Introduction

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This chapter contains some general information about the Clowder Project.

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1.1 Introduction

1.1.1 Project Description and Goals

In short, the Clowder Project is an online reference work and wiki for category theory and mathematics that aims to essentially become a Stacks Project for category theory.

The project arose from a desire to improve upon a number of issues with the existing category theory literature, as well as fill several gaps in it.

In this section, we list and discuss the goals of the Clowder Project.

1.1.1.1 Provide a Unified and Complete Reference for Category Theory

The category theory literature is at times rather fragmented, and often it takes a long while for book-long treatments on a given subject to appear.

For example, although the theory of bicategories dates back to the late 1960s, it was not until 2020 that the subject would receive its first textbook in the topic, namely [JY21].

The Clowder Project aims to bridge this gap, providing a complete overview of the foundational material on category theory (see also Section 1.2.1).

1.1.1.2 Gather Hard to Find Results

As an extension of the previous goal, the Clowder Project also aims to gather in a single place results that are hard to find in the literature. These tend to be recorded only on original sources, which often means papers, notes or theses from the 1970s.

Since the Clowder Project is organized as a wiki, it becomes rather easy to search and find such results, as one merely needs to go to the page for a given concept and then look at the properties listed there.

1.1.1.3 Elaborate on Details That Are Often Left Out

Another goal of the Clowder Project is to include all kinds of details and intuitions that often don't make their way into textbooks, papers, monographs, etc.

For instance, one sometimes finds claims that a given diagram commutes and that it is "easy" to fill in the details. This also tends to happen particularly when the details are rather unwieldy.

One of the goals of the Clowder Project is to provide such proofs in great detail, including discussions of technical results, even when these are indeed "obvious".

1.1.1.4 Homogenize Conventions, Notation, and Terminology

Another issue with practice in the field is that there are often a number of conflicting conventions, notations, and terminology.

Being organized as a comprehensive and encyclopedic wiki, the Clowder Project tries to homogenize these conventions, notations, and terminology.

1.1.1.5 Fill Gaps in the Category Theory Literature

There are quite a few significant gaps in the category theory literature, some of which we hope to fill with the Clowder Project. For a list of (some of) these gaps, see Section 1.3.4.

1.1.1.6 Provide a Citable Reference for All Kinds of Results

It is a common situation to require a well-known result for a paper. Although proving it might be straightforward, it is often more convenient to cite a reference instead. Finding such a reference, however, may be hard and/or time-consuming.

With its encyclopedic nature, the Clowder Project hopes to serve as that convenient reference.

1.1.2 Navigating the Clowder Project

Hopefully, it should be intuitive to navigate through the web version of project. Nevertheless, here we mention a couple things that might be useful to know.

1.1.2.1 Preferences

You can change the font of the site, the style of the PDFs, as well as turn on dark mode by clicking the gear button located at the top right corner of the page.

1.1.2.2 Large Diagrams and the Zoom in Feature

This work features many diagrams that are unfortunately a bit too large to be comfortably legible in their native size.

To compensate for this, it's possible to click on them to expand their size by

200%.

In addition, you may also right-click on diagrams and then select "Open image in new tab" to allow for even higher amounts of zoom.

1.1.2.3 PDF Styles

The PDFs for each chapter as well as for the whole book are generated using twelve different styles, as summarised in the following table:

Typeface	Theorem Environments
Alegreya Sans	`tcbthm`
Alegreya	`tcbthm`
EB Garamond	`tcbthm`
Crimson Pro	`tcbthm`
XCharter	`tcbthm`
Computer Modern	`tcbthm`
Alegreya Sans	`amsthm`
Alegreya	`amsthm`
EB Garamond	`amsthm`
Crimson Pro	`amsthm`
XCharter	`amsthm`
Computer Modern	`amsthm`

The default style uses Alegreya Sans and `tcbthm`.

1.1.3 Prerequisites/Assumed Background

The Clowder Project assumes at least a background on basic category theory corresponding to e.g. [Rie16], as well as some comfort in working with category-theoretic notions.

In particular, it should be viewed as a reference work/wiki, and *not* as a textbook. This, however, doesn't mean it shouldn't be pedagogical. Indeed, a number of stylistic choices are made aiming to make the material as easily digestible as possible.

For an outline of several introductory references for different topics in category theory, see A Guide to the Literature.

1.1.4 Community Engagement, Contributions and Collaboration

All kinds of feedback and contributions to the Clowder Project are extremely welcome: pointing out typos, errors, historical remarks, references, layout of webpages, spelling errors, improvements to the overall structure, missing lemmas, etc.

The Clowder Project has an official Discord server in which people can ask questions, carry out discussions and give feedback. Please join it if you'd like to contribute to the Clowder Project. Alternatively, you may also reach out to the project maintainer at `emily.de.oliveira.santos.tmf@gmail.com`.

1.1.4.1 How to Contribute

There's a number of ways to contribute to the Clowder Project, some of which will be detailed a bit below. However, please keep in mind that they are not just examples, and are most definitely not meant to be exhaustive.

If there's another way in which you'd like to contribute, by all means feel free to drop by the project's Discord (or, alternatively, reach out to the project maintainer).

1.1.4.2 Ways to Contribute: Missing Proofs

There is a large number of missing proofs in the project, ranging from trivial proofs to simple lemmas to more involved results.

Missing proofs are listed in Section 1.3.1.

Note: The following chapters are undergoing revision. If you're interested in contributing, please disregard them for now:

- Relations
- Constructions With Relations
- Conditions on Relations
- Categories
- Constructions With Monoidal Categories
- Types of Morphisms in Bicategories

1.1.4.3 Ways to Contribute: Missing Examples

New examples to the Clowder Project are always welcome. These could be examples illustrating a new concept, examples showing why certain conditions are necessary in a given proof, counterexamples to be aware of, etc.

Some examples which would be particularly nice to have in Clowder are listed in Section 1.3.2. Please do keep in mind however that ***all examples are welcome***, even if they fall outside the examples listed in Section 1.3.2.

1.1.4.4 Ways to Contribute: Questions

A number of questions appear throughout the Clowder Project; tackling these would be an amazing way to contribute to the project.

The questions appearing throughout the Clowder Project are listed in Section 1.3.3.

1.1.5 Frequently Asked Questions

1.1.5.1 How does Clowder differ from the nLab?

Clowder is meant to be much more comprehensive than the nLab, which includes even filling a number of gaps in the category theory literature. Additionally, it also has a different set of goals and stylistic choices. For a more in-depth explanation, see Section 1.1.1.

1.1.5.2 Why not just use the nLab instead?

There are a number of reasons why Clowder was built as a separate project, instead of e.g. just editing the nLab:

1. *Curation.* All content on Clowder is personally curated by the project maintainer. This ensures an even quality to everything in the project.

2. *Cohesion.* As a consequence of Item 1, the Clowder Project ends up being much more cohesive than the nLab, having a clear and coherent organization, consistent notation and conventions, as well as a consistent style.

3. *Referenceability.* Clowder employs Gerby's Tag system, meaning that every citable statement in Clowder (e.g. definitions, examples, constructions, propositions, remarks, even individual items in lists, etc.) carries a corresponding tag.

This makes the project easy to cite and reference, since although the numbering of e.g. a given definition may change, its associated tag will forever be the same. See also Clowder — The Tag System.

4. *Crowdsourcing and Crowdfunding.* Clowder is meant to be a crowdfunded project in which the community can help directly finance its development. As a result, the project has a dedicated project maintainer whose role is to continuously take care of the project, coordinating contributions, developing infrastructure, and expanding the content of the project.

5. *Infrastructure.* The Clowder Project makes use of several very specific features which simply wouldn't be possible to implement in the nLab. This includes:

   (a) An elaborate fork of `gerby-website`, implementing a variety of new features and quality-of-life additions.

   (b) Another elaborate fork, this time of Gerby (which is itself a fork of plasTeX), implement a number of similarly needed features for the website to work as intended.

   See Section 1.2.3.2 for a (slightly) more in-depth description of the features and additions that have been created specifically for Clowder.

1.1.6 Goodies

In this section we list a few sample nice results and things from the Clowder Project.

1.1.6.1 General Utility

- Notes, Section 15.1 contains several `tikz-cd` snippets producing somewhat-hard-to-draw diagrams. Examples include cube, pentagon, and hexagon diagrams, as well as e.g. co/product diagrams with perfectly circular arrows.

1.1.6.2 Set Theory Through a Categorical Lens

Sets:

- Constructions With Sets, Section 4.4.7 contains a discussion of internal Homs in powersets viewed as categories.

- More generally, Constructions With Sets, Section 4.4 discusses several properties of powersets that are analogous to those of presheaf categories.
- Constructions With Sets, ?? discusses the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ induced by a function $f\colon X \to Y$.
- Constructions With Sets, Section 4.6.4 constructs a kind of "six functor formalism for (power)sets".
- Monoidal Structures on the Category of Sets contains explicit proofs that product/coproduct of sets form a monoidal structure.
- Monoidal Structures on the Category of Sets, Section 5.1.10 gives a completely 1-categorical proof of the universal property of $(\mathsf{Sets}, \times, \mathrm{pt})$.

Pointed Sets:

- Tensor Products of Pointed Sets constructs several tensor products of pointed sets, including a few unusual ones giving rise to skew monoidal structures on Sets_*.
- Tensor Products of Pointed Sets, Section 7.5.10 gives a completely 1-categorical proof of the universal property of $(\mathsf{Sets}_*, \wedge, S^0)$.
- Tensor Products of Pointed Sets, Definition 7.5.12.1.1 contains a description of comonoids in Sets_* with respect to $\wedge$.

Relations:

- Relations, Section 8.5 contains a discussion of several properties of the 2-category of relations like descriptions of internal adjunctions and internal monads.
- Relations, Sections 8.8 and 8.9 contains a discussion of two skew monoidal structures on the category $\mathbf{Rel}(A, B)$ of relations from a set A to a set B.
- Constructions With Relations, ?? contains a description of left/right Kan extensions and lifts internal to the 2-category of relations.

1.1.6.3 Category Theory

- Categories contains a description of several properties of functors, including somewhat lesser known ones such as dominant functors or pseudoepic functors.

1.2 Project Overview

1.2.1 Content and Scope

In this section, we outline what content is expected to be covered in the Clowder Project.

1.2.1.1 Elementary Category Theory

First and foremost, the Clowder Project aims to cover the foundations of category theory. This comprises all the usual topics treated in basic textbooks in category theory, such as [Mac98] or [Rie16], like adjunctions, co/limits, Kan extensions, co/ends, monoidal categories, etc.

1.2.1.2 Variants of Category Theory

Second, the Clowder Project aims to cover variants of category theory such as internal, fibred, or enriched category theory. The literature on these topics is often quite scattered and scarce, and so having a comprehensive discussion of them in Clowder aims to fill a large gap in the literature. See also Definition 1.3.4.1.14.

1.2.1.3 Higher Category Theory

Third, a detailed presentation of the theories of bicategories and double categories is planned, along with *some* material on tricategories.

Bicategories are another topic for which the literature is rather scattered, and, for some topics, scarce. As mentioned in the introduction, only recently has a proper textbook on bicategories appeared, [JY21]. Moreover, one finds several gaps in the literature, with a number of important results missing. As one particular example, one could look at the theory of 2-dimensional co/ends, in which case a comprehensive treatment based upon lax/oplax/pseudo dinatural transformations seems to be missing.

All of the elementary and not-so-elementary topics in the theory of bicategories are planned to appear in Clowder, and the same holds true for the theory of double categories.

1.2.1.4 ∞-Categories

Lastly, some material on ∞-categories is planned, although the precise scope of this remains to be defined. Ideally, this would include both model categories as well as synthetic and concrete models for ∞-categories (e.g. quasicategories, complete Segal spaces, cubical quasicategories, etc.).

In this way, we view Clowder as a good *complement* to [Lur25].

1.2.1.5 Other Topics

Occasionally, material on topics not a-priori related to category theory will be included. This may be done for a variety of reasons, including:

- Illustrating general theory.
- Comparison with classical concepts, such as e.g. ionads vs.topological spaces.
- Providing a more consistent and unified treatment of a particular topic, with hyperlinks to relevant concepts or examples.

1.2.2 Style

The Clowder Project makes several unusual stylistic choices, aligned with its goals.

1.2.2.1 Presentation of Topics

The presentation of topics is encyclopedic, non-linear, and sometimes idiosyncratic.

In particular, there's some amount of repetition throughout the project. This is a result of simultaneously wanting to cover as much material as possible while still allowing Clowder to be used as an online reference work/wiki.

1.2.2.2 Provable Items Come With Proofs

Every proposition, theorem, lemma, etc. needs to come with a proof. In case a proof has not been written yet, it shall read as "Omitted". This is to ensure results without proof are clearly labelled as such.

1.2.2.3 Proper Justification of Proofs

Every proof must read either "Omitted" or be properly justified, no matter how trivial the details are.

Expressions like "it is clear that", "it is straightforward to show that", "it is obvious", etc. inside proofs should not be used.

1.2.3 Infrastructure and Technical Implementation

1.2.3.1 Removed Features (in Comparison With the Stacks Project)

A few features present in the general infrastructure of the Stacks Project were removed in Clowder, including:

1. The python back-end, in favour of static pages.

2. The comment system, as a result of the static nature of the website.

1.2.3.2 Gerby and the Tags System

Clowder is built using Gerby, similarly to the Stacks Project. However, a number of additional features and quality-of-life additions not implemented in plasTeX or Gerby were required by Clowder, including:

1. Clowder uses `tcbtheorem`-like environments, which affects the placement of footnotes (which are often used).

2. Clowder implements a dangerous bend symbol to help visually highlight warnings (example).

3. There are a few aesthetic changes in Clowder's HTML/CSS structure, including font selection as well as a dark mode.

4. `tikz-cd` diagrams are very frequently used, and they need to be separately compiled and converted to `svg` files.

5. Code in Clowder can be copied easily using a "Copy" button, with code for bibliography entries also having proper syntax highlighting (example).

6. Clowder is automatically built using GitHub actions.

7. Non-sectioning tags are rendered differently and shown in context (example).

These have been implemented using a fork of Gerby along with a few build scripts.

1.2.3.3 Placeholder Symbols and Future Style

Currently, a number of macros have been defined using placeholder symbols, and look very ugly as a result.

They will eventually be replaced with proper symbols coming from the math fonts of Hundar, a free and open-source typeface project currently being worked on.

Hundar

A free, open-source typeface optimized
for academic, scientific, and book writing.

You can find more details about Hundar at its GitHub repository or website.

1.3 Lists

1.3.1 List of Omitted Proofs

Не так благотворна истина, как зловредна ее видимость.

Даниил Данковский

Truth does not do as much good in the world as the appearance of truth does evil.

Daniil Dankovsky

There's a very large number of omitted proofs throughout these notes. In this section we list them in order of decreasing importance.

- If a proof relies on material that has yet to be developed on Clowder, we mark it by a ⚠ sign. If you're interested in contributing, please disregard those for now.
- The following chapters are undergoing revision. If you're interested in contributing, please disregard them for now:
  - Relations
  - Constructions With Relations
  - Conditions on Relations
  - Categories
  - Constructions With Monoidal Categories
  - Types of Morphisms in Bicategories
- This list is under construction.

Remark 1.3.1.1.1. Proofs that *need* to be added at some point:

- Extra proof of Tensor Products of Pointed Sets, Definition 7.5.10.1.1 using the machinery of presentable categories, following Maxime Ranzi's answer to MO 466593 ⚠.
- Horizontal composition of natural transformations is associative: Categories, Item 2 of Definition 11.9.5.1.3.
- Fully faithful functors are essentially injective: Categories, Item 4 of Definition 11.6.3.1.2.

Proofs that *would be very nice* to be added at some point:

- Properties of pseudomonic functors: Categories, Definition 11.7.4.1.2 ⚠.
- Characterisation of fully faithful functors: Categories, Item 1 of Definition 11.6.3.1.2.
- The quadruple adjunction between categories and sets: Categories, Definition 11.3.1.1.1.
- F_* faithful iff F faithful: Categories, Item 2 of Definition 11.6.1.1.2.
- Properties of groupoid completions: Categories, Definition 11.4.3.1.3.
- Properties of cores: Categories, Definition 11.4.4.1.4.
- Rel is isomorphic to the category of free algebras of the powerset monad: Relations, Definition 8.5.20.1.1 ⚠.
- Non/existence of left Kan extensions in **Rel**:
  - Constructions With Relations, **??** of **??**.
  - Constructions With Relations, **??** of **??**.
- Non/existence of left Kan lifts in **Rel**:
  - Constructions With Relations, **??** of **??**.
  - Constructions With Relations, **??** of **??**.

Proofs that *would be nice* to be added at some point:

- Properties of posetal categories: Categories, Definition 11.2.7.1.2.
- Injective on objects functors are precisely the isocofibrations in Cats_2: Categories, Item 1 of Definition 11.8.1.1.2 ⚠.
- Characterisations of monomorphisms of categories: Categories, Item 1 of Definition 11.7.2.1.2.
- Epimorphisms of categories are surjective on objects: Categories, Item 2 of Definition 11.7.3.1.2.

- Properties of pseudoepic functors: Categories, Definition 11.7.5.1.2 ⚠.

Proofs that *would be nice but not essential* to be added at some point:

- Proof that $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category: Constructions With Sets, Item 15 of Definition 4.1.3.1.3 ⚠.
- Proof that $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category: Monoidal Structures on the Category of Sets, Definition 5.3.5.1.1 ⚠.
- Proof that $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category: Constructions With Sets, Item 11 of Definition 4.1.4.1.5 ⚠.
- Proof that $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category: Constructions With Sets, Item 6 of Definition 4.2.3.1.3 ⚠.
- Proof that $(\mathsf{Sets}, \coprod_X, X)$ is a symmetric monoidal category: Constructions With Sets, Item 8 of Definition 4.2.4.1.6 ⚠.

Proofs that have been (temporarily) omitted because they are "clear", "straightforward", or "tedious":

- Properties of pushouts of sets:
  - Associativity: Constructions With Sets, Item 3 of Definition 4.2.4.1.6.
  - Unitality: Constructions With Sets, Item 5 of Definition 4.2.4.1.6.
  - Commutativity: Constructions With Sets, Item 6 of Definition 4.2.4.1.6.
  - Pushout of sets over the empty set recovers the coproduct of sets: Constructions With Sets, Item 7 of Definition 4.2.4.1.6.
- Properties of coequalisers of sets:
  - Associativity: Constructions With Sets, Item 1 of Definition 4.2.5.1.5.
  - Unitality: Constructions With Sets, Item 4 of Definition 4.2.5.1.5.
  - Commutativity: Constructions With Sets, Item 5 of Definition 4.2.5.1.5.
  - Interaction with composition: Constructions With Sets, Item 6 of Definition 4.2.5.1.5.
- Properties of set differences:

- Constructions With Sets, Item 4 of Definition 4.3.10.1.2.
- Constructions With Sets, Item 11 of Definition 4.3.10.1.2.
- Constructions With Sets, Item 13 of Definition 4.3.10.1.2.
- Constructions With Sets, Item 15 of Definition 4.3.10.1.2.

- Complements and characteristic functions: Constructions With Sets, Item 4 of Definition 4.3.11.1.2.
- Properties of symmetric differences:
  - Constructions With Sets, Item 1 of Definition 4.3.12.1.2.
  - Constructions With Sets, Item 16 of Definition 4.3.12.1.2.
- Properties of direct images:
  - Functoriality: Constructions With Sets, Item 1 of Definition 4.6.1.1.5.
  - Interaction with coproducts: Constructions With Sets, Item 15 of Definition 4.6.1.1.5.
  - Interaction with products: Constructions With Sets, Item 16 of Definition 4.6.1.1.5.
- Properties of inverse images:
  - Functoriality: Constructions With Sets, Item 1 of Definition 4.6.2.1.3.
  - Interaction with coproducts: Constructions With Sets, Item 15 of Definition 4.6.2.1.3.
  - Interaction with products; Constructions With Sets, Item 16 of Definition 4.6.2.1.3.
- Properties of codirect images:
  - Functoriality: Constructions With Sets, Item 1 of Definition 4.6.3.1.7.
  - Lax preservation of colimits: Constructions With Sets, Item 10 of Definition 4.6.3.1.7.
  - Interaction with coproducts: Constructions With Sets, Item 14 of Definition 4.6.3.1.7.

  - Interaction with products: Constructions With Sets, Item 15 of Definition 4.6.3.1.7.

- Left distributor of $\times$ over $\coprod$ is a natural isomorphism: Monoidal Structures on the Category of Sets, Definition 5.3.1.1.1.

- Right distributor of $\times$ over $\coprod$ is a natural isomorphism: Monoidal Structures on the Category of Sets, Definition 5.3.2.1.1.

- Left annihilator of $\times$ is a natural isomorphism: Monoidal Structures on the Category of Sets, Definition 5.3.3.1.1.

- Right annihilator of $\times$ is a natural isomorphism: Monoidal Structures on the Category of Sets, Definition 5.3.4.1.1.

- Properties of wedge products of pointed sets:
  - Associativity: Pointed Sets, Item 2 of Definition 6.3.3.1.3.
  - Unitality: Pointed Sets, Item 3 of Definition 6.3.3.1.3.
  - Commutativity: Pointed Sets, Item 4 of Definition 6.3.3.1.3.
  - Symmetric monoidality: Pointed Sets, Item 5 of Definition 6.3.3.1.3.

- Properties of pushouts of pointed sets:
  - Interaction with coproducts: Pointed Sets, Item 5 of Definition 6.3.4.1.3.
  - Symmetric monoidality: Pointed Sets, Item 6 of Definition 6.3.4.1.3.

1.3.2 List of Missing Examples

Adding new examples is always welcome! In this section, we list some subjects and sections which could do with more examples:

Remark 1.3.2.1.1. Potentially interesting examples to add include, but are definitely not limited to:

- Examples of 2-categorical monomorphisms in **Rel**, following Relations, Section 8.5.11.

- Examples of 2-categorical epimorphisms in **Rel**, following Relations, Section 8.5.13.

- Examples of left Kan extensions and left Kan lifts in **Rel**.
- Examples of functors satisfying the conditions described in Categories.

1.3.3 List of Questions

There's a number of questions listed throughout this project. Here we collect them in a single place.

Remark 1.3.3.1.1. On relations:

- Relations, Definition 8.5.11.1.2, on better characterisations of representably full morphisms in **Rel**. This question also appears as [MO 467527].
- Relations, Definition 8.5.13.1.2, on better characterisations of corepresentably full morphisms in **Rel**. This question also appears as [MO 467527].
- Relations, **??**, seeking a characterisation of which left Kan extensions exist in **Rel**. This question also appears as [MO 461592].
- Relations, **??**, seeking an explicit descriptions of left Kan extensions along relations of the form f^{-1} (which always exist in **Rel**). This question also appears as [MO 461592].
- Relations, **??**, seeking a characterisation of which left Kan lifts exist in **Rel**. This question also appears as [MO 461592].
- Relations, **??**, seeking an explicit descriptions of left Kan lifts along relations of the form $\mathrm{Gr}(f)$ (which always exist in **Rel**). This question also appears as [MO 461592].

On categories:

- Categories, Definition 11.6.2.1.3, seeking a better characterisation of necessary and sufficient conditions on F for F^* to always be full. This question also appears as [MO 468121b].
- Categories, Definition 11.6.4.1.3, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be conservative. This question also appears as [MO 468121a].

- Categories, Definition 11.6.5.1.2, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be essentially injective. This question also appears as [MO 468121a].
- Categories, Definition 11.6.6.1.2, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be essentially surjective. This question also appears as [MO 468121a].
- Categories, Definition 11.7.1.1.3, , seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be dominant. This question also appears as [MO 468121a].
- Categories, Definition 11.7.2.1.3, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be monic. This question also appears as [MO 468121a].
- Categories, Definition 11.7.3.1.3, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be epic. This question also appears as [MO 468121a].
- Categories, Definition 11.7.5.1.5, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be pseudoepic. This question also appears as [MO 468121a].
- Categories, Definition 11.7.5.1.3, seeking a characterisation of pseudoepic functors. This question also appears as [MO 321971].
- Categories, Definition 11.7.5.1.4, which asks whether a pseudomonic and pseudoepic functor must necessarily be an equivalence of categories. This question also appears as [MO 468334].
- Categories, Definition 11.8.4.1.3, seeking a characterisation of functors representably faithful on cores.
- Categories, Definition 11.8.5.1.3, seeking a characterisation of functors representably full on cores.
- Categories, Definition 11.8.6.1.3, seeking a characterisation of functors representably fully faithful on cores.

- Categories, Definition 11.8.7.1.3, seeking a characterisation of functors corepresentably faithful on cores.
- Categories, Definition 11.8.8.1.3, seeking a characterisation of functors corepresentably full on cores.
- Categories, Definition 11.8.9.1.3, seeking a characterisation of functors corepresentably fully faithful on cores.

1.3.4 List of Gaps in the Category Theory Literature

The Clowder Project aims to address several significant gaps in the existing literature on category theory, as detailed below. See also [MO 494959].

Gap 1.3.4.1.1. Even though its analogue for ∞-categories has for years been a widely used tool[1], a comprehensive treatment of the tensor product of presentable categories seems to be currently missing.

Gap 1.3.4.1.2. An exhaustive concrete description of the various limits and colimits of categories, including 2-dimensional ones, is missing.

Gap 1.3.4.1.3. There seems to be no unified presentation of dinatural transformation co/classifiers in the literature. These are characterised by isomorphisms of the form

$$\mathrm{Nat}(F, G) \cong \mathrm{DiNat}(\Gamma(F), G),$$

and were originally studied in Dubuc–Street's paper introducing dinatural transformations, [DS06].

Even though these arguably form a fundamental piece of the framework of co/end calculus, it seems that all foundational treatments that followed after ended up not covering this concept.

Gap 1.3.4.1.4. The tensor product of symmetric monoidal categories had been a missing concept from the literature for years. Recently, [GJO24] covered the case of permutative categories. It would be nice, however, to also have a treatment of the non-strict case available.

Gap 1.3.4.1.5. A comprehensive and exhaustive treatment of the theory of promonoidal categories is currently missing. There are several important notions undefined, like:

[1] See [MO 490557].

- Promonoidal profunctors.
- Dualisability internal to a promonoidal category.
- Invertibility internal to a promonoidal category.

Moreover, it would be nice to record how promonoidal categories may be viewed as categorifications of "hypermonoids" (i.e. monoids in Rel).

Gap 1.3.4.1.6. A comprehensive and exhaustive treatment of the theory of multicategories is currently missing. There are several important notions undefined, like:

- Co/limits internal to multicategories.

See [MO 484647].

Gap 1.3.4.1.7. It would be nice to have an extensive collection of examples of what a given 2-categorical notion looks like in a 2-category. For instance, it would be nice to explicitly list what internal adjunctions look like in **Rel**, Span, Prof, etc.

See Relations, Section 8.5 for a concrete example of what is meant by this gap.

Gap 1.3.4.1.8. The literature on centres and traces of categories is really small. There are lots of results missing[2] and very few worked examples[3].

Gap 1.3.4.1.9. Natural transformations satisfy an isomorphism of the form

$$\mathrm{Nat}(F, G) \cong \int_{A\in C} \mathrm{Hom}_{\mathcal{D}}(F_A, G_A).$$

It is then exceedingly natural to define *natural cotransformations* via an isomorphism of the form

$$\mathrm{CoNat}(F, G) \cong \int^{A\in C} \mathrm{Hom}_{\mathcal{D}}(F_A, G_A)$$

and study their properties. This generalises traces of categories, since we have

$$\mathrm{Tr}(C) = \mathrm{CoNat}(\mathrm{id}_C, \mathrm{id}_C),$$

much like $Z(C) = \mathrm{Nat}(\mathrm{id}_C, \mathrm{id}_C)$.

[2] E.g. There's a certain interaction between traces of categories and Leinster's eventual image.
[3] E.g. what is the trace of Connes's cycle category? Such a computation doesn't seem to be

Gap 1.3.4.1.10. There are several results, notions, and examples in the theory of Isbell duality missing from the literature, and a truly comprehensive treatment is still lacking.[4]

Gap 1.3.4.1.11. The currently available treatments of 2-dimensional co/ends are unsatisfactory.[5]

Gap 1.3.4.1.12. A comprehensive treatment of factorisation systems is currently missing; see [MO 495003].

Gap 1.3.4.1.13. Several proofs of coherence theorems for string diagrams currently have gaps; see [MO 497309].

Gap 1.3.4.1.14. The currently available treatments of variants of category theory such as fibred category theory, enriched category theory, or internal category theory are unsatisfactory for a number of reasons.

Ideally, there should be a comprehensive and (simultaneously) approachable treatment for these topics. See also [MO 497419].

Appendices

available.

[4]For instance, there appears to be no mention of the duality pairings

$$\begin{aligned} \mathsf{Spec}(F) \boxtimes F &\to \mathrm{Tr}(C), \\ \mathcal{F} \boxtimes \mathsf{O}(\mathcal{F}) &\to \mathrm{Tr}(C) \end{aligned}$$

in the currently available literature.

[5]For instance, none of them define 2-dimensional co/ends via 2-dimensional dinatural transformations and then go on to develop a general theory from there.

A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

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