Constructions With Monoidal Categories

The Clowder Project Authors

July 21, 2025

This chapter contains some material on constructions with monoidal categories.

Contents

13.1 Moduli Categories of Monoidal Structures	1
13.1.1 The Moduli Category of Monoidal Structures on a Category	1
13.1.2 The Moduli Category of Braided Monoidal Structures on a Category	15
egory	15
Category	15
13.2 Moduli Categories of Closed Monoidal Structures	15
13.3 Moduli Categories of Refinements of Monoidal Structures	15
Structure	15
A Other Chapters	15
13.1 Moduli Categories of Monoidal Structures	
13.1.1 The Moduli Category of Monoidal Structures on a Categ	ory
Let C be a category.	

Definition 13.1.1.1.1. The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{B}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathsf{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, \qquad \qquad \downarrow \qquad \qquad \downarrow \ rac{}{\mathbb{E}} \ \mathsf{pt} \stackrel{\mathsf{Golder}}{\longrightarrow} \mathsf{Cats}.$$

Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{B}_1}(C)$ where:

- · Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
- Morphisms. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^C)$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{I}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C: C \to C$ of C.

· *Identities*. For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at M is defined by

$$\operatorname{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \stackrel{\text{\tiny def}}{=} \Big(\operatorname{id}_C^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes} \Big),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of ??.

· Composition. For each $M,N,P\in \mathsf{Obj}\big(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})\big)$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,P)$$
 of $\mathcal{M}_{\mathbb{B}_{1}}(C)$ at (M,N,P) is defined by

$$\left(\operatorname{id}_{\mathcal{C}}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(\mathcal{C})}\left(\operatorname{id}_{\mathcal{C}}^{\otimes},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)\stackrel{\text{\tiny def}}{=}\left(\operatorname{id}_{\mathcal{C}}^{\otimes,\prime}\circ\operatorname{id}_{\mathcal{C}}^{\otimes},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right).$$

Remark 13.1.1.13. In particular, a morphism in $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

1. Naturality. For each pair $f:A\to X$ and $g:B\to Y$ of morphisms of C, the diagram

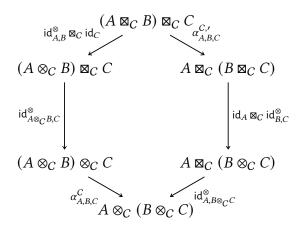
$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow_{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow_{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

commutes.

2. Monoidality. For each $A, B, C \in Obj(C)$, the diagram



commutes.

3. Left Monoidal Unity. For each $A \in Obj(C)$, the diagram

$$\mathbb{1}_{C}\boxtimes_{C}A\overset{\operatorname{id}_{\mathbb{1}'_{C},A}^{\otimes}}{\longrightarrow}\mathbb{1}_{C}\otimes_{C}A$$

$$\operatorname{id}_{\mathbb{1}}^{\otimes}\boxtimes_{C}\operatorname{id}_{A}\overset{\lambda_{A}^{C}}{\longrightarrow}A$$

$$\mathbb{1}'_{C}\boxtimes_{C}A\overset{\lambda_{A}^{C,\prime}}{\longrightarrow}A$$

commutes.

4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,\prime}} A$$

commutes.

Proposition 13.1.1.1.4. Let C be a category.

- 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
 - (a) The diagram

commutes.

(b) The diagram

commutes.

2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}_C', \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

(a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. Mixed Associators. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathsf{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes}: (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \downarrow^{\operatorname{id}_{A \otimes_C B,C}^{\otimes}} & & \downarrow^{\operatorname{id}_A \otimes_C \operatorname{id}_{B,C}^{\otimes}} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_C (B \boxtimes_C C) \\ \operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,r}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{C}B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A}\boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A\boxtimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes\otimes}} A\otimes_C (B\boxtimes_C C) \\ \mathrm{id}_{A,B}^{\otimes}\otimes_C \mathrm{id}_C & & & & & & \\ (A\otimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} A\otimes_C (B\otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

Proof. Item 1, Extra Monoidality Conditions: We claim that Items 1a and 1b are indeed true:

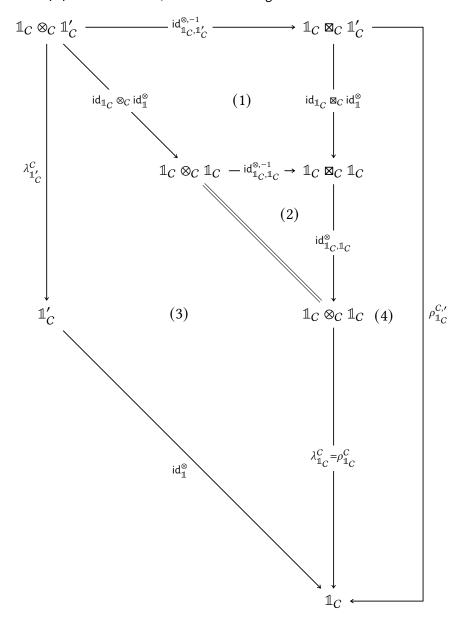
1. Proof of Item 1a: This follows from the naturality of id^{\otimes} with respect to the morphisms id_{AB}^{\otimes} and id_{C} .

2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and id_{RC}^{\otimes} .

This finishes the proof.

Item 2, *Extra Monoidal Unity Constraints*: We claim that *Items* 2a and 2b are indeed true:

1. Proof of Item 1a: Indeed, consider the diagram

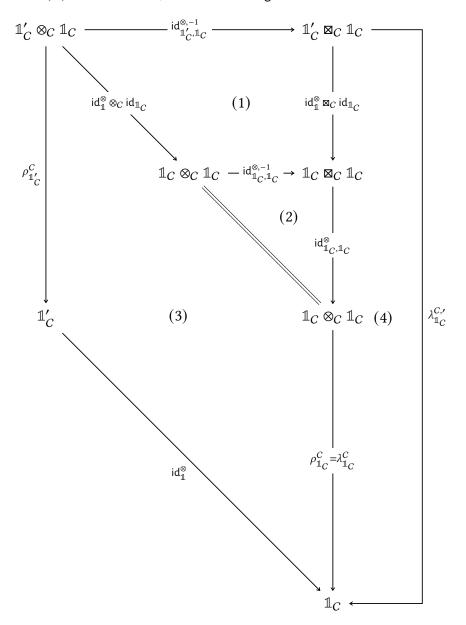


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- · Subdiagram (2) commutes trivially;
- · Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:condition}$;
- $\cdot \ \, \text{Subdiagram} \, (4) \, \text{commutes by the right monoidal unity of} \Big(\mathrm{id}_C, \mathrm{id}_C^\otimes, \mathrm{id}_{C|\mathbb{1}}^\otimes \Big);$

so does the boundary diagram, and we are done.

2. Proof of Item 1b: Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- · Subdiagram (1) commutes by the naturality of $\mathrm{id}_{C}^{\otimes,-1}$;
- · Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from $\ref{eq:condition}$;
- $\cdot \; \mathsf{Subdiagram}\,(4)\,\mathsf{commutes}\,\mathsf{by}\,\mathsf{the}\,\mathsf{left}\,\mathsf{monoidal}\,\mathsf{unity}\,\mathsf{of}\Big(\mathsf{id}_{\mathcal{C}},\mathsf{id}_{\mathcal{C}}^{\otimes},\mathsf{id}_{\mathcal{C}|\mathbb{1}}^{\otimes}\Big);$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- · The boundary diagram commutes trivially;
- · Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- · The boundary diagram commutes trivially;
- · Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \\
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}} \\
\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}^{\otimes,-1}}^{\otimes,-1}} \mathbb{1}_{C}$$

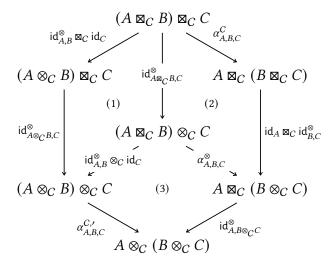
commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for id $^{\otimes}$ of Item 2

of Definition 13.1.1.1.3 as follows:

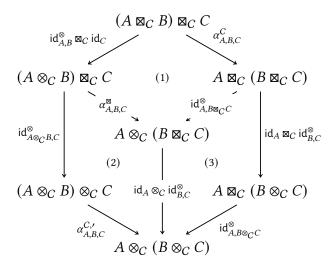


Since:

- · Subdiagram (1) commutes by Item 1a of Item 1.
- · Subdiagram (2) commutes by assumption.
- · Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for id $^{\otimes}$ of Item 2 of Definition 13.1.1.1.3 as follows:

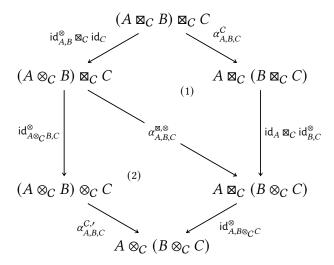


Since:

- · Subdiagram (1) commutes by assumption.
- · Subdiagram (2) commutes by assumption.
- · Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item* 3c: We may partition the monoidality diagram for id $^{\otimes}$ of Item 2 of Definition 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicate- Extra Part gories

15. Notes