Constructions With Monoidal Categories

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01UF   This chapter contains some material on constructions with monoidal categories.

Contents

01UG ## 13.1 Moduli Categories of Monoidal Structures

01UH ### 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let C be a category.

01UJ
> **DEFINITION 13.1.1.1.1 ▶ THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY**
>
> The **moduli category of monoidal structures on** C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by
>
> $$\mathcal{M}_{\mathbb{E}_1}(C) \overset{\text{def}}{=} \mathrm{pt} \times_{\mathsf{Cats}} \mathsf{MonCats},$$
>
> $$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \mathsf{MonCats} \\ \downarrow & \ulcorner & \downarrow {\scriptstyle 忘} \\ \mathrm{pt} & \underset{[C]}{\longrightarrow} & \mathsf{Cats.} \end{array}$$

01UK
> **REMARK 13.1.1.1.2 ▶ UNWINDING DEFINITION 13.1.1.1.1, I**
>
> In detail, **the moduli category of monoidal structures on** C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:
>
> - *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
>
> - *Morphisms.* A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure
>
> $$\mathrm{id}_C^\otimes \colon A \boxtimes_C B \overset{\sim}{\dashrightarrow} A \otimes_C B,$$
> $$\mathrm{id}_{\mathbb{1}|C}^\otimes \colon \mathbb{1}'_C \overset{\sim}{\dashrightarrow} \mathbb{1}_C$$
>
> on the identity functor $\mathrm{id}_C \colon C \to C$ of C.
>
> - *Identities.* For each $M \overset{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map
>
> $$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\mathrm{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \overset{\text{def}}{=} (\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}),$$

where $\left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes} \right)$ is the identity monoidal functor of C of **??**.

- *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N, P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, P)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at (M, N, P) is defined by

$$\left(\mathrm{id}_C^{\otimes,\prime}, \mathrm{id}_{\mathbb{1}|C}^{\otimes,\prime} \right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes} \right) \overset{\text{def}}{=} \left(\mathrm{id}_C^{\otimes,\prime} \circ \mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes,\prime} \circ \mathrm{id}_{\mathbb{1}|C}^{\otimes} \right).$$

REMARK 13.1.1.1.3 ▶ UNWINDING DEFINITION 13.1.1.1.1, II

In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C \right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime} \right)$ satisfies the following conditions:

1. *Naturality.* For each pair $f \colon A \to X$ and $g \colon B \to Y$ of morphisms of C, the diagram

$$
\begin{array}{ccc}
A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\
{\scriptstyle \mathrm{id}_{A,B}^{\otimes}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{X,Y}^{\otimes}} \\
A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y
\end{array}
$$

commutes.

01UN   2. *Monoidality.* For each $A, B, C \in \mathrm{Obj}(C)$, the diagram

$$
\begin{array}{ccc}
& (A \boxtimes_C B) \boxtimes_C C & \\
{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C} \swarrow & & \searrow {\scriptstyle \alpha^{C,'}_{A,B,C}} \\
(A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A \otimes_C B, C}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}} \\
(A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
{\scriptstyle \alpha^{C}_{A,B,C}} \searrow & & \swarrow {\scriptstyle \mathrm{id}^{\otimes}_{A, B \otimes_C C}} \\
& A \otimes_C (B \otimes_C C) &
\end{array}
$$

commutes.

01UP   3. *Left Monoidal Unity.* For each $A \in \mathrm{Obj}(C)$, the diagram

$$
\begin{array}{ccc}
& \mathbb{1}_C \boxtimes_C A \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}'_C, A}} \mathbb{1}_C \otimes_C A & \\
{\scriptstyle \mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_C \mathrm{id}_A} \nearrow & & \searrow {\scriptstyle \lambda^{C}_{A}} \\
\mathbb{1}'_C \boxtimes_C A & \xrightarrow[\lambda^{C,'}_{A}]{} & A
\end{array}
$$

commutes.

01UQ   4. *Right Monoidal Unity.* For each $A \in \mathrm{Obj}(C)$, the diagram

$$
\begin{array}{ccc}
& A \boxtimes_C \mathbb{1}_C \xrightarrow{\mathrm{id}^{\otimes}_{A, \mathbb{1}'_C}} A \otimes_C \mathbb{1}_C & \\
{\scriptstyle \mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}} \nearrow & & \searrow {\scriptstyle \rho^{C}_{A}} \\
A \boxtimes_C \mathbb{1}'_C & \xrightarrow[\rho^{C,'}_{A}]{} & A
\end{array}
$$

commutes.

PROPOSITION 13.1.1.1.4 ▶ PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

01UR

Let C be a category.

01US

1. *Extra Monoidality Conditions.* Let $\left(\mathrm{id}_C^\otimes, \mathrm{id}_{\mathbb{1}|C}^\otimes\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

01UT

   (a) The diagram

$$
\begin{array}{ccc}
(A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\mathrm{id}_{A,B}^\otimes \boxtimes_C \mathrm{id}_C} & (A \otimes_C B) \boxtimes_C C \\
{\scriptstyle \mathrm{id}_{A\boxtimes_C B,C}^\otimes} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}_{A\otimes_C B,C}^\otimes} \\
(A \boxtimes_C B) \otimes_C C & \xrightarrow[\mathrm{id}_{A,B}^\otimes \otimes_C \mathrm{id}_C]{} & (A \otimes_C B) \otimes_C C
\end{array}
$$

   commutes.

01UU

   (b) The diagram

$$
\begin{array}{ccc}
A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\mathrm{id}_A \boxtimes_C \mathrm{id}_{B,C}^\otimes} & A \boxtimes_C (B \otimes_C C) \\
{\scriptstyle \mathrm{id}_{A,B\boxtimes_C C}^\otimes} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{id}_{A,B\otimes_C C}^\otimes} \\
A \otimes_C (B \boxtimes_C C) & \xrightarrow[\mathrm{id}_A \otimes_C \mathrm{id}_{B,C}^\otimes]{} & A \otimes_C (B \otimes_C C)
\end{array}
$$

   commutes.

01WB

2. *Extra Monoidal Unity Constraints.* Let $\left(\mathrm{id}_C^\otimes, \mathrm{id}_{\mathbb{1}|C}^\otimes\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

01WC   (a) The diagram

$$
\begin{CD}
\mathbb{1}_C \otimes_C \mathbb{1}'_C @>{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}}>> \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \\
@V{\lambda^C_{\mathbb{1}'_C}}VV @VV{\rho^{C,\prime}_{\mathbb{1}_C}}V \\
\mathbb{1}'_C @>>{\mathrm{id}^{\otimes}_{\mathbb{1}}}> \mathbb{1}_C
\end{CD}
$$

commutes.

01WD   (b) The diagram

$$
\begin{CD}
\mathbb{1}'_C \otimes_C \mathbb{1}_C @>{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}}>> \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\
@V{\rho^C_{\mathbb{1}'_C}}VV @VV{\lambda^{C,\prime}_{\mathbb{1}_C}}V \\
\mathbb{1}'_C @>>{\mathrm{id}^{\otimes}_{\mathbb{1}}}> \mathbb{1}_C
\end{CD}
$$

commutes.

01WE   (c) The diagram

$$
\begin{CD}
\mathbb{1}'_C \boxtimes_C \mathbb{1}_C @>{\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}}>> \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
@V{\lambda^{C,\prime}_{\mathbb{1}_C}}VV @VV{\rho^C_{\mathbb{1}'_C}}V \\
\mathbb{1}_C @>>{\mathrm{id}^{\otimes,-1}_{\mathbb{1}}}> \mathbb{1}'_C
\end{CD}
$$

commutes.

01WF   (d) The diagram

$$
\begin{CD}
\mathbb{1}_C \boxtimes_C \mathbb{1}'_C @>{\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}}>> \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
@V{\rho^{C,\prime}_{\mathbb{1}_C}}VV @VV{\lambda^C_{\mathbb{1}'_C}}V \\
\mathbb{1}_C @>>{\mathrm{id}^{\otimes,-1}_{\mathbb{1}}}> \mathbb{1}'_C
\end{CD}
$$

commutes.

01UV

3. *Mixed Associators.* Let $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ and $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$ be monoidal structures on C and let

$$\mathrm{id}^{\otimes}_{-_1, -_2} \colon -_1 \boxtimes_C -_2 \to -_1 \otimes_C -_2$$

be a natural transformation.

01UW

    (a) If there exists a natural transformation

$$\alpha^{\otimes}_{A,B,C} \colon (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$
\begin{array}{ccc}
(A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha^{\otimes}_{A,B,C}} & A \otimes_C (B \boxtimes_C C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A\otimes_C B, C}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}} \\
(A \otimes_C B) \otimes_C C & \xrightarrow[\alpha^C_{A,B,C}]{} & A \otimes_C (B \otimes_C C)
\end{array}
$$

and

$$
\begin{array}{ccc}
(A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha^{C,\prime}_{A,B,C}} & A \boxtimes_C (B \boxtimes_C C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{A, B\boxtimes_C C}} \\
(A \otimes_C B) \boxtimes_C C & \xrightarrow[\alpha^{\otimes}_{A,B,C}]{} & A \otimes_C (B \boxtimes_C C)
\end{array}
$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.3.

01UX

    (b) If there exists a natural transformation

$$\alpha^{\boxtimes}_{A,B,C} \colon (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$
\begin{array}{ccc}
(A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha^{\boxtimes}_{A,B,C}} & A \boxtimes_C (B \otimes_C C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C} \downarrow & & \downarrow {\scriptstyle \mathrm{id}^{\otimes}_{A, B\otimes_C C}} \\
(A \otimes_C B) \otimes_C C & \xrightarrow[\alpha^C_{A,B,C}]{} & A \otimes_C (B \otimes_C C)
\end{array}
$$

and

$$(A \boxtimes_C B) \boxtimes_C C \xrightarrow{\alpha^{C,\prime}_{A,B,C}} A \boxtimes_C (B \boxtimes_C C)$$

$$\mathrm{id}^{\otimes}_{A \boxtimes_C B,C} \Big\downarrow \qquad\qquad\qquad \Big\downarrow \mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}$$

$$(A \boxtimes_C B) \otimes_C C \xrightarrow[\alpha^{\boxtimes}_{A,B,C}]{} A \boxtimes_C (B \otimes_C C)$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UY

(c) If there exists a natural transformation

$$\alpha^{\boxtimes,\otimes}_{A,B,C} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$(A \boxtimes_C B) \otimes_C C \xrightarrow{\alpha^{\boxtimes,\otimes}_{A,B,C}} A \otimes_C (B \boxtimes_C C)$$

$$\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C \Big\downarrow \qquad\qquad\qquad \Big\downarrow \mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}$$

$$(A \otimes_C B) \otimes_C C \xrightarrow[\alpha^{C}_{A,B,C}]{} A \otimes_C (B \otimes_C C)$$

and

$$(A \boxtimes_C B) \boxtimes_C C \xrightarrow{\alpha^{C,\prime}_{A,B,C}} A \boxtimes_C (B \boxtimes_C C)$$

$$\mathrm{id}^{\otimes}_{A \boxtimes_C B,C} \Big\downarrow \qquad\qquad\qquad \Big\downarrow \mathrm{id}^{\otimes}_{A,B \boxtimes_C C}$$

$$(A \boxtimes_C B) \otimes_C C \xrightarrow[\alpha^{\boxtimes,\otimes}_{A,B,C}]{} A \otimes_C (B \boxtimes_C C)$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

PROOF 13.1.1.1.5 ▶ PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

We claim that Items 1a and 1b are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}^{\otimes}_{A,B}$ and id_{C}.

2. *Proof of Item 1b:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms id_{A} and $\mathrm{id}^{\otimes}_{B,C}$.

This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:
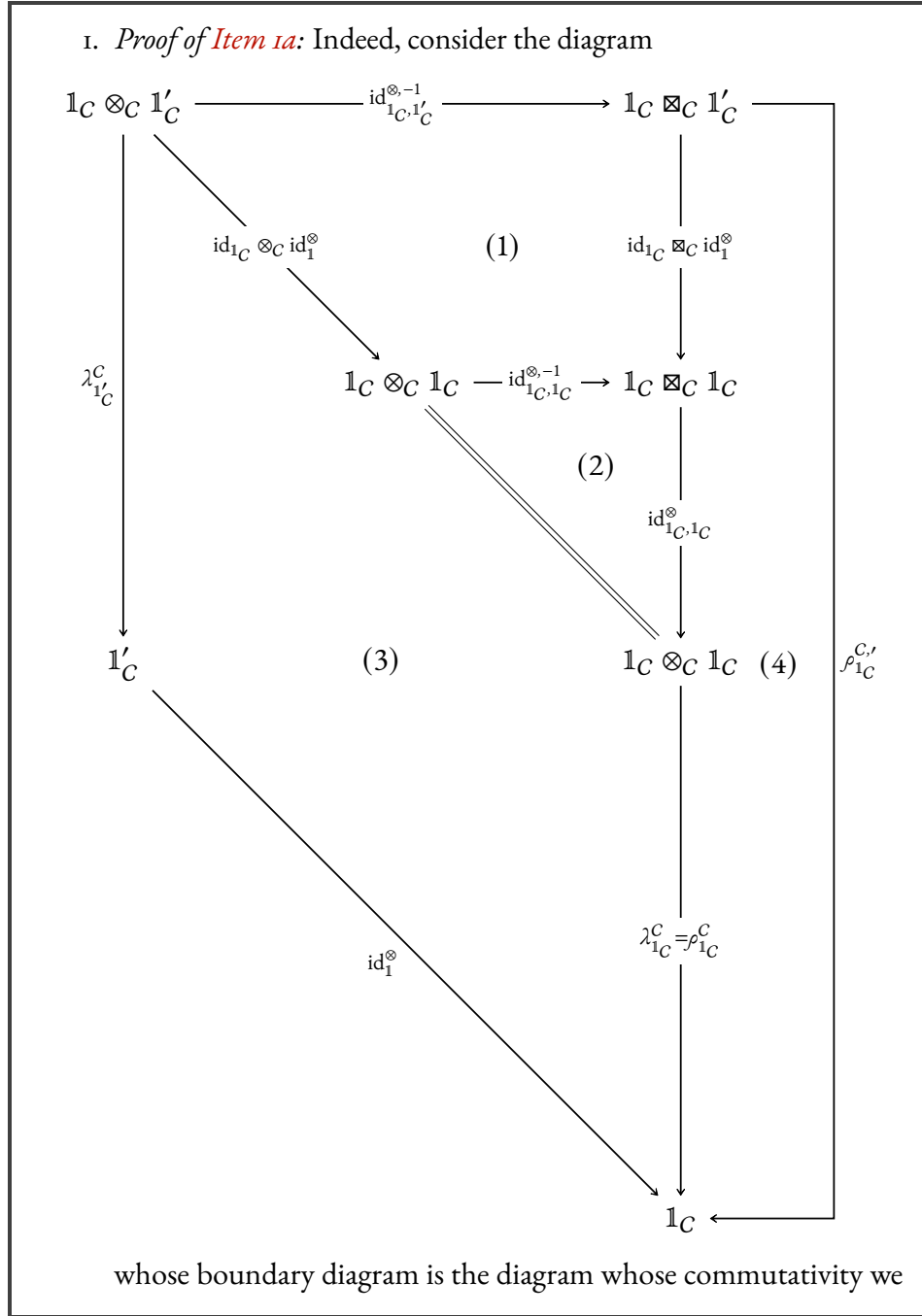
1. *Proof of Item 1a:* Indeed, consider the diagram

$$
\begin{array}{c}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \xrightarrow{\quad \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C} \quad} \mathbb{1}_C \boxtimes_C \mathbb{1}'_C
\end{array}
$$

$$\mathrm{id}_{\mathbb{1}_C} \otimes_C \mathrm{id}^{\otimes}_{\mathbb{1}} \qquad (1) \qquad \mathrm{id}_{\mathbb{1}_C} \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}$$

$$\lambda^C_{\mathbb{1}'_C}$$

$$\mathbb{1}_C \otimes_C \mathbb{1}_C \xrightarrow{\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}\ } \mathbb{1}_C \boxtimes_C \mathbb{1}_C$$

$$(2)$$

$$\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}$$

$$\mathbb{1}'_C \qquad (3) \qquad \mathbb{1}_C \otimes_C \mathbb{1}_C \quad (4) \quad \rho^{C,\prime}_{\mathbb{1}_C}$$

$$\mathrm{id}^{\otimes}_{\mathbb{1}} \qquad \lambda^C_{\mathbb{1}_C} = \rho^C_{\mathbb{1}_C}$$

$$\mathbb{1}_C \longleftarrow$$

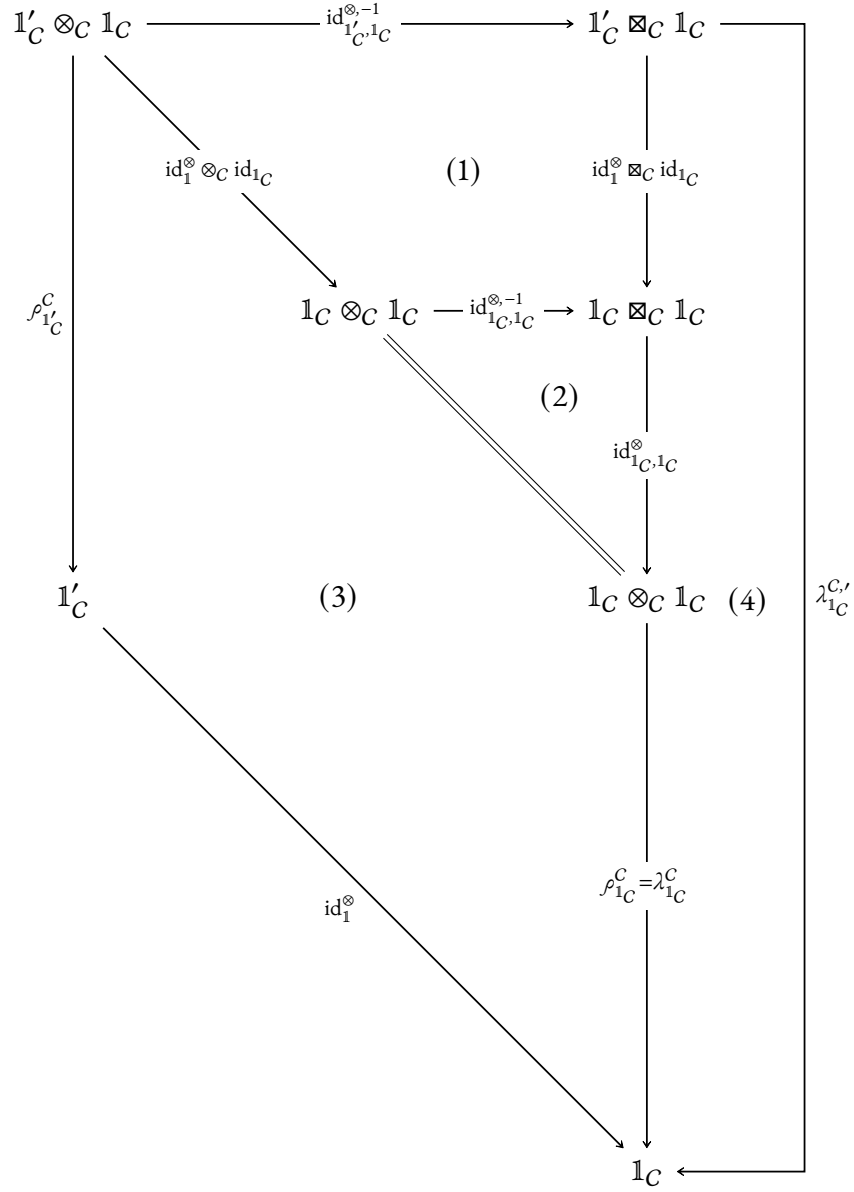whose boundary diagram is the diagram whose commutativity we

wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;

- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of λ^C, where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;

- Subdiagram (4) commutes by the right monoidal unity of $(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}_{C|\mathbb{1}}^{\otimes})$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram

$$
\begin{array}{c}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \xrightarrow{\quad \mathrm{id}^{\otimes,-1}_{\mathbb{1}'_C,\mathbb{1}_C} \quad} \mathbb{1}'_C \boxtimes_C \mathbb{1}_C
\end{array}
$$

$$\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_C \mathrm{id}_{\mathbb{1}_C} \qquad (1) \qquad \mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_C \mathrm{id}_{\mathbb{1}_C}$$

$$\rho^C_{\mathbb{1}'_C}$$

$$\mathbb{1}_C \otimes_C \mathbb{1}_C \xrightarrow{\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}\ } \mathbb{1}_C \boxtimes_C \mathbb{1}_C$$

$$(2)$$

$$\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}$$

$$\mathbb{1}'_C \qquad (3) \qquad \mathbb{1}_C \otimes_C \mathbb{1}_C \quad (4) \quad \lambda^{C,\prime}_{\mathbb{1}_C}$$

$$\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$$

$$\mathrm{id}^{\otimes}_{\mathbb{1}}$$

$$\mathbb{1}_C$$

whose boundary diagram is the diagram whose commutativity we

wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C, where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}_{C|\mathbb{1}}^{\otimes})$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$
\begin{array}{ccccc}
\mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}\ } & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\ \mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}\ } & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
\Big\downarrow{\scriptstyle \rho^C_{\mathbb{1}'_C}} & (1) & \Big\downarrow{\scriptstyle \lambda^{C,\prime}_{\mathbb{1}_C}} & (\dagger) & \Big\downarrow{\scriptstyle \rho^C_{\mathbb{1}'_C}} \\
\mathbb{1}'_C & \xrightarrow[\ \mathrm{id}^{\otimes}_{\mathbb{1}}\ ]{} & \mathbb{1}_C & \xrightarrow[\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}}\ ]{} & \mathbb{1}'_C.
\end{array}
$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$
\begin{array}{ccccc}
\mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}\ } & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\ \mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}\ } & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
& & \Big\downarrow{\scriptstyle \lambda^{C,\prime}_{\mathbb{1}_C}} & (\dagger) & \Big\downarrow{\scriptstyle \rho^C_{\mathbb{1}'_C}} \\
& & \mathbb{1}_C & \xrightarrow[\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}}\ ]{} & \mathbb{1}'_C
\end{array}
$$

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$
\begin{array}{ccccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
\downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} & (1) & \downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} & (\dagger) & \downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} \\
\mathbb{1}'_C & \xrightarrow{\mathrm{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}}^{\otimes,-1}} & \mathbb{1}_C
\end{array}
$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$
\begin{array}{ccccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
& & \downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} & (\dagger) & \downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} \\
& & \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}}^{\otimes,-1}} & \mathbb{1}_C
\end{array}
$$

commutes. But since $\mathrm{id}_{\mathbb{1}}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.
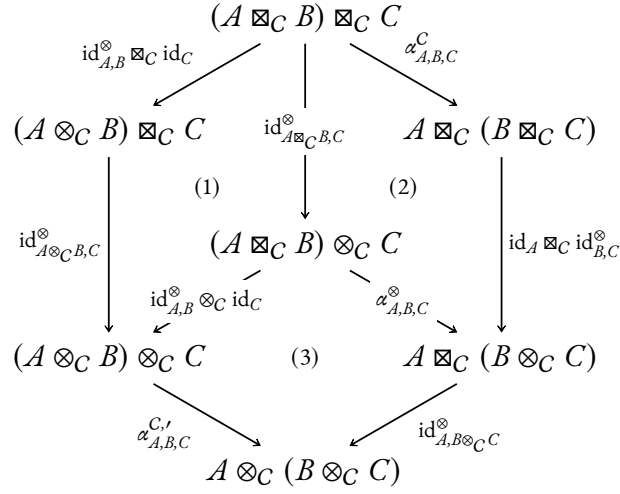
This finishes the proof.

Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

01UZ   1. *Proof of Item 3a:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of Item 2 of Remark 13.1.1.1.3 as follows:

$$
\begin{array}{c}
(A \boxtimes_C B) \boxtimes_C C \\
\end{array}
$$

$$
\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C \qquad \alpha^C_{A,B,C}
$$

$$
(A \otimes_C B) \boxtimes_C C \qquad \mathrm{id}^{\otimes}_{A\boxtimes_C B, C} \qquad A \boxtimes_C (B \boxtimes_C C)
$$

$$
(1) \qquad\qquad (2)
$$

$$
\mathrm{id}^{\otimes}_{A\otimes_C B, C} \qquad (A \boxtimes_C B) \otimes_C C \qquad \mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}
$$

$$
\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C \qquad \alpha^{\otimes}_{A,B,C}
$$

$$
(A \otimes_C B) \otimes_C C \qquad (3) \qquad A \boxtimes_C (B \otimes_C C)
$$

$$
\alpha^{C,\prime}_{A,B,C} \qquad\qquad \mathrm{id}^{\otimes}_{A,B\otimes_C C}
$$

$$
A \otimes_C (B \otimes_C C)
$$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01V0   2. *Proof of Item 3b:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:

$$(A \boxtimes_C B) \boxtimes_C C$$

$\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C$ \qquad\qquad $\alpha^C_{A,B,C}$

$$(A \otimes_C B) \boxtimes_C C \qquad (1) \qquad A \boxtimes_C (B \boxtimes_C C)$$

$\alpha^{\boxtimes}_{A,B,C}$ \qquad $\mathrm{id}^{\otimes}_{A,B\boxtimes_C C}$

$\mathrm{id}^{\otimes}_{A\otimes_C B,C}$ \qquad $A \otimes_C (B \boxtimes_C C)$ \qquad $\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}$

$$(2) \qquad\qquad (3)$$

$$(A \otimes_C B) \otimes_C C \quad \mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C} \quad A \boxtimes_C (B \otimes_C C)$$

$\alpha^{C,\prime}_{A,B,C}$ \qquad\qquad $\mathrm{id}^{\otimes}_{A,B\otimes_C C}$

$$A \otimes_C (B \otimes_C C)$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01V1   3. *Proof of Item 3c:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:

$$(A \boxtimes_C B) \boxtimes_C C$$

$$\mathrm{id}^\otimes_{A,B} \boxtimes_C \mathrm{id}_C \qquad\qquad \alpha^C_{A,B,C}$$

$$(A \otimes_C B) \boxtimes_C C \qquad\qquad\qquad A \boxtimes_C (B \boxtimes_C C)$$

$$(1)$$

$$\mathrm{id}^\otimes_{A\otimes_C B,C} \qquad\qquad \alpha^{\boxtimes,\otimes}_{A,B,C} \qquad\qquad \mathrm{id}_A \boxtimes_C \mathrm{id}^\otimes_{B,C}$$

$$(2)$$

$$(A \otimes_C B) \otimes_C C \qquad\qquad\qquad A \boxtimes_C (B \otimes_C C)$$

$$\alpha^{C,\prime}_{A,B,C} \qquad\qquad \mathrm{id}^\otimes_{A,B\otimes_C C}$$

$$A \otimes_C (B \otimes_C C)$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^\otimes$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

Appendices

A   Other Chapters