# Conditions on Relations

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00TJ This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

## Contents

# 02D0 10.1 Functional and Total Relations

## 00JC 10.1.1 Functional Relations

Let $A$ and $B$ be sets.

00JD **Definition 10.1.1.1.1.** A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

00JE **Proposition 10.1.1.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JF 1. *Characterisations.* The following conditions are equivalent:

00JG (a) The relation $R$ is functional.

00JH (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

*Proof.* *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b*: Let $(b, b') \in B \times B$. We need to show that

$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since $R$ is functional.

- *Item 1b* $\Longrightarrow$ *Item 1a*: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  - Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  - Since $R \diamond R^{\dagger} \subset \chi_B$, we have

$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

    and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^{\dagger}\right](b, b') = \mathrm{true}$, and thus $\chi_B(b, b') = \mathrm{true}$ as well, i.e. $b = b'$.

This finishes the proof. ◻

## 00JJ 10.1.2 Total Relations

Let $A$ and $B$ be sets.

00JK **Definition 10.1.2.1.1.** A relation $R\colon A \rightarrow\!\!\!\!\!\!+\;\, B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

00JL **Proposition 10.1.2.1.2.** Let $R\colon A \rightarrow\!\!\!\!\!\!+\;\, B$ be a relation.

00JM 1. *Characterisations.* The following conditions are equivalent:

00JN (a) The relation $R$ is total.

00JP (b) We have $\chi_A \subset R^\dagger \diamond R$.

*Proof.* *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b*: We have to show that, for each $(a, a') \in A$, we have

$$\chi_A(a, a') \preceq_{\{\mathrm{t},\mathrm{f}\}} \left[R^\dagger \diamond R\right](a, a'),$$

  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of $R$.

- *Item 1b* $\Longrightarrow$ *Item 1a*: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$

  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

# 00TK 10.2 Reflexive Relations

## 00TL 10.2.1 Foundations

Let $A$ be a set.

00TM **Definition 10.2.1.1.1.** A **reflexive relation** is equivalently:[1]

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A,A), \chi_A)$.

00TN **Remark 10.2.1.1.2.** In detail, a relation $R$ on $A$ is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

00TP **Definition 10.2.1.1.3.** Let $A$ be a set.

00TQ 1. The **set of reflexive relations on** $A$ is the subset $\mathrm{Rel}^{\mathrm{refl}}(A,A)$ of $\mathrm{Rel}(A,A)$ spanned by the reflexive relations.

00TR 2. The **poset of relations on** $A$ is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A,A)$ of $\mathbf{Rel}(A,A)$ spanned by the reflexive relations.

00TS **Proposition 10.2.1.1.4.** Let $R$ and $S$ be relations on $A$.

00TT 1. *Interaction With Inverses.* If $R$ is reflexive, then so is $R^\dagger$.

00TU 2. *Interaction With Composition.* If $R$ and $S$ are reflexive, then so is $S \diamond R$.

*Proof.* *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: Clear. □

## 00TV 10.2.2 The Reflexive Closure of a Relation

Let $R$ be a relation on $A$.

00TW **Definition 10.2.2.1.1.** The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[2] satisfying the following universal property:[3]

(★) Given another reflexive relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

structure.

[2] *Further Notation:* Also written $R^{\mathrm{refl}}$.

[3] *Slogan:* The reflexive closure of $R$ is the smallest reflexive relation containing $R$.

00TX **Construction 10.2.2.1.2.** Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on $R$ in $(\mathbf{Rel}(A,A),\chi_A)$[4], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

*Proof.* Clear. □

00TY **Proposition 10.2.2.1.3.** Let $R$ be a relation on $A$.

00TZ 1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathrm{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00U0 2. *The Reflexive Closure of a Reflexive Relation.* If $R$ is reflexive, then $R^{\mathrm{refl}} = R$.

00U1 3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

00U2 4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

[4] Or, equivalently, the free $\mathbb{E}_0$-monoid on $R$ in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)),\chi_A)$.

00U3 5. *Interaction With Composition*. We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

*Proof.* *Item 1, Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 10.2.2.1.1.
*Item 2, The Reflexive Closure of a Reflexive Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2.
*Item 4, Interaction With Inverses*: Clear.
*Item 5, Interaction With Composition*: This follows from Item 2 of Definition 10.2.1.1.4.
□

## 00U4 10.3 Symmetric Relations

### 00U5 10.3.1 Foundations

Let $A$ be a set.

00U6 **Definition 10.3.1.1.1.** A relation $R$ on $A$ is **symmetric** if we have $R^{\dagger} = R$.

00U7 **Remark 10.3.1.1.2.** In detail, a relation $R$ is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

00U8 **Definition 10.3.1.1.3.** Let $A$ be a set.

00U9 1. The **set of symmetric relations on** $A$ is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

00UA 2. The **poset of relations on** $A$ is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UB **Proposition 10.3.1.1.4.** Let $R$ and $S$ be relations on $A$.

00UC 1. *Interaction With Inverses*. If $R$ is symmetric, then so is $R^{\dagger}$.

00UD 2. *Interaction With Composition*. If $R$ and $S$ are symmetric, then so is $S \diamond R$.

*Proof.* *Item 1, Interaction With Inverses*: Clear.
*Item 2, Interaction With Composition*: Clear. □

## 00UE 10.3.2 The Symmetric Closure of a Relation

Let $R$ be a relation on $A$.

00UF **Definition 10.3.2.1.1.** The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5] satisfying the following universal property:[6]

($\star$) Given another symmetric relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

00UG **Construction 10.3.2.1.2.** Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on $A$ defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

*Proof.* Clear. □

00UH **Proposition 10.3.2.1.3.** Let $R$ be a relation on $A$.

00UJ 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00UK 2. *The Symmetric Closure of a Symmetric Relation.* If $R$ is symmetric, then $R^{\mathrm{symm}} = R$.

00UL 3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

[5] *Further Notation:* Also written $R^{\mathrm{symm}}$.

[6] *Slogan:* The symmetric closure of $R$ is the smallest symmetric relation containing $R$.

00UM 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger},\qquad \begin{tikzcd} \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

00UN 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}},\qquad \begin{tikzcd} \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

*Proof.* *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 10.3.2.1.1.
*Item 2, The Symmetric Closure of a Symmetric Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2.
*Item 4, Interaction With Inverses*: Clear.
*Item 5, Interaction With Composition*: This follows from Item 2 of Definition 10.3.1.1.4.
□

# 00UP 10.4 Transitive Relations

## 00UQ 10.4.1 Foundations

Let $A$ be a set.

00UR **Definition 10.4.1.1.1.** A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

[7] Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

00US **Remark 10.4.1.1.2.** In detail, a relation $R$ on $A$ is **transitive** if we have an inclusion

$$\mu_R : R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00UT **Definition 10.4.1.1.3.** Let $A$ be a set.

00UU 1. The **set of transitive relations from** $A$ **to** $B$ is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.

00UV 2. The **poset of relations from** $A$ **to** $B$ is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UW **Proposition 10.4.1.1.4.** Let $R$ and $S$ be relations on $A$.

00UX 1. *Interaction With Inverses.* If $R$ is transitive, then so is $R^\dagger$.

00UY 2. *Interaction With Composition.* If $R$ and $S$ are transitive, then $S \diamond R$ ***may fail to be transitive***.

*Proof.* *Item 1*, *Interaction With Inverses*: Clear.
*Item 2*, *Interaction With Composition*: See [MSE 2096272].[8] □

## 00UZ 10.4.2 The Transitive Closure of a Relation

Let $R$ be a relation on $A$.

[8] *Intuition:* Transitivity for $R$ and $S$ fails to imply that of $S \diamond R$ because the composition operation for relations intertwines $R$ and $S$ in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
  - There is some $b \in A$ such that:
    - $a \sim_R b$;
    - $b \sim_S c$;
  - There is some $d \in A$ such that:
    - $c \sim_R d$;
    - $d \sim_S e$.

00V0 **Definition 10.4.2.1.1.** The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[9] satisfying the following universal property:[10]

(★) Given another transitive relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

00V1 **Construction 10.4.2.1.2.** Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on $R$ in $(\mathbf{Rel}(A, A), \diamond)$[11], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \,\middle|\, \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

*Proof.* Clear. □

00V2 **Proposition 10.4.2.1.3.** Let $R$ be a relation on $A$.

00V3 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}\left(R^{\mathrm{trans}}, S\right) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{trans}}(A, A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

00V4 2. *The Transitive Closure of a Transitive Relation.* If $R$ is transitive, then $R^{\mathrm{trans}} = R$.

00V5 3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

[9] *Further Notation:* Also written $R^{\mathrm{trans}}$.
[10] *Slogan:* The transitive closure of $R$ is the smallest transitive relation containing $R$.
[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on $R$ in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

00V6 4. *Interaction With Inverses*. We have

$$\left(R^{\dagger}\right)^{\mathsf{trans}} = \left(R^{\mathsf{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathsf{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

00V7 5. *Interaction With Composition*. We have

$$(S \diamond R)^{\mathsf{trans}} \overset{\text{poss.}}{\neq} S^{\mathsf{trans}} \diamond R^{\mathsf{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{trans}} \times (-)^{\mathsf{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathsf{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

*Proof.* *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 10.4.2.1.1.
*Item 2, The Transitive Closure of a Transitive Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2.
*Item 4, Interaction With Inverses*: We have

$$\begin{aligned} \left(R^{\dagger}\right)^{\mathsf{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} \\ &= \bigcup_{n=1}^{\infty} \left(R^{\diamond n}\right)^{\dagger} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} \\ &= \left(R^{\mathsf{trans}}\right)^{\dagger}, \end{aligned}$$

where we have used, respectively:

- Definition 10.4.2.1.2.
- Constructions With Relations, **??** of **??**.
- Constructions With Relations, **??** of Definition 9.2.3.1.2.

- Definition 10.4.2.1.2.

This finishes the proof.

*Item 5, Interaction With Composition*: This follows from Item 2 of Definition 10.4.1.1.4.

□

00V8 ## 10.5 Equivalence Relations

00V9 ### 10.5.1 Foundations

Let $A$ be a set.

00VA **Definition 10.5.1.1.1.** A relation $R$ is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

00VB **Example 10.5.1.1.2.** The **kernel of a function** $f\colon A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on $A$ obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

00VC **Definition 10.5.1.1.3.** Let $A$ and $B$ be sets.

00VD 1. The **set of equivalence relations from $A$ to $B$** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

00VE 2. The **poset of relations from $A$ to $B$** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

00VF ### 10.5.2 The Equivalence Closure of a Relation

Let $R$ be a relation on $A$.

00VG **Definition 10.5.2.1.1.** The **equivalence closure**[14] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[15] satisfying the following universal property:[16]

[12] *Further Terminology:* If instead $R$ is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of $f$ is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Constructions With Relations, **??** of **??**.

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written $R^{\mathrm{eq}}$.

[16] *Slogan:* The equivalence closure of $R$ is the smallest equivalence relation containing $R$.

(★) Given another equivalence relation $\sim_S$ on $A$ such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

00VH **Construction 10.5.2.1.2.** Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on $A$ defined by

$$
\begin{aligned}
R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\
&= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \\
&= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1,\ldots,x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \quad\quad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \quad\quad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \quad\quad\quad\ \text{for each } 1 \leq i \leq n-1; \\ \quad\quad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}.
\end{aligned}
$$

*Proof.* From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

00VJ 1. The symmetric closure of a reflexive relation is still reflexive.

00VK 2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

00VL **Proposition 10.5.2.1.3.** Let $R$ be a relation on $A$.

00VM 1. *Adjointness.* We have an adjunction

$$
\left((-)^{\mathrm{eq}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}
$$

witnessed by a bijection of sets

$$
\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),
$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

00VN 2. *The Equivalence Closure of an Equivalence Relation*. If $R$ is an equivalence relation, then $R^{\mathrm{eq}} = R$.

00VP 3. *Idempotency*. We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

*Proof.* *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 10.5.2.1.1.
*Item 2, The Equivalence Closure of an Equivalence Relation*: Clear.
*Item 3, Idempotency*: This follows from Item 2. □

## 00VQ 10.6 Quotients by Equivalence Relations

### 00VR 10.6.1 Equivalence Classes

Let $A$ be a set, let $R$ be a relation on $A$, and let $a \in A$.

02B1 **Definition 10.6.1.1.1.** The **equivalence class associated to** $a$ is the set $[a]$ defined by

$$\begin{aligned} [a] &\overset{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \qquad \text{(since } R \text{ is symmetric)} \end{aligned}$$

### 02B2 10.6.2 Quotients of Sets by Equivalence Relations

Let $A$ be a set and let $R$ be a relation on $A$.

02B3 **Definition 10.6.2.1.1.** The **quotient of** $X$ **by** $R$ is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00VV **Remark 10.6.2.1.2.** The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of $X$ under $R$ are well-behaved:

- *Reflexivity*. If $R$ is reflexive, then, for each $a \in X$, we have $a \in [a]$.

- *Symmetry*. The equivalence class $[a]$ of an element $a$ of $X$ is defined by

$$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when $R$ is symmetric, as we then have $[a] = [a]'$.[17]

- *Transitivity*. If $R$ is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

00VW **Proposition 10.6.2.1.3.** Let $f\colon X \to Y$ be a function and let $R$ be a relation on $X$.

02B4 1. *As a Coequaliser*. We have an isomorphism of sets

$$X/\sim^{\text{eq}}_R \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} X\right),$$

where $\sim^{\text{eq}}_R$ is the equivalence relation generated by $\sim_R$.

02B5 2. *As a Pushout*. We have an isomorphism of sets[18]

$$X/\sim^{\text{eq}}_R \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/\sim^{\text{eq}}_R \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

where $\sim^{\text{eq}}_R$ is the equivalence relation generated by $\sim_R$.

[17] When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

[18] Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/\sim^{\text{eq}}_R} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/\sim^{\text{eq}}_R. \end{tikzcd}$$

02B6 3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]

$$X/{\sim_{\mathrm{Ker}(f)}} \cong \mathrm{Im}(f).$$

00W0 4. *Descending Functions to Quotient Sets, I.* Let $R$ be an equivalence relation on $X$. The following conditions are equivalent:

02B7 (a) There exists a map

$$\bar{f} \colon X/{\sim_R} \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\exists"', "\bar{f}"] & \end{tikzcd}$$

commute.

02B8 (b) We have $R \subset \mathrm{Ker}(f)$.

02B9 (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

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[19] *Further Terminology:* The set $X/{\sim_{\mathrm{Ker}(f)}}$ is often called the **coimage of** $f$, and denoted by $\mathrm{CoIm}(f)$.

[20] In a sense this is a result relating the monad in **Rel** induced by $f$ with the comonad in **Rel** induced by $f$, as the kernel and image

$$\mathrm{Ker}(f) \colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of $f$ are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Constructions With Relations, ?? of ??.

00W1 5. *Descending Functions to Quotient Sets, II.* Let $R$ be an equivalence relation on $X$. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\overline{f}"', "\exists!"] & \end{tikzcd}$$

commute.

00W2 6. *Descending Functions to Quotient Sets, III.* Let $R$ be an equivalence relation on $X$. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/{\sim_R}, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

02BA 7. *Descending Functions to Quotient Sets, IV.* Let $R$ be an equivalence relation on $X$. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BB (a) The map $\overline{f}$ is an injection.

02BC (b) We have $R = \mathrm{Ker}(f)$.

02BD (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

02BE 8. *Descending Functions to Quotient Sets, V.* Let $R$ be an equivalence relation on $X$. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BF (a) The map $f \colon X \to Y$ is surjective.

02BG (b) The map $\overline{f} \colon X/{\sim_R} \to Y$ is surjective.

02BH 9. *Descending Functions to Quotient Sets, VI.* Let $R$ be a relation on $X$ and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to $R$. The following conditions are equivalent:

02BJ (a) The map $f$ satisfies the equivalent conditions of Item 4:

- There exists a map
$$\overline{f}\colon X/\sim_R^{\mathrm{eq}} \to Y$$
making the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R^{\mathrm{eq}} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$
commute.
- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

02BK (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

*Proof.* *Item 1, As a Coequaliser*: Omitted.
*Item 2, As a Pushout*: Omitted.
*Item 3, The First Isomorphism Theorem for Sets*: Clear.
*Item 4, Descending Functions to Quotient Sets, I*: See [Pro25c].
*Item 5, Descending Functions to Quotient Sets, II*: See [Pro25d].
*Item 6, Descending Functions to Quotient Sets, III*: This follows from Items 5 and 6.
*Item 7, Descending Functions to Quotient Sets, IV*: See [Pro25b].
*Item 8, Descending Functions to Quotient Sets, V*: See [Pro25a].
*Item 9, Descending Functions to Quotient Sets, VI*: The implication Item 8a $\implies$ Item 8b is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of $R$, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

- The following conditions are satisfied:
  - We have $x \sim_R x_1$ or $x_1 \sim_R x$;
  - We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
  - We have $y \sim_R x_n$ or $x_n \sim_R y$;

- We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

# Appendices

## A Other Chapters

# References


[MSE 2096272] Akiva Weinberger. *Is composition of two transitive relations transitive? If not, can you give me a counterexample?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2096272 (cit. on p. 9).

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[Pro25c] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be Well-Defined — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Well-Defined (cit. on p. 18).

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