.* The set  $A \times B$  defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \text{Sets}(\{0, 1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\ &\cong \left\{ \begin{array}{l} \text{ordered pairs } (a, b) \text{ with} \\ a \in A \text{ and } b \in B \end{array} \right\}. \end{aligned}$$

2. *The Cone.* The maps

$$\begin{aligned} \text{pr}_1 &: A \times B \rightarrow A, \\ \text{pr}_2 &: A \times B \rightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(a, b) &\stackrel{\text{def}}{=} a, \\ \text{pr}_2(a, b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each  $(a, b) \in A \times B$ .

*Proof.* We claim that  $A \times B$  is the categorical product of  $A$  and  $B$  in the category of sets. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & P & \\ p_1 \swarrow & & \searrow p_2 \\ A & \xleftarrow{\text{pr}_1} A \times B \xrightarrow{\text{pr}_2} & B \end{array}$$

in Sets. Then there exists a unique map  $\phi: P \rightarrow A \times B$  making the diagram

$$\begin{array}{ccc} & P & \\ p_1 \swarrow & \downarrow \phi \downarrow \exists! & \searrow p_2 \\ A & \xleftarrow{\text{pr}_1} A \times B \xrightarrow{\text{pr}_2} & B \end{array}$$

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<sup>2</sup>*Further Terminology:* Also called the **Cartesian product of  $A$  and  $B$** .

commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ . □

**Proposition 4.1.3.1.3.** Let  $A, B, C$ , and  $X$  be sets.

1. *Functoriality.* The assignments  $A, B, (A, B) \mapsto A \times B$  define functors

$$\begin{aligned} A \times -: \quad \text{Sets} &\rightarrow \text{Sets}, \\ - \times B: \quad \text{Sets} &\rightarrow \text{Sets}, \\ -_1 \times -_2: \text{Sets} \times \text{Sets} &\rightarrow \text{Sets}, \end{aligned}$$

where  $-_1 \times -_2$  is the functor where

- *Action on Objects.* For each  $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$ , we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each  $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$ , the action on Hom-sets

$$\times_{(A,B),(X,Y)}: \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \times B, X \times Y)$$

of  $\times$  at  $((A, B), (X, Y))$  is defined by sending  $(f, g)$  to the function

$$f \times g: A \times B \rightarrow X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each  $(a, b) \in A \times B$ .

and where  $A \times -$  and  $- \times B$  are the partial functors of  $-_1 \times -_2$  at  $A, B \in \text{Obj}(\text{Sets})$ .

2. *Adjointness I.* We have adjunctions

$$\begin{aligned} (A \times - \dashv \text{Sets}(A, -)): \quad \text{Sets} &\begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets}, \\ (- \times B \dashv \text{Sets}(B, -)): \quad \text{Sets} &\begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets}, \end{aligned}$$



witnessed by bijections

$$\begin{aligned}\text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)),\end{aligned}$$

natural in  $A, B, C \in \text{Obj}(\text{Sets})$ .

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\text{Sets}} \dashv -_1 \times -_2): \text{Sets} \begin{array}{c} \xrightarrow{\Delta_{\text{Sets}}} \\ \perp \\ \xleftarrow{-_1 \times -_2} \end{array} \text{Sets} \times \text{Sets},$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets} \times \text{Sets}}((A, A), (B, C)) \cong \text{Sets}(A, B \times C),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and in  $(B, C) \in \text{Obj}(\text{Sets} \times \text{Sets})$ .

4. *Associativity.* We have an isomorphism of sets

$$\alpha_{A,B,C}^{\text{Sets}}: (A \times B) \times C \xrightarrow{\sim} A \times (B \times C),$$

natural in  $A, B, C \in \text{Obj}(\text{Sets})$ .

5. *Unitality.* We have isomorphisms of sets

$$\begin{aligned}\lambda_A^{\text{Sets}}: \text{pt} \times A &\xrightarrow{\sim} A, \\ \rho_A^{\text{Sets}}: A \times \text{pt} &\xrightarrow{\sim} A,\end{aligned}$$

natural in  $A \in \text{Obj}(\text{Sets})$ .

6. *Commutativity.* We have an isomorphism of sets

$$\sigma_{A,B}^{\text{Sets}}: A \times B \xrightarrow{\sim} B \times A,$$

natural in  $A, B \in \text{Obj}(\text{Sets})$ .

7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\begin{aligned}\delta_\ell^{\text{Sets}}: A \times (B \amalg C) &\xrightarrow{\sim} (A \times B) \amalg (A \times C), \\ \delta_r^{\text{Sets}}: (A \amalg B) \times C &\xrightarrow{\sim} (A \times C) \amalg (B \times C),\end{aligned}$$

natural in  $A, B, C \in \text{Obj}(\text{Sets})$ .

8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned}\zeta_\ell^{\text{Sets}} &: \emptyset \times A \xrightarrow{\sim} \emptyset, \\ \zeta_r^{\text{Sets}} &: A \times \emptyset \xrightarrow{\sim} \emptyset,\end{aligned}$$

natural in  $A \in \text{Obj}(\text{Sets})$ .

9. *Distributivity Over Unions.* Let  $X$  be a set. For each  $U, V, W \in \mathcal{P}(X)$ , we have equalities

$$\begin{aligned}U \times (V \cup W) &= (U \times V) \cup (U \times W), \\ (U \cup V) \times W &= (U \times W) \cup (V \times W)\end{aligned}$$

of subsets of  $\mathcal{P}(X \times X)$ .

10. *Distributivity Over Intersections.* Let  $X$  be a set. For each  $U, V, W \in \mathcal{P}(X)$ , we have equalities

$$\begin{aligned}U \times (V \cap W) &= (U \times V) \cap (U \times W), \\ (U \cap V) \times W &= (U \times W) \cap (V \times W)\end{aligned}$$

of subsets of  $\mathcal{P}(X \times X)$ .

11. *Distributivity Over Differences.* Let  $X$  be a set. For each  $U, V, W \in \mathcal{P}(X)$ , we have equalities

$$\begin{aligned}U \times (V \setminus W) &= (U \times V) \setminus (U \times W), \\ (U \setminus V) \times W &= (U \times W) \setminus (V \times W)\end{aligned}$$

of subsets of  $\mathcal{P}(X \times X)$ .

12. *Distributivity Over Symmetric Differences.* Let  $X$  be a set. For each  $U, V, W \in \mathcal{P}(X)$ , we have equalities

$$\begin{aligned}U \times (V \triangle W) &= (U \times V) \triangle (U \times W), \\ (U \triangle V) \times W &= (U \times W) \triangle (V \times W)\end{aligned}$$

of subsets of  $\mathcal{P}(X \times X)$ .

13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$\begin{array}{ccc}(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \xrightarrow{\cap \times \cap} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \mathcal{P}_{X,X}^\times \times \mathcal{P}_{X,X}^\times \downarrow & & \downarrow \mathcal{P}_{X,X}^\times \\ \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) & \xrightarrow{\cap} & \mathcal{P}(X \times X)\end{array}$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap W) \times (V \cap T).$$

for each  $U, V, W, T \in \mathcal{P}(X)$ .

14. *Symmetric Monoidality*. The 8-tuple  $(\mathbf{Sets}, \times, \text{pt}, \mathbf{Sets}(-_1, -_2), \alpha^{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$  is a closed symmetric monoidal category.

15. *Symmetric Bimonoidality*. The 18-tuple

$$\left( \mathbf{Sets}, \coprod, \times, \emptyset, \text{pt}, \mathbf{Sets}(-_1, -_2), \alpha^{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}}, \right. \\ \left. \alpha^{\mathbf{Sets}, \coprod}, \lambda^{\mathbf{Sets}, \coprod}, \rho^{\mathbf{Sets}, \coprod}, \sigma^{\mathbf{Sets}, \coprod}, \delta_\ell^{\mathbf{Sets}}, \delta_r^{\mathbf{Sets}}, \zeta_\ell^{\mathbf{Sets}}, \zeta_r^{\mathbf{Sets}} \right),$$

is a symmetric closed bimonoidal category, where  $\alpha^{\mathbf{Sets}, \coprod}$ ,  $\lambda^{\mathbf{Sets}, \coprod}$ ,  $\rho^{\mathbf{Sets}, \coprod}$ , and  $\sigma^{\mathbf{Sets}, \coprod}$  are the natural transformations from [Items 3 to 5 of Definition 4.2.3.1.3](#).

*Proof. Item 1, Functoriality*: This follows from Limits and Colimits, ?? of ??.

*Item 2, Adjointness*: We prove only that there's an adjunction  $- \times B \dashv \mathbf{Sets}(B, -)$ , witnessed by a bijection

$$\mathbf{Sets}(A \times B, C) \cong \mathbf{Sets}(A, \mathbf{Sets}(B, C)),$$

natural in  $B, C \in \text{Obj}(\mathbf{Sets})$ , as the proof of the existence of the adjunction  $A \times - \dashv \mathbf{Sets}(A, -)$  follows almost exactly in the same way.

- *Map I*. We define a map

$$\Phi_{B,C}: \mathbf{Sets}(A \times B, C) \rightarrow \mathbf{Sets}(A, \mathbf{Sets}(B, C)),$$

by sending a function

$$\xi: A \times B \rightarrow C$$

to the function

$$\xi^\dagger: A \longrightarrow \mathbf{Sets}(B, C), \\ a \mapsto (\xi_a^\dagger: B \rightarrow C),$$

where we define

$$\xi_a^\dagger(b) \stackrel{\text{def}}{=} \xi(a, b)$$

for each  $b \in B$ . In terms of the  $\llbracket a \mapsto f(a) \rrbracket$  notation of [Sets, Definition 3.1.1.1.2](#), we have

$$\xi^\dagger \stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket.$$

- *Map II.* We define a map

$$\Psi_{B,C} : \text{Sets}(A, \text{Sets}(B, C)) \rightarrow \text{Sets}(A \times B, C)$$

given by sending a function

$$\begin{aligned} \xi : A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a : B \rightarrow C), \end{aligned}$$

to the function

$$\xi^\dagger : A \times B \rightarrow C$$

defined by

$$\begin{aligned} \xi^\dagger(a, b) &\stackrel{\text{def}}{=} \text{ev}_b(\text{ev}_a(\xi)) \\ &\stackrel{\text{def}}{=} \text{ev}_b(\xi_a) \\ &\stackrel{\text{def}}{=} \xi_a(b) \end{aligned}$$

for each  $(a, b) \in A \times B$ .

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \text{id}_{\text{Sets}(A \times B, C)}.$$

Indeed, given a function  $\xi : A \times B \rightarrow C$ , we have

$$\begin{aligned} [\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\ &= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\ &= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\ &= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket) \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\text{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\ &= \llbracket (a, b) \mapsto \xi(a, b) \rrbracket \\ &= \xi. \end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \text{id}_{\text{Sets}(A, \text{Sets}(B, C))}.$$

Indeed, given a function

$$\begin{aligned} \xi : A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a : B \rightarrow C), \end{aligned}$$

we have

$$\begin{aligned}
 [\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\stackrel{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\
 &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a, b) \mapsto \xi_a(b) \rrbracket) \\
 &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \\
 &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \text{ev}_{(a,b)}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\
 &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\
 &\stackrel{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\
 &\stackrel{\text{def}}{=} \xi.
 \end{aligned}$$

- *Naturality for  $\Phi$ , Part I.* We need to show that, given a function  $g: B \rightarrow B'$ , the diagram

$$\begin{array}{ccc}
 \text{Sets}(A \times B', C) & \xrightarrow{\Phi_{B',C}} & \text{Sets}(A, \text{Sets}(B', C)), \\
 \text{id}_A \times g^* \downarrow & & \downarrow (g^*)! \\
 \text{Sets}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Sets}(A, \text{Sets}(B, C))
 \end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B' \rightarrow C,$$

we have

$$\begin{aligned}
 [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\
 &= \Phi_{B,C}(\xi(-1, g(-2))) \\
 &= [\xi(-1, g(-2))]^\dagger \\
 &= \xi_{-1}^\dagger(g(-2)) \\
 &= (g^*)!(\xi^\dagger) \\
 &= (g^*)!(\Phi_{B',C}(\xi)) \\
 &= [(g^*)! \circ \Phi_{B',C}](\xi).
 \end{aligned}$$

Alternatively, using the  $\llbracket a \mapsto f(a) \rrbracket$  notation of [Sets, Definition 3.1.1.1.2](#), we have

$$\begin{aligned}
 [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\
 &= \Phi_{B,C}(\llbracket [\text{id}_A \times g^*](\llbracket (a, b') \mapsto \xi(a, b') \rrbracket) \rrbracket) \\
 &= \Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, g(b)) \rrbracket)
 \end{aligned}$$

$$\begin{aligned}
&= \llbracket a \mapsto \llbracket b \mapsto \xi(a, g(b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto g^*(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\
&= (g^*)_!(\llbracket a \mapsto \llbracket b' \mapsto \xi(a, b') \rrbracket \rrbracket) \\
&= (g^*)_!(\Phi_{B',C}(\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

- *Naturality for  $\Phi$ , Part II.* We need to show that, given a function  $h: C \rightarrow C'$ , the diagram

$$\begin{array}{ccc}
\text{Sets}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Sets}(A, \text{Sets}(B, C)), \\
\downarrow h! & & \downarrow (h_1)! \\
\text{Sets}(A \times B, C') & \xrightarrow{\Phi_{B,C'}} & \text{Sets}(A, \text{Sets}(B, C'))
\end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B \rightarrow C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ h!](\xi) &= \Phi_{B,C}(h_1(\xi)) \\
&= \Phi_{B,C}(h_1(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\
&= \Phi_{B,C}(\llbracket (a, b) \mapsto h(\xi(a, b)) \rrbracket) \\
&= \llbracket a \mapsto \llbracket b \mapsto h(\xi(a, b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto h_1(\llbracket b \mapsto \xi(a, b) \rrbracket) \rrbracket \\
&= (h_1)_!(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\
&= (h_1)_!(\Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\
&= (h_1)_!(\Phi_{B,C}(\xi)) \\
&= [(h_1)! \circ \Phi_{B,C}](\xi).
\end{aligned}$$

- *Naturality for  $\Psi$ .* Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from [Categories, Item 2](#) of [Definition 11.9.7.1.2](#) that  $\Psi$  is also natural in each argument.

This finishes the proof.

*Item 3, Adjointness II:* This follows from the universal property of the product.

*Item 4, Associativity:* This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.1.4.1.1.

*Item 5, Unitality:* This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.1.5.1.1 and 5.1.6.1.1.

*Item 6, Commutativity:* This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.1.7.1.1.

*Item 7, Distributivity Over Coproducts:* This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.3.1.1.1 and 5.3.2.1.1.

*Item 8, Annihilation With the Empty Set:* This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.3.3.1.1 and 5.3.4.1.1.

*Item 9, Distributivity Over Unions:* See [Pro25c].

*Item 10, Distributivity Over Intersections:* See [Pro25d, Corollary 1].

*Item 11, Distributivity Over Differences:* See [Pro25a].

*Item 12, Distributivity Over Symmetric Differences:* See [Pro25b].

*Item 13, Middle-Four Exchange With Respect to Intersections:* See [Pro25d, Corollary 1].

*Item 14, Symmetric Monoidality:* This is a repetition of **Monoidal Structures on the Category of Sets**, Definition 5.1.9.1.1, and is proved there.

*Item 15, Symmetric Bimonoidality:* This is a repetition of **Monoidal Structures on the Category of Sets**, Definition 5.3.5.1.1, and is proved there.  $\square$

**Remark 4.1.3.1.4.** As shown in **Item 1** of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

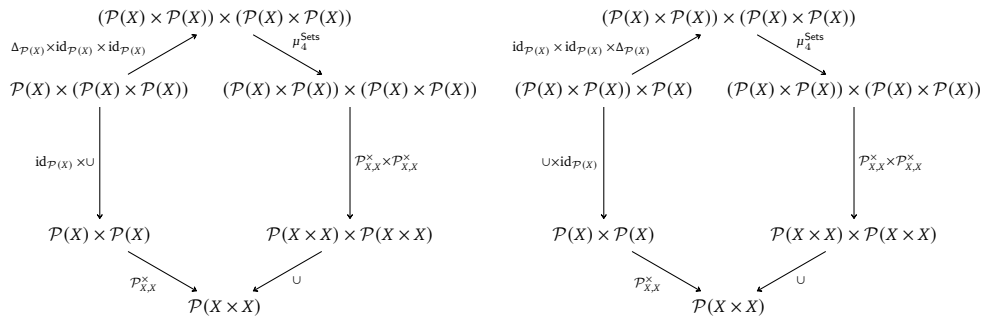
$$-_1 \times -_2 : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}.$$

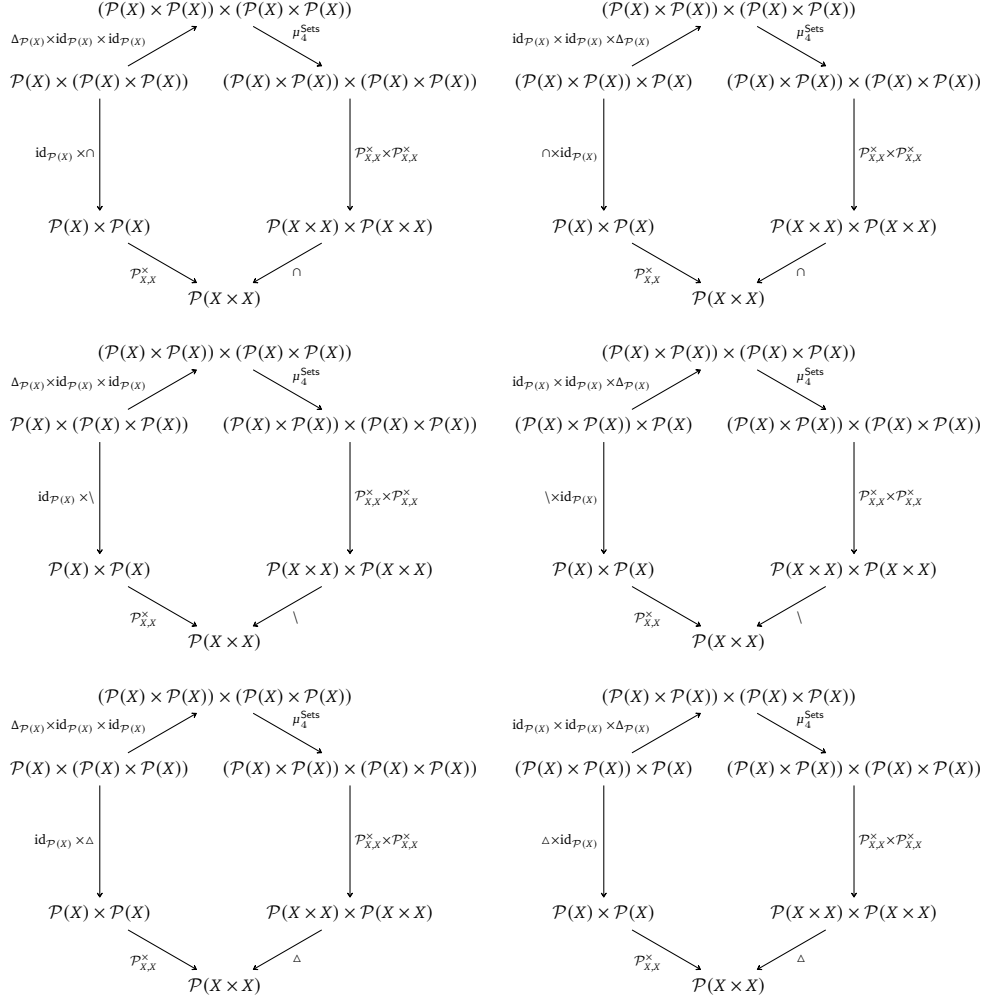
This functor is the  $(k, \ell) = (-1, -1)$  case of a family of functors

$$\otimes_{k,\ell} : \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{B}_\ell}(\mathbf{Sets}) \rightarrow \mathbf{Mon}_{\mathbb{B}_{k+\ell}}(\mathbf{Sets})$$

of tensor products of  $\mathbb{B}_k$ -monoid objects on  $\mathbf{Sets}$  with  $\mathbb{B}_\ell$ -monoid objects on  $\mathbf{Sets}$ ; see ??.

**Remark 4.1.3.1.5.** We may state the equalities in **Items 9 to 12** of Definition 4.1.3.1.3 as the commutativity of the following diagrams:





#### 4.1.4 Pullbacks

Let  $A$ ,  $B$ , and  $C$  be sets and let  $f: A \rightarrow C$  and  $g: B \rightarrow C$  be functions.

**Definition 4.1.4.1.1.** The **pullback of  $A$  and  $B$  over  $C$  along  $f$  and  $g$** <sup>3</sup> is the pullback of  $A$  and  $B$  over  $C$  along  $f$  and  $g$  in **Sets** as in **Limits and Colimits**, ??.

**Construction 4.1.4.1.2.** Concretely, the pullback of  $A$  and  $B$  over  $C$  along  $f$  and  $g$  is the pair  $(A \times_C B, \{\text{pr}_1, \text{pr}_2\})$  consisting of:

<sup>3</sup>*Further Terminology:* Also called the **fibre product of  $A$  and  $B$  over  $C$  along  $f$  and  $g$** .



1. *The Limit.* The set  $A \times_C B$  defined by

$$A \times_C B \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

2. *The Cone.* The maps<sup>4</sup>

$$\begin{aligned} \text{pr}_1 &: A \times_C B \rightarrow A, \\ \text{pr}_2 &: A \times_C B \rightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(a, b) &\stackrel{\text{def}}{=} a, \\ \text{pr}_2(a, b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each  $(a, b) \in A \times_C B$ .

*Proof.* We claim that  $A \times_C B$  is the categorical pullback of  $A$  and  $B$  over  $C$  with respect to  $(f, g)$  in **Sets**. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \text{pr}_1 = g \circ \text{pr}_2, \quad \begin{array}{ccc} A \times_C B & \xrightarrow{\text{pr}_2} & B \\ \text{pr}_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C. \end{array}$$

Indeed, given  $(a, b) \in A \times_C B$ , we have

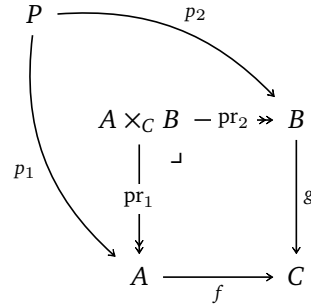
$$\begin{aligned} [f \circ \text{pr}_1](a, b) &= f(\text{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\text{pr}_2(a, b)) \\ &= [g \circ \text{pr}_2](a, b), \end{aligned}$$

where  $f(a) = g(b)$  since  $(a, b) \in A \times_C B$ . Next, we prove that  $A \times_C B$  satisfies the

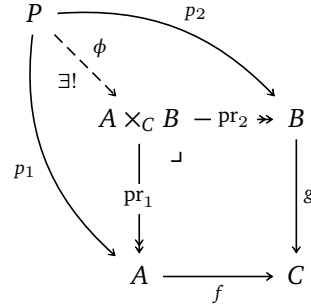
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<sup>4</sup>Further Notation: Also written  $\text{pr}_1^{A \times_C B}$  and  $\text{pr}_2^{A \times_C B}$ .

universal property of the pullback. Suppose we have a diagram of the form



in Sets. Then there exists a unique map  $\phi: P \rightarrow A \times_C B$  making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ , where we note that  $(p_1(x), p_2(x)) \in A \times B$  indeed lies in  $A \times_C B$  by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each  $x \in P$ , so that  $(p_1(x), p_2(x)) \in A \times_C B$ . □

**Remark 4.1.4.1.3.** It is common practice to write  $A \times_C B$  for the pullback of  $A$  and  $B$  over  $C$  along  $f$  and  $g$ , omitting the maps  $f$  and  $g$  from the notation and instead leaving them implicit, to be understood from the context.

However, the set  $A \times_C B$  depends very much on the maps  $f$  and  $g$ , and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write  $A \times_{f,C,g} B$  or  $A \times_C^{f,g} B$  for  $A \times_C B$ .

**Example 4.1.4.1.4.** Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let  $X$  be a set. We have

$$A \cap B \cong A \times_{A \cup B} B,$$

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow \lrcorner & & \downarrow \iota_B \\ A & \xrightarrow{\iota_A} & A \cup B \end{array}$$

for each  $A, B \in \mathcal{P}(X)$ .

*Proof.* **Item 1, Unions via Intersections:** Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof.  $\square$

**Proposition 4.1.4.1.5.** Let  $A, B, C$ , and  $X$  be sets.

1. *Functoriality.* The assignment  $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$  defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where  $\mathcal{P}$  is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of  $-_1 \times_{-3} -_1$  is given by sending a morphism

$$\begin{array}{ccccc} A \times_C B & \longrightarrow & B & & \\ \downarrow \lrcorner & & \downarrow g & \searrow \psi & \\ & A' \times_{C'} B' & \longrightarrow & B' & \\ \downarrow \lrcorner & & \downarrow & & \downarrow g' \\ A & \xrightarrow{f} & C & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \downarrow \\ & A' & \xrightarrow{f'} & C' & \end{array}$$

in  $\text{Fun}(\mathcal{P}, \text{Sets})$  to the map  $\xi: A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$  given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each  $(a, b) \in A \times_C B$ , which is the unique map making the diagram

$$\begin{array}{ccccc}
 A \times_C B & \xrightarrow{\quad} & B & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & A' \times_{C'} B' & \xrightarrow{\quad} & B' & \\
 & \downarrow & \lrcorner & \downarrow & \\
 A & \xrightarrow{f} & C & & \\
 \searrow \phi & & \searrow \chi & & \\
 & A' & \xrightarrow{f'} & C' & \\
 & & & \downarrow g' &
 \end{array}$$

commute.

2. *Adjointness I.* We have adjunctions

$$\begin{aligned}
 (A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)) &: \mathbf{Sets}_{/X} \begin{array}{c} \xrightarrow{A \times_X -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_{/X}(A, -)} \end{array} \mathbf{Sets}_{/X}, \\
 (- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)) &: \mathbf{Sets}_{/X} \begin{array}{c} \xrightarrow{- \times_X B} \\ \perp \\ \xleftarrow{\mathbf{Sets}_{/X}(B, -)} \end{array} \mathbf{Sets}_{/X},
 \end{aligned}$$

witnessed by bijections

$$\begin{aligned}
 \mathbf{Sets}_{/X}(A \times_X B, C) &\cong \mathbf{Sets}_{/X}(A, \mathbf{Sets}_{/X}(B, C)), \\
 \mathbf{Sets}_{/X}(A \times_X B, C) &\cong \mathbf{Sets}_{/X}(B, \mathbf{Sets}_{/X}(A, C)),
 \end{aligned}$$

natural in  $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \text{Obj}(\text{Sets}_{/X})$ , where  $\mathbf{Sets}_{/X}(A, B)$  is the object of  $\text{Sets}_{/X}$  consisting of (see Fibred Sets, ??):

- *The Set.* The set  $\mathbf{Sets}_{/X}(A, B)$  defined by

$$\mathbf{Sets}_{/X}(A, B) \stackrel{\text{def}}{=} \coprod_{x \in X} \text{Sets}(\phi_A^{-1}(x), \phi_B^{-1}(x))$$

- *The Map to X.* The map

$$\phi_{\mathbf{Sets}_{/X}(A, B)}: \mathbf{Sets}_{/X}(A, B) \rightarrow X$$

defined by

$$\phi_{\mathbf{Sets}/X}(A,B)(x, f) \stackrel{\text{def}}{=} x$$

for each  $(x, f) \in \mathbf{Sets}/X(A, B)$ .

3. *Adjointness II.* We have an adjunction

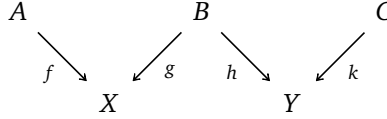
$$\left( \Delta_{\mathbf{Sets}/X} \dashv -1 \times -2 \right): \mathbf{Sets}/X \xrightleftharpoons[-1 \times -2]{\Delta_{\mathbf{Sets}/X}} \mathbf{Sets}/X \times \mathbf{Sets}/X,$$

witnessed by a bijection

$$\text{Hom}_{\mathbf{Sets}/X \times \mathbf{Sets}/X}((A, A), (B, C)) \cong \mathbf{Sets}/X(A, B \times_X C),$$

natural in  $A \in \text{Obj}(\mathbf{Sets}/X)$  and in  $(B, C) \in \text{Obj}(\mathbf{Sets}/X \times \mathbf{Sets}/X)$ .

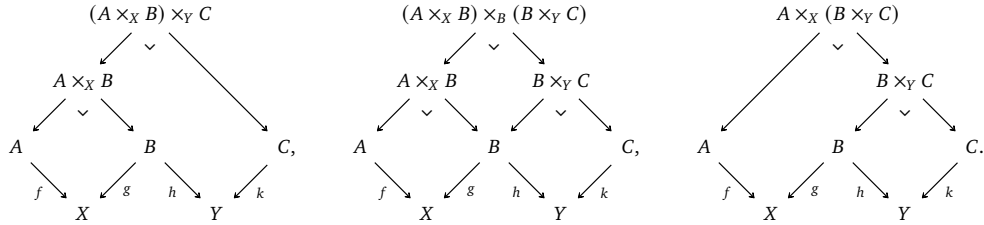
4. *Associativity.* Given a diagram



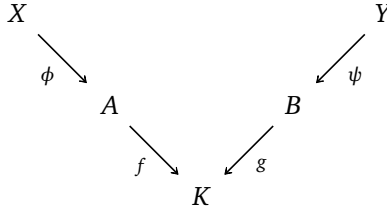
in  $\mathbf{Sets}$ , we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams



5. *Interaction With Composition.* Given a diagram



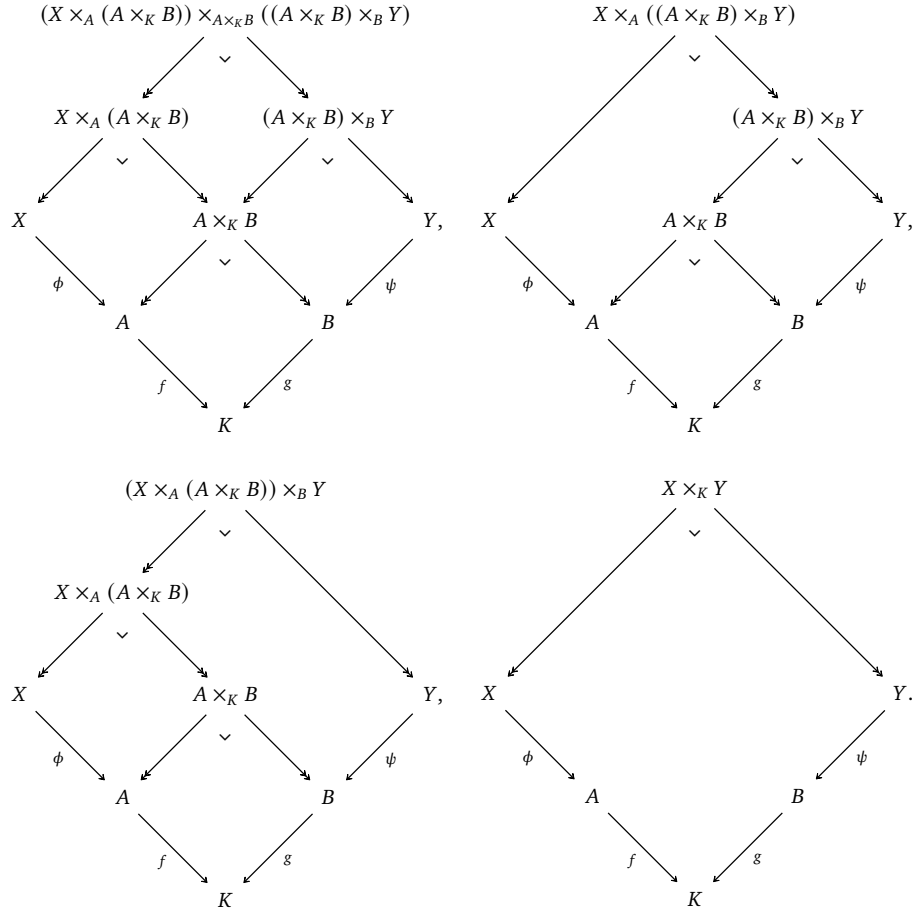
in Sets, we have isomorphisms of sets

$$\begin{aligned}
 X \times_K^{f \circ \phi, g \circ \psi} Y &\cong (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \times_{A \times_K^{f, g} B}^{p_2, p_1} ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y) \\
 &\cong X \times_A^{\phi, p} ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y) \\
 &\cong (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \times_B^{q, \psi} Y
 \end{aligned}$$

where

$$\begin{aligned}
 q_1 &= \text{pr}_1^{A \times_K^{f, g} B}, & q_2 &= \text{pr}_2^{A \times_K^{f, g} B}, \\
 p_1 &= \text{pr}_1^{(A \times_K^{f, g} B) \times_Y^{q_2, \psi}}, & p_2 &= \text{pr}_2^{X \times_{A \times_K^{f, g} B}^{\phi, q_1} (A \times_K^{f, g} B)}, \\
 p &= q_1 \circ \text{pr}_1^{(A \times_K^{f, g} B) \times_B^{q_2, \psi} Y}, & q &= q_2 \circ \text{pr}_2^{X \times_A^{\phi, q_1} (A \times_K^{f, g} B)},
 \end{aligned}$$

and where these pullbacks are built as in the following diagrams:



6. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow f & \lrcorner & \downarrow f \\ X & \xlongequal{\quad} & X \end{array} & \begin{array}{l} \lambda_A^{\text{Sets}/X} : X \times_X A \xrightarrow{\sim} A, \\ \rho_A^{\text{Sets}/X} : A \times_X X \xrightarrow{\sim} A, \end{array} & \begin{array}{ccc} A & \xrightarrow{f} & X \\ \parallel & \lrcorner & \parallel \\ X & \xrightarrow{f} & X, \end{array}
 \end{array}$$

natural in  $(A, f) \in \text{Obj}(\text{Sets}/X)$ .

7. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 \begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & X, \end{array} & \sigma_{A,B}^{\text{Sets}/X} : A \times_X B \xrightarrow{\sim} B \times_X A & \begin{array}{ccc} B \times_X A & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ B & \xrightarrow{g} & X, \end{array}
 \end{array}$$

natural in  $(A, f), (B, g) \in \text{Obj}(\text{Sets}/X)$ .

8. *Distributivity Over Coproducts.* Let  $A, B$ , and  $C$  be sets and let  $\phi_A : A \rightarrow X$ ,  $\phi_B : B \rightarrow X$ , and  $\phi_C : C \rightarrow X$  be morphisms of sets. We have isomorphisms of sets

$$\begin{aligned}
 \delta_\ell^{\text{Sets}/X} : A \times_X (B \amalg C) &\xrightarrow{\sim} (A \times_X B) \amalg (A \times_X C), \\
 \delta_r^{\text{Sets}/X} : (A \amalg B) \times_X C &\xrightarrow{\sim} (A \times_X C) \amalg (B \times_X C),
 \end{aligned}$$

as in the diagrams

$$\begin{array}{ccc}
 (A \times_X B) \amalg (A \times_X C) & \longrightarrow & B \amalg C \\
 \downarrow & \lrcorner & \downarrow \phi_B \amalg \phi_C \\
 A & \xrightarrow{\phi_A} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 (A \times_X C) \amalg (B \times_X C) & \longrightarrow & C \\
 \downarrow & \lrcorner & \downarrow \phi_C \\
 A \amalg B & \xrightarrow{\phi_A \amalg \phi_B} & X
 \end{array}$$

natural in  $A, B, C \in \text{Obj}(\text{Sets}/X)$ .

9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{array}{ccc}
 \begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{f} & X, \end{array} & \begin{array}{l} \zeta_\ell^{\text{Sets}/X} : A \times_X \emptyset \xrightarrow{\sim} \emptyset, \\ \zeta_r^{\text{Sets}/X} : \emptyset \times_X A \xrightarrow{\sim} \emptyset, \end{array} & \begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ \emptyset & \longrightarrow & X, \end{array}
 \end{array}$$

natural in  $(A, f) \in \text{Obj}(\text{Sets}/X)$ .

10. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\text{pt}} B \cong A \times B,$$

$$\begin{array}{ccc} A \times B & \longrightarrow & B \\ \downarrow \lrcorner & & \downarrow !_B \\ A & \longrightarrow & \text{pt.} \\ & \downarrow !_A & \end{array}$$

11. *Symmetric Monoidality.* The 8-tuple  $(\mathbf{Sets}_{/X}, \times_X, X, \mathbf{Sets}_{/X}, \alpha^{\mathbf{Sets}_{/X}}, \lambda^{\mathbf{Sets}_{/X}}, \rho^{\mathbf{Sets}_{/X}}, \sigma^{\mathbf{Sets}_{/X}})$  is a symmetric closed monoidal category.

*Proof.* **Item 1, Functoriality:** This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pullback diagram.

**Item 2, Adjointness I:** This is a repetition of Fibred Sets, ?? of ??, and is proved there.

**Item 3, Adjointness II:** This follows from the universal property of the product (pullbacks are products in  $\mathbf{Sets}_{/X}$ ).

**Item 4, Associativity:** We have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{(a, b), c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{(a, b), c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a, (b, c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a, (b, c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned} (A \times_X B) \times_B (B \times_Y C) &\cong \{(a, b), (b', c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\ &\cong \left\{ ((a, b), (b', c)) \in (A \times B) \times (B \times C) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\ &\cong \left\{ (a, (b, (b', c))) \in A \times (B \times (B \times C)) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\ &\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times B) \times C) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\ &\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times_B B) \times C) \mid \begin{array}{l} f(a) = g(b) \text{ and } \\ h(b') = k(c) \end{array} \right\} \\ &\cong \{(a, (b, c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong A \times_X (B \times_Y C), \end{aligned}$$

where we have used **Item 6** for the isomorphism  $B \times_B B \cong B$ .

**Item 5, Interaction With Composition:** By **Item 4**, it suffices to construct only the



isomorphism

$$X \times_K^{f \circ \phi, g \circ \psi} Y \cong (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \times_{A \times_K^{f, g} B}^{p_2, p_1} ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y).$$

We have

$$\begin{aligned} (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) &\stackrel{\text{def}}{=} \{(x, (a, b)) \in X \times (A \times_K^{f, g} B) \mid \phi(x) = q_1(a, b)\} \\ &\stackrel{\text{def}}{=} \{(x, (a, b)) \in X \times (A \times_K^{f, g} B) \mid \phi(x) = a\} \\ &\cong \{(x, (a, b)) \in X \times (A \times B) \mid \phi(x) = a \text{ and } f(a) = g(b)\}, \\ ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y) &\stackrel{\text{def}}{=} \{((a, b), y) \in (A \times_K^{f, g} B) \times Y \mid q_2(a, b) = \psi(y)\} \\ &\stackrel{\text{def}}{=} \{((a, b), y) \in (A \times_K^{f, g} B) \times Y \mid b = \psi(y)\} \\ &\cong \{((a, b), y) \in (A \times B) \times Y \mid b = \psi(y) \text{ and } f(a) = g(b)\}, \end{aligned}$$

so writing

$$\begin{aligned} S &= (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \\ S' &= ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y), \end{aligned}$$

we have

$$\begin{aligned} S \times_{A \times_K^{f, g} B}^{p_2, p_1} S' &\stackrel{\text{def}}{=} \{((x, (a, b)), ((a', b'), y)) \in S \times S' \mid p_1(x, (a, b)) = p_2((a', b'), y)\} \\ &\stackrel{\text{def}}{=} \{((x, (a, b)), ((a', b'), y)) \in S \times S' \mid (a, b) = (a', b')\} \\ &\cong \{(x, (a, b, y)) \in X \times A \times B \times Y \mid \phi(x) = a, \psi(y) = b, \text{ and } f(a) = g(b)\} \\ &\cong \{(x, (a, b, y)) \in X \times A \times B \times Y \mid f(\phi(x)) = g(\psi(y))\} \\ &\stackrel{\text{def}}{=} X \times_K Y. \end{aligned}$$

This finishes the proof.

**Item 6, Unitality:** We have

$$\begin{aligned} X \times_X A &\cong \{(x, a) \in X \times A \mid f(a) = x\}, \\ A \times_X X &\cong \{(a, x) \in X \times A \mid f(a) = x\}, \end{aligned}$$

which are isomorphic to  $A$  via the maps  $(x, a) \mapsto a$  and  $(a, x) \mapsto a$ . The proof of the naturality of  $\lambda^{\text{Sets}/X}$  and  $\rho^{\text{Sets}/X}$  is omitted.

**Item 7, Commutativity:** We have

$$\begin{aligned} A \times_C B &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\} \\ &= \{(a, b) \in A \times B \mid g(b) = f(a)\} \\ &\cong \{(b, a) \in B \times A \mid g(b) = f(a)\} \\ &\stackrel{\text{def}}{=} B \times_C A. \end{aligned}$$

The proof of the naturality of  $\sigma^{\text{Sets}/X}$  is omitted.

*Item 8, Distributivity Over Coproducts:* We have

$$\begin{aligned}
 A \times_X (B \amalg C) &\stackrel{\text{def}}{=} \left\{ (a, z) \in A \times (B \amalg C) \mid \phi_A(a) = \phi_{B \amalg C}(z) \right\} \\
 &= \left\{ (a, z) \in A \times (B \amalg C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b) \right\} \\
 &\quad \cup \left\{ (a, z) \in A \times (B \amalg C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c) \right\} \\
 &= \left\{ (a, z) \in A \times (B \amalg C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b) \right\} \\
 &\quad \cup \left\{ (a, z) \in A \times (B \amalg C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c) \right\} \\
 &\cong \left\{ (a, b) \in A \times B \mid \phi_A(a) = \phi_B(b) \right\} \\
 &\quad \cup \left\{ (a, c) \in A \times C \mid \phi_A(a) = \phi_C(c) \right\} \\
 &\stackrel{\text{def}}{=} (A \times_X B) \cup (A \times_X C) \\
 &\cong (A \times_X B) \amalg (A \times_X C),
 \end{aligned}$$

with the construction of the isomorphism

$$\delta_r^{\text{Sets}/X} : (A \amalg B) \times_X C \xrightarrow{\sim} (A \times_X C) \amalg (B \times_X C)$$

being similar. The proof of the naturality of  $\delta_\ell^{\text{Sets}/X}$  and  $\delta_r^{\text{Sets}/X}$  is omitted.

*Item 9, Annihilation With the Empty Set:* We have

$$\begin{aligned}
 A \times_X \emptyset &\stackrel{\text{def}}{=} \{ (a, b) \in A \times \emptyset \mid f(a) = g(b) \} \\
 &= \{ k \in \emptyset \mid f(a) = g(b) \} \\
 &= \emptyset,
 \end{aligned}$$

and similarly for  $\emptyset \times_X A$ , where we have used *Item 8* of *Definition 4.1.3.1.3*. The proof of the naturality of  $\zeta_\ell^{\text{Sets}/X}$  and  $\zeta_r^{\text{Sets}/X}$  is omitted.

*Item 10, Interaction With Products:* We have

$$\begin{aligned}
 A \times_{\text{pt}} B &\stackrel{\text{def}}{=} \{ (a, b) \in A \times B \mid !_A(a) = !_B(b) \} \\
 &\stackrel{\text{def}}{=} \{ (a, b) \in A \times B \mid \star = \star \} \\
 &= \{ (a, b) \in A \times B \} \\
 &= A \times B.
 \end{aligned}$$

*Item 11, Symmetric Monoidality:* Omitted. □

### 4.1.5 Equalisers

Let  $A$  and  $B$  be sets and let  $f, g : A \rightrightarrows B$  be functions.

**Definition 4.1.5.1.1.** The **equaliser of  $f$  and  $g$**  is the equaliser of  $f$  and  $g$  in Sets as in Limits and Colimits, ??.

**Construction 4.1.5.1.2.** Concretely, the equaliser of  $f$  and  $g$  is the pair  $(\text{Eq}(f, g), \text{eq}(f, g))$  consisting of:

1. *The Limit.* The set  $\text{Eq}(f, g)$  defined by

$$\text{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

2. *The Cone.* The inclusion map

$$\text{eq}(f, g) : \text{Eq}(f, g) \hookrightarrow A.$$

*Proof.* We claim that  $\text{Eq}(f, g)$  is the categorical equaliser of  $f$  and  $g$  in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set  $\text{Eq}(f, g)$ . Next, we prove that  $\text{Eq}(f, g)$  satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A \xrightleftharpoons[f]{f} B \\ & \nearrow e & \\ E & & \end{array}$$

in Sets. Then there exists a unique map  $\phi : E \rightarrow \text{Eq}(f, g)$  making the diagram

$$\begin{array}{ccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A \xrightleftharpoons[f]{f} B \\ \uparrow \phi \exists! & \nearrow e & \\ E & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in A$  indeed lies in  $\text{Eq}(f, g)$  by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f, g)$ . □

**Proposition 4.1.5.1.3.** Let  $A$ ,  $B$ , and  $C$  be sets.

1. *Associativity.* We have isomorphisms of sets<sup>5</sup>

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

<sup>5</sup>That is, the following three ways of forming “the” equaliser of  $(f, g, h)$  agree:

1. Take the equaliser of  $(f, g, h)$ , i.e. the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

in Sets.

2. First take the equaliser of  $f$  and  $g$ , forming a diagram

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) = \text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))$$

of  $\text{Eq}(f, g)$ .

3. First take the equaliser of  $g$  and  $h$ , forming a diagram

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

and then take the equaliser of the composition

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) = \text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))$$

of  $\text{Eq}(g, h)$ .

where  $\text{Eq}(f, g, h)$  is the limit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g} \\ \xrightarrow{h} \end{array} B$$

in Sets, being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\text{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have an inclusion of sets

$$\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) \subset \text{Eq}(h \circ f, k \circ g),$$

where  $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$  is the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C.$$

*Proof.* **Item 1, Associativity:** We first prove that  $\text{Eq}(f, g, h)$  is indeed given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} \text{Eq}(f, g, h) & \xrightarrow{\text{eq}(f, g, h)} & A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g} \\ \xrightarrow{h} \end{array} B \\ & \nearrow e & \\ E & & \end{array}$$

in Sets. Then there exists a unique map  $\phi: E \rightarrow \text{Eq}(f, g, h)$ , uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in A$  indeed lies in  $\text{Eq}(f, g, h)$  by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f, g, h)$ .

We now check the equalities

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) \cong \text{Eq}(f, g, h) \cong \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)).$$

Indeed, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) &\cong \{x \in \text{Eq}(g, h) \mid [f \circ \text{eq}(g, h)](a) = [g \circ \text{eq}(g, h)](a)\} \\ &\cong \{x \in \text{Eq}(g, h) \mid f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) &\cong \{x \in \text{Eq}(f, g) \mid [f \circ \text{eq}(f, g)](a) = [h \circ \text{eq}(f, g)](a)\} \\ &\cong \{x \in \text{Eq}(f, g) \mid f(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

*Item 4, Unitality:* Indeed, we have

$$\begin{aligned} \text{Eq}(f, f) &\stackrel{\text{def}}{=} \{a \in A \mid f(a) = f(a)\} \\ &= A. \end{aligned}$$

*Item 5, Commutativity:* Indeed, we have

$$\begin{aligned} \text{Eq}(f, g) &\stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\} \\ &= \{a \in A \mid g(a) = f(a)\} \\ &\stackrel{\text{def}}{=} \text{Eq}(g, f). \end{aligned}$$

**Item 6, Interaction With Composition:** Indeed, we have

$$\begin{aligned} \text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) &\cong \{a \in \text{Eq}(f, g) \mid h(f(a)) = k(g(a))\} \\ &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}. \end{aligned}$$

and

$$\text{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from  $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$  to  $\text{Eq}(h \circ f, k \circ g)$ .  $\square$

### 4.1.6 Inverse Limits

Let  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I} : (I, \preceq) \rightarrow \text{Sets}$  be an inverse system of sets.

**Definition 4.1.6.1.1.** The **inverse limit** of  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$  is the inverse limit of  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$  in  $\text{Sets}$  as in Limits and Colimits, ??.

**Construction 4.1.6.1.2.** Concretely, the inverse limit of  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$  is the pair  $\left(\lim_{\leftarrow \alpha \in I} (X_\alpha), \{\text{pr}_\alpha\}_{\alpha \in I}\right)$  consisting of:

1. *The Limit.* The set  $\lim_{\leftarrow \alpha \in I} (X_\alpha)$  defined by

$$\lim_{\leftarrow \alpha \in I} (X_\alpha) \stackrel{\text{def}}{=} \left\{ (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \mid \begin{array}{l} \text{for each } \alpha, \beta \in I, \text{ if } \alpha \preceq \beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta) \end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \text{pr}_\gamma : \lim_{\leftarrow \alpha \in I} (X_\alpha) \rightarrow X_\gamma \right\}_{\gamma \in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{ \text{pr}_\gamma : \prod_{\alpha \in I} X_\alpha \rightarrow X_\gamma \right\}_{\gamma \in I}$$

of **Item 2** of **Definition 4.1.2.1.2** to  $\lim_{\leftarrow \alpha \in I} (X_\alpha)$  and hence given by

$$\text{pr}_\gamma((x_\alpha)_{\alpha \in I}) \stackrel{\text{def}}{=} x_\gamma$$

for each  $\gamma \in I$  and each  $(x_\alpha)_{\alpha \in I} \in \lim_{\leftarrow \alpha \in I} (X_\alpha)$ .

*Proof.* We claim that  $\lim_{\leftarrow \alpha \in I} (X_\alpha)$  is the limit of the inverse system of sets  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ . First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta} \circ \text{pr}_\alpha = \text{pr}_\beta,$$

$$\begin{array}{ccc} & \lim_{\leftarrow \alpha \in I} (X_\alpha) & \\ \text{pr}_\alpha \swarrow & & \searrow \text{pr}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

for each  $\alpha, \beta \in I$  with  $\alpha \preceq \beta$ . Indeed, given  $(x_\gamma)_{\gamma \in I} \in \lim_{\leftarrow \gamma \in I} (X_\gamma)$ , we have

$$\begin{aligned} [f_{\alpha\beta} \circ \text{pr}_\alpha]((x_\gamma)_{\gamma \in I}) &\stackrel{\text{def}}{=} f_{\alpha\beta}(\text{pr}_\alpha((x_\gamma)_{\gamma \in I})) \\ &\stackrel{\text{def}}{=} f_{\alpha\beta}(x_\alpha) \\ &= x_\beta \\ &\stackrel{\text{def}}{=} \text{pr}_\beta((x_\gamma)_{\gamma \in I}), \end{aligned}$$

where the third equality comes from the definition of  $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ . Next, we prove that  $\lim_{\leftarrow \alpha \in I} (X_\alpha)$  satisfies the universal property of an inverse limit. Suppose that we have, for each  $\alpha, \beta \in I$  with  $\alpha \preceq \beta$ , a diagram of the form

$$\begin{array}{ccc} & L & \\ p_\alpha \swarrow & & \searrow p_\beta \\ & \lim_{\leftarrow \alpha \in I} (X_\alpha) & \\ \text{pr}_\alpha \swarrow & & \searrow \text{pr}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

in Sets. Then there indeed exists a unique map  $\phi: L \xrightarrow{\exists!} \lim_{\leftarrow \alpha \in I} (X_\alpha)$  making the diagram

$$\begin{array}{ccc} & L & \\ \phi \downarrow \exists! & & \\ p_\alpha \swarrow & \lim_{\leftarrow \alpha \in I} (X_\alpha) & \searrow p_\beta \\ \text{pr}_\alpha \swarrow & & \searrow \text{pr}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$



commute, being uniquely determined by the family of conditions

$$\{p_\alpha = \text{pr}_\alpha \circ \phi\}_{\alpha \in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha \in I}$$

for each  $\ell \in L$ , where we note that  $(p_\alpha(\ell))_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$  indeed lies in  $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ , as we have

$$\begin{aligned} f_{\alpha\beta}(p_\alpha(\ell)) &\stackrel{\text{def}}{=} [f_{\alpha\beta} \circ p_\alpha](\ell) \\ &\stackrel{\text{def}}{=} p_\beta(\ell) \end{aligned}$$

for each  $\beta \in I$  with  $\alpha \preceq \beta$  by the commutativity of the diagram for  $(L, \{p_\alpha\}_{\alpha \in I})$ .  $\square$

**Example 4.1.6.1.3.** Here are some examples of inverse limits of sets.

1. *The  $p$ -Adic Integers.* The ring of  $p$ -adic integers  $\mathbb{Z}_p$  of ?? is the inverse limit

$$\mathbb{Z}_p \cong \lim_{\leftarrow n \in \mathbb{N}} (\mathbb{Z}/p^n);$$

see ??.

2. *Rings of Formal Power Series.* The ring  $R[[t]]$  of formal power series in a variable  $t$  is the inverse limit

$$R[[t]] \cong \lim_{\leftarrow n \in \mathbb{N}} (R[t]/t^n R[t]);$$

see ??.

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see ??.

## 4.2 Colimits of Sets

### 4.2.1 The Initial Set

**Definition 4.2.1.1.1.** The **initial set** is the initial object of  $\mathbf{Sets}$  as in Limits and Colimits, ??.

**Construction 4.2.1.1.2.** Concretely, the initial set is the pair  $(\emptyset, \{\iota_A\}_{A \in \text{Obj}(\mathbf{Sets})})$  consisting of:

1. *The Colimit.* The empty set  $\emptyset$  of [Definition 4.3.1.1.1](#).
2. *The Cocone.* The collection of maps

$$\{\iota_A : \emptyset \rightarrow A\}_{A \in \text{Obj}(\text{Sets})}$$

given by the inclusion maps from  $\emptyset$  to  $A$ .

*Proof.* We claim that  $\emptyset$  is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \quad A$$

in Sets. Then there exists a unique map  $\phi : \emptyset \rightarrow A$  making the diagram

$$\emptyset \xrightarrow[\exists!]{\phi} A$$

commute, namely the inclusion map  $\iota_A$ . □

## 4.2.2 Coproducts of Families of Sets

Let  $\{A_i\}_{i \in I}$  be a family of sets.

**Definition 4.2.2.1.1.** The **coproduct** of  $\{A_i\}_{i \in I}$ <sup>6</sup> is the coproduct of  $\{A_i\}_{i \in I}$  in Sets as in Limits and Colimits, ??.

**Construction 4.2.2.1.2.** Concretely, the disjoint union of  $\{A_i\}_{i \in I}$  is the pair  $(\coprod_{i \in I} A_i, \{\text{inj}_i\}_{i \in I})$  consisting of:

1. *The Colimit.* The set  $\coprod_{i \in I} A_i$  defined by

$$\coprod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ (i, x) \in I \times \left( \bigcup_{i \in I} A_i \right) \mid x \in A_i \right\}.$$

2. *The Cocone.* The collection

$$\left\{ \text{inj}_i : A_i \rightarrow \coprod_{i \in I} A_i \right\}_{i \in I}$$

of maps given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each  $x \in A_i$  and each  $i \in I$ .

---

<sup>6</sup>*Further Terminology:* Also called the **disjoint union of the family**  $\{A_i\}_{i \in I}$ .

*Proof.* We claim that  $\coprod_{i \in I} A_i$  is the categorical coproduct of  $\{A_i\}_{i \in I}$  in Sets. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

in Sets. Then there exists a unique map  $\phi: \coprod_{i \in I} A_i \rightarrow C$  making the diagram

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \uparrow \exists! \phi \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

commute, being uniquely determined by the condition  $\phi \circ \text{inj}_i = \iota_i$  for each  $i \in I$  via

$$\phi((i, x)) = \iota_i(x)$$

for each  $(i, x) \in \coprod_{i \in I} A_i$ . □

**Proposition 4.2.2.1.3.** Let  $\{A_i\}_{i \in I}$  be a family of sets.

1. *Functoriality.* The assignment  $\{A_i\}_{i \in I} \mapsto \coprod_{i \in I} A_i$  defines a functor

$$\coprod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}) \rightarrow \text{Sets}$$

where

- *Action on Objects.* For each  $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$ , we have

$$\left[ \coprod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \coprod_{i \in I} A_i$$

- *Action on Morphisms.* For each  $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$ , the action on Hom-sets

$$\left( \coprod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}}: \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \text{Sets} \left( \coprod_{i \in I} A_i, \coprod_{i \in I} B_i \right)$$

of  $\coprod_{i \in I}$  at  $((A_i)_{i \in I}, (B_i)_{i \in I})$  is defined by sending a map

$$\{f_i: A_i \rightarrow B_i\}_{i \in I}$$

in  $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$  to the map of sets

$$\coprod_{i \in I} f_i: \coprod_{i \in I} A_i \rightarrow \coprod_{i \in I} B_i$$

defined by

$$\left[ \coprod_{i \in I} f_i \right] (i, a) \stackrel{\text{def}}{=} f_i(a)$$

for each  $(i, a) \in \coprod_{i \in I} A_i$ .

*Proof.* **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??.  $\square$

### 4.2.3 Binary Coproducts

Let  $A$  and  $B$  be sets.

**Definition 4.2.3.1.1.** The **coproduct of  $A$  and  $B$** <sup>7</sup> is the coproduct of  $A$  and  $B$  in Sets as in Limits and Colimits, ??.

**Construction 4.2.3.1.2.** Concretely, the coproduct of  $A$  and  $B$  is the pair  $(A \coprod B, \{\text{inj}_1, \text{inj}_2\})$  consisting of:

1. *The Colimit.* The set  $A \coprod B$  defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in A\} \cup \{(1, b) \in S \mid b \in B\}, \end{aligned}$$

where  $S = \{0, 1\} \times (A \cup B)$ .

2. *The Cocone.* The maps

$$\begin{aligned} \text{inj}_1: A &\rightarrow A \coprod B, \\ \text{inj}_2: B &\rightarrow A \coprod B, \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(a) &\stackrel{\text{def}}{=} (0, a), \\ \text{inj}_2(b) &\stackrel{\text{def}}{=} (1, b), \end{aligned}$$

for each  $a \in A$  and each  $b \in B$ .

---

<sup>7</sup>*Further Terminology:* Also called the **disjoint union of  $A$  and  $B$** .

*Proof.* We claim that  $A \amalg B$  is the categorical coproduct of  $A$  and  $B$  in **Sets**. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & C & \\ \iota_1 \nearrow & & \nwarrow \iota_2 \\ A & \xrightarrow{\text{inj}_1} & A \amalg B \xleftarrow{\text{inj}_2} B \end{array}$$

in **Sets**. Then there exists a unique map  $\phi: A \amalg B \rightarrow C$  making the diagram

$$\begin{array}{ccc} & C & \\ \iota_1 \nearrow & \uparrow \phi \mid \exists! & \nwarrow \iota_2 \\ A & \xrightarrow{\text{inj}_1} & A \amalg B \xleftarrow{\text{inj}_2} B \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \text{inj}_A &= \iota_A, \\ \phi \circ \text{inj}_B &= \iota_B \end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each  $x \in A \amalg B$ . □

**Proposition 4.2.3.1.3.** Let  $A$ ,  $B$ ,  $C$ , and  $X$  be sets.

1. *Functoriality.* The assignment  $A, B, (A, B) \mapsto A \amalg B$  defines functors

$$\begin{aligned} A \amalg -: \quad \text{Sets} &\rightarrow \text{Sets}, \\ - \amalg B: \quad \text{Sets} &\rightarrow \text{Sets}, \\ -_1 \amalg -_2: \text{Sets} \times \text{Sets} &\rightarrow \text{Sets}, \end{aligned}$$

where  $-_1 \amalg -_2$  is the functor where

- *Action on Objects.* For each  $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$ , we have

$$[-_1 \amalg -_2](A, B) \stackrel{\text{def}}{=} A \amalg B.$$

- *Action on Morphisms.* For each  $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$ , the action on Hom-sets

$$\amalg_{(A,B),(X,Y)}: \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \amalg B, X \amalg Y)$$

of  $\coprod$  at  $((A, B), (X, Y))$  is defined by sending  $(f, g)$  to the function

$$f \coprod g: A \coprod B \rightarrow X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each  $x \in A \coprod B$ .

and where  $A \coprod -$  and  $- \coprod B$  are the partial functors of  $-_1 \coprod -_2$  at  $A, B \in \text{Obj}(\text{Sets})$ .

2. *Adjointness.* We have an adjunction

$$(-_1 \coprod -_2 \dashv \Delta_{\text{Sets}}): \text{Sets} \times \text{Sets} \begin{matrix} \xrightarrow{-_1 \coprod -_2} \\ \perp \\ \xleftarrow{\Delta_{\text{Sets}}} \end{matrix} \text{Sets},$$

witnessed by a bijection

$$\text{Sets}(A \coprod B, C) \cong \text{Hom}_{\text{Sets} \times \text{Sets}}((A, B), (C, C))$$

natural in  $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$  and in  $C \in \text{Obj}(\text{Sets})$ .

3. *Associativity.* We have an isomorphism of sets

$$\alpha_{X,Y,Z}^{\text{Sets}, \coprod}: (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z),$$

natural in  $X, Y, Z \in \text{Obj}(\text{Sets})$ .

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \lambda_X^{\text{Sets}, \coprod}: \emptyset \coprod X &\xrightarrow{\sim} X, \\ \rho_X^{\text{Sets}, \coprod}: X \coprod \emptyset &\xrightarrow{\sim} X, \end{aligned}$$

natural in  $X \in \text{Obj}(\text{Sets})$ .

5. *Commutativity.* We have an isomorphism of sets

$$\sigma_{X,Y}^{\text{Sets}, \coprod}: X \coprod Y \xrightarrow{\sim} Y \coprod X,$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ .

6. *Symmetric Monoidality*. The 7-tuple  $(\mathbf{Sets}, \coprod, \emptyset, \alpha_{\coprod}^{\mathbf{Sets}}, \lambda_{\coprod}^{\mathbf{Sets}}, \rho_{\coprod}^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$  is a symmetric monoidal category.

*Proof. Item 1, Functoriality*: This follows from Limits and Colimits, ?? of ??.

*Item 2, Adjointness*: This follows from the universal property of the coproduct.

*Item 3, Associativity*: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.2.3.1.1.

*Item 4, Unitality*: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.2.4.1.1 and 5.2.5.1.1.

*Item 5, Commutativity*: This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.2.6.1.1.

*Item 6, Symmetric Monoidality*: This is a repetition of **Monoidal Structures on the Category of Sets**, Definition 5.2.7.1.1, and is proved there.  $\square$

#### 4.2.4 Pushouts

Let  $A$ ,  $B$ , and  $C$  be sets and let  $f: C \rightarrow A$  and  $g: C \rightarrow B$  be functions.

**Definition 4.2.4.1.1.** The **pushout of  $A$  and  $B$  over  $C$  along  $f$  and  $g$** <sup>8</sup> is the pushout of  $A$  and  $B$  over  $C$  along  $f$  and  $g$  in  $\mathbf{Sets}$  as in Limits and Colimits, ??.

**Construction 4.2.4.1.2.** Concretely, the pushout of  $A$  and  $B$  over  $C$  along  $f$  and  $g$  is the pair  $(A \coprod_C B, \{\text{inj}_1, \text{inj}_2\})$  consisting of:

1. *The Colimit*. The set  $A \coprod_C B$  defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B / \sim_C,$$

where  $\sim_C$  is the equivalence relation on  $A \coprod B$  generated by  $(0, f(c)) \sim_C (1, g(c))$ .

2. *The Cocone*. The maps

$$\begin{aligned} \text{inj}_1: A &\rightarrow A \coprod_C B, \\ \text{inj}_2: B &\rightarrow A \coprod_C B \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(a) &\stackrel{\text{def}}{=} [(0, a)] \\ \text{inj}_2(b) &\stackrel{\text{def}}{=} [(1, b)] \end{aligned}$$

for each  $a \in A$  and each  $b \in B$ .

---

<sup>8</sup>*Further Terminology*: Also called the **fibre coproduct of  $A$  and  $B$  over  $C$  along  $f$  and  $g$** .

*Proof.* We claim that  $A \amalg_C B$  is the categorical pushout of  $A$  and  $B$  over  $C$  with respect to  $(f, g)$  in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$$

$$\begin{array}{ccc} A \amalg_C B & \xleftarrow{\text{inj}_2} & B \\ \text{inj}_1 \uparrow & & \uparrow g \\ A & \xleftarrow{f} & C. \end{array}$$

Indeed, given  $c \in C$ , we have

$$\begin{aligned} [\text{inj}_1 \circ f](c) &= \text{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \text{inj}_2(g(c)) \\ &= [\text{inj}_2 \circ g](c), \end{aligned}$$

where  $[(0, f(c))] = [(1, g(c))]$  by the definition of the relation  $\sim$  on  $A \amalg B$ . Next, we prove that  $A \amalg_C B$  satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{array}{ccc} & & P \\ & \nwarrow \iota_2 & \\ & A \amalg_C B \xleftarrow{\text{inj}_2} B & \\ & \uparrow \text{inj}_1 \quad \uparrow g & \\ & A \xleftarrow{f} C & \\ & \nearrow \iota_1 & \end{array}$$

in Sets. Then there exists a unique map  $\phi: A \amalg_C B \rightarrow P$  making the diagram

$$\begin{array}{ccc} & & P \\ & \nwarrow \iota_2 & \\ & A \amalg_C B \xleftarrow{\text{inj}_2} B & \\ & \uparrow \text{inj}_1 \quad \uparrow g & \\ & A \xleftarrow{f} C & \\ & \nearrow \iota_1 & \\ & \nwarrow \phi & \\ & \exists! & \end{array}$$



commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_1 &= \iota_1, \\ \phi \circ \text{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each  $x \in A \amalg_C B$ , where the well-definedness of  $\phi$  is guaranteed by the equality  $\iota_1 \circ f = \iota_2 \circ g$  and the definition of the relation  $\sim$  on  $A \amalg B$  as follows:

1. *Case 1:* Suppose we have  $x = [(0, a)] = [(0, a')]$  for some  $a, a' \in A$ . Then, by [Definition 4.2.4.1.3](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

2. *Case 2:* Suppose we have  $x = [(1, b)] = [(1, b')]$  for some  $b, b' \in B$ . Then, by [Definition 4.2.4.1.3](#), we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

3. *Case 3:* Suppose we have  $x = [(0, a)] = [(1, b)]$  for some  $a \in A$  and  $b \in B$ . Then, by [Definition 4.2.4.1.3](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare  $x \sim' y$  iff there exists some  $c \in C$  such that  $x = (0, f(c))$  and  $y = (1, g(c))$  or  $x = (1, g(c))$  and  $y = (0, f(c))$ . Then, the equality  $\iota_1 \circ f = \iota_2 \circ g$  gives

$$\begin{aligned}\phi([x]) &= \phi([(0, f(c))]) \\ &\stackrel{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]),\end{aligned}$$

with the case where  $x = (1, g(c))$  and  $y = (0, f(c))$  similarly giving  $\phi([x]) = \phi([y])$ . Thus, if  $x \sim' y$ , then  $\phi([x]) = \phi([y])$ . Applying this equality pairwise to the sequences

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'),$$

$$\begin{aligned}(1, b) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)\end{aligned}$$

gives

$$\begin{aligned}\phi([(0, a)]) &= \phi([(0, a')]), \\ \phi([(1, b)]) &= \phi([(1, b')]), \\ \phi([(0, a)]) &= \phi([(1, b)]),\end{aligned}$$

showing  $\phi$  to be well-defined.  $\square$

**Remark 4.2.4.1.3.** In detail, by [Conditions on Relations](#), [Definition 10.5.2.1.2](#), the relation  $\sim$  of [Definition 4.2.4.1.1](#) is given by declaring  $a \sim b$  iff one of the following conditions is satisfied:

1. We have  $a, b \in A$  and  $a = b$ .
2. We have  $a, b \in B$  and  $a = b$ .
3. There exist  $x_1, \dots, x_n \in A \amalg B$  such that  $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$ , where we declare  $x \sim' y$  if one of the following conditions is satisfied:
  - (a) There exists  $c \in C$  such that  $x = (0, f(c))$  and  $y = (1, g(c))$ .
  - (b) There exists  $c \in C$  such that  $x = (1, g(c))$  and  $y = (0, f(c))$ .

In other words, there exist  $x_1, \dots, x_n \in A \amalg B$  satisfying the following conditions:

- (c) There exists  $c_0 \in C$  satisfying one of the following conditions:
  - i. We have  $a = f(c_0)$  and  $x_1 = g(c_0)$ .
  - ii. We have  $a = g(c_0)$  and  $x_1 = f(c_0)$ .
- (d) For each  $1 \leq i \leq n - 1$ , there exists  $c_i \in C$  satisfying one of the following conditions:
  - i. We have  $x_i = f(c_i)$  and  $x_{i+1} = g(c_i)$ .
  - ii. We have  $x_i = g(c_i)$  and  $x_{i+1} = f(c_i)$ .
- (e) There exists  $c_n \in C$  satisfying one of the following conditions:
  - i. We have  $x_n = f(c_n)$  and  $b = g(c_n)$ .
  - ii. We have  $x_n = g(c_n)$  and  $b = f(c_n)$ .

**Remark 4.2.4.1.4.** It is common practice to write  $A \amalg_C B$  for the pushout of  $A$  and  $B$  over  $C$  along  $f$  and  $g$ , omitting the maps  $f$  and  $g$  from the notation and

instead leaving them implicit, to be understood from the context.

However, the set  $A \amalg_c B$  depends very much on the maps  $f$  and  $g$ , and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write  $A \amalg_{f,C,g} B$  or  $A \amalg_C^{f,g} B$  for  $A \amalg_c B$ .

**Example 4.2.4.1.5.** Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **Pointed Sets**, **Definition 6.3.3.1.1** is an example of a pushout of sets.
2. *Intersections via Unions.* Let  $X$  be a set. We have

$$A \cup B \cong A \amalg_{A \cap B} B,$$

$$\begin{array}{ccc} A \cup B & \longleftarrow & B \\ \uparrow \ulcorner & & \uparrow \\ A & \longleftarrow & A \cap B \end{array}$$

for each  $A, B \in \mathcal{P}(X)$ .

*Proof. Item 1, Wedge Sums of Pointed Sets:* This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.

*Item 2, Intersections via Unions:* Indeed,  $A \amalg_{A \cap B} B$  is the quotient of  $A \amalg B$  by the equivalence relation obtained by declaring  $(0, a) \sim (1, b)$  iff  $a = b \in A \cap B$ , which is in bijection with  $A \cup B$  via the map with  $[(0, a)] \mapsto a$  and  $[(1, b)] \mapsto b$ .  $\square$

**Proposition 4.2.4.1.6.** Let  $A, B, C$ , and  $X$  be sets.

1. *Functoriality.* The assignment  $(A, B, C, f, g) \mapsto A \amalg_{f,C,g} B$  defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where  $\mathcal{P}$  is the category that looks like this:

$$\begin{array}{ccc} & & \bullet \\ & \uparrow & \\ \bullet & \longleftarrow & \bullet \end{array}$$

In particular, the action on morphisms of  ${}_1 \amalg_{-3} -_1$  is given by sending a

morphism

$$\begin{array}{ccccc}
 A \amalg_C B & \xleftarrow{\quad} & B & & \\
 \uparrow \scriptstyle \ulcorner & & \uparrow & \searrow \scriptstyle \psi & \\
 & & A' \amalg_{C'} B' & \xleftarrow{\quad} & B' \\
 & & \uparrow \scriptstyle \ulcorner & & \uparrow \scriptstyle g' \\
 A & \xleftarrow{\quad} & C & \xrightarrow{\quad} & B' \\
 \searrow \scriptstyle \phi & & \uparrow \scriptstyle f & \searrow \scriptstyle \chi & \\
 & & A' & \xleftarrow{\quad} & C' \\
 & & \uparrow \scriptstyle f' & & 
 \end{array}$$

in  $\text{Fun}(\mathcal{P}, \text{Sets})$  to the map  $\xi: A \amalg_C B \xrightarrow{\exists!} A' \amalg_{C'} B'$  given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each  $x \in A \amalg_C B$ , which is the unique map making the diagram

$$\begin{array}{ccccc}
 A \amalg_C B & \xleftarrow{\quad} & B & & \\
 \uparrow \scriptstyle \ulcorner & \searrow \scriptstyle \xi & \uparrow & \searrow \scriptstyle \psi & \\
 & & A' \amalg_{C'} B' & \xleftarrow{\quad} & B' \\
 & & \uparrow \scriptstyle \ulcorner & & \uparrow \scriptstyle g' \\
 A & \xleftarrow{\quad} & C & \xrightarrow{\quad} & B' \\
 \searrow \scriptstyle \phi & & \uparrow \scriptstyle f & \searrow \scriptstyle \chi & \\
 & & A' & \xleftarrow{\quad} & C' \\
 & & \uparrow \scriptstyle f' & & 
 \end{array}$$

commute.

2. *Adjointness.* We have an adjunction

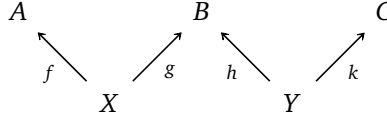
$$\left( -1 \amalg_{X^{-2}} \dashv \Delta_{\text{Sets}_{X/}} \right): \text{Sets}_{X/} \times \text{Sets}_{X/} \begin{array}{c} \xrightarrow{-1 \amalg_{X^{-2}}} \\ \perp \\ \xleftarrow{\Delta_{\text{Sets}_{X/}}} \end{array} \text{Sets}_{X/},$$

witnessed by a bijection

$$\text{Sets}_{X/}(A \amalg_X B, C) \cong \text{Hom}_{\text{Sets}_{X/} \times \text{Sets}_{X/}}((A, B), (C, C))$$

natural in  $(A, B) \in \text{Obj}(\text{Sets}_{X/} \times \text{Sets}_{X/})$  and in  $C \in \text{Obj}(\text{Sets}_{X/})$ .

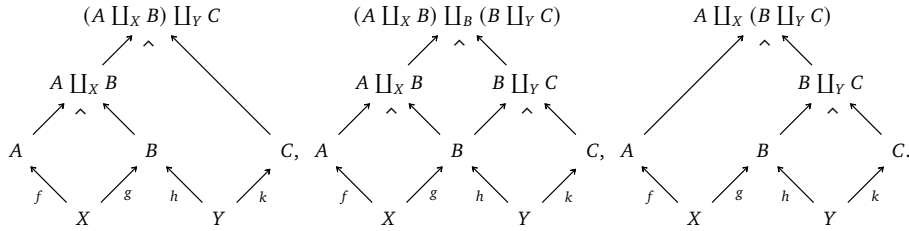
3. *Associativity.* Given a diagram



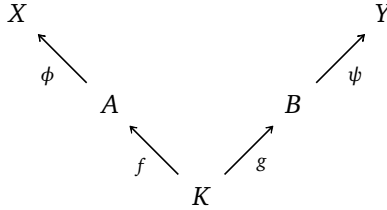
in Sets, we have isomorphisms of sets

$$(A \amalg_X B) \amalg_Y C \cong (A \amalg_X B) \amalg_B (B \amalg_Y C) \cong A \amalg_X (B \amalg_Y C)$$

where these pullbacks are built as in the diagrams



4. *Interaction With Composition.* Given a diagram



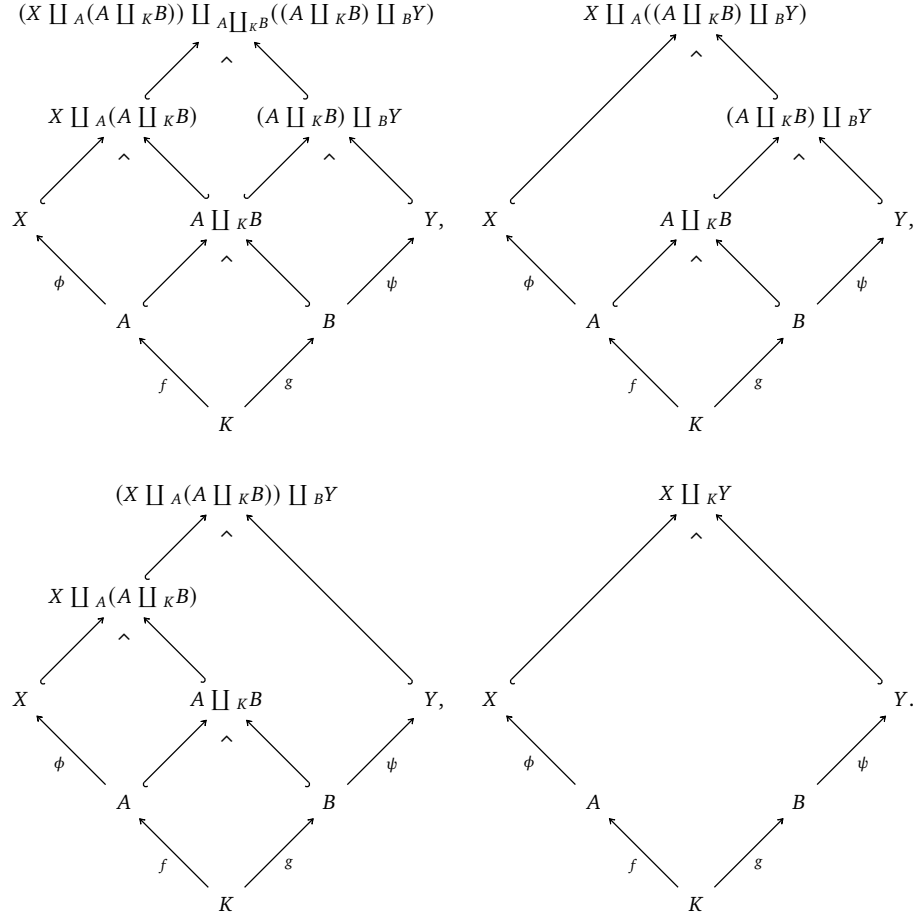
in Sets, we have isomorphisms of sets

$$\begin{aligned} X \amalg_K^{\phi \circ f, \psi \circ g} Y &\cong (X \amalg_A^{\phi, j_1} (A \amalg_K^{f, g} B)) \amalg_{A \amalg_K^{f, g} B}^{i_2, i_1} ((A \amalg_K^{f, g} B) \amalg_B^{j_2, \psi} Y) \\ &\cong X \amalg_A^{\phi, i} ((A \amalg_K^{f, g} B) \amalg_B^{j_2, \psi} Y) \\ &\cong (X \amalg_A^{\phi, i_1} (A \amalg_K^{f, g} B)) \amalg_B^{j, \psi} Y \end{aligned}$$

where

$$\begin{aligned} j_1 &= \text{inj}_1^{A \times_K^{f, g} B}, & j_2 &= \text{inj}_2^{A \times_K^{f, g} B}, \\ i_1 &= \text{inj}_1^{(A \times_K^{f, g} B) \times_Y^{q_2, \psi}}, & i_2 &= \text{inj}_2^{X \times_{A \times_K^{f, g} B}^{\phi, q_1} (A \times_K^{f, g} B)}, \\ i &= j_1 \circ \text{inj}_1^{(A \times_K^{f, g} B) \times_B^{q_2, \psi} Y}, & j &= j_2 \circ \text{inj}_2^{X \times_A^{\phi, q_1} (A \times_K^{f, g} B)}, \end{aligned}$$

and where these pullbacks are built as in the diagrams



5. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 f \uparrow & \lrcorner & \uparrow f \\
 X & \xlongequal{\quad} & X
 \end{array} & \begin{array}{l}
 \lambda_A^{\mathbf{Sets}_X/} : X \amalg_X A \xrightarrow{\sim} A, \\
 \rho_A^{\mathbf{Sets}_X/} : A \amalg_X X \xrightarrow{\sim} A,
 \end{array} & \begin{array}{ccc}
 A & \xleftarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xleftarrow{f} & X,
 \end{array}
 \end{array}$$

natural in  $(A, f) \in \mathbf{Obj}(\mathbf{Sets}_X/)$ .

6. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 A \amalg_X B & \longleftarrow & B \\
 \uparrow \ulcorner & & \uparrow g \\
 A & \xleftarrow{f} & X,
 \end{array}
 \quad
 \sigma_A^{\mathbf{Sets}_X/} : A \amalg_X B \xrightarrow{\sim} B \amalg_X A
 \quad
 \begin{array}{ccc}
 B \amalg_X A & \longleftarrow & A \\
 \uparrow \ulcorner & & \uparrow f \\
 B & \xleftarrow{g} & X.
 \end{array}$$

natural in  $(A, f), (B, g) \in \mathbf{Obj}(\mathbf{Sets}_X/)$ .

7. *Interaction With Coproducts.* We have

$$A \amalg_{\emptyset} B \cong A \amalg B, \quad
 \begin{array}{ccc}
 A \amalg B & \longleftarrow & B \\
 \uparrow \ulcorner & & \uparrow \iota_B \\
 A & \xleftarrow{\iota_A} & \emptyset.
 \end{array}$$

8. *Symmetric Monoidality.* The triple  $(\mathbf{Sets}_X/, \amalg_X, X)$  is a symmetric monoidal category.

*Proof.* **Item 1, Functoriality:** This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pushout diagram.

**Item 2, : Adjointness:** This follows from the universal property of the coproduct (pushouts are coproducts in  $\mathbf{Sets}_X/$ ).

**Item 3, Associativity:** Omitted.

**Item 4, Interaction With Composition:** Omitted.

**Item 5, Unitality:** Omitted.

**Item 6, Commutativity:** Omitted.

**Item 7, Interaction With Coproducts:** Omitted.

**Item 8, Symmetric Monoidality:** Omitted. □

## 4.2.5 Coequalisers

Let  $A$  and  $B$  be sets and let  $f, g : A \rightrightarrows B$  be functions.

**Definition 4.2.5.1.1.** The **coequaliser of  $f$  and  $g$**  is the coequaliser of  $f$  and  $g$  in  $\mathbf{Sets}$  as in Limits and Colimits, ??.

**Construction 4.2.5.1.2.** Concretely, the coequaliser of  $f$  and  $g$  is the pair  $(\mathbf{CoEq}(f, g), \mathbf{coeq}(f, g))$  consisting of:

1. *The Colimit.* The set  $\text{CoEq}(f, g)$  defined by

$$\text{CoEq}(f, g) \stackrel{\text{def}}{=} B/\sim,$$

where  $\sim$  is the equivalence relation on  $B$  generated by  $f(a) \sim g(a)$ .

2. *The Cocone.* The map

$$\text{coeq}(f, g): B \twoheadrightarrow \text{CoEq}(f, g)$$

given by the quotient map  $\pi: B \twoheadrightarrow B/\sim$  with respect to the equivalence relation generated by  $f(a) \sim g(a)$ .

*Proof.* We claim that  $\text{CoEq}(f, g)$  is the categorical coequaliser of  $f$  and  $g$  in **Sets**. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](a) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(a)) \\ &\stackrel{\text{def}}{=} [f(a)] \\ &= [g(a)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(a)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](a) \end{aligned}$$

for each  $a \in A$ . Next, we prove that  $\text{CoEq}(f, g)$  satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\ & \searrow c & \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & C \end{array}$$

in **Sets**. Then, since  $c(f(a)) = c(g(a))$  for each  $a \in A$ , it follows from **Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3** that there exists a unique map  $\text{CoEq}(f, g) \xrightarrow{\exists!} C$  making the diagram

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\ & \searrow c & \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & \downarrow \exists! \\ & & C \end{array}$$

commute.

□



**Remark 4.2.5.1.3.** In detail, by [Conditions on Relations](#), [Definition 10.5.2.1.2](#), the relation  $\sim$  of [Definition 4.2.5.1.1](#) is given by declaring  $a \sim b$  iff one of the following conditions is satisfied:

1. We have  $a = b$ ;
2. There exist  $x_1, \dots, x_n \in B$  such that  $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$ , where we declare  $x \sim' y$  if one of the following conditions is satisfied:
  - (a) There exists  $z \in A$  such that  $x = f(z)$  and  $y = g(z)$ .
  - (b) There exists  $z \in A$  such that  $x = g(z)$  and  $y = f(z)$ .

In other words, there exist  $x_1, \dots, x_n \in B$  satisfying the following conditions:

- (a) There exists  $z_0 \in A$  satisfying one of the following conditions:
  - i. We have  $a = f(z_0)$  and  $x_1 = g(z_0)$ .
  - ii. We have  $a = g(z_0)$  and  $x_1 = f(z_0)$ .
- (b) For each  $1 \leq i \leq n - 1$ , there exists  $z_i \in A$  satisfying one of the following conditions:
  - i. We have  $x_i = f(z_i)$  and  $x_{i+1} = g(z_i)$ .
  - ii. We have  $x_i = g(z_i)$  and  $x_{i+1} = f(z_i)$ .
- (c) There exists  $z_n \in A$  satisfying one of the following conditions:
  - i. We have  $x_n = f(z_n)$  and  $b = g(z_n)$ .
  - ii. We have  $x_n = g(z_n)$  and  $b = f(z_n)$ .

**Example 4.2.5.1.4.** Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let  $R$  be an equivalence relation on a set  $X$ . We have a bijection of sets

$$X/\sim_R \cong \text{CoEq}(R \hookrightarrow X \times X \begin{matrix} \xrightarrow{\text{pr}_1} \\ \rightrightarrows \\ \xleftarrow{\text{pr}_2} \end{matrix} X).$$

*Proof.* [Item 1](#), *Quotients by Equivalence Relations*: See [\[Pro25z\]](#). □

**Proposition 4.2.5.1.5.** Let  $A$ ,  $B$ , and  $C$  be sets.

1. *Associativity.* We have isomorphisms of sets<sup>9</sup>

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where  $\text{CoEq}(f, g, h)$  is the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\text{CoEq}(f, f) \cong B.$$

---

<sup>9</sup>That is, the following three ways of forming “the” coequaliser of  $(f, g, h)$  agree:

1. Take the coequaliser of  $(f, g, h)$ , i.e. the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} B$$

in Sets.

2. First take the coequaliser of  $f$  and  $g$ , forming a diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} B \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h) = \text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)$$

of  $\text{CoEq}(f, g)$

3. First take the coequaliser of  $g$  and  $h$ , forming a diagram

$$A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B \xrightarrow{\text{coeq}(g, h)} \text{CoEq}(g, h)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{\text{coeq}(g, h)} \text{CoEq}(g, h),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g) = \text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)$$

of  $\text{CoEq}(g, h)$ .

5. *Commutativity.* We have an isomorphism of sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have a surjection

$$\text{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$$

exhibiting  $\text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$  as a quotient of  $\text{CoEq}(h \circ f, k \circ g)$  by the relation generated by declaring  $h(y) \sim k(y)$  for each  $y \in B$ .

*Proof.* **Item 1**, *Associativity*: Omitted.

**Item 4**, *Unitality*: Omitted.

**Item 5**, *Commutativity*: Omitted.

**Item 6**, *Interaction With Composition*: Omitted. □

#### 4.2.6 Direct Colimits

Let  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}: (I, \preceq) \rightarrow \Pi$  be a direct system of sets.

**Definition 4.2.6.1.1.** The **direct colimit** of  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$  is the direct colimit of  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$  in Sets as in Limits and Colimits, ??.

**Construction 4.2.6.1.2.** Concretely, the direct colimit of  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$  is the pair  $\left( \varinjlim_{\alpha \in I} X_\alpha, \{\text{inj}_\alpha\}_{\alpha \in I} \right)$  consisting of:

1. *The Colimit.* The set  $\varinjlim_{\alpha \in I} X_\alpha$  defined by

$$\varinjlim_{\alpha \in I} X_\alpha \stackrel{\text{def}}{=} \left( \coprod_{\alpha \in I} X_\alpha \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\coprod_{\alpha \in I} X_\alpha$  generated by declaring  $(\alpha, x) \sim (\beta, y)$  iff there exists some  $\gamma \in I$  satisfying the following conditions:

- (a) We have  $\alpha \preceq \gamma$ .
- (b) We have  $\beta \preceq \gamma$ .
- (c) We have  $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$ .

2. *The Cocone.* The collection

$$\left\{ \text{inj}_\gamma : X_\gamma \rightarrow \underset{\alpha \in I}{\text{colim}}(X_\alpha) \right\}_{\gamma \in I}$$

of maps of sets defined by

$$\text{inj}_\gamma(x) \stackrel{\text{def}}{=} [(\gamma, x)]$$

for each  $\gamma \in I$  and each  $x \in X_\gamma$ .

*Proof.* We will prove **Definition 4.2.6.1.2** below in a bit, but first we need a lemma (which is interesting in its own right).  $\square$

**Lemma 4.2.6.1.3.** For each  $\alpha, \beta \in I$  and each  $x \in X_\alpha$ , if  $\alpha \preceq \beta$ , then we have

$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in  $\underset{\alpha \in I}{\text{colim}}(X_\alpha)$ .

*Proof.* Taking  $\gamma = \beta$ , we have  $f_{\alpha\gamma} = f_{\alpha\beta}$ , we have  $f_{\beta\gamma} = f_{\beta\beta} \stackrel{\text{def}}{=} \text{id}_{X_\beta}$ , and we have

$$\begin{aligned} f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} \text{id}_{X_\beta}(f_{\alpha\beta}(x)), \\ &= f_{\alpha\beta}(x). \end{aligned}$$

As a result, since  $\alpha \preceq \beta$  and  $\beta \preceq \beta$  as well, **Items 1a to 1c** of **Definition 4.2.6.1.2** are met. Thus we have  $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$ .  $\square$

We can now prove **Definition 4.2.6.1.2**:

*Proof.* We claim that  $\underset{\alpha \in I}{\text{colim}}(X_\alpha)$  is the colimit of the direct system of sets  $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ .

*Commutativity of the Colimit Diagram:* First, we need to check that the colimit diagram defined by  $\underset{\alpha \in I}{\text{colim}}(X_\alpha)$  commutes, i.e. that we have

$$\begin{array}{ccc} & \underset{\alpha \in I}{\text{colim}}(X_\alpha) & \\ & \uparrow \text{inj}_\alpha \quad \nwarrow \text{inj}_\beta & \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

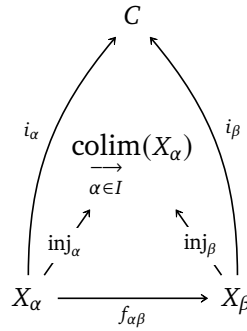
$\text{inj}_\alpha = \text{inj}_\beta \circ f_{\alpha\beta},$

for each  $\alpha, \beta \in I$  with  $\alpha \preceq \beta$ . Indeed, given  $x \in X_\alpha$ , we have

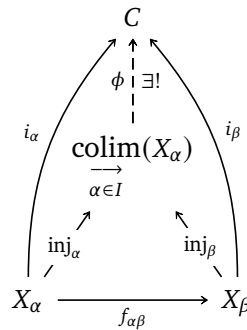
$$\begin{aligned} [\text{inj}_\beta \circ f_{\alpha\beta}](x) &\stackrel{\text{def}}{=} \text{inj}_\beta(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\ &= [(\alpha, x)] \\ &\stackrel{\text{def}}{=} \text{inj}_\alpha(x), \end{aligned}$$

where we have used [Definition 4.2.6.1.3](#) for the third equality.

*Proof of the Universal Property of the Colimit:* Next, we prove that  $\text{colim}_{\alpha \in I} (X_\alpha)$  as constructed in [Definition 4.2.6.1.2](#) satisfies the universal property of a direct colimit. Suppose that we have, for each  $\alpha, \beta \in I$  with  $\alpha \preceq \beta$ , a diagram of the form



in Sets. We claim that there exists a unique map  $\phi: \text{colim}(X_\alpha) \xrightarrow{\exists!} C$  making the diagram



commute. To this end, first consider the diagram

$$\begin{array}{ccc} \coprod_{\alpha \in I} X_\alpha & \xrightarrow{\text{pr}} & \text{colim}(X_\alpha) \\ & \searrow & \downarrow \\ & & \coprod_{\alpha \in I} i_\alpha \\ & & C. \end{array}$$

**Lemma.** If  $(\alpha, x) \sim (\beta, y)$ , then we have

$$\left[ \coprod_{\alpha \in I} i_{\alpha} \right] (x) = \left[ \coprod_{\alpha \in I} i_{\alpha} \right] (y).$$

*Proof.* Indeed, if  $(\alpha, x) \sim (\beta, y)$ , then there exists some  $\gamma \in I$  satisfying the following conditions:

1. We have  $\alpha \preceq \gamma$ .
2. We have  $\beta \preceq \gamma$ .
3. We have  $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$ .

We then have

$$\begin{aligned} \left[ \coprod_{\alpha \in I} i_{\alpha} \right] (x) &\stackrel{\text{def}}{=} i_{\alpha}(x) \\ &\stackrel{\text{def}}{=} [i_{\gamma} \circ f_{\alpha\gamma}](x) \\ &\stackrel{\text{def}}{=} i_{\gamma}(f_{\alpha\gamma}(x)) \\ &= i_{\gamma}(f_{\beta\gamma}(y)) \\ &\stackrel{\text{def}}{=} [i_{\gamma} \circ f_{\beta\gamma}](y) \\ &= i_{\beta}(y) \\ &\stackrel{\text{def}}{=} \left[ \coprod_{\alpha \in I} i_{\alpha} \right] (y). \end{aligned}$$

This finishes the proof of the lemma. Continuing, by **Conditions on Relations**, ?? of **Definition 10.6.2.1.3**, there then exists a map  $\phi: \text{colim}(X_{\alpha}) \xrightarrow{\exists!} C$  making the diagram

$$\begin{array}{ccc} \coprod_{\alpha \in I} X_{\alpha} & \xrightarrow{\text{pr}} & \text{colim}(X_{\alpha}) \\ & \searrow & \downarrow \phi \\ \coprod_{\alpha \in I} i_{\alpha} & & C \end{array}$$

commute. In particular, this implies that the diagram

$$\begin{array}{ccc} X_{\alpha} & \xrightarrow{\text{inj}_{\alpha}} & \text{colim}(X_{\alpha}) \\ & \searrow i_{\alpha} & \downarrow \phi \\ & & C \end{array}$$

also commutes, and thus so does the diagram

$$\begin{array}{ccc}
 & C & \\
 i_\alpha \nearrow & \uparrow \phi \mid \exists! & \nwarrow i_\beta \\
 & \text{colim}(X_\alpha) & \\
 \text{inj}_\alpha \nearrow & \xrightarrow{\alpha \in I} & \nwarrow \text{inj}_\beta \\
 X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta
 \end{array}$$

This finishes the proof.<sup>10</sup>

□

**Example 4.2.6.1.4.** Here are some examples of direct colimits of sets.

1. *The Prüfer Group.* The Prüfer group  $\mathbb{Z}(p^\infty)$  is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \stackrel{\text{def}}{=} \text{colim}_{n \in \mathbb{N}} (\mathbb{Z}/p^n);$$

see ??.

## 4.3 Operations With Sets

### 4.3.1 The Empty Set

**Definition 4.3.1.1.1.** The **empty set** is the set  $\emptyset$  defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where  $X$  is the set in the set existence axiom, ?? of ??.

---

<sup>10</sup>Incidentally, the conditions

$$\{i_\alpha = \phi \circ \text{inj}_\alpha\}_{\alpha \in I}$$

show that  $\phi$  must be given by

$$\phi([\langle \alpha, x \rangle]) = (i_\alpha(x))_{\alpha \in I}$$

for each  $[\langle \alpha, x \rangle] \in \text{colim}_{\alpha \in I} (X_\alpha)$ , although we would need to show that this assignment is well-defined were we to prove **Definition 4.2.6.1.2** in this way. Instead, invoking **Conditions on Relations**, ?? of **Definition 10.6.2.1.3** gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

### 4.3.2 Singleton Sets

Let  $X$  be a set.

**Definition 4.3.2.1.1.** The **singleton set containing**  $X$  is the set  $\{X\}$  defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where  $\{X, X\}$  is the pairing of  $X$  with itself of [Definition 4.3.3.1.1](#).

### 4.3.3 Pairings of Sets

Let  $X$  and  $Y$  be sets.

**Definition 4.3.3.1.1.** The **pairing of  $X$  and  $Y$**  is the set  $\{X, Y\}$  defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where  $A$  is the set in the axiom of pairing, ?? of ??.

### 4.3.4 Ordered Pairs

Let  $A$  and  $B$  be sets.

**Definition 4.3.4.1.1.** The **ordered pair associated to  $A$  and  $B$**  is the set  $(A, B)$  defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

**Proposition 4.3.4.1.2.** Let  $A$  and  $B$  be sets.

1. *Uniqueness.* Let  $A$ ,  $B$ ,  $C$ , and  $D$  be sets. The following conditions are equivalent:

- (a) We have  $(A, B) = (C, D)$ .
- (b) We have  $A = C$  and  $B = D$ .

*Proof.* [Item 1](#), *Uniqueness*: See [[Cie97](#), Theorem 1.2.3].

□

### 4.3.5 Sets of Maps

Let  $A$  and  $B$  be sets.



**Definition 4.3.5.1.1.** The **set of maps from  $A$  to  $B$** <sup>11</sup> is the set  $\text{Sets}(A, B)$ <sup>12</sup> whose elements are the functions from  $A$  to  $B$ .

**Proposition 4.3.5.1.2.** Let  $A$  and  $B$  be sets.

1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto \text{Hom}_{\text{Sets}}(X, Y)$  define functors

$$\begin{aligned} \text{Sets}(X, -) &: \text{Sets} \rightarrow \text{Sets}, \\ \text{Sets}(-, Y) &: \text{Sets}^{\text{op}} \rightarrow \text{Sets}, \\ \text{Sets}(-_1, -_2) &: \text{Sets}^{\text{op}} \times \text{Sets} \rightarrow \text{Sets}. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (A \times - \dashv \text{Sets}(A, -)) &: \text{Sets} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets}, \\ (- \times B \dashv \text{Sets}(B, -)) &: \text{Sets} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets}, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)), \end{aligned}$$

natural in  $A, B, C \in \text{Obj}(\text{Sets})$ .

3. *Maps From the Punctual Set.* We have a bijection

$$\text{Sets}(\text{pt}, A) \cong A,$$

natural in  $A \in \text{Obj}(\text{Sets})$ .

4. *Maps to the Punctual Set.* We have a bijection

$$\text{Sets}(A, \text{pt}) \cong \text{pt},$$

natural in  $A \in \text{Obj}(\text{Sets})$ .

*Proof. Item 1, Functoriality:* This follows from **Categories, Items 2 and 5** of **Definition 11.1.4.1.2**.

<sup>11</sup>*Further Terminology:* Also called the **Hom set from  $A$  to  $B$** .

<sup>12</sup>*Further Notation:* Also written  $\text{Hom}_{\text{Sets}}(A, B)$ .

*Item 2, Adjointness:* This is a repetition of *Item 2* of *Definition 4.1.3.1.3* and is proved there.

*Item 3, Maps From the Punctual Set:* The bijection

$$\Phi_A : \text{Sets}(\text{pt}, A) \xrightarrow{\sim} A$$

is given by

$$\Phi_A(f) \stackrel{\text{def}}{=} f(\star)$$

for each  $f \in \text{Sets}(\text{pt}, A)$ , admitting an inverse

$$\Phi_A^{-1} : A \xrightarrow{\sim} \text{Sets}(\text{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \stackrel{\text{def}}{=} [\star \mapsto a]$$

for each  $a \in A$ . Indeed, we have

$$\begin{aligned} [\Phi_A^{-1} \circ \Phi_A](f) &\stackrel{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\ &\stackrel{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\ &\stackrel{\text{def}}{=} [\star \mapsto f(\star)] \\ &\stackrel{\text{def}}{=} f \\ &\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(\text{pt}, A)}](f) \end{aligned}$$

for each  $f \in \text{Sets}(\text{pt}, A)$  and

$$\begin{aligned} [\Phi_A \circ \Phi_A^{-1}](a) &\stackrel{\text{def}}{=} \Phi_A(\Phi_A^{-1}(a)) \\ &\stackrel{\text{def}}{=} \Phi_A([\star \mapsto a]) \\ &\stackrel{\text{def}}{=} \text{ev}_\star([\star \mapsto a]) \\ &\stackrel{\text{def}}{=} a \\ &\stackrel{\text{def}}{=} [\text{id}_A](a) \end{aligned}$$

for each  $a \in A$ , and thus we have

$$\begin{aligned} \Phi_A^{-1} \circ \Phi_A &= \text{id}_{\text{Sets}(\text{pt}, A)} \\ \Phi_A \circ \Phi_A^{-1} &= \text{id}_A. \end{aligned}$$

To prove naturality, we need to show that the diagram

$$\begin{array}{ccc} \text{Sets}(\text{pt}, A) & \xrightarrow{f_!} & \text{Sets}(\text{pt}, B) \\ \Phi_A \downarrow & & \downarrow \Phi_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes. Indeed, we have

$$\begin{aligned}
 [f \circ \Phi_A](\phi) &\stackrel{\text{def}}{=} f(\Phi_A(\phi)) \\
 &\stackrel{\text{def}}{=} f(\phi(\star)) \\
 &\stackrel{\text{def}}{=} [f \circ \phi](\star) \\
 &\stackrel{\text{def}}{=} \Phi_B(f \circ \phi) \\
 &\stackrel{\text{def}}{=} \Phi_B(f_!(\phi)) \\
 &\stackrel{\text{def}}{=} [\Phi_B \circ f_!](\phi)
 \end{aligned}$$

for each  $\phi \in \text{Sets}(\text{pt}, A)$ . This finishes the proof.

**Item 4, Maps to the Punctual Set:** This follows from the universal property of  $\text{pt}$  as the terminal set, **Definition 4.1.1.1.1**.  $\square$

### 4.3.6 Unions of Families of Subsets

Let  $X$  be a set and let  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

**Definition 4.3.6.1.1.** The **union of  $\mathcal{U}$**  is the set  $\bigcup_{U \in \mathcal{U}} U$  defined by

$$\bigcup_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}.$$

**Proposition 4.3.6.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignment  $\mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$  defines a functor

$$\bigcup: (\mathcal{P}(\mathcal{P}(X)), \subset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ , the following condition is satisfied:

$$(\star) \text{ If } \mathcal{U} \subset \mathcal{V}, \text{ then } \bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

2. *Associativity.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \bigcup} & \mathcal{P}(\mathcal{P}(X)) \\
 \bigcup \star \text{id}_{\mathcal{P}(X)} \downarrow & & \downarrow \bigcup \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\bigcup_{A \in \mathcal{A}} \bigcup_{U \in A} U = \bigcup_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right)$$

for each  $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ .

3. *Left Unitality*. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \downarrow \chi_{\mathcal{P}(X)} & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \{U\}} V = U$$

for each  $U \in \mathcal{P}(X)$ .

4. *Right Unitality*. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \downarrow \mathcal{P}(\chi_X) & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{\{u\} \in \chi_X(U)} \{u\} = U$$

for each  $U \in \mathcal{P}(X)$ .

5. *Interaction With Unions I*. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \bigcup \times \bigcup & & \downarrow \bigcup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\bigcup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W = \left( \bigcup_{U \in \mathcal{U}} U \right) \cup \left( \bigcup_{V \in \mathcal{V}} V \right)$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

6. *Interaction With Unions II.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{U \cup -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{- \cup V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \cup \left( \bigcup_{V \in \mathcal{V}} V \right) &= \bigcup_{V \in \mathcal{V}} (U \cup V), \\ \left( \bigcup_{U \in \mathcal{U}} U \right) \cup V &= \bigcup_{U \in \mathcal{U}} (U \cup V) \end{aligned}$$

for each nonempty  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$  and each  $U, V \in \mathcal{P}(X)$ .

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup \times \cup & \wr & \downarrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X), \end{array}$$

with components

$$\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \subset \left( \bigcup_{U \in \mathcal{U}} U \right) \cap \left( \bigcup_{V \in \mathcal{V}} V \right)$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

8. *Interaction With Intersections II.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$U \cap \left( \bigcup_{V \in \mathcal{V}} V \right) = \bigcup_{V \in \mathcal{V}} (U \cap V),$$

$$\left( \bigcup_{U \in \mathcal{U}} U \right) \cap V = \bigcup_{U \in \mathcal{U}} (U \cap V)$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$  and each  $U, V \in \mathcal{P}(X)$ .

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\setminus} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup \times \cup & \text{X} & \downarrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\setminus} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left( \bigcup_{U \in \mathcal{U}} U \right) \setminus \left( \bigcup_{V \in \mathcal{V}} V \right)$$

in general, where  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup^{\text{op}} & \text{X} & \downarrow \cup \\ \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{U \in \mathcal{U}^c} U \neq \bigcup_{U \in \mathcal{U}} U^c$$

in general, where  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

11. *Interaction With Complements II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \searrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cup & & \downarrow \cap^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left( \bigcup_{U \in \mathcal{U}} U \right)^c = \bigcap_{U \in \mathcal{U}} U^c$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

12. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \searrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cap & & \downarrow \cup^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left( \bigcap_{U \in \mathcal{U}} U \right)^c = \bigcup_{U \in \mathcal{U}} U^c$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup \times \cup & \text{X} & \downarrow \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W \neq \left( \bigcup_{U \in \mathcal{U}} U \right) \Delta \left( \bigcup_{V \in \mathcal{V}} V \right)$$

in general, where  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

14. *Interaction With Internal Homs I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \cup^{\text{op}} \times \cup^{\text{op}} & \text{X} & \downarrow \cup \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[ \bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

15. *Interaction With Internal Homs II.* The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\ & \nearrow \text{dashed} & & \searrow \cup^{\text{op}} & \\ \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & & & \mathcal{P}(X)^{\text{op}} \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & & & \downarrow [-, V]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & & & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[ \bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$  and each  $V \in \mathcal{P}(X)$ .



16. *Interaction With Internal Homs III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \\ \text{id}_{\mathcal{P}(X)} \star [U, -]_X \downarrow & & \downarrow [U, -]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[ U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each  $U \in \mathcal{P}(X)$  and each  $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

17. *Interaction With Direct Images.* Let  $f: X \rightarrow Y$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ , where  $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$ .

18. *Interaction With Inverse Images.* Let  $f: X \rightarrow Y$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{V})} U$$

for each  $\mathcal{V} \in \mathcal{P}(\mathcal{P}(Y))$ , where  $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$ .

19. *Interaction With Codirect Images.* Let  $f: X \rightarrow Y$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(X)$ , where  $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$ .

20. *Interaction With Intersections of Families I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(x)) \\ \downarrow \cup \star \text{id}_{\mathcal{P}(X)} & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{\cap} & X \end{array}$$

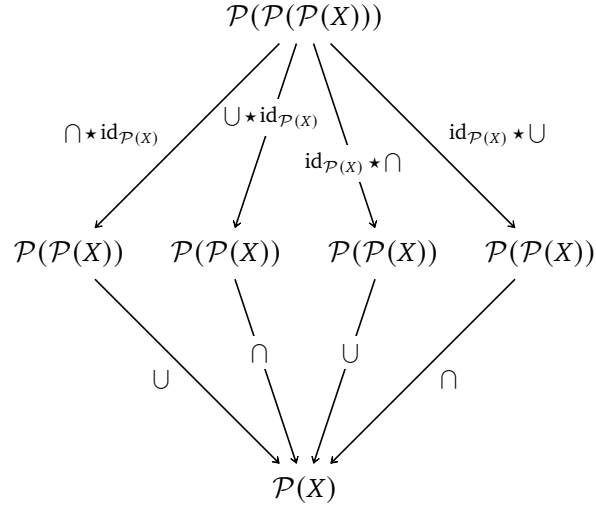
commutes, i.e. we have

$$\bigcap_{\substack{U \in \\ A \in \mathcal{A}}} U = \bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right)$$

for each  $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$ .

21. *Interaction With Intersections of Families II.* Let  $X$  be a set and consider the

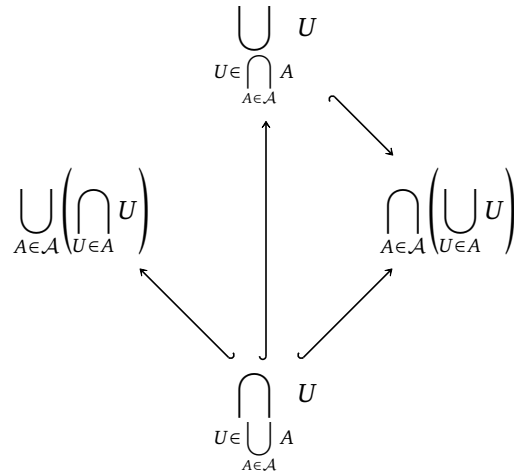
compositions



given by

$$\begin{aligned} \mathcal{A} &\mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, & \mathcal{A} &\mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U, \\ \mathcal{A} &\mapsto \bigcup_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right), & \mathcal{A} &\mapsto \bigcap_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right) \end{aligned}$$

for each  $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ . We have the following inclusions:



All other possible inclusions fail to hold in general.

*Proof. Item 1, Functoriality:* Since  $\mathcal{P}(X)$  is posetal, it suffices to prove the condition  $(\star)$ . So let  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$  with  $\mathcal{U} \subset \mathcal{V}$ . We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given  $x \in \bigcup_{U \in \mathcal{U}} U$ , there exists some  $U \in \mathcal{U}$  such that  $x \in U$ , but since  $\mathcal{U} \subset \mathcal{V}$ , we have  $U \in \mathcal{V}$  as well, and thus  $x \in \bigcup_{V \in \mathcal{V}} V$ , which gives our desired inclusion.

*Item 2, Associativity:* We have

$$\begin{aligned} \bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right\} \\ &\stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right). \end{aligned}$$

This finishes the proof.

*Item 3, Left Unitality:* We have

$$\begin{aligned} \bigcup_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in V \end{array} \right\} \\ &= \{x \in X \mid x \in U\} \\ &= U. \end{aligned}$$

This finishes the proof.

*Item 4, Right Unitality:* We have

$$\begin{aligned} \bigcup_{\{u\} \in \chi_X(U)} \{u\} &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

*Item 5, Interaction With Unions I:* We have

$$\begin{aligned}
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \\ \text{such that we have } x \in W \end{array} \right\} \\
&\quad \cup \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left( \bigcup_{W \in \mathcal{U}} W \right) \cup \left( \bigcup_{W \in \mathcal{V}} W \right) \\
&= \left( \bigcup_{U \in \mathcal{U}} U \right) \cup \left( \bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}$$

This finishes the proof.

*Item 6, Interaction With Unions II:* Assume  $\mathcal{V}$  is nonempty. We have

$$\begin{aligned}
U \cup \bigcup_{V \in \mathcal{V}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid x \in U \text{ or } x \in \bigcup_{V \in \mathcal{V}} V \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} x \in U \text{ or there exists some} \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ or } x \in V \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cup V \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cup V.
\end{aligned}$$

This concludes the proof of the first statement. For the second statement, use **Item 4** of **Definition 4.3.8.1.2** to rewrite

$$\begin{aligned} \left( \bigcup_{U \in \mathcal{U}} U \right) \cup V &= V \cup \left( \bigcup_{U \in \mathcal{U}} U \right), \\ \bigcup_{U \in \mathcal{U}} (U \cup V) &= \bigcup_{U \in \mathcal{U}} (V \cup U). \end{aligned}$$

But these two sets are equal by the first statement.

**Item 7, Interaction With Intersections I:** We have

$$\begin{aligned} \bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cap \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\ &\subset \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \text{ and some } V \in \mathcal{V} \\ \text{such that we have } x \in U \text{ and } x \in V \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\} \\ &\quad \cup \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that we have } x \in V \end{array} \right\} \\ &\stackrel{\text{def}}{=} \left( \bigcup_{U \in \mathcal{U}} U \right) \cap \left( \bigcup_{V \in \mathcal{V}} V \right). \end{aligned}$$

This finishes the proof.

**Item 8, Interaction With Intersections II:** We have

$$\begin{aligned} U \cap \bigcup_{V \in \mathcal{V}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid x \in U \text{ and } x \in \bigcup_{V \in \mathcal{V}} V \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} x \in U \text{ and there exists some} \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ and } x \in V \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cap V \end{array} \right\} \\ &\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cap V. \end{aligned}$$

This concludes the proof of the first statement. For the second statement, use **Item 5** of **Definition 4.3.9.1.2** to rewrite

$$\begin{aligned} \left( \bigcup_{U \in \mathcal{U}} U \right) \cap V &= V \cap \left( \bigcup_{U \in \mathcal{U}} U \right), \\ \bigcup_{U \in \mathcal{U}} (U \cap V) &= \bigcup_{U \in \mathcal{U}} (V \cap U). \end{aligned}$$

But these two sets are equal by the first statement.

**Item 9, Interaction With Differences:** Let  $X = \{0, 1\}$ , let  $\mathcal{U} = \{\{0, 1\}\}$ , and let  $\mathcal{V} = \{\{0\}\}$ . We have

$$\begin{aligned} \bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W &= \bigcup_{W \in \{\{0, 1\}\}} W \\ &= \{0, 1\}, \end{aligned}$$

whereas

$$\begin{aligned} \left( \bigcup_{U \in \mathcal{U}} U \right) \setminus \left( \bigcup_{V \in \mathcal{V}} V \right) &= \{0, 1\} \setminus \{0\} \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \{1\} = \left( \bigcup_{U \in \mathcal{U}} U \right) \setminus \left( \bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

**Item 10, Interaction With Complements I:** Let  $X = \{0, 1\}$  and let  $\mathcal{U} = \{0\}$ . We have

$$\begin{aligned} \bigcup_{U \in \mathcal{U}^c} U &= \bigcup_{U \in \{\emptyset, \{1\}, \{0, 1\}\}} U \\ &= \{0, 1\}, \end{aligned}$$

whereas

$$\begin{aligned} \bigcup_{U \in \mathcal{U}} U^c &= \{0\}^c \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcup_{U \in \mathcal{U}^c} U = \{0, 1\} \neq \{1\} = \bigcup_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

*Item 11, Interaction With Complements II:* We have

$$\begin{aligned}
 \left( \bigcup_{U \in \mathcal{U}} U \right)^c &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists no } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{for all } U \in \mathcal{U} \\ \text{we have } x \notin U \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for all } U \in \mathcal{U} \\ \text{we have } x \in U^c \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \bigcap_{U \in \mathcal{U}} U^c.
 \end{aligned}$$

*Item 12, Interaction With Complements III:* By *Item 11 Item 3* of *Definition 4.3.11.1.2*, we have

$$\begin{aligned}
 \left( \bigcap_{U \in \mathcal{U}} U \right)^c &= \left( \bigcap_{U \in \mathcal{U}} (U^c)^c \right)^c \\
 &= \left( \left( \bigcup_{U \in \mathcal{U}} U^c \right)^c \right)^c \\
 &= \bigcup_{U \in \mathcal{U}} U^c.
 \end{aligned}$$

*Item 13, Interaction With Symmetric Differences:* Let  $X = \{0, 1\}$ , let  $\mathcal{U} = \{\{0, 1\}\}$ , and let  $\mathcal{V} = \{\{0\}, \{0, 1\}\}$ . We have

$$\begin{aligned}
 \bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W &= \bigcup_{W \in \{\{0\}\}} W \\
 &= \{0\},
 \end{aligned}$$

whereas

$$\begin{aligned}
 \left( \bigcup_{U \in \mathcal{U}} U \right) \Delta \left( \bigcup_{V \in \mathcal{V}} V \right) &= \{0, 1\} \Delta \{0, 1\} \\
 &= \emptyset,
 \end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W = \{0\} \neq \emptyset = \left( \bigcup_{U \in \mathcal{U}} U \right) \Delta \left( \bigcup_{V \in \mathcal{V}} V \right).$$



This finishes the proof.

*Item 14, Interaction With Internal Homs I:* This is a repetition of **Item 7** of **Definition 4.4.7.1.3** and is proved there.

*Item 15, Interaction With Internal Homs II:* This is a repetition of **Item 8** of **Definition 4.4.7.1.3** and is proved there.

*Item 16, Interaction With Internal Homs III:* This is a repetition of **Item 9** of **Definition 4.4.7.1.3** and is proved there.

*Item 17, Interaction With Direct Images:* This is a repetition of **Item 3** of **Definition 4.6.1.1.5** and is proved there.

*Item 18, Interaction With Inverse Images:* This is a repetition of **Item 3** of **Definition 4.6.2.1.3** and is proved there.

*Item 19, Interaction With Codirect Images:* This is a repetition of **Item 3** of **Definition 4.6.3.1.7** and is proved there.

*Item 20, Interaction With Intersections of Families I:* We have

$$\begin{aligned} \bigcap_{A \in \mathcal{A}} U &\stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right. \right\} \\ &= \left\{ x \in X \left| \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right. \right\} \\ &\stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right). \end{aligned}$$

This finishes the proof.

*Item 21, Interaction With Intersections of Families II:* Omitted. □

### 4.3.7 Intersections of Families of Subsets

Let  $X$  be a set and let  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

**Definition 4.3.7.1.1.** The **intersection of  $\mathcal{U}$**  is the set  $\bigcap_{U \in \mathcal{U}} U$  defined by

$$\bigcap_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \left| \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right. \right\}.$$

**Proposition 4.3.7.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignment  $\mathcal{U} \mapsto \bigcap_{U \in \mathcal{U}} U$  defines a functor

$$\bigcap: (\mathcal{P}(\mathcal{P}(X)), \supset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ , the following condition is satisfied:

(★) If  $\mathcal{U} \subset \mathcal{V}$ , then  $\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U$ .

2. *Oplax Associativity.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \star \text{id}_{\mathcal{P}(X)} \downarrow & \wr & \downarrow \cap \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

with components

$$\bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right) \subset \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

for each  $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ .

3. *Left Unitality.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \chi_{\mathcal{P}(X)} \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \{U\}} V = U.$$

for each  $U \in \mathcal{P}(X)$ .

4. *Oplax Right Unitality.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & & \\ \mathcal{P}(\chi_X) \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

✗

does not commute in general, i.e. we may have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \neq U$$

in general, where  $U \in \mathcal{P}(X)$ . However, when  $U$  is nonempty, we have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \subset U.$$

5. *Interaction With Unions I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W = \left( \bigcap_{U \in \mathcal{U}} U \right) \cap \left( \bigcap_{V \in \mathcal{V}} V \right)$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

6. *Interaction With Unions II.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{U \cup -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{- \cup V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \cup \left( \bigcap_{V \in \mathcal{V}} V \right) &= \bigcap_{V \in \mathcal{V}} (U \cup V), \\ \left( \bigcap_{U \in \mathcal{U}} U \right) \cup V &= \bigcap_{U \in \mathcal{U}} (U \cup V) \end{aligned}$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$  and each  $U, V \in \mathcal{P}(X)$ .

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \wr & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X), \end{array}$$

with components

$$\left( \bigcap_{U \in \mathcal{U}} U \right) \cap \left( \bigcap_{V \in \mathcal{V}} V \right) \subset \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

8. *Interaction With Intersections II.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \cup \left( \bigcap_{V \in \mathcal{V}} V \right) &= \bigcap_{V \in \mathcal{V}} (U \cup V), \\ \left( \bigcap_{U \in \mathcal{U}} U \right) \cup V &= \bigcap_{U \in \mathcal{U}} (U \cup V) \end{aligned}$$

for each  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$  and each  $U, V \in \mathcal{P}(X)$ .

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\setminus} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\setminus} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left( \bigcap_{U \in \mathcal{U}} U \right) \setminus \left( \bigcap_{V \in \mathcal{V}} V \right)$$

in general, where  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\
 \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U}^c} W \neq \bigcap_{U \in \mathcal{U}} U^c$$

in general, where  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

11. *Interaction With Complements II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \searrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \cap \searrow & & \swarrow \cup^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left( \bigcap_{U \in \mathcal{U}} U \right)^c = \bigcup_{U \in \mathcal{U}} U^c$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

12. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \searrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \cup \searrow & & \swarrow \cap^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left( \bigcup_{U \in \mathcal{U}} U \right)^c = \bigcap_{U \in \mathcal{U}} U^c$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W \neq \left( \bigcap_{U \in \mathcal{U}} U \right) \Delta \left( \bigcap_{V \in \mathcal{V}} V \right)$$

in general, where  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

14. *Interaction With Internal Homs I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap^{\text{op}} \times \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[ \bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

15. *Interaction With Internal Homs II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\
 \swarrow \sim & & \searrow \cap^{\text{op}} \\
 \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & \mathcal{P}(X)^{\text{op}} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & \downarrow [-, V]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[ \bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$  and each  $V \in \mathcal{P}(X)$ .

16. *Interaction With Internal Homs III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[ U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each  $U \in \mathcal{P}(X)$  and each  $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

17. *Interaction With Direct Images.* Let  $f: X \rightarrow Y$  be a map of sets. The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)^!} & \mathcal{P}(\mathcal{P}(Y)) \\
 \downarrow \cap & & \downarrow \cap \\
 \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
 \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(X)$ , where  $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$ .

18. *Interaction With Inverse Images.* Let  $f: X \rightarrow Y$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each  $\mathcal{V} \in \mathcal{P}(Y)$ , where  $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$ .

19. *Interaction With Codirect Images.* Let  $f: X \rightarrow Y$  be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(X)$ , where  $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$ .

20. *Interaction With Unions of Families I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(x)) \\ \cup \star \text{id}_{\mathcal{P}(X)} \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{\cap} & X \end{array}$$

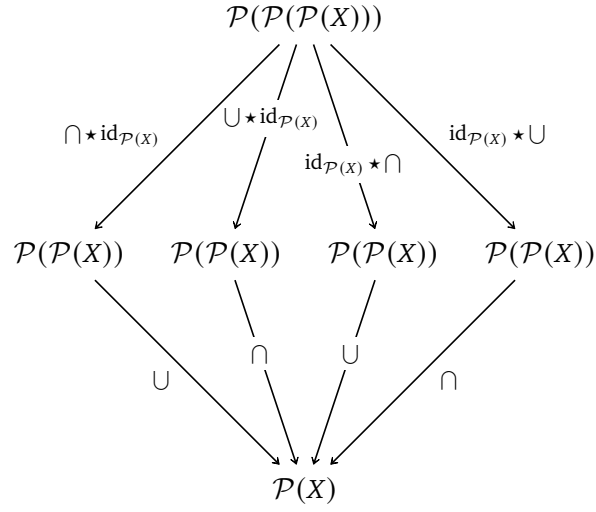


commutes, i.e. we have

$$\bigcap_{A \in \mathcal{A}} \bigcup_{U \in A} U = \bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right)$$

for each  $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$ .

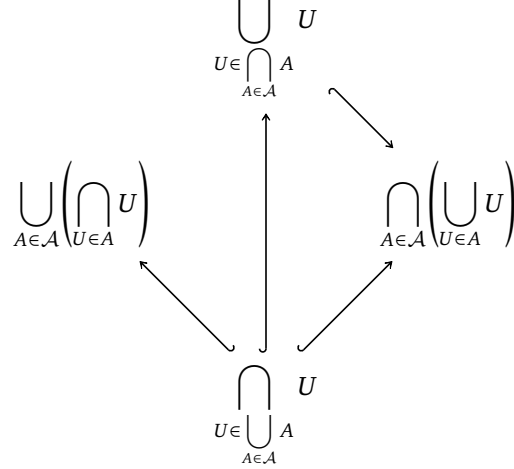
21. *Interaction With Unions of Families II.* Let  $X$  be a set and consider the compositions



given by

$$\begin{aligned} \mathcal{A} &\mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, & \mathcal{A} &\mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U, \\ \mathcal{A} &\mapsto \bigcup_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right), & \mathcal{A} &\mapsto \bigcap_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right) \end{aligned}$$

for each  $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ . We have the following inclusions:



All other possible inclusions fail to hold in general.

*Proof. Item 1, Functoriality:* Since  $\mathcal{P}(X)$  is posetal, it suffices to prove the condition  $(\star)$ . So let  $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$  with  $\mathcal{U} \subset \mathcal{V}$ . We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if  $x \in \bigcap_{V \in \mathcal{V}} V$ , then  $x \in V$  for all  $V \in \mathcal{V}$ . But since  $\mathcal{U} \subset \mathcal{V}$ , it follows that  $x \in U$  for all  $U \in \mathcal{U}$  as well. Thus  $x \in \bigcap_{U \in \mathcal{U}} U$ , which gives our desired inclusion.

*Item 2, Oplax Associativity:* We have

$$\begin{aligned}
 \bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right) &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \bigcap_{U \in A} U \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
 &\subset \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.
 \end{aligned}$$

Since  $\mathcal{P}(X)$  is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

*Item 3, Left Unitality:* We have

$$\begin{aligned} \bigcap_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in V \end{array} \right\} \\ &= \{x \in X \mid x \in U\} \\ &= U. \end{aligned}$$

This finishes the proof.

*Item 4, Oplax Right Unitality:* If  $U = \emptyset$ , then we have

$$\begin{aligned} \bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \bigcap_{\{u\} \in \emptyset} \{u\} \\ &= X, \end{aligned}$$

so  $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$ . When  $U$  is nonempty, we have two cases:

1. If  $U$  is a singleton, say  $U = \{u\}$ , we have

$$\begin{aligned} \bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \{u\} \\ &\stackrel{\text{def}}{=} U. \end{aligned}$$

2. If  $U$  contains at least two elements, we have

$$\begin{aligned} \bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \emptyset \\ &\subset U. \end{aligned}$$

This finishes the proof.

*Item 5, Interaction With Unions I:* We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \text{ and each} \\ W \in \mathcal{V}, \text{ we have } x \in W \end{array} \right\} \\ &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U}, \\ \text{we have } x \in W \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
& \cap \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
& \stackrel{\text{def}}{=} \left( \bigcap_{W \in \mathcal{U}} W \right) \cap \left( \bigcap_{W \in \mathcal{V}} W \right) \\
& = \left( \bigcap_{U \in \mathcal{U}} U \right) \cap \left( \bigcap_{V \in \mathcal{V}} V \right).
\end{aligned}$$

This finishes the proof.

*Item 6, Interaction With Unions II:* Omitted.

*Item 7, Interaction With Intersections I:* We have

$$\begin{aligned}
\left( \bigcap_{U \in \mathcal{U}} U \right) \cap \left( \bigcap_{V \in \mathcal{V}} V \right) & \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\} \\
& \cup \left\{ x \in X \mid \begin{array}{l} \text{for each } V \in \mathcal{V}, \\ \text{we have } x \in V \end{array} \right\} \\
& = \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \cap \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
& \subset \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
& \stackrel{\text{def}}{=} \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W.
\end{aligned}$$

Since  $\mathcal{P}(X)$  is posetal, naturality is automatic ([Categories](#), [Item 4 of Definition 11.2.7.1.2](#)). This finishes the proof.

*Item 8, Interaction With Intersections II:* Omitted.

*Item 9, Interaction With Differences:* Let  $X = \{0, 1\}$ , let  $\mathcal{U} = \{\{0\}, \{0, 1\}\}$ , and let  $\mathcal{V} = \{\{0\}\}$ . We have

$$\begin{aligned}
\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W & = \bigcap_{W \in \{\{0, 1\}\}} W \\
& = \{0, 1\},
\end{aligned}$$

whereas

$$\begin{aligned}
\left( \bigcap_{U \in \mathcal{U}} U \right) \setminus \left( \bigcap_{V \in \mathcal{V}} V \right) & = \{0\} \setminus \{0\} \\
& = \emptyset.
\end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \emptyset = \left( \bigcap_{U \in \mathcal{U}} U \right) \setminus \left( \bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

**Item 10, Interaction With Complements I:** Let  $X = \{0, 1\}$  and let  $\mathcal{U} = \{\{0\}\}$ . We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U}^c} U &= \bigcap_{W \in \{\emptyset, \{1\}, \{0, 1\}\}} W \\ &= \emptyset, \end{aligned}$$

whereas

$$\begin{aligned} \bigcap_{U \in \mathcal{U}} U^c &= \{0\}^c \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U}^c} U = \emptyset \neq \{1\} = \bigcap_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

**Item 11, Interaction With Complements II:** This is a repetition of **Item 12** of **Definition 4.3.6.1.2** and is proved there.

**Item 12, Interaction With Complements III:** This is a repetition of **Item 11** of **Definition 4.3.6.1.2** and is proved there.

**Item 13, Interaction With Symmetric Differences:** Let  $X = \{0, 1\}$ , let  $\mathcal{U} = \{\{0, 1\}\}$ , and let  $\mathcal{V} = \{\{0\}, \{0, 1\}\}$ . We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W &= \bigcap_{W \in \{\{0\}\}} W \\ &= \{0\}, \end{aligned}$$

whereas

$$\begin{aligned} \left( \bigcap_{U \in \mathcal{U}} U \right) \Delta \left( \bigcap_{V \in \mathcal{V}} V \right) &= \{0, 1\} \Delta \{0\} \\ &= \emptyset, \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W = \{0\} \neq \emptyset = \left( \bigcap_{U \in \mathcal{U}} U \right) \Delta \left( \bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

*Item 14, Interaction With Internal Homs I:* This is a repetition of **Item 10** of **Definition 4.4.7.1.3** and is proved there.

*Item 15, Interaction With Internal Homs II:* This is a repetition of **Item 11** of **Definition 4.4.7.1.3** and is proved there.

*Item 16, Interaction With Internal Homs III:* This is a repetition of **Item 12** of **Definition 4.4.7.1.3** and is proved there.

*Item 17, Interaction With Direct Images:* This is a repetition of **Item 4** of **Definition 4.6.1.1.5** and is proved there.

*Item 18, Interaction With Inverse Images:* This is a repetition of **Item 4** of **Definition 4.6.2.1.3** and is proved there.

*Item 19, Interaction With Codirect Images:* This is a repetition of **Item 4** of **Definition 4.6.3.1.7** and is proved there.

*Item 20, Interaction With Unions of Families I:* This is a repetition of **Item 20** of **Definition 4.3.6.1.2** and is proved there.

*Item 21, Interaction With Unions of Families II:* This is a repetition of **Item 21** of **Definition 4.3.6.1.2** and is proved there.  $\square$

### 4.3.8 Binary Unions

Let  $X$  be a set and let  $U, V \in \mathcal{P}(X)$ .

**Definition 4.3.8.1.1.** The **union of  $U$  and  $V$**  is the set  $U \cup V$  defined by

$$\begin{aligned} U \cup V &\stackrel{\text{def}}{=} \bigcup_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

**Proposition 4.3.8.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignments  $U, V, (U, V) \mapsto U \cup V$  define functors

$$\begin{aligned} U \cup -: & (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ - \cup V: & (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cup -_2: & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each  $U, V, A, B \in \mathcal{P}(X)$ :

- (a) If  $U \subset A$ , then  $U \cup V \subset A \cup V$ .
- (b) If  $V \subset B$ , then  $U \cup V \subset U \cup B$ .
- (c) If  $U \subset A$  and  $V \subset B$ , then  $U \cup V \subset A \cup B$ .

2. *Associativity.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \nearrow \sim & & \searrow \text{id}_{\mathcal{P}(X)} \times \cup \\
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \downarrow \cup \times \text{id}_{\mathcal{P}(X)} & & \downarrow \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\quad \cup \quad} & \mathcal{P}(X),
 \end{array}$$

commutes, i.e. we have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

3. *Unitality.* The diagrams

$$\begin{array}{ccc}
 \text{pt} \times \mathcal{P}(X) & \xrightarrow{[\emptyset] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \sim \lambda_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [\emptyset]} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \sim \rho_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$\emptyset \cup U = U,$$

$$U \cup \emptyset = U$$

for each  $U \in \mathcal{P}(X)$ .

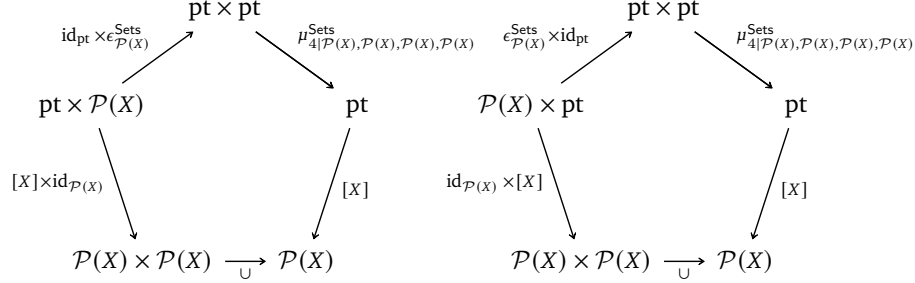
4. *Commutativity.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \cup & & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup V = V \cup U$$

for each  $U, V \in \mathcal{P}(X)$ .

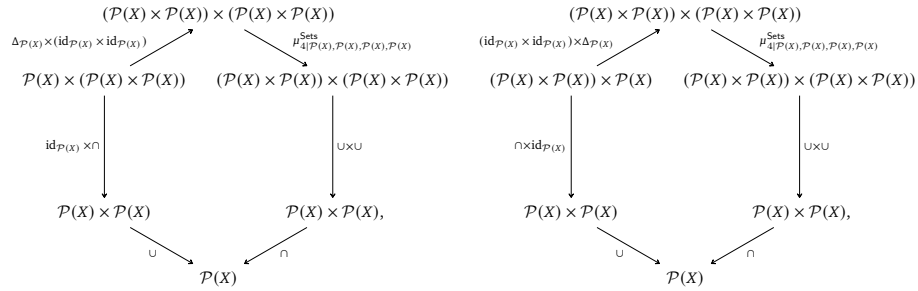
5. *Annihilation With X.* The diagrams

commute, i.e. we have equalities of sets

$$U \cup X = X,$$

$$X \cup V = X$$

for each  $U, V \in \mathcal{P}(X)$ .

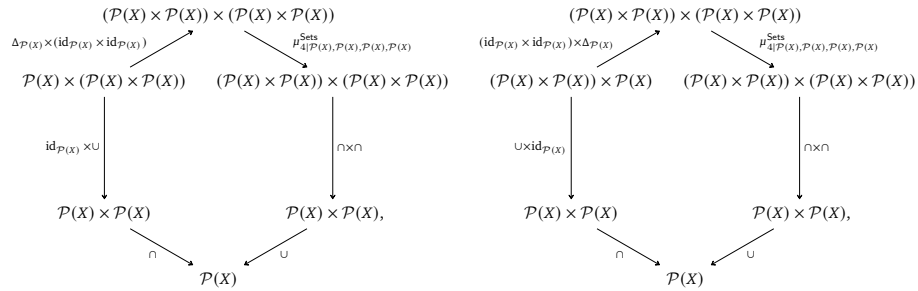
6. *Distributivity of Unions Over Intersections.* The diagrams

commute, i.e. we have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$

$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

7. *Distributivity of Intersections Over Unions.* The diagrams



commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

8. *Idempotency.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\Delta_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)} & \downarrow \cup \\ & & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each  $U \in \mathcal{P}(X)$ .

9. *Via Intersections and Symmetric Differences.* The diagram

$$\begin{array}{ccc} & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \xrightarrow{\Delta \times \cap} \mathcal{P}(X) \times \mathcal{P}(X) \\ & \nearrow \Delta_{\mathcal{P}(X) \times \mathcal{P}(X)} & \searrow \Delta \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup V = (U \Delta V) \Delta (U \cap V)$$

for each  $U, V \in \mathcal{P}(X)$ .

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each  $U, V \in \mathcal{P}(X)$ .

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each  $U, V \in \mathcal{P}(X)$ .

12. *Interaction With Direct Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each  $U, V \in \mathcal{P}(X)$ .

13. *Interaction With Inverse Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each  $U, V \in \mathcal{P}(Y)$ .

14. *Interaction With Codirect Images.* Let  $f: X \rightarrow Y$  be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cup & \subset & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

15. *Interaction With Powersets and Semirings.* The quintuple  $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$  is an idempotent commutative semiring.

*Proof.* **Item 1, Functoriality:** See [Pro25an].

**Item 2, Associativity:** See [Pro25ba].

**Item 3, Unitality:** This follows from [Pro25bd] and **Item 4**.

**Item 4, Commutativity:** See [Pro25bb].

**Item 5, Annihilation With  $X$ :** We have

$$\begin{aligned} U \cup X &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\ &= \{x \in X \mid x \in X\}, \\ &= X \end{aligned}$$

and

$$\begin{aligned} X \cup V &\stackrel{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\ &= \{x \in X \mid x \in X\} \\ &= X. \end{aligned}$$

This finishes the proof.

**Item 6, Distributivity of Unions Over Intersections:** See [Pro25az].

**Item 7, Distributivity of Intersections Over Unions:** See [Pro25aj].

**Item 8, Idempotency:** See [Pro25am].

**Item 9, Via Intersections and Symmetric Differences:** See [Pro25ay].

**Item 10, Interaction With Characteristic Functions I:** See [Pro25h].

**Item 11, Interaction With Characteristic Functions II:** See [Pro25h].

**Item 12, Interaction With Direct Images:** See [Pro25p].

**Item 13, Interaction With Inverse Images:** See [Pro25y].

**Item 14, Interaction With Codirect Images:** This is a repetition of **Item 5** of **Definition 4.6.3.1.7** and is proved there.

**Item 15, Interaction With Powersets and Semirings:** This follows from **Items 2** to **4** and **8** of this proposition and **Items 3** to **6** and **8** of **Definition 4.3.9.1.2**.  $\square$

### 4.3.9 Binary Intersections

Let  $X$  be a set and let  $U, V \in \mathcal{P}(X)$ .

**Definition 4.3.9.1.1.** The **intersection of  $U$  and  $V$**  is the set  $U \cap V$  defined by

$$\begin{aligned} U \cap V &\stackrel{\text{def}}{=} \bigcap_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

**Proposition 4.3.9.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignments  $U, V, (U, V) \mapsto U \cap V$  define functors

$$\begin{aligned} U \cap -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \cap V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cap -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each  $U, V, A, B \in \mathcal{P}(X)$ :

- (a) If  $U \subset A$ , then  $U \cap V \subset A \cap V$ .
- (b) If  $V \subset B$ , then  $U \cap V \subset U \cap B$ .
- (c) If  $U \subset A$  and  $V \subset B$ , then  $U \cap V \subset A \cap B$ .

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (U \cap - \dashv [U, -]_X): \mathcal{P}(X) &\overset{U \cap -}{\underset{[U, -]_X}{\rightleftarrows}} \mathcal{P}(X), \\ (- \cap V \dashv [V, -]_X): \mathcal{P}(X) &\overset{- \cap V}{\underset{[V, -]_X}{\rightleftarrows}} \mathcal{P}(X), \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(V, [U, W]_X), \end{aligned}$$

natural in  $U, V, W \in \mathcal{P}(X)$ , where

$$[-_1, -_2]_X: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is the bifunctor of [Section 4.4.7](#). In particular, the following statements hold for each  $U, V, W \in \mathcal{P}(X)$ :

- (a) The following conditions are equivalent:
  - i. We have  $U \cap V \subset W$ .
  - ii. We have  $U \subset [V, W]_X$ .
- (b) The following conditions are equivalent:
  - i. We have  $U \cap V \subset W$ .
  - ii. We have  $V \subset [U, W]_X$ .

3. *Associativity.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \nearrow & & \searrow \text{id}_{\mathcal{P}(X)} \times \cap \\
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \downarrow \cap \times \text{id}_{\mathcal{P}(X)} & & \downarrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

4. *Unitality.* The diagrams

$$\begin{array}{ccc}
 \text{pt} \times \mathcal{P}(X) & \xrightarrow{[X] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \lambda_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [X]} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \rho_{\mathcal{P}(X)}^{\text{Sets}} & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$X \cap U = U,$$

$$U \cap X = U$$

for each  $U \in \mathcal{P}(X)$ .

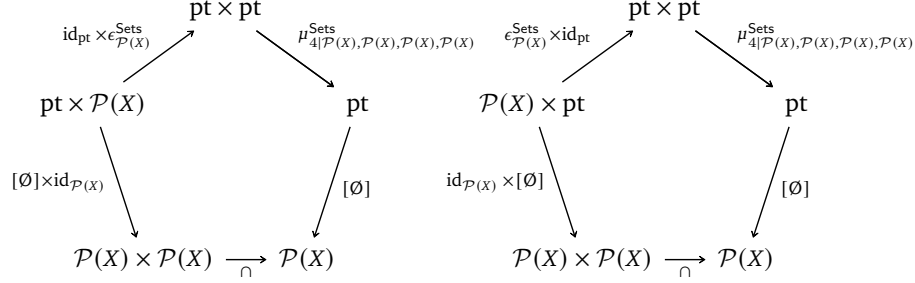
5. *Commutativity.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \searrow \cap & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each  $U, V \in \mathcal{P}(X)$ .

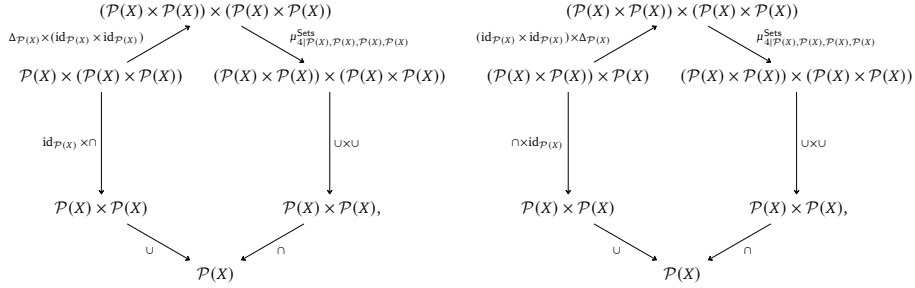
6. *Annihilation With the Empty Set.* The diagrams

commute, i.e. we have equalities of sets

$$\emptyset \cap X = \emptyset,$$

$$X \cap \emptyset = \emptyset$$

for each  $U \in \mathcal{P}(X)$ .

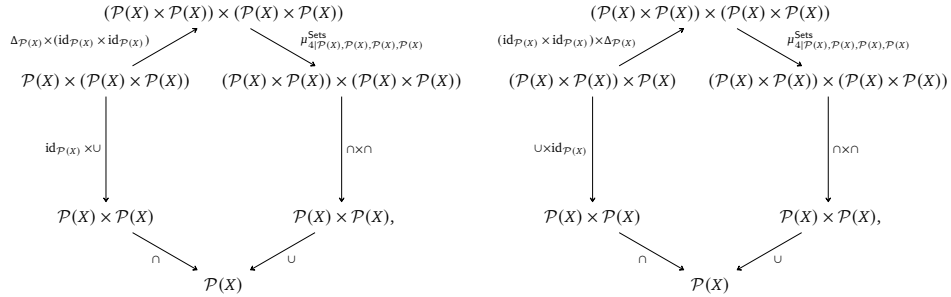
7. *Distributivity of Unions Over Intersections.* The diagrams

commute, i.e. we have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$

$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

8. *Distributivity of Intersections Over Unions.* The diagrams

commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

9. *Idempotency.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\Delta_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)} & \downarrow \cap \\ & & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each  $U \in \mathcal{P}(X)$ .

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each  $U, V \in \mathcal{P}(X)$ .

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each  $U, V \in \mathcal{P}(X)$ .

12. *Interaction With Direct Images.* Let  $f: X \rightarrow Y$  be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & \subset & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

13. *Interaction With Inverse Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each  $U, V \in \mathcal{P}(Y)$ .

14. *Interaction With Codirect Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each  $U, V \in \mathcal{P}(X)$ .

15. *Interaction With Powersets and Monoids With Zero.* The quadruple  $((\mathcal{P}(X), \emptyset), \cap, X)$  is a commutative monoid with zero.

16. *Interaction With Powersets and Semirings.* The quintuple  $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$  is an idempotent commutative semiring.

*Proof.* **Item 1, Functoriality:** See [Pro25al].

**Item 2, Adjointness:** See [MSE 267469].

**Item 3, Associativity:** See [Pro25r].

**Item 4, Unitality:** This follows from [Pro25v] and **Item 5**.

**Item 5, Commutativity:** See [Pro25s].

**Item 6, Annihilation With the Empty Set:** This follows from [Pro25t] and **Item 5**.

**Item 7, Distributivity of Unions Over Intersections:** See [Pro25az].

**Item 8, Distributivity of Intersections Over Unions:** See [Pro25aj].

**Item 9, Idempotency:** See [Pro25ak].



*Item 10, Interaction With Characteristic Functions I:* See [Pro25e].

*Item 11, Interaction With Characteristic Functions II:* See [Pro25e].

*Item 12, Interaction With Direct Images:* See [Pro25n].

*Item 13, Interaction With Inverse Images:* See [Pro25w].

*Item 14, Interaction With Codirect Images:* This is a repetition of *Item 6* of *Definition 4.6.3.1.7* and is proved there.

*Item 15, Interaction With Powersets and Monoids With Zero:* This follows from *Items 3* to *6*.

*Item 16, Interaction With Powersets and Semirings:* This follows from *Items 2* to *4* and *8* and *Items 3* to *6* and *8* of *Definition 4.3.9.1.2*.  $\square$

### 4.3.10 Differences

Let  $X$  and  $Y$  be sets.

**Definition 4.3.10.1.1.** The **difference of  $X$  and  $Y$**  is the set  $X \setminus Y$  defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

**Proposition 4.3.10.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignments  $U, V, (U, V) \mapsto U \cap V$  define functors

$$\begin{aligned} U \setminus - &: (\mathcal{P}(X), \supset) && \rightarrow (\mathcal{P}(X), \subset), \\ - \setminus V &: (\mathcal{P}(X), \subset) && \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each  $U, V, A, B \in \mathcal{P}(X)$ :

- (a) If  $U \subset A$ , then  $U \setminus V \subset A \setminus V$ .
- (b) If  $V \subset B$ , then  $U \setminus B \subset U \setminus V$ .
- (c) If  $U \subset A$  and  $V \subset B$ , then  $U \setminus B \subset A \setminus V$ .

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each  $U, V \in \mathcal{P}(X)$ .

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^c$$

for each  $U, V \in \mathcal{P}(X)$ .

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \Delta (U \cap V)$$

for each  $U, V \in \mathcal{P}(X)$ .

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each  $U \in \mathcal{P}(X)$ .

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each  $U \in \mathcal{P}(X)$ .

13. *Right Annihilation.* We have

$$U \setminus X = \emptyset$$

for each  $U \in \mathcal{P}(X)$ .

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each  $U \in \mathcal{P}(X)$ .

15. *Interaction With Containment.* The following conditions are equivalent:

- (a) We have  $V \setminus U \subset W$ .
- (b) We have  $V \setminus W \subset U$ .

16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each  $U, V \in \mathcal{P}(X)$ .

17. *Interaction With Direct Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \setminus & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

18. *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\
 \downarrow \setminus & & \downarrow \setminus \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each  $U, V \in \mathcal{P}(X)$ .

19. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{array}{ccc}
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\
 \downarrow \setminus & \supset & \downarrow \setminus \\
 \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
 \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

*Proof.* **Item 1, Functoriality:** See [Pro25ad] and [Pro25ah].

**Item 2, De Morgan's Laws:** See [Pro25k].

**Item 3, Interaction With Unions I:** See [Pro25l].

**Item 4, Interaction With Unions II:** We have

$$\begin{aligned}
 (U \setminus V) \cup W &\stackrel{\text{def}}{=} \{x \in X \mid (x \in U \text{ and } x \notin V) \text{ or } x \in W\} \\
 &= \{x \in X \mid (x \in U \text{ or } x \in W) \text{ and } (x \notin V \text{ or } x \in W)\} \\
 &= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \text{ and } x \notin W)\} \\
 &= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \setminus W)\} \\
 &= \{x \in X \mid (x \in (U \cup W) \setminus (V \setminus W))\} \\
 &= (U \cup W) \setminus (V \setminus W).
 \end{aligned}$$

**Item 5, Interaction With Unions III:** See [Pro25ai].

**Item 6, Interaction With Unions IV:** See [Pro25ac].

**Item 7, Interaction With Intersections:** See [Pro25u].

*Item 8, Interaction With Complements:* See [Pro25aa].

*Item 9, Interaction With Symmetric Differences:* See [Pro25ab].

*Item 10, Triple Differences:* See [Pro25ag].

*Item 11, Left Annihilation:* The direction  $\emptyset \subset \emptyset \setminus U$  always holds. Now assume  $x \in \emptyset \setminus U$ . Then,  $x \in \emptyset$  and  $x \notin U$ . Hence  $\emptyset \setminus U \subset \emptyset$  must hold and the sets are equal.

*Item 12, Right Unitality:* See [Pro25ae].

*Item 13, Right Annihilation:* It suffices to show that no  $x \in X$  can be an element of  $U \setminus X$ . Assume  $x \in U \setminus X$ . Then  $x \notin X$ , contradicting  $x \in X$ . This completes the proof.

*Item 14, Invertibility:* See [Pro25af].

*Item 15, Interaction With Containment:* The conditions are symmetric in  $U, W$ , hence it suffices to show that  $V \setminus U \subset W$  implies  $V \setminus W \subset U$ . So assume  $V \setminus U \subset W, x \in V \setminus W$ . Then  $x \in V, x \notin W$ . So by contraposition,  $x \notin V \setminus U$ . But  $x \in V$ , so we must have  $x \in U$ , completing the proof.

*Item 16, Interaction With Characteristic Functions:* See [Pro25f].

*Item 17, Interaction With Direct Images:* See [Pro25o].

*Item 18, Interaction With Inverse Images:* See [Pro25x]. □

### 4.3.11 Complements

Let  $X$  be a set and let  $U \in \mathcal{P}(X)$ .

**Definition 4.3.11.1.1.** The **complement** of  $U$  is the set  $U^c$  defined by

$$\begin{aligned} U^c &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

**Proposition 4.3.11.1.2.** Let  $X$  be a set.

1. *Functoriality.* The assignment  $U \mapsto U^c$  defines a functor

$$(-)^c: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X).$$

In particular, the following statements hold for each  $U, V \in \mathcal{P}(X)$ :

- (★) If  $U \subset V$ , then  $V^c \subset U^c$ .

2. *De Morgan's Laws.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cup^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array} \qquad \begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}(U \cup V)^c &= U^c \cap V^c, \\ (U \cap V)^c &= U^c \cup V^c\end{aligned}$$

for each  $U, V \in \mathcal{P}(X)$ .

3. *Involutority.* The diagram

$$\begin{array}{ccc}\mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)^{\text{op}}} & \downarrow (-)^{c, \text{op}} \\ & & \mathcal{P}(X)^{\text{op}}\end{array}$$

commutes, i.e. we have

$$(U^c)^c = U$$

for each  $U \in \mathcal{P}(X)$ .

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each  $U \in \mathcal{P}(X)$ .

5. *Interaction With Direct Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc}\mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)\end{array}$$

commutes, i.e. we have

$$f_!(U^c) = f_*(U)^c$$

for each  $U \in \mathcal{P}(X)$ .

6. *Interaction With Inverse Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc}\mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)\end{array}$$

commutes, i.e. we have

$$f^{-1}(U^c) = f^{-1}(U)^c$$

for each  $U \in \mathcal{P}(X)$ .

7. *Interaction With Codirect Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U^c) = f_!(U)^c$$

for each  $U \in \mathcal{P}(X)$ .

*Proof.* **Item 1, Functoriality:** This follows from **Item 1** of **Definition 4.3.10.1.2**.

**Item 2, De Morgan's Laws:** See [Pro25k].

**Item 3, Involutority:** See [Pro25i].

**Item 4, Interaction With Characteristic Functions:** We consider the two cases  $x \in U, x \notin U$ .

1. If  $x \in U$ , then  $x \notin U^c$ . So  $\chi_U(x) = 1$  and

$$\begin{aligned} \chi_{U^c}(x) &= 0 \\ &= 1 - \chi_U(x). \end{aligned}$$

2. If  $x \notin U$ , then  $x \in U^c$ . So  $\chi_U(x) = 0$  and

$$\begin{aligned} \chi_{U^c}(x) &= 1 \\ &= 1 - \chi_U(x). \end{aligned}$$

Hence, the equation holds for all  $x \in X$ .

**Item 5, Interaction With Direct Images:** This is a repetition of **Item 8** of **Definition 4.6.1.1.5** and is proved there.

**Item 6, Interaction With Inverse Images:** This is a repetition of **Item 8** of **Definition 4.6.2.1.3** and is proved there.

**Item 7, Interaction With Codirect Images:** This is a repetition of **Item 7** of **Definition 4.6.3.1.7** and is proved there.  $\square$

### 4.3.12 Symmetric Differences

Let  $X$  be a set and let  $U, V \in \mathcal{P}(X)$ .

**Definition 4.3.12.1.1.** The **symmetric difference of  $U$  and  $V$**  is the set  $U \Delta V$  defined by<sup>13</sup>

$$U \Delta V \stackrel{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

**Proposition 4.3.12.1.2.** Let  $X$  be a set.

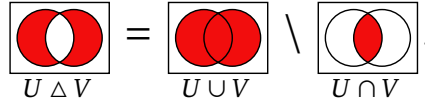
1. *Lack of Functoriality.* The assignment  $(U, V) \mapsto U \Delta V$  **does not** in general define functors

$$\begin{aligned} U \Delta - &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ - \Delta V &: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \Delta -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have

$$U \Delta V = (U \cup V) \setminus (U \cap V)$$

for each  $U, V \in \mathcal{P}(X)$ , as in the Venn diagram



3. *Symmetric Differences of Disjoint Sets.* If  $U$  and  $V$  are disjoint, then we have

$$U \Delta V = U \cup V.$$

4. *Associativity.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\ \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \nearrow & & \searrow \text{id}_{\mathcal{P}(X)} \times \Delta \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\ \Delta \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \Delta \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X), \end{array}$$

<sup>13</sup>Illustration:

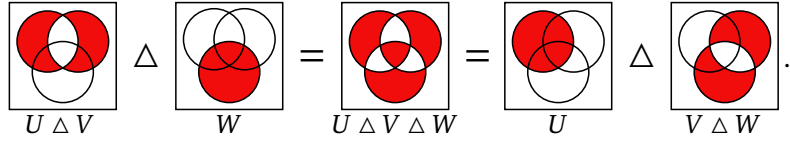




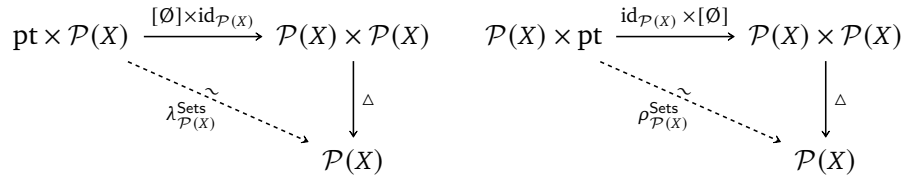
commutes, i.e. we have

$$(U \Delta V) \Delta W = U \Delta (V \Delta W)$$

for each  $U, V, W \in \mathcal{P}(X)$ , as in the Venn diagram



5. *Unitality.* The diagrams



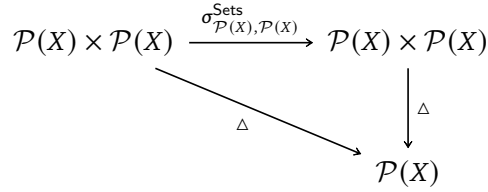
commute, i.e. we have

$$U \Delta \emptyset = U,$$

$$\emptyset \Delta U = U$$

for each  $U \in \mathcal{P}(X)$ .

6. *Commutativity.* The diagram



commutes, i.e. we have

$$U \Delta V = V \Delta U$$

for each  $U, V \in \mathcal{P}(X)$ .

7. *Invertibility.* We have

$$U \Delta U = \emptyset$$

for each  $U \in \mathcal{P}(X)$ .

8. *Interaction With Unions.* We have

$$(U \Delta V) \cup (V \Delta T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each  $U, V, W \in \mathcal{P}(X)$ .

9. *Interaction With Complements I.* We have

$$U \Delta U^c = X$$

for each  $U \in \mathcal{P}(X)$ .

10. *Interaction With Complements II.* We have

$$U \Delta X = U^c,$$

$$X \Delta U = U^c$$

for each  $U \in \mathcal{P}(X)$ .

11. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X) \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$U^c \Delta V^c = U \Delta V$$

for each  $U, V \in \mathcal{P}(X)$ .

12. *“Transitivity”.* We have

$$(U \Delta V) \Delta (V \Delta W) = U \Delta W$$

for each  $U, V, W \in \mathcal{P}(X)$ .

13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \Delta W \subset U \Delta V \cup V \Delta W$$

for each  $U, V, W \in \mathcal{P}(X)$ .

---

14. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each  $U, V \in \mathcal{P}(X)$ .

16. *Bijectivity.* Given  $U, V \in \mathcal{P}(X)$ , the maps

$$\begin{aligned} U \triangle -: \mathcal{P}(X) &\rightarrow \mathcal{P}(X), \\ - \triangle V: \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \end{aligned}$$

are self-inverse bijections. Moreover, the map

$$\begin{aligned} \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ C &\longmapsto C \triangle (U \triangle V) \end{aligned}$$

is a bijection of  $\mathcal{P}(X)$  onto itself sending  $U$  to  $V$  and  $V$  to  $U$ .

17. *Interaction With Powersets and Groups.* Let  $X$  be a set.

(a) The quadruple  $(\mathcal{P}(X), \triangle, \emptyset, \text{id}_{\mathcal{P}(X)})$  is an abelian group.<sup>14</sup>

<sup>14</sup>Here are some examples:

1. When  $X = \emptyset$ , we have an isomorphism of groups between  $\mathcal{P}(\emptyset)$  and the trivial group:

$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \text{id}_{\mathcal{P}(\emptyset)}) \cong \text{pt.}$$

2. When  $X = \text{pt}$ , we have an isomorphism of groups between  $\mathcal{P}(\text{pt})$  and  $\mathbb{Z}_2$ :

$$(\mathcal{P}(\text{pt}), \triangle, \emptyset, \text{id}_{\mathcal{P}(\text{pt})}) \cong \mathbb{Z}_2.$$

3. When  $X = \{0, 1\}$ , we have an isomorphism of groups between  $\mathcal{P}(\{0, 1\})$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ :

$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \text{id}_{\mathcal{P}(\{0, 1\})}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- (b) Every element of  $\mathcal{P}(X)$  has order 2 with respect to  $\Delta$ , and thus  $\mathcal{P}(X)$  is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair  $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$  consisting of

- The group  $\mathcal{P}(X)$  of **Item 17**;
- The map  $\alpha_{\mathcal{P}(X)} : \mathbb{F}_2 \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  defined by

$$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

is an  $\mathbb{F}_2$ -vector space.

5. *Interaction With Powersets and Vector Spaces II.* If  $X$  is finite, then:

- (a) The set of singletons sets on the elements of  $X$  forms a basis for the  $\mathbb{F}_2$ -vector space  $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$  of **Item 4**.
- (b) We have

$$\dim(\mathcal{P}(X)) = \#X.$$

6. *Interaction With Powersets and Rings.* The quintuple  $(\mathcal{P}(X), \Delta, \cap, \emptyset, X)$  is a commutative ring.<sup>15</sup>

7. *Interaction With Direct Images.* We have a natural transformation


$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \supset & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \Delta f_!(V) \subset f_!(U \Delta V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

---

<sup>15</sup>  **Warning:** The analogous statement replacing intersections by unions (i.e. that the quintuple  $(\mathcal{P}(X), \Delta, \cup, \emptyset, X)$  is a ring) is false, however. See [Pro25aw] for a proof.

8. *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \Delta \downarrow & & \downarrow \Delta \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

i.e. we have

$$f^{-1}(U) \Delta f^{-1}(V) = f^{-1}(U \Delta V)$$

for each  $U, V \in \mathcal{P}(Y)$ .

9. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \wr & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U \Delta V) \subset f_*(U) \Delta f_*(V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

*Proof. Item 1, Lack of Functoriality:* Let  $X = \{0, 1\}, U = \{0\}$ . Then  $\emptyset \subset U$ , but  $U \Delta \emptyset = U \not\subset \emptyset = U \Delta U$  from [Item 5](#) and [Item 7](#). This gives a counterexample to the first statement. By using [Item 6](#), we can adapt it to the second and third statement.

*Item 2, Via Unions and Intersections:* See [\[Pro25m\]](#).

*Item 3, Symmetric Differences of Disjoint Sets:* Since  $U$  and  $V$  are disjoint, we have  $U \cap V = \emptyset$ , and therefore we have

$$\begin{aligned} U \Delta V &= (U \cup V) \setminus (U \cap V) \\ &= (U \cup V) \setminus \emptyset \\ &= U \cup V, \end{aligned}$$

where we've used [Item 2](#) and [Item 12](#) of [Definition 4.3.10.1.2](#).

*Item 4, Associativity:* See [\[Pro25ao\]](#).

*Item 5, Unitality:* This follows from [Item 6](#) and [\[Pro25at\]](#).

*Item 6, Commutativity:* See [Pro25ap].

*Item 7, Invertibility:* See [Pro25av].

*Item 8, Interaction With Unions:* See [Pro25bc].

*Item 9, Interaction With Complements I:* See [Pro25as].

*Item 10, Interaction With Complements II:* This follows from *Item 6* and [Pro25ax].

*Item 11, Interaction With Complements III:* See [Pro25aq].

*Item 12, “Transitivity”:* We have

$$\begin{aligned}
 (U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 4)} \\
 &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 4)} \\
 &= U \triangle (\emptyset \triangle W) && \text{(by Item 7)} \\
 &= U \triangle W. && \text{(by Item 5)}
 \end{aligned}$$

This finishes the proof.

*Item 13, The Triangle Inequality for Symmetric Differences:* This follows from *Items 2* and *12*.

*Item 14, Distributivity Over Intersections:* See [Pro25q].

*Item 15, Interaction With Characteristic Functions:* See [Pro25g].

*Item 16, Bijectivity:*

- We show that

$$(U \triangle -): \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is self-inverse.

Let  $W \in \mathcal{P}(X)$ . Then,

$$\begin{aligned}
 U \triangle (U \triangle W) &= (U \triangle U) \triangle W && \text{(by Item 4)} \\
 &= \emptyset \triangle W && \text{(by Item 7)} \\
 &= W. && \text{(by Item 5)}
 \end{aligned}$$

- By *Item 6*,  $(- \triangle V) = (V \triangle -)$ , hence the former is also self-inverse by the first point.
- The map  $- \triangle (U \triangle V)$  is a bijection as a special case of the second point. From the first two points and *Item 6*, we get

$$U \triangle (U \triangle V) = V, \quad V \triangle (U \triangle V) = V \triangle (V \triangle U) = U.$$

Hence the function maps  $U$  to  $V$  and  $V$  to  $U$ .

*Item 17, Interaction With Powersets and Groups:* *Item 17a* follows from *Items 4* to *7*, while *Item 3b* follows from *Item 7*.<sup>16</sup>

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<sup>16</sup>Reference: [Pro25ar].

*Item 4, Interaction With Powersets and Vector Spaces I:* See [MSE 2719059].

*Item 5, Interaction With Powersets and Vector Spaces II:* See [MSE 2719059].

*Item 6, Interaction With Powersets and Rings:* This follows from [Items 6 and 15](#) of [Definition 4.3.9.1.2](#) and [Items 14 and 17](#).<sup>17</sup>

*Item 7, Interaction With Direct Images:* This is a repetition of [Item 9](#) of [Definition 4.6.1.1.5](#) and is proved there.

*Item 8, Interaction With Inverse Images:* This is a repetition of [Item 9](#) of [Definition 4.6.2.1.3](#) and is proved there.

*Item 9, Interaction With Codirect Images:* This is a repetition of [Item 8](#) of [Definition 4.6.3.1.7](#) and is proved there.  $\square$

## 4.4 Powersets

### 4.4.1 Foundations

Let  $X$  be a set.

**Definition 4.4.1.1.1.** The **powerset of  $X$**  is the set  $\mathcal{P}(X)$  defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where  $P$  is the set in the axiom of powerset, ?? of ??.

**Remark 4.4.1.1.2.** Under the analogy that  $\{t, f\}$  should be the  $(-1)$ -categorical analogue of  $\text{Sets}$ , we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set  $X$  is equivalently ([Item 2](#) of [Definition 4.5.1.1.4](#)) the set

$$\text{Sets}(X, \{t, f\})$$

of functions from  $X$  to the set  $\{t, f\}$  of classical truth values.

- The category of presheaves on a category  $C$  is the category

$$\text{Fun}(C^{\text{op}}, \text{Sets})$$

of functors from  $C^{\text{op}}$  to the category  $\text{Sets}$  of sets.

**Notation 4.4.1.1.3.** Let  $X$  be a set.

1. We write  $\mathcal{P}_0(X)$  for the set of nonempty subsets of  $X$ .

---

<sup>17</sup>Reference: [\[Pro25au\]](#).

2. We write  $\mathcal{P}_{\text{fin}}(X)$  for the set of finite subsets of  $X$ .

**Proposition 4.4.1.1.4.** Let  $X$  be a set.

1. *Co/Completeness.* The (posetal) category (associated to)  $(\mathcal{P}(X), \subset)$  is complete and cocomplete:
  - (a) *Products.* The products in  $\mathcal{P}(X)$  are given by intersection of subsets.
  - (b) *Coproducts.* The coproducts in  $\mathcal{P}(X)$  are given by union of subsets.
  - (c) *Co/Equalisers.* Being a posetal category,  $\mathcal{P}(X)$  only has at most one morphisms between any two objects, so co/equalisers are trivial.
2. *Cartesian Closedness.* The category  $\mathcal{P}(X)$  is Cartesian closed.
3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \text{Rel}(\text{pt}, X), \\ \mathcal{P}(X) &\cong \text{Rel}(X, \text{pt}),\end{aligned}$$

natural in  $X \in \text{Obj}(\text{Sets})$ .

4. *Interaction With Products I.* The map

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \amalg Y) \\ (U, V) &\longmapsto U \cup V\end{aligned}$$

is an isomorphism of sets, natural in  $X, Y \in \text{Obj}(\text{Sets})$  with respect to each of the functor structures  $\mathcal{P}_!$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  on  $\mathcal{P}$  of [Definition 4.4.2.1.1](#). Moreover, this makes each of  $\mathcal{P}_!$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  into a symmetric monoidal functor.

5. *Interaction With Products II.* The map

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \amalg Y) \\ (U, V) &\longmapsto U \boxtimes_{X \times Y} V,\end{aligned}$$

where<sup>18</sup>

$$U \boxtimes_{X \times Y} V \stackrel{\text{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

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<sup>18</sup>The set  $U \boxtimes_{X \times Y} V$  is usually denoted simply  $U \times V$ . Here we denote it in this somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see also [Section 4.6.4](#)).



is an inclusion of sets, natural in  $X, Y \in \text{Obj}(\text{Sets})$  with respect to each of the functor structures  $\mathcal{P}_!$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  on  $\mathcal{P}$  of [Definition 4.4.2.1.1](#). Moreover, this makes each of  $\mathcal{P}_!$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  into a symmetric monoidal functor.

6. *Interaction With Products III*. We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$  with respect to each of the functor structures  $\mathcal{P}_!$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  on  $\mathcal{P}$  of [Definition 4.4.2.1.1](#), where  $\otimes$  denotes the tensor product of suplattices of  $\mathbb{P}$ . Moreover, this makes each of  $\mathcal{P}_!$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  into a symmetric monoidal functor.

*Proof.* [Item 1](#), *Co/Completeness*: Omitted.

[Item 2](#), *Cartesian Closedness*: See [Section 4.4.7](#).

[Item 3](#), *Powersets as Sets of Relations*: Indeed, we have

$$\begin{aligned} \text{Rel}(\text{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\text{pt} \times X) \\ &\cong \mathcal{P}(X) \end{aligned}$$

and

$$\begin{aligned} \text{Rel}(X, \text{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \text{pt}) \\ &\cong \mathcal{P}(X), \end{aligned}$$

where we have used [Item 5](#) of [Definition 4.1.3.1.3](#).

[Item 4](#), *Interaction With Products I*: The inverse of the map in the statement is the map

$$\Phi: \mathcal{P}(X \amalg Y) \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \stackrel{\text{def}}{=} (S_X, S_Y)$$

for each  $S \in \mathcal{P}(X \amalg Y)$ , where

$$\begin{aligned} S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\} \\ S_Y &\stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}. \end{aligned}$$

The rest of the proof is omitted.

[Item 5](#), *Interaction With Products II*: Omitted.

[Item 6](#), *Interaction With Products III*: Omitted. □

### 4.4.2 Functoriality of Powersets

**Proposition 4.4.2.1.1.** Let  $X$  be a set.

1. *Functoriality I.* The assignment  $X \mapsto \mathcal{P}(X)$  defines a functor

$$\mathcal{P}_!: \mathbf{Sets} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each  $A \in \mathbf{Obj}(\mathbf{Sets})$ , we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each  $A, B \in \mathbf{Obj}(\mathbf{Sets})$ , the action on morphisms

$$\mathcal{P}_{*|A,B}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of  $\mathcal{P}_!$  at  $(A, B)$  is the map defined by sending a map of sets  $f: A \rightarrow B$  to the map

$$\mathcal{P}_!(f): \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in [Definition 4.6.1.1.1](#).

2. *Functoriality II.* The assignment  $X \mapsto \mathcal{P}(X)$  defines a functor

$$\mathcal{P}^{-1}: \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each  $A \in \mathbf{Obj}(\mathbf{Sets})$ , we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each  $A, B \in \mathbf{Obj}(\mathbf{Sets})$ , the action on morphisms

$$\mathcal{P}_{A,B}^{-1}: \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of  $\mathcal{P}^{-1}$  at  $(A, B)$  is the map defined by sending a map of sets  $f: A \rightarrow B$  to the map

$$\mathcal{P}^{-1}(f): \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in [Definition 4.6.2.1.1](#).

3. *Functoriality III.* The assignment  $X \mapsto \mathcal{P}(X)$  defines a functor

$$\mathcal{P}_* : \mathbf{Sets} \rightarrow \mathbf{Sets},$$

where

- *Action on Objects.* For each  $A \in \mathbf{Obj}(\mathbf{Sets})$ , we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each  $A, B \in \mathbf{Obj}(\mathbf{Sets})$ , the action on morphisms

$$\mathcal{P}_{!|A,B} : \mathbf{Sets}(A, B) \rightarrow \mathbf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of  $\mathcal{P}_*$  at  $(A, B)$  is the map defined by sending a map of sets  $f : A \rightarrow B$  to the map

$$\mathcal{P}_*(f) : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in [Definition 4.6.3.1.1](#).

*Proof.* [Item 1, Functoriality I](#): This follows from [Items 3 and 4 of Definition 4.6.1.1.6](#).  
[Item 2, Functoriality II](#): This follows from [Items 3 and 4 of Definition 4.6.2.1.4](#).  
[Item 3, Functoriality III](#): This follows from [Items 3 and 4 of Definition 4.6.3.1.8](#).

□

### 4.4.3 Adjointness of Powersets I

**Proposition 4.4.3.1.1.** We have an adjunction

$$\left( \mathcal{P}^{-1} \dashv \mathcal{P}^{-1, \text{op}} \right) : \mathbf{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{P}^{-1}} \\ \perp \\ \xleftarrow{\mathcal{P}^{-1, \text{op}}} \end{array} \mathbf{Sets},$$

witnessed by a bijection

$$\underbrace{\mathbf{Sets}^{\text{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathbf{Sets}(Y, \mathcal{P}(X))} \cong \mathbf{Sets}(X, \mathcal{P}(Y)),$$

natural in  $X \in \mathbf{Obj}(\mathbf{Sets})$  and  $Y \in \mathbf{Obj}(\mathbf{Sets}^{\text{op}})$ .

*Proof.* We have

$$\begin{aligned}
\text{Sets}^{\text{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \text{Sets}(B, \mathcal{P}(A)) \\
&\cong \text{Sets}(B, \text{Sets}(A, \{t, f\})) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\
&\cong \text{Sets}(A \times B, \{t, f\}) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \text{Sets}(A, \text{Sets}(B, \{t, f\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \text{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{aligned}$$

where all bijections are natural in  $A$  and  $B$ .<sup>19</sup>  $\square$

#### 4.4.4 Adjointness of Powersets II

**Proposition 4.4.4.1.1.** We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_!): \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_!} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(X), Y) \cong \text{Sets}(X, \mathcal{P}(Y))$$

natural in  $X \in \text{Obj}(\text{Sets})$  and  $Y \in \text{Obj}(\text{Rel})$ , where  $\text{Gr}$  is the graph functor of [Relations](#), [Item 1](#) of [Definition 8.2.2.1.2](#) and  $\mathcal{P}_!$  is the functor of [Relations](#), [Definition 8.7.5.1.1](#).

*Proof.* We have

$$\begin{aligned}
\text{Rel}(\text{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
&\cong \text{Sets}(A \times B, \{t, f\}) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\
&\cong \text{Sets}(A, \text{Sets}(B, \{t, f\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \text{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{aligned}$$

where all bijections are natural in  $A$ , (where we are using [Item 3](#) of [Definition 4.5.1.1.4](#)). Explicitly, this isomorphism is given by sending a relation  $R: \text{Gr}(A) \rightarrow B$  to the map  $R^\dagger: A \rightarrow \mathcal{P}(B)$  sending  $a$  to the subset  $R(a)$  of  $B$ , as in [Relations](#), [Definition 8.1.1.1.1](#).

Naturality in  $B$  is then the statement that given a relation  $R: B \rightarrow B'$ , the diagram

$$\begin{array}{ccc}
\text{Rel}(\text{Gr}(A), B) & \xrightarrow{R \diamond -} & \text{Rel}(\text{Gr}(A), B') \\
\downarrow \wr & & \downarrow \wr \\
\text{Sets}(A, \mathcal{P}(B)) & \xrightarrow{R_!} & \text{Sets}(A, \mathcal{P}(B'))
\end{array}$$

commutes, which follows from [Relations](#), [Definition 8.7.1.1.3](#).  $\square$

<sup>19</sup>Here we are using [Item 3](#) of [Definition 4.5.1.1.4](#).

### 4.4.5 Powersets as Free Cocompletions

Let  $X$  be a set.

**Proposition 4.4.5.1.1.** The pair  $(\mathcal{P}(X), \chi_{(-)})$  consisting of

- The powerset  $(\mathcal{P}(X), \subset)$  of  $X$  of [Definition 4.4.1.1.1](#);
- The characteristic embedding  $\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$  of  $X$  into  $\mathcal{P}(X)$  of [Definition 4.5.4.1.1](#);

satisfies the following universal property:

(★) Given another pair  $(Y, f)$  consisting of

- A suplattice  $(Y, \preceq)$ ;
- A function  $f: X \rightarrow Y$ ;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc} & & \mathcal{P}(X) \\ & \nearrow \chi_X & \downarrow \exists! \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

*Proof.* This is a rephrasing of [Definition 4.4.5.1.2](#), which we prove below.<sup>20</sup>  $\square$

**Proposition 4.4.5.1.2.** We have an adjunction

$$(\mathcal{P} \dashv \text{forget}): \text{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\text{forget}} \end{array} \text{SupLat},$$

witnessed by a bijection

$$\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in  $X \in \text{Obj}(\text{Sets})$  and  $(Y, \preceq) \in \text{Obj}(\text{SupLat})$ , where:

<sup>20</sup>Here we only remark that the unique morphism of suplattices in the statement is given by

- The category  $\text{SupLat}$  is the category of suplattices of ??.
- The map

$$\chi_X^*: \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of suplattices  $f: \mathcal{P}(X) \rightarrow Y$  to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

$$\text{Lan}_{\chi_X}: \text{Sets}(X, Y) \rightarrow \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function  $f: X \rightarrow Y$  to its left Kan extension along  $\chi_X$ ,

$$\text{Lan}_{\chi_X}(f): \mathcal{P}(X) \rightarrow Y, \quad \begin{array}{ccc} & \mathcal{P}(X) & \\ \nearrow \chi_X & \uparrow & \downarrow \text{Lan}_{\chi_X}(f) \\ X & \xrightarrow{f} & Y. \end{array}$$

Moreover, invoking the bijection  $\mathcal{P}(X) \cong \text{Sets}(X, \{t, f\})$  of [Item 2 of Definition 4.5.1.1.4](#),  $\text{Lan}_{\chi_X}(f)$  can be explicitly computed by

$$\begin{aligned} [\text{Lan}_{\chi_X}(f)](U) &= \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &= \int^{x \in X} \chi_U(x) \odot f(x) \\ &= \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \\ &= \left( \bigvee_{x \in U} (\chi_U(x) \odot f(x)) \right) \vee \left( \bigvee_{x \in U^c} (\chi_U(x) \odot f(x)) \right) \\ &= \left( \bigvee_{x \in U} f(x) \right) \vee \left( \bigvee_{x \in U^c} \emptyset_Y \right) \end{aligned}$$


---

$$= \bigvee_{x \in U} f(x)$$

for each  $U \in \mathcal{P}(X)$ , where:

- We have used ?? for the first equality.
- We have used **Definition 4.5.5.1.1** for the second equality.
- We have used ?? for the third equality.
- The symbol  $\bigvee$  denotes the join in  $(Y, \preceq)$ .
- The symbol  $\odot$  denotes the tensor of an element of  $Y$  by a truth value as in ?. In particular, we have

$$\begin{aligned} \text{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \odot f(x) &\stackrel{\text{def}}{=} \emptyset_Y, \end{aligned}$$

where  $\emptyset_Y$  is the bottom element of  $(Y, \preceq)$ .

In particular, when  $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$  for some set  $B$ , the Kan extension  $\text{Lan}_{\chi_X}(f)$  is given by

$$\begin{aligned} [\text{Lan}_{\chi_X}(f)](U) &= \bigvee_{x \in U} f(x) \\ &= \bigcup_{x \in U} f(x) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

*Proof. Map I:* We define a map

$$\Phi_{X,Y}: \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

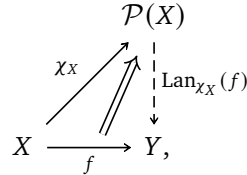
for each  $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$ .

*Map II:* We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f),$$



for each  $f \in \text{Sets}(X, Y)$ .

*Invertibility I:* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{SupLat}((\mathcal{P}(X), \subset), (Y, \leq))}.$$

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each  $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \leq))$ . We now claim that

$$\text{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each  $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \leq))$ . Indeed, we have

$$\begin{aligned} [\text{Lan}_{\chi_X}(f \circ \chi_X)](U) &= \bigvee_{x \in U} f(\chi_X(x)) \\ &= f\left(\bigvee_{x \in U} \chi_X(x)\right) \\ &= f\left(\bigcup_{x \in U} \{x\}\right) \\ &= f(U) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ , where we have used that  $f$  is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each  $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \leq))$ , it follows that  $\Psi_{X,Y} \circ \Phi_{X,Y}$  must be equal to the identity map  $\text{id}_{\text{SupLat}((\mathcal{P}(X), \subset), (Y, \leq))}$  of  $\text{SupLat}((\mathcal{P}(X), \subset), (Y, \leq))$ .

*Invertibility II:* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)}.$$

We have

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) \stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))$$

---

the left Kan extension  $\text{Lan}_{\chi_X}(f)$  of  $f$  along  $\chi_X$ .



$$\begin{aligned}
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Lan}_{\chi_X}(f)) \\
&\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f) \circ \chi_X
\end{aligned}$$

for each  $f \in \text{Sets}(X, Y)$ . We now claim that

$$\text{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each  $f \in \text{Sets}(X, Y)$ . Indeed, we have

$$\begin{aligned}
[\text{Lan}_{\chi_X}(f) \circ \chi_X](x) &= \bigvee_{y \in \{x\}} f(y) \\
&= f(x)
\end{aligned}$$

for each  $x \in X$ . This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each  $f \in \text{Sets}(X, Y)$ , it follows that  $\Phi_{X,Y} \circ \Psi_{X,Y}$  must be equal to the identity map  $\text{id}_{\text{Sets}(X,Y)}$  of  $\text{Sets}(X, Y)$ .

*Naturality for  $\Phi$ , Part I:* We need to show that, given a function  $f: X \rightarrow X'$ , the diagram

$$\begin{array}{ccc}
\text{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\
\mathcal{P}_!(f)^* \downarrow & & \downarrow f^* \\
\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y)
\end{array}$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
&\stackrel{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
&= \xi \circ (f_! \circ \chi_X) \\
&\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

for each  $\xi \in \text{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$ , where we have used [Item 1 of Definition 4.5.4.1.3](#) for the fifth equality above.

*Naturality for  $\Phi$ , Part II:* We need to show that, given a morphism of suplattices

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc} \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\ \downarrow g! & & \downarrow g! \\ \text{SupLat}((\mathcal{P}(X), \subset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y') \end{array}$$

commutes. Indeed, we have

$$\begin{aligned} [\Phi_{X,Y'} \circ g!](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g!(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\ &\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\ &= g \circ (\xi \circ \chi_X) \\ &\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} g!(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} [g! \circ \Phi_{X,Y}](\xi). \end{aligned}$$

for each  $\xi \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$ .

*Naturality for  $\Psi$ :* Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from [Categories, Item 2](#) of [Definition 11.9.7.1.2](#) that  $\Psi$  is also natural in each argument.  $\square$

**Warning 4.4.5.1.3.** Although the assignment  $X \mapsto \mathcal{P}(X)$  is called the *free cocompletion* of  $X$ , it is not an idempotent operation, i.e. we have  $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$ .

## 4.4.6 Powersets as Free Completions

Let  $X$  be a set.

**Proposition 4.4.6.1.1.** The pair  $(\mathcal{P}(X), \chi_{(-)})$  consisting of

- The powerset of  $X$  together with reverse inclusion  $\mathcal{P}(X)^{\text{op}} = (\mathcal{P}(X), \supset)$  of [Definition 4.4.1.1.1](#);
- The characteristic embedding  $\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$  of  $X$  into  $\mathcal{P}(X)$  of [Definition 4.5.4.1.1](#);

satisfies the following universal property:

(★) Given another pair  $(Y, \preceq)$  consisting of

- An inflattice  $(Y, \preceq)$ ;
- A function  $f: X \rightarrow Y$ ;

there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc} & & \mathcal{P}(X)^{\text{op}} \\ & \nearrow \chi_X & \downarrow \exists! \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

*Proof.* This is a rephrasing of [Definition 4.4.6.1.2](#), which we prove below.<sup>21</sup>  $\square$

**Proposition 4.4.6.1.2.** We have an adjunction

$$(\mathcal{P} \dashv \text{忘}): \text{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\text{忘}} \end{array} \text{InfLat},$$

witnessed by a bijection

$$\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in  $X \in \text{Obj}(\text{Sets})$  and  $(Y, \preceq) \in \text{Obj}(\text{InfLat})$ , where:

- The category  $\text{InfLat}$  is the category of inflattices of ??.
- The map

$$\chi_X^*: \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of inflattices  $f: \mathcal{P}(X)^{\text{op}} \rightarrow Y$  to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X)^{\text{op}} \xrightarrow{f} Y.$$

---

<sup>21</sup>Here we only remark that the unique morphism of inflattices in the statement is given by the

- The map

$$\text{Ran}_{\chi_X} : \text{Sets}(X, Y) \rightarrow \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function  $f : X \rightarrow Y$  to its right Kan extension along  $\chi_X$ ,

$$\text{Ran}_{\chi_X}(f) : \mathcal{P}(X)^{\text{op}} \rightarrow Y,$$

Moreover, invoking the bijection  $\mathcal{P}(X) \cong \text{Sets}(X, \{\mathbf{t}, \mathbf{f}\})$  of **Item 2** of **Definition 4.5.1.1.4**,  $\text{Ran}_{\chi_X}(f)$  can be explicitly computed by

$$\begin{aligned} [\text{Ran}_{\chi_X}(f)](U) &= \int_{x \in X} \chi_{\mathcal{P}(X)^{\text{op}}}(\chi_x, U) \multimap f(x) \\ &= \int_{x \in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \multimap f(x) \\ &= \int_{x \in X} \chi_U(x) \multimap f(x) \\ &= \bigwedge_{x \in X} \chi_U(x) \multimap f(x) \\ &= \left( \bigwedge_{x \in U} \chi_U(x) \multimap f(x) \right) \wedge \left( \bigwedge_{x \in U^c} \chi_U(x) \multimap f(x) \right) \\ &= \left( \bigwedge_{x \in U} f(x) \right) \wedge \left( \bigwedge_{x \in U^c} \infty_Y \right) \\ &= \left( \bigwedge_{x \in U} f(x) \right) \wedge \infty_Y \\ &= \bigwedge_{x \in U} f(x) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ , where:

- We have used ?? for the first equality.
- We have used **Definition 4.5.5.1.1** for the second equality.
- We have used ?? for the third equality.

- The symbol  $\wedge$  denotes the meet in  $(Y, \preceq)$ .
- The symbol  $\pitchfork$  denotes the cotensor of an element of  $Y$  by a truth value as in ???. In particular, we have

$$\begin{aligned}\text{true} \pitchfork f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \pitchfork f(x) &\stackrel{\text{def}}{=} \infty_Y,\end{aligned}$$

where  $\infty_Y$  is the top element of  $(Y, \preceq)$ .

In particular, when  $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$  for some set  $B$ , the Kan extension  $\text{Ran}_{\chi_X}(f)$  is given by

$$\begin{aligned}[\text{Ran}_{\chi_X}(f)](U) &= \bigwedge_{x \in U} f(x) \\ &= \bigcap_{x \in U} f(x)\end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

*Proof. Map I:* We define a map

$$\Phi_{X,Y}: \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each  $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$ .

*Map II:* We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f),$$

for each  $f \in \text{Sets}(X, Y)$ .

*Invertibility I:* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}.$$


---

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each  $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$ . We now claim that

$$\text{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each  $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$ . Indeed, we have

$$\begin{aligned} [\text{Ran}_{\chi_X}(f \circ \chi_X)](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\ &= f\left(\bigwedge_{x \in U} \chi_X(x)\right) \\ &= f\left(\bigcup_{x \in U} \{x\}\right) \\ &= f(U) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ , where we have used that  $f$  is a morphism of inflattices and hence preserves meets in  $(\mathcal{P}(X), \supset)$  (i.e. joins in  $(\mathcal{P}(X), \subset)$ ) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each  $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$ , it follows that  $\Psi_{X,Y} \circ \Phi_{X,Y}$  must be equal to the identity map  $\text{id}_{\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}$  of  $\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$ .

*Invertibility II:* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)} .$$

We have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Ran}_{\chi_X}(f)) \\ &\stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f) \circ \chi_X \end{aligned}$$

---

right Kan extension  $\text{Ran}_{\chi_X}(f)$  of  $f$  along  $\chi_X$ .

for each  $f \in \text{Sets}(X, Y)$ . We now claim that

$$\text{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each  $f \in \text{Sets}(X, Y)$ . Indeed, we have

$$\begin{aligned} [\text{Ran}_{\chi_X}(f) \circ \chi_X](x) &= \bigwedge_{y \in \{x\}} f(y) \\ &= f(x) \end{aligned}$$

for each  $x \in X$ . This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each  $f \in \text{Sets}(X, Y)$ , it follows that  $\Phi_{X,Y} \circ \Psi_{X,Y}$  must be equal to the identity map  $\text{id}_{\text{Sets}(X,Y)}$  of  $\text{Sets}(X, Y)$ .

*Naturality for  $\Phi$ , Part I:* We need to show that, given a function  $f: X \rightarrow X'$ , the diagram

$$\begin{array}{ccc} \text{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ \mathcal{P}_!(f)^* \downarrow & & \downarrow f^* \\ \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, we have

$$\begin{aligned} [\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\ &\stackrel{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\ &= \xi \circ (f_! \circ \chi_X) \\ &\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\ &= (\xi \circ \chi_{X'}) \circ f \\ &\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\ &\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\ &\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi), \end{aligned}$$

for each  $\xi \in \text{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$ , where we have used **Item 1** of **Definition 4.5.4.1.3** for the fifth equality above.

*Naturality for  $\Phi$ , Part II:* We need to show that, given a cocontinuous morphism of posets

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc}
 \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\
 \downarrow g! & & \downarrow g! \\
 \text{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y')
 \end{array}$$

commutes. Indeed, we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g!](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g!(\xi)) \\
 &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
 &\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
 &= g \circ (\xi \circ \chi_X) \\
 &\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
 &\stackrel{\text{def}}{=} g!(\Phi_{X,Y}(\xi)) \\
 &\stackrel{\text{def}}{=} [g! \circ \Phi_{X,Y}](\xi).
 \end{aligned}$$

for each  $\xi \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$ .

*Naturality for  $\Psi$* : Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\Psi$  is also natural in each argument.  $\square$

**Warning 4.4.6.1.3.** Although the assignment  $X \mapsto \mathcal{P}(X)^{\text{op}}$  is called the *free completion of  $X$* , it is not an idempotent operation, i.e. we have  $\mathcal{P}(\mathcal{P}(X)^{\text{op}})^{\text{op}} \neq \mathcal{P}(X)^{\text{op}}$ .

#### 4.4.7 The Internal Hom of a Powerset

Let  $X$  be a set and let  $U, V \in \mathcal{P}(X)$ .

**Proposition 4.4.7.1.1.** The internal Hom of  $\mathcal{P}(X)$  from  $U$  to  $V$  is the subset  $[U, V]_X$ <sup>22</sup> of  $X$  given by

$$\begin{aligned}
 [U, V]_X &= U^c \cup V \\
 &= (U \setminus V)^c
 \end{aligned}$$

where  $U^c$  is the complement of  $U$  of **Definition 4.3.11.1.1**.

<sup>22</sup>*Further Notation:* Also written  $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$ .



*Proof. Proof of the Equality  $U^c \cup V = (U \setminus V)^c$ :* We have

$$\begin{aligned}
 (U \setminus V)^c &\stackrel{\text{def}}{=} X \setminus (U \setminus V) \\
 &= (X \cap V) \cup (X \setminus U) \\
 &= V \cup (X \setminus U) \\
 &\stackrel{\text{def}}{=} V \cup U^c \\
 &= U^c \cup V,
 \end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.10.1.2** for the second equality.
2. **Item 4** of **Definition 4.3.9.1.2** for the third equality.
3. **Item 4** of **Definition 4.3.8.1.2** for the last equality.

This finishes the proof.

*Proof that  $U^c \cup V$  Is Indeed the Internal Hom:* This follows from **Item 2** of **Definition 4.3.9.1.2**.  $\square$

**Remark 4.4.7.1.2.** Henning Makholm suggests the following heuristic intuition for the internal Hom of  $\mathcal{P}(X)$  from  $U$  to  $V$  ([MSE 267365]):

1. Since products in  $\mathcal{P}(X)$  are given by binary intersections (**Item 1** of **Definition 4.4.1.1.4**), the right adjoint  $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$  of  $U \cap -$  may be thought of as a function type  $[U, V]$ .
2. Under the Curry–Howard correspondence (??), the function type  $[U, V]$  corresponds to implication  $U \Rightarrow V$ .
3. Implication  $U \Rightarrow V$  is logically equivalent to  $\neg U \vee V$ .
4. The expression  $\neg U \vee V$  then corresponds to the set  $U^c \cup V$  in  $\mathcal{P}(X)$ .
5. The set  $U^c \cup V$  turns out to indeed be the internal Hom of  $\mathcal{P}(X)$ .

**Proposition 4.4.7.1.3.** Let  $X$  be a set.

1. *Functoriality.* The assignments  $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$  define functors

$$\begin{aligned}
 [U, -]_X &: (\mathcal{P}(X), \supset) &\rightarrow (\mathcal{P}(X), \subset), \\
 [-, V]_X &: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\
 [-_1, -_2]_X &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\rightarrow (\mathcal{P}(X), \subset).
 \end{aligned}$$

In particular, the following statements hold for each  $U, V, A, B \in \mathcal{P}(X)$ :

- (a) If  $U \subset A$ , then  $[A, V]_X \subset [U, V]_X$ .
- (b) If  $V \subset B$ , then  $[U, V]_X \subset [U, B]_X$ .
- (c) If  $U \subset A$  and  $V \subset B$ , then  $[A, V]_X \subset [U, B]_X$ .

2. *Adjointness.* We have adjunctions

$$\begin{aligned}
 (U \cap - \dashv [U, -]_X) : \mathcal{P}(X) &\overset{U \cap -}{\underset{[U, -]_X}{\rightleftarrows}} \mathcal{P}(X), \\
 (- \cap V \dashv [V, -]_X) : \mathcal{P}(X) &\overset{- \cap V}{\underset{[V, -]_X}{\rightleftarrows}} \mathcal{P}(X),
 \end{aligned}$$

witnessed by bijections

$$\begin{aligned}
 \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\
 \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(V, [U, W]_X).
 \end{aligned}$$

In particular, the following statements hold for each  $U, V, W \in \mathcal{P}(X)$ :

- (a) The following conditions are equivalent:
  - i. We have  $U \cap V \subset W$ .
  - ii. We have  $U \subset [V, W]_X$ .
- (b) The following conditions are equivalent:
  - i. We have  $U \cap V \subset W$ .
  - ii. We have  $V \subset [U, W]_X$ .

3. *Interaction With the Empty Set  $I$ .* We have

$$\begin{aligned}
 [U, \emptyset]_X &= U^c, \\
 [\emptyset, V]_X &= X,
 \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

4. *Interaction With  $X$ .* We have

$$\begin{aligned}
 [U, X]_X &= X, \\
 [X, V]_X &= V,
 \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

5. *Interaction With the Empty Set II.* The functor

$$D_X: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X)$$

defined by

$$\begin{aligned} D_X &\stackrel{\text{def}}{=} [-, \emptyset]_X \\ &= (-)^c \end{aligned}$$

is an involutory isomorphism of categories, making  $\emptyset$  into a dualising object for  $(\mathcal{P}(X), \cap, X, [-, -]_X)$  in the sense of ???. In particular:

(a) The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)^{\text{op}}} & \downarrow D_X \\ & & \mathcal{P}(X)^{\text{op}} \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X} = U$$

for each  $U \in \mathcal{P}(X)$ .

(b) The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ & \nearrow \text{id}_{\mathcal{P}(X)^{\text{op}}} \times D_X & & & \searrow D_X \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{\quad \quad \quad} & & & \mathcal{P}(X) \\ & & [-1, -2]_X & & \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each  $U, V \in \mathcal{P}(X)$ .

6. *Interaction With the Empty Set III.* Let  $f: X \rightarrow Y$  be a function.

(a) *Interaction With Direct Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each  $U \in \mathcal{P}(X)$ .

(b) *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ D_Y \downarrow & & \downarrow D_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each  $U \in \mathcal{P}(X)$ .

(c) *Interaction With Codirect Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each  $U \in \mathcal{P}(X)$ .

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup^{\text{op}} \times \cup^{\text{op}} & \text{X} & \downarrow \cup \\
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[ \bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{array}{ccccc}
 & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\
 & \nearrow \sim & & \searrow \cup^{\text{op}} & \\
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & & & \mathcal{P}(X)^{\text{op}} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & & & \downarrow [-, V]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) & & 
 \end{array}$$

commutes, i.e. we have

$$\left[ \bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$  and each  $V \in \mathcal{P}(X)$ .

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[ U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each  $U \in \mathcal{P}(X)$  and each  $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap^{\text{op}} \times \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[ \bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ .

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\ & \nearrow \cap^{\text{op}} & & \searrow \cap^{\text{op}} & \\ \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & & & \mathcal{P}(X)^{\text{op}} \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & & & \downarrow [-, V]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) & & \end{array}$$

commutes, i.e. we have

$$\left[ \bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$  and each  $V \in \mathcal{P}(X)$ .

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \\
 \text{id}_{\mathcal{P}(X)} \star [U, -]_X \downarrow & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[ U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each  $U \in \mathcal{P}(X)$  and each  $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ .

13. *Interaction With Binary Unions.* We have equalities of sets

$$\begin{aligned}
 [U \cap V, W]_X &= [U, W]_X \cup [V, W]_X, \\
 [U, V \cap W]_X &= [U, V]_X \cap [U, W]_X
 \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

14. *Interaction With Binary Intersections.* We have equalities of sets

$$\begin{aligned}
 [U \cup V, W]_X &= [U, W]_X \cap [V, W]_X, \\
 [U, V \cup W]_X &= [U, V]_X \cup [U, W]_X
 \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

15. *Interaction With Differences.* We have equalities of sets

$$\begin{aligned}
 [U \setminus V, W]_X &= [U, W]_X \cup [V^c, W]_X \\
 &= [U, W]_X \cup [U, V]_X, \\
 [U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)
 \end{aligned}$$

for each  $U, V, W \in \mathcal{P}(X)$ .

16. *Interaction With Complements.* We have equalities of sets

$$\begin{aligned}
 [U^c, V]_X &= U \cup V, \\
 [U, V^c]_X &= U \cap V, \\
 [U, V]_X^c &= U \setminus V
 \end{aligned}$$

for each  $U, V \in \mathcal{P}(X)$ .

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod{2}, \chi_V)$$

for each  $U, V \in \mathcal{P}(X)$ .

18. *Interaction With Direct Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in  $U, V \in \mathcal{P}(X)$ .

19. *Interaction With Inverse Images.* Let  $f: X \rightarrow Y$  be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in  $U, V \in \mathcal{P}(X)$ .

20. *Interaction With Codirect Images.* Let  $f: X \rightarrow Y$  be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & \wr & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by  $U, V \in \mathcal{P}(X)$ .



*Proof. Item 1, Functoriality:* Since  $\mathcal{P}(X)$  is posetal, it suffices to prove **Items 1a** to **1c**.

1. *Proof of Item 1a:* We have

$$\begin{aligned} [A, V]_X &\stackrel{\text{def}}{=} A^c \cup V \\ &\subset U^c \cup V \\ &\stackrel{\text{def}}{=} [U, V]_X, \end{aligned}$$

where we have used:

- (a) **Item 1** of **Definition 4.3.11.1.2**, which states that if  $U \subset A$ , then  $A^c \subset U^c$ .
- (b) **Item 1a** of **Item 1** of **Definition 4.3.11.1.2**, which states that if  $A^c \subset U^c$ , then  $A^c \cup K \subset U^c \cup K$  for any  $K \in \mathcal{P}(X)$ .

2. *Proof of Item 1b:* We have

$$\begin{aligned} [U, V]_X &\stackrel{\text{def}}{=} U^c \cup V \\ &\subset U^c \cup B \\ &\stackrel{\text{def}}{=} [U, B]_X, \end{aligned}$$

where we have used **Item 1b** of **Item 1** of **Definition 4.3.11.1.2**, which states that if  $V \subset B$ , then  $K \cup V \subset K \cup B$  for any  $K \in \mathcal{P}(X)$ .

3. *Proof of Item 1c:* We have

$$\begin{aligned} [A, V]_X &\subset [U, V]_X \\ &\subset [U, B]_X, \end{aligned}$$

where we have used **Items 1a** and **1b**.

This finishes the proof.

**Item 2, Adjointness:** This is a repetition of **Item 2** of **Definition 4.3.9.1.2** and is proved there.

**Item 3, Interaction With the Empty Set I:** We have

$$\begin{aligned} [U, \emptyset]_X &\stackrel{\text{def}}{=} U^c \cup \emptyset \\ &= U^c, \end{aligned}$$

where we have used **Item 3** of **Definition 4.3.8.1.2**, and we have

$$[\emptyset, V]_X \stackrel{\text{def}}{=} \emptyset^c \cup V$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} (X \setminus \emptyset) \cup V \\
&= X \cup V \\
&= X,
\end{aligned}$$

where we have used:

1. [Item 12 of Definition 4.3.10.1.2](#) for the first equality.
2. [Item 5 of Definition 4.3.8.1.2](#) for the last equality.

Since  $\mathcal{P}(X)$  is posetal, naturality is automatic ([Categories, Item 4 of Definition 11.2.7.1.2](#)).

[Item 4, Interaction With  \$X\$](#) : We have

$$\begin{aligned}
[U, X]_X &\stackrel{\text{def}}{=} U^c \cup X \\
&= X,
\end{aligned}$$

where we have used [Item 5 of Definition 4.3.8.1.2](#), and we have

$$\begin{aligned}
[X, V]_X &\stackrel{\text{def}}{=} X^c \cup V \\
&\stackrel{\text{def}}{=} (X \setminus X) \cup V \\
&= \emptyset \cup V \\
&= V,
\end{aligned}$$

where we have used [Item 3 of Definition 4.3.8.1.2](#) for the last equality. Since  $\mathcal{P}(X)$  is posetal, naturality is automatic ([Categories, Item 4 of Definition 11.2.7.1.2](#)).

[Item 5, Interaction With the Empty Set II](#): We have

$$\begin{aligned}
D_X(D_X(U)) &\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X \\
&= [U^c, \emptyset]_X \\
&= (U^c)^c \\
&= U,
\end{aligned}$$

where we have used:

1. [Item 3](#) for the second and third equalities.
2. [Item 3 of Definition 4.3.11.1.2](#) for the fourth equality.

Since  $\mathcal{P}(X)$  is posetal, naturality is automatic ([Categories, Item 4 of Definition 11.2.7.1.2](#)), and thus we have

$$[[ -, \emptyset]_X, \emptyset]_X \cong \text{id}_{\mathcal{P}(X)}$$

This finishes the proof.

*Item 6, Interaction With the Empty Set III:* Since  $D_X = (-)^c$ , this is essentially a repetition of the corresponding results for  $(-)^c$ , namely *Items 5 to 7 of Definition 4.3.11.1.2*.

*Item 7, Interaction With Unions of Families of Subsets I:* By *Item 3 of Definition 4.4.7.1.3*, we have

$$\begin{aligned} [\mathcal{U}, \emptyset]_{\mathcal{P}(X)} &= \mathcal{U}^c, \\ [U, \emptyset]_X &= U^c. \end{aligned}$$

With this, the counterexample given in the proof of *Item 10 of Definition 4.3.6.1.2* then applies.

*Item 8, Interaction With Unions of Families of Subsets II:* We have

$$\begin{aligned} \left[ \bigcup_{U \in \mathcal{U}} U, V \right]_X &\stackrel{\text{def}}{=} \left( \bigcup_{U \in \mathcal{U}} U \right)^c \cup V \\ &= \left( \bigcap_{U \in \mathcal{U}} U^c \right) \cup V \\ &= \bigcap_{U \in \mathcal{U}} (U^c \cup V) \\ &\stackrel{\text{def}}{=} \bigcap_{U \in \mathcal{U}} [U, V]_X, \end{aligned}$$

where we have used:

1. *Item 11 of Definition 4.3.6.1.2* for the second equality.
2. *Item 6 of Definition 4.3.7.1.2* for the third equality.

This finishes the proof.

*Item 9, Interaction With Unions of Families of Subsets III:* We have

$$\begin{aligned} \bigcup_{V \in \mathcal{V}} [U, V]_X &\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} (U^c \cup V) \\ &= U^c \cup \left( \bigcup_{V \in \mathcal{V}} V \right) \\ &\stackrel{\text{def}}{=} \left[ U, \bigcup_{V \in \mathcal{V}} V \right]_X. \end{aligned}$$

where we have used *Item 6*. This finishes the proof.

*Item 10, Interaction With Intersections of Families of Subsets I:* Let  $X = \{0, 1\}$ , let  $\mathcal{U} = \{\{0, 1\}\}$ , and let  $\mathcal{V} = \{\{0\}, \{0, 1\}\}$ . We have

$$\begin{aligned} \bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W &= \bigcap_{W \in \mathcal{P}(X)} W \\ &= \{0, 1\}, \end{aligned}$$

whereas

$$\begin{aligned} \left[ \bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X &= [\{0, 1\}, \{0\}] \\ &= \{0\}, \end{aligned}$$

Thus we have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[ \bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X.$$

This finishes the proof.

*Item 11, Interaction With Intersections of Families of Subsets II:* We have

$$\begin{aligned} \left[ \bigcap_{U \in \mathcal{U}} U, V \right]_X &\stackrel{\text{def}}{=} \left( \bigcap_{U \in \mathcal{U}} U \right)^c \cup V \\ &= \left( \bigcup_{U \in \mathcal{U}} U^c \right) \cup V \\ &= \bigcup_{U \in \mathcal{U}} (U^c \cup V) \\ &\stackrel{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X, \end{aligned}$$

where we have used:

1. *Item 12* of *Definition 4.3.6.1.2* for the second equality.
2. *Item 6* of *Definition 4.3.7.1.2* for the third equality.

This finishes the proof.

*Item 12, Interaction With Intersections of Families of Subsets III:* We have

$$\bigcap_{V \in \mathcal{V}} [U, V]_X \stackrel{\text{def}}{=} \bigcap_{V \in \mathcal{V}} (U^c \cup V)$$

$$\begin{aligned}
&= U^c \cup \left( \bigcap_{V \in \mathcal{V}} V \right) \\
&\stackrel{\text{def}}{=} \left[ U, \bigcap_{V \in \mathcal{V}} V \right]_X.
\end{aligned}$$

where we have used **Item 6**. This finishes the proof.

**Item 13, Interaction With Binary Unions:** We have

$$\begin{aligned}
[U \cap V, W]_X &\stackrel{\text{def}}{=} (U \cap V)^c \cup W \\
&= (U^c \cup V^c) \cup W \\
&= (U^c \cup V^c) \cup (W \cup W) \\
&= (U^c \cup W) \cup (V^c \cup W) \\
&\stackrel{\text{def}}{=} [U, W]_X \cup [V, W]_X,
\end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.3.11.1.2** for the second equality.
2. **Item 8** of **Definition 4.3.8.1.2** for the third equality.
3. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the fourth equality.

For the second equality in the statement, we have

$$\begin{aligned}
[U, V \cap W]_X &\stackrel{\text{def}}{=} U^c \cup (V \cap W) \\
&= (U^c \cup V) \cap (U^c \cap W) \\
&\stackrel{\text{def}}{=} [U, V]_X \cap [U, W]_X,
\end{aligned}$$

where we have used **Item 6** of **Definition 4.3.8.1.2** for the second equality.

**Item 14, Interaction With Binary Intersections:** We have

$$\begin{aligned}
[U \cup V, W]_X &\stackrel{\text{def}}{=} (U \cup V)^c \cup W \\
&= (U^c \cap V^c) \cup W \\
&= (U^c \cup W) \cap (V^c \cup W) \\
&\stackrel{\text{def}}{=} [U, W]_X \cap [V, W]_X,
\end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.3.11.1.2** for the second equality.
2. **Item 6** of **Definition 4.3.8.1.2** for the third equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
 [U, V \cup W]_X &\stackrel{\text{def}}{=} U^c \cup (V \cup W) \\
 &= (U^c \cup U^c) \cup (V \cup W) \\
 &= (U^c \cup V) \cup (U^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, V]_X \cup [U, W]_X,
 \end{aligned}$$

where we have used:

1. **Item 8** of **Definition 4.3.8.1.2** for the second equality.
2. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the third equality.

This finishes the proof.

**Item 15, Interaction With Differences:** We have

$$\begin{aligned}
 [U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^c \cup W \\
 &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
 &= ((X \cap V) \cup (X \setminus U)) \cup W \\
 &= (V \cup (X \setminus U)) \cup W \\
 &\stackrel{\text{def}}{=} (V \cup U^c) \cup W \\
 &= (V \cup (U^c \cup U^c)) \cup W \\
 &= (U^c \cup W) \cup (U^c \cup V) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cup [U, V]_X,
 \end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.10.1.2** for the third equality.
2. **Item 4** of **Definition 4.3.9.1.2** for the fourth equality.
3. **Item 8** of **Definition 4.3.8.1.2** for the sixth equality.
4. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the seventh equality.

We also have

$$\begin{aligned}
 [U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^c \cup W \\
 &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
 &= ((X \cap V) \cup (X \setminus U)) \cup W
 \end{aligned}$$

$$\begin{aligned}
&= (V \cup (X \setminus U)) \cup W \\
&\stackrel{\text{def}}{=} (V \cup U^c) \cup W \\
&= (V \cup U^c) \cup (W \cup W) \\
&= (U^c \cup W) \cup (V \cup W) \\
&= (U^c \cup W) \cup ((V^c)^c \cup W) \\
&\stackrel{\text{def}}{=} [U, W]_X \cup [V^c, W]_X,
\end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.10.1.2** for the third equality.
2. **Item 4** of **Definition 4.3.9.1.2** for the fourth equality.
3. **Item 8** of **Definition 4.3.8.1.2** for the sixth equality.
4. Several applications of **Items 2** and **4** of **Definition 4.3.8.1.2** and for the seventh equality.
5. **Item 3** of **Definition 4.3.11.1.2** for the eighth equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
[U, V \setminus W]_X &\stackrel{\text{def}}{=} U^c \cup (V \setminus W) \\
&= (V \setminus W) \cup U^c \\
&= (V \cup U^c) \setminus (W \setminus U^c) \\
&\stackrel{\text{def}}{=} (V \cup U^c) \setminus (W \setminus (X \setminus U)) \\
&= (V \cup U^c) \setminus ((W \cap U) \cup (W \setminus X)) \\
&= (V \cup U^c) \setminus ((W \cap U) \cup \emptyset) \\
&= (V \cup U^c) \setminus (W \cap U) \\
&= (V \cup U^c) \setminus (U \cap W) \\
&\stackrel{\text{def}}{=} [U, V]_X \setminus (U \cap W)
\end{aligned}$$

where we have used:

1. **Item 4** of **Definition 4.3.8.1.2** for the second equality.
2. **Item 4** of **Definition 4.3.10.1.2** for the third equality.
3. **Item 10** of **Definition 4.3.10.1.2** for the fifth equality.
4. **Item 13** of **Definition 4.3.10.1.2** for the sixth equality.
5. **Item 3** of **Definition 4.3.8.1.2** for the seventh equality.

6. **Item 5** of **Definition 4.3.9.1.2** for the eighth equality.

This finishes the proof.

**Item 16, Interaction With Complements:** We have

$$\begin{aligned} [U^c, V]_X &\stackrel{\text{def}}{=} (U^c)^c \cup V, \\ &= U \cup V, \end{aligned}$$

where we have used **Item 3** of **Definition 4.3.11.1.2**. We also have

$$\begin{aligned} [U, V^c]_X &\stackrel{\text{def}}{=} U^c \cup V^c \\ &= U \cap V \end{aligned}$$

where we have used **Item 2** of **Definition 4.3.11.1.2**. Finally, we have

$$\begin{aligned} [U, V]_X^c &= ((U \setminus V)^c)^c \\ &= U \setminus V, \end{aligned}$$

where we have used **Item 2** of **Definition 4.3.11.1.2**.

**Item 17, Interaction With Characteristic Functions:** We have

$$\begin{aligned} \chi_{[U, V]_{\mathcal{P}(X)}}(x) &\stackrel{\text{def}}{=} \chi_{U^c \cup V}(x) \\ &= \max(\chi_{U^c}, \chi_V) \\ &= \max(1 - \chi_U \pmod{2}, \chi_V), \end{aligned}$$

where we have used:

1. **Item 10** of **Definition 4.3.8.1.2** for the second equality.
2. **Item 4** of **Definition 4.3.11.1.2** for the third equality.

This finishes the proof.

**Item 18, Interaction With Direct Images:** This is a repetition of **Item 10** of **Definition 4.6.1.1.5** and is proved there.

**Item 19, Interaction With Inverse Images:** This is a repetition of **Item 10** of **Definition 4.6.2.1.3** and is proved there.

**Item 20, Interaction With Codirect Images:** This is a repetition of **Item 9** of **Definition 4.6.3.1.7** and is proved there.  $\square$



### 4.4.8 Isbell Duality for Sets

Let  $X$  be a set.

**Definition 4.4.8.1.1.** The **Isbell function** of  $X$  is the map

$$l: \mathcal{P}(X) \rightarrow \text{Sets}(X, \mathcal{P}(X))$$

defined by

$$l(U) \stackrel{\text{def}}{=} \llbracket x \mapsto [U, \{x\}]_X \rrbracket$$

for each  $U \in \mathcal{P}(X)$ .

**Remark 4.4.8.1.2.** Recall from [Definition 4.4.1.1.2](#) that we may view the powerset  $\mathcal{P}(X)$  of a set  $X$  as the decategorification of the category of presheaves  $\text{PSh}(C)$  of a category  $C$ . Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\text{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \text{Nat}(\mathcal{F}, h_{(-)})$$

for each  $\mathcal{F} \in \text{Obj}(\text{PSh}(C))$ . To this end, we could define

$$l(U) \stackrel{\text{def}}{=} [U, \chi_{(-)}]_X,$$

replacing:

- The Yoneda embedding  $X \mapsto h_X$  of  $C$  into  $\text{PSh}(C)$  with the characteristic embedding  $x \mapsto \chi_x$  of  $X$  into  $\mathcal{P}(X)$  of [Definition 4.5.4.1.1](#).
- The internal Hom  $\text{Nat}$  of  $\text{PSh}(C)$  with the internal Hom  $[-, -]_X$  of  $\mathcal{P}(X)$  of [Definition 4.4.7.1.1](#).

However, since  $[U, \chi_x]_X$  is a subset of  $U$  instead of a truth value, we get a function

$$l: \mathcal{P}(X) \rightarrow \text{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$l: \mathcal{P}(X) \rightarrow \mathcal{P}(X).$$

This makes some of the properties involving  $l$  a bit more cumbersome to state, although we still have an analogue of Isbell duality in that  $l_! \circ l$  evaluates to  $\text{id}_{\mathcal{P}(X)}$  in the sense of [Definition 4.4.8.1.3](#).

**Proposition 4.4.8.1.3.** The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) & \xrightarrow{I!} & \text{Sets}(X, \mathcal{P}(X)) \\
 & \searrow \Delta_{\Delta_{\text{id}_{\mathcal{P}(X)}}} & \downarrow I! \\
 & & \text{Sets}(X, \text{Sets}(X, \mathcal{P}(X)))
 \end{array}$$

commutes, i.e. we have

$$I!(I(U)) = \llbracket x \mapsto \llbracket y \mapsto U \rrbracket \rrbracket$$

for each  $U \in \mathcal{P}(X)$ .

*Proof.* We have

$$\begin{aligned}
 I!(I(U)) &\stackrel{\text{def}}{=} I!(\llbracket x \mapsto U^c \cup \{x\} \rrbracket) \\
 &\stackrel{\text{def}}{=} \llbracket x \mapsto I(U^c \cup \{x\}) \rrbracket \\
 &\stackrel{\text{def}}{=} \llbracket x \mapsto \llbracket y \mapsto (U^c \cup \{x\})^c \cup \{x\} \rrbracket \rrbracket \\
 &= \llbracket x \mapsto \llbracket y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\} \rrbracket \rrbracket \\
 &= \llbracket x \mapsto \llbracket y \mapsto (U \setminus \{x\}) \cup \{x\} \rrbracket \rrbracket \\
 &= \llbracket x \mapsto \llbracket y \mapsto U \rrbracket \rrbracket,
 \end{aligned}$$

where we have used **Item 2** of **Definition 4.3.11.1.2** for the fourth equality above.  $\square$

## 4.5 Characteristic Functions

### 4.5.1 The Characteristic Function of a Subset

Let  $X$  be a set and let  $U \in \mathcal{P}(X)$ .

**Definition 4.5.1.1.1.** The **characteristic function of  $U$** <sup>23</sup> is the function  $\chi_U : X \rightarrow \{\text{t}, \text{f}\}$ <sup>24</sup> defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each  $x \in X$ .

<sup>23</sup>*Further Terminology:* Also called the **indicator function of  $U$** .

<sup>24</sup>*Further Notation:* Also written  $\chi_X(U, -)$  or  $\chi_X(-, U)$ .

**Remark 4.5.1.1.2.** Under the analogy that  $\{t, f\}$  should be the  $(-1)$ -categorical analogue of Sets, we may view a function

$$f: X \rightarrow \{t, f\}$$

as a decategorification of presheaves and copresheaves

$$\begin{aligned} \mathcal{F}: C^{\text{op}} &\rightarrow \text{Sets}, \\ F: C &\rightarrow \text{Sets}. \end{aligned}$$

The characteristic functions  $\chi_U$  of the subsets of  $X$  are then the primordial examples of such functions (and, in fact, all of them).

**Notation 4.5.1.1.3.** We will often employ the bijection  $\{t, f\} \cong \{0, 1\}$  to make use of the arithmetical operations defined on  $\{0, 1\}$  when discussing characteristic functions.

Examples of this include **Items 4 to 11** of **Definition 4.5.1.1.4** below.

**Proposition 4.5.1.1.4.** Let  $X$  be a set.

1. *Functionality.* The assignment  $U \mapsto \chi_U$  defines a function

$$\chi_{(-)}: \mathcal{P}(X) \rightarrow \text{Sets}(X, \{t, f\}).$$

2. *Bijectivity.* The function  $\chi_{(-)}$  from **Item 1** is bijective.
3. *Naturality.* The collection

$$\{\chi_{(-)}: \mathcal{P}(X) \rightarrow \text{Sets}(X, \{t, f\})\}_{X \in \text{Obj}(\text{Sets})}$$

defines a natural isomorphism between  $\mathcal{P}^{-1}$  and  $\text{Sets}(-, \{t, f\})$ . In particular, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc} \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \\ \chi_{(-)} \downarrow \wr & & \wr \downarrow \chi_{(-)} \\ \text{Sets}(Y, \{t, f\}) & \xrightarrow{f^*} & \text{Sets}(X, \{t, f\}) \end{array}$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each  $V \in \mathcal{P}(Y)$ .

4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each  $U, V \in \mathcal{P}(X)$ .

5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each  $U, V \in \mathcal{P}(X)$ .

6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each  $U, V \in \mathcal{P}(X)$ .

7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each  $U, V \in \mathcal{P}(X)$ .

8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each  $U, V \in \mathcal{P}(X)$ .

9. *Interaction With Complements.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each  $U \in \mathcal{P}(X)$ .

10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \Delta V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \Delta V} \equiv \chi_U + \chi_V \pmod{2}$$

for each  $U, V \in \mathcal{P}(X)$ .

11. *Interaction With Internal Homs.* We have

$$\chi_{[U,V]\mathcal{P}(X)} = \max(1 - \chi_U \pmod{2}, \chi_V)$$

for each  $U, V \in \mathcal{P}(X)$ .

*Proof.* **Item 1, Functionality:** There is nothing to prove.

**Item 2, Bijectivity:** We proceed in three steps:

1. *The Inverse of  $\chi_{(-)}$ .* The inverse of  $\chi_{(-)}$  is the map

$$\Phi: \text{Sets}(X, \{\text{t}, \text{f}\}) \xrightarrow{\sim} \mathcal{P}(X),$$

defined by

$$\begin{aligned} \Phi(f) &\stackrel{\text{def}}{=} U_f \\ &\stackrel{\text{def}}{=} f^{-1}(\text{true}) \\ &\stackrel{\text{def}}{=} \{x \in X \mid f(x) = \text{true}\} \end{aligned}$$

for each  $f \in \text{Sets}(X, \{\text{t}, \text{f}\})$ .

2. *Invertibility I.* We have

$$\begin{aligned} [\Phi \circ \chi_{(-)}](U) &\stackrel{\text{def}}{=} \Phi(\chi_U) \\ &\stackrel{\text{def}}{=} \chi_U^{-1}(\text{true}) \\ &\stackrel{\text{def}}{=} \{x \in X \mid \chi_U(x) = \text{true}\} \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U\} \\ &= U \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathcal{P}(X)}](U) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ . Thus, we have

$$\Phi \circ \chi_{(-)} = \text{id}_{\mathcal{P}(X)}.$$

3. *Invertibility II.* We have

$$\begin{aligned} [\chi_{(-)} \circ \Phi](U) &\stackrel{\text{def}}{=} \chi_{\Phi(f)} \\ &\stackrel{\text{def}}{=} \chi_{f^{-1}(\text{true})} \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \text{true} & \text{if } x \in f^{-1}(\text{true}) \\ \text{false} & \text{otherwise} \end{cases} \rrbracket \\ &= \llbracket x \mapsto f(x) \rrbracket \end{aligned}$$

$$\begin{aligned}
&= f \\
&\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(X, \{t, f\})}](f)
\end{aligned}$$

for each  $f \in \text{Sets}(X, \{t, f\})$ . Thus, we have

$$\chi(-) \circ \Phi = \text{id}_{\text{Sets}(X, \{t, f\})}.$$

This finishes the proof.

**Item 3, Naturality:** We proceed in two steps:

1. *Naturality of  $\chi(-)$ .* We have

$$\begin{aligned}
[\chi_V \circ f](v) &\stackrel{\text{def}}{=} \chi_V(f(v)) \\
&= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases} \\
&\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v)
\end{aligned}$$

for each  $v \in V$ .

2. *Naturality of  $\Phi$ .* Since  $\chi(-)$  is natural and a componentwise inverse to  $\Phi$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\Phi$  is also natural in each argument.

This finishes the proof.

**Item 4, Interaction With Unions I:** This is a repetition of **Item 10** of **Definition 4.3.8.1.2** and is proved there.

**Item 5, Interaction With Unions II:** This is a repetition of **Item 11** of **Definition 4.3.8.1.2** and is proved there.

**Item 6, Interaction With Intersections I:** This is a repetition of **Item 10** of **Definition 4.3.9.1.2** and is proved there.

**Item 7, Interaction With Intersections II:** This is a repetition of **Item 11** of **Definition 4.3.9.1.2** and is proved there.

**Item 8, Interaction With Differences:** This is a repetition of **Item 16** of **Definition 4.3.10.1.2** and is proved there.

**Item 9, Interaction With Complements:** This is a repetition of **Item 4** of **Definition 4.3.11.1.2** and is proved there.

**Item 10, Interaction With Symmetric Differences:** This is a repetition of **Item 15** of **Definition 4.3.12.1.2** and is proved there.

**Item 11, Interaction With Internal Homs:** This is a repetition of **Item 17** of **Definition 4.4.7.1.3** and is proved there.  $\square$

**Remark 4.5.1.1.5.** The bijection

$$\mathcal{P}(X) \cong \text{Sets}(X, \{\text{t}, \text{f}\})$$

of **Item 2** of **Definition 4.5.1.1.4**, which

- Takes a subset  $U \hookrightarrow X$  of  $X$  and *straightens* it to a function  $\chi_U: X \rightarrow \{\text{true}, \text{false}\}$ ;
- Takes a function  $f: X \rightarrow \{\text{true}, \text{false}\}$  and *unstraightens* it to a subset  $f^{-1}(\text{true}) \hookrightarrow X$  of  $X$ ;

may be viewed as the  $(-1)$ -categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\text{FibSets}_X}_{\stackrel{\text{def}}{=} \text{Sets}_{/X}} \cong \underbrace{\text{ISets}_X}_{\stackrel{\text{def}}{=} \text{Fun}(X_{\text{disc}}, \text{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, ???. Here we view:

- Subsets  $U \hookrightarrow X$  as being analogous to  $X$ -fibred sets  $\phi_X: A \rightarrow X$ .
- Functions  $f: X \rightarrow \{\text{t}, \text{f}\}$  as being analogous to  $X$ -indexed sets  $A: X_{\text{disc}} \rightarrow \text{Sets}$ .

## 4.5.2 The Characteristic Function of a Point

Let  $X$  be a set and let  $x \in X$ .

**Definition 4.5.2.1.1.** The **characteristic function of  $x$**  is the function<sup>25</sup>

$$\chi_x: X \rightarrow \{\text{t}, \text{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each  $y \in X$ .

**Remark 4.5.2.1.2.** Expanding upon **Definition 4.5.1.1.2**, we may think of the

<sup>25</sup>*Further Notation:* Also written  $\chi^x$ ,  $\chi_X(x, -)$ , or  $\chi_X(-, x)$ .

characteristic function

$$\chi_x: X \rightarrow \{t, f\}$$

of an *element*  $x$  of  $X$  as a decategorification of the representable presheaf and of the representable copresheaf

$$\begin{aligned} h_X: C^{\text{op}} &\rightarrow \text{Sets}, \\ h^X: C &\rightarrow \text{Sets} \end{aligned}$$

associated of an *object*  $X$  of a category  $C$ .

### 4.5.3 The Characteristic Relation of a Set

Let  $X$  be a set.

**Definition 4.5.3.1.1.** The **characteristic relation on  $X$** <sup>26</sup> is the relation<sup>27</sup>

$$\chi_X(-, -): X \times X \rightarrow \{t, f\}$$

on  $X$  defined by<sup>28</sup>

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each  $x, y \in X$ .

**Remark 4.5.3.1.2.** Expanding upon **Definitions 4.5.1.1.2** and **4.5.2.1.2**, we may view the characteristic relation

$$\chi_X(-, -): X \times X \rightarrow \{t, f\}$$

of  $X$  as a decategorification of the Hom profunctor

$$\text{Hom}_C(-, -): C^{\text{op}} \times C \rightarrow \text{Sets}$$

of a category  $C$ .

**Proposition 4.5.3.1.3.** Let  $f: X \rightarrow Y$  be a function.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let  $f: A \rightarrow$

<sup>26</sup>*Further Terminology:* Also called the **identity relation on  $X$** .

<sup>27</sup>*Further Notation:* Also written  $\chi_X^{-1}$ , or  $\sim_{\text{id}}$  in the context of relations.

<sup>28</sup>Under the bijection  $\text{Sets}(X \times X, \{t, f\}) \cong \mathcal{P}(X \times X)$  of **Item 2** of **Definition 4.5.1.1.4**, the relation  $\chi_X$  corresponds to the diagonal  $\Delta_X \subset X \times X$  of  $X$ .



$B$  be a function. We have an inclusion<sup>29</sup>

$$\chi_B \circ (f \times f) \subset \chi_A,$$

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \chi_A \searrow & \circlearrowleft & \swarrow \chi_B \\ & \{t, f\}. & \end{array}$$

*Proof.* **Item 1**, *The Inclusion of Characteristic Relations Associated to a Function*: The inclusion  $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$  is equivalent to the statement “if  $a = b$ , then  $f(a) = f(b)$ ”, which is true.  $\square$

#### 4.5.4 The Characteristic Embedding of a Set

Let  $X$  be a set.

**Definition 4.5.4.1.1.** The **characteristic embedding**<sup>30</sup> of  $X$  into  $\mathcal{P}(X)$  is the function

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

defined by<sup>31</sup>

$$\begin{aligned} \chi_{(-)}(x) &\stackrel{\text{def}}{=} \chi_x \\ &= \{x\} \end{aligned}$$

for each  $x \in X$ .

**Remark 4.5.4.1.2.** Expanding upon **Definitions 4.5.1.1.2**, **4.5.2.1.2** and **4.5.3.1.2**, we may view the characteristic embedding

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

of  $X$  into  $\mathcal{P}(X)$  as a decategorification of the Yoneda embedding

$$\mathfrak{Y} : C^{\text{op}} \hookrightarrow \text{PSh}(C)$$

of a category  $C$  into  $\text{PSh}(C)$ .

**Proposition 4.5.4.1.3.** Let  $f : X \rightarrow Y$  be a map of sets.

<sup>29</sup>Note: This is the 0-categorical version of **Categories**, **Definition 11.5.4.1.1**.

<sup>30</sup>The name “characteristic embedding” is justified by **Definition 4.5.5.1.2**, which gives an analogue of fully faithfulness for  $\chi_{(-)}$ .

<sup>31</sup>Here we are identifying  $\mathcal{P}(X)$  with  $\text{Sets}(X, \{t, f\})$  as per **Item 2** of **Definition 4.5.1.1.4**.

1. *Interaction With Functions.* We have

$$f! \circ \chi_X = \chi_Y \circ f, \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \chi_X \downarrow & & \downarrow \chi_Y \\ \mathcal{P}(X) & \xrightarrow{f!} & \mathcal{P}(Y). \end{array}$$

*Proof.* **Item 1**, *Interaction With Functions*: Indeed, we have

$$\begin{aligned} [f! \circ \chi_X](x) &\stackrel{\text{def}}{=} f!(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f!(\{x\}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x), \end{aligned}$$

for each  $x \in X$ , showing the desired equality. □

### 4.5.5 The Yoneda Lemma for Sets

Let  $X$  be a set and let  $U \subset X$  be a subset of  $X$ .

**Proposition 4.5.5.1.1.** We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each  $x \in X$ , giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi(-), \chi_U) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } U \subset V, \\ \text{false} & \text{otherwise.} \end{cases}$$

*Proof.* We have

$$\begin{aligned} \chi_{\mathcal{P}(X)}(\chi_x, \chi_U) &\stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } \{x\} \subset U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } x \in U \\ \text{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_U(x). \end{aligned}$$

This finishes the proof. □

**Corollary 4.5.5.1.2.** The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) \cong \chi_X(x, y)$$

for each  $x, y \in X$ .

*Proof.* We have

$$\begin{aligned} \chi_{\mathcal{P}(X)}(\chi_x, \chi_y) &= \chi_y(x) \\ &\stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in \{y\} \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } x = y \\ \text{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_X(x, y). \end{aligned}$$

where we have used [Definition 4.5.5.1.1](#) for the first equality.  $\square$

## 4.6 The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

### 4.6.1 Direct Images

Let  $f: X \rightarrow Y$  be a function.

**Definition 4.6.1.1.1.** The **direct image function associated to  $f$**  is the function<sup>32</sup>

$$f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by<sup>33</sup>

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ y \in Y \mid \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\} \\ &= \{f(x) \in Y \mid x \in U\} \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

**Notation 4.6.1.1.2.** Sometimes one finds the notation

$$\exists_f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

for  $f_!$ . This notation comes from the fact that the following statements are equivalent, where  $y \in Y$  and  $U \in \mathcal{P}(X)$ :

<sup>32</sup>*Further Notation:* Also written simply  $f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ .

<sup>33</sup>*Further Terminology:* The set  $f(U)$  is called the **direct image of  $U$  by  $f$** .

- We have  $y \in \exists_f(U)$ .
- There exists some  $x \in U$  such that  $f(x) = y$ .

We will not make use of this notation elsewhere in Clowder.

**Warning 4.6.1.1.3.** Notation for direct images between powersets is tricky:

1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:

- (a) *Powersets.* Given a function  $f: X \rightarrow Y$ , we have an inverse image functor

$$f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X).$$

The *left* adjoint of this functor is the usual direct image, defined above in [Definition 4.6.1.1.1](#).

- (b) *Presheaves.* Given a morphism of topological spaces  $f: X \rightarrow Y$ , we have an inverse image functor

$$f^{-1}: \text{PSh}(Y) \rightarrow \text{PSh}(X).$$

The *right* adjoint of this functor is the direct image functor of presheaves, defined in ??.

2. The presheaf direct image functor is denoted  $f_*$ , but the direct image functor for powersets is denoted  $f_!$  (as it's a left adjoint).
3. Adding to the confusion, it's somewhat common for  $f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  to be denoted  $f_*$ .

We chose to write  $f_!$  for the direct image to keep the notation aligned with the following similar adjoint situations:

SITUATION	ADJOINT STRING
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*): \mathcal{P}(X) \xrightarrow{\cong} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*): \text{PSh}(X) \xrightarrow{\cong} \text{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*): \mathcal{C}_{/X} \xrightarrow{\cong} \mathcal{C}_{/Y}$
Kan Extensions	$(F_! \dashv F^* \dashv F_*): \text{Fun}(C, \mathcal{E}) \xrightarrow{\cong} \text{Fun}(\mathcal{D}, \mathcal{E})$

**Remark 4.6.1.1.4.** Identifying  $\mathcal{P}(X)$  with  $\text{Sets}(X, \{t, f\})$  via **Item 2 of Definition 4.5.1.1.4**, we see that the direct image function associated to  $f$  is equivalently the function

$$f_! : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \text{Lan}_f(\chi_U) \\ &= \text{colim}((f \times \underline{(-1)}) \xrightarrow{\text{pr}} A \xrightarrow{\chi_U} \{t, f\}) \\ &= \text{colim}_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)) \\ &= \bigvee_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)), \end{aligned}$$

where we have used ?? for the second equality. In other words, we have

$$\begin{aligned} [f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \text{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each  $y \in Y$ .

**Proposition 4.6.1.1.5.** Let  $f : X \rightarrow Y$  be a function.

1. *Functoriality.* The assignment  $U \mapsto f_!(U)$  defines a functor

$$f_! : (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset).$$

In particular, for each  $U, V \in \mathcal{P}(X)$ , the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f_!(U) \subset f_!(V).$$

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \quad \mathcal{P}(X) \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{array} \mathcal{P}(Y),$$

witnessed by:

(a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ .

(b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$  and (respectively)  $V \in \mathcal{P}(X)$  and  $U \in \mathcal{P}(Y)$ . In particular:

- i. The following conditions are equivalent:
  - A. We have  $f_!(U) \subset V$ .
  - B. We have  $U \subset f^{-1}(V)$ .
- ii. The following conditions are equivalent:
  - A. We have  $f^{-1}(U) \subset V$ .
  - B. We have  $U \subset f_*(V)$ .

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(X)$ , where  $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$ .

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ , where  $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$ .

5. *Interaction With Binary Unions.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each  $U, V \in \mathcal{P}(X)$ .

6. *Interaction With Binary Intersections.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & \subset & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

7. *Interaction With Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \setminus & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

8. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U^c) = f_*(U)^c$$

for each  $U \in \mathcal{P}(X)$ .

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \Delta & \supset & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \Delta f_!(V) \subset f_!(U \Delta V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .



10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in  $U, V \in \mathcal{P}(X)$ .

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_!(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} f_!(U) \cup f_!(V) &= f_!(U \cup V), \\ f_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_!(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have inclusions

$$\begin{aligned} f_!(U \cap V) &\subset f_!(U) \cap f_!(V), \\ f_!(X) &\subset Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_!, f_!^{\otimes}, f_{!1}^{\otimes}): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^{\otimes} &: f_!(U) \cup f_!(V) \xrightarrow{=} f_!(U \cup V), \\ f_{!|1}^{\otimes} &: \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|1}^{\otimes}) : (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} &: f_!(U \cap V) \hookrightarrow f_!(U) \cap f_!(V), \\ f_{!|1}^{\otimes} &: f_!(X) \hookrightarrow Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

15. *Interaction With Coproducts.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be maps of sets. We have

$$(f \amalg g)_!(U \amalg V) = f_!(U) \amalg g_!(V)$$

for each  $U \in \mathcal{P}(X)$  and each  $V \in \mathcal{P}(Y)$ .

16. *Interaction With Products.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each  $U \in \mathcal{P}(X)$  and each  $V \in \mathcal{P}(Y)$ .

17. *Relation to Codirect Images.* We have

$$\begin{aligned} f_!(U) &= f_*(U^c)^c \\ &\stackrel{\text{def}}{=} Y \setminus f_*(X \setminus U) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

*Proof.* **Item 1**, *Functoriality*: Omitted.

**Item 2**, *Triple Adjointness*: This follows from **Definition 4.6.1.1.4**, **Definition 4.6.2.1.2**, **Definition 4.6.3.1.4**, and Kan Extensions, ?? of ??.

**Item 3**, *Interaction With Unions of Families of Subsets*: We have

$$\bigcup_{V \in f_!(\mathcal{U})} V = \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$

$$= \bigcup_{U \in \mathcal{U}} f_!(U).$$

This finishes the proof.

*Item 4, Interaction With Intersections of Families of Subsets:* We have

$$\begin{aligned} \bigcap_{V \in f_!(\mathcal{U})} V &= \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.

*Item 5, Interaction With Binary Unions:* See [Pro25p].

*Item 6, Interaction With Binary Intersections:* See [Pro25n].

*Item 7, Interaction With Differences:* See [Pro25o].

*Item 8, Interaction With Complements:* Applying *Item 17* to  $X \setminus U$ , we have

$$\begin{aligned} f_!(U^c) &= f_!(X \setminus U) \\ &= Y \setminus f_*(X \setminus (X \setminus U)) \\ &= Y \setminus f_*(U) \\ &= f_*(U)^c. \end{aligned}$$

This finishes the proof.

*Item 9, Interaction With Symmetric Differences:* We have

$$\begin{aligned} f_!(U) \Delta f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\ &\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\ &= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\ &\subset f_!((U \cup V) \setminus (U \cap V)) \\ &= f_!(U \Delta V), \end{aligned}$$

where we have used:

1. *Item 2* of *Definition 4.3.12.1.2* for the first equality.
2. *Item 6* of this proposition together with *Item 1* of *Definition 4.3.10.1.2* for the first inclusion.
3. *Item 5* for the second equality.
4. *Item 7* for the second inclusion.
5. *Item 2* of *Definition 4.3.12.1.2* for the third equality.

Since  $\mathcal{P}(Y)$  is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Definition 11.2.7.1.2](#)). This finishes the proof.

[Item 10](#), *Interaction With Internal Homs of Powersets*: We have

$$\begin{aligned} f_!([U, V]_X) &\stackrel{\text{def}}{=} f_!(U^c \cup V) \\ &= f_!(U^c) \cup f_!(V) \\ &= f_*(U)^c \cup f_!(V) \\ &\stackrel{\text{def}}{=} [f_*(U), f_!(V)]_Y, \end{aligned}$$

where we have used:

1. [Item 5](#) for the second equality.
2. [Item 17](#) for the third equality.

Since  $\mathcal{P}(Y)$  is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Definition 11.2.7.1.2](#)). This finishes the proof.

[Item 11](#), *Preservation of Colimits*: This follows from [Item 2](#) and ??, ?? of ??.<sup>34</sup>

[Item 12](#), *Oplax Preservation of Limits*: The inclusion  $f_!(X) \subset Y$  is automatic. See [\[Pro25n\]](#) for the other inclusions.

[Item 13](#), *Symmetric Strict Monoidality With Respect to Unions*: This follows from [Item 11](#).

[Item 14](#), *Symmetric Oplax Monoidality With Respect to Intersections*: The inclusions in the statement follow from [Item 12](#). Since  $\mathcal{P}(Y)$  is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic ([Categories](#), [Item 4](#) of [Definition 11.2.7.1.2](#)).

[Item 15](#), *Interaction With Coproducts*: Omitted.

[Item 16](#), *Interaction With Products*: Omitted.

[Item 17](#), *Relation to Codirect Images*: Applying [Item 16](#) of [Definition 4.6.3.1.7](#) to  $X \setminus U$ , we have

$$\begin{aligned} f_*(X \setminus U) &= B \setminus f_!(X \setminus (X \setminus U)) \\ &= B \setminus f_!(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_!(U) &= B \setminus (B \setminus f_!(U)), \\ &= B \setminus f_*(X \setminus U), \end{aligned}$$

which finishes the proof. □

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<sup>34</sup>Reference: [\[Pro25p\]](#).

**Proposition 4.6.1.1.6.** Let  $f: X \rightarrow Y$  be a function.

1. *Functionality I.* The assignment  $f \mapsto f_!$  defines a function

$$(-)_{*|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment  $f \mapsto f_!$  defines a function

$$(-)_{*|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have

$$(\text{id}_X)_! = \text{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \\ & \searrow (g \circ f)_! & \downarrow g_! \\ & & \mathcal{P}(Z). \end{array}$$

*Proof.* **Item 1, Functionality I:** There is nothing to prove.

**Item 2, Functionality II:** This follows from **Item 1** of **Definition 4.6.1.1.5**.

**Item 3, Interaction With Identities:** This follows from **Definition 4.6.1.1.4** and Kan Extensions, ?? of ??.

**Item 4, Interaction With Composition:** This follows from **Definition 4.6.1.1.4** and Kan Extensions, ?? of ??.

□

## 4.6.2 Inverse Images

Let  $f: X \rightarrow Y$  be a function.

**Definition 4.6.2.1.1.** The **inverse image function associated to  $f$**  is the function<sup>35</sup>

$$f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by<sup>36</sup>

$$f^{-1}(V) \stackrel{\text{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each  $V \in \mathcal{P}(Y)$ .

<sup>35</sup>*Further Notation:* Also written  $f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ .

<sup>36</sup>*Further Terminology:* The set  $f^{-1}(V)$  is called the **inverse image of  $V$  by  $f$** .

**Remark 4.6.2.1.2.** Identifying  $\mathcal{P}(Y)$  with  $\text{Sets}(Y, \{t, f\})$  via **Item 2** of **Definition 4.5.1.1.4**, we see that the inverse image function associated to  $f$  is equivalently the function

$$f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each  $\chi_V \in \mathcal{P}(Y)$ , where  $\chi_V \circ f$  is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in  $\text{Sets}$ .

**Proposition 4.6.2.1.3.** Let  $f: X \rightarrow Y$  be a function.

1. *Functoriality.* The assignment  $V \mapsto f^{-1}(V)$  defines a functor

$$f^{-1}: (\mathcal{P}(Y), \subset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each  $U, V \in \mathcal{P}(Y)$ , the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f^{-1}(U) \subset f^{-1}(V).$$

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \quad \mathcal{P}(X) \begin{array}{c} \xleftarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xleftarrow{f_*} \end{array} \mathcal{P}(Y),$$

witnessed by:

- (a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ .

(b) Bijections of sets

$$\begin{aligned}\text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)),\end{aligned}$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$  and (respectively)  $V \in \mathcal{P}(X)$  and  $U \in \mathcal{P}(Y)$ . In particular:

- i. The following conditions are equivalent:
  - A. We have  $f_!(U) \subset V$ .
  - B. We have  $U \subset f^{-1}(V)$ .
- ii. The following conditions are equivalent:
  - A. We have  $f^{-1}(U) \subset V$ .
  - B. We have  $U \subset f_*(V)$ .

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc}\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)\end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each  $\mathcal{V} \in \mathcal{P}(Y)$ , where  $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$ .

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc}\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)\end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each  $\mathcal{V} \in \mathcal{P}(Y)$ , where  $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$ .

5. *Interaction With Binary Unions.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each  $U, V \in \mathcal{P}(Y)$ .

6. *Interaction With Binary Intersections.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \downarrow \cap & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each  $U, V \in \mathcal{P}(Y)$ .

7. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow \setminus & & \downarrow \setminus \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each  $U, V \in \mathcal{P}(X)$ .



8. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U^c) = f^{-1}(U)^c$$

for each  $U \in \mathcal{P}(X)$ .

9. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \Delta \downarrow & & \downarrow \Delta \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

i.e. we have

$$f^{-1}(U) \Delta f^{-1}(V) = f^{-1}(U \Delta V)$$

for each  $U, V \in \mathcal{P}(Y)$ .

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ [-1, -2]_Y \downarrow & & \downarrow [-1, -2]_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in  $U, V \in \mathcal{P}(X)$ .

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(Y) &= X, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{1}}^{-1, \otimes}): (\mathcal{P}(Y), \cup, \emptyset) \rightarrow (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U, V}^{-1, \otimes}: f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{1}}^{-1, \otimes}: \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{1}}^{-1, \otimes}): (\mathcal{P}(Y), \cap, Y) \rightarrow (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$\begin{aligned} f_{U, V}^{-1, \otimes}: f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cap V), \\ f_{\mathbb{1}}^{-1, \otimes}: X &\xrightarrow{=} f^{-1}(Y), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

15. *Interaction With Coproducts.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be maps of sets.

We have

$$(f \amalg g)^{-1}(U' \amalg V') = f^{-1}(U') \amalg g^{-1}(V')$$

for each  $U' \in \mathcal{P}(X')$  and each  $V' \in \mathcal{P}(Y')$ .

16. *Interaction With Products.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be maps of sets.

We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each  $U' \in \mathcal{P}(X')$  and each  $V' \in \mathcal{P}(Y')$ .

*Proof.* **Item 1, Functoriality:** Omitted.

**Item 2, Triple Adjointness:** This follows from **Definition 4.6.1.1.4**, **Definition 4.6.2.1.2**, **Definition 4.6.3.1.4**, and Kan Extensions, ?? of ??.

**Item 3, Interaction With Unions of Families of Subsets:** We have

$$\begin{aligned} \bigcup_{U \in f^{-1}(\mathcal{V})} U &= \bigcup_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcup_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.

**Item 4, Interaction With Intersections of Families of Subsets:** We have

$$\begin{aligned} \bigcap_{U \in f^{-1}(\mathcal{V})} U &= \bigcap_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcap_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.

**Item 5, Interaction With Binary Unions:** See **[Pro25y]**.

**Item 6, Interaction With Binary Intersections:** See **[Pro25w]**.

**Item 7, Interaction With Differences:** See **[Pro25x]**.

**Item 8, Interaction With Complements:** See **[Pro25j]**.

**Item 9, Interaction With Symmetric Differences:** We have

$$\begin{aligned} f^{-1}(U \triangle V) &= f^{-1}((U \cup V) \setminus (U \cap V)) \\ &= f^{-1}(U \cup V) \setminus f^{-1}(U \cap V) \\ &= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U \cap V) \\ &= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U) \cap f^{-1}(V) \\ &= f^{-1}(U) \triangle f^{-1}(V), \end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.3.12.1.2** for the first equality.
2. **Item 7** for the second equality.
3. **Item 5** for the third equality.
4. **Item 6** for the fourth equality.
5. **Item 2** of **Definition 4.3.12.1.2** for the fifth equality.

This finishes the proof.

**Item 10**, *Interaction With Internal Homs of Powersets*: We have

$$\begin{aligned}
 f^{-1}([U, V]_Y) &\stackrel{\text{def}}{=} f^{-1}(U^c \cup V) \\
 &= f^{-1}(U^c) \cup f^{-1}(V) \\
 &= f^{-1}(U)^c \cup f^{-1}(V) \\
 &\stackrel{\text{def}}{=} [f^{-1}(U), f^{-1}(V)]_X,
 \end{aligned}$$

where we have used:

1. **Item 8** for the second equality.
2. **Item 5** for the third equality.

Since  $\mathcal{P}(Y)$  is posetal, naturality is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**). This finishes the proof.

**Item 11**, *Preservation of Colimits*: This follows from **Item 2** and ??, ?? of ??.<sup>37</sup>

**Item 12**, *Preservation of Limits*: This follows from **Item 2** and ??, ?? of ??.<sup>38</sup>

**Item 13**, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **Item 11**.

**Item 14**, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **Item 12**.

**Item 15**, *Interaction With Coproducts*: Omitted.

**Item 16**, *Interaction With Products*: Omitted. □

**Proposition 4.6.2.1.4**. Let  $f: X \rightarrow Y$  be a function.

1. *Functionality I*. The assignment  $f \mapsto f^{-1}$  defines a function

$$(-)_{X,Y}^{-1}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

---

<sup>37</sup>Reference: [Pro25y].

<sup>38</sup>Reference: [Pro25w].

2. *Functionality II.* The assignment  $f \mapsto f^{-1}$  defines a function

$$(-)_{X,Y}^{-1}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have

$$\text{id}_X^{-1} = \text{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y) \\ & \searrow (g \circ f)^{-1} & \downarrow f^{-1} \\ & & \mathcal{P}(X). \end{array}$$

*Proof.* **Item 1, Functionality I:** There is nothing to prove.

**Item 2, Functionality II:** This follows from **Item 1** of **Definition 4.6.2.1.3**.

**Item 3, Interaction With Identities:** This follows from **Definition 4.6.2.1.2** and **Categories, Item 5** of **Definition 11.1.4.1.2**.

**Item 4, Interaction With Composition:** This follows from **Definition 4.6.2.1.2** and **Categories, Item 2** of **Definition 11.1.4.1.2**.  $\square$

### 4.6.3 Codirect Images

Let  $f: X \rightarrow Y$  be a function.

**Definition 4.6.3.1.1.** The **codirect image function** associated to  $f$  is the function

$$f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by<sup>39,40</sup>

$$f_*(U) \stackrel{\text{def}}{=} \left\{ y \in Y \left| \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right. \right\}$$

<sup>39</sup>*Further Terminology:* The set  $f_*(U)$  is called the **codirect image of  $U$  by  $f$** .

<sup>40</sup>We also have

$$\begin{aligned} f_*(U) &= f_!(U^c)^c \\ &\stackrel{\text{def}}{=} Y \setminus f_!(X \setminus U); \end{aligned}$$

see **Item 16** of **Definition 4.6.3.1.7**.

$$= \{y \in Y \mid \text{we have } f^{-1}(y) \subset U\}$$

for each  $U \in \mathcal{P}(X)$ .

**Notation 4.6.3.1.2.** Sometimes one finds the notation

$$\forall_f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

for  $f_!$ . This notation comes from the fact that the following statements are equivalent, where  $y \in Y$  and  $U \in \mathcal{P}(X)$ :

- We have  $y \in \forall_f(U)$ .
- For each  $x \in X$ , if  $y = f(x)$ , then  $x \in U$ .

We will not make use of this notation elsewhere in Clowder.

**Warning 4.6.3.1.3.** See [Definition 4.6.1.1.3](#).

**Remark 4.6.3.1.4.** Identifying  $\mathcal{P}(X)$  with  $\text{Sets}(X, \{t, f\})$  via [Item 2 of Definition 4.5.1.1.4](#), we see that the codirect image function associated to  $f$  is equivalently the function

$$f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \text{Ran}_f(\chi_U) \\ &= \lim(((-1) \times f) \xrightarrow{\text{pr}} X \xrightarrow{\chi_U} \{\text{true}, \text{false}\}) \\ &= \lim_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)) \\ &= \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)). \end{aligned}$$

where we have used ?? for the second equality. In other words, we have

$$\begin{aligned} [f_*(\chi_U)](y) &= \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \text{true} & \text{if, for each } x \in X \text{ such that} \\ & f(x) = y, \text{ we have } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } f^{-1}(y) \subset U \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each  $y \in Y$ .

**Definition 4.6.3.1.5.** Let  $U$  be a subset of  $X$ .<sup>41,42</sup>

1. The **image part of the codirect image**  $f_*(U)$  of  $U$  is the set  $f_{*,\text{im}}(U)$  defined by

$$\begin{aligned} f_{*,\text{im}}(U) &\stackrel{\text{def}}{=} f_*(U) \cap \text{Im}(f) \\ &= \left\{ y \in Y \mid \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{array} \right\}. \end{aligned}$$

2. The **complement part of the codirect image**  $f_*(U)$  of  $U$  is the set  $f_{*,\text{cp}}(U)$  defined by

$$\begin{aligned} f_{*,\text{cp}}(U) &\stackrel{\text{def}}{=} f_*(U) \cap (Y \setminus \text{Im}(f)) \\ &= Y \setminus \text{Im}(f) \\ &= \left\{ y \in Y \mid \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{array} \right\} \\ &= \{ y \in Y \mid f^{-1}(y) = \emptyset \}. \end{aligned}$$

**Example 4.6.3.1.6.** Here are some examples of codirect images.

1. *Multiplication by Two.* Consider the function  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by

$$f(n) \stackrel{\text{def}}{=} 2n$$

for each  $n \in \mathbb{N}$ . Since  $f$  is injective, we have

$$f_{*,\text{im}}(U) = f!(U)$$

---

<sup>41</sup>Note that we have

$$f_*(U) = f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U),$$

as

$$\begin{aligned} f_*(U) &= f_*(U) \cap Y \\ &= f_*(U) \cap (\text{Im}(f) \cup (Y \setminus \text{Im}(f))) \\ &= (f_*(U) \cap \text{Im}(f)) \cup (f_*(U) \cap (Y \setminus \text{Im}(f))) \\ &\stackrel{\text{def}}{=} f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U). \end{aligned}$$

<sup>42</sup>In terms of the meet computation of  $f_*(U)$  of **Definition 4.6.3.1.4**, namely

$$f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),$$

$$f_{*,\text{cp}}(U) = \{\text{odd natural numbers}\}$$

for any  $U \subset \mathbb{N}$ . In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

2. *Parabolas.* Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) \stackrel{\text{def}}{=} x^2$$

for each  $x \in \mathbb{R}$ . We have

$$f_{*,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any  $U \subset \mathbb{R}$ . Moreover, since  $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$ , we have e.g.:

$$\begin{aligned} f_{*,\text{im}}([0, 1]) &= \{0\}, \\ f_{*,\text{im}}([-1, 1]) &= [0, 1], \\ f_{*,\text{im}}([1, 2]) &= \emptyset, \\ f_{*,\text{im}}([-2, -1] \cup [1, 2]) &= [1, 4]. \end{aligned}$$

3. *Circles.* Consider the function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$f(x, y) \stackrel{\text{def}}{=} x^2 + y^2$$

for each  $(x, y) \in \mathbb{R}^2$ . We have

$$f_{*,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any  $U \subset \mathbb{R}^2$ , and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0, 0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{*,\text{im}}([-1, 1] \times [-1, 1]) &= [0, 1], \\ f_{*,\text{im}}(([-1, 1] \times [-1, 1]) \setminus [-1, 1] \times \{0\}) &= \emptyset. \end{aligned}$$

---

we see that  $f_{*,\text{im}}$  corresponds to meets indexed over nonempty sets, while  $f_{*,\text{cp}}$  corresponds to meets indexed over the empty set.



**Proposition 4.6.3.1.7.** Let  $f: X \rightarrow Y$  be a function.

1. *Functoriality.* The assignment  $U \mapsto f_*(U)$  defines a functor

$$f_*: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset).$$

In particular, for each  $U, V \in \mathcal{P}(X)$ , the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f_*(U) \subset f_*(V).$$

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \mathcal{P}(X) \begin{matrix} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{matrix} \mathcal{P}(Y),$$

witnessed by:

- (a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ .

- (b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$  and (respectively)  $V \in \mathcal{P}(X)$  and  $U \in \mathcal{P}(Y)$ . In particular:

- i. The following conditions are equivalent:
  - A. We have  $f_!(U) \subset V$ .
  - B. We have  $U \subset f^{-1}(V)$ .
- ii. The following conditions are equivalent:
  - A. We have  $f^{-1}(U) \subset V$ .

B. We have  $U \subset f_*(V)$ .

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(X)$ , where  $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$ .

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cap & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each  $\mathcal{U} \in \mathcal{P}(X)$ , where  $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$ .

5. *Interaction With Binary Unions.* Let  $f: X \rightarrow Y$  be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \downarrow \cup & \subset & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

6. *Interaction With Binary Intersections.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each  $U, V \in \mathcal{P}(X)$ .

7. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U^c) = f_!(U)^c$$

for each  $U \in \mathcal{P}(X)$ .

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \supset & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U \Delta V) \subset f_*(U) \Delta f_*(V)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & \wr & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by  $U, V \in \mathcal{P}(X)$ .

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i\right),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have inclusions

$$\begin{aligned} f_*(U) \cup f_*(V) &\hookrightarrow f_*(U \cup V), \\ \emptyset &\hookrightarrow f_*(\emptyset), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

11. *Preservation of Limits.* We have an equality of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_*(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_*(U) \cap f^{-1}(V), \\ f_*(X) &= Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of [Item 1](#) has a symmetric lax monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) : (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^{\otimes} &: f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &: \emptyset \hookrightarrow f_*(\emptyset), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) : (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &: f_*(U \cap V) \xrightarrow{=} f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} &: f_*(X) \xrightarrow{=} Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

14. *Interaction With Coproducts.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be maps of sets. We have

$$(f \amalg g)_*(U \amalg V) = f_*(U) \amalg g_*(V)$$

for each  $U \in \mathcal{P}(X)$  and each  $V \in \mathcal{P}(Y)$ .

15. *Interaction With Products.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_*(U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each  $U \in \mathcal{P}(X)$  and each  $V \in \mathcal{P}(Y)$ .

16. *Relation to Direct Images.* We have

$$\begin{aligned} f_*(U) &= f_!(U^c)^c \\ &= Y \setminus f_!(X \setminus U) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

17. *Interaction With Injections.* If  $f$  is injective, then we have

$$\begin{aligned} f_{*,\text{im}}(U) &= f_!(U), \\ f_{*,\text{cp}}(U) &= Y \setminus \text{Im}(f), \end{aligned}$$

and so

$$\begin{aligned} f_*(U) &= f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U) \\ &= f_!(U) \cup (Y \setminus \text{Im}(f)) \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

18. *Interaction With Surjections.* If  $f$  is surjective, then we have

$$\begin{aligned} f_{*,\text{im}}(U) &\subset f_!(U), \\ f_{*,\text{cp}}(U) &= \emptyset, \end{aligned}$$

and so

$$f_*(U) \subset f_!(U)$$

for each  $U \in \mathcal{P}(X)$ .

*Proof. Item 1, Functoriality:* Omitted.

*Item 2, Triple Adjointness:* This follows from [Definition 4.6.1.1.4](#), [Definition 4.6.2.1.2](#), [Definition 4.6.3.1.4](#), and Kan Extensions, ?? of ??.

*Item 3, Interaction With Unions of Families of Subsets:* We have

$$\begin{aligned} \bigcup_{V \in f_*(\mathcal{U})} V &= \bigcup_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.

*Item 4, Interaction With Intersections of Families of Subsets:* We have

$$\begin{aligned} \bigcap_{V \in f_*(\mathcal{U})} V &= \bigcap_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.

*Item 5, Interaction With Binary Unions:* We have

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_!(U^c)^c \cup f_!(V^c)^c \\ &= (f_!(U^c) \cap f_!(V^c))^c \\ &\subset (f_!(U^c \cap V^c))^c \\ &= f_!((U \cup V)^c)^c \\ &= f_*(U \cup V), \end{aligned}$$

where:

1. We have used [Item 16](#) for the first equality.
2. We have used [Item 2](#) of [Definition 4.3.11.1.2](#) for the second equality.
3. We have used [Item 6](#) of [Definition 4.6.1.1.5](#) for the third equality.
4. We have used [Item 2](#) of [Definition 4.3.11.1.2](#) for the fourth equality.
5. We have used [Item 16](#) for the last equality.

This finishes the proof.

[Item 6](#), *Interaction With Binary Intersections*: This follows from [Item 11](#).

[Item 7](#), *Interaction With Complements*: Omitted.

[Item 8](#), *Interaction With Symmetric Differences*: Omitted.

[Item 9](#), *Interaction With Internal Homs of Powersets*: We have

$$\begin{aligned}
 [f_!(U), f^!(V)]_X &\stackrel{\text{def}}{=} f_!(U)^c \cup f_*(V) \\
 &= f_*(U^c) \cup f_*(V) \\
 &\subset f_*(U^c \cup V) \\
 &\stackrel{\text{def}}{=} f_*([U, V]_X),
 \end{aligned}$$

where we have used:

1. [Item 7](#) of [Definition 4.6.3.1.7](#) for the second equality.
2. [Item 5](#) of [Definition 4.6.3.1.7](#) for the inclusion.

Since  $\mathcal{P}(X)$  is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Definition 11.2.7.1.2](#)). This finishes the proof.

[Item 10](#), *Lax Preservation of Colimits*: Omitted.

[Item 11](#), *Preservation of Limits*: This follows from [Item 2](#) and ??, ?? of ??.

[Item 12](#), *Symmetric Lax Monoidality With Respect to Unions*: This follows from [Item 10](#).

[Item 13](#), *Symmetric Strict Monoidality With Respect to Intersections*: This follows from [Item 11](#).

[Item 14](#), *Interaction With Coproducts*: Omitted.

[Item 15](#), *Interaction With Products*: Omitted.

[Item 16](#), *Relation to Direct Images*: We claim that  $f_*(U) = Y \setminus f_!(X \setminus U)$ .

- *The First Implication*. We claim that

$$f_*(U) \subset Y \setminus f_!(X \setminus U).$$

Let  $y \in f_*(U)$ . We need to show that  $y \notin f_!(X \setminus U)$ , i.e. that there is no  $x \in X \setminus U$  such that  $f(x) = y$ .

This is indeed the case, as otherwise we would have  $x \in f^{-1}(y)$  and  $x \notin U$ , contradicting  $f^{-1}(y) \subset U$  (which holds since  $y \in f_*(U)$ ).

Thus  $y \in Y \setminus f_!(X \setminus U)$ .

- *The Second Implication.* We claim that

$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$

Let  $y \in Y \setminus f_!(X \setminus U)$ . We need to show that  $y \in f_*(U)$ , i.e. that  $f^{-1}(y) \subset U$ .

Since  $y \notin f_!(X \setminus U)$ , there exists no  $x \in X \setminus U$  such that  $y = f(x)$ , and hence  $f^{-1}(y) \subset U$ .

Thus  $y \in f_*(U)$ .

This finishes the proof of **Item 16**.

**Item 17, Interaction With Injections:** Omitted.

**Item 18, Interaction With Surjections:** Omitted. □

**Proposition 4.6.3.1.8.** Let  $f: X \rightarrow B$  be a function.

1. *Functionality I.* The assignment  $f \mapsto f_*$  defines a function

$$(-)_{||X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment  $f \mapsto f_*$  defines a function

$$(-)_{||X,Y}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have

$$(\text{id}_X)_* = \text{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$ , we have

$$(g \circ f)_* = g_* \circ f_*,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \\ & \searrow (g \circ f)_* & \downarrow g_* \\ & & \mathcal{P}(Z). \end{array}$$

*Proof.* **Item 1, Functionality I:** There is nothing to prove.

**Item 2, Functionality II:** This follows from **Item 1** of **Definition 4.6.3.1.7**.

**Item 3, Interaction With Identities:** This follows from **Definition 4.6.3.1.4** and Kan Extensions, ?? of ??.

**Item 4, Interaction With Composition:** This follows from **Definition 4.6.3.1.4** and Kan Extensions, ?? of ??. □



### 4.6.4 A Six-Functor Formalism for Sets

**Remark 4.6.4.1.1.** The assignment  $X \mapsto \mathcal{P}(X)$  together with the functors  $f_*$ ,  $f^{-1}$ , and  $f_!$  of [Item 1 of Definition 4.6.1.1.5](#), [Item 1 of Definition 4.6.2.1.3](#), and [Item 1 of Definition 4.6.3.1.7](#), and the functors

$$\begin{aligned} -_1 \cap -_2 &: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X), \\ [-_1, -_2]_X &: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X) \end{aligned}$$

of [Item 1 of Definition 4.3.9.1.2](#) and [Item 1 of Definition 4.4.7.1.3](#) satisfy several properties reminiscent of a six functor formalism in the sense of ??.

We collect these properties in [Definition 4.6.4.1.2](#) below.<sup>43</sup>

**Proposition 4.6.4.1.2.** Let  $X$  be a set.

1. *The Beck–Chevalley Condition.* Let

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\text{pr}_2} & Y \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

be a pullback diagram in Sets. We have

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{pr}_1^{-1}} & \mathcal{P}(X \times_Z Y) \\ f_! \downarrow & & \downarrow (\text{pr}_2)_! \\ \mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y), \end{array} \quad \begin{array}{l} g^{-1} \circ f_! = (\text{pr}_2)_! \circ \text{pr}_1^{-1}, \\ f^{-1} \circ g_! = (\text{pr}_1)_! \circ \text{pr}_2^{-1}, \end{array} \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{pr}_2^{-1}} & \mathcal{P}(X \times_Z Y) \\ g_! \downarrow & & \downarrow (\text{pr}_1)_! \\ \mathcal{P}(Z) & \xrightarrow{f^{-1}} & \mathcal{P}(Y). \end{array}$$

2. *The Projection Formula I.* The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(X) \times \mathcal{P}(X) & & \\ & \text{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap & \\ \mathcal{P}(X) \times \mathcal{P}(Y) & & & & \mathcal{P}(X) \\ & f_! \times \text{id}_{\mathcal{P}(Y)} \searrow & & \swarrow f_! & \\ & \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\cap} & \mathcal{P}(Y), & \end{array}$$

<sup>43</sup>See also [\[nLa25\]](#).

commutes, i.e. we have

$$f_!(U \cap f^{-1}(V)) = f_!(U) \cap V$$

for each  $U \in \mathcal{P}(X)$  and each  $V \in \mathcal{P}(Y)$ .

3. *The Projection Formula II.* We have a natural transformation

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times \mathcal{P}(X) & \\
 \text{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
 f_* \times \text{id}_{\mathcal{P}(Y)} \searrow & \cup & \downarrow f_* \\
 \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\cap} & \mathcal{P}(Y),
 \end{array}$$

with components

$$f_*(U) \cap V \subset f_*(U \cap f^{-1}(V))$$

indexed by  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ .

4. *Strong Closed Monoidality.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\
 \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in  $U, V \in \mathcal{P}(X)$ .

5. *The External Tensor Product.* We have an external tensor product

$$-1 \boxtimes_{X \times Y} -2: \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathcal{P}(X \times Y)$$

given by

$$U \boxtimes_{X \times Y} V \stackrel{\text{def}}{=} \text{pr}_1^{-1}(U) \cap \text{pr}_2^{-1}(V)$$

$$= \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}.$$

This is the same map as the one in [Item 5 of Definition 4.4.1.1.4](#). Moreover, the following conditions are satisfied:

- (a) *Interaction With Direct Images.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_! \times g_!} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \boxtimes_{X \times Y} \downarrow & & \downarrow \boxtimes_{X' \times Y'} \\ \mathcal{P}(X \times Y) & \xrightarrow{f_! \times g_!} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each  $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ .

- (b) *Interaction With Inverse Images.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X') \times \mathcal{P}(Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X) \times \mathcal{P}(Y) \\ \boxtimes_{X' \times Y'} \downarrow & & \downarrow \boxtimes_{X \times Y} \\ \mathcal{P}(X' \times Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X \times Y) \end{array}$$

commutes, i.e. we have

$$[f^{-1} \times g^{-1}](U \boxtimes_{X' \times Y'} V) = f^{-1}(U) \boxtimes_{X \times Y} g^{-1}(V)$$

for each  $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$ .

- (c) *Interaction With Codirect Images.* Let  $f: X \rightarrow X'$  and  $g: Y \rightarrow Y'$  be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \boxtimes_{X \times Y} \downarrow & & \downarrow \boxtimes_{X' \times Y'} \\ \mathcal{P}(X \times Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, i.e. we have

$$[f_* \times g_*](U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each  $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ .

(d) *Interaction With Diagonals.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\boxtimes_{X \times X}} & \mathcal{P}(X \times X) \\ & \searrow \cap & \downarrow \Delta_X^{-1} \\ & & \mathcal{P}(X), \end{array}$$

i.e. we have

$$U \cap V = \Delta_X^{-1}(U \boxtimes_{X \times X} V)$$

for each  $U, V \in \mathcal{P}(X)$ .

6. *The Dualisation Functor.* We have a functor

$$D_X: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X)$$

given by

$$D_X(U) \stackrel{\text{def}}{=} [U, \emptyset]_X \\ \stackrel{\text{def}}{=} U^c$$

for each  $U \in \mathcal{P}(X)$ , as in [Item 5](#) of [Definition 4.4.7.1.3](#), satisfying the following conditions:

(a) *Duality.* We have

$$D_X(D_X(U)) = U, \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)} & \downarrow D_X \\ & & \mathcal{P}(X). \end{array}$$

(b) *Duality.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} \mathcal{P}(X)^{\text{op}} \\ \text{id}_{\mathcal{P}(X)^{\text{op}}} \times D_X \nearrow & & \searrow D_X \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each  $U, V \in \mathcal{P}(X)$ .

(c) *Interaction With Direct Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each  $U \in \mathcal{P}(X)$ .

(d) *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ D_Y \downarrow & & \downarrow D_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each  $U \in \mathcal{P}(X)$ .

(e) *Interaction With Codirect Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each  $U \in \mathcal{P}(X)$ .

*Proof.* **Item 1, The Beck–Chevalley Condition:** We have

$$\begin{aligned}
 [g^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} g^{-1}(f_!(U)) \\
 &\stackrel{\text{def}}{=} \{y \in Y \mid g(y) \in f_!(U)\} \\
 &= \left\{ y \in Y \mid \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
 &= \left\{ y \in Y \mid \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \end{array} \right\} \\
 &= \left\{ y \in Y \mid \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } y = y \end{array} \right\} \\
 &= \left\{ y \in Y \mid \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } \text{pr}_2(x, y) = y \end{array} \right\} \\
 &\stackrel{\text{def}}{=} (\text{pr}_2)_!(\{(x, y) \in X \times_Z Y \mid x \in U\}) \\
 &= (\text{pr}_2)_!(\{(x, y) \in X \times_Z Y \mid \text{pr}_1(x, y) \in U\}) \\
 &\stackrel{\text{def}}{=} (\text{pr}_2)_!(\text{pr}_1^{-1}(U)) \\
 &\stackrel{\text{def}}{=} [(\text{pr}_2)_! \circ \text{pr}_1^{-1}](U)
 \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ . Therefore, we have

$$g^{-1} \circ f_! = (\text{pr}_2)_! \circ \text{pr}_1^{-1}.$$

For the second equality, we have

$$\begin{aligned}
 [f^{-1} \circ g_!](U) &\stackrel{\text{def}}{=} f^{-1}(g_!(U)) \\
 &\stackrel{\text{def}}{=} \{x \in X \mid f(x) \in g_!(U)\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } x = x \end{array} \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } \text{pr}_1(x, y) = x \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} (\text{pr}_1)_! (\{(x, y) \in X \times_Z Y \mid y \in V\}) \\
&= (\text{pr}_1)_! (\{(x, y) \in X \times_Z Y \mid \text{pr}_2(x, y) \in V\}) \\
&\stackrel{\text{def}}{=} (\text{pr}_1)_! (\text{pr}_2^{-1}(V)) \\
&\stackrel{\text{def}}{=} [(\text{pr}_1)_! \circ \text{pr}_2^{-1}](V)
\end{aligned}$$

for each  $V \in \mathcal{P}(Y)$ . Therefore, we have

$$f^{-1} \circ g_! = (\text{pr}_1)_! \circ \text{pr}_2^{-1}.$$

This finishes the proof.

**Item 2, The Projection Formula I:** We claim that

$$f_!(U) \cap V \subset f_!(U \cap f^{-1}(V)).$$

Indeed, we have

$$\begin{aligned}
f_!(U) \cap V &\subset f_!(U) \cap f_!(f^{-1}(V)) \\
&= f_!(U \cap f^{-1}(V)),
\end{aligned}$$

where we have used:

1. **Item 2** of **Definition 4.6.1.1.5** for the inclusion.
2. **Item 6** of **Definition 4.6.1.1.5** for the equality.

Conversely, we claim that

$$f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V.$$

Indeed:

1. Let  $y \in f_!(U \cap f^{-1}(V))$ .
2. Since  $y \in f_!(U \cap f^{-1}(V))$ , there exists some  $x \in U \cap f^{-1}(V)$  such that  $f(x) = y$ .
3. Since  $x \in U \cap f^{-1}(V)$ , we have  $x \in U$ , and thus  $f(x) \in f_!(U)$ .
4. Since  $x \in U \cap f^{-1}(V)$ , we have  $x \in f^{-1}(V)$ , and thus  $f(x) \in V$ .
5. Since  $f(x) \in f_!(U)$  and  $f(x) \in V$ , we have  $f(x) \in f_!(U) \cap V$ .

6. But  $y = f(x)$ , so  $y \in f_!(U) \cap V$ .

7. Thus  $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$ .

This finishes the proof.

*Item 3, The Projection Formula II:* We have

$$\begin{aligned} f_*(U) \cap V &\subset f_*(U) \cap f_*(f^{-1}(V)) \\ &= f_*(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

1. *Item 2* of *Definition 4.6.3.1.7* for the inclusion.
2. *Item 6* of *Definition 4.6.3.1.7* for the equality.

Since  $\mathcal{P}(Y)$  is posetal, naturality is automatic (*Categories*, *Item 4* of *Definition 11.2.7.1.2*).

*Item 4, Strong Closed Monoidality:* This is a repetition of *Item 19* of *Definition 4.4.7.1.3* and is proved there.

*Item 5, The External Tensor Product:* We have

$$\begin{aligned} U \boxtimes_{X \times Y} V &\stackrel{\text{def}}{=} \text{pr}_1^{-1}(U) \cap \text{pr}_2^{-1}(V) \\ &\stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid \text{pr}_1(x, y) \in U\} \\ &\quad \cup \{(x, y) \in X \times Y \mid \text{pr}_2(x, y) \in V\} \\ &= \{(x, y) \in X \times Y \mid x \in U\} \\ &\quad \cup \{(x, y) \in X \times Y \mid y \in V\} \\ &= \{(x, y) \in X \times Y \mid x \in U \text{ and } y \in V\} \\ &\stackrel{\text{def}}{=} U \times V. \end{aligned}$$

Next, we claim that *Items 5a* to *5d* are indeed true:

1. *Proof of Item 5a:* This is a repetition of *Item 16* of *Definition 4.6.1.1.5* and is proved there.
2. *Proof of Item 5b:* This is a repetition of *Item 16* of *Definition 4.6.2.1.3* and is proved there.
3. *Proof of Item 5c:* This is a repetition of *Item 15* of *Definition 4.6.3.1.7* and is proved there.
4. *Proof of Item 5d:* We have

$$\begin{aligned} \Delta_X^{-1}(U \boxtimes_{X \times X} V) &\stackrel{\text{def}}{=} \{x \in X \mid (x, x) \in U \boxtimes_{X \times X} V\} \\ &= \{x \in X \mid (x, x) \in \{(u, v) \in X \times X \mid u \in U \text{ and } v \in V\}\} \\ &= U \cap V. \end{aligned}$$



This finishes the proof.

*Item 6, The Dualisation Functor:* This is a repetition of **Items 5** and **6** of **Definition 4.4.7.1.3** and is proved there.  $\square$

## 4.7 Miscellany

### 4.7.1 Injective Functions

Let  $A$  and  $B$  be sets.

**Definition 4.7.1.1.1.** A function  $f: A \rightarrow B$  is **injective** if it satisfies the following condition:

( $\star$ ) For each  $a, a' \in A$ , if  $f(a) = f(a')$ , then  $a = a'$ .

**Proposition 4.7.1.1.2.** Let  $f: A \rightarrow B$  be a function.

1. *Characterisations.* The following conditions are equivalent:<sup>44</sup>

- (a) The function  $f$  is injective.
- (b) The function  $f$  is a monomorphism in Sets.
- (c) The direct image function

$$f_! : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to  $f$  is injective.

- (d) The codirect image function

$$f_* : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to  $f$  is injective.

- (e) The direct image functor

$$f_! : (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

associated to  $f$  is full.

---

<sup>44</sup>**Items 1c** to **1f** unwind respectively to the following statements:

- For each  $U, V \in \mathcal{P}(A)$ , if  $f_!(U) = f_!(V)$ , then  $U = V$ .
- For each  $U, V \in \mathcal{P}(A)$ , if  $f_*(U) = f_*(V)$ , then  $U = V$ .
- For each  $U, V \in \mathcal{P}(A)$ , if  $f_!(U) \subset f_!(V)$ , then  $U \subset V$ .
- For each  $U, V \in \mathcal{P}(A)$ , if  $f_*(U) \subset f_*(V)$ , then  $U \subset V$ .

(f) The codirect image function

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to  $f$  is full.

(g) The diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \chi_A & \downarrow f^{-1} \\ & & \mathcal{P}(A) \end{array}$$

commutes. That is, we have

$$f^{-1}(f(a)) = \{a\}$$

for each  $a \in A$ .

(h) We have

$$\begin{array}{ccc} & \mathcal{P}(A) & \xrightarrow{f_!} \mathcal{P}(B) \\ f^{-1} \circ f_! = \text{id}_{\mathcal{P}(A)} & \searrow & \downarrow f^{-1} \\ & & \mathcal{P}(A). \end{array}$$

In other words, we have

$$\{a \in A \mid f(a) \in f(U)\} = U$$

for each  $U \in \mathcal{P}(A)$ .

(i) We have

$$\begin{array}{ccc} & \mathcal{P}(A) & \xrightarrow{f_*} \mathcal{P}(B) \\ f^{-1} \circ f_* = \text{id}_{\mathcal{P}(A)} & \searrow & \downarrow f^{-1} \\ & & \mathcal{P}(A). \end{array}$$

In other words, we have

$$\{a \in A \mid f^{-1}(f(a)) \subset U\} = U$$

for each  $U \in \mathcal{P}(A)$ .

*Proof. Item 1, Characterisations:* We will proceed by showing:

- Step 1: **Item 1a**  $\iff$  **Item 1b**.
- Step 2: **Item 1a**  $\iff$  **Item 1c**.
- Step 3: **Item 1a**  $\iff$  **Item 1d**.
- Step 4: **Item 1c**  $\iff$  **Item 1e**.
- Step 5: **Item 1e**  $\iff$  **Item 1f**.
- Step 6: **Item 1a**  $\iff$  **Item 1g**.
- Step 7: **Item 1g**  $\iff$  **Item 1h**.
- Step 8: **Item 1a**  $\iff$  **Item 1i**.

**Step 1: Item 1a**  $\iff$  **Item 1b**. We claim that **Items 1a** and **1b** are equivalent:

- **Item 1a**  $\implies$  **Item 1b**: We proceed in a few steps:
  - Proceeding by contrapositive, we claim that given a pair of maps  $g, h: C \rightrightarrows A$  such that  $g \neq h$ , we have  $f \circ g \neq f \circ h$ .
  - Indeed, as  $g$  and  $h$  are different maps, there must exist at least one element  $x \in C$  such that  $g(x) \neq h(x)$ .
  - But then we have  $f(g(x)) \neq f(h(x))$ , as  $f$  is injective.
  - Thus  $f \circ g \neq f \circ h$ , and we are done.
- **Item 1b**  $\implies$  **Item 1a**: We proceed in a few steps:
  - Consider the diagram

$$\text{pt} \begin{array}{c} \xrightarrow{[x]} \\ \xrightarrow{[y]} \end{array} A \xrightarrow{f} B,$$

where  $[x]$  and  $[y]$  are the morphisms picking the elements  $x$  and  $y$  of  $A$ .

- Note that we have  $f(x) = f(y)$  iff  $f \circ [x] = f \circ [y]$ .
- Since  $f$  is assumed to be a monomorphism, if  $f(x) = f(y)$ , then  $f \circ [x] = f \circ [y]$  and therefore  $[x] = [y]$ .
- This shows that if  $f(x) = f(y)$ , then  $x = y$ , so  $f$  is injective.

**Step 2: Item 1a**  $\iff$  **Item 1c**. We claim that **Items 1a** and **1c** are indeed equivalent:

- *Item 1a*  $\implies$  *Item 1c*: We proceed in a few steps:
  - Assume that  $f$  is injective and let  $U, V \in \mathcal{P}(A)$  such that  $f_!(U) = f_!(V)$ . We wish to show that  $U = V$ .
  - To show that  $U \subset V$ , let  $u \in U$ .
  - By the definition of the direct image, we have  $f(u) \in f_!(U)$ .
  - Since  $f_!(U) = f_!(V)$ , it follows that  $f(u) \in f_!(V)$ .
  - Thus, there exists some  $v \in V$  such that  $f(v) = f(u)$ .
  - Since  $f$  is injective, the equality  $f(v) = f(u)$  implies that  $v = u$ .
  - Thus  $u \in V$  and  $U \subset V$ .
  - A symmetric argument shows that  $V \subset U$ .
  - Therefore  $U = V$ , showing  $f_!$  to be injective.
- *Item 1c*  $\implies$  *Item 1a*: We proceed in a few steps:
  - Assume that the direct image function  $f_!$  is injective and let  $a, a' \in A$  such that  $f(a) = f(a')$ . We wish to show that  $a = a'$ .
  - Since

$$\begin{aligned}
 f_!({a}) &= \{f(a)\} \\
 &= \{f(a')\} \\
 &= f_!({a'}),
 \end{aligned}$$

we must have  $\{a\} = \{a'\}$ , as  $f_!$  is injective, so  $a = a'$ , showing  $f$  to be injective.

**Step 3:** *Item 1c*  $\iff$  *Item 1d*. This follows from *Item 17* of *Definition 4.6.1.1.5*.

**Step 4:** *Item 1c*  $\iff$  *Item 1e*. We claim that *Items 1c* and *1e* are equivalent:

- *Item 1c*  $\implies$  *Item 1e*: We proceed in a few steps:
  - Let  $U, V \in \mathcal{P}(A)$  such that  $f_!(U) \subset f_!(V)$ , assume  $f_!$  to be injective, and consider the set  $U \cup V$ .
  - Since  $f_!(U) \subset f_!(V)$ , we have

$$\begin{aligned}
 f_!(U \cup V) &= f_!(U) \cup f_!(V) \\
 &= f_!(V),
 \end{aligned}$$

where we have used *Item 5* of *Definition 4.6.1.1.5* for the first equality.

- Since  $f_!$  is injective, this implies  $U \cup V = V$ .

– Thus  $U \subset V$ , as we wished to show.

• **Item 1c**  $\implies$  **Item 1e**: We proceed in a few steps:

- Suppose **Item 1e** holds, and let  $U, V \in \mathcal{P}(A)$  such that  $f_!(U) = f_!(V)$ .
- Since  $f_!(U) = f_!(V)$ , we have  $f_!(U) \subset f_!(V)$  and  $f_!(V) \subset f_!(U)$ .
- By assumption, this implies  $U \subset V$  and  $V \subset U$ .
- Thus  $U = V$ , showing  $f_!$  to be injective.

**Step 5: Item 1e**  $\iff$  **Item 1f**. This follows from **Item 17** of **Definition 4.6.1.1.5**.

**Step 6: Item 1a**  $\iff$  **Item 1g**. We have

$$f^{-1}(f(a)) = \{a' \in A \mid f(a') = f(a)\}$$

so the condition  $f^{-1}(f(a)) = \{a\}$  states precisely that if  $f(a') = f(a)$ , then  $a' = a$ .

**Step 7: Item 1g**  $\iff$  **Item 1h**. We claim that **Items 1g** and **1h** are indeed equivalent:

• **Item 1g**  $\implies$  **Item 1h**: We have

$$\begin{aligned} [f^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} f^{-1}(f_!(U)) \\ &= f^{-1}\left(f_!\left(\bigcup_{u \in U} \{u\}\right)\right) \\ &= f^{-1}\left(\bigcup_{u \in U} f_!(\{u\})\right) \\ &= \bigcup_{u \in U} f^{-1}(f_!(\{u\})) \\ &= \bigcup_{u \in U} f^{-1}(f_!(u)) \\ &= \bigcup_{u \in U} \{u\} \\ &= U \end{aligned}$$

for each  $U \in \mathcal{P}(A)$ , where we have used **Item 5** of **Definition 4.6.1.1.5** for the third equality and **Item 5** of **Definition 4.6.2.1.3** for the fourth equality.

• **Item 1h**  $\implies$  **Item 1g**: Applying the condition  $f^{-1} \circ f_! = \text{id}_{\mathcal{P}(A)}$  to  $U = \{a\}$  gives

$$f^{-1}(f_!(\{a\})) = \{a\}.$$

**Step 8: Item 1a**  $\iff$  **Item 1i**. We claim that **Items 1a** and **1i** are equivalent:

- *Item 1a*  $\implies$  *Item 1i*: If  $f$  is injective, then  $f^{-1}(f(a)) = \{a\}$ , so we have

$$\begin{aligned} f^{-1}(f_*(a)) &= \{a \in A \mid \{a\} \subset U\} \\ &= U. \end{aligned}$$

- *Item 1i*  $\implies$  *Item 1a*: For  $U = \{a\}$ , the condition  $f^{-1}(f_*(U)) = U$  becomes

$$\{a' \in A \mid f^{-1}(f(a')) \subset \{a\}\} = \{a\}.$$

Since the set  $f^{-1}(f(a'))$  is given by

$$\{a \in A \mid f(a) = f(a')\},$$

it follows that  $f$  is injective.

This finishes the proof. □

## 4.7.2 Surjective Functions

Let  $A$  and  $B$  be sets.

**Definition 4.7.2.1.1.** A function  $f: A \rightarrow B$  is **surjective** if it satisfies the following condition:

- (★) For each  $b \in B$ , there exists some  $a \in A$  such that  $f(a) = b$ .

**Proposition 4.7.2.1.2.** Let  $f: A \rightarrow B$  be a function.

1. *Characterisations.* The following conditions are equivalent:

- (a) The function  $f$  is surjective.
- (b) The function  $f$  is an epimorphism in Sets.
- (c) The inverse image function

$$f^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

associated to  $f$  is injective.

- (d) The inverse image functor

$$f^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

associated to  $f$  is full.

(e) The diagram

$$\begin{array}{ccc} B & \xrightarrow{f^{-1}} & \mathcal{P}(A) \\ & \searrow \chi_B & \downarrow f_! \\ & & \mathcal{P}(B) \end{array}$$

commutes. That is, we have

$$f_!(f^{-1}(b)) = \{b\}$$

for each  $b \in B$ .

(f) We have

$$f_! \circ f^{-1} = \text{id}_{\mathcal{P}(B)} \quad \begin{array}{ccc} \mathcal{P}(B) & \xrightarrow{f^{-1}} & \mathcal{P}(A) \\ & \searrow & \downarrow f_! \\ & & \mathcal{P}(B). \end{array}$$

In other words, we have

$$\left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in f^{-1}(U) \\ \text{such that } f(a) = b \end{array} \right. \right\} = U$$

for each  $U \in \mathcal{P}(A)$ .

(g) We have

$$f_* \circ f^{-1} = \text{id}_{\mathcal{P}(B)} \quad \begin{array}{ccc} \mathcal{P}(B) & \xrightarrow{f^{-1}} & \mathcal{P}(A) \\ & \searrow & \downarrow f_* \\ & & \mathcal{P}(B). \end{array}$$

In other words, we have

$$\{b \in B \mid f^{-1}(b) \subset f^{-1}(U)\} = U$$

for each  $U \in \mathcal{P}(B)$ .

*Proof.* **Item 1**, *Characterisations*: We will proceed by showing:

- Step 1: **Item 1a**  $\iff$  **Item 1b**.

- Step 2: **Item 1a**  $\iff$  **Item 1c**.
- Step 3: **Item 1c**  $\iff$  **Item 1d**.
- Step 4: **Item 1a**  $\iff$  **Item 1e**.
- Step 5: **Item 1e**  $\iff$  **Item 1f**.
- Step 6: **Item 1a**  $\iff$  **Item 1g**.

**Step 1: Item 1a**  $\iff$  **Item 1b**. We claim **Items 1a** and **1b** are indeed equivalent:

- **Item 1a**  $\implies$  **Item 1b**: We proceed in a few steps:
  - Let  $g, h: B \rightrightarrows C$  be morphisms such that  $g \circ f = h \circ f$ .
  - For each  $a \in A$ , we have

$$g(f(a)) = h(f(a)).$$

- However, this implies that

$$g(b) = h(b)$$

for each  $b \in B$ , as  $f$  is surjective.

- Thus  $g = h$  and  $f$  is an epimorphism.

- **Item 1b**  $\implies$  **Item 1a**: We proceed by contrapositive. Consider the diagram

$$A \xrightarrow{f} B \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} C,$$

where  $h$  is the map defined by  $h(b) = 0$  for each  $b \in B$  and  $g$  is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \text{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then  $h \circ f = g \circ f$ , as  $h(f(a)) = 1 = g(f(a))$  for each  $a \in A$ . However, for any  $b \in B \setminus \text{Im}(f)$ , we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore  $g \neq h$  and  $f$  is not an epimorphism.

**Step 2: Item 1a**  $\iff$  **Item 1c**. We claim **Items 1a** and **1c** are indeed equivalent:

- **Item 1a**  $\implies$  **Item 1c**: We proceed in a few steps:



- Assume that  $f$  is surjective. Let  $U, V \in \mathcal{P}(B)$  such that  $f^{-1}(U) = f^{-1}(V)$ . We wish to show that  $U = V$ .
  - To show that  $U \subset V$ , let  $b \in U$ .
  - Since  $f$  is surjective, there must exist some  $a \in A$  such that  $f(a) = b$ .
  - By the definition of the inverse image, since  $f(a) = b$  and  $b \in U$ , we have  $a \in f^{-1}(U)$ .
  - By our initial assumption,  $f^{-1}(U) = f^{-1}(V)$ , so it follows that  $a \in f^{-1}(V)$ .
  - Again, by the definition of the inverse image,  $a \in f^{-1}(V)$  means that  $f(a) \in V$ .
  - Since  $f(a) = b$ , we have shown that  $b \in V$ .
  - This establishes that  $U \subset V$ . A symmetric argument shows that  $V \subset U$ .
  - Thus  $U = V$ , proving that  $f^{-1}$  is injective.
- *Item 1c*  $\implies$  *Item 1a*: We proceed in a few steps:
    - Assume that the inverse image function  $f^{-1}$  is injective. Suppose, for the sake of contradiction, that  $f$  is not surjective.
    - The assumption that  $f$  is not surjective means there exists some  $b_0 \in B$  such that for all  $a \in A$ , we have  $f(a) \neq b_0$ .
    - By the definition of the inverse image, this is equivalent to stating that  $f^{-1}(\{b_0\}) = \emptyset$ .
    - Since  $f^{-1}(\emptyset) = \emptyset$ , we have  $f^{-1}(\{b_0\}) = f^{-1}(\emptyset)$ .
    - Since  $f^{-1}$  is injective, this implies that  $\{b_0\} = \emptyset$ .
    - This is a contradiction, as the singleton set  $\{b_0\}$  is non-empty.
    - Therefore,  $f$  is surjective.

**Step 3:** *Item 1c*  $\iff$  *Item 1d*. We claim that *Items 1c* and *1d* are equivalent:

- *Item 1c*  $\implies$  *Item 1d*: We proceed in a few steps:
  - Let  $U, V \in \mathcal{P}(B)$  such that  $f^{-1}(U) \subset f^{-1}(V)$ , assume  $f^{-1}$  to be injective, and consider the set  $U \cup V$ .
  - Since  $f^{-1}(U) \subset f^{-1}(V)$ , we have

$$\begin{aligned} f^{-1}(U \cup V) &= f^{-1}(U) \cup f^{-1}(V) \\ &= f^{-1}(V), \end{aligned}$$

where we have used *Item 5* of *Definition 4.6.2.1.3* for the first equality.

- Since  $f^{-1}$  is injective, this implies  $U \cup V = V$ .
- Thus  $U \subset V$ , as we wished to show.
- **Item 1d**  $\implies$  **Item 1c**: We proceed in a few steps:
  - Suppose **Item 1d** holds, and let  $U, V \in \mathcal{P}(B)$  such that  $f^{-1}(U) = f^{-1}(V)$ .
  - Since  $f^{-1}(U) = f^{-1}(V)$ , we have  $f^{-1}(U) \subset f^{-1}(V)$  and  $f^{-1}(V) \subset f^{-1}(U)$ .
  - By assumption, this implies  $U \subset V$  and  $V \subset U$ .
  - Thus  $U = V$ , showing  $f^{-1}$  to be injective.

**Step 4: Item 1a**  $\iff$  **Item 1e**. We have

$$f_!(f^{-1}(b)) = \left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in f^{-1}(b) \\ \text{such that } f(a) = b \end{array} \right. \right\},$$

so the condition  $f_!(f^{-1}(b)) = \{b\}$  holds iff  $f$  is surjective.

**Step 5: Item 1e**  $\iff$  **Item 1f**. We claim that **Items 1e** and **1f** are indeed equivalent:

- **Item 1e**  $\implies$  **Item 1f**: We have

$$\begin{aligned} [f_! \circ f^{-1}](U) &\stackrel{\text{def}}{=} f_!(f^{-1}(U)) \\ &= f_!\left(f^{-1}\left(\bigcup_{u \in U} \{u\}\right)\right) \\ &= f_!\left(\bigcup_{u \in U} f^{-1}(\{u\})\right) \\ &= \bigcup_{u \in U} f_!(f^{-1}(\{u\})) \\ &= \bigcup_{u \in U} f_!(f^{-1}(u)) \\ &= \bigcup_{u \in U} \{u\} \\ &= U \end{aligned}$$

for each  $U \in \mathcal{P}(B)$ , where we have used **Item 5** of **Definition 4.6.1.1.5** for the third equality and **Item 5** of **Definition 4.6.2.1.3** for the fourth equality.

- **Item 1f**  $\implies$  **Item 1e**: Applying the condition  $f_! \circ f^{-1} = \text{id}_{\mathcal{P}(B)}$  to  $U = \{b\}$  gives

$$f_!(f^{-1}(\{b\})) = \{b\}.$$

**Step 6: Item 1a**  $\iff$  **Item 1g**. First, note that for the condition  $f^{-1}(b) \subset f^{-1}(U)$  to hold, we must have  $b \in U$  or  $f^{-1}(b) = \emptyset$ . Thus

$$f_*(f^{-1}(U)) = (U \cap \text{Im}(f)) \cup (B \setminus \text{Im}(f)).$$

We now claim that **Items 1a** and **1g** are indeed equivalent:

- **Item 1a**  $\implies$  **Item 1g**: If  $f$  is surjective, we have

$$\begin{aligned} (U \cap \text{Im}(f)) \cup (B \setminus \text{Im}(f)) &= U \cup \emptyset \\ &= U, \end{aligned}$$

$$\text{so } f_* \circ f^{-1} = \text{id}_{\mathcal{P}(B)}.$$

- **Item 1g**  $\implies$  **Item 1a**: Taking  $U = \emptyset$  gives

$$\begin{aligned} f_*(f^{-1}(\emptyset)) &= (\emptyset \cap \text{Im}(f)) \cup (B \setminus \text{Im}(f)) \\ &= B \setminus \text{Im}(f), \end{aligned}$$

so the condition  $f_*(f^{-1}(\emptyset)) = \emptyset$  implies  $B \setminus \text{Im}(f) = \emptyset$ . Thus  $\text{Im}(f) = B$  and  $f$  is surjective.

This finishes the proof.  $\square$

## Appendices

### A Other Chapters

#### Preliminaries

1. **Introduction**
2. **A Guide to the Literature**

#### Sets

3. **Sets**

#### 4. **Constructions With Sets**

#### 5. **Monoidal Structures on the Category of Sets**

#### 6. **Pointed Sets**

#### 7. **Tensor Products of Pointed Sets**

#### Relations

- |                                     |  |
|-------------------------------------|--|
| 8. Relations                        | 13. Constructions With Monoidal Categories |
| 9. Constructions With Relations     |  |
| 10. Conditions on Relations         | Bicategories                               |
| Categories                          |  |
| 11. Categories                      | 14. Types of Morphisms in Bicategories     |
| 12. Presheaves and the Yoneda Lemma | Extra Part                                 |
| Monoidal Categories                 | 15. Notes                                  |

## References

- [MSE 267365] J. B. Show that the powerset partial order is a cartesian closed category. Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/267365> (cit. on p. 129).
- [MSE 267469] Zhen Lin. Show that the powerset partial order is a cartesian closed category. Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/267469> (cit. on p. 96).
- [MSE 2719059] Vinny Chase.  $\mathcal{P}(X)$  with symmetric difference as addition as a vector space over  $\mathbb{Z}_2$ . Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2719059> (cit. on p. 111).
- [Cie97] Krzysztof Ciesielski. *Set Theory for the Working Mathematician*. Vol. 39. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. xii+236. ISBN: 0-521-59441-3; 0-521-59465-0. DOI: [10.1017/CB09781139173131](https://doi.org/10.1017/CB09781139173131). URL: <https://doi.org/10.1017/CB09781139173131> (cit. on p. 56).
- [nLa25] nLab Authors. *Interactions of Images and Pre-images with Unions and Intersections*. <https://ncatlab.org/nlab/show/interactions+of+images+and+pre-images+with+unions+and+intersections>. Oct. 2025 (cit. on p. 185).
- [Pro25a] Proof Wiki Contributors. *Cartesian Product Distributes Over Set Difference — Proof Wiki*. 2025. URL: [https://proofwiki.org/wiki/Cartesian\\_Product\\_Distributes\\_over\\_Set\\_Difference](https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Set_Difference) (cit. on p. 15).

- [Pro25b] Proof Wiki Contributors. *Cartesian Product Distributes Over Symmetric Difference* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Cartesian\\_Product\\_Distributes\\_over\\_Symmetric\\_Difference](https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Symmetric_Difference) (cit. on p. 15).
- [Pro25c] Proof Wiki Contributors. *Cartesian Product Distributes Over Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Cartesian\\_Product\\_Distributes\\_over\\_Union](https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Union) (cit. on p. 15).
- [Pro25d] Proof Wiki Contributors. *Cartesian Product of Intersections* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Cartesian\\_Product\\_of\\_Intersections](https://proofwiki.org/wiki/Cartesian_Product_of_Intersections) (cit. on p. 15).
- [Pro25e] Proof Wiki Contributors. *Characteristic Function of Intersection* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Characteristic\\_Function\\_of\\_Intersection](https://proofwiki.org/wiki/Characteristic_Function_of_Intersection) (cit. on p. 97).
- [Pro25f] Proof Wiki Contributors. *Characteristic Function of Set Difference* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Characteristic\\_Function\\_of\\_Set\\_Difference](https://proofwiki.org/wiki/Characteristic_Function_of_Set_Difference) (cit. on p. 101).
- [Pro25g] Proof Wiki Contributors. *Characteristic Function of Symmetric Difference* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Characteristic\\_Function\\_of\\_Symmetric\\_Difference](https://proofwiki.org/wiki/Characteristic_Function_of_Symmetric_Difference) (cit. on p. 110).
- [Pro25h] Proof Wiki Contributors. *Characteristic Function of Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Characteristic\\_Function\\_of\\_Union](https://proofwiki.org/wiki/Characteristic_Function_of_Union) (cit. on p. 91).
- [Pro25i] Proof Wiki Contributors. *Complement of Complement* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Complement\\_of\\_Complement](https://proofwiki.org/wiki/Complement_of_Complement) (cit. on p. 103).
- [Pro25j] Proof Wiki Contributors. *Complement of Preimage equals Preimage of Complement* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Complement\\_of\\_Preimage\\_equals\\_Preimage\\_of\\_Complement](https://proofwiki.org/wiki/Complement_of_Preimage_equals_Preimage_of_Complement) (cit. on p. 171).
- [Pro25k] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/De\\_Morgan's\\_Laws\\_\(Set\\_Theory\)](https://proofwiki.org/wiki/De_Morgan's_Laws_(Set_Theory)) (cit. on pp. 100, 103).
- [Pro25l] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)/Set Difference/Difference with Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/De\\_Morgan's\\_Laws\\_\(Set\\_Theory\)/Set\\_Difference/Difference\\_with\\_Union](https://proofwiki.org/wiki/De_Morgan's_Laws_(Set_Theory)/Set_Difference/Difference_with_Union)

- [//proofwiki.org/wiki/De\\_Morgan%27s\\_Laws\\_\(Set\\_Theory\)/Set\\_Difference/Difference\\_with\\_Union](https://proofwiki.org/wiki/De_Morgan%27s_Laws_(Set_Theory)/Set_Difference/Difference_with_Union) (cit. on p. 100).
- [Pro25m] Proof Wiki Contributors. *Equivalence of Definitions of Symmetric Difference* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Equivalence\\_of\\_Definitions\\_of\\_Symmetric\\_Difference](https://proofwiki.org/wiki/Equivalence_of_Definitions_of_Symmetric_Difference) (cit. on p. 109).
- [Pro25n] Proof Wiki Contributors. *Image of Intersection Under Mapping* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Image\\_of\\_Intersection\\_under\\_Mapping](https://proofwiki.org/wiki/Image_of_Intersection_under_Mapping) (cit. on pp. 97, 163, 164).
- [Pro25o] Proof Wiki Contributors. *Image of Set Difference Under Mapping* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Image\\_of\\_Set\\_Difference\\_under\\_Mapping](https://proofwiki.org/wiki/Image_of_Set_Difference_under_Mapping) (cit. on pp. 101, 163).
- [Pro25p] Proof Wiki Contributors. *Image of Union Under Mapping* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Image\\_of\\_Union\\_under\\_Mapping](https://proofwiki.org/wiki/Image_of_Union_under_Mapping) (cit. on pp. 91, 163, 164).
- [Pro25q] Proof Wiki Contributors. *Intersection Distributes Over Symmetric Difference* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_Distributes\\_over\\_Symmetric\\_Difference](https://proofwiki.org/wiki/Intersection_Distributes_over_Symmetric_Difference) (cit. on p. 110).
- [Pro25r] Proof Wiki Contributors. *Intersection Is Associative* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_is\\_Associative](https://proofwiki.org/wiki/Intersection_is_Associative) (cit. on p. 96).
- [Pro25s] Proof Wiki Contributors. *Intersection Is Commutative* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_is\\_Commutative](https://proofwiki.org/wiki/Intersection_is_Commutative) (cit. on p. 96).
- [Pro25t] Proof Wiki Contributors. *Intersection With Empty Set* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_with\\_Empty\\_Set](https://proofwiki.org/wiki/Intersection_with_Empty_Set) (cit. on p. 96).
- [Pro25u] Proof Wiki Contributors. *Intersection With Set Difference Is Set Difference With Intersection* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_with\\_Set\\_Difference\\_is\\_Set\\_Difference\\_with\\_Intersection](https://proofwiki.org/wiki/Intersection_with_Set_Difference_is_Set_Difference_with_Intersection) (cit. on p. 100).
- [Pro25v] Proof Wiki Contributors. *Intersection With Subset Is Subset* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_with\\_Subset\\_is\\_Subset](https://proofwiki.org/wiki/Intersection_with_Subset_is_Subset) (cit. on p. 96).

- [Pro25w] Proof Wiki Contributors. *Preimage of Intersection Under Mapping* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Preimage\\_of\\_Intersection\\_under\\_Mapping](https://proofwiki.org/wiki/Preimage_of_Intersection_under_Mapping) (cit. on pp. 97, 171, 172).
- [Pro25x] Proof Wiki Contributors. *Preimage of Set Difference Under Mapping* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Preimage\\_of\\_Set\\_Difference\\_under\\_Mapping](https://proofwiki.org/wiki/Preimage_of_Set_Difference_under_Mapping) (cit. on pp. 101, 171).
- [Pro25y] Proof Wiki Contributors. *Preimage of Union Under Mapping* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Preimage\\_of\\_Union\\_under\\_Mapping](https://proofwiki.org/wiki/Preimage_of_Union_under_Mapping) (cit. on pp. 91, 171, 172).
- [Pro25z] Proof Wiki Contributors. *Quotient Mapping Is Coequalizer* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Quotient\\_Mapping\\_is\\_Coequalizer](https://proofwiki.org/wiki/Quotient_Mapping_is_Coequalizer) (cit. on p. 49).
- [Pro25aa] Proof Wiki Contributors. *Set Difference as Intersection With Complement* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_as\\_Intersection\\_with\\_Complement](https://proofwiki.org/wiki/Set_Difference_as_Intersection_with_Complement) (cit. on p. 101).
- [Pro25ab] Proof Wiki Contributors. *Set Difference as Symmetric Difference With Intersection* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_as\\_Symmetric\\_Difference\\_with\\_Intersection](https://proofwiki.org/wiki/Set_Difference_as_Symmetric_Difference_with_Intersection) (cit. on p. 101).
- [Pro25ac] Proof Wiki Contributors. *Set Difference Is Right Distributive Over Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_is\\_Right\\_Distributive\\_over\\_Union](https://proofwiki.org/wiki/Set_Difference_is_Right_Distributive_over_Union) (cit. on p. 100).
- [Pro25ad] Proof Wiki Contributors. *Set Difference Over Subset* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_over\\_Subset](https://proofwiki.org/wiki/Set_Difference_over_Subset) (cit. on p. 100).
- [Pro25ae] Proof Wiki Contributors. *Set Difference With Empty Set Is Self* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_with\\_Empty\\_Set\\_is\\_Self](https://proofwiki.org/wiki/Set_Difference_with_Empty_Set_is_Self) (cit. on p. 101).
- [Pro25af] Proof Wiki Contributors. *Set Difference With Self Is Empty Set* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_with\\_Self\\_is\\_Empty\\_Set](https://proofwiki.org/wiki/Set_Difference_with_Self_is_Empty_Set) (cit. on p. 101).
- [Pro25ag] Proof Wiki Contributors. *Set Difference With Set Difference Is Union of Set Difference With Intersection* — Proof Wiki. 2025.

- URL: [https://proofwiki.org/wiki/Set\\_Difference\\_with\\_Set\\_Difference\\_is\\_Union\\_of\\_Set\\_Difference\\_with\\_Intersection](https://proofwiki.org/wiki/Set_Difference_with_Set_Difference_is_Union_of_Set_Difference_with_Intersection) (cit. on p. 101).
- [Pro25ah] Proof Wiki Contributors. *Set Difference With Subset Is Superset of Set Difference* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_with\\_Subset\\_is\\_Superset\\_of\\_Set\\_Difference](https://proofwiki.org/wiki/Set_Difference_with_Subset_is_Superset_of_Set_Difference) (cit. on p. 100).
- [Pro25ai] Proof Wiki Contributors. *Set Difference With Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Difference\\_with\\_Union](https://proofwiki.org/wiki/Set_Difference_with_Union) (cit. on p. 100).
- [Pro25aj] Proof Wiki Contributors. *Set Intersection Distributes Over Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Intersection\\_Distributes\\_over\\_Union](https://proofwiki.org/wiki/Intersection_Distributes_over_Union) (cit. on pp. 91, 96).
- [Pro25ak] Proof Wiki Contributors. *Set Intersection Is Idempotent* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Intersection\\_is\\_Idempotent](https://proofwiki.org/wiki/Set_Intersection_is_Idempotent) (cit. on p. 96).
- [Pro25al] Proof Wiki Contributors. *Set Intersection Preserves Subsets* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Intersection\\_Preserves\\_Subsets](https://proofwiki.org/wiki/Set_Intersection_Preserves_Subsets) (cit. on p. 96).
- [Pro25am] Proof Wiki Contributors. *Set Union Is Idempotent* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Union\\_is\\_Idempotent](https://proofwiki.org/wiki/Set_Union_is_Idempotent) (cit. on p. 91).
- [Pro25an] Proof Wiki Contributors. *Set Union Preserves Subsets* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Set\\_Union\\_Preserves\\_Subsets](https://proofwiki.org/wiki/Set_Union_Preserves_Subsets) (cit. on p. 91).
- [Pro25ao] Proof Wiki Contributors. *Symmetric Difference Is Associative* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_is\\_Associative](https://proofwiki.org/wiki/Symmetric_Difference_is_Associative) (cit. on p. 109).
- [Pro25ap] Proof Wiki Contributors. *Symmetric Difference Is Commutative* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_is\\_Commutative](https://proofwiki.org/wiki/Symmetric_Difference_is_Commutative) (cit. on p. 110).
- [Pro25aq] Proof Wiki Contributors. *Symmetric Difference of Complements* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_of\\_Complements](https://proofwiki.org/wiki/Symmetric_Difference_of_Complements) (cit. on p. 110).
- [Pro25ar] Proof Wiki Contributors. *Symmetric Difference on Power Set Forms Abelian Group* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_on\\_Power\\_Set\\_forms\\_Abelian\\_Group](https://proofwiki.org/wiki/Symmetric_Difference_on_Power_Set_forms_Abelian_Group) (cit. on p. 110).



- [Pro25as] Proof Wiki Contributors. *Symmetric Difference With Complement* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_with\\_Complement](https://proofwiki.org/wiki/Symmetric_Difference_with_Complement) (cit. on p. 110).
- [Pro25at] Proof Wiki Contributors. *Symmetric Difference With Empty Set* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_with\\_Empty\\_Set](https://proofwiki.org/wiki/Symmetric_Difference_with_Empty_Set) (cit. on p. 109).
- [Pro25au] Proof Wiki Contributors. *Symmetric Difference With Intersection Forms Ring* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_with\\_Intersection\\_forms\\_Ring](https://proofwiki.org/wiki/Symmetric_Difference_with_Intersection_forms_Ring) (cit. on p. 111).
- [Pro25av] Proof Wiki Contributors. *Symmetric Difference With Self Is Empty Set* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_with\\_Self\\_is\\_Empty\\_Set](https://proofwiki.org/wiki/Symmetric_Difference_with_Self_is_Empty_Set) (cit. on p. 110).
- [Pro25aw] Proof Wiki Contributors. *Symmetric Difference With Union Does Not Form Ring* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_with\\_Union\\_does\\_not\\_form\\_Ring](https://proofwiki.org/wiki/Symmetric_Difference_with_Union_does_not_form_Ring) (cit. on p. 108).
- [Pro25ax] Proof Wiki Contributors. *Symmetric Difference With Universe* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Symmetric\\_Difference\\_with\\_Universe](https://proofwiki.org/wiki/Symmetric_Difference_with_Universe) (cit. on p. 110).
- [Pro25ay] Proof Wiki Contributors. *Union as Symmetric Difference With Intersection* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Union\\_as\\_Symmetric\\_Difference\\_with\\_Intersection](https://proofwiki.org/wiki/Union_as_Symmetric_Difference_with_Intersection) (cit. on p. 91).
- [Pro25az] Proof Wiki Contributors. *Union Distributes Over Intersection* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Union\\_Distributes\\_over\\_Intersection](https://proofwiki.org/wiki/Union_Distributes_over_Intersection) (cit. on pp. 91, 96).
- [Pro25ba] Proof Wiki Contributors. *Union Is Associative* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Union\\_is\\_Associative](https://proofwiki.org/wiki/Union_is_Associative) (cit. on p. 91).
- [Pro25bb] Proof Wiki Contributors. *Union Is Commutative* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Union\\_is\\_Commutative](https://proofwiki.org/wiki/Union_is_Commutative) (cit. on p. 91).
- [Pro25bc] Proof Wiki Contributors. *Union of Symmetric Differences* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/Union\\_of\\_Symmetric\\_Differences](https://proofwiki.org/wiki/Union_of_Symmetric_Differences) (cit. on p. 110).

- [Pro25bd] Proof Wiki Contributors. *Union With Empty Set* — *Proof Wiki*. 2025. URL: [https://proofwiki.org/wiki/Union\\_with\\_Empty\\_Set](https://proofwiki.org/wiki/Union_with_Empty_Set) (cit. on p. 91).