

# Types of Morphisms in Bicategories

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In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories* ([Sections 14.1 and 14.2](#)). There is a large number of different notions capturing the idea of a “monomorphism” or of an “epimorphism” in a bicategory.

Arguably, the notion that best captures these concepts is that of a *pseudomononic morphism* ([Definition 14.1.10.1.1](#)) and of a *pseudoepic morphism* ([Definition 14.2.10.1.1](#)), although the other notions introduced in [Sections 14.1](#) and [14.2](#) are also interesting on their own.

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## 14.1 Monomorphisms in Bicategories

### 14.1.1 Representably Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### DEFINITION 14.1.1.1 ► REPRESENTABLY FAITHFUL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably faithful**<sup>1</sup> if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is faithful.

<sup>1</sup>*Further Terminology:* Also called simply a **faithful morphism**, based on [Item 1](#) of [Example 14.1.1.1.3](#).

#### REMARK 14.1.1.1.2 ► UNWINDING DEFINITION 14.1.1.1

In detail,  $f$  is representably faithful if, for all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

**EXAMPLE 14.1.1.1.3 ► EXAMPLES OF REPRESENTABLY FAITHFUL MORPHISMS**

Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in  $\mathbf{Cats}_2$ .* The representably faithful morphisms in  $\mathbf{Cats}_2$  are precisely the faithful functors; see [Categories, Item 2](#) of [Proposition 11.6.1.1.2](#).
2. *Representably Faithful Morphisms in  $\mathbf{Rel}$ .* Every morphism of  $\mathbf{Rel}$  is representably faithful; see [Relations, Item 1](#) of [Proposition 8.5.11.1.1](#).

**14.1.2 Representably Full Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.1.2.1.1 ► REPRESENTABLY FULL MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably full**<sup>1</sup> if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is full.

<sup>1</sup>*Further Terminology:* Also called simply a **full morphism**, based on [Item 1](#) of [Example 14.1.2.1.3](#).

**REMARK 14.1.2.1.2 ► UNWINDING DEFINITION 14.1.2.1.1**

In detail,  $f$  is representably full if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: f \circ \phi \Longrightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

#### EXAMPLE 14.1.2.1.3 ► EXAMPLES OF REPRESENTABLY FULL MORPHISMS

Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in  $\mathbf{Cats}_2$ .* The representably full morphisms in  $\mathbf{Cats}_2$  are precisely the full functors; see [Categories](#), ?? of [Proposition 11.6.2.1.2](#).
2. *Representably Full Morphisms in  $\mathbf{Rel}$ .* The representably full morphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), [Item 2](#) of [Proposition 8.5.11.1.1](#).

### 14.1.3 Representably Fully Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### DEFINITION 14.1.3.1.1 ► REPRESENTABLY FULLY FAITHFUL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably fully faithful**<sup>1</sup> if the following equivalent conditions are satisfied:

1. The 1-morphism  $f$  is representably faithful ([Definition 14.1.1.1.1](#)) and representably full ([Definition 14.1.2.1.1](#)).
2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is fully faithful.

<sup>1</sup>*Further Terminology:* Also called simply a **fully faithful morphism**, based on [Item 1](#) of [Example 14.1.3.1.3](#).

**REMARK 14.1.3.1.2 ► UNWINDING REPRESENTABLY FULLY FAITHFUL MORPHISMS**

In detail,  $f$  is representably fully faithful if the conditions in [Remark 14.1.1.1.2](#) and [Remark 14.1.2.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $X \in \mathrm{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: f \circ \phi \Longrightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B \quad = \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

**EXAMPLE 14.1.3.1.3 ► EXAMPLES OF REPRESENTABLY FULLY FAITHFUL MORPHISMS**

Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in  $\mathbf{Cats}_2$ .* The representably fully faithful morphisms in  $\mathbf{Cats}_2$  are precisely the fully faithful functors; see [Categories, Item 6](#) of [Proposition 11.6.3.1.2](#).
2. *Representably Fully Faithful Morphisms in  $\mathbf{Rel}$ .* The representably fully faithful morphisms of  $\mathbf{Rel}$  coincide ([Relations, Item 3](#) of [Proposition 8.5.11.1.1](#)) with the representably full morphisms in  $\mathbf{Rel}$ , which are characterised in [Relations, Item 2](#) of [Proposition 8.5.11.1.1](#).

**14.1.4 Morphisms Representably Faithful on Cores**

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.1.4.1.1 ► MORPHISMS REPRESENTABLY FAITHFUL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably faithful on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is faithful.

**REMARK 14.1.4.1.2 ► UNWINDING DEFINITION 14.1.4.1.1**

In detail,  $f$  is representably faithful on cores if, for all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

### 14.1.5 Morphisms Representably Full on Cores

Let  $\mathcal{C}$  be a bicategory.

#### DEFINITION 14.1.5.1.1 ► MORPHISMS REPRESENTABLY FULL ON CORES

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably full on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is full.

#### REMARK 14.1.5.1.2 ► UNWINDING DEFINITION 14.1.5.1.1

In detail,  $f$  is representably full on cores if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

### 14.1.6 Morphisms Representably Fully Faithful on Cores

Let  $\mathcal{C}$  be a bicategory.





A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism  $f$  is representably faithful on cores ([Definition 14.1.5.1.1](#)) and representably full on cores ([Definition 14.1.4.1.1](#)).
2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is fully faithful.

**REMARK 14.1.6.1.2** ► UNWINDING DEFINITION 14.1.6.1.1

In detail,  $f$  is representably fully faithful on cores if the conditions in [Remark 14.1.4.1.2](#) and [Remark 14.1.5.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

### 14.1.7 Representably Essentially Injective Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### DEFINITION 14.1.7.1.1 ► REPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably essentially injective** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is essentially injective.

#### REMARK 14.1.7.1.2 ► UNWINDING DEFINITION 14.1.7.1.1

In detail,  $f$  is representably essentially injective if, for each pair of morphisms  $\phi, \psi: X \rightrightarrows A$  of  $\mathcal{C}$ , the following condition is satisfied:

( $\star$ ) If  $f \circ \phi \cong f \circ \psi$ , then  $\phi \cong \psi$ .

### 14.1.8 Representably Conservative Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### DEFINITION 14.1.8.1.1 ► REPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably conservative** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is conservative.

**REMARK 14.1.8.1.2 ► UNWINDING DEFINITION 14.1.8.1.1**

In detail,  $f$  is representably conservative if, for each pair of morphisms  $\phi, \psi: X \rightrightarrows A$  and each 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$ , if the 2-morphism

$$\mathrm{id}_f \star \alpha: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \parallel \\ \mathrm{id}_f \star \alpha \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

is a 2-isomorphism, then so is  $\alpha$ .

## 14.1.9 Strict Monomorphisms

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.1.9.1.1 ► STRICT MONOMORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is a **strict monomorphism** if, for each  $X \in \mathrm{Obj}(\mathcal{C})$ , the functor

$$f_*: \mathrm{Hom}_{\mathcal{C}}(X, A) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is injective on objects, i.e. its action on objects

$$f_*: \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \rightarrow \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

is injective.

**REMARK 14.1.9.1.2 ► UNWINDING DEFINITION 14.1.9.1.1**

In detail,  $f$  is a strict monomorphism in  $\mathcal{C}$  if, for each diagram in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $f \circ \phi = f \circ \psi$ , then  $\phi = \psi$ .

**EXAMPLE 14.1.9.1.3 ► EXAMPLES OF STRICT MONOMORPHISMS**

Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in  $\mathbf{Cats}_2$* . The strict monomorphisms in  $\mathbf{Cats}_2$  are precisely the functors which are injective on objects and injective on morphisms; see [Categories](#), [Item 1](#) of [Proposition 11.7.2.1.2](#).
2. *Strict Monomorphisms in  $\mathbf{Rel}$* . The strict monomorphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), [Proposition 8.5.10.1.1](#).

## 14.1.10 Pseudomonic Morphisms

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.1.10.1.1 ► PSEUDOMONIC MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **pseudomonic** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is pseudomonic.

**REMARK 14.1.10.1.2 ► UNWINDING DEFINITION 14.1.10.1.1**

In detail, a 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is pseudomonic if it satisfies the following conditions:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $X \in \mathrm{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

**PROPOSITION 14.1.10.1.3** ► PROPERTIES OF PSEUDOMONIC MORPHISMS

Let  $f: A \rightarrow B$  be a 1-morphism of  $\mathcal{C}$ .

1. *Characterisations.* The following conditions are equivalent:
  - (a) The morphism  $f$  is pseudomononic.

- (b) The morphism  $f$  is representably full on cores and representably faithful.
- (c) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_B A, \quad \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & \nearrow & \downarrow F \\ A & \xrightarrow{F} & B \end{array}$$

in  $\mathcal{C}$  up to equivalence.

2. *Interaction With Cotensors.* If  $\mathcal{C}$  has cotensors with  $\mathbb{1}$ , then the following conditions are equivalent:

- (a) The morphism  $f$  is pseudomonic.
- (b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_{\mathbb{1} \pitchfork F} B, \quad \begin{array}{ccc} A & \hookrightarrow & \mathbb{1} \pitchfork A \\ F \downarrow & \nearrow & \downarrow \mathbb{1} \pitchfork F \\ B & \hookrightarrow & \mathbb{1} \pitchfork B \end{array}$$

in  $\mathcal{C}$  up to equivalence.

#### PROOF 14.1.10.1.4 ► PROOF OF PROPOSITION 14.1.10.1.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted. 

## 14.2 Epimorphisms in Bicategories

### 14.2.1 Corepresentably Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.2.1.1 ► COREPRESENTABLY FAITHFUL MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably faithful** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is faithful.

**REMARK 14.2.1.1.2 ► UNWINDING DEFINITION 14.2.1.1.1**

In detail,  $f$  is corepresentably faithful if, for all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

**EXAMPLE 14.2.1.1.3 ► EXAMPLES OF COREPRESENTABLY FAITHFUL MORPHISMS**

Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in  $\mathbf{Cats}_2$* . The corepresentably faithful morphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 5](#) of [Proposition 11.6.1.1.2](#).
2. *Corepresentably Faithful Morphisms in  $\mathbf{Rel}$* . Every morphism of  $\mathbf{Rel}$  is corepresentably faithful; see [Relations, Item 1](#) of [Proposition 8.5.13.1.1](#).

## 14.2.2 Corepresentably Full Morphisms

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.2.2.1.1 ► COREPRESENTABLY FULL MORPHISMS**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably full** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is full.

**REMARK 14.2.2.1.2 ► UNWINDING DEFINITION 14.2.2.1.1**

In detail,  $f$  is corepresentably full if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

**EXAMPLE 14.2.2.1.3 ► EXAMPLES OF COREPRESENTABLY FULL MORPHISMS**

Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in  $\mathbf{Cats}_2$ .* The corepresentably full morphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 7 of Proposition 11.6.2.1.2](#).



2. *Corepresentably Full Morphisms in **Rel***. The corepresentably full morphisms in **Rel** are characterised in [Relations](#), [Item 2 of Proposition 8.5.13.1.1](#).

### 14.2.3 Corepresentably Fully Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

#### DEFINITION 14.2.3.1.1 ► COREPRESENTABLY FULLY FAITHFUL MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably fully faithful**<sup>1</sup> if the following equivalent conditions are satisfied:

1. The 1-morphism  $f$  is corepresentably full ([Definition 14.2.2.1.1](#)) and corepresentably faithful ([Definition 14.2.1.1.1](#)).
2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is fully faithful.

<sup>1</sup>*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [\[Adá+01\]](#)), though we will always use the name “corepresentably fully faithful morphism” instead in this work.

#### REMARK 14.2.3.1.2 ► UNWINDING DEFINITION 14.2.3.1.1

In detail,  $f$  is corepresentably fully faithful if the conditions in [Remark 14.2.1.1.2](#) and [Remark 14.2.2.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X \quad = \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

#### EXAMPLE 14.2.3.1.3 ► EXAMPLES OF COREPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in  $\mathbf{Cats}_2$ .* The fully faithful epimorphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 10](#) of [Proposition 11.6.3.1.2](#).
2. *Corepresentably Fully Faithful Morphisms in  $\mathbf{Rel}$ .* The corepresentably fully faithful morphisms of  $\mathbf{Rel}$  coincide ([Relations, Item 3](#) of [Proposition 8.5.13.1.1](#)) with the corepresentably full morphisms in  $\mathbf{Rel}$ , which are characterised in [Relations, Item 2](#) of [Proposition 8.5.13.1.1](#).

## 14.2.4 Morphisms Corepresentably Faithful on Cores

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.2.4.1.1 ► MORPHISMS COREPRESENTABLY FAITHFUL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably faithful on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is faithful.

**REMARK 14.2.4.1.2 ► UNWINDING DEFINITION 14.2.4.1.1**

In detail,  $f$  is corepresentably faithful on cores if, for all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

**14.2.5 Morphisms Corepresentably Full on Cores**

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.2.5.1.1 ► MORPHISMS COREPRESENTABLY FULL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably full on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is full.

**REMARK 14.2.5.1.2 ► UNWINDING DEFINITION 14.2.5.1.1**

In detail,  $f$  is corepresentably full on cores if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

## 14.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let  $\mathcal{C}$  be a bicategory.

**DEFINITION 14.2.6.1.1 ► MORPHISMS COREPRESENTABLY FULLY FAITHFUL ON CORES**

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism  $f$  is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is fully faithful.

**REMARK 14.2.6.1.2 ► UNWINDING DEFINITION 14.2.6.1.1**

In detail,  $f$  is corepresentably fully faithful on cores if the conditions in [Remark 14.2.4.1.2](#) and [Remark 14.2.5.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xrightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

## 14.2.7 Corepresentably Essentially Injective Morphisms

Let  $\mathcal{C}$  be a bicategory.

### DEFINITION 14.2.7.1.1 ► COREPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably essentially injective** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is essentially injective.

### REMARK 14.2.7.1.2 ► UNWINDING DEFINITION 14.2.7.1.1

In detail,  $f$  is corepresentably essentially injective if, for each pair of morphisms  $\phi, \psi: B \rightrightarrows X$  of  $\mathcal{C}$ , the following condition is satisfied:

( $\star$ ) If  $\phi \circ f \cong \psi \circ f$ , then  $\phi \cong \psi$ .

## 14.2.8 Corepresentably Conservative Morphisms

Let  $\mathcal{C}$  be a bicategory.

### DEFINITION 14.2.8.1.1 ► COREPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably conservative** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is conservative.

### REMARK 14.2.8.1.2 ► UNWINDING DEFINITION 14.2.8.1.1

In detail,  $f$  is corepresentably conservative if, for each pair of morphisms  $\phi, \psi: B \rightrightarrows X$  and each 2-morphism

$$\alpha: \phi \rightrightarrows \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$ , if the 2-morphism

$$\alpha \star \text{id}_f: \phi \circ f \Rightarrow \psi \circ f, \quad \begin{array}{ccc} & \phi \circ f & \\ \curvearrowright & \parallel & \curvearrowright \\ A & \alpha \star \text{id}_f & X \\ \curvearrowleft & \Downarrow & \curvearrowleft \\ & \psi \circ f & \end{array}$$

is a 2-isomorphism, then so is  $\alpha$ .

## 14.2.9 Strict Epimorphisms

Let  $\mathcal{C}$  be a bicategory.

### DEFINITION 14.2.9.1.1 ► STRICT EPIMORPHISMS

A 1-morphism  $f: A \rightarrow B$  is a **strict epimorphism in  $\mathcal{C}$**  if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is injective on objects, i.e. its action on objects

$$f_*: \text{Obj}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Obj}(\text{Hom}_{\mathcal{C}}(A, X))$$

is injective.

### REMARK 14.2.9.1.2 ► UNWINDING DEFINITION 14.2.9.1.1

In detail,  $f$  is a strict epimorphism if, for each diagram in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \xrightleftharpoons[\psi]{\phi} X,$$

if  $\phi \circ f = \psi \circ f$ , then  $\phi = \psi$ .

### EXAMPLE 14.2.9.1.3 ► EXAMPLES OF STRICT EPIMORPHISMS

Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in  $\text{Cats}_2$* . The strict epimorphisms in  $\text{Cats}_2$  are characterised in [Categories, Item 1](#) of [Proposition 11.7.3.1.2](#).

2. *Strict Epimorphisms in Rel*. The strict epimorphisms in **Rel** are characterised in [Relations](#), [Proposition 8.5.12.1.1](#).

## 14.2.10 Pseudoepic Morphisms

Let  $\mathcal{C}$  be a bicategory.

### DEFINITION 14.2.10.1.1 ► PSEUDOEPIC MORPHISMS

A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **pseudoepic** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is pseudomonic.

### REMARK 14.2.10.1.2 ► UNWINDING DEFINITION 14.2.10.1.1

In detail, a 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is pseudoepic if it satisfies the following conditions:

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$



of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

#### PROPOSITION 14.2.10.1.3 ► PROPERTIES OF PSEUDOEPIC MORPHISMS

Let  $f: A \rightarrow B$  be a 1-morphism of  $\mathcal{C}$ .

1. *Characterisations.* The following conditions are equivalent:
  - (a) The morphism  $f$  is pseudoepic.
  - (b) The morphism  $f$  is corepresentably full on cores and corepresentably faithful.
  - (c) We have an isococcomma square of the form

$$B \stackrel{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \quad \begin{array}{ccc} B & \xleftarrow{\text{id}_B} & B \\ \text{id}_B \uparrow & \nearrow \text{dashed} & \uparrow F \\ B & \xleftarrow{F} & A \end{array}$$

in  $\mathcal{C}$  up to equivalence.

#### PROOF 14.2.10.1.4 ► PROOF OF PROPOSITION 14.2.10.1.3

Item 1: Characterisations

Omitted.



## Appendices

## A Other Chapters

### Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

### Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

### Relations

8. [Relations](#)
9. [Constructions With Relations](#)

10. [Conditions on Relations](#)

### Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

### Monoidal Categories

13. [Constructions With Monoidal Categories](#)

### Bicategories

14. [Types of Morphisms in Bicategories](#)

### Extra Part

15. [Notes](#)

## References

- [Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. “On Functors Which Are Lax Epimorphisms”. In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. [17](#)).