Pointed Sets

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July 29, 2025

This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

Definition 6.1.1.1.1. A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$.
- A pointed object in (Sets, pt).

Remark 6.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0) .
- The Basepoint. A morphism

$$[x_0]: \mathsf{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 6.1.1.13. The 0-sphere² is the pointed set $(S^0, 0)^3$ consisting of:

• The Underlying Set. The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

• *The Basepoint*. The element 0 of S^0 .

Example 6.1.1.1.4. The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- *The Underlying Set.* The punctual set $pt \stackrel{\text{def}}{=} \{ \star \}$.
- *The Basepoint*. The element \star of pt.

Example 6.1.1.1.5. The **standard pointed set with** n + 1 **elements** is the pointed set $\langle n \rangle$ consisting of

• *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

• *The Basepoint*. The element * of $\langle n \rangle$.

¹Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -modules.

²Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element.**

³Further Notation: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1. A morphism of pointed sets^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$.
- A morphism of pointed objects in (Sets, pt).

Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f: (X, x_0) \to (Y, y_0)$ is a morphism of sets $f: X \to Y$ such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & & [y_0] \\
X & \xrightarrow{f} Y
\end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1. The **category of pointed sets** is the category Sets_{*} defined equivalently as:

- The homotopy category of the ∞ -category $\mathsf{Mon}_{\mathbb{E}_0}(\mathsf{N}_{\bullet}(\mathsf{Sets}),\mathsf{pt})$ of $\ref{eq:sets}$??.
- The category Sets* of Constructions With Categories, ??.

Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category Sets_{*} where:

- *Objects*. The objects of Sets* are pointed sets.
- *Morphisms*. The morphisms of Sets_{*} are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in Obj(Sets_*)$, the unit map

$$\mathbb{1}_{(X,x_0)}^{\mathsf{Sets}_*} : \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets_{*} at (X, x_0) is defined by⁶

$$id_{(X,x_0)}^{Sets_*} \stackrel{\text{def}}{=} id_X$$
.

denoted $(\mathbb{F}_1, 0)$.

⁴Further Terminology: Also called a **pointed function**.

⁵Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of** \mathbb{F}_1 -**modules**.

⁶Note that id_X is indeed a morphism of pointed sets, as we have id_X(x_0) = x_0 .

• *Composition*. For each (X, x_0) , (Y, y_0) , $(Z, z_0) \in Obj(\mathsf{Sets}_*)$, the composition map

6.1.4 Elementary Properties of Pointed Sets

Proposition 6.1.4.1.1. Let (X, x_0) be a pointed set.

- 1. *Completeness*. The category Sets_{*} of pointed sets and morphisms between them is complete, having in particular:
 - (a) Products, described as in Definition 6.2.3.1.1.
 - (b) Pullbacks, described as in Definition 6.2.4.1.1.
 - (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets_{*} of pointed sets and morphisms between them is cocomplete, having in particular:
 - (a) Coproducts, described as in Definition 6.3.3.1.1.
 - (b) Pushouts, described as in Definition 6.3.4.1.1;
 - (c) Coequalisers, described as in Definition 6.3.5.1.1.
- 3. Failure To Be Cartesian Closed. The category Sets, is not Cartesian closed.8

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

⁸The category Sets_{*} does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

4. Morphisms From the Monoidal Unit. We have a bijection of sets⁹

$$\mathsf{Sets}_*(S^0, X) \cong X$$
,

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

Sets_{*}(
$$S^0, X$$
) $\cong (X, x_0),$

again natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

5. Relation to Partial Functions. We have an equivalence of categories 10

$$\mathsf{Sets}_* \stackrel{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

- i. A pointed set (X, x_0) to X.
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1}$$
: Sets^{part.} $\stackrel{\cong}{\to}$ Sets_{*}

sends:

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

⁹In other words, the forgetful functor

 $^{^{10}}$ Warning: This is not an isomorphism of categories, only an equivalence.

- i. A set *X* is to the pointed set (X, \star) with \star an element that is not in *X*.
- ii. A partial function

$$f: X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, Completeness: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X. The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, *Relation to Partial Functions*: See [MSE 884460].

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1. Let $f: (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

- 1. The morphism f is active if $f^{-1}(y_0) = x_0$.
- 2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2. We write $Sets^{actv}_*$ for the wide subcategory of $Sets_*$ spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

1. The map $\mu: \langle 2 \rangle \to \langle 1 \rangle$ given by

$$1 \longmapsto 1$$

$$2 \qquad \qquad * \longmapsto *$$

is active but not inert.

2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by

$$1 \longmapsto 1$$

$$2 \longmapsto 2$$

$$* \longmapsto *$$

is inert but not active.

3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by

is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i: \langle 3 \rangle \to \langle 2 \rangle,$$

 $a: \langle 2 \rangle \to \langle 1 \rangle$

are the morphisms of pointed sets given by

with *i* being inert and *a* being active.

Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation*. Every morphism of pointed sets $f:(X,x_0) \to (Y,y_0)$ factors uniquely as

$$f = a \circ i$$
,

where:

- (a) The map $i: (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets
- (b) The map $a: (K, k_0) \to (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets*.

Proof. Item 1, *Active-Inert Factorisation*: Let $f: X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\rightarrow} K \stackrel{a}{\rightarrow} Y$$
.

where:

• *K* is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$

= $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$

• $i: X \to K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

• $a: K \to Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
f & & \downarrow g \\
A & \xrightarrow{} & B
\end{array}$$

be a commutative diagram in Sets_{*}. Consider the morphism $\phi: Y \to A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{array}{c|c}
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow & \downarrow g \\
A & \xrightarrow{} & B.
\end{array}$$

Indeed, this diagram commutes, as we have

$$[\phi \circ i](x) \stackrel{\text{def}}{=} \phi(i(x))$$
$$\stackrel{\text{def}}{=} f(i^{-1}(i(x)))$$
$$= f(x)$$

for each $x \in X$ and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
f & & \downarrow & \downarrow g \\
A & \xrightarrow{a} & B
\end{array}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1. The **terminal pointed set** is the terminal object of Sets_{*} as in Limits and Colimits, **??**.

Construction 6.2.1.1.2. Concretely, the **terminal pointed set** is the pair $((pt, \star), \{!_X\}_{(X,x_0) \in Obj(Sets_*)})$ consisting of:

- *The Limit*. The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathsf{pt}, \star)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$!_{X}(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in Obj(Sets)$.

Proof. We claim that (pt, \star) is the terminal object of Sets_{*}. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt, \star)

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathsf{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists !}} (pt, \star)$$

commute, namely $!_X$.

6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 6.2.2.1.1. The **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.2.1.2. Concretely, the **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$ consisting of:

- The Limit. The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- The Cone. The collection

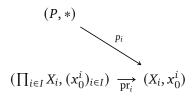
$$\left\{ \operatorname{pr}_i \colon \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \to (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_{i}((x_{i})_{i \in I}) \stackrel{\operatorname{def}}{=} x_{i}$$

for each $(x_i)_{i \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*}. Indeed, suppose we have, for each $i \in I$, a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*)\to (\prod_{i\in I}X_i,(x_0^i)_{i\in I})$$

making the diagram

$$(P, *)$$

$$\phi \downarrow \exists !$$

$$(\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \xrightarrow{\operatorname{pr}_i} (X_i, x_0^i)$$

commute, being uniquely determined by the condition $\operatorname{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I} = (x_0^i)_{i \in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$.

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Proposition 6.2.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, Functoriality: This follows from Limits and Colimits, ?? of ??. □

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.3.1.2. Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone*. The morphisms of pointed sets

$$\operatorname{pr}_1 : (X \times Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 : (X \times Y, (x_0, y_0)) \to (Y, y_0)$

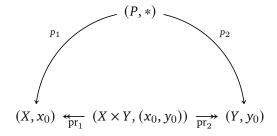
defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form

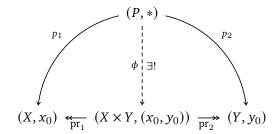


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in Sets. Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$

 $\operatorname{pr}_2 \circ \phi = p_2$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 6.2.3.1.3. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$A \times -:$$
 Sets_{*} \rightarrow Sets_{*},
 $- \times B:$ Sets_{*} \rightarrow Sets_{*},
 $-_1 \times -_2:$ Sets_{*} \times Sets_{*} \rightarrow Sets_{*},

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

- 2. *Lack of Adjointness*. The functors $X \times -$ and $\times Y$ do not admit right adjoints.
- 3. Associativity. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathsf{Sets}_*).$

4. Unitality. We have isomorphisms of pointed sets

$$(pt, \star) \times (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \times (pt, \star) \cong (X, x_0),$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

5. Commutativity. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\mathsf{Sets}_*).$

6. *Symmetric Monoidality*. The triple (Sets_{*}, ×, (pt, ★)) is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2, Lack of Adjointness: See [MSE 2855868].

Item 3, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.

Item 4, Unitality: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.

Item 5, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.

Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3. □

6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.4.1.2. Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 : (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 : (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad (X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$

$$\downarrow^{g} \qquad \qquad \downarrow^{g} \qquad (X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$[f \circ pr_1](x, y) = f(pr_1(x, y))$$

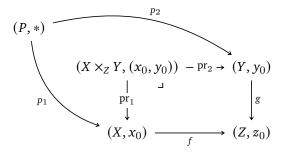
$$= f(x)$$

$$= g(y)$$

$$= g(pr_2(x, y))$$

$$= [g \circ pr_2](x, y),$$

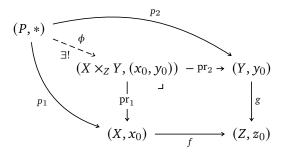
where f(x) = g(y) since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$

 $\operatorname{pr}_2 \circ \phi = p_2$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2$$
,

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 6.2.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

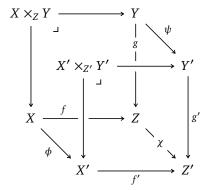
1. Functoriality. The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-1 \times_{-3} -1$$
: Fun(\mathcal{P} , Sets_{*}) \rightarrow Sets_{*},

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \times_{-3} -1$ is given by sending a morphism



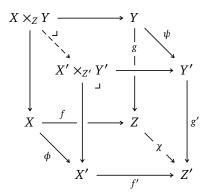
in $Fun(\mathcal{P}, Sets_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

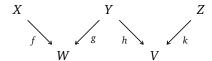
$$\xi(x,y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

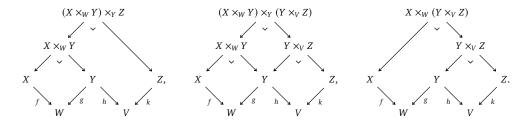
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



3. Unitality. We have isomorphisms of pointed sets



4. Commutativity. We have an isomorphism of pointed sets

$$A \times_X B \longrightarrow B$$

$$\downarrow \qquad \qquad \downarrow g \qquad A \times_X B \cong B \times_X A \qquad \qquad \downarrow f$$

$$A \xrightarrow{f} X, \qquad \qquad B \xrightarrow{g} X.$$

5. Interaction With Products. We have an isomorphism of pointed sets

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \xrightarrow{!_X} pt.$$

6. *Symmetric Monoidality*. The triple (Sets_{*}, \times_X , X) is a symmetric monoidal category.

Proof. Item 1, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.

Item 3, Unitality: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.

Item 4, Commutativity: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.

Item 5, Interaction With Products: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.

Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5.

6.2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1. The **equaliser of** (f, g) is the equaliser of f and g in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(Eq(f,g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$eq(f,g): (Eq(f,g),x_0) \hookrightarrow (X,x_0)$$

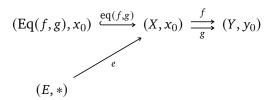
given by the canonical inclusion eq $(f,g) \hookrightarrow \text{Eq}(f,g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f,g),x_0)$ is the categorical equaliser of f and g in Sets_{*}. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ eq(f,g) = g \circ eq(f,g),$$

which indeed holds by the definition of the set Eq(f,g). Next, we prove that Eq(f,g) satisfies the universal property of the equaliser. Suppose we have a

diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\text{Eq}(f, g), x_0)$$

making the diagram

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow \downarrow \downarrow \downarrow e$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in Eq(f,g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

Proposition 6.2.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}$$

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f \atop -g \Rightarrow} (Y, y_0)$$

in Sets*, being explicitly given by

$$Eq(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, f) \cong X$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,g) \cong \operatorname{Eq}(g,f)$$
.

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3.

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1. The **initial pointed set** is the initial object of Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.1.1.2. Concretely, the **initial pointed set** is the pair $((pt, \star), \{\iota_X\}_{(X,x_0) \in Obj(Sets_*)})$ consisting of:

• *The Limit.* The pointed set (pt, \star) .

• *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathsf{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_{*}. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
 (X, x_0)

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(pt, \star) \xrightarrow{-\frac{\phi}{\exists 1}} (X, x_0)$$

commute, namely ι_X .

6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 6.3.2.1.1. The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \{\inf_i\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
 - The Underlying Set. The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\coprod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i,x_0^i)\sim (j,x_0^j)$$

for each $i, j \in I$.

¹¹Further Terminology: Also called the **wedge sum of the family** $\{(X_i, x_0^i)\}_{i \in I}$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \stackrel{\text{def}}{=} [(i, x_0^i)]$$
$$= [(j, x_0^j)]$$

for any $i, j \in I$.

• The Cocone. The collection

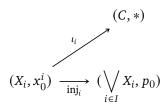
$$\left\{ \operatorname{inj}_i \colon (X_i, x_0^i) \to (\bigvee_{i \in I} X_i, p_0) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\operatorname{inj}_{i}(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*}. Indeed, suppose we have, for each $i \in I$, a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi\colon (\bigvee_{i\in I} X_i, p_0) \to (C, *)$$

making the diagram

$$(X_i, x_0^i) \xrightarrow[\text{inj}_i]{(C, *)} \begin{pmatrix} (C, *) \\ \phi \mid \exists ! \\ \vdots \\ (X_i, p_0) \end{pmatrix}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$
= *,

as ι_i is a morphism of pointed sets.

Proposition 6.3.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??. □

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** $(Y, y_0)^{12}$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
 - The Underlying Set. The set $X \vee Y$ defined by

$$(X \lor Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \qquad X \lor Y \longleftarrow Y$$

$$\cong (X \coprod_{pt} Y, p_0) \qquad \uparrow \qquad \uparrow \qquad \downarrow [y_0]$$

$$\cong (X \coprod Y/\sim, p_0), \qquad X \longleftarrow_{[x_0]} pt,$$

where \sim is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- The Basepoint. The element p_0 of $X \vee Y$ defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$

= $[(1, y_0)].$

¹²Further Terminology: Also called the **wedge sum of** (X, x_0) **and** (Y, y_0) .

• The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \vee Y, p_0),$$

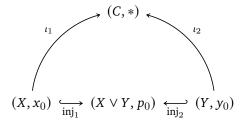
 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \vee Y, p_0),$

given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)],
inj_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

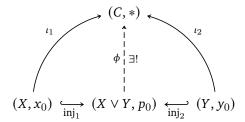
Proof. We claim that $(X \lor Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$

$$\phi \circ \operatorname{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

= $\iota_Y([(1, y_0)])$
= *.

as ι_X and ι_Y are morphisms of pointed sets.

Proposition 6.3.3.1.3. Let (X, x_0) and (Y, y_0) be pointed sets.

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$

2. Associativity. We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in
$$(X, x_0), (Y, y_0), (Z, z_0) \in Sets_*$$
.

3. Unitality. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \lor (pt, *) \cong (X, x_0),$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality*. The triple (Sets_{*}, ∨, pt) is a symmetric monoidal category.

6. The Fold Map. We have a natural transformation

$$\nabla\colon \vee\circ\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}\Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{array}{c} \mathsf{Sets}_*\times\mathsf{Sets}_*\\ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}& & \downarrow \\ \mathsf{Sets}_*& & \downarrow \\ \mathsf{id}_{\mathsf{Sets}_*}& & \mathsf{Sets}_*, \end{array}$$

called the **fold map**, whose component

$$\nabla_X : X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. Item 1, Functoriality: This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

Item 5, *Symmetric Monoidality*: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \to (Y, y_0)$, we have

$$\nabla_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad X \xrightarrow{\nabla_{X}} X$$

$$V_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad f \vee f \downarrow \qquad \downarrow f$$

$$Y \vee Y \xrightarrow{\nabla_{Y}} Y.$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation.

6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.4.1.2. Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
 - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
 - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone*. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$

given by

$$inj1(x) \stackrel{\text{def}}{=} [(0, x)]
inj2(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check that the relevant pushout diagram commutes, i.e. that we have

Indeed, given $z \in Z$, we have

$$[inj_1 \circ f](z) = inj_1(f(z))$$

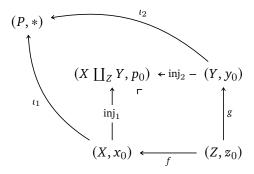
$$= [(0, f(z))]$$

$$= [(1, g(z))]$$

$$= inj_2(g(z))$$

$$= [inj_2 \circ g](z),$$

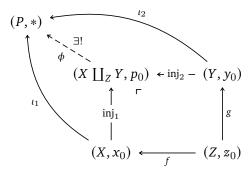
where [(0, f(z))] = [(1, g(z))] by the definition of the relation \sim on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi: (X \mid I_Z Y, p_0) \rightarrow (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1,$$

 $\phi \circ \operatorname{inj}_2 = \iota_2$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$

= $\iota_1(x_0)$
= *,

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$

= $\iota_2(y_0)$
= *,

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

Proposition 6.3.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

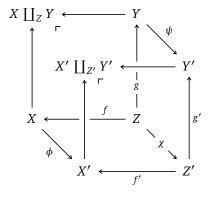
1. Functoriality. The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f, Z, g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 : \mathsf{Fun}(\mathcal{P},\mathsf{Sets}) \to \mathsf{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \coprod_{-3} -1$ is given by sending a morphism



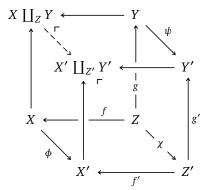
in $Fun(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

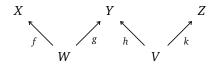
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

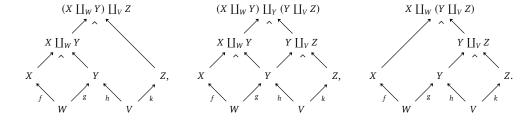
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

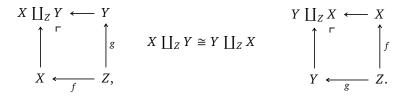
where these pullbacks are built as in the diagrams



3. *Unitality*. We have isomorphisms of sets



4. Commutativity. We have an isomorphism of sets



5. Interaction With Coproducts. We have

$$X \coprod_{\mathsf{pt}} Y \cong X \vee Y, \qquad \bigwedge^{\mathsf{r}} \bigvee^{\mathsf{r}} \bigvee^{\mathsf{r}} [y_0]$$

$$X \longleftrightarrow_{[x_0]} \mathsf{pt}.$$

6. *Symmetric Monoidality*. The triple (Sets_{*}, \coprod_X , (X, x_0)) is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.

Item 3, Unitality: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.

Item 4, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.

Item 5, *Interaction With Coproducts*: Omitted.

Item 6, Symmetric Monoidality: Omitted.

6.3.5 Coequalisers

Let $f, g: (X, x_0) \Rightarrow (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f,g) is the pointed set $(CoEq(f,g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f, g) is the pair $((CoEq(f, g), [y_0]), coeq(f, g))$ consisting of:

- *The Colimit.* The pointed set $(CoEq(f,g), [y_0])$, where CoEq(f,g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
- The Cocone. The map

$$coeq(f,g): Y \rightarrow (CoEq(f,g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(CoEq(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_{*}. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f, g) \circ f = coeq(f, g) \circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g) \circ g](x)$$

for each $x \in X$. Next, we prove that CoEq(f, g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f, g)} (\operatorname{CoEq}(f, g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique

map $\phi \colon \operatorname{CoEq}(f,g) \xrightarrow{\exists !} C$ making the diagram

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \operatorname{coeq}(f, g)]([y_0])$$
$$= c([y_0])$$
$$= *$$

where we have used that c is a morphism of pointed sets.

Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathsf{CoEq}(\mathsf{coeq}(f,g)\circ f,\mathsf{coeq}(f,g)\circ h)}_{=\mathsf{CoEq}(\mathsf{coeq}(f,g)\circ g,\mathsf{coeq}(f,g)\circ h)} \cong \mathsf{CoEq}(f,g,h) \cong \underbrace{\mathsf{CoEq}(\mathsf{coeq}(g,h)\circ f,\mathsf{coeq}(g,h)\circ g)}_{=\mathsf{CoEq}(\mathsf{coeq}(g,h)\circ f,\mathsf{coeq}(g,h)\circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f \atop -g \Rightarrow} (Y, y_0)$$

in Sets_{*}.

2. *Unitality*. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f, g) \cong CoEq(g, f)$$
.

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5.

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let *X* be a set.

Definition 6.4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

• The Underlying Set. The set X^+ defined by 13

$$X^{+} \stackrel{\text{def}}{=} X \coprod \text{pt}$$

$$\stackrel{\text{def}}{=} X \coprod \{ \star \}.$$

• *The Basepoint*. The element \star of X^+ .

Proposition 6.4.1.1.2. Let *X* be a set.

1. Functoriality. The assignment $X \mapsto X^+$ defines a functor

$$(-)^+$$
: Sets \rightarrow Sets_{*},

where:

• *Action on Objects.* For each $X \in Obj(Sets)$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

• *Action on Morphisms*. For each morphism $f: X \to Y$ of Sets, the image

$$f^+: X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. Adjointness. We have an adjunction

$$((-)^+ \dashv \stackrel{\leftarrow}{\kappa}): \operatorname{Sets} \stackrel{(-)^+}{\underset{\stackrel{\leftarrow}{\kappa}}{\smile}} \operatorname{Sets}_*,$$

¹³Further Notation: We sometimes write \star_X for the basepoint of X^+ for clarity, specially when

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums*. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+,(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}})\colon (\mathsf{Sets},\coprod,\emptyset) \to (\mathsf{Sets}_*,\vee,\mathsf{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod}: X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod}: \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in $X, Y \in Obj(Sets)$.

4. *Symmetric Strong Monoidality With Respect to Smash Products*. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)^+_1): (Sets, \times, pt) \to (Sets_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_{1}^+ \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+,$$

natural in $X, Y \in Obj(Sets)$.

Proof. Item 1, *Functoriality*: We claim that $(-)^+$ is indeed a functor:

• Preservation of Identities. Let $X \in Obj(Sets)$. We have

$$\operatorname{id}_{X}^{+}(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_{X} & \text{if } x = \star_{X}, \end{cases}$$

for each $x \in X^+$, so $id_X^+ = id_{X^+}$.

• Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$

$$g: Y \to Z$$

we have

$$[g^+ \circ f^+](x) \stackrel{\text{def}}{=} g^+(f^+(x))$$
$$\stackrel{\text{def}}{=} g^+(f(x))$$
$$\stackrel{\text{def}}{=} g(f(x))$$
$$\stackrel{\text{def}}{=} [g \circ f]^+(x)$$

for each $x \in X$ and

$$[g^+ \circ f^+](\star_X) \stackrel{\text{def}}{=} g^+(f^+(\star_X))$$

$$\stackrel{\text{def}}{=} g^+(\star_Y)$$

$$\stackrel{\text{def}}{=} \star_Z$$

$$\stackrel{\text{def}}{=} [g \circ f]^+(\star_X),$$

so
$$(g \circ f)^{+} = g^{+} \circ f^{+}$$
.

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

• Map I. We define a map

$$\Phi_{X,Y} : \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger}: X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

• Map II. We define a map

$$\Psi_{X,Y} : \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function $\xi: X \to Y$ to the morphism of pointed sets

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

• Invertibility I. Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^{\dagger}) \\ &\stackrel{\text{def}}{=} \left[\! \left[\! x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \right] \\ &= \left[\! \left[\! x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \right] \\ &= \xi \\ \stackrel{\text{def}}{=} \left[\text{id}_{\mathsf{Sets}_*(X^+,Y)} \right](\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}$$
.

• *Invertibility II.* Given a map of sets $\xi: X \to Y$, we have

$$\begin{split} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\xi^{\dagger}) \\ &= \Phi_{X,Y}([x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}]) \\ &= [x \mapsto \xi(x)] \\ &= \xi \\ &\stackrel{\text{def}}{=} [\text{idsets}(x,y)](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)} \;.$$

• *Naturality for* Φ , *Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \to (X', x_0'),$$

the diagram

commutes. Indeed, given a morphism of pointed sets $\xi: X'^{+} \to Y$, we have

$$[\Phi_{X,Y} \circ f^*](\xi) = \Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \xi \circ f$$

$$= \Phi_{X',Y}(\xi) \circ f$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= [f^* \circ \Phi_{X',Y}(\xi)](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

• Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \to (Y', y_0'),$$

the diagram

$$\begin{split} \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X,Y) \\ & & \downarrow^{g_*} & & \downarrow^{g_*} \\ \mathsf{Sets}_*(X^+,Y'), & \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}(X,Y') \end{split}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger} \colon X^+ \to Y,$$

we have

$$\begin{split} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'}\circ g_*=g_*\circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

Naturality for Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to \coprod and \lor as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{X,Y}^{+,\coprod}: X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+,\coprod,-1}\colon (X\coprod Y)^+\stackrel{\sim}{\longrightarrow} X^+\vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each $z \in (X \mid Y)^+$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} \colon \operatorname{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_{\emptyset} .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)^{+,-1}_{XY} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $pt^+ = {\star, \star_{pt}}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

there are multiple free pointed sets involved in the current discussion.

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1. The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^{-} \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2. Let (X, x_0) be a pointed set.

1. Functoriality. The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^-: \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets},$$

where:

• Action on Objects. For each $X \in Obj(Sets^{actv}_*)$, we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where X^- is the set of Definition 6.4.2.1.1.

• *Action on Morphisms*. For each morphism $f: X \to Y$ of $\mathsf{Sets}^\mathsf{actv}_*$, the image

$$f^-: X^- \to Y^-$$

of f by $(-)^-$ is the map defined by

$$f^-(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

2. Adjoint Equivalence. We have an adjoint equivalence of categories

$$((-)^- + (-)^+)$$
: Sets** $(-)^-$ Sets,

witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*(X, Y^+),$$

natural in $X \in \text{Obj}(\mathsf{Sets}_*)$ and $Y \in \text{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^-)^+ \cong X,$$

$$(Y^+)^- \cong Y$$

once again natural in $X \in Obj(Sets_*)$ and $Y \in Obj(Sets)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums*. The functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-,(-)^{-,\vee},(-)^{-,\vee}_{\mathbb{1}})\colon (\mathsf{Sets}^{\mathsf{actv}}_*,\vee,\mathsf{pt}),\to (\mathsf{Sets}, {\textstyle\coprod}, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^{-} \coprod Y^{-} \xrightarrow{\sim} (X \vee Y)^{-},$$
$$(-)_{1}^{-,\vee} \colon \varnothing \xrightarrow{\sim} \mathsf{pt}^{-},$$

natural in $X, Y \in Obj(Sets)$.

4. *Symmetric Strong Monoidality With Respect to Smash Products*. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)^{-,\times}_1) \colon (\mathsf{Sets}^{\mathsf{actv}}_*, \wedge, S^0), \to (\mathsf{Sets}, \times, \mathsf{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-},$$
$$(-)^{-}_{1} \colon \operatorname{pt} \xrightarrow{\sim} (S^{0})^{-},$$

natural in $X, Y \in Obj(Sets)$.

Proof. Item 1, *Functoriality*: We claim that $(-)^-$ is indeed a functor:

• Preservation of Identities. Let $X \in Obj(Sets)$. We have

$$id_X^-(x) \stackrel{\text{def}}{=} x$$

for each $x \in X^-$, so $id_X^- = id_{X^-}$.

• Preservation of Composition. Given morphisms of pointed sets

$$f: (X, x_0) \to (Y, y_0),$$

 $g: (Y, y_0) \to (Z, z_0),$

we have

$$[g^- \circ f^-](x) \stackrel{\text{def}}{=} g^-(f^-(x))$$
$$\stackrel{\text{def}}{=} g^-(f(x))$$
$$\stackrel{\text{def}}{=} g(f(x))$$
$$\stackrel{\text{def}}{=} [g \circ f]^-(x)$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2, Adjoint Equivalence: We proceed in a few steps:

1. Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-,Y) \to \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

by sending a map $\xi \colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^{\dagger}: X \to Y^{+}$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$

given by sending an active morphism of pointed sets $\xi \colon X \to Y^+$ to the map

$$\xi^{\dagger}: X^{-} \to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi: X^- \to Y$, we have

$$\begin{split} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}([x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}]) \\ &= [x \mapsto \xi(x)] \\ &= \xi \\ &= [\text{id}_{\mathsf{Sets}(X^-,Y)}](\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-Y)}$$
.

4. Invertibility II. Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$\begin{split} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= [id_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)}](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} .$$

5. Naturality for Φ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \to (X', x_0'),$$

the diagram

$$\begin{array}{ccccc} \mathsf{Sets}(X^{',-},Y) & \xrightarrow{\Phi_{X',Y}} & \mathsf{Sets}^{\mathsf{actv}}_*(X',Y^+) \\ & & & \downarrow f^* & & \downarrow f^* \\ & & & \mathsf{Sets}_*(X^-,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \end{array}$$

commutes. Indeed, given a map of sets $\xi: X' \to Y$, we have

$$\begin{split} [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= [\![x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases}]\!] \\ &= f^*([\![x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x'_0 \end{cases}]\!]) \\ &= f^*(\Phi_{X',Y}(\xi)) \\ &= [f^* \circ \Phi_{X',Y}](\xi). \end{split}$$

Therefore we have

$$\Phi_{XY} \circ f^* = f^* \circ \Phi_{X'Y}$$

and the naturality diagram for Φ above indeed commutes.

6. Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \to (Y', y_0'),$$

the diagram

$$\begin{array}{ccc} \mathsf{Sets}(X^-,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \\ & \downarrow^{g_*} & & \downarrow^{g_*} \\ \mathsf{Sets}(X^-,Y') & \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}^{\mathsf{actv}}_*(X,Y'^{,+}) \end{array}$$

commutes. Indeed, given a map of sets $\xi: X^- \to Y$, we have

$$\begin{split} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= [\![x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}]\!] \\ &= g_*([\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases}]\!]) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

- 7. Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.
- 8. Fully Faithfulness of $(-)^-$. We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{X,Y}^- \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y) \overset{\sim}{\dashrightarrow} \mathsf{Sets}(X^-,Y^-).$$

Indeed, the inverse map

$$(-)_{XY}^{-,-1} : \mathsf{Sets}(X^-, Y^-) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X, Y)$$

is given by sending a map of sets $f: X^- \to Y^-$ to the active morphism of pointed sets $f^{\dagger}: X \to Y$ defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_{0} & \text{if } x = x_{0} \end{cases}$$

for each $x \in X$.

9. *Essential Surjectivity of* $(-)^-$. We need to show that, given an object $X \in \text{Obj}(\mathsf{Sets})$, there exists some $X' \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}}_*)$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$(X^+)^- \stackrel{\text{def}}{=} (X \cup \{\star_X\})^-$$
$$\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\}$$
$$= X,$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. *The Functor* (−)[−] *Is an Equivalence*. Since (−)[−] is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on (-)⁻ with respect to \lor and \coprod as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{-,\vee}_{X,Y}\colon X^-\coprod Y^-\stackrel{\sim}{\dashrightarrow} (X\vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \coprod Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1} \colon (X \vee Y)^{-} \xrightarrow{\sim} X^{-} \coprod Y^{-}$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{XY}^{+,\vee,1}: \not O \xrightarrow{\sim} pt^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{X,Y}^- \colon X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{X,Y}(x,y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{-,\mathbb{1}} : \operatorname{pt} \xrightarrow{\sim} (S^0)^-$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Appendices

A Other Chapters

Preliminaries	

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

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com/q/2855868 (cit. on pp. 6, 14).

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