

# Pointed Sets

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This chapter contains some foundational material on pointed sets.

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## 6.1 Pointed Sets

### 6.1.1 Foundations

#### DEFINITION 6.1.1.1.1 ► POINTED SETS

A **pointed set**<sup>1</sup> is equivalently:

- An  $\mathbb{E}_0$ -monoid in  $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ .
- A pointed object in  $(\mathbf{Sets}, \text{pt})$ .

<sup>1</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, pointed sets are viewed as  $\mathbb{F}_1$ -**modules**.

#### REMARK 6.1.1.1.2 ► UNWINDING DEFINITION 6.1.1.1.1

In detail, a **pointed set** is a pair  $(X, x_0)$  consisting of:

- *The Underlying Set.* A set  $X$ , called the **underlying set of**  $(X, x_0)$ .
- *The Basepoint.* A morphism

$$[x_0]: \text{pt} \rightarrow X$$

in  $\mathbf{Sets}$ , determining an element  $x_0 \in X$ , called the **basepoint of**  $X$ .

#### EXAMPLE 6.1.1.1.3 ► THE ZERO SPHERE

The **0-sphere**<sup>1</sup> is the pointed set  $(S^0, 0)$ <sup>2</sup> consisting of:

- *The Underlying Set.* The set  $S^0$  defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of  $S^0$ .

<sup>1</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

<sup>2</sup>*Further Notation:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras,  $S^0$  is also denoted  $(\mathbb{F}_1, 0)$ .

#### EXAMPLE 6.1.1.1.4 ► THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set  $(\text{pt}, \star)$  consisting of:

- *The Underlying Set.* The punctual set  $\text{pt} \stackrel{\text{def}}{=} \{\star\}$ .
- *The Basepoint.* The element  $\star$  of  $\text{pt}$ .

#### EXAMPLE 6.1.1.1.5 ► THE STANDARD POINTED SET WITH $n + 1$ ELEMENTS

The **standard pointed set with  $n + 1$  elements** is the pointed set  $\langle n \rangle$  consisting of

- *The Underlying Set.* The set  $\langle n \rangle$  defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{\star\} \cup \{1, \dots, n\}.$$

- *The Basepoint.* The element  $\star$  of  $\langle n \rangle$ .

## 6.1.2 Morphisms of Pointed Sets

#### DEFINITION 6.1.2.1.1 ► MORPHISMS OF POINTED SETS

A **morphism of pointed sets**<sup>1,2</sup> is equivalently:

- A morphism of  $\mathbb{E}_0$ -monoids in  $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ .
- A morphism of pointed objects in  $(\mathbf{Sets}, \text{pt})$ .

<sup>1</sup>*Further Terminology:* Also called a **pointed function**.

<sup>2</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, morphisms of pointed sets are also called **morphism of  $\mathbb{F}_1$ -modules**.

#### REMARK 6.1.2.1.2 ► UNWINDING DEFINITION 6.1.2.1.1

In detail, a **morphism of pointed sets**  $f: (X, x_0) \rightarrow (Y, y_0)$  is a

morphism of sets  $f: X \rightarrow Y$  such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

### 6.1.3 The Category of Pointed Sets

#### DEFINITION 6.1.3.1.1 ► THE CATEGORY OF POINTED SETS

The **category of pointed sets** is the category  $\mathbf{Sets}_*$  defined equivalently as:

- The homotopy category of the  $\infty$ -category  $\mathbf{Mon}_{\mathbb{E}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$  of ??, ??.
- The category  $\mathbf{Sets}_*$  of Constructions With Categories, ??.

#### REMARK 6.1.3.1.2 ► UNWINDING DEFINITION 6.1.3.1.1

In detail, the **category of pointed sets** is the category  $\mathbf{Sets}_*$  where:

- *Objects.* The objects of  $\mathbf{Sets}_*$  are pointed sets.
- *Morphisms.* The morphisms of  $\mathbf{Sets}_*$  are morphisms of pointed sets.
- *Identities.* For each  $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ , the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*}: \text{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

of  $\mathbf{Sets}_*$  at  $(X, x_0)$  is defined by<sup>1</sup>

$$\text{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\text{def}}{=} \text{id}_X.$$

- *Composition.* For each  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ , the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*}: \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of  $\mathbf{Sets}_*$  at  $((X, x_0), (Y, y_0), (Z, z_0))$  is defined by<sup>2</sup>

$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\text{def}}{=} g \circ f.$$

<sup>1</sup>Note that  $\text{id}_X$  is indeed a morphism of pointed sets, as we have  $\text{id}_X(x_0) = x_0$ .

<sup>2</sup>Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & \downarrow [y_0] & \searrow [z_0] \\ X & \xrightarrow{f} Y & \xrightarrow{g} Z \end{array}$$

$g(f(x_0)) = g(y_0) = z_0,$

## 6.1.4 Elementary Properties of Pointed Sets

### PROPOSITION 6.1.4.1.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let  $(X, x_0)$  be a pointed set.

1. *Completeness.* The category  $\mathbf{Sets}_*$  of pointed sets and morphisms between them is complete, having in particular:
  - (a) Products, described as in [Definition 6.2.3.1.1](#).
  - (b) Pullbacks, described as in [Definition 6.2.4.1.1](#).
  - (c) Equalisers, described as in [Definition 6.2.5.1.1](#).
2. *Cocompleteness.* The category  $\mathbf{Sets}_*$  of pointed sets and morphisms between them is cocomplete, having in particular:
  - (a) Coproducts, described as in [Definition 6.3.3.1.1](#).
  - (b) Pushouts, described as in [Definition 6.3.4.1.1](#);
  - (c) Coequalisers, described as in [Definition 6.3.5.1.1](#).
3. *Failure To Be Cartesian Closed.* The category  $\mathbf{Sets}_*$  is not Cartesian closed.<sup>1</sup>
4. *Morphisms From the Monoidal Unit.* We have a bijection of sets<sup>2</sup>

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in  $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ , internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in  $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

5. *Relation to Partial Functions.* We have an equivalence of categories<sup>3</sup>

$$\mathbf{Sets}_* \xrightarrow{\text{eq.}} \mathbf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

- (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \mathbf{Sets}_* \xrightarrow{\cong} \mathbf{Sets}^{\text{part.}}$$

sends:

- i. A pointed set  $(X, x_0)$  to  $X$ .
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on  $f^{-1}(Y \setminus y_0)$  and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in f^{-1}(Y \setminus y_0)$ .

- (b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \mathbf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathbf{Sets}_*$$

sends:

- i. A set  $X$  is to the pointed set  $(X, \star)$  with  $\star$  an element that is not in  $X$ .
- ii. A partial function

$$f: X \rightarrow Y$$

defined on  $U \subset X$  to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$


for each  $x \in X$ .

<sup>1</sup>The category  $\mathbf{Sets}_*$  does admit a natural monoidal closed structure, however; see [Tensor Products of Pointed Sets](#).

<sup>2</sup>In other words, the forgetful functor

$$\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by  $S^0$ .

<sup>3</sup> *Warning:* This is not an isomorphism of categories, only an equivalence.

#### PROOF 6.1.4.1.2 ► PROOF OF PROPOSITION 6.1.4.1.1

##### Item 1: Completeness

This follows from (the proofs) of [Definitions 6.2.3.1.1](#), [6.2.4.1.1](#) and [6.2.5.1.1](#) and ??.

##### Item 2: Cocompleteness

This follows from (the proofs) of [Definitions 6.3.3.1.1](#), [6.3.4.1.1](#) and [6.3.5.1.1](#) and ??.

##### Item 3: Failure To Be Cartesian Closed

See [\[MSE 2855868\]](#).

**Item 4: Morphisms From the Monoidal Unit**

Since a morphism from  $S^0$  to a pointed set  $(X, x_0)$  sends  $0 \in S^0$  to  $x_0$  and then can send  $1 \in S^0$  to any element of  $X$ , we obtain a bijection between pointed maps  $S^0 \rightarrow X$  and the elements of  $X$ .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that  $\Delta_{x_0}: S^0 \rightarrow X$ , the basepoint of  $\mathbf{Sets}_*(S^0, X)$ , corresponds to the pointed map  $S^0 \rightarrow X$  picking the element  $x_0$  of  $X$ , and thus we see that the bijection between pointed maps  $S^0 \rightarrow X$  and elements of  $X$  is compatible with basepoints, lifting to an isomorphism of pointed sets.

**Item 5: Relation to Partial Functions**

See [\[MSE 884460\]](#).



## 6.1.5 Active and Inert Morphisms of Pointed Sets

### DEFINITION 6.1.5.1.1 ► ACTIVE AND INERT MORPHISMS OF POINTED SETS

Let  $f: (X, x_0) \rightarrow (Y, y_0)$  be a morphism of pointed sets.

1. The morphism  $f$  is **active** if  $f^{-1}(y_0) = x_0$ .
2. The morphism  $f$  is **inert** if, for each  $y \in Y$ , the set  $f^{-1}(y)$  has exactly one element.

### NOTATION 6.1.5.1.2 ► THE CATEGORY OF POINTED SETS AND ACTIVE MORPHISMS

We write  $\mathbf{Sets}_*^{\text{actv}}$  for the wide subcategory of  $\mathbf{Sets}_*$  spanned by pointed sets and the active maps between them.

### EXAMPLE 6.1.5.1.3 ► EXAMPLES OF ACTIVE AND INERT MAPS OF POINTED SETS

Here are some examples of active and inert maps of pointed sets.



1. The map  $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$  given by

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \searrow & \\ * & \xrightarrow{\quad} & * \end{array}$$

is active but not inert.

2. The map  $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$  given by

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \searrow & 2 \\ * & \xrightarrow{\quad} & * \end{array}$$

is inert but not active.

3. The map  $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$  given by

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \searrow & \\ 3 & \searrow & \\ * & \xrightarrow{\quad} & * \end{array}$$

is neither inert nor active. However, it factors as  $f = a \circ i$ , where

$$\begin{aligned} i: \langle 3 \rangle &\rightarrow \langle 2 \rangle, \\ a: \langle 2 \rangle &\rightarrow \langle 1 \rangle \end{aligned}$$

are the morphisms of pointed sets given by

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \xrightarrow{\quad} & 2 \\ 3 & \searrow & \\ * & \xrightarrow{\quad} & * \end{array} \quad \begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \searrow & \\ * & \xrightarrow{\quad} & * \end{array},$$

with  $i$  being inert and  $a$  being active.

**PROPOSITION 6.1.5.1.4 ► PROPERTIES OF ACTIVE AND INERT MAPS OF POINTED SETS**

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets  $f: (X, x_0) \rightarrow (Y, y_0)$  factors uniquely as

$$f = a \circ i,$$

where:

- (a) The map  $i: (X, x_0) \rightarrow (K, k_0)$  is an inert morphism of pointed sets
- (b) The map  $a: (K, k_0) \rightarrow (Y, y_0)$  is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in  $\mathbf{Sets}_*$ .

**PROOF 6.1.5.1.5 ► PROOF OF PROPOSITION 6.1.5.1.4****Item 1: Active-Inert Factorisation**

Let  $f: X \rightarrow Y$  be a morphism of pointed sets. We can factor  $f$  as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- $K$  is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= (X \setminus f^{-1}(y_0)) \cup \{x_0\}; \end{aligned}$$

- $i: X \rightarrow K$  is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each  $x \in X$ ;

- $a: K \rightarrow Y$  is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in K$ .

Next, let

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{a} & B \end{array}$$

be a commutative diagram in  $\mathbf{Sets}_*$ . Consider the morphism  $\phi: Y \rightarrow A$  given by

$$\phi(y) = f(i^{-1}(y))$$

for each  $y \in Y$  (which is well-defined since, as  $i$  is inert,  $i^{-1}(y)$  is a singleton for all  $y \in Y$ ). We claim that  $\phi$  is the unique diagonal filler in the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & \exists! \nearrow \phi & \downarrow g \\ A & \xrightarrow{a} & B. \end{array}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\stackrel{\text{def}}{=} \phi(i(x)) \\ &\stackrel{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each  $x \in X$  and

$$\begin{aligned} [a \circ \phi](y) &\stackrel{\text{def}}{=} a(\phi(y)) \\ &\stackrel{\text{def}}{=} a(f(i^{-1}(y))) \\ &\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y)) \end{aligned}$$

$$\begin{aligned}
&= [g \circ i](i^{-1}(y)) \\
&\stackrel{\text{def}}{=} g(i(i^{-1}(y))) \\
&\stackrel{\text{def}}{=} g(y)
\end{aligned}$$

for each  $y \in Y$ . Moreover, given another morphism  $\psi$  such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
f \downarrow & \swarrow \psi & \downarrow g \\
A & \xrightarrow{a} & B
\end{array}$$

commutes, it follows that we must have  $\psi = \phi$ , since, given  $y \in Y$ , there exists a unique  $x \in X$  such that  $i(x) = y$ , so we have

$$\begin{aligned}
\psi(y) &= \psi(i(x)) \\
&= f(x) \\
&= f(i^{-1}(y)) \\
&\stackrel{\text{def}}{=} \phi(y).
\end{aligned}$$

This finishes the proof. 

## 6.2 Limits of Pointed Sets

### 6.2.1 The Terminal Pointed Set

#### DEFINITION 6.2.1.1.1 ► THE TERMINAL POINTED SET

The **terminal pointed set** is the terminal object of  $\mathbf{Sets}_*$  as in Limits and Colimits, ??.

**CONSTRUCTION 6.2.1.1.2 ► CONSTRUCTION OF THE TERMINAL POINTED SET**

Concretely, the **terminal pointed set** is the pair  $((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)})$  consisting of:

- *The Limit.* The pointed set  $(\text{pt}, \star)$ .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each  $x \in X$  and each  $(X, x_0) \in \text{Obj}(\text{Sets})$ .

**PROOF 6.2.1.1.3 ► PROOF OF CONSTRUCTION 6.2.1.1.2**

We claim that  $(\text{pt}, \star)$  is the terminal object of  $\text{Sets}_*$ . Indeed, suppose we have a diagram of the form


$$(X, x_0) \quad (\text{pt}, \star)$$

in  $\text{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely  $!_X$ . 

**6.2.2 Products of Families of Pointed Sets**

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

**DEFINITION 6.2.2.1.1 ► THE PRODUCT OF A FAMILY OF POINTED SETS**

The **product** of  $\{(X_i, x_0^i)\}_{i \in I}$  is the product of  $\{(X_i, x_0^i)\}_{i \in I}$  in  $\text{Sets}_*$  as in Limits and Colimits, ??.

**CONSTRUCTION 6.2.2.1.2 ► CONSTRUCTION OF THE PRODUCT OF A FAMILY OF POINTED SETS**

Concretely, the **product** of  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $\left(\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right), \{\text{pr}_i\}_{i \in I}\right)$  consisting of:

- *The Limit.* The pointed set  $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ .
- *The Cone.* The collection

$$\left\{ \text{pr}_i: \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i((x_j)_{j \in I}) \stackrel{\text{def}}{=} x_i$$

for each  $(x_j)_{j \in I} \in \prod_{i \in I} X_i$  and each  $i \in I$ .

**PROOF 6.2.2.1.3 ► PROOF OF CONSTRUCTION 6.2.2.1.2**

We claim that  $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$  is the categorical product of  $\{(X_i, x_0^i)\}_{i \in I}$  in  $\mathbf{Sets}_*$ . Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ & \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right) & \xrightarrow{\text{pr}_i} (X_i, x_0^i) \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$$

making the diagram

$$\begin{array}{ccc}
 (P, *) & & \\
 \downarrow \phi \exists! & \searrow p_i & \\
 \left( \prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i)
 \end{array}$$

commute, being uniquely determined by the condition  $\text{pr}_i \circ \phi = p_i$  for each  $i \in I$  via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_i(*))_{i \in I} \\
 &= (x_0^i)_{i \in I},
 \end{aligned}$$

where we have used that  $p_i$  is a morphism of pointed sets for each  $i \in I$ .



#### PROPOSITION 6.2.2.1.4 ► PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

1. *Functoriality.* The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto \left( \prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$  defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

#### PROOF 6.2.2.1.5 ► PROOF OF PROPOSITION 6.2.2.1.4

##### Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.



## 6.2.3 Products

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

\_\_\_\_\_



The **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the product of  $(X, x_0)$  and  $(Y, y_0)$  in  $\mathbf{Sets}_*$  as in Limits and Colimits, ??.

**CONSTRUCTION 6.2.3.1.2 ► CONSTRUCTION OF PRODUCTS OF POINTED SETS**

Concretely, the **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pair consisting of:

- *The Limit.* The pointed set  $(X \times Y, (x_0, y_0))$ .
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1 &: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0), \\ \text{pr}_2 &: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each  $(x, y) \in X \times Y$ .

**PROOF 6.2.3.1.3 ► PROOF OF CONSTRUCTION 6.2.3.1.2**

We claim that  $(X \times Y, (x_0, y_0))$  is the categorical product of  $(X, x_0)$  and  $(Y, y_0)$  in  $\mathbf{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & (P, *) & \\ p_1 \swarrow & & \searrow p_2 \\ (X, x_0) & \xleftarrow{\text{pr}_1} (X \times Y, (x_0, y_0)) \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccc}
 & (P, *) & \\
 p_1 \swarrow & \downarrow \phi \mid \exists! & \searrow p_2 \\
 (X, x_0) & \xleftarrow{\text{pr}_1} (X \times Y, (x_0, y_0)) \xrightarrow{\text{pr}_2} & (Y, y_0)
 \end{array}$$

commute, being uniquely determined by the conditions


$$\begin{aligned}
 \text{pr}_1 \circ \phi &= p_1, \\
 \text{pr}_2 \circ \phi &= p_2
 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_1(*), p_2(*)) \\
 &= (x_0, y_0),
 \end{aligned}$$

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets. 

#### PROPOSITION 6.2.3.1.4 ► PROPERTIES OF PRODUCTS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned}
 A \times -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\
 - \times B: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\
 -_1 \times -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*,
 \end{aligned}$$

defined in the same way as the functors of **Constructions With Sets**, Item 1 of Proposition 4.1.3.1.4.

2. *Lack of Adjointness.* The functors  $X \times -$  and  $- \times Y$  do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

4. *Unitality.* We have isomorphisms of pointed sets

$$(\text{pt}, \star) \times (X, x_0) \cong (X, x_0),$$

$$(X, x_0) \times (\text{pt}, \star) \cong (X, x_0),$$

natural in  $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in  $(X, x_0), (Y, y_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

6. *Symmetric Monoidality.* The triple  $(\mathbf{Sets}_*, \times, (\text{pt}, \star))$  is a symmetric monoidal category.

#### PROOF 6.2.3.1.5 ► PROOF OF PROPOSITION 6.2.3.1.4

##### Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

##### Item 2: Lack of Adjointness

See [MSE 2855868].

##### Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

##### Item 4: Unitality

This follows from **Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.**

Item 5: Commutativity

This follows from **Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.**

Item 6: Symmetric Monoidality

This follows from **Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.** 

## 6.2.4 Pullbacks

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (X, x_0) \rightarrow (Z, z_0)$  and  $g: (Y, y_0) \rightarrow (Z, z_0)$  be morphisms of pointed sets.

### DEFINITION 6.2.4.1.1 ► PULLBACKS OF POINTED SETS

The **pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$**  is the pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$  in **Sets<sub>\*</sub>** as in Limits and Colimits, ??.

### CONSTRUCTION 6.2.4.1.2 ► CONSTRUCTION OF PULLBACKS OF POINTED SETS

Concretely, the **pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$**  is the pair consisting of:

- *The Limit.* The pointed set  $(X \times_Z Y, (x_0, y_0))$ .
- *The Cone.* The morphisms of pointed sets

$$\text{pr}_1: (X \times_Z Y, (x_0, y_0)) \rightarrow (X, x_0),$$

$$\text{pr}_2: (X \times_Z Y, (x_0, y_0)) \rightarrow (Y, y_0)$$

defined by

$$\text{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

$$\text{pr}_2(x, y) \stackrel{\text{def}}{=} y$$

for each  $(x, y) \in X \times_Z Y$ .

**PROOF 6.2.4.1.3 ► PROOF OF CONSTRUCTION 6.2.4.1.2**

We claim that  $X \times_Z Y$  is the categorical pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to  $(f, g)$  in  $\mathbf{Sets}_*$ . First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$\begin{array}{ccc}
 (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\
 \text{pr}_1 \downarrow & & \downarrow g \\
 (X, x_0) & \xrightarrow{f} & (Z, z_0).
 \end{array}$$

$f \circ \text{pr}_1 = g \circ \text{pr}_2,$

Indeed, given  $(x, y) \in X \times_Z Y$ , we have

$$\begin{aligned}
 [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\
 &= f(x) \\
 &= g(y) \\
 &= g(\text{pr}_2(x, y)) \\
 &= [g \circ \text{pr}_2](x, y),
 \end{aligned}$$

where  $f(x) = g(y)$  since  $(x, y) \in X \times_Z Y$ . Next, we prove that  $X \times_Z Y$  satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{array}{ccccc}
 (P, *) & & \xrightarrow{p_2} & & (Y, y_0) \\
 & \searrow p_1 & & \swarrow \text{pr}_2 & \\
 & & (X \times_Z Y, (x_0, y_0)) & & \\
 & & \downarrow \text{pr}_1 & \lrcorner & \downarrow g \\
 & & (X, x_0) & \xrightarrow{f} & (Z, z_0)
 \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc}
 (P, *) & & & & \\
 \downarrow \phi & \nearrow p_2 & & & \\
 (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) & & \\
 \downarrow \text{pr}_1 & \lrcorner & \downarrow g & & \\
 (X, x_0) & \xrightarrow{f} & (Z, z_0) & & \\
 \uparrow p_1 & & & & 
 \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
 \text{pr}_1 \circ \phi &= p_1, \\
 \text{pr}_2 \circ \phi &= p_2
 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ , where we note that  $(p_1(x), p_2(x)) \in X \times Y$  indeed lies in  $X \times_Z Y$  by the condition


$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each  $x \in P$ , so that  $(p_1(x), p_2(x)) \in X \times_Z Y$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_1(*), p_2(*)) \\
 &= (x_0, y_0),
 \end{aligned}$$

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets. 

#### PROPOSITION 6.2.4.1.4 ► PROPERTIES OF PULLBACKS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

1. *Functoriality.* The assignment  $(X, Y, Z, f, g) \mapsto X \times_{f, Z, g} Y$  defines

a functor

$$-_1 \times_{-3} -_1 : \mathbf{Fun}(\mathcal{P}, \mathbf{Sets}_*) \rightarrow \mathbf{Sets}_*,$$

where  $\mathcal{P}$  is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of  $-_1 \times_{-3} -_1$  is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \\ & X' & \xrightarrow{f'} & Z' & \end{array}$$

in  $\mathbf{Fun}(\mathcal{P}, \mathbf{Sets}_*)$  to the morphism of pointed sets

$$\xi : (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each  $(x, y) \in X \times_Z Y$ , which is the unique morphism of pointed

sets making the diagram

$$\begin{array}{ccccc}
 X \times_Z Y & \longrightarrow & Y & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & X' \times_{Z'} Y' & \longrightarrow & Y' & \\
 & \downarrow & \downarrow & \downarrow g' & \\
 X & \xrightarrow{f} & Z & & \\
 \searrow \phi & & \searrow \chi & & \\
 & X' & \xrightarrow{f'} & Z' &
 \end{array}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{array}{ccccc}
 X & & Y & & Z \\
 & \searrow f & \swarrow g & \searrow h & \swarrow k \\
 & W & & V &
 \end{array}$$

in  $\mathbf{Sets}_*$ , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} (X \times_W Y) \times_Y Z \\ \swarrow \quad \searrow \\ X \times_W Y \quad Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} & 
 \begin{array}{c} (X \times_W Y) \times_Y (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} & 
 \begin{array}{c} X \times_W (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array}
 \end{array}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow f & \lrcorner & \downarrow f \\ X & \xlongequal{\quad} & X \end{array} & 
 \begin{array}{l} X \times_X A \cong A, \\ A \times_X X \cong A, \end{array} & 
 \begin{array}{ccc} A & \xrightarrow{f} & X \\ \parallel & \lrcorner & \parallel \\ X & \xrightarrow{f} & X. \end{array}
 \end{array}$$



4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow \lrcorner & & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad X \times_{\text{pt}} Y \cong X \times Y,$$

6. *Symmetric Monoidality.* The triple  $(\mathbf{Sets}_*, \times_X, X)$  is a symmetric monoidal category.

#### PROOF 6.2.4.1.5 ► PROOF OF PROPOSITION 6.2.4.1.4

##### Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pullback diagram.

##### Item 2: Associativity

This follows from [Constructions With Sets, Item 4 of Proposition 4.1.4.1.7](#).

##### Item 3: Unitality

This follows from [Constructions With Sets, Item 6 of Proposition 4.1.4.1.7](#).

##### Item 4: Commutativity

This follows from [Constructions With Sets, Item 7 of Proposition 4.1.4.1.7](#).

### Item 5: Interaction With Products

This follows from **Constructions With Sets**, **Item 10** of **Proposition 4.1.4.1.7**.

### Item 6: Symmetric Monoidality

This follows from **Constructions With Sets**, **Item 11** of **Proposition 4.1.4.1.7**. 

## 6.2.5 Equalisers

Let  $f, g: (X, x_0) \rightrightarrows (Y, y_0)$  be morphisms of pointed sets.

### DEFINITION 6.2.5.1.1 ► EQUALISERS OF POINTED SETS

The **equaliser of**  $(f, g)$  is the equaliser of  $f$  and  $g$  in  $\mathbf{Sets}_*$  as in Limits and Colimits, ??.

### CONSTRUCTION 6.2.5.1.2 ► CONSTRUCTION OF EQUALISERS OF POINTED SETS

Concretely, the **equaliser of**  $(f, g)$  is the pair consisting of:

- *The Limit.* The pointed set  $(\mathrm{Eq}(f, g), x_0)$ .
- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g): (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion  $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$ .

### PROOF 6.2.5.1.3 ► PROOF OF CONSTRUCTION 6.2.5.1.2

We claim that  $(\mathrm{Eq}(f, g), x_0)$  is the categorical equaliser of  $f$  and  $g$  in  $\mathbf{Sets}_*$ . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set  $\mathrm{Eq}(f, g)$ . Next, we prove that  $\mathrm{Eq}(f, g)$  satisfies the universal property of the equaliser. Suppose

we have a diagram of the form

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \\ & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \\ \uparrow \phi \exists! & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in A$  indeed lies in  $\text{Eq}(f, g)$  by the condition


$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f, g)$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that  $e$  is a morphism of pointed sets. 

**PROPOSITION 6.2.5.1.4 ► PROPERTIES OF EQUALISERS OF POINTED SETS**

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where  $\text{Eq}(f, g, h)$  is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{h} \end{array} (Y, y_0)$$

in  $\mathbf{Sets}_*$ , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

**PROOF 6.2.5.1.5 ► PROOF OF PROPOSITION 6.2.5.1.4****Item 1: Associativity**

This follows from **Constructions With Sets, Item 1 of Proposition 4.1.5.1.4.**

**Item 2: Unitality**

This follows from **Constructions With Sets, Item 2 of Proposition 4.1.5.1.4.**

**Item 3: Commutativity**

This follows from [Constructions With Sets, Item 3 of Proposition 4.1.5.1.4](#). 

## 6.3 Colimits of Pointed Sets

### 6.3.1 The Initial Pointed Set

#### DEFINITION 6.3.1.1.1 ► THE INITIAL POINTED SET

The **initial pointed set** is the initial object of  $\mathbf{Sets}_*$  as in Limits and Colimits, ??.

#### CONSTRUCTION 6.3.1.1.2 ► CONSTRUCTION OF THE INITIAL POINTED SET

Concretely, the **initial pointed set** is the pair  $((\text{pt}, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$  consisting of:

- *The Limit.* The pointed set  $(\text{pt}, \star)$ .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (\text{pt}, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

#### PROOF 6.3.1.1.3 ► PROOF OF CONSTRUCTION 6.3.1.1.2

We claim that  $(\text{pt}, \star)$  is the initial object of  $\mathbf{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$(\text{pt}, \star) \quad (X, x_0)$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi : (\text{pt}, \star) \rightarrow (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely  $\iota_X$ .



## 6.3.2 Coproducts of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

### DEFINITION 6.3.2.1.1 ► COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$ <sup>1</sup> is the coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in  $\mathbf{Sets}_*$  as in Limits and Colimits, ??.

<sup>1</sup>*Further Terminology:* Also called the **wedge sum of the family**  $\{(X_i, x_0^i)\}_{i \in I}$ .

### CONSTRUCTION 6.3.2.1.2 ► CONSTRUCTION OF COPRODUCTS OF FAMILIES OF POINTED SETS

Concretely, the **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $((\bigvee_{i \in I} X_i, p_0), \{\text{inj}_i\}_{i \in I})$  consisting of:

- *The Colimit.* The pointed set  $(\bigvee_{i \in I} X_i, p_0)$  consisting of:
  - *The Underlying Set.* The set  $\bigvee_{i \in I} X_i$  defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left( \coprod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\coprod_{i \in I} X_i$  given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each  $i, j \in I$ .

- *The Basepoint.* The element  $p_0$  of  $\bigvee_{i \in I} X_i$  defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any  $i, j \in I$ .

- *The Cocone.* The collection

$$\left\{ \text{inj}_i : (X_i, x_0^i) \rightarrow \left( \bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each  $x \in X_i$  and each  $i \in I$ .

**PROOF 6.3.2.1.3 ► PROOF OF CONSTRUCTION 6.3.2.1.2**

We claim that  $(\bigvee_{i \in I} X_i, p_0)$  is the categorical coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in  $\mathbf{Sets}_*$ . Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left( \bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi : \left( \bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram


$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \uparrow \exists! \phi \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left( \bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

commute, being uniquely determined by the condition  $\phi \circ \text{inj}_i = \iota_i$  for each  $i \in I$  via

$$\phi([i, x]) = \iota_i(x)$$

for each  $[(i, x)] \in \bigvee_{i \in I} X_i$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \iota_i\left(\left[(i, x_0^i)\right]\right) \\ &= *,\end{aligned}$$

as  $\iota_i$  is a morphism of pointed sets. 

#### PROPOSITION 6.3.2.1.4 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

1. *Functoriality.* The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$  defines a functor

$$\bigvee : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

#### PROOF 6.3.2.1.5 ► PROOF OF PROPOSITION 6.3.2.1.4

##### Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??. 

### 6.3.3 Coproducts

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

#### DEFINITION 6.3.3.1.1 ► COPRODUCTS OF POINTED SETS

The **coproduct of  $(X, x_0)$  and  $(Y, y_0)$** <sup>1</sup> is the coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in  $\text{Sets}_*$  as in Limits and Colimits, ??.

<sup>1</sup>*Further Terminology:* Also called the **wedge sum of  $(X, x_0)$  and  $(Y, y_0)$** .



**CONSTRUCTION 6.3.3.1.2 ► CONSTRUCTION OF COPRODUCTS OF POINTED SETS**

Concretely, the **coproduct of**  $(X, x_0)$  **and**  $(Y, y_0)$ , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set  $(X \vee Y, p_0)$  consisting of:
  - *The Underlying Set.* The set  $X \vee Y$  defined by

$$\begin{aligned}
 (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) & X \vee Y &\longleftarrow Y \\
 &\cong (X \amalg_{\text{pt}} Y, p_0) & \uparrow \lrcorner & \uparrow [y_0] \\
 &\cong (X \amalg Y / \sim, p_0), & X &\xleftarrow{[x_0]} \text{pt},
 \end{aligned}$$

where  $\sim$  is the equivalence relation on  $X \amalg Y$  obtained by declaring  $(0, x_0) \sim (1, y_0)$ .

- *The Basepoint.* The element  $p_0$  of  $X \vee Y$  defined by

$$\begin{aligned}
 p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\
 &= [(1, y_0)].
 \end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}
 \text{inj}_1 &: (X, x_0) \rightarrow (X \vee Y, p_0), \\
 \text{inj}_2 &: (Y, y_0) \rightarrow (X \vee Y, p_0),
 \end{aligned}$$

given by

$$\begin{aligned}
 \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\
 \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],
 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ .

**PROOF 6.3.3.1.3 ► PROOF OF CONSTRUCTION 6.3.3.1.2**

We claim that  $(X \vee Y, p_0)$  is the categorical coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in  $\mathbf{Sets}_*$ . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_1 & & \nwarrow \iota_2 & \\ (X, x_0) & \xrightarrow{\text{inj}_1} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \end{array}$$

in  $\mathbf{Sets}$ . Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_1 & \uparrow \phi \exists! & \nwarrow \iota_2 & \\ (X, x_0) & \xrightarrow{\text{inj}_1} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \text{inj}_X &= \iota_X, \\ \phi \circ \text{inj}_Y &= \iota_Y \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each  $z \in X \vee Y$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

$$\begin{aligned}
&= \iota_Y([(1, y_0)]) \\
&= *,
\end{aligned}$$

as  $\iota_X$  and  $\iota_Y$  are morphisms of pointed sets.



**PROPOSITION 6.3.3.1.4 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS**

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned}
X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\
- \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\
-_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*.
\end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$ .

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}
(\text{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\
(X, x_0) \vee (\text{pt}, *) &\cong (X, x_0),
\end{aligned}$$

natural in  $(X, x_0) \in \mathbf{Sets}_*$ .

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in  $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$ .

5. *Symmetric Monoidality.* The triple  $(\mathbf{Sets}_*, \vee, \text{pt})$  is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at  $X$  is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each  $p \in X \vee X$ .

#### PROOF 6.3.3.1.5 ► PROOF OF PROPOSITION 6.3.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation  $\nabla$  is the statement that, given a

morphism of pointed sets  $f: (X, x_0) \rightarrow (Y, y_0)$ , we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \quad \begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each  $[(i, x)] \in X \vee X$ , and thus  $\nabla$  is indeed a natural transformation.



### 6.3.4 Pushouts

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (Z, z_0) \rightarrow (X, x_0)$  and  $g: (Z, z_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

#### DEFINITION 6.3.4.1.1 ► PUSHOUTS OF POINTED SETS

The **pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$**  is the pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$  in  $\mathbf{Sets}_*$  as in Limits and Colimits, ??.

#### CONSTRUCTION 6.3.4.1.2 ► CONSTRUCTION OF PUSHOUTS OF POINTED SETS

Concretely, the **pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along  $(f, g)$**  is the pair consisting of:

- *The Colimit.* The pointed set  $(X \amalg_{f, Z, g} Y, p_0)$ , where:
  - The set  $X \amalg_{f, Z, g} Y$  is the pushout (of unpointed sets) of  $X$

- and  $Y$  over  $Z$  with respect to  $f$  and  $g$ ;
- We have  $p_0 = [x_0] = [y_0]$ .
  - *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1 &: (X, x_0) \rightarrow (X \amalg_Z Y, p_0), \\ \text{inj}_2 &: (Y, y_0) \rightarrow (X \amalg_Z Y, p_0) \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)] \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ .

#### PROOF 6.3.4.1.3 ► PROOF OF ??

Firstly, we note that indeed  $[x_0] = [y_0]$ , as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since  $f$  and  $g$  are morphisms of pointed sets, with the relation  $\sim$  on  $X \amalg_Z Y$  then identifying  $x_0 = f(z_0) \sim g(z_0) = y_0$ .

We now claim that  $(X \amalg_Z Y, p_0)$  is the categorical pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to  $(f, g)$  in  $\mathbf{Sets}_*$ . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} & (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} (Y, y_0) \\ \text{inj}_1 \circ f = \text{inj}_2 \circ g, & \uparrow \text{inj}_1 & \uparrow g \\ & (X, x_0) & \xleftarrow{f} (Z, z_0). \end{array}$$

Indeed, given  $z \in Z$ , we have

$$[\text{inj}_1 \circ f](z) = \text{inj}_1(f(z))$$

$$\begin{aligned}
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \text{inj}_2(g(z)) \\
&= [\text{inj}_2 \circ g](z),
\end{aligned}$$

where  $[(0, f(z))] = [(1, g(z))]$  by the definition of the relation  $\sim$  on  $X \amalg Y$  (the coproduct of unpointed sets of  $X$  and  $Y$ ). Next, we prove that  $X \amalg_Z Y$  satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{array}{ccc}
(P, *) & \xleftarrow{\quad \iota_2 \quad} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\
& \nwarrow \iota_1 & \uparrow \text{inj}_1 \quad \sqcap \quad \uparrow g \\
& & (X, x_0) \xleftarrow{\quad f \quad} (Z, z_0)
\end{array}$$

in  $\mathbf{Sets}_*$ . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram

$$\begin{array}{ccc}
(P, *) & \xleftarrow{\quad \iota_2 \quad} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\
& \nwarrow \iota_1 \quad \swarrow \phi \quad \exists! & \uparrow \text{inj}_1 \quad \sqcap \quad \uparrow g \\
& & (X, x_0) \xleftarrow{\quad f \quad} (Z, z_0)
\end{array}$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via


$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each  $p \in X \amalg_Z Y$ , where the well-definedness of  $\phi$  is proven in the same way as in the proof of **Constructions With Sets, Definition 4.2.4.1.1**. Finally, we show that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

where we use that  $\iota_1$  (resp.  $\iota_2$ ) is a morphism of pointed sets. 

#### PROPOSITION 6.3.4.1.4 ► PROPERTIES OF PUSHOUTS OF POINTED SETS

Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

1. *Functoriality.* The assignment  $(X, Y, Z, f, g) \mapsto X \amalg_{f, Z, g} Y$  defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where  $\mathcal{P}$  is the category that looks like this:

$$\begin{array}{ccc} & & \bullet \\ & \uparrow & \\ \bullet & \leftarrow & \bullet \end{array}$$

In particular, the action on morphisms of  ${}_1 \amalg_{-3} -_1$  is given by



sending a morphism

$$\begin{array}{ccccc}
 X \amalg_Z Y & \longleftarrow & Y & & \\
 \uparrow & & \uparrow & \searrow \psi & \\
 & X' \amalg_{Z'} Y' & \longleftarrow & Y' & \\
 & \uparrow & \downarrow g & & \uparrow g' \\
 X & \longleftarrow & f & Z & \\
 \searrow \phi & & & \searrow \chi & \\
 & X' & \longleftarrow & f' & Z'
 \end{array}$$

in  $\text{Fun}(\mathcal{P}, \text{Sets}_*)$  to the morphism of pointed sets

$$\xi: (X \amalg_Z Y, p_0) \xrightarrow{\exists!} (X' \amalg_{Z'} Y', p'_0)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each  $p \in X \amalg_Z Y$ , which is the unique morphism of pointed sets making the diagram

$$\begin{array}{ccccc}
 X \amalg_Z Y & \longleftarrow & Y & & \\
 \uparrow & \searrow \xi & \uparrow & \searrow \psi & \\
 & X' \amalg_{Z'} Y' & \longleftarrow & Y' & \\
 & \uparrow & \downarrow g & & \uparrow g' \\
 X & \longleftarrow & f & Z & \\
 \searrow \phi & & & \searrow \chi & \\
 & X' & \longleftarrow & f' & Z'
 \end{array}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{array}{ccccc} X & & Y & & Z \\ & \nwarrow f & \nearrow g & \nwarrow h & \nearrow k \\ & W & & V & \end{array}$$

in **Sets**, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc} \begin{array}{ccccc} & (X \amalg_W Y) \amalg_V Z & & & \\ & \nearrow \wedge & \nwarrow & & \\ X \amalg_W Y & & Y & & Z \\ \nearrow \wedge & & \nwarrow & & \\ X & & Y & & Z \\ \nwarrow f & & \nearrow g & & \nwarrow h & & \nearrow k \\ & W & & V & \end{array} & \begin{array}{ccccc} & (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) & & & \\ & \nearrow \wedge & \nwarrow & & \\ X \amalg_W Y & & Y \amalg_V Z & & Z \\ \nearrow \wedge & & \nwarrow & & \\ X & & Y & & Z \\ \nwarrow f & & \nearrow g & & \nwarrow h & & \nearrow k \\ & W & & V & \end{array} & \begin{array}{ccccc} & X \amalg_W (Y \amalg_V Z) & & & \\ & \nearrow \wedge & \nwarrow & & \\ X & & Y \amalg_V Z & & Z \\ \nearrow \wedge & & \nwarrow & & \\ X & & Y & & Z \\ \nwarrow f & & \nearrow g & & \nwarrow h & & \nearrow k \\ & W & & V & \end{array} \end{array}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc} \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \uparrow f & \lrcorner & \uparrow f \\ X & \xlongequal{\quad} & X \end{array} & \begin{array}{c} X \amalg_X A \cong A, \\ A \amalg_X X \cong A, \end{array} & \begin{array}{ccc} A & \xleftarrow{f} & X \\ \parallel \lrcorner & & \parallel \\ X & \xleftarrow{f} & X. \end{array} \end{array}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc} \begin{array}{ccc} X \amalg_Z Y & \xleftarrow{\quad} & Y \\ \uparrow \lrcorner & & \uparrow g \\ X & \xleftarrow{f} & Z, \end{array} & X \amalg_Z Y \cong Y \amalg_Z X & \begin{array}{ccc} Y \amalg_Z X & \xleftarrow{\quad} & X \\ \uparrow \lrcorner & & \uparrow f \\ Y & \xleftarrow{g} & Z. \end{array} \end{array}$$

5. *Interaction With Coproducts.* We have

$$\begin{array}{ccc} X \amalg_{\text{pt}} Y \cong X \vee Y, & \begin{array}{ccc} X \vee Y & \xleftarrow{\quad} & Y \\ \uparrow \lrcorner & & \uparrow [y_0] \\ X & \xleftarrow{[x_0]} & \text{pt.} \end{array} \end{array}$$

6. *Symmetric Monoidality.* The triple  $(\mathbf{Sets}_*, \amalg_X, (X, x_0))$  is a symmetric monoidal category.

**PROOF 6.3.4.1.5 ► PROOF OF PROPOSITION 6.3.4.1.4**

**Item 1: Functoriality**

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pushout diagram.

**Item 2: Associativity**

This follows from **Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.**

**Item 3: Unitality**

This follows from **Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.**


**Item 4: Commutativity**

This follows from **Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.**

**Item 5: Interaction With Coproducts**

Omitted.

**Item 6: Symmetric Monoidality**

Omitted. 

## 6.3.5 Coequalisers

Let  $f, g: (X, x_0) \rightrightarrows (Y, y_0)$  be morphisms of pointed sets.

**DEFINITION 6.3.5.1.1 ► COEQUALISERS OF POINTED SETS**

The **coequaliser of  $(f, g)$**  is the pointed set  $(\text{CoEq}(f, g), [y_0])$ .

**CONSTRUCTION 6.3.5.1.2 ► CONSTRUCTION OF COEQUALISERS OF POINTED SETS**

The **coequaliser of**  $(f, g)$  is the pair  $((\text{CoEq}(f, g), [y_0]), \text{coeq}(f, g))$  consisting of:

- *The Colimit.* The pointed set  $(\text{CoEq}(f, g), [y_0])$ , where  $\text{CoEq}(f, g)$  is the coequaliser of  $f$  and  $g$  as in **Constructions With Sets, Definition 4.2.5.1.1**.
- *The Cocone.* The map

$$\text{coeq}(f, g): Y \twoheadrightarrow (\text{CoEq}(f, g), [y_0])$$

given by the quotient map, as in **Constructions With Sets, Item 2 of Construction 4.2.5.1.2**.

**PROOF 6.3.5.1.3 ► PROOF OF CONSTRUCTION 6.3.5.1.2**

We claim that  $(\text{CoEq}(f, g), [y_0])$  is the categorical coequaliser of  $f$  and  $g$  in  $\mathbf{Sets}_*$ . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\ &\stackrel{\text{def}}{=} [f(x)] \\ &= [g(x)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x) \end{aligned}$$

for each  $x \in X$ . Next, we prove that  $\text{CoEq}(f, g)$  satisfies the universal property of the coequaliser. Suppose we have a diagram of the form


$$\begin{array}{ccc} (X, x_0) & \xrightarrow[f]{g} & (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} & (\text{CoEq}(f, g), [y_0]) \\ & & & \searrow c & \\ & & & & (C, *) \end{array}$$

in **Sets**. Then, since  $c(f(a)) = c(g(a))$  for each  $a \in A$ , it follows from **Conditions on Relations**, **Items 4** and **5** of **Proposition 10.6.2.1.3** that there exists a unique map  $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$  making the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\ & \searrow c & \downarrow \phi \exists! \\ & & (C, *) \end{array}$$

commute, where we note that  $\phi$  is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that  $c$  is a morphism of pointed sets. 

#### PROPOSITION 6.3.5.1.4 ► PROPERTIES OF COEQUALISERS OF POINTED SETS

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where  $\text{CoEq}(f, g, h)$  is the colimit of the diagram

$$(X, x_0) \xrightarrow[g]{f} (Y, y_0) \xrightarrow[h]{g}$$

in **Sets**<sub>\*</sub>.

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

|

|

#### Item 1: Associativity

This follows from **Constructions With Sets**, Item 1 of Proposition 4.2.5.1.7.

#### Item 2: Unitality

This follows from **Constructions With Sets**, Item 2 of Proposition 4.2.5.1.7.

#### Item 3: Commutativity

This follows from **Constructions With Sets**, Item 3 of Proposition 4.2.5.1.7. 

## 6.4 Constructions With Pointed Sets

### 6.4.1 Free Pointed Sets

Let  $X$  be a set.

#### DEFINITION 6.4.1.1.1 ► FREE POINTED SETS

The **free pointed set on  $X$**  is the pointed set  $X^+$  consisting of:

- *The Underlying Set.* The set  $X^+$  defined by<sup>1</sup>

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \amalg \text{pt} \\ &\stackrel{\text{def}}{=} X \amalg \{\star\}. \end{aligned}$$

- *The Basepoint.* The element  $\star$  of  $X^+$ .

<sup>1</sup>*Further Notation:* We sometimes write  $\star_X$  for the basepoint of  $X^+$  for clarity, specially when there are multiple free pointed sets involved in the current discussion.

#### PROPOSITION 6.4.1.1.2 ► PROPERTIES OF FREE POINTED SETS

Let  $X$  be a set.

1. *Functoriality.* The assignment  $X \mapsto X^+$  defines a functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where:

- *Action on Objects.* For each  $X \in \text{Obj}(\mathbf{Sets})$ , we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where  $X^+$  is the pointed set of [Definition 6.4.1.1.1](#).

- *Action on Morphisms.* For each morphism  $f: X \rightarrow Y$  of  $\mathbf{Sets}$ , the image

$$f^+: X^+ \rightarrow Y^+$$

of  $f$  by  $(-)^+$  is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left( (-)^+ \dashv \overset{\sim}{\omega} \right): \quad \mathbf{Sets} \begin{array}{c} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\overset{\sim}{\omega}} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathbf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathbf{Sets}(X, Y),$$

natural in  $X \in \text{Obj}(\mathbf{Sets})$  and  $(Y, y_0) \in \text{Obj}(\mathbf{Sets}_*)$ .

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of [Item 1](#) has a symmetric strong monoidal structure

$$\left( (-)^+, (-)^+, \amalg, (-)^+_1 \right): (\mathbf{Sets}, \amalg, \emptyset) \rightarrow (\mathbf{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} \amalg: X^+ \vee Y^+ &\xrightarrow{\sim} (X \amalg Y)^+, \\ (-)^+_1 \amalg: \text{pt} &\xrightarrow{\sim} \emptyset^+, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\mathbf{Sets})$ .



4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, (-)_{\mathbb{1}}^+\right): (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^+ &: X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+, \\ (-)_{\mathbb{1}}^+ &: S^0 \xrightarrow{\sim} \text{pt}^+, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\mathbf{Sets})$ .

#### PROOF 6.4.1.1.3 ► PROOF OF PROPOSITION 6.4.1.1.2

##### Item 1: Functoriality

We claim that  $(-)^+$  is indeed a functor:

- *Preservation of Identities.* Let  $X \in \text{Obj}(\mathbf{Sets})$ . We have

$$\text{id}_X^+(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each  $x \in X^+$ , so  $\text{id}_X^+ = \text{id}_{X^+}$ .

- *Preservation of Composition.* Given morphisms of sets

$$\begin{aligned} f &: X \rightarrow Y, \\ g &: Y \rightarrow Z, \end{aligned}$$

we have

$$\begin{aligned} [g^+ \circ f^+](x) &\stackrel{\text{def}}{=} g^+(f^+(x)) \\ &\stackrel{\text{def}}{=} g^+(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^+(x) \end{aligned}$$

for each  $x \in X$  and

$$\begin{aligned} [g^+ \circ f^+](\star_X) &\stackrel{\text{def}}{=} g^+(f^+(\star_X)) \\ &\stackrel{\text{def}}{=} g^+(\star_Y) \\ &\stackrel{\text{def}}{=} \star_Z \\ &\stackrel{\text{def}}{=} [g \circ f]^+(\star_X), \end{aligned}$$

so  $(g \circ f)^+ = g^+ \circ f^+$ .

This finishes the proof.

#### Item 2: Adjointness

We proceed in a few steps:

- *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}_*(X^+, Y) \rightarrow \mathbf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger: X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X$ .

- *Map II.* We define a map

$$\Psi_{X,Y}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Sets}_*(X^+, Y)$$

given by sending a function  $\xi: X \rightarrow Y$  to the morphism of pointed sets

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each  $x \in X^+$ .

- *Invertibility I.* Given a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0),$$

we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^\dagger) \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\ &= \xi \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathbf{Sets}_*(X^+, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\mathbf{Sets}_*(X^+, Y)}.$$

- *Invertibility II.* Given a map of sets  $\xi: X \rightarrow Y$ , we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\xi^\dagger) \\ &= \Phi_{X,Y}\left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket\right) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathbf{Sets}(X, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\mathbf{Sets}(X, Y)}.$$

- *Naturality for  $\Phi$ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc} \mathbf{Sets}_*(X', +, Y) & \xrightarrow{\Phi_{X', Y}} & \mathbf{Sets}(X', Y) \\ f^* \downarrow & & \downarrow f^* \\ \mathbf{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X, Y}} & \mathbf{Sets}(X, Y) \end{array}$$

commutes. Indeed, given a morphism of pointed sets  $\xi: X'^+ \rightarrow Y$ , we have

$$\begin{aligned} [\Phi_{X, Y} \circ f^*](\xi) &= \Phi_{X, Y}(f^*(\xi)) \\ &= \Phi_{X, Y}(\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X', Y}(\xi) \circ f \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= [f^* \circ \Phi_{X', Y}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X, Y} \circ f^* = f^* \circ \Phi_{X', Y}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

- *Naturality for  $\Phi$ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
 \mathbf{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \mathbf{Sets}(X, Y) \\
 g_* \downarrow & & \downarrow g_* \\
 \mathbf{Sets}_*(X^+, Y'), & \xrightarrow{\Phi_{X,Y'}} & \mathbf{Sets}(X, Y')
 \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger: X^+ \rightarrow Y,$$

we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
 &= \Phi_{X,Y'}(g \circ \xi) \\
 &= g \circ \xi \\
 &= g \circ \Phi_{X,Y'}(\xi) \\
 &= g_*(\Phi_{X,Y'}(\xi)) \\
 &= [g_* \circ \Phi_{X,Y'}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

- *Naturality for  $\Psi$ .* Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that  $\Psi$  is also natural in each argument.

This finishes the proof.

### Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on  $(-)^+$  with respect to  $\amalg$  and  $\vee$  as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{+, \amalg}: X^+ \vee Y^+ \xrightarrow{\sim} (X \amalg Y)^+$$

is given by

$$(-)_{X,Y}^{+, \amalg}(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each  $z \in X^+ \vee Y^+$ , with inverse

$$(-)_{X,Y}^{+, \amalg, -1}: (X \amalg Y)^+ \xrightarrow{\sim} X^+ \vee Y^+$$

given by

$$(-)_{X,Y}^{+, \amalg, -1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(1, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg_Y \end{cases}$$

for each  $z \in (X \amalg Y)^+$ .

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+, \amalg, 1}: \text{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending  $\star_X$  to  $\star_\emptyset$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

#### Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on  $(-)^+$  with respect to  $\times$  and  $\wedge$  as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+ : X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each  $x \wedge y \in X^+ \wedge Y^+$ , with inverse

$$(-)_{X,Y}^{+,-1} : (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by


$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each  $z \in (X \times Y)^+$ .

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\mathbb{1}} : S^0 \xrightarrow{\sim} \text{pt}^+$$

is given by sending 0 to  $\star_{\text{pt}}$  and 1 to  $\star$ , where  $\text{pt}^+ = \{\star, \star_{\text{pt}}\}$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted. 

## 6.4.2 Deleting Basepoints

Let  $(X, x_0)$  be a pointed set.

### DEFINITION 6.4.2.1.1 ► SETS WITH DELETED BASEPOINTS

The **set with deleted basepoint associated to  $X$**  is the set  $X^-$  defined by

$$X^- \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

**PROPOSITION 6.4.2.1.2 ► PROPERTIES OF SETS WITH DELETED BASEPOINTS**

Let  $(X, x_0)$  be a pointed set.

1. *Functoriality.* The assignment  $(X, x_0) \mapsto X^-$  defines a functor

$$X^- : \mathbf{Sets}_*^{\text{actv}} \rightarrow \mathbf{Sets},$$

where:

- *Action on Objects.* For each  $X \in \text{Obj}(\mathbf{Sets}_*^{\text{actv}})$ , we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where  $X^-$  is the set of [Definition 6.4.2.1.1](#).

- *Action on Morphisms.* For each morphism  $f : X \rightarrow Y$  of  $\mathbf{Sets}_*^{\text{actv}}$ , the image

$$f^- : X^- \rightarrow Y^-$$

of  $f$  by  $(-)^-$  is the map defined by

$$f^-(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in X^-$ .

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$(((-)^- \dashv (-)^+) : \mathbf{Sets}_*^{\text{actv}} \begin{array}{c} \xrightarrow{(-)^-} \\ \perp_{\text{eq}} \\ \xleftarrow{(-)^+} \end{array} \mathbf{Sets},$$

witnessed by a bijection of sets

$$\mathbf{Sets}(X^-, Y) \cong \mathbf{Sets}_*(X, Y^+),$$

natural in  $X \in \text{Obj}(\mathbf{Sets}_*)$  and  $Y \in \text{Obj}(\mathbf{Sets})$ , and by isomorphisms

$$\begin{aligned} (X^-)^+ &\cong X, \\ (Y^+)^- &\cong Y, \end{aligned}$$

once again natural in  $X \in \text{Obj}(\mathbf{Sets}_*)$  and  $Y \in \text{Obj}(\mathbf{Sets})$ .



3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^-, (-)^{-,\vee}, (-)_{\mathbb{1}}^{-,\vee}\right): (\mathbf{Sets}_*^{\text{actv}}, \vee, \text{pt}) \rightarrow (\mathbf{Sets}, \amalg, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^{-,\vee}: X^- \amalg Y^- &\xrightarrow{\sim} (X \vee Y)^-, \\ (-)_{\mathbb{1}}^{-,\vee}: \emptyset &\xrightarrow{\sim} \text{pt}^-, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\mathbf{Sets})$ .

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^-, (-)^{-,\times}, (-)_{\mathbb{1}}^{-,\times}\right): (\mathbf{Sets}_*^{\text{actv}}, \wedge, S^0) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^-: X^- \times Y^- &\xrightarrow{\sim} (X \wedge Y)^-, \\ (-)_{\mathbb{1}}^-: \text{pt} &\xrightarrow{\sim} (S^0)^-, \end{aligned}$$

natural in  $X, Y \in \text{Obj}(\mathbf{Sets})$ .

#### PROOF 6.4.2.1.3 ► PROOF OF PROPOSITION 6.4.2.1.2

##### Item 1: Functoriality

We claim that  $(-)^-$  is indeed a functor:

- *Preservation of Identities.* Let  $X \in \text{Obj}(\mathbf{Sets})$ . We have

$$\text{id}_X^-(x) \stackrel{\text{def}}{=} x$$

for each  $x \in X^-$ , so  $\text{id}_X^- = \text{id}_X^-$ .

- *Preservation of Composition.* Given morphisms of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

$$g: (Y, y_0) \rightarrow (Z, z_0),$$

we have

$$\begin{aligned} [g^- \circ f^-](x) &\stackrel{\text{def}}{=} g^-(f^-(x)) \\ &\stackrel{\text{def}}{=} g^-(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^-(x) \end{aligned}$$

for each  $x \in X$ , so  $(g \circ f)^- = g^- \circ f^-$ .

This finishes the proof.

### Item 2: Adjoint Equivalence

We proceed in a few steps:

1. *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}(X^-, Y) \rightarrow \mathbf{Sets}_*^{\text{actv}}(X, Y^+)$$

by sending a map  $\xi: X^- \rightarrow Y$  to the active morphism of pointed sets

$$\xi^\dagger: X \rightarrow Y^+$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each  $x \in X$ , where this morphism is indeed active since  $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$  for all  $x \in X^-$ .

2. *Map II.* We define a map

$$\Psi_{X,Y}: \mathbf{Sets}_*^{\text{actv}}(X, Y^+) \rightarrow \mathbf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets  $\xi: X \rightarrow Y^+$  to the map

$$\xi^\dagger: X^- \rightarrow Y$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X^-$ , which is indeed well-defined (in that  $\xi(x) \in Y$  for all  $x \in X^-$ ) since  $\xi$  is active.

3. *Invertibility I.* Given a map of sets  $\xi: X^- \rightarrow Y$ , we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \left( \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \right) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}(X^-, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}(X^-, Y)}.$$

4. *Invertibility II.* Given a morphism of pointed sets

$$\xi: (X, x_0) \rightarrow (Y^+, \star_Y),$$

we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}.$$

5. *Naturality for  $\Phi$ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc} \mathbf{Sets}(X', -, Y) & \xrightarrow{\Phi_{X', Y}} & \mathbf{Sets}_*^{\text{actv}}(X', Y^+) \\ f^* \downarrow & & \downarrow f^* \\ \mathbf{Sets}_*(X^-, Y) & \xrightarrow{\Phi_{X, Y}} & \mathbf{Sets}_*^{\text{actv}}(X, Y^+) \end{array}$$

commutes. Indeed, given a map of sets  $\xi: X' \rightarrow Y$ , we have

$$\begin{aligned} [\Phi_{X, Y} \circ f^*](\xi) &= \Phi_{X, Y}(f^*(\xi)) \\ &= \Phi_{X, Y}(\xi \circ f) \\ &= \llbracket x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases} \rrbracket \\ &= f^* \left( \llbracket x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{-} \\ \star_Y & \text{if } x' = x'_0 \end{cases} \rrbracket \right) \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= [f^* \circ \Phi_{X', Y}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X, Y} \circ f^* = f^* \circ \Phi_{X', Y},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

6. *Naturality for  $\Phi$ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
 \mathbf{Sets}(X^-, Y) & \xrightarrow{\Phi_{X,Y}} & \mathbf{Sets}_*^{\text{actv}}(X, Y^+) \\
 g_* \downarrow & & \downarrow g_* \\
 \mathbf{Sets}(X^-, Y') & \xrightarrow{\Phi_{X,Y'}} & \mathbf{Sets}_*^{\text{actv}}(X, Y'^+)
 \end{array}$$

commutes. Indeed, given a map of sets  $\xi: X^- \rightarrow Y$ , we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
 &= \Phi_{X,Y'}(g \circ \xi) \\
 &= \llbracket x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \rrbracket \\
 &= g_* \left( \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \right) \\
 &= g_*(\Phi_{X,Y}(\xi)) \\
 &= [g_* \circ \Phi_{X,Y}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

7. *Naturality for  $\Psi$* . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that  $\Psi$  is also natural in each argument.
8. *Fully Faithfulness of  $(-)^-$* . We aim to show that the assignment  $f \mapsto f^-$  sets up a bijection

$$(-)_{X,Y}^-: \mathbf{Sets}_*^{\text{actv}}(X, Y) \xrightarrow{\sim} \mathbf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)_{X,Y}^{-,-1}: \mathbf{Sets}(X^-, Y^-) \xrightarrow{\sim} \mathbf{Sets}_*^{\text{actv}}(X, Y)$$

is given by sending a map of sets  $f: X^- \rightarrow Y^-$  to the active morphism of pointed sets  $f^\dagger: X \rightarrow Y$  defined by

$$f^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each  $x \in X$ .

9. *Essential Surjectivity of  $(-)^-$ .* We need to show that, given an object  $X \in \text{Obj}(\mathbf{Sets})$ , there exists some  $X' \in \text{Obj}(\mathbf{Sets}_*^{\text{actv}})$  such that  $(X')^- \cong X$ . Indeed, taking  $X' = X^+$ , we have

$$\begin{aligned} (X^+)^- &\stackrel{\text{def}}{=} (X \cup \{\star_X\})^- \\ &\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality*  $(X^+)^- = X$ , showing  $(-)^-$  to be essentially surjective.

10. *The Functor  $(-)^-$  Is an Equivalence.* Since  $(-)^-$  is fully faithful and essentially surjective, it is an equivalence by [Categories, Item 1](#) of [Proposition 11.6.7.1.2](#).

This finishes the proof.

### Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on  $(-)^-$  with respect to  $\vee$  and  $\amalg$  as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{-,\vee}: X^- \amalg Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0, x)] & \text{if } z = (0, x) \text{ with } x \in X, \\ [(1, y)] & \text{if } z = (1, y) \text{ with } y \in Y \end{cases}$$

for each  $z \in X^- \amalg Y^-$ , with inverse

$$(-)_{X,Y}^{-,\vee,-1}: (X \vee Y)^- \xrightarrow{\sim} X^- \amalg Y^-$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = [(0, x)], \\ (1, y) & \text{if } z = [(1, y)], \end{cases}$$

for each  $z \in (X \vee Y)^-$ .

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\vee,1}: \emptyset \xrightarrow{\sim} \text{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^-$  into a symmetric strong monoidal functor is omitted.

#### Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on  $(-)^+$  with respect to  $\wedge$  and  $\times$  as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^-: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)_{X,Y}^-(x, y) = x \wedge y$$

for each  $(x, y) \in X^- \times Y^-$ , with inverse

$$(-)_{X,Y}^{-,-1}: (X \wedge Y)^- \xrightarrow{\sim} X^- \times Y^-$$

given by


$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x, y)$$

for each  $x \wedge y \in (X \wedge Y)^-$ .

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{-,1} : \text{pt} \xrightarrow{\sim} (S^0)^{-}$$

is given by sending  $\star$  to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted. 

# Appendices

## A Other Chapters

### Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

### Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
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### Relations

8. [Relations](#)
9. [Constructions With Relations](#)

### 10. [Conditions on Relations](#)

### Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

### Monoidal Categories

13. [Constructions With Monoidal Categories](#)

### Bicategories

14. [Types of Morphisms in Bicategories](#)

### Extra Part

15. [Notes](#)



## References

- [MSE 2855868] **Qiaochu Yuan**. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on pp. 7, 19).
- [MSE 884460] **Martin Brandenburg**. *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 8).