# Constructions With Sets

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000J This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

01YT 1. Explicit descriptions of the major types of co/limits in $\mathsf{Sets}$, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 4.2.4.1.1, 4.2.4.1.3, 4.2.5.1.1 and 4.2.5.1.3).

01YU 2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:

01YV (a) A discussion of the internal Hom of a powerset (Section 4.4.7).

01YW (b) A 0-categorical version of the Yoneda lemma (Presheaves and the Yoneda Lemma, Definition 12.1.5.1.1), which we term the *Yoneda lemma for sets* (Definition 4.5.5.1.1).

01YX (c) A characterisation of powersets as free cocompletions (Section 4.4.5), mimicking the corresponding statement for categories of presheaves (**??**).

01YY (d) A characterisation of powersets as free completions (Section 4.4.6), mimicking the corresponding statement for categories of copresheaves (**??**).

01YZ (e) A $(-1)$-categorical version of un/straightening (Item 2 of Definition 4.5.1.1.4 and Definition 4.5.1.1.5).

01Z0 (f) A 0-categorical form of Isbell duality internal to powersets (Section 4.4.8).

01Z1 3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f : A \to B$, including in particular:

01Z2 (a) How $f^{-1}$ can be described as a precomposition while $f_!$ and $f_*$ can be described as Kan extensions (Definitions 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).

01Z3 (b) An extensive list of the properties of $f_!$, $f^{-1}$, and $f_*$ (Definitions 4.6.1.1.5, 4.6.1.1.6, 4.6.2.1.3, 4.6.2.1.4, 4.6.3.1.7 and 4.6.3.1.8).

01Z4 (c) How the functors $f_!$, $f^{-1}$, $f_*$, along with the functors

$$\begin{aligned} -_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) &\to \mathcal{P}(X), \\ [-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) &\to \mathcal{P}(X) \end{aligned}$$

may be viewed as a six-functor formalism with the empty set $\emptyset$ as the dualising object (Section 4.6.4).

# Contents

000K
# 4.1 Limits of Sets

000L
## 4.1.1 The Terminal Set

000M **Definition 4.1.1.1.1.** The **terminal set** is the terminal object of Sets as in Limits and Colimits, **??**.

01DB **Construction 4.1.1.1.2.** Concretely, the terminal set is the pair $(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

01DC 1. *The Limit.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.

01DD 2. *The Cone.* The collection of maps

$$\{!_A : A \to \mathrm{pt}\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_A(a) \overset{\mathrm{def}}{=} \star$$

for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

*Proof.* We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi : A \to \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. □

## 000N 4.1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

000P **Definition 4.1.2.1.1.** The **product**[1] **of** $\{A_i\}_{i\in I}$ is the product of $\{A_i\}_{i\in I}$ in $\mathsf{Sets}$ as in Limits and Colimits, **??**.

01DE **Construction 4.1.2.1.2.** Concretely, the product of $\{A_i\}_{i\in I}$ is the pair $(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I})$ consisting of:

01DF 1. *The Limit.* The set $\prod_{i\in I} A_i$ defined by

$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \,\middle|\, \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

01DG 2. *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i \right\}_{i\in I}$$

of maps given by

$$\mathrm{pr}_i(f) \overset{\text{def}}{=} f(i)$$

for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

*Proof.* We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in $\mathsf{Sets}$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. □

01DH **Remark 4.1.2.1.3.** Less formally, we may think of Cartesian products and projection maps as follows:

01DJ 1. We think of $\prod_{i \in I} A_i$ as the set whose elements are $I$-indexed collections $(a_i)_{i \in I}$ with $a_i \in A_i$ for each $i \in I$.

01DK 2. We view the projection maps

$$\left\{\mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i\right\}_{i \in I}$$

as being given by

$$\mathrm{pr}_i((a_j)_{j \in I}) \stackrel{\text{def}}{=} a_i$$

for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

000Q **Proposition 4.1.2.1.4.** Let $\{A_i\}_{i \in I}$ be a family of sets.

000R 1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i \in I}\right]((A_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I}\right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} \colon \mathrm{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \to \mathsf{Sets}\left(\prod_{i \in I} A_i, \prod_{i \in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

*Proof.* *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

## 000S 4.1.3 Binary Products of Sets

Let $A$ and $B$ be sets.

000T **Definition 4.1.3.1.1.** The **product of** $A$ **and** $B$[2] is the product of $A$ and $B$ in Sets as in Limits and Colimits, **??**.

01DL **Construction 4.1.3.1.2.** Concretely, the product of $A$ and $B$ is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

01DM 1. *The Limit.* The set $A \times B$ defined by

$$\begin{aligned} A \times B &\overset{\text{def}}{=} \prod_{z\in\{A,B\}} z \\ &\overset{\text{def}}{=} \left\{f \in \mathsf{Sets}(\{0,1\}, A\cup B) \,\middle|\, \text{we have } f(0) \in A \text{ and } f(1) \in B\right\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A\cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\ &\cong \left\{\begin{matrix}\text{ordered pairs } (a,b) \text{ with} \\ a \in A \text{ and } b \in B\end{matrix}\right\}. \end{aligned}$$

[2] *Further Terminology:* Also called the **Cartesian product of** $A$ **and** $B$.

01DN 2. *The Cone.* The maps

$$\begin{aligned}\mathrm{pr}_1 &\colon A \times B \to A,\\ \mathrm{pr}_2 &\colon A \times B \to B\end{aligned}$$

defined by

$$\begin{aligned}\mathrm{pr}_1(a,b) &\overset{\mathrm{def}}{=} a,\\ \mathrm{pr}_2(a,b) &\overset{\mathrm{def}}{=} b\end{aligned}$$

for each $(a,b) \in A \times B$.

*Proof.* We claim that $A \times B$ is the categorical product of $A$ and $B$ in the category of sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon P \to A \times B$ making the diagram

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

000U **Proposition 4.1.3.1.3.** Let $A$, $B$, $C$, and $X$ be sets.

000V 1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times - &\colon \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\ - \times B &\colon \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\ -_1 \times -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

  of $\times$ at $((A, B), (X, Y))$ is defined by sending $(f, g)$ to the function

$$f \times g \colon A \times B \to X \times Y$$

  defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

  for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

000W 2. *Adjointness I.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

01Z5 3. *Adjointness II.* We have an adjunction

$$(\Delta_{\mathsf{Sets}} \dashv -_1 \times -_2)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\Delta_{\mathsf{Sets}}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \times \mathsf{Sets}, \arrow[l, bend left, "-_1\times -_2"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}\times\mathsf{Sets}}((A, A), (B, C)) \cong \mathsf{Sets}(A, B \times C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$.

000X 4. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets}}_{A,B,C}\colon (A \times B) \times C \xrightarrow{\sim} A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000Y 5. *Unitality.* We have isomorphisms of sets

$$\lambda^{\mathsf{Sets}}_{A}\colon \mathrm{pt} \times A \xrightarrow{\sim} A,$$
$$\rho^{\mathsf{Sets}}_{A}\colon A \times \mathrm{pt} \xrightarrow{\sim} A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

000Z 6. *Commutativity.* We have an isomorphism of sets

$$\sigma^{\mathsf{Sets}}_{A,B}\colon A \times B \xrightarrow{\sim} B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

01DP 7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\delta^{\mathsf{Sets}}_{\ell}\colon A \times \left(B \coprod C\right) \xrightarrow{\sim} (A \times B) \coprod (A \times C),$$
$$\delta^{\mathsf{Sets}}_{r}\colon \left(A \coprod B\right) \times C \xrightarrow{\sim} (A \times C) \coprod (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

0010 8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\zeta_\ell^{\mathsf{Sets}} \colon \emptyset \times A \xrightarrow{\sim} \emptyset,$$
$$\zeta_r^{\mathsf{Sets}} \colon A \times \emptyset \xrightarrow{\sim} \emptyset,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

0011 9. *Distributivity Over Unions.* Let $X$ be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cup W) = (U \times V) \cup (U \times W),$$
$$(U \cup V) \times W = (U \times W) \cup (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

0012 10. *Distributivity Over Intersections.* Let $X$ be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cap W) = (U \times V) \cap (U \times W),$$
$$(U \cap V) \times W = (U \times W) \cap (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

0014 11. *Distributivity Over Differences.* Let $X$ be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \setminus W) = (U \times V) \setminus (U \times W),$$
$$(U \setminus V) \times W = (U \times W) \setminus (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

0015 12. *Distributivity Over Symmetric Differences.* Let $X$ be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \mathbin{\triangle} W) = (U \times V) \mathbin{\triangle} (U \times W),$$
$$(U \mathbin{\triangle} V) \times W = (U \times W) \mathbin{\triangle} (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

0013 13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$\begin{tikzcd}
(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[r, "\cap\times\cap"] \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[r, "\cap"'] & \mathcal{P}(X \times X)
\end{tikzcd}$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).$$

for each $U, V, W, T \in \mathcal{P}(X)$.

0016 14. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}, \times, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\right)$ is a closed symmetric monoidal category.

0017 15. *Symmetric Bimonoidality.* The 18-tuple

$$\left(\begin{gathered}\mathsf{Sets}, \coprod, \times, \varnothing, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}, \\ \alpha^{\mathsf{Sets},\coprod}, \lambda^{\mathsf{Sets},\coprod}, \rho^{\mathsf{Sets},\coprod}, \sigma^{\mathsf{Sets},\coprod}, \delta^{\mathsf{Sets}}_{\ell}, \delta^{\mathsf{Sets}}_{r}, \zeta^{\mathsf{Sets}}_{\ell}, \zeta^{\mathsf{Sets}}_{r}\end{gathered}\right),$$

is a symmetric closed bimonoidal category, where $\alpha^{\mathsf{Sets},\coprod}$, $\lambda^{\mathsf{Sets},\coprod}$, $\rho^{\mathsf{Sets},\coprod}$, and $\sigma^{\mathsf{Sets},\coprod}$ are the natural transformations from Items 3 to 5 of Definition 4.2.3.1.3.

*Proof.* *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**.
*Item 2*, *Adjointness*: We prove only that there's an adjunction $- \times B \dashv \mathsf{Sets}(B, -)$, witnessed by a bijection

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathsf{Sets}(A, -)$ follows almost exactly in the same way.

01Z6 • *Map I.* We define a map

$$\Phi_{B,C} \colon \mathsf{Sets}(A \times B, C) \to \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

by sending a function

$$\xi \colon A \times B \to C$$

to the function

$$\begin{aligned} \xi^{\dagger} \colon A &\longrightarrow \mathsf{Sets}(B, C), \\ a &\longmapsto (\xi^{\dagger}_{a} \colon B \to C), \end{aligned}$$

where we define

$$\xi^{\dagger}_{a}(b) \stackrel{\text{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$\xi^{\dagger} \stackrel{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

01Z7 • *Map II*. We define a map

$$\Psi_{B,C} \colon \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \to \mathsf{Sets}(A \times B, C)$$

given by sending a function

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}(B, C), \\ a &\longmapsto (\xi_{a} \colon B \to C), \end{aligned}$$

to the function

$$\xi^{\dagger} \colon A \times B \to C$$

defined by

$$\begin{aligned} \xi^{\dagger}(a, b) &\stackrel{\text{def}}{=} \mathrm{ev}_{b}(\mathrm{ev}_{a}(\xi)) \\ &\stackrel{\text{def}}{=} \mathrm{ev}_{b}(\xi_{a}) \\ &\stackrel{\text{def}}{=} \xi_{a}(b) \end{aligned}$$

for each $(a, b) \in A \times B$.

01Z8 • *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A \times B, C)}.$$

Indeed, given a function $\xi\colon A \times B \to C$, we have

$$\begin{aligned}
[\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\
&= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b') \rrbracket \rrbracket) \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b') \rrbracket \rrbracket)) \rrbracket \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\llbracket b' \mapsto \xi(a,b') \rrbracket) \rrbracket \\
&= \llbracket (a,b) \mapsto \xi(a,b) \rrbracket \\
&= \xi.
\end{aligned}$$

01Z9 • *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A, \mathsf{Sets}(B,C))}.$$

Indeed, given a function

$$\begin{aligned}
\xi\colon A &\longrightarrow \mathsf{Sets}(B,C), \\
a &\longmapsto (\xi_a\colon B \to C),
\end{aligned}$$

we have

$$\begin{aligned}
[\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a,b) \mapsto \xi_a(b) \rrbracket) \\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \mathrm{ev}_{(a,b)}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\
&\overset{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

01ZA • *Naturality for $\Phi$, Part I.* We need to show that, given a function $g\colon B \to$

$B'$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A \times B', C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A \times g^*"'] & \mathsf{Sets}(A, \mathsf{Sets}(B', C)), \arrow[d, "(g^*)_!"] \\
\mathsf{Sets}(A \times B, C) \arrow[r, "\Phi_{B,C}"'] & \mathsf{Sets}(A, \mathsf{Sets}(B, C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A \times B' \to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}(\xi(-_1, g(-_2))) \\
&= [\xi(-_1, g(-_2))]^{\dagger} \\
&= \xi^{\dagger}_{-_1}(g(-_2)) \\
&= (g^*)_!(\xi^{\dagger}) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)_! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

Alternatively, using the $[\![a \mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}([\mathrm{id}_A \times g^*]([\![(a, b') \mapsto \xi(a, b')]\!])) \\
&= \Phi_{B,C}([\![(a, b) \mapsto \xi(a, g(b))]\!]) \\
&= [\![a \mapsto [\![b \mapsto \xi(a, g(b))]\!]]\!] \\
&= [\![a \mapsto g^*([\![b' \mapsto \xi(a, b')]\!])]\!] \\
&= (g^*)_!([\![a \mapsto [\![b' \mapsto \xi(a, b')]\!]]\!]) \\
&= (g^*)_!(\Phi_{B',C}([\![(a, b') \mapsto \xi(a, b')]\!])) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)_! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

01ZB • *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C \to C'$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A \times B, C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_!"'] & \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \arrow[d, "(h_!)_!"] \\
\mathsf{Sets}(A \times B, C') \arrow[r, "\Phi_{B,C'}"'] & \mathsf{Sets}(A, \mathsf{Sets}(B, C'))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A \times B \to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ h_!](\xi) &= \Phi_{B,C}(h_!(\xi)) \\
&= \Phi_{B,C}(h_!(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= \Phi_{B,C}(\llbracket (a,b) \mapsto h(\xi(a,b)) \rrbracket) \\
&= \llbracket a \mapsto \llbracket b \mapsto h(\xi(a,b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto h_!(\llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= (h_!)_!(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= (h_!)_!(\Phi_{B,C}(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= (h_!)_!(\Phi_{B,C}(\xi)) \\
&= [(h_!)_! \circ \Phi_{B,C}](\xi).
\end{aligned}$$

01ZC • *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.
*Item 3, Adjointness II*: This follows from the universal property of the product.
*Item 4, Associativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.4.1.1.
*Item 5, Unitality*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.1.5.1.1 and 5.1.6.1.1.
*Item 6, Commutativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.7.1.1.

*Item 7, Distributivity Over Coproducts*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.1.1.1 and 5.3.2.1.1.
*Item 8, Annihilation With the Empty Set*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.3.1.1 and 5.3.4.1.1.
*Item 9, Distributivity Over Unions*: See [Pro25c].
*Item 10, Distributivity Over Intersections*: See [Pro25d, Corollary 1].
*Item 11, Distributivity Over Differences*: See [Pro25a].
*Item 12, Distributivity Over Symmetric Differences*: See [Pro25b].
*Item 13, Middle-Four Exchange With Respect to Intersections*: See [Pro25d, Corollary 1].
*Item 14, Symmetric Monoidality*: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.1.9.1.1, and is proved there.
*Item 15, Symmetric Bimonoidality*: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.3.5.1.1, and is proved there. □

01DR **Remark 4.1.3.1.4.** As shown in Item 1 of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

$$-_1 \times -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}.$$

This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k,\ell} \colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets})$$

of tensor products of $\mathbb{E}_k$-monoid objects on $\mathsf{Sets}$ with $\mathbb{E}_\ell$-monoid objects on $\mathsf{Sets}$; see **??**.

01DS **Remark 4.1.3.1.5.** We may state the equalities in Items 9 to 12 of Definition 4.1.3.1.3 as the commutativity of the following diagrams:

$$\begin{tikzcd}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[dl, "\cup"] \\
 & \mathcal{P}(X \times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \times \Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[dl, "\cup"] \\
 & \mathcal{P}(X \times X) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cap"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cap"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\setminus"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\setminus"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\setminus\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\setminus"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\triangle"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\triangle"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_4"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\triangle\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\triangle"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

## 0018 4.1.4 Pullbacks

Let $A$, $B$, and $C$ be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

0019 **Definition 4.1.4.1.1.** The **pullback of** $A$ **and** $B$ **over** $C$ **along** $f$ **and** $g$[3] is the pullback of $A$ and $B$ over $C$ along $f$ and $g$ in Sets as in Limits and Colimits, **??**.

01DT **Construction 4.1.4.1.2.** Concretely, the pullback of $A$ and $B$ over $C$ along $f$ and $g$ is the pair $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

[3] *Further Terminology:* Also called the **fibre product of** $A$ **and** $B$ **over** $C$ **along** $f$ **and** $g$.

01DU 1. *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \left\{(a, b) \in A \times B \,\middle|\, f(a) = g(b)\right\}.$$

01DV 2. *The Cone.* The maps[4]

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times_C B \to A, \\ \mathrm{pr}_2 &\colon A \times_C B \to B \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a, \\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b \end{aligned}$$

for each $(a, b) \in A \times_C B$.

*Proof.* We claim that $A \times_C B$ is the categorical pullback of $A$ and $B$ over $C$ with respect to $(f, g)$ in $\mathsf{Sets}$. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\mathrm{pr}_2(a, b)) \\ &= [g \circ \mathrm{pr}_2](a, b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies

[4] *Further Notation:* Also written $\mathrm{pr}_1^{A \times_C B}$ and $\mathrm{pr}_2^{A \times_C B}$.

the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon P \to A \times_C B$ making the diagram

$$\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

01DW **Remark 4.1.4.1.3.** It is common practice to write $A \times_C B$ for the pullback of $A$ and $B$ over $C$ along $f$ and $g$, omitting the maps $f$ and $g$ from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \times_C B$ depends very much on the maps $f$ and $g$, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

001A **Example 4.1.4.1.4.** Here are some examples of pullbacks of sets.

001B 1. *Unions via Intersections.* Let $X$ be a set. We have

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook, "\iota_B"] \\ A \arrow[r, hook, "\iota_A"'] & A \cup B \end{tikzcd}$$

for each $A, B \in \mathcal{P}(X)$.

*Proof.* *Item 1, Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \left\{ (x, y) \in A \times B \;\middle|\; x = y \right\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

001C **Proposition 4.1.4.1.5.** Let $A$, $B$, $C$, and $X$ be sets.

001D 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

commute.

01DX 2. *Adjointness I.* We have adjunctions

$$\left(A\times_X - \dashv \mathbf{Sets}_{/X}(A,-)\right)\colon\quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "A\times_X -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X} \arrow[l, bend left, "\mathbf{Sets}_{/X}(A,-)"] \end{tikzcd},$$

$$\left(-\times_X B \dashv \mathbf{Sets}_{/X}(B,-)\right)\colon\quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "-\times_X B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X} \arrow[l, bend left, "\mathbf{Sets}_{/X}(B,-)"] \end{tikzcd},$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}_{/X}(A\times_X B, C) &\cong \mathsf{Sets}_{/X}(A, \mathbf{Sets}_{/X}(B,C)),\\ \mathsf{Sets}_{/X}(A\times_X B, C) &\cong \mathsf{Sets}_{/X}(B, \mathbf{Sets}_{/X}(A,C)), \end{aligned}$$

natural in $(A,\phi_A), (B,\phi_B), (C,\phi_C) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$, where $\mathbf{Sets}_{/X}(A,B)$ is the object of $\mathsf{Sets}_{/X}$ consisting of (see Fibred Sets, **??**):

- *The Set.* The set $\mathbf{Sets}_{/X}(A,B)$ defined by

$$\mathbf{Sets}_{/X}(A,B) \stackrel{\text{def}}{=} \coprod_{x\in X} \mathsf{Sets}(\phi_A^{-1}(x), \phi_Y^{-1}(x))$$

- *The Map to $X$.* The map

$$\phi_{\mathbf{Sets}_{/X}(A,B)}\colon \mathbf{Sets}_{/X}(A,B) \longrightarrow X$$

  defined by

$$\phi_{\mathbf{Sets}_{/X}(A,B)}(x,f) \stackrel{\text{def}}{=} x$$

  for each $(x,f) \in \mathbf{Sets}_{/X}(A,B)$.

01ZD 3. *Adjointness II.* We have an adjunction

$$\left(\Delta_{\mathsf{Sets}_{/X}} \dashv -_1 \times -_2\right)\colon\quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "\Delta_{\mathsf{Sets}_{/X}}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X}\times\mathsf{Sets}_{/X} \arrow[l, bend left, "-_1\times -_2"] \end{tikzcd},$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_{/X}\times\mathsf{Sets}_{/X}}((A,A),(B,C)) \cong \mathsf{Sets}_{/X}(A, B\times_X C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets}_{/X})$ and in $(B,C) \in \mathrm{Obj}(\mathsf{Sets}_{/X}\times\mathsf{Sets}_{/X})$.

001E 4. *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in $\mathsf{Sets}$, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\vee", very near start] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & & \\
& & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

01DY 5. *Interaction With Composition.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "\phi"'] & & & & Y \arrow[dl, "\psi"] \\
& A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
& & K & &
\end{tikzcd}$$

in $\mathsf{Sets}$, we have isomorphisms of sets

$$\begin{aligned}
X \times_K^{f\circ\phi, g\circ\psi} Y &\cong (X \times_A^{\phi,q_1} (A \times_K^{f,g} B)) \times^{p_2,p_1}_{A\times_K^{f,g} B} ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y) \\
&\cong X \times_A^{\phi,p} ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y) \\
&\cong (X \times_A^{\phi,q_1} (A \times_K^{f,g} B)) \times_B^{q,\psi} Y
\end{aligned}$$

where

$$\begin{aligned}
q_1 &= \mathrm{pr}_1^{A\times_K^{f,g} B}, & q_2 &= \mathrm{pr}_2^{A\times_K^{f,g} B}, \\
p_1 &= \mathrm{pr}_1^{(A\times_K^{f,g} B)\times_Y^{q_2,\psi}}, & p_2 &= \mathrm{pr}_2^{X\times^{\phi,q_1}_{A\times_K^{f,g} B}(A\times_K^{f,g} B)}, \\
p &= q_1 \circ \mathrm{pr}_1^{(A\times_K^{f,g} B)\times_B^{q_2,\psi} Y}, & q &= q_2 \circ \mathrm{pr}_2^{X\times_A^{\phi,q_1}(A\times_K^{f,g} B)},
\end{aligned}$$

and where these pullbacks are built as in the following diagrams:

$$
\begin{tikzcd}[column sep=tiny]
 & & (X \times_A (A \times_K B)) \times_{A \times_K B} ((A \times_K B) \times_B Y) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & X \times_A (A \times_K B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & (A \times_K B) \times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
X \arrow[dr, "\phi"'] & & A \times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & X \times_A ((A \times_K B) \times_B Y) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & & & (A \times_K B) \times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
X \arrow[dr, "\phi"'] & & A \times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
 & & (X \times_A (A \times_K B)) \times_B Y \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & X \times_A (A \times_K B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & & \\
X \arrow[dr, "\phi"'] & & A \times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & X \times_K Y \arrow[ddll] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & & & & \\
X \arrow[dr, "\phi"'] & & & & Y. \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

001F 6. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
\lambda^{\mathsf{Sets}_{/X}}_{A} &\colon X \times_X A \overset{\sim}{\dashrightarrow} A, \\
\rho^{\mathsf{Sets}_{/X}}_{A} &\colon A \times_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X,
\end{tikzcd}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

001G 7. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \sigma^{\mathsf{Sets}_{/X}}_{A,B} \colon A \times_X B \overset{\sim}{\dashrightarrow} B \times_X A \qquad \begin{tikzcd} B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X, \end{tikzcd}$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

01DZ 8. *Distributivity Over Coproducts.* Let $A$, $B$, and $C$ be sets and let $\phi_A \colon A \to X$, $\phi_B \colon B \to X$, and $\phi_C \colon C \to X$ be morphisms of sets. We have isomorphisms of sets

$$\delta^{\mathsf{Sets}_{/X}}_{\ell} \colon A \times_X \left(B \coprod C\right) \overset{\sim}{\dashrightarrow} (A \times_X B) \coprod (A \times_X C),$$
$$\delta^{\mathsf{Sets}_{/X}}_{r} \colon \left(A \coprod B\right) \times_X C \overset{\sim}{\dashrightarrow} (A \times_X C) \coprod (B \times_X C),$$

as in the diagrams

$$\begin{tikzcd} (A \times_X B) \coprod (A \times_X C) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \coprod C \arrow[d, "\phi_B \coprod \phi_C"] \\ A \arrow[r, "\phi_A"'] & X \end{tikzcd} \qquad \begin{tikzcd} (A \times_X C) \coprod (B \times_X C) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\phi_C"] \\ A \coprod B \arrow[r, "\phi_A \coprod \phi_B"'] & X \end{tikzcd}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

001H 9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \begin{aligned} \zeta^{\mathsf{Sets}_{/X}}_{\ell} &\colon A \times_X \emptyset \overset{\sim}{\dashrightarrow} \emptyset, \\ \zeta^{\mathsf{Sets}_{/X}}_{r} &\colon \emptyset \times_X A \overset{\sim}{\dashrightarrow} \emptyset, \end{aligned} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X, \end{tikzcd}$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

001J 10. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

001K 11. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}_{/X}, \times_X, X, \mathbf{Sets}_{/X}, \alpha^{\mathsf{Sets}_{/X}}, \lambda^{\mathsf{Sets}_{/X}}, \rho^{\mathsf{Sets}_{/X}}, \sigma^{\mathsf{Sets}_{/X}}\right)$ is a symmetric closed monoidal category.

*Proof.* *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for $\xi$ following from the commutativity of the cube pullback diagram.
*Item 2, Adjointness I*: This is a repetition of Fibred Sets, **??** of **??**, and is proved there.
*Item 3, Adjointness II*: This follows from the universal property of the product (pullbacks are products in $\mathsf{Sets}_{/X}$).
*Item 4, Associativity*: We have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \left\{((a,b),c) \in (A \times B) \times C \,\middle|\, f(a) = g(b) \text{ and } h(b) = k(c)\right\} \\ &\cong \left\{(a,(b,c)) \in A \times (B \times C) \,\middle|\, f(a) = g(b) \text{ and } h(b) = k(c)\right\} \\ &\cong \left\{(a,(b,c)) \in A \times (B \times_Y C) \,\middle|\, f(a) = g(b)\right\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned} (A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\ &\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{aligned} & f(a) = g(b), b = b', \\ & \text{and } h(b') = k(c) \end{aligned}\right\} \\ &\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{aligned} & f(a) = g(b), b = b', \\ & \text{and } h(b') = k(c) \end{aligned}\right\} \\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{aligned} & f(a) = g(b), b = b', \\ & \text{and } h(b') = k(c) \end{aligned}\right\} \\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{aligned} & f(a) = g(b) \text{ and} \\ & h(b') = k(c) \end{aligned}\right\} \\ &\cong \left\{(a,(b,c)) \in A \times (B \times C) \,\middle|\, f(a) = g(b) \text{ and } h(b) = k(c)\right\} \end{aligned}$$

$$\cong A \times_X (B \times_Y C),$$

where we have used Item 6 for the isomorphism $B \times_B B \cong B$.
*Item 5, Interaction With Composition*: By Item 4, it suffices to construct only the isomorphism

$$X \times_K^{f\circ\phi, g\circ\psi} Y \cong (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \times_{A \times_K^{f,g} B}^{p_2, p_1} ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y).$$

We have

$$\begin{aligned}
(X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) &\stackrel{\text{def}}{=} \left\{(x,(a,b)) \in X \times (A \times_K^{f,g} B) \,\middle|\, \phi(x) = q_1(a,b)\right\} \\
&\stackrel{\text{def}}{=} \left\{(x,(a,b)) \in X \times (A \times_K^{f,g} B) \,\middle|\, \phi(x) = a\right\} \\
&\cong \left\{(x,(a,b)) \in X \times (A \times B) \,\middle|\, \phi(x) = a \text{ and } f(a) = g(b)\right\}, \\
((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y) &\stackrel{\text{def}}{=} \left\{((a,b),y) \in (A \times_K^{f,g} B) \times Y \,\middle|\, q_2(a,b) = \psi(y)\right\} \\
&\stackrel{\text{def}}{=} \left\{((a,b),y) \in (A \times_K^{f,g} B) \times Y \,\middle|\, b = \psi(y)\right\} \\
&\cong \left\{((a,b),y) \in (A \times B) \times Y \,\middle|\, b = \psi(y) \text{ and } f(a) = g(b)\right\},
\end{aligned}$$

so writing

$$\begin{aligned}
S &= (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \\
S' &= ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y),
\end{aligned}$$

we have

$$\begin{aligned}
S \times_{A \times_K^{f,g} B}^{p_2,p_1} S' &\stackrel{\text{def}}{=} \left\{((x,(a,b)),((a',b'),y)) \in S \times S' \,\middle|\, p_1(x,(a,b)) = p_2((a',b'),y)\right\} \\
&\stackrel{\text{def}}{=} \left\{((x,(a,b)),((a',b'),y)) \in S \times S' \,\middle|\, (a,b) = (a',b')\right\} \\
&\cong \left\{((x,a,b,y)) \in X \times A \times B \times Y \,\middle|\, \phi(x) = a, \psi(y) = b, \text{ and } f(a) = g(b)\right\} \\
&\cong \left\{((x,a,b,y)) \in X \times A \times B \times Y \,\middle|\, f(\phi(x)) = g(\psi(y))\right\} \\
&\stackrel{\text{def}}{=} X \times_K Y.
\end{aligned}$$

This finishes the proof.
*Item 6, Unitality*: We have

$$\begin{aligned}
X \times_X A &\cong \left\{(x,a) \in X \times A \,\middle|\, f(a) = x\right\}, \\
A \times_X X &\cong \left\{(a,x) \in X \times A \,\middle|\, f(a) = x\right\},
\end{aligned}$$

which are isomorphic to $A$ via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$. The proof of the naturality of $\lambda^{\mathsf{Sets}/X}$ and $\rho^{\mathsf{Sets}/X}$ is omitted.

*Item 7, Commutativity*: We have

$$\begin{aligned} A \times_C B &\overset{\text{def}}{=} \{(a, b) \in A \times B \,|\, f(a) = g(b)\} \\ &= \{(a, b) \in A \times B \,|\, g(b) = f(a)\} \\ &\cong \{(b, a) \in B \times A \,|\, g(b) = f(a)\} \\ &\overset{\text{def}}{=} B \times_C A. \end{aligned}$$

The proof of the naturality of $\sigma^{\mathsf{Sets}/X}$ is omitted.

*Item 8, Distributivity Over Coproducts*: We have

$$\begin{aligned} A \times_X \left(B \textstyle\coprod C\right) &\overset{\text{def}}{=} \left\{(a, z) \in A \times \left(B \textstyle\coprod C\right) \,\middle|\, \phi_A(a) = \phi_{B \coprod C}(z)\right\} \\ &= \left\{(a, z) \in A \times \left(B \textstyle\coprod C\right) \,\middle|\, z = (0, b) \text{ and } \phi_A(a) = \phi_{B \coprod C}(z)\right\} \\ &\quad \cup \left\{(a, z) \in A \times \left(B \textstyle\coprod C\right) \,\middle|\, z = (1, c) \text{ and } \phi_A(a) = \phi_{B \coprod C}(z)\right\} \\ &= \left\{(a, z) \in A \times \left(B \textstyle\coprod C\right) \,\middle|\, z = (0, b) \text{ and } \phi_A(a) = \phi_B(b)\right\} \\ &\quad \cup \left\{(a, z) \in A \times \left(B \textstyle\coprod C\right) \,\middle|\, z = (1, c) \text{ and } \phi_A(a) = \phi_C(c)\right\} \\ &\cong \{(a, b) \in A \times B \,|\, \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, c) \in A \times C \,|\, \phi_A(a) = \phi_C(c)\} \\ &\overset{\text{def}}{=} (A \times_X B) \cup (A \times_X C) \\ &\cong (A \times_X B) \textstyle\coprod (A \times_X C), \end{aligned}$$

with the construction of the isomorphism

$$\delta_r^{\mathsf{Sets}/X} \colon \left(A \textstyle\coprod B\right) \times_X C \overset{\sim}{\dashrightarrow} (A \times_X C) \textstyle\coprod (B \times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\mathsf{Sets}/X}$ and $\delta_r^{\mathsf{Sets}/X}$ is omitted.

*Item 9, Annihilation With the Empty Set*: We have

$$\begin{aligned} A \times_X \varnothing &\overset{\text{def}}{=} \{(a, b) \in A \times \varnothing \,|\, f(a) = g(b)\} \\ &= \{k \in \varnothing \,|\, f(a) = g(b)\} \\ &= \varnothing, \end{aligned}$$

and similarly for $\varnothing \times_X A$, where we have used Item 8 of Definition 4.1.3.1.3. The proof of the naturality of $\zeta_\ell^{\mathsf{Sets}/X}$ and $\zeta_r^{\mathsf{Sets}/X}$ is omitted.

*Item 10*, *Interaction With Products*: We have

$$\begin{aligned} A \times_{\mathrm{pt}} B &\overset{\text{def}}{=} \{(a, b) \in A \times B \mid !_A(a) = !_B(b)\} \\ &\overset{\text{def}}{=} \{(a, b) \in A \times B \mid \star = \star\} \\ &= \{(a, b) \in A \times B\} \\ &= A \times B. \end{aligned}$$

*Item 11*, *Symmetric Monoidality*: Omitted. □

## 001L 4.1.5 Equalisers

Let $A$ and $B$ be sets and let $f, g \colon A \rightrightarrows B$ be functions.

001M **Definition 4.1.5.1.1.** The **equaliser of** $f$ **and** $g$ is the equaliser of $f$ and $g$ in $\mathsf{Sets}$ as in Limits and Colimits, **??**.

01E0 **Construction 4.1.5.1.2.** Concretely, the equaliser of $f$ and $g$ is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

01E1 1. *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \overset{\text{def}}{=} \left\{a \in A \,\middle|\, f(a) = g(a)\right\}.$$

01E2 2. *The Cone.* The inclusion map

$$\mathrm{eq}(f,g) \colon \mathrm{Eq}(f,g) \to A.$$

*Proof.* We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of $f$ and $g$ in $\mathsf{Sets}$. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g)$ making the diagram

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. □

001N **Proposition 4.1.5.1.3.** Let $A$, $B$, and $C$ be sets.

001P 1. *Associativity.* We have isomorphisms of sets[5]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

[5]That is, the following three ways of forming "the" equaliser of $(f,g,h)$ agree:

01ZE 1. Take the equaliser of $(f,g,h)$, i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in $\mathsf{Sets}$.

01ZF 2. First take the equaliser of $f$ and $g$, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\to} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\to} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

of $\mathrm{Eq}(f,g)$.

01ZG 3. First take the equaliser of $g$ and $h$, forming a diagram

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\to} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g,h) \overset{\mathrm{eq}(g,h)}{\to} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

of $\mathrm{Eq}(g,h)$.

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \underset{h}{\overset{f}{\underset{g}{\rightrightarrows}}}\!\!\rightarrow B$$

in $\mathsf{Sets}$, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

001Q 4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

001R 5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

001S 6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\rightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

*Proof.* *Item 1*, *Associativity*: We first prove that $\mathrm{Eq}(f, g, h)$ is indeed given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f, g, h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g,h)$.

We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) \cong \mathrm{Eq}(f,g,h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) &\cong \left\{x \in \mathrm{Eq}(g,h) \,\middle|\, [f \circ \mathrm{eq}(g,h)](a) = [g \circ \mathrm{eq}(g,h)](a)\right\} \\
&\cong \left\{x \in \mathrm{Eq}(g,h) \,\middle|\, f(a) = g(a)\right\} \\
&\cong \left\{x \in A \,\middle|\, f(a) = g(a) \text{ and } g(a) = h(a)\right\} \\
&\cong \left\{x \in A \,\middle|\, f(a) = g(a) = h(a)\right\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) &\cong \left\{x \in \mathrm{Eq}(f,g) \,\middle|\, [f \circ \mathrm{eq}(f,g)](a) = [h \circ \mathrm{eq}(f,g)](a)\right\} \\
&\cong \left\{x \in \mathrm{Eq}(f,g) \,\middle|\, f(a) = h(a)\right\} \\
&\cong \left\{x \in A \,\middle|\, f(a) = h(a) \text{ and } f(a) = g(a)\right\} \\
&\cong \left\{x \in A \,\middle|\, f(a) = g(a) = h(a)\right\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

*Item 4*, *Unitality*: Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f,f) &\overset{\text{def}}{=} \left\{a \in A \,\middle|\, f(a) = f(a)\right\} \\
&= A.
\end{aligned}$$

*Item 5*, *Commutativity*: Indeed, we have

$$\mathrm{Eq}(f,g) \overset{\text{def}}{=} \left\{a \in A \,\middle|\, f(a) = g(a)\right\}$$

$$\begin{aligned} &= \left\{a \in A \,\middle|\, g(a) = f(a)\right\} \\ &\stackrel{\text{def}}{=} \mathrm{Eq}(g, f). \end{aligned}$$

*Item 6, Interaction With Composition*: Indeed, we have

$$\begin{aligned} \mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) &\cong \left\{a \in \mathrm{Eq}(f,g) \,\middle|\, h(f(a)) = k(g(a))\right\} \\ &\cong \left\{a \in A \,\middle|\, f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\right\}. \end{aligned}$$

and

$$\mathrm{Eq}(h \circ f, k \circ g) \cong \left\{a \in A \,\middle|\, h(f(a)) = k(g(a))\right\},$$

and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$. □

### 01E3 4.1.6 Inverse Limits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \longrightarrow \mathsf{Sets}$ be an inverse system of sets.

01E4 **Definition 4.1.6.1.1.** The **inverse limit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in $\mathsf{Sets}$ as in Limits and Colimits, **??**.

01E5 **Construction 4.1.6.1.2.** Concretely, the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the pair $\left(\varprojlim_{\alpha \in I}(X_\alpha), \{\mathrm{pr}_\alpha\}_{\alpha \in I}\right)$ consisting of:

01E6 1. *The Limit.* The set $\varprojlim_{\alpha \in I}(X_\alpha)$ defined by

$$\varprojlim_{\alpha \in I}(X_\alpha) \stackrel{\text{def}}{=} \left\{(x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \,\middle|\, \begin{array}{l}\text{for each } \alpha, \beta \in I, \text{ if } \alpha \preceq \beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta)\end{array}\right\}.$$

01E7 2. *The Cone.* The collection

$$\left\{\mathrm{pr}_\gamma \colon \varprojlim_{\alpha \in I}(X_\alpha) \to X_\gamma\right\}_{\gamma \in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{\mathrm{pr}_\gamma \colon \prod_{\alpha \in I} X_\alpha \to X_\gamma\right\}_{\gamma \in I}$$

of Item 2 of Definition 4.1.2.1.2 to $\lim\limits_{\overleftarrow{\alpha\in I}}(X_\alpha)$ and hence given by

$$\mathrm{pr}_\gamma((x_\alpha)_{\alpha\in I}) \overset{\text{def}}{=} x_\gamma$$

for each $\gamma \in I$ and each $(x_\alpha)_{\alpha\in I} \in \lim\limits_{\overleftarrow{\alpha\in I}}(X_\alpha)$.

*Proof.* We claim that $\lim_{\overleftarrow{\alpha\in I}}(X_\alpha)$ is the limit of the inverse system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$. First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta}\circ \mathrm{pr}_\alpha = \mathrm{pr}_\beta, \qquad \begin{tikzcd} & \lim\limits_{\overleftarrow{\alpha\in I}}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\ X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \end{tikzcd}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma\in I} \in \lim_{\overleftarrow{\gamma\in I}}(X_\gamma)$, we have

$$\begin{aligned}
[f_{\alpha\beta}\circ \mathrm{pr}_\alpha]((x_\gamma)_{\gamma\in I}) &\overset{\text{def}}{=} f_{\alpha\beta}(\mathrm{pr}_\alpha((x_\gamma)_{\gamma\in I}))\\
&\overset{\text{def}}{=} f_{\alpha\beta}(x_\alpha)\\
&= x_\beta\\
&\overset{\text{def}}{=} \mathrm{pr}_\beta((x_\gamma)_{\gamma\in I}),
\end{aligned}$$

where the third equality comes from the definition of $\lim_{\overleftarrow{\alpha\in I}}(X_\alpha)$. Next, we prove that $\lim_{\overleftarrow{\alpha\in I}}(X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{tikzcd} & L \arrow[ddl, bend right, "p_\alpha"'] \arrow[ddr, bend left, "p_\beta"] & \\ & \lim\limits_{\overleftarrow{\alpha\in I}}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\ X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \end{tikzcd}$$

in Sets. Then there indeed exists a unique map $\phi\colon L \xrightarrow{\exists!} \varprojlim_{\alpha\in I}(X_\alpha)$ making the diagram

$$\begin{tikzcd}
 & L \arrow[d, dashed, "\phi"', "\exists!"] \arrow[ddl, bend right, "p_\alpha"'] \arrow[ddr, bend left, "p_\beta"] & \\
 & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\
X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta
\end{tikzcd}$$

commute, being uniquely determined by the family of conditions

$$\left\{p_\alpha = \mathrm{pr}_\alpha \circ \phi\right\}_{\alpha\in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha\in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha$ indeed lies in $\varprojlim_{\alpha\in I}(X_\alpha)$, as we have

$$\begin{aligned} f_{\alpha\beta}(p_\alpha(\ell)) &\overset{\text{def}}{=} [f_{\alpha\beta} \circ p_\alpha](\ell) \\ &\overset{\text{def}}{=} p_\beta(\ell) \end{aligned}$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $(L, \left\{p_\alpha\right\}_{\alpha\in I})$. □

01E8 **Example 4.1.6.1.3.** Here are some examples of inverse limits of sets.

01E9 1. *The p-Adic Integers.* The ring of $p$-adic integers $\mathbb{Z}_p$ of **??** is the inverse limit

$$\mathbb{Z}_p \cong \varprojlim_{n\in\mathbb{N}}(\mathbb{Z}_{/p^n});$$

see **??**.

01EA 2. *Rings of Formal Power Series.* The ring $R[\![t]\!]$ of formal power series in a variable $t$ is the inverse limit

$$R[\![t]\!] \cong \varprojlim_{n\in\mathbb{N}}(R[t]/t^n R[t]);$$

see **??**.

01EB 3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see **??**.

## 001T 4.2 Colimits of Sets

### 001U 4.2.1 The Initial Set

001V **Definition 4.2.1.1.1.** The **initial set** is the initial object of Sets as in Limits and Colimits, **??**.

01EC **Construction 4.2.1.1.2.** Concretely, the initial set is the pair $(\emptyset, \{\iota_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

01ED 1. *The Colimit.* The empty set $\emptyset$ of Definition 4.3.1.1.1.

01EE 2. *The Cocone.* The collection of maps

$$\{\iota_A \colon \emptyset \to A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

given by the inclusion maps from $\emptyset$ to $A$.

*Proof.* We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi\colon \emptyset \to A$ making the diagram

$$\begin{tikzcd} \emptyset \arrow[r, dashed, "\phi", "\exists!"'] & A \end{tikzcd}$$

commute, namely the inclusion map $\iota_A$. □

### 001W 4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

001X **Definition 4.2.2.1.1.** The **coproduct of** $\{A_i\}_{i\in I}$[6] is the coproduct of $\{A_i\}_{i\in I}$ in Sets as in Limits and Colimits, **??**.

[6] *Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i\in I}$.

01EF **Construction 4.2.2.1.2.** Concretely, the disjoint union of $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of:

01EG 1. *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \overset{\mathrm{def}}{=} \left\{ (i,x) \in I \times \left( \bigcup_{i\in I} A_i \right) \;\middle|\; x \in A_i \right\}.$$

01EH 2. *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i \right\}_{i\in I}$$

of maps given by

$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i,x)$$

for each $x \in A_i$ and each $i \in I$.

*Proof.* We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in $\mathsf{Sets}$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi \colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i,x)) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$. $\square$

001Y **Proposition 4.2.2.1.3.** Let $\{A_i\}_{i\in I}$ be a family of sets.

001Z 1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have
$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$
- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets
$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$
of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map
$$\{f_i \colon A_i \to B_i\}_{i\in I}$$
in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets
$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$
defined by
$$\left[\coprod_{i\in I} f_i\right](i, a) \stackrel{\text{def}}{=} f_i(a)$$
for each $(i, a) \in \coprod_{i\in I} A_i$.

*Proof.* *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

### 0020 4.2.3 Binary Coproducts

Let $A$ and $B$ be sets.

0021 **Definition 4.2.3.1.1.** The **coproduct of** $A$ **and** $B$[7] is the coproduct of $A$ and $B$ in $\mathsf{Sets}$ as in Limits and Colimits, **??**.

01EJ **Construction 4.2.3.1.2.** Concretely, the coproduct of $A$ and $B$ is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

01EK 1. *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\overset{\mathrm{def}}{=} \coprod_{z \in \{A,B\}} z \\ &\overset{\mathrm{def}}{=} \{(0, a) \in S \mid a \in A\} \cup \{(1, b) \in S \mid b \in B\}, \end{aligned}$$

   where $S = \{0, 1\} \times (A \cup B)$.

01EL 2. *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &\colon A \to A \coprod B, \\ \mathrm{inj}_2 &\colon B \to A \coprod B, \end{aligned}$$

   given by

$$\begin{aligned} \mathrm{inj}_1(a) &\overset{\mathrm{def}}{=} (0, a), \\ \mathrm{inj}_2(b) &\overset{\mathrm{def}}{=} (1, b), \end{aligned}$$

   for each $a \in A$ and each $b \in B$.

*Proof.* We claim that $A \coprod B$ is the categorical coproduct of $A$ and $B$ in $\mathsf{Sets}$. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & A \coprod B & B \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"'] \end{tikzcd}$$

[7] *Further Terminology:* Also called the **disjoint union of** $A$ **and** $B$.

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_A = \iota_A,$$
$$\phi \circ \mathrm{inj}_B = \iota_B$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \coprod B$. $\square$

0022 **Proposition 4.2.3.1.3.** Let $A$, $B$, $C$, and $X$ be sets.

0023 1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod - \colon & \quad \mathsf{Sets} && \to \mathsf{Sets}, \\ - \coprod B \colon & \quad \mathsf{Sets} && \to \mathsf{Sets}, \\ -_1 \coprod -_2 \colon & \mathsf{Sets} \times \mathsf{Sets} && \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending $(f, g)$ to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \coprod B$.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

01ZH 2. *Adjointness.* We have an adjunction

$$(-_1 \coprod -_2 \dashv \Delta_{\mathsf{Sets}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \times \mathsf{Sets} \arrow[r, bend left, "-_1 \coprod -_2"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\Delta_{\mathsf{Sets}}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}(A \coprod B, C), \cong \mathrm{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A, B), (C, C))$$

natural in $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$ and in $C \in \mathrm{Obj}(\mathsf{Sets})$.

0024 3. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets}, \coprod}_{X,Y,Z} \colon (X \coprod Y) \coprod Z \overset{\sim}{\dashrightarrow} X \coprod (Y \coprod Z),$$

natural in $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets})$.

0025 4. *Unitality.* We have isomorphisms of sets

$$\lambda^{\mathsf{Sets}, \coprod}_{X} \colon \emptyset \coprod X \overset{\sim}{\dashrightarrow} X,$$
$$\rho^{\mathsf{Sets}, \coprod}_{X} \colon X \coprod \emptyset \overset{\sim}{\dashrightarrow} X,$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

0026 5. *Commutativity.* We have an isomorphism of sets

$$\sigma^{\mathsf{Sets}, \coprod}_{X,Y} \colon X \coprod Y \overset{\sim}{\dashrightarrow} Y \coprod X,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

0027 6. *Symmetric Monoidality.* The 7-tuple $\left(\mathsf{Sets}, \coprod, \emptyset, \alpha^{\mathsf{Sets}}_{\coprod}, \lambda^{\mathsf{Sets}}_{\coprod}, \rho^{\mathsf{Sets}}_{\coprod}, \sigma^{\mathsf{Sets}}\right)$ is a symmetric monoidal category.

*Proof.* *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.
*Item 2, Adjointness*: This follows from the universal property of the coproduct.
*Item 3, Associativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.3.1.1.
*Item 4, Unitality*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.2.4.1.1 and 5.2.5.1.1.
*Item 5, Commutativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.6.1.1.
*Item 6, Symmetric Monoidality*: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.2.7.1.1, and is proved there. □

## 0028 4.2.4 Pushouts

Let $A$, $B$, and $C$ be sets and let $f: C \to A$ and $g: C \to B$ be functions.

0029 **Definition 4.2.4.1.1.** The **pushout of $A$ and $B$ over $C$ along $f$ and $g$**[8] is the pushout of $A$ and $B$ over $C$ along $f$ and $g$ in $\mathsf{Sets}$ as in Limits and Colimits, **??**.

01EM **Construction 4.2.4.1.2.** Concretely, the pushout of $A$ and $B$ over $C$ along $f$ and $g$ is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

01EN 1. *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

01EP 2. *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &: A \to A \textstyle\coprod_C B, \\ \mathrm{inj}_2 &: B \to A \textstyle\coprod_C B \end{aligned}$$

given by

$$\mathrm{inj}_1(a) \stackrel{\text{def}}{=} [(0, a)]$$

[8] *Further Terminology:* Also called the **fibre coproduct of $A$ and $B$ over $C$ along $f$ and $g$**.

$$\mathrm{inj}_2(b) \stackrel{\mathrm{def}}{=} [(1, b)]$$

for each $a \in A$ and each $b \in B$.

*Proof.* We claim that $A \coprod_C B$ is the categorical pushout of $A$ and $B$ over $C$ with respect to $(f, g)$ in $\mathsf{Sets}$. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned} [\mathrm{inj}_1 \circ f](c) &= \mathrm{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathrm{inj}_2(g(c)) \\ &= [\mathrm{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} P & & \\ & A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in $\mathsf{Sets}$. Then there exists a unique map $\phi\colon A\coprod_C B\to P$ making the diagram

$$\begin{tikzcd}
P & & \\
& A\coprod_C B \arrow[ul, dashed, "\phi", "\exists!"'] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi\circ\mathrm{inj}_1&=\iota_1,\\ \phi\circ\mathrm{inj}_2&=\iota_2\end{aligned}$$

via

$$\phi(x)=\begin{cases}\iota_1(a) & \text{if } x=[(0,a)],\\ \iota_2(b) & \text{if } x=[(1,b)]\end{cases}$$

for each $x\in A\coprod_C B$, where the well-definedness of $\phi$ is guaranteed by the equality $\iota_1\circ f=\iota_2\circ g$ and the definition of the relation $\sim$ on $A\coprod B$ as follows:

01EQ 1. *Case 1:* Suppose we have $x=[(0,a)]=[(0,a')]$ for some $a,a'\in A$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0,a)\sim' x_1\sim'\cdots\sim' x_n\sim'(0,a').$$

01ER 2. *Case 2:* Suppose we have $x=[(1,b)]=[(1,b')]$ for some $b,b'\in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(1,b)\sim' x_1\sim'\cdots\sim' x_n\sim'(1,b').$$

01ES 3. *Case 3:* Suppose we have $x=[(0,a)]=[(1,b)]$ for some $a\in A$ and $b\in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0,a)\sim' x_1\sim'\cdots\sim' x_n\sim'(1,b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned}\phi([x]) &= \phi([(0, f(c))]) \\ &\stackrel{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]),\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)\end{aligned}$$

gives

$$\begin{aligned}\phi([(0, a)]) = \phi([(0, a')]), \\ \phi([(1, b)]) = \phi([(1, b')]), \\ \phi([(0, a)]) = \phi([(1, b)]),\end{aligned}$$

showing $\phi$ to be well-defined. □

002A **Remark 4.2.4.1.3.** In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

01ET 1. We have $a, b \in A$ and $a = b$.

01EU 2. We have $a, b \in B$ and $a = b$.

01EV 3. There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

01EW (a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.

01EX (b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

In other words, there exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

01EY (c) There exists $c_0 \in C$ satisfying one of the following conditions:

01ZJ i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.

01ZK ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.

01EZ (d) For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:

01ZL i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.

01ZM ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.

01F0 (e) There exists $c_n \in C$ satisfying one of the following conditions:

01F1 i. We have $x_n = f(c_n)$ and $b = g(c_n)$.

01F2 ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

01ZN **Remark 4.2.4.1.4.** It is common practice to write $A \coprod_C B$ for the pushout of $A$ and $B$ over $C$ along $f$ and $g$, omitting the maps $f$ and $g$ from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \coprod_C B$ depends very much on the maps $f$ and $g$, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \coprod_{f,C,g} B$ or $A \coprod_C^{f,g} B$ for $A \coprod_C B$.

002B **Example 4.2.4.1.5.** Here are some examples of pushouts of sets.

002C 1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 6.3.3.1.1 is an example of a pushout of sets.

002D 2. *Intersections via Unions.* Let $X$ be a set. We have

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B & B \arrow[l] \\ A \arrow[u] & A \cap B \arrow[l, hook'] \arrow[u, hook] \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

for each $A, B \in \mathcal{P}(X)$.

*Proof.* *Item 1, Wedge Sums of Pointed Sets*: This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.
*Item 2, Intersections via Unions*: Indeed, $A \coprod_{A\cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. □

002E **Proposition 4.2.4.1.6.** Let $A$, $B$, $C$, and $X$ be sets.

002F 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd} A \coprod_C B & & B \arrow[ll] \arrow[dr, "\psi"] & \\ & A' \coprod_{C'} B' & & B' \arrow[ll] \\ A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\ & A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"'] \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \overset{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

01ZP 2. *Adjointness.* We have an adjunction

$$\left(-_1 \coprod{}_X -_2 \dashv \Delta_{\mathsf{Sets}_{X/}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/} \arrow[r, bend left, "-_1 \coprod_X -_2"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{X/}, \arrow[l, bend left, "\Delta_{\mathsf{Sets}_{X/}}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_{X/}(A \coprod{}_X B, C), \cong \mathrm{Hom}_{\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/}}((A, B), (C, C))$$

natural in $(A, B) \in \mathrm{Obj}(\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/})$ and in $C \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

002G 3. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in $\mathsf{Sets}$, we have isomorphisms of sets

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C)$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=small]
 & & (A \coprod_X B) \coprod_Y C & & \\
 & A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
 & A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & & A \coprod_X (B \coprod_Y C) & \\
 & & & & B \coprod_Y C \arrow[ul] \\
A \arrow[uurrr] & & B \arrow[urr] & & C. \arrow[u] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

01F4 4. *Interaction With Composition.* Given a diagram

$$
\begin{tikzcd}
X & & & & Y \\
 & A \arrow[ul, "\phi"] & & B \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] & &
\end{tikzcd}
$$

in Sets, we have isomorphisms of sets

$$
\begin{aligned}
X \coprod\nolimits_K^{\phi\circ f,\psi\circ g} Y &\cong (X \coprod\nolimits_A^{\phi,j_1} (A \coprod\nolimits_K^{f,g} B)) \coprod\nolimits_{A\coprod_K^{f,g} B}^{i_2,i_1} ((A \coprod\nolimits_K^{f,g} B) \coprod\nolimits_B^{j_2,\psi} Y) \\
&\cong X \coprod\nolimits_A^{\phi,i} ((A \coprod\nolimits_K^{f,g} B) \coprod\nolimits_B^{j_2,\psi} Y) \\
&\cong (X \coprod\nolimits_A^{\phi,i_1} (A \coprod\nolimits_K^{f,g} B)) \coprod\nolimits_B^{j,\psi} Y
\end{aligned}
$$

where

$$
\begin{aligned}
j_1 &= \mathrm{inj}_1^{A\times_K^{f,g}B}, & j_2 &= \mathrm{inj}_2^{A\times_K^{f,g}B}, \\
i_1 &= \mathrm{inj}_1^{(A\times_K^{f,g}B)\times_Y^{q_2,\psi}}, & i_2 &= \mathrm{inj}_2^{X\times^{\phi,q_1}_{A\times_K^{f,g}B}(A\times_K^{f,g}B)}, \\
i &= j_1 \circ \mathrm{inj}_1^{(A\times_K^{f,g}B)\times_B^{q_2,\psi}Y}, & j &= j_2 \circ \mathrm{inj}_2^{X\times_A^{\phi,q_1}(A\times_K^{f,g}B)},
\end{aligned}
$$

and where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=tiny]
& & (X \coprod_A (A \coprod_K B)) \coprod_{A \coprod_K B} ((A \coprod_K B) \coprod_B Y) & & \\
& X \coprod_A (A \coprod_K B) \arrow[ur] & & (A \coprod_K B) \coprod_B Y \arrow[ul] & \\
X \arrow[ur] & & A \coprod_K B \arrow[ul] \arrow[ur] \arrow[uu, phantom, "\ulcorner"] & & Y, \arrow[ul] \\
& A \arrow[ul, "\phi"] \arrow[ur] \arrow[uu, phantom, "\ulcorner"] & & B \arrow[ul] \arrow[ur, "\psi"'] \arrow[uu, phantom, "\ulcorner"] & \\
& & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\ulcorner"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & X \coprod_A ((A \coprod_K B) \coprod_B Y) & & \\
& & & (A \coprod_K B) \coprod_B Y \arrow[ul] & \\
X \arrow[uurr] & & A \coprod_K B \arrow[ur] \arrow[uu, phantom, "\ulcorner"] & & Y, \arrow[ul] \\
& A \arrow[ul, "\phi"] \arrow[ur] & & B \arrow[ul] \arrow[ur, "\psi"'] \arrow[uu, phantom, "\ulcorner"] & \\
& & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\ulcorner"] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
& & (X \coprod_A (A \coprod_K B)) \coprod_B Y & & \\
& X \coprod_A (A \coprod_K B) \arrow[ur] & & & \\
X \arrow[ur] & & A \coprod_K B \arrow[ul] \arrow[uu, phantom, "\ulcorner"] & & Y, \arrow[uull] \\
& A \arrow[ul, "\phi"] \arrow[ur] \arrow[uu, phantom, "\ulcorner"] & & B \arrow[ul] \arrow[ur, "\psi"'] & \\
& & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\ulcorner"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& & X \coprod_K Y & & \\
& & & & \\
X \arrow[uurr] & & & & Y. \arrow[uull] \\
& A \arrow[ul, "\phi"] & & B \arrow[ur, "\psi"'] & \\
& & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uuuu, phantom, "\ulcorner"] & &
\end{tikzcd}
$$

002H 5. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equals] & A \\
X \arrow[u, "f"] \arrow[r, equals] \arrow[ur, phantom, "\ulcorner", very near end] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
\lambda^{\mathsf{Sets}_{X/}}_A &\colon X \textstyle\coprod_X A \overset{\sim}{\dashrightarrow} A,\\
\rho^{\mathsf{Sets}_{X/}}_A &\colon A \textstyle\coprod_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l, "f"'] \\
X \arrow[u, equals] & X, \arrow[l, "f"] \arrow[u, equals]
\end{tikzcd}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

002J 6. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \coprod_X B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \quad \sigma^{\mathsf{Sets}_{X/}}_{A} : A \coprod_X B \overset{\sim}{\dashrightarrow} B \coprod_X A \quad \begin{tikzcd} B \coprod_X A & A \arrow[l] \\ B \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

002K 7. *Interaction With Coproducts.* We have

$$A \coprod_{\emptyset} B \cong A \coprod B, \qquad \begin{tikzcd} A \coprod B & B \arrow[l] \\ A \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"'] \end{tikzcd}$$

002L 8. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_{X/}, \coprod_X, X)$ is a symmetric monoidal category.

*Proof.* *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for $\xi$ following from the commutativity of the cube pushout diagram.
*Item 2, : Adjointness*: This follows from the universal property of the coproduct (pushouts are coproducts in $\mathsf{Sets}_{X/}$).
*Item 3, Associativity*: Omitted.
*Item 4, Interaction With Composition*: Omitted.
*Item 5, Unitality*: Omitted.
*Item 6, Commutativity*: Omitted.
*Item 7, Interaction With Coproducts*: Omitted.
*Item 8, Symmetric Monoidality*: Omitted. □

### 002M 4.2.5 Coequalisers

Let $A$ and $B$ be sets and let $f, g : A \rightrightarrows B$ be functions.

002N **Definition 4.2.5.1.1.** The **coequaliser of $f$ and $g$** is the coequaliser of $f$ and $g$ in $\mathsf{Sets}$ as in Limits and Colimits, **??**.

01F5 **Construction 4.2.5.1.2.** Concretely, the coequaliser of $f$ and $g$ is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

01F6 1. *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \overset{\text{def}}{=} B/\sim,$$

where $\sim$ is the equivalence relation on $B$ generated by $f(a) \sim g(a)$.

01F7 2. *The Cocone.* The map

$$\mathrm{coeq}(f,g) \colon B \twoheadrightarrow \mathrm{CoEq}(f,g)$$

given by the quotient map $\pi \colon B \twoheadrightarrow B/\sim$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

*Proof.* We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of $f$ and $g$ in $\mathsf{Sets}$. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\overset{\text{def}}{=} [f(a)] \\
&= [g(a)] \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \\
& & C
\end{tikzcd}$$

in $\mathsf{Sets}$. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

commute. □

002P **Remark 4.2.5.1.3.** In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

01ZQ 1. We have $a = b$;

01ZR 2. There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

01ZS (a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.

01ZT (b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

In other words, there exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

01ZU (a) There exists $z_0 \in A$ satisfying one of the following conditions:

01ZV i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.

01ZW ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.

01ZX (b) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:

01ZY i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.

01ZZ ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.

0200 (c) There exists $z_n \in A$ satisfying one of the following conditions:

0201 i. We have $x_n = f(z_n)$ and $b = g(z_n)$.

0202 ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

002Q **Example 4.2.5.1.4.** Here are some examples of coequalisers of sets.

002R 1. *Quotients by Equivalence Relations.* Let $R$ be an equivalence relation on a set $X$. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}(R \to X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X).$$

*Proof.* *Item 1*, *Quotients by Equivalence Relations*: See [Pro25z]. □

002S **Proposition 4.2.5.1.5.** Let $A$, $B$, and $C$ be sets.

002T 1. *Associativity.* We have isomorphisms of sets[9]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

[9]That is, the following three ways of forming "the" coequaliser of $(f,g,h)$ agree:

0203 1. Take the coequaliser of $(f,g,h)$, i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in $\mathsf{Sets}$.

0204 2. First take the coequaliser of $f$ and $g$, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

of $\mathrm{CoEq}(f,g)$

0205 3. First take the coequaliser of $g$ and $h$, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

of $\mathrm{CoEq}(g,h)$.

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{g}{\rightrightarrows\!\!\!\to}}} B$$

in Sets.

002U 4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, f) \cong B.$$

002V 5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

002W 6. *Interaction With Composition.* Let

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

*Proof.* *Item 1*, *Associativity*: Omitted.
*Item 4*, *Unitality*: Omitted.
*Item 5*, *Commutativity*: Omitted.
*Item 6*, *Interaction With Composition*: Omitted. □

## 01F8 4.2.6 Direct Colimits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} : (I, \preceq) \to \mathsf{Sets}$ be a direct system of sets.

01F9 **Definition 4.2.6.1.1.** The **direct colimit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in Limits and Colimits, **??**.

01FA **Construction 4.2.6.1.2.** Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ is the pair $\left(\underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha), \{\operatorname{inj}_\alpha\}_{\alpha\in I}\right)$ consisting of:

01FB 1. *The Colimit.* The set $\underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha)$ defined by

$$\underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha) \overset{\text{def}}{=} \left(\coprod_{\alpha\in I} X_\alpha\right)\Big/\sim,$$

where $\sim$ is the equivalence relation on $\coprod_{\alpha\in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

01FC (a) We have $\alpha \preceq \gamma$.

01FD (b) We have $\beta \preceq \gamma$.

01FE (c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

01FF 2. *The Cocone.* The collection

$$\left\{\operatorname{inj}_\gamma \colon X_\gamma \to \underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha)\right\}_{\gamma\in I}$$

of maps of sets defined by

$$\operatorname{inj}_\gamma(x) \overset{\text{def}}{=} [(\gamma, x)]$$

for each $\gamma \in I$ and each $x \in X_\gamma$.

*Proof.* We will prove Definition 4.2.6.1.2 below in a bit, but first we need a lemma (which is interesting in its own right). □

01FG **Lemma 4.2.6.1.3.** For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have

$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in $\underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha)$.

*Proof.* Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \overset{\text{def}}{=} \mathrm{id}_{X_\beta}$, and we have

$$\begin{aligned} f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\ &\overset{\text{def}}{=} \mathrm{id}_{X_\beta}(f_{\alpha\beta}(x)), \\ &= f_{\alpha\beta}(x). \end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, Items 1a to 1c of Definition 4.2.6.1.2 are met. Thus we have $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$. □

We can now prove Definition 4.2.6.1.2:

*Proof.* We claim that $\underset{\longrightarrow}{\operatorname{colim}}_{\alpha \in I}(X_\alpha)$ is the colimit of the direct system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$.

*Commutativity of the Colimit Diagram*: First, we need to check that the colimit diagram defined by $\underset{\longrightarrow}{\operatorname{colim}}_{\alpha \in I}(X_\alpha)$ commutes, i.e. that we have

$$\mathrm{inj}_\alpha = \mathrm{inj}_\beta \circ f_{\alpha\beta}, \qquad \begin{tikzcd} & \underset{\longrightarrow}{\operatorname{colim}}_{\alpha \in I}(X_\alpha) & \\ X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[ul, "\mathrm{inj}_\beta"'] \end{tikzcd}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$\begin{aligned} [\mathrm{inj}_\beta \circ f_{\alpha\beta}](x) &\overset{\text{def}}{=} \mathrm{inj}_\beta(f_{\alpha\beta}(x)) \\ &\overset{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\ &= [(\alpha, x)] \\ &\overset{\text{def}}{=} \mathrm{inj}_\alpha(x), \end{aligned}$$

where we have used Definition 4.2.6.1.3 for the third equality.

*Proof of the Universal Property of the Colimit*: Next, we prove that $\underset{\longrightarrow}{\operatorname{colim}}_{\alpha \in I}(X_\alpha)$ as constructed in Definition 4.2.6.1.2 satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{tikzcd}
 & C & \\
 & \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) & \\
X_\alpha \arrow[uur, bend left, "i_\alpha"] \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[uul, bend right, "i_\beta"'] \arrow[ul, "\mathrm{inj}_\beta"']
\end{tikzcd}$$

in Sets. We claim that there exists a unique map $\phi\colon \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
 & C & \\
 & \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) \arrow[u, dashed, "\phi", "\exists!"'] & \\
X_\alpha \arrow[uur, bend left, "i_\alpha"] \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[uul, bend right, "i_\beta"'] \arrow[ul, "\mathrm{inj}_\beta"']
\end{tikzcd}$$

commute. To this end, first consider the diagram

$$\begin{tikzcd}
\coprod_{\alpha\in I} X_\alpha \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, "\coprod_{\alpha\in I} i_\alpha"'] & \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) \\
 & C.
\end{tikzcd}$$

**Lemma.** If $(\alpha, x) \sim (\beta, y)$, then we have

$$\left[\coprod_{\alpha\in I} i_\alpha\right](x) = \left[\coprod_{\alpha\in I} i_\alpha\right](y).$$

*Proof.* Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

0206 1. We have $\alpha \preceq \gamma$.

0207 2. We have $\beta \preceq \gamma$.

0208 3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$\begin{aligned}\left[\coprod_{\alpha\in I} i_\alpha\right](x) &\stackrel{\text{def}}{=} i_\alpha(x)\\ &\stackrel{\text{def}}{=} [i_\gamma \circ f_{\alpha\gamma}](x)\\ &\stackrel{\text{def}}{=} i_\gamma(f_{\alpha\gamma}(x))\\ &= i_\gamma(f_{\beta\gamma}(x))\\ &\stackrel{\text{def}}{=} [i_\gamma \circ f_{\beta\gamma}](x)\\ &= i_\beta(y)\\ &\stackrel{\text{def}}{=} \left[\coprod_{\alpha\in I} i_\alpha\right](y).\end{aligned}$$

This finishes the proof of the lemma. Continuing, by Conditions on Relations, **??** of Definition 10.6.2.1.3, there then exists a map $\phi\colon \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd} \coprod_{\alpha\in I} X_\alpha \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, "\coprod_{\alpha\in I} i_\alpha"'] & \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) \arrow[d, "\phi"] \\ & C \end{tikzcd}$$

commute. In particular, this implies that the diagram

$$\begin{tikzcd} X_\alpha \arrow[r, two heads, "\mathrm{inj}_\alpha"] \arrow[dr, "i_\alpha"'] & \underset{\alpha\in I}{\underrightarrow{\operatorname{colim}}}(X_\alpha) \arrow[d, "\phi"] \\ & C \end{tikzcd}$$

also commutes, and thus so does the diagram

$$\begin{tikzcd} & C & \\ & \underset{\overrightarrow{\alpha\in I}}{\mathrm{colim}}(X_\alpha) \arrow[u, dashed, "\phi", "\exists!"'] & \\ X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] \arrow[uur, bend left, "i_\alpha"] & & X_\beta. \arrow[ul, "\mathrm{inj}_\beta"'] \arrow[uul, bend right, "i_\beta"'] \end{tikzcd}$$

This finishes the proof.[10] □

01FH **Example 4.2.6.1.4.** Here are some examples of direct colimits of sets.

01FJ 1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \overset{\text{def}}{=} \underset{\overrightarrow{n\in\mathbb{N}}}{\mathrm{colim}}(\mathbb{Z}_{/p^n});$$

see **??**.

## 002X 4.3 Operations With Sets

### 002Y 4.3.1 The Empty Set

002Z **Definition 4.3.1.1.1.** The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where $X$ is the set in the set existence axiom, **??** of **??**.

[10]Incidentally, the conditions

$$\{i_\alpha = \phi \circ \mathrm{inj}_\alpha\}_{\alpha\in I}$$

show that $\phi$ must be given by

$$\phi([(\alpha, x)]) = (i_\alpha(x))_{\alpha\in I}$$

for each $[(\alpha, x)] \in \underset{\overrightarrow{\alpha\in I}}{\mathrm{colim}}(X_\alpha)$, although we would need to show that this assignment is

## 0030 4.3.2 Singleton Sets

Let $X$ be a set.

0031 **Definition 4.3.2.1.1.** The **singleton set containing** $X$ is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of $X$ with itself of Definition 4.3.3.1.1.

## 0032 4.3.3 Pairings of Sets

Let $X$ and $Y$ be sets.

0033 **Definition 4.3.3.1.1.** The **pairing of** $X$ **and** $Y$ is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where $A$ is the set in the axiom of pairing, **??** of **??**.

## 0034 4.3.4 Ordered Pairs

Let $A$ and $B$ be sets.

0035 **Definition 4.3.4.1.1.** The **ordered pair associated to** $A$ **and** $B$ is the set $(A, B)$ defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

0036 **Proposition 4.3.4.1.2.** Let $A$ and $B$ be sets.

0037 1. *Uniqueness.* Let $A$, $B$, $C$, and $D$ be sets. The following conditions are equivalent:

0038 (a) We have $(A, B) = (C, D)$.

0039 (b) We have $A = C$ and $B = D$.

*Proof.* *Item 1*, *Uniqueness*: See [Cie97, Theorem 1.2.3]. □

well-defined were we to prove Definition 4.2.6.1.2 in this way. Instead, invoking Conditions on Relations, **??** of Definition 10.6.2.1.3 gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

## 003A 4.3.5 Sets of Maps

Let $A$ and $B$ be sets.

003B **Definition 4.3.5.1.1.** The **set of maps from $A$ to $B$**[11] is the set $\mathsf{Sets}(A, B)$[12] whose elements are the functions from $A$ to $B$.

003C **Proposition 4.3.5.1.2.** Let $A$ and $B$ be sets.

003D 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathsf{Sets}(X, -)\colon & \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\ \mathsf{Sets}(-, Y)\colon & \quad \mathsf{Sets}^{\mathsf{op}} \quad \to \mathsf{Sets}, \\ \mathsf{Sets}(-_1, -_2)\colon & \ \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}. \end{aligned}$$

01FK 2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

01FL 3. *Maps From the Punctual Set.* We have a bijection

$$\mathsf{Sets}(\mathrm{pt}, A) \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

---

[11] *Further Terminology:* Also called the **Hom set from $A$ to $B$**.
[12] *Further Notation:* Also written $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$.

01FM 4. *Maps to the Punctual Set.* We have a bijection

$$\mathsf{Sets}(A, \mathrm{pt}) \cong \mathrm{pt},$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

*Proof.* *Item 1, Functoriality*: This follows from Categories, Items 2 and 5 of Definition 11.1.4.1.2.
*Item 2, Adjointness*: This is a repetition of Item 2 of Definition 4.1.3.1.3 and is proved there.
*Item 3, Maps From the Punctual Set*: The bijection

$$\Phi_A \colon \mathsf{Sets}(\mathrm{pt}, A) \overset{\sim}{\dashrightarrow} A$$

is given by

$$\Phi_A(f) \overset{\mathrm{def}}{=} f(\star)$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$, admitting an inverse

$$\Phi_A^{-1} \colon A \overset{\sim}{\dashrightarrow} \mathsf{Sets}(\mathrm{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \overset{\mathrm{def}}{=} [\![\star \mapsto a]\!]$$

for each $a \in A$. Indeed, we have

$$\begin{aligned}
[\Phi_A^{-1} \circ \Phi_A](f) &\overset{\mathrm{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\
&\overset{\mathrm{def}}{=} \Phi_A^{-1}(f(\star)) \\
&\overset{\mathrm{def}}{=} [\![\star \mapsto f(\star)]\!] \\
&\overset{\mathrm{def}}{=} f \\
&\overset{\mathrm{def}}{=} [\mathrm{id}_{\mathsf{Sets}(\mathrm{pt}, A)}](f)
\end{aligned}$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$ and

$$\begin{aligned}
[\Phi_A \circ \Phi_A^{-1}](a) &\overset{\mathrm{def}}{=} \Phi_A(\Phi_A^{-1}(a)) \\
&\overset{\mathrm{def}}{=} \Phi_A([\![\star \mapsto a]\!]) \\
&\overset{\mathrm{def}}{=} \mathrm{ev}_\star([\![\star \mapsto a]\!]) \\
&\overset{\mathrm{def}}{=} a \\
&\overset{\mathrm{def}}{=} [\mathrm{id}_A](a)
\end{aligned}$$

for each $a \in A$, and thus we have

$$\begin{aligned}\Phi_A^{-1} \circ \Phi_A &= \mathrm{id}_{\mathsf{Sets}(\mathrm{pt},A)}\\ \Phi_A \circ \Phi_A^{-1} &= \mathrm{id}_A.\end{aligned}$$

To prove naturality, we need to show that the diagram

$$\begin{tikzcd} \mathsf{Sets}(\mathrm{pt},A) \arrow[r, "f_!"] \arrow[d, dashed, "\Phi_A"', "\wr"] & \mathsf{Sets}(\mathrm{pt},B) \arrow[d, dashed, "\Phi_B", "\wr"'] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}[f \circ \Phi_A](\phi) &\overset{\text{def}}{=} f(\Phi_A(\phi))\\ &\overset{\text{def}}{=} f(\phi(\star))\\ &\overset{\text{def}}{=} [f \circ \phi](\star)\\ &\overset{\text{def}}{=} \Phi_B(f \circ \phi)\\ &\overset{\text{def}}{=} \Phi_B(f_!(\phi))\\ &\overset{\text{def}}{=} [\Phi_B \circ f_!](\phi)\end{aligned}$$

for each $\phi \in \mathsf{Sets}(\mathrm{pt},A)$. This finishes the proof.
*Item 4, Maps to the Punctual Set*: This follows from the universal property of pt as the terminal set, Definition 4.1.1.1.1. □

## 003E 4.3.6 Unions of Families of Subsets

Let $X$ be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

003F **Definition 4.3.6.1.1.** The **union of** $\mathcal{U}$ is the set $\bigcup_{U \in \mathcal{U}} U$ defined by

$$\bigcup_{U \in \mathcal{U}} U \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } U \in \mathcal{U}\\ \text{such that we have } x \in U\end{array} \right\}.$$

01FN **Proposition 4.3.6.1.2.** Let $X$ be a set.

01FP 1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U\in\mathcal{U}} U$ defines a functor

$$\bigcup\colon (\mathcal{P}(\mathcal{P}(X)), \subset) \to (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

($\star$) If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcup_{U\in\mathcal{U}} U \subset \bigcup_{V\in\mathcal{V}} V$.

01FQ 2. *Associativity.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U\in\bigcup_{A\in\mathcal{A}} A} U = \bigcup_{A\in\mathcal{A}} \left(\bigcup_{U\in A} U\right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

01FR 3. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\chi_{\mathcal{P}(X)}"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V\in\{U\}} V = U$$

for each $U \in \mathcal{P}(X)$.

01FS 4. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\mathcal{P}(\chi_X)"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{\{u\}\in\chi_X(U)} \{u\} = U$$

for each $U \in \mathcal{P}(X)$.

01FT 5. *Interaction With Unions I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cup"] \arrow[d, "\bigcup\times\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{W\in\mathcal{U}\cup\mathcal{V}} W = \left(\bigcup_{U\in\mathcal{U}} U\right) \cup \left(\bigcup_{V\in\mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01FU 6. *Interaction With Unions II.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "U\cup-"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "-\cup V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U \cup \left(\bigcup_{V\in\mathcal{V}} V\right) = \bigcup_{V\in\mathcal{V}} (U \cup V),$$

$$\left(\bigcup_{U\in\mathcal{U}} U\right)\cup V = \bigcup_{U\in\mathcal{U}} (U\cup V)$$

for each nonempty $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U, V\in\mathcal{P}(X)$.

01FV 7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))\times\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cap"] \arrow[d, "\cup\times\cup"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X),
\end{tikzcd}$$

with components

$$\bigcup_{W\in\mathcal{U}\cap\mathcal{V}} W \subset \left(\bigcup_{U\in\mathcal{U}} U\right)\cap\left(\bigcup_{V\in\mathcal{V}} V\right)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

01FW 8. *Interaction With Intersections II.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cap-)"] \arrow[d, "\cup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "U\cap-"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cap V)"] \arrow[d, "\cup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "-\cap V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U\cap\left(\bigcup_{V\in\mathcal{V}} V\right) = \bigcup_{V\in\mathcal{V}} (U\cap V),$$

$$\left(\bigcup_{U\in\mathcal{U}} U\right)\cap V = \bigcup_{U\in\mathcal{U}} (U\cap V)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U, V\in\mathcal{P}(X)$.

01FX 9. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\setminus"] \arrow[d, "\bigcup \times \bigcup"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\setminus"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U\right) \setminus \left(\bigcup_{V \in \mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01FY 10. *Interaction With Complements I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[d, "\bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcup_{U \in \mathcal{U}^{\mathsf{c}}} U \neq \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01FZ 11. *Interaction With Complements II.* The diagram

$$\begin{tikzcd}
& \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[dr, dashed, "\sim"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[d, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[d, "\bigcap^{\mathsf{op}}"] \\
\mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}},
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01G0 12. *Interaction With Complements III.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[dr, dashed, "\sim"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[d, "\bigcap"'] & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[d, "\bigcup^{\mathsf{op}}"] \\
\mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}},
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcap_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01G1 13. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\triangle"] \arrow[d, "\bigcup\times\bigcup"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01G2 14. *Interaction With Internal Homs I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01G3 15. *Interaction With Internal Homs II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcup^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01G4 16. *Interaction With Internal Homs III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X}"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V\in\mathcal{V}} V\right]_X = \bigcup_{V\in\mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01G5 17. *Interaction With Direct Images.* Let $f: X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U\in\mathcal{U}} f_!(U) = \bigcup_{V\in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01G6 18. *Interaction With Inverse Images.* Let $f: X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V\in\mathcal{V}} f^{-1}(V) = \bigcup_{U\in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

01G7 19. *Interaction With Codirect Images.* Let $f: X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01G8 20. *Interaction With Intersections of Families I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(x)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "\bigcap"'] & X
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup\limits_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

01G9 21. *Interaction With Intersections of Families II.* Let $X$ be a set and consider the compositions

$$
\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[dll, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dl, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}" description] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap" description] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[drr, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[dr, "\bigcap" description] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup" description] & \mathcal{P}(\mathcal{P}(X)) \arrow[dl, "\bigcap"] \\
 & & \mathcal{P}(X) &
\end{tikzcd}
$$

given by

$$
\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,
$$

$$
\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right)
$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\begin{tikzcd}
& \displaystyle\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U \arrow[dr, hook] & \\
\displaystyle\bigcup_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right) & & \displaystyle\bigcap_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right) \\
& \displaystyle\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \arrow[ul, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

All other possible inclusions fail to hold in general.

*Proof.* *Item 1*, *Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition $(\star)$. So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

*Item 2*, *Associativity*: We have

$$\begin{aligned}
\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U &\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } U \in \displaystyle\bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right\} \\
&\overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right).
\end{aligned}
$$

This finishes the proof.

*Item 3, Left Unitality*: We have

$$
\begin{aligned}
\bigcup_{V \in \{U\}} V &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in U \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.

*Item 4, Right Unitality*: We have

$$
\begin{aligned}
\bigcup_{\{u\} \in \chi_X(U)} \{u\} &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.

*Item 5, Interaction With Unions I*: We have

$$
\begin{aligned}
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{U}\\ \text{such that we have } x \in W\end{array} \right\}\\
&\quad \cup \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{V}\\ \text{such that we have } x \in W\end{array} \right\}\\
&\stackrel{\text{def}}{=} \left( \bigcup_{W \in \mathcal{U}} W \right) \cup \left( \bigcup_{W \in \mathcal{V}} W \right)\\
&= \left( \bigcup_{U \in \mathcal{U}} U \right) \cup \left( \bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}
$$

This finishes the proof.

*Item 6, Interaction With Unions II*: Assume $\mathcal{V}$ is nonempty. We have

$$
\begin{aligned}
U \cup \bigcup_{V \in \mathcal{V}} V &\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; x \in U \text{ or } x \in \bigcup_{V \in \mathcal{V}} V \right\}\\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} x \in U \text{ or there exists some}\\ V \in \mathcal{V} \text{ such that } x \in V\end{array} \right\}\\
&= \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } V \in \mathcal{V}\\ \text{such that } x \in U \text{ or } x \in V\end{array} \right\}\\
&= \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } V \in \mathcal{V}\\ \text{such that } x \in U \cup V\end{array} \right\}\\
&\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cup V.
\end{aligned}
$$

This concludes the proof of the first statement. For the second statement, use Item 4 of Definition 4.3.8.1.2 to rewrite

$$
\begin{aligned}
\left( \bigcup_{U \in \mathcal{U}} U \right) \cup V &= V \cup \left( \bigcup_{U \in \mathcal{U}} U \right),\\
\bigcup_{U \in \mathcal{U}} (U \cup V) &= \bigcup_{U \in \mathcal{U}} (V \cup U).
\end{aligned}
$$

But these two sets are equal by the first statement.

*Item 7, Interaction With Intersections I*: We have

$$\begin{aligned}
\bigcup_{W\in\mathcal{U}\cap\mathcal{V}} W &\overset{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists some } W\in\mathcal{U}\cap\mathcal{V}\\ \text{such that we have } x\in W\end{array}\right\}\\
&\subset \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists some } U\in\mathcal{U} \text{ and some } V\in\mathcal{V}\\ \text{such that we have } x\in U \text{ and } x\in V\end{array}\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists some } U\in\mathcal{U}\\ \text{such that we have } x\in U\end{array}\right\}\\
&\quad\cup \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists some } V\in\mathcal{V}\\ \text{such that we have } x\in V\end{array}\right\}\\
&\overset{\text{def}}{=} \left(\bigcup_{U\in\mathcal{U}} U\right)\cap\left(\bigcup_{V\in\mathcal{V}} V\right).
\end{aligned}$$

This finishes the proof.
*Item 8, Interaction With Intersections II*: We have

$$\begin{aligned}
U\cap\bigcup_{V\in\mathcal{V}} V &\overset{\text{def}}{=} \left\{x\in X \;\middle|\; x\in U \text{ and } x\in \bigcup_{V\in\mathcal{V}} V\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}x\in U \text{ and there exists some}\\ V\in\mathcal{V} \text{ such that } x\in V\end{array}\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists some } V\in\mathcal{V}\\ \text{such that } x\in U \text{ and } x\in V\end{array}\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists some } V\in\mathcal{V}\\ \text{such that } x\in U\cap V\end{array}\right\}\\
&\overset{\text{def}}{=} \bigcup_{V\in\mathcal{V}} U\cap V.
\end{aligned}$$

This concludes the proof of the first statement. For the second statement, use Item 5 of Definition 4.3.9.1.2 to rewrite

$$\left(\bigcup_{U\in\mathcal{U}} U\right)\cap V = V\cap\left(\bigcup_{U\in\mathcal{U}} U\right),$$

$$\bigcup_{U\in\mathcal{U}}(U\cap V)=\bigcup_{U\in\mathcal{U}}(V\cap U).$$

But these two sets are equal by the first statement.

*Item 9, Interaction With Differences*: Let $X=\{0,1\}$, let $\mathcal{U}=\{\{0,1\}\}$, and let $\mathcal{V}=\{\{0\}\}$. We have

$$\begin{aligned}\bigcup_{W\in\mathcal{U}\setminus\mathcal{V}}U&=\bigcup_{W\in\{\{0,1\}\}}W\\&=\{0,1\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcup_{U\in\mathcal{U}}U\right)\setminus\left(\bigcup_{V\in\mathcal{V}}V\right)&=\{0,1\}\setminus\{0\}\\&=\{1\}.\end{aligned}$$

Thus we have

$$\bigcup_{W\in\mathcal{U}\setminus\mathcal{V}}W=\{0,1\}\neq\{1\}=\left(\bigcup_{U\in\mathcal{U}}U\right)\setminus\left(\bigcup_{V\in\mathcal{V}}V\right).$$

This finishes the proof.

*Item 10, Interaction With Complements I*: Let $X=\{0,1\}$ and let $\mathcal{U}=\{0\}$. We have

$$\begin{aligned}\bigcup_{U\in\mathcal{U}^{\mathsf{c}}}U&=\bigcup_{U\in\{\emptyset,\{1\},\{0,1\}\}}U\\&=\{0,1\},\end{aligned}$$

whereas

$$\begin{aligned}\bigcup_{U\in\mathcal{U}}U^{\mathsf{c}}&=\{0\}^{\mathsf{c}}\\&=\{1\}.\end{aligned}$$

Thus we have

$$\bigcup_{U\in\mathcal{U}^{\mathsf{c}}}U=\{0,1\}\neq\{1\}=\bigcup_{U\in\mathcal{U}}U^{\mathsf{c}}.$$

This finishes the proof.
*Item 11, Interaction With Complements II*: We have

$$\begin{aligned}\left(\bigcup_{U\in\mathcal{U}}U\right)^{\mathsf{c}} &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{aligned}&\text{there exists no } U\in\mathcal{U}\\ &\text{such that we have } x\in U\end{aligned}\right\}\\ &= \left\{x\in X \,\middle|\, \begin{aligned}&\text{for all } U\in\mathcal{U}\\ &\text{we have } x\notin U\end{aligned}\right\}\\ &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{aligned}&\text{for all } U\in\mathcal{U}\\ &\text{we have } x\in U^{\mathsf{c}}\end{aligned}\right\}\\ &\overset{\text{def}}{=} \bigcap_{U\in\mathcal{U}}U^{\mathsf{c}}.\end{aligned}$$

*Item 12, Interaction With Complements III*: By Item 11 Item 3 of Definition 4.3.11.1.2, we have

$$\begin{aligned}\left(\bigcap_{U\in\mathcal{U}}U\right)^{\mathsf{c}} &= \left(\bigcap_{U\in\mathcal{U}}(U^{\mathsf{c}})^{\mathsf{c}}\right)^{\mathsf{c}}\\ &= \left(\left(\bigcup_{U\in\mathcal{U}}U^{\mathsf{c}}\right)^{\mathsf{c}}\right)^{\mathsf{c}}\\ &= \bigcup_{U\in\mathcal{U}}U^{\mathsf{c}}.\end{aligned}$$

*Item 13, Interaction With Symmetric Differences*: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$\begin{aligned}\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}}W &= \bigcup_{W\in\{\{0\}\}}W\\ &= \{0\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcup_{U\in\mathcal{U}}U\right)\triangle\left(\bigcup_{V\in\mathcal{V}}V\right) &= \{0,1\}\triangle\{0,1\}\\ &= \emptyset,\end{aligned}$$

Thus we have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right).$$

This finishes the proof.
*Item 14, Interaction With Internal Homs I*: This is a repetition of Item 7 of Definition 4.4.7.1.3 and is proved there.
*Item 15, Interaction With Internal Homs II*: This is a repetition of Item 8 of Definition 4.4.7.1.3 and is proved there.
*Item 16, Interaction With Internal Homs III*: This is a repetition of Item 9 of Definition 4.4.7.1.3 and is proved there.
*Item 17, Interaction With Direct Images*: This is a repetition of Item 3 of Definition 4.6.1.1.5 and is proved there.
*Item 18, Interaction With Inverse Images*: This is a repetition of Item 3 of Definition 4.6.2.1.3 and is proved there.
*Item 19, Interaction With Codirect Images*: This is a repetition of Item 3 of Definition 4.6.3.1.7 and is proved there.
*Item 20, Interaction With Intersections of Families I*: We have

$$\begin{aligned}\bigcap_{U\in\bigcup_{A\in\mathcal{A}} A} U &\stackrel{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } U\in\bigcup_{A\in\mathcal{A}} A,\\ \text{we have } x\in U\end{array}\right\}\\ &= \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } A\in\mathcal{A} \text{ and each}\\ U\in A, \text{ we have } x\in U\end{array}\right\}\\ &\stackrel{\text{def}}{=} \bigcap_{A\in\mathcal{A}}\left(\bigcap_{U\in A} U\right).\end{aligned}$$

This finishes the proof.
*Item 21, Interaction With Intersections of Families II*: Omitted. □

## 003V 4.3.7 Intersections of Families of Subsets

Let $X$ be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

003W **Definition 4.3.7.1.1.** The **intersection of** $\mathcal{U}$ is the set $\bigcap_{U\in\mathcal{U}} U$ defined by

$$\bigcap_{U\in\mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{for each } U \in \mathcal{U},\\ \text{we have } x \in U\end{array} \right\}.$$

01GA **Proposition 4.3.7.1.2.** Let $X$ be a set.

01GB 1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcap_{U\in\mathcal{U}} U$ defines a functor

$$\bigcap \colon (\mathcal{P}(\mathcal{P}(X)), \supset) \to (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

(★) If $\mathcal{U} \subset \mathcal{V}$, then $\bigcap_{V\in\mathcal{V}} V \subset \bigcap_{U\in\mathcal{U}} U$.

01GC 2. *Oplax Associativity.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

with components

$$\bigcap_{A\in\mathcal{A}} \left( \bigcap_{U\in A} U \right) \subset \bigcap_{U\in\bigcap_{A\in\mathcal{A}} A} U$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

01GD 3. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\chi_{\mathcal{P}(X)}"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V \in \{U\}} V = U.$$

for each $U \in \mathcal{P}(X)$.

01GE 4. *Oplax Right Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\mathcal{P}(\chi_X)"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, phantom, "\times"'] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \neq U$$

in general, where $U \in \mathcal{P}(X)$. However, when $U$ is nonempty, we have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \subset U.$$

01GF 5. *Interaction With Unions I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cup"] \arrow[d, "\bigcap \times \bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcap_{U \in \mathcal{U}} U\right) \cap \left(\bigcap_{V \in \mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GG 6. *Interaction With Unions II.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "U\cup-"'] & \mathcal{P}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "-\cup V"'] & \mathcal{P}(X) \end{tikzcd}$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V\in\mathcal{V}} V\right) = \bigcap_{V\in\mathcal{V}} (U\cup V),$$

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \cup V = \bigcap_{U\in\mathcal{U}} (U\cup V)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U,V\in\mathcal{P}(X)$.

01GH 7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X))\times\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cap"] \arrow[d, "\cap\times\cap"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X), \end{tikzcd}$$

with components

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right) \subset \bigcap_{W\in\mathcal{U}\cap\mathcal{V}} W$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

01GJ 8. *Interaction With Intersections II.* The diagrams

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cap-)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "U\cap-"'] & \mathcal{P}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cap V)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "-\cap V"'] & \mathcal{P}(X) \end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned}U \cup \left(\bigcap_{V\in\mathcal{V}} V\right) &= \bigcap_{V\in\mathcal{V}} (U \cup V),\\ \left(\bigcap_{U\in\mathcal{U}} U\right) \cup V &= \bigcap_{U\in\mathcal{U}} (U \cup V)\end{aligned}$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

01GK 9. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\setminus"] \arrow[d, "\cap\times\cap"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\setminus"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} W \neq \left(\bigcap_{U\in\mathcal{U}} U\right) \setminus \left(\bigcap_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GL 10. *Interaction With Complements I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[d, "\cap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} W \neq \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GM 11. *Interaction With Complements II.* The diagram

$$\begin{tikzcd}[row sep=small] & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[dr, dashed, "\sim"] & \\ \mathcal{P}(\mathcal{P}(X)) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[d, "\bigcap"'] & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[d, "\bigcup^{\mathsf{op}}"] \\ \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, \end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U\right)^{\mathsf{c}} = \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GN 12. *Interaction With Complements III.* The diagram

$$\begin{tikzcd}[row sep=small] & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[dr, dashed, "\sim"] & \\ \mathcal{P}(\mathcal{P}(X)) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[d, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[d, "\bigcap^{\mathsf{op}}"] \\ \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, \end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GP 13. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\triangle"] \arrow[d, "\bigcap \times \bigcap"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \triangle \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U\right) \triangle \left(\bigcap_{V \in \mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GQ 14. *Interaction With Internal Homs I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GR 15. *Interaction With Internal Homs II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcap^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcup"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcap_{U\in\mathcal{U}} U, V\right]_X = \bigcup_{U\in\mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01GS 16. *Interaction With Internal Homs III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V\in\mathcal{V}} V\right]_X = \bigcap_{V\in\mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GT 17. *Interaction With Direct Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U\in\mathcal{U}} f_!(U) = \bigcap_{V\in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01GU 18. *Interaction With Inverse Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

01GV 19. *Interaction With Codirect Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01GW 20. *Interaction With Unions of Families I.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(x)) \arrow[d, "\bigcap"] \\ \mathcal{P}(X) \arrow[r, "\bigcap"'] & X \end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

01GX 21. *Interaction With Unions of Families II.* Let $X$ be a set and consider the compositions

$$\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[dll, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dl, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[drr, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[dr, "\bigcap"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] & \mathcal{P}(\mathcal{P}(X)) \arrow[dl, "\bigcap"] \\
 & & \mathcal{P}(X) &
\end{tikzcd}$$

given by

$$\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\begin{tikzcd}
 & \displaystyle\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U \arrow[dr] & \\
\displaystyle\bigcup_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right) & & \displaystyle\bigcap_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right) \\
 & \displaystyle\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \arrow[ul, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

All other possible inclusions fail to hold in general.

*Proof.* *Item 1*, *Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition ($\star$). So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion.

*Item 2*, *Oplax Associativity*: We have

$$\begin{aligned}
\bigcap_{A \in \mathcal{A}} \left( \bigcap_{U \in A} U \right) &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \bigcap_{U \in A} U \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
&\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.
\end{aligned}$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

*Item 3, Left Unitality*: We have

$$\begin{aligned}
\bigcap_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in U \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

*Item 4, Oplax Right Unitality*: If $U = \emptyset$, then we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \bigcap_{\{u\} \in \emptyset} \{u\} \\
&= X,
\end{aligned}$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$. When $U$ is nonempty, we have two cases:

020B 1. If $U$ is a singleton, say $U = \{u\}$, we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \{u\} \\
&\stackrel{\text{def}}{=} U.
\end{aligned}$$

020C 2. If $U$ contains at least two elements, we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \emptyset \\
&\subset U.
\end{aligned}$$

This finishes the proof.

*Item 5, Interaction With Unions I*: We have

$$
\begin{aligned}
\bigcap_{W\in\mathcal{U}\cup\mathcal{V}} W &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } W\in\mathcal{U}\cup\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } W\in\mathcal{U} \text{ and each}\\ W\in\mathcal{V}, \text{ we have } x\in W\end{array}\right\}\\
&\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } W\in\mathcal{U},\\ \text{we have } x\in W\end{array}\right\}\\
&\quad\cap \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } W\in\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&\overset{\text{def}}{=} \left(\bigcap_{W\in\mathcal{U}} W\right)\cap\left(\bigcap_{W\in\mathcal{V}} W\right)\\
&= \left(\bigcap_{U\in\mathcal{U}} U\right)\cap\left(\bigcap_{V\in\mathcal{V}} V\right).
\end{aligned}
$$

This finishes the proof.

*Item 6, Interaction With Unions II*: Omitted.

*Item 7, Interaction With Intersections I*: We have

$$
\begin{aligned}
\left(\bigcap_{U\in\mathcal{U}} U\right)\cap\left(\bigcap_{V\in\mathcal{V}} V\right) &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } U\in\mathcal{U},\\ \text{we have } x\in U\end{array}\right\}\\
&\quad\cup \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } V\in\mathcal{V},\\ \text{we have } x\in V\end{array}\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } W\in\mathcal{U}\cap\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&\subset \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } W\in\mathcal{U}\cup\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&\overset{\text{def}}{=} \bigcap_{W\in\mathcal{U}\cap\mathcal{V}} W.
\end{aligned}
$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
*Item 8, Interaction With Intersections II*: Omitted.
*Item 9, Interaction With Differences*: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0\},\{0,1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned}\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} U &= \bigcap_{W\in\{\{0,1\}\}} W\\ &= \{0,1\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcap_{U\in\mathcal{U}} U\right)\setminus\left(\bigcap_{V\in\mathcal{V}} V\right) &= \{0\}\setminus\{0\}\\ &= \emptyset.\end{aligned}$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} W = \{0,1\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right)\setminus\left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.
*Item 10, Interaction With Complements I*: Let $X = \{0,1\}$ and let $\mathcal{U} = \{\{0\}\}$. We have

$$\begin{aligned}\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U &= \bigcap_{W\in\{\emptyset,\{1\},\{0,1\}\}} W\\ &= \emptyset,\end{aligned}$$

whereas

$$\begin{aligned}\bigcap_{U\in\mathcal{U}} U^{\mathsf{c}} &= \{0\}^{\mathsf{c}}\\ &= \{1\}.\end{aligned}$$

Thus we have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U = \emptyset \neq \{1\} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.

*Item 11, Interaction With Complements II*: This is a repetition of Item 12 of Definition 4.3.6.1.2 and is proved there.

*Item 12, Interaction With Complements III*: This is a repetition of Item 11 of Definition 4.3.6.1.2 and is proved there.

*Item 13, Interaction With Symmetric Differences*: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$\begin{aligned}\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W &= \bigcap_{W\in\{\{0\}\}} W\\ &= \{0\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcap_{U\in\mathcal{U}} U\right)\triangle\left(\bigcap_{V\in\mathcal{V}} V\right) &= \{0,1\}\triangle\{0\}\\ &= \emptyset,\end{aligned}$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right)\triangle\left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

*Item 14, Interaction With Internal Homs I*: This is a repetition of Item 10 of Definition 4.4.7.1.3 and is proved there.

*Item 15, Interaction With Internal Homs II*: This is a repetition of Item 11 of Definition 4.4.7.1.3 and is proved there.

*Item 16, Interaction With Internal Homs III*: This is a repetition of Item 12 of Definition 4.4.7.1.3 and is proved there.

*Item 17, Interaction With Direct Images*: This is a repetition of Item 4 of Definition 4.6.1.1.5 and is proved there.

*Item 18, Interaction With Inverse Images*: This is a repetition of Item 4 of Definition 4.6.2.1.3 and is proved there.

*Item 19, Interaction With Codirect Images*: This is a repetition of Item 4 of Definition 4.6.3.1.7 and is proved there.

*Item 20, Interaction With Unions of Families I*: This is a repetition of Item 20 of Definition 4.3.6.1.2 and is proved there.

*Item 21, Interaction With Unions of Families II*: This is a repetition of Item 21 of Definition 4.3.6.1.2 and is proved there. □

## 003G 4.3.8 Binary Unions

Let $X$ be a set and let $U, V \in \mathcal{P}(X)$.

003H **Definition 4.3.8.1.1.** The **union of** $U$ **and** $V$ is the set $U \cup V$ defined by

$$\begin{aligned} U \cup V &\overset{\text{def}}{=} \bigcup_{z \in \{U,V\}} z \\ &\overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

003J **Proposition 4.3.8.1.2.** Let $X$ be a set.

003K 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon \quad & (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ - \cup V\colon \quad & (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon \quad & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01GY (a) If $U \subset A$, then $U \cup V \subset A \cup V$.

01GZ (b) If $V \subset B$, then $U \cup V \subset U \cup B$.

01H0 (c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

003M 2. *Associativity.* The diagram

$$\begin{tikzcd} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"] & \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}", "\sim"'] \arrow[d, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X), \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

003N 3. *Unitality*. The diagrams

$$\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[\emptyset]\times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\sim" above, "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
 & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [\emptyset]}"] \arrow[dr, dashed, "\sim" above, "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
\emptyset \cup U &= U, \\
U \cup \emptyset &= U
\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

003P 4. *Commutativity*. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup V = V \cup U$$

for each $U, V \in \mathcal{P}(X)$.

01H1 5. *Annihilation With $X$*. The diagrams

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathrm{pt} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \varepsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}"] \arrow[d, "{[X]\times \mathrm{id}_{\mathcal{P}(X)}}"'] & & \mathrm{pt} \arrow[d, "{[X]}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times \mathrm{pt} \arrow[ur, "\varepsilon^{\mathsf{Sets}}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \times [X]}"'] & & \mathrm{pt} \arrow[d, "{[X]}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
U \cup X &= X,\\
X \cup V &= X
\end{aligned}
$$

for each $U, V \in \mathcal{P}(X)$.

003R 6. *Distributivity of Unions Over Intersections.* The diagrams

$$
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
U \cup (V \cap W) &= (U \cup V) \cap (U \cup W),\\
(U \cap V) \cup W &= (U \cup W) \cap (V \cup W)
\end{aligned}
$$

for each $U, V, W \in \mathcal{P}(X)$.

01H2 7. *Distributivity of Intersections Over Unions.* The diagrams

$$
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cup"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
U \cap (V \cup W) &= (U \cap V) \cup (U \cap W),\\
(U \cup V) \cap W &= (U \cap W) \cup (V \cap W)
\end{aligned}
$$

for each $U, V, W \in \mathcal{P}(X)$.

003Q 8. *Idempotency.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\Delta_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each $U \in \mathcal{P}(X)$.

003L 9. *Via Intersections and Symmetric Differences.* The diagram

$$\begin{tikzcd} (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[r, "\triangle \times \cap"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\triangle"] & \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[u, "\Delta_{\mathcal{P}(X) \times \mathcal{P}(X)}"] \arrow[rr, "\cup"'] & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

003S 10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

003T 11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

01H3 12. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cup"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

01H4 13. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01H5 14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cup"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

003U 15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

*Proof.* *Item 1, Functoriality*: See [Pro25an].
*Item 2, Associativity*: See [Pro25ba].
*Item 3, Unitality*: This follows from [Pro25bd] and Item 4.
*Item 4, Commutativity*: See [Pro25bb].
*Item 5, Annihilation With $X$*: We have

$$\begin{aligned} U \cup X &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\ &= \{x \in X \mid x \in X\}, \\ &= X \end{aligned}$$

and

$$\begin{aligned} X \cup V &\stackrel{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\ &= \{x \in X \mid x \in X\} \\ &= X. \end{aligned}$$

This finishes the proof.
*Item 6, Distributivity of Unions Over Intersections*: See [Pro25az].
*Item 7, Distributivity of Intersections Over Unions*: See [Pro25aj].
*Item 8, Idempotency*: See [Pro25am].
*Item 9, Via Intersections and Symmetric Differences*: See [Pro25ay].
*Item 10, Interaction With Characteristic Functions I*: See [Pro25h].
*Item 11, Interaction With Characteristic Functions II*: See [Pro25h].
*Item 12, Interaction With Direct Images*: See [Pro25p].
*Item 13, Interaction With Inverse Images*: See [Pro25y].
*Item 14, Interaction With Codirect Images*: This is a repetition of Item 5 of Definition 4.6.3.1.7 and is proved there.
*Item 15, Interaction With Powersets and Semirings*: This follows from Items 2 to 4 and 8 of this propostition and Items 3 to 6 and 8 of Definition 4.3.9.1.2. □

## 003X 4.3.9 Binary Intersections

Let $X$ be a set and let $U, V \in \mathcal{P}(X)$.

003Y **Definition 4.3.9.1.1.** The **intersection of** $U$ **and** $V$ is the set $U \cap V$ defined by

$$\begin{aligned} U \cap V &\overset{\text{def}}{=} \bigcap_{z \in \{U, V\}} z \\ &\overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

003Z **Proposition 4.3.9.1.2.** Let $X$ be a set.

0040 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -&\colon & (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \cap V&\colon & (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2&\colon & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01H6 (a) If $U \subset A$, then $U \cap V \subset A \cap V$.

01H7 (b) If $V \subset B$, then $U \cap V \subset U \cap B$.

01H8 (c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

0041 2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U \cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[U,-]_X}"] \end{tikzcd}$$

$$(- \cap V \dashv [V, -]_X)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "- \cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[V,-]_X}"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor of Section 4.4.7. In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

01H9 (a) The following conditions are equivalent:

01HA i. We have $U \cap V \subset W$.

01HB ii. We have $U \subset [V, W]_X$.

01HC (b) The following conditions are equivalent:

01HD i. We have $U \cap V \subset W$.

01HE ii. We have $V \subset [U, W]_X$.

0042 3. *Associativity.* The diagram

$$
\begin{tikzcd}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \cap"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}", "\sim"'] \arrow[d, "\cap \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X),
\end{tikzcd}
$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

0043 4. *Unitality.* The diagrams

$$
\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[X]} \times \mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \times {[X]}"] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
X \cap U &= U, \\
U \cap X &= U
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

0044 5. *Commutativity.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\cap"'] & \mathcal{P}(X)\times\mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each $U, V \in \mathcal{P}(X)$.

0047 6. *Annihilation With the Empty Set.* The diagrams

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathrm{pt}\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}"] \arrow[d, "{[\emptyset]\times\mathrm{id}_{\mathcal{P}(X)}}"'] & & \mathrm{pt} \arrow[d, "{[\emptyset]}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times\mathrm{pt} \arrow[ur, "\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)}\times[\emptyset]}"'] & & \mathrm{pt} \arrow[d, "{[\emptyset]}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
\emptyset \cap X &= \emptyset, \\
X \cap \emptyset &= \emptyset
\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

01HF 7. *Distributivity of Unions Over Intersections.* The diagrams

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

0046 8. *Distributivity of Intersections Over Unions.* The diagrams

$$\begin{tikzcd}[column sep=tiny]
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)} \times (\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cap \times \cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)}) \times \Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cap \times \cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

0045 9. *Idempotency.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\Delta_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

0048 10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U\cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

0049 11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U\cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

01HG 12. *Interaction With Direct Images.* Let $f : X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cap"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01HH 13. *Interaction With Inverse Images.* Let $f : X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01HJ 14. *Interaction With Codirect Images.* Let $f : X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cap"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

004A 15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

004B 16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

*Proof.* *Item 1*, *Functoriality*: See [Pro25al].
*Item 2*, *Adjointness*: See [MSE 267469].
*Item 3*, *Associativity*: See [Pro25r].
*Item 4*, *Unitality*: This follows from [Pro25v] and Item 5.
*Item 5*, *Commutativity*: See [Pro25s].
*Item 6*, *Annihilation With the Empty Set*: This follows from [Pro25t] and Item 5.
*Item 7*, *Distributivity of Unions Over Intersections*: See [Pro25az].
*Item 8*, *Distributivity of Intersections Over Unions*: See [Pro25aj].
*Item 9*, *Idempotency*: See [Pro25ak].
*Item 10*, *Interaction With Characteristic Functions I*: See [Pro25e].
*Item 11*, *Interaction With Characteristic Functions II*: See [Pro25e].
*Item 12*, *Interaction With Direct Images*: See [Pro25n].
*Item 13*, *Interaction With Inverse Images*: See [Pro25w].
*Item 14*, *Interaction With Codirect Images*: This is a repetition of Item 6 of Definition 4.6.3.1.7 and is proved there.
*Item 15*, *Interaction With Powersets and Monoids With Zero*: This follows from Items 3 to 6.
*Item 16*, *Interaction With Powersets and Semirings*: This follows from Items 2 to 4 and 8 and Items 3 to 6 and 8 of Definition 4.3.9.1.2. □

## 004D 4.3.10 Differences

Let $X$ and $Y$ be sets.

004E **Definition 4.3.10.1.1.** The **difference of** $X$ **and** $Y$ is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

004F **Proposition 4.3.10.1.2.** Let $X$ be a set.

004G 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon & \quad (\mathcal{P}(X), \supset) && \to (\mathcal{P}(X), \subset), \\ - \setminus V\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01HK (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.

01HL (b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.

01HM (c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

004H 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

004J 3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

004K 4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

004L 5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

004M 6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

004N 7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

004P 8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $U, V \in \mathcal{P}(X)$.

004Q 9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \bigtriangleup (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

004R 10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

004S 11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

004T 12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

01HN 13. *Right Annihilation.* We have

$$U \setminus X = \emptyset$$

for each $U \in \mathcal{P}(X)$.

004U 14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

004V 15. *Interaction With Containment.* The following conditions are equivalent:

004W (a) We have $V \setminus U \subset W$.

004X (b) We have $V \setminus W \subset U$.

004Y 16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

01HP 17. *Interaction With Direct Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01HQ 18. *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathrm{op},-1} \times f^{-1}"] \arrow[d, "\setminus"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\setminus"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

01HR 19. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathrm{op}} \times f_!"] \arrow[d, "\setminus"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] \arrow[ur, phantom, "\supset"] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

*Proof.* *Item 1, Functoriality*: See [Pro25ad] and [Pro25ah].
*Item 2, De Morgan's Laws*: See [Pro25k].
*Item 3, Interaction With Unions I*: See [Pro25l].
*Item 4, Interaction With Unions II*: We have

$$\begin{aligned}
(U \setminus V) \cup W &\overset{\text{def}}{=} \{x \in X \mid (x \in U \text{ and } x \notin V) \text{ or } x \in W\} \\
&= \{x \in X \mid (x \in U \text{ or } x \in W) \text{ and } (x \notin V \text{ or } x \in W)\} \\
&= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \text{ and } x \notin W)\} \\
&= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \setminus W)\}
\end{aligned}$$

$$\begin{aligned}&= \{x \in X \mid (x \in (U \cup W) \setminus (V \setminus W))\}\\ &= (U \cup W) \setminus (V \setminus W).\end{aligned}$$

*Item 5, Interaction With Unions III*: See [Pro25ai].
*Item 6, Interaction With Unions IV*: See [Pro25ac].
*Item 7, Interaction With Intersections*: See [Pro25u].
*Item 8, Interaction With Complements*: See [Pro25aa].
*Item 9, Interaction With Symmetric Differences*: See [Pro25ab].
*Item 10, Triple Differences*: See [Pro25ag].
*Item 11, Left Annihilation*: The direction $\emptyset \subset \emptyset \setminus U$ always holds. Now assume $x \in \emptyset \setminus U$. Then, $x \in \emptyset$ and $x \notin U$. Hence $\emptyset \setminus U \subset \emptyset$ must hold and the sets are equal.
*Item 12, Right Unitality*: See [Pro25ae].
*Item 13, Right Annihilation*: It suffices to show that no $x \in X$ can be an element of $U \setminus X$. Assume $x \in U \setminus X$. Then $x \notin X$, contradicting $x \in X$. This completes the proof.
*Item 14, Invertibility*: See [Pro25af].
*Item 15, Interaction With Containment*: The conditions are symmetric in $U, W$, hence it suffices to show that $V \setminus U \subset W$ implies $V \setminus W \subset U$. So assume $V \setminus U \subset W, x \in V \setminus W$. Then $x \in V, x \notin W$. So by contraposition, $x \notin V \setminus U$. But $x \in V$, so we must have $x \in U$, completing the proof.
*Item 16, Interaction With Characteristic Functions*: See [Pro25f].
*Item 17, Interaction With Direct Images*: See [Pro25o].
*Item 18, Interaction With Inverse Images*: See [Pro25x]. □

## 004Z 4.3.11 Complements

Let $X$ be a set and let $U \in \mathcal{P}(X)$.

0050 **Definition 4.3.11.1.1.** The **complement of** $U$ is the set $U^{\mathsf{c}}$ defined by

$$\begin{aligned}U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U\\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}.\end{aligned}$$

0051 **Proposition 4.3.11.1.2.** Let $X$ be a set.

0052 1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X).$$

In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

(★) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

0053 2. *De Morgan's Laws.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cup^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}\times(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}\times(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
(U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}}, \\
(U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}}
\end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

0054 3. *Involutority.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}"'] & \mathcal{P}(X) \arrow[d, "(-)^{\mathsf{c},\mathsf{op}}"] \\
& \mathcal{P}(X)^{\mathsf{op}}
\end{tikzcd}$$

commutes, i.e. we have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $U \in \mathcal{P}(X)$.

0055 4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

01HS 5. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U^\mathsf{c}) = f_*(U)^\mathsf{c}$$

for each $U \in \mathcal{P}(X)$.

01HT 6. *Interaction With Inverse Images.* Let $f: X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "(-)^\mathsf{c}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^\mathsf{c}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U^\mathsf{c}) = f^{-1}(U)^\mathsf{c}$$

for each $U \in \mathcal{P}(X)$.

01HU 7. *Interaction With Codirect Images.* Let $f: X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "(-)^\mathsf{c}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^\mathsf{c}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U^\mathsf{c}) = f_!(U)^\mathsf{c}$$

for each $U \in \mathcal{P}(X)$.

*Proof.* *Item 1, Functoriality*: This follows from Item 1 of Definition 4.3.10.1.2.
*Item 2, De Morgan's Laws*: See [Pro25k].
*Item 3, Involutority*: See [Pro25i].
*Item 4, Interaction With Characteristic Functions*: We consider the two cases $x \in U, x \notin U$.

1. If $x \in U$, then $x \notin U^{\mathsf{c}}$. So $\chi_U(x) = 1$ and

$$\begin{aligned}\chi_{U^{\mathsf{c}}}(x) &= 0\\ &= 1 - \chi_U(x).\end{aligned}$$

2. If $x \notin U$, then $x \in U^{\mathsf{c}}$. So $\chi_U(x) = 0$ and

$$\begin{aligned}\chi_{U^{\mathsf{c}}}(x) &= 1\\ &= 1 - \chi_U(x).\end{aligned}$$

Hence, the equation holds for all $x \in X$.
*Item 5, Interaction With Direct Images*: This is a repetition of Item 8 of Definition 4.6.1.1.5 and is proved there.
*Item 6, Interaction With Inverse Images*: This is a repetition of Item 8 of Definition 4.6.2.1.3 and is proved there.
*Item 7, Interaction With Codirect Images*: This is a repetition of Item 7 of Definition 4.6.3.1.7 and is proved there. □

## 0056 4.3.12 Symmetric Differences

Let $X$ be a set and let $U, V \in \mathcal{P}(X)$.

0057 **Definition 4.3.12.1.1.** The **symmetric difference of** $U$ **and** $V$ is the set $U \triangle V$ defined by[13]

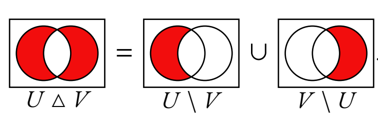
$$U \triangle V \overset{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

0058 **Proposition 4.3.12.1.2.** Let $X$ be a set.

0059 1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***does not*** in general define functors

$$\begin{aligned}U \triangle -\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset),\\ - \triangle V\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset),\\ -_1 \triangle -_2\colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) && \to (\mathcal{P}(X), \subset).\end{aligned}$$

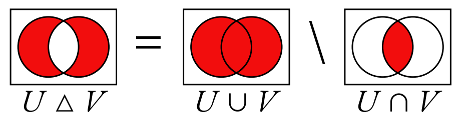
[13] *Illustration:*



005A 2. *Via Unions and Intersections.* We have

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram




01HV 3. *Symmetric Differences of Disjoint Sets.* If $U$ and $V$ are disjoint, then we have

$$U \triangle V = U \cup V.$$

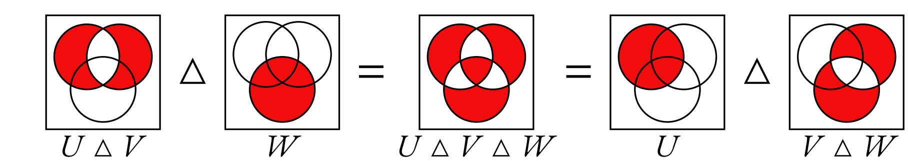
005B 4. *Associativity.* The diagram

$$\begin{tikzcd}
& \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \triangle"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}", "\sim"'] \arrow[d, "\triangle \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\triangle"'] & & \mathcal{P}(X),
\end{tikzcd}$$

commutes, i.e. we have

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram



005D 5. *Unitality.* The diagrams

$$\begin{tikzcd} \mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[\varnothing] \times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ & \mathcal{P}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [\varnothing]}"] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned} U \triangle \varnothing &= U, \\ \varnothing \triangle U &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

005C 6. *Commutativity.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\triangle"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$U \triangle V = V \triangle U$$

for each $U, V \in \mathcal{P}(X)$.

005E 7. *Invertibility.* We have

$$U \triangle U = \varnothing$$

for each $U \in \mathcal{P}(X)$.

005F 8. *Interaction With Unions.* We have

$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

005G 9. *Interaction With Complements I.* We have

$$U \triangle U^{\mathsf{c}} = X$$

for each $U \in \mathcal{P}(X)$.

005H 10. *Interaction With Complements II.* We have

$$\begin{aligned} U \triangle X &= U^{\mathsf{c}}, \\ X \triangle U &= U^{\mathsf{c}} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

005J 11. *Interaction With Complements III.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X) \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$U^{\mathsf{c}} \triangle V^{\mathsf{c}} = U \triangle V$$

for each $U, V \in \mathcal{P}(X)$.

005K 12. *"Transitivity".* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $U, V, W \in \mathcal{P}(X)$.

005L 13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $U, V, W \in \mathcal{P}(X)$.

005M 14. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

005N 15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

005P 16. *Bijectivity.* Given $U, V \in \mathcal{P}(X)$, the maps

$$\begin{aligned} U \triangle -&\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle V&\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are self-inverse bijections. Moreover, the map

$$\begin{aligned} \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ C &\longmapsto C \triangle (U \triangle V) \end{aligned}$$

is a bijection of $\mathcal{P}(X)$ onto itself sending $U$ to $V$ and $V$ to $U$.

005Q 17. *Interaction With Powersets and Groups.* Let $X$ be a set.

005R (a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[14]

005S (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

[14] Here are some examples:

020H 1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}) \cong \mathrm{pt}.$$

020J 2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}) \cong \mathbb{Z}_{/2}.$$

020K 3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0,1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$(\mathcal{P}(\{0,1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

005T 4. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

- The group $\mathcal{P}(X)$ of Item 17;
- The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

is an $\mathbb{F}_2$-vector space.

005U 5. *Interaction With Powersets and Vector Spaces II.* If $X$ is finite, then:

020L (a) The set of singletons sets on the elements of $X$ forms a basis for the $\mathbb{F}_2$-vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of Item 4.

020M (b) We have

$$\dim(\mathcal{P}(X)) = \#X.$$

005V 6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[15]

01HW 7. *Interaction With Direct Images.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_!(U) \triangle f_!(V) \subset f_!(U \triangle V)$$

indexed by $U, V \in \mathcal{P}(X)$.

[15] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25aw] for a proof.

01HX 8. *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathrm{op},-1} \times f^{-1}"] \arrow[d, "\triangle"'] & \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

i.e. we have

$$f^{-1}(U) \triangle f^{-1}(V) = f^{-1}(U \triangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

01HY 9. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathrm{op}} \times f_*"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U \triangle V) \subset f_*(U) \triangle f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

*Proof. Item 1, Lack of Functoriality*: Let $X = \{0, 1\}$, $U = \{0\}$. Then $\emptyset \subset U$, but $U \triangle \emptyset = U \not\subset \emptyset = U \triangle U$ from Item 5 and Item 7. This gives a counterexample to the first statement. By using Item 6, we can adapt it to the second and third statement.

*Item 2, Via Unions and Intersections*: See [Pro25m].

*Item 3, Symmetric Differences of Disjoint Sets*: Since $U$ and $V$ are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$\begin{aligned} U \triangle V &= (U \cup V) \setminus (U \cap V) \\ &= (U \cup V) \setminus \emptyset \end{aligned}$$

$$= U \cup V,$$

where we've used Item 2 and Item 12 of Definition 4.3.10.1.2.
*Item 4, Associativity*: See [Pro25ao].
*Item 5, Unitality*: This follows from Item 6 and [Pro25at].
*Item 6, Commutativity*: See [Pro25ap].
*Item 7, Invertibility*: See [Pro25av].
*Item 8, Interaction With Unions*: See [Pro25bc].
*Item 9, Interaction With Complements I*: See [Pro25as].
*Item 10, Interaction With Complements II*: This follows from Item 6 and [Pro25ax].
*Item 11, Interaction With Complements III*: See [Pro25aq].
*Item 12, "Transitivity"*: We have

$$\begin{aligned}
(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 4)}\\
&= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 4)}\\
&= U \triangle (\emptyset \triangle W) && \text{(by Item 7)}\\
&= U \triangle W. && \text{(by Item 5)}
\end{aligned}$$

This finishes the proof.
*Item 13, The Triangle Inequality for Symmetric Differences*: This follows from Items 2 and 12.
*Item 14, Distributivity Over Intersections*: See [Pro25q].
*Item 15, Interaction With Characteristic Functions*: See [Pro25g].
*Item 16, Bijectivity*:

- We show that
$$(U \triangle -) \colon \mathcal{P}(X) \to \mathcal{P}(X)$$
is self-inverse.

  Let $W \in \mathcal{P}(X)$. Then,
$$\begin{aligned}
U \triangle (U \triangle W) &= (U \triangle U) \triangle W && \text{(by Item 4)}\\
&= \emptyset \triangle W && \text{(by Item 7)}\\
&= W. && \text{(by Item 5)}
\end{aligned}$$

- By Item 6, $(- \triangle V) = (V \triangle -)$, hence the former is also self-inverse by the first point.

- The map $- \triangle (U \triangle V)$ is a bijection as a special case of the second point.

From the first two points and Item 6, we get

$$U \triangle (U \triangle V) = V, \; V \triangle (U \triangle V) = V \triangle (V \triangle U) = U.$$

Hence the function maps $U$ to $V$ and $V$ to $U$.

*Item 17, Interaction With Powersets and Groups*: Item 17a follows from Items 4 to 7, while Item 3b follows from Item 7.[16]
*Item 4, Interaction With Powersets and Vector Spaces I*: See [MSE 2719059].
*Item 5, Interaction With Powersets and Vector Spaces II*: See [MSE 2719059].
*Item 6, Interaction With Powersets and Rings*: This follows from Items 6 and 15 of Definition 4.3.9.1.2 and Items 14 and 17.[17]
*Item 7, Interaction With Direct Images*: This is a repetition of Item 9 of Definition 4.6.1.1.5 and is proved there.
*Item 8, Interaction With Inverse Images*: This is a repetition of Item 9 of Definition 4.6.2.1.3 and is proved there.
*Item 9, Interaction With Codirect Images*: This is a repetition of Item 8 of Definition 4.6.3.1.7 and is proved there. □

## 005W 4.4 Powersets

### 01HZ 4.4.1 Foundations

Let $X$ be a set.

006P **Definition 4.4.1.1.1.** The **powerset of** $X$ is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where $P$ is the set in the axiom of powerset, **??** of **??**.

006Q **Remark 4.4.1.1.2.** Under the analogy that $\{\mathsf{t}, \mathsf{f}\}$ should be the $(-1)$-categorical analogue of Sets, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set $X$ is equivalently (Item 2 of Definition 4.5.1.1.4) the

[16] *Reference:* [Pro25ar].
[17] *Reference:* [Pro25au].

set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of functions from $X$ to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category $C$ is the category

$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$

of functors from $C^{\mathsf{op}}$ to the category $\mathsf{Sets}$ of sets.

01J0 **Notation 4.4.1.1.3.** Let $X$ be a set.

01J1 1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of $X$.

01J2 2. We write $\mathcal{P}_{\mathrm{fin}}(X)$ for the set of finite subsets of $X$.

01J3 **Proposition 4.4.1.1.4.** Let $X$ be a set.

01J4 1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

020P (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

020Q (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

020R (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

01J5 2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.

01J6 3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned} \mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X), \\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}), \end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

01J7 4. *Interaction With Products I.* The map

$$\begin{aligned} \mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \textstyle\coprod Y) \\ (U, V) &\longmapsto U \cup V \end{aligned}$$

is an isomorphism of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

01J8 5. *Interaction With Products II.* The map

$$\begin{aligned} \mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \textstyle\coprod Y) \\ (U, V) &\longmapsto U \boxtimes_{X \times Y} V, \end{aligned}$$

where[18]

$$U \boxtimes_{X \times Y} V \stackrel{\text{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

is an inclusion of sets, natural in $X, Y \in \text{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

01J9 6. *Interaction With Products III.* We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1, where $\otimes$ denotes the tensor product of suplattices of **??**. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

*Proof.* *Item 1, Co/Completeness*: Omitted.
*Item 2, Cartesian Closedness*: See Section 4.4.7.
*Item 3, Powersets as Sets of Relations*: Indeed, we have

$$\begin{aligned} \text{Rel}(\text{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\text{pt} \times X) \\ &\cong \mathcal{P}(X) \end{aligned}$$

and

$$\begin{aligned} \text{Rel}(X, \text{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \text{pt}) \\ &\cong \mathcal{P}(X), \end{aligned}$$

where we have used Item 5 of Definition 4.1.3.1.3.

[18] The set $U \boxtimes_{X \times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see

*Item 4, Interaction With Products I*: The inverse of the map in the statement is the map

$$\Phi\colon \mathcal{P}(X \coprod Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \stackrel{\text{def}}{=} (S_X, S_Y)$$

for each $S \in \mathcal{P}(X \coprod Y)$, where

$$\begin{aligned} S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\} \\ S_Y &\stackrel{\text{def}}{=} \left\{y \in Y \,\middle|\, (1, y) \in S\right\}. \end{aligned}$$

The rest of the proof is omitted.
*Item 5, Interaction With Products II*: Omitted.
*Item 6, Interaction With Products III*: Omitted. □

## 01JA 4.4.2 Functoriality of Powersets

01JB **Proposition 4.4.2.1.1.** Let $X$ be a set.

01JC 1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_!\colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at $(A, B)$ is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}_!(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in Definition 4.6.1.1.1.

01JD 2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of $\mathcal{P}^{-1}$ at $(A, B)$ is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in Definition 4.6.2.1.1.

01JE 3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

also Section 4.6.4).

of $\mathcal{P}_*$ at $(A, B)$ is the map defined by by sending a map of sets $f : A \to B$ to the map

$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in Definition 4.6.3.1.1.

*Proof.* *Item 1, Functoriality I*: This follows from Items 3 and 4 of Definition 4.6.1.1.6.
*Item 2, Functoriality II*: This follows from Items 3 and 4 of Definition 4.6.2.1.4.
*Item 3, Functoriality III*: This follows from Items 3 and 4 of Definition 4.6.3.1.8. □

## 01JF 4.4.3 Adjointness of Powersets I

01JG **Proposition 4.4.3.1.1.** We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

*Proof.* We have

$$\begin{aligned} \mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\ &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)} \end{aligned}$$

where all bijections are natural in $A$ and $B$.[19] □

[19] Here we are using Item 3 of Definition 4.5.1.1.4.

## 01JH 4.4.4 Adjointness of Powersets II

01JJ **Proposition 4.4.4.1.1.** We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_!)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_!"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(X), Y) \cong \mathsf{Sets}(X, \mathcal{P}(Y))$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, Item 1 of Definition 8.2.2.1.2 and $\mathcal{P}_!$ is the functor of Relations, Definition 8.7.5.1.1.

*Proof.* We have

$$\begin{aligned} \mathrm{Rel}(\mathrm{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)} \end{aligned}$$

where all bijections are natural in $A$, (where we are using Item 3 of Definition 4.5.1.1.4). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending $a$ to the subset $R(a)$ of $B$, as in Relations, Definition 8.1.1.1.1.

Naturality in $B$ is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$\begin{tikzcd} \mathrm{Rel}(\mathrm{Gr}(A), B) \arrow[r, "R\diamond-"] \arrow[d, dashed, "\wr"'] & \mathrm{Rel}(\mathrm{Gr}(A), B') \arrow[d, dashed, "\wr"] \\ \mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_!"'] & \mathsf{Sets}(A, \mathcal{P}(B')) \end{tikzcd}$$

commutes, which follows from Relations, Definition 8.7.1.1.3. □

## 01JK 4.4.5 Powersets as Free Cocompletions

Let $X$ be a set.

01JL **Proposition 4.4.5.1.1.** The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of $X$ of Definition 4.4.1.1.1;
- The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of $X$ into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

(★) Given another pair $(Y, f)$ consisting of

- A suplattice $(Y, \preceq)$;
- A function $f \colon X \to Y$;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{tikzcd}
 & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\
X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

commute.

*Proof.* This is a rephrasing of Definition 4.4.5.1.2, which we prove below.[20] □

01JM **Proposition 4.4.5.1.2.** We have an adjunction

$$\begin{tikzcd}
(\mathcal{P} \dashv 忘) \colon \quad \mathsf{Sets} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathsf{SupLat}, \arrow[l, bend left, "忘"]
\end{tikzcd}$$

[20] Here we only remark that the unique morphism of suplattices in the statement is given by

witnessed by a bijection

$$\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{SupLat})$, where:

- The category $\mathsf{SupLat}$ is the category of suplattices of **??**.
- The map

  $$\chi_X^* \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  witnessing the above bijection is defined by

  $$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of suplattices $f \colon \mathcal{P}(X) \to Y$ to the composition

  $$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  witnessing the above bijection is given by sending a function $f \colon X \to Y$ to its left Kan extension along $\chi_X$,

  $$\mathrm{Lan}_{\chi_X}(f) \colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, ""] & Y. \end{tikzcd}$$

  Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$[\mathrm{Lan}_{\chi_X}(f)](U) = \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)$$

---

the left Kan extension $\mathrm{Lan}_{\chi_X}(f)$ of $f$ along $\chi_X$.

$$
\begin{aligned}
&= \int^{x\in X} \chi_U(x) \odot f(x)\\
&= \bigvee_{x\in X} (\chi_U(x) \odot f(x))\\
&= \left(\bigvee_{x\in U} (\chi_U(x) \odot f(x))\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} (\chi_U(x) \odot f(x))\right)\\
&= \left(\bigvee_{x\in U} f(x)\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} \varnothing_Y\right)\\
&= \bigvee_{x\in U} f(x)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigvee$ denotes the join in $(Y, \preceq)$.
- The symbol $\odot$ denotes the tensor of an element of $Y$ by a truth value as in **??**. In particular, we have

  $$
  \begin{aligned}
  \mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x),\\
  \mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y,
  \end{aligned}
  $$

  where $\varnothing_Y$ is the bottom element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set $B$, the Kan extension $\mathrm{Lan}_{\chi_X}(f)$ is given by

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f)](U) &= \bigvee_{x\in U} f(x)\\
&= \bigcup_{x\in U} f(x)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

*Proof. Map I*: We define a map

$$\Phi_{X,Y}\colon \mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq)) \to \mathsf{Sets}(X,Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \overset{\text{def}}{=} f\circ\chi_X$$

for each $f\in\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))$.
*Map II*: We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}(X,Y) \to \mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow", near end] & Y, \end{tikzcd}$$

for each $f\in\mathsf{Sets}(X,Y)$.
*Invertibility I*: We claim that

$$\Psi_{X,Y}\circ\Phi_{X,Y} = \mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))}.$$

We have

$$\begin{aligned}[\Psi_{X,Y}\circ\Phi_{X,Y}](f) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f))\\ &\overset{\text{def}}{=} \Psi_{X,Y}(f\circ\chi_X)\\ &\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f\circ\chi_X)\end{aligned}$$

for each $f\in\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f\circ\chi_X) = f$$

for each $f\in\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))$. Indeed, we have

$$\left[\mathrm{Lan}_{\chi_X}(f\circ\chi_X)\right](U) = \bigvee_{x\in U} f(\chi_X(x))$$

$$
\begin{aligned}
&= f\left(\bigvee_{x\in U} \chi_X(x)\right)\\
&= f\left(\bigcup_{x\in U} \{x\}\right)\\
&= f(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we have used that $f$ is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))}$ of $\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

*Invertibility II*: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

We have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Lan}_{\chi_X}(f))\\
&\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f) \circ \chi_X
\end{aligned}
$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f) \circ \chi_X](x) &= \bigvee_{y\in\{x\}} f(y)\\
&= f(x)
\end{aligned}
$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

*Naturality for* $\Phi$*, Part I*: We need to show that, given a function $f\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_!(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
&\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
&= \xi \circ (f_! \circ \chi_X) \\
&\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\overset{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

*Naturality for* $\Phi$*, Part II*: We need to show that, given a morphism of suplattices

$$g\colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_!"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_!"] \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, we have

$$[\Phi_{X,Y'} \circ g_!](\xi) \overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi))$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\
&\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X\\
&= g \circ (\xi \circ \chi_X)\\
&\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))\\
&\stackrel{\text{def}}{=} g_!(\Phi_{X,Y}(\xi))\\
&\stackrel{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}
$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

*Naturality for* $\Psi$: Since $\Phi$ is natural in each argument and $\Phi$ is a componentwise inverse to $\Psi$ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\Psi$ is also natural in each argument. □

01JN **Warning 4.4.5.1.3.** Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion of* $X$, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.

## 01JP 4.4.6 Powersets as Free Completions

Let $X$ be a set.

01JQ **Proposition 4.4.6.1.1.** The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset of $X$ together with reverse inclusion $\mathcal{P}(X)^{\mathsf{op}} = (\mathcal{P}(X), \supset)$ of Definition 4.4.1.1.1;
- The characteristic embedding $\chi_{(-)} \colon X \longrightarrow \mathcal{P}(X)$ of $X$ into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

(★) Given another pair $(Y, f)$ consisting of

- An inflattice $(Y, \preceq)$;
- A function $f \colon X \to Y$;

there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

*Proof.* This is a rephrasing of Definition 4.4.6.1.2, which we prove below.[21] □

01JR **Proposition 4.4.6.1.2.** We have an adjunction

$$(\mathcal{P} \dashv 忘): \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathsf{InfLat}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathsf{Obj}(\mathsf{InfLat})$, where:

- The category $\mathsf{InfLat}$ is the category of inflattices of **??**.
- The map

$$\chi_X^*\colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of inflattices $f\colon \mathcal{P}(X)^{\mathsf{op}} \to Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X)^{\mathsf{op}} \xrightarrow{f} Y.$$

[21] Here we only remark that the unique morphism of inflattices in the statement is given by the right Kan extension $\mathrm{Ran}_{\chi_X}(f)$ of $f$ along $\chi_X$.

- The map

$$\mathrm{Ran}_{\chi_X}\colon \mathsf{Sets}(X,Y) \to \mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))$$

witnessing the above bijection is given by sending a function $f\colon X \to Y$ to its right Kan extension along $\chi_X$,

$$\mathrm{Ran}_{\chi_X}(f)\colon \mathcal{P}(X)^{\mathsf{op}} \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\mathrm{Ran}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow"] & Y. \end{tikzcd}$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned}
[\mathrm{Ran}_{\chi_X}(f)](U) &= \int_{x\in X} \chi_{\mathcal{P}(X)^{\mathsf{op}}}(\chi_x, U) \pitchfork f(x)\\
&= \int_{x\in X} \chi_{\mathcal{P}(X)}(U,\chi_x) \pitchfork f(x)\\
&= \int_{x\in X} \chi_U(x) \pitchfork f(x)\\
&= \bigwedge_{x\in X} \chi_U(x) \pitchfork f(x)\\
&= \left(\bigwedge_{x\in U} \chi_U(x) \pitchfork f(x)\right) \wedge \left(\bigwedge_{x\in U^{\mathsf{c}}} \chi_U(x) \pitchfork f(x)\right)\\
&= \left(\bigwedge_{x\in U} f(x)\right) \wedge \left(\bigwedge_{x\in U^{\mathsf{c}}} \infty_Y\right)\\
&= \left(\bigwedge_{x\in U} f(x)\right) \wedge \infty_Y\\
&= \bigwedge_{x\in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

  – We have used **??** for the first equality.

- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigwedge$ denotes the meet in $(Y, \preceq)$.
- The symbol $\pitchfork$ denotes the cotensor of an element of $Y$ by a truth value as in **??**. In particular, we have

$$\begin{aligned} \mathsf{true} \pitchfork f(x) &\stackrel{\text{def}}{=} f(x), \\ \mathsf{false} \pitchfork f(x) &\stackrel{\text{def}}{=} \infty_Y, \end{aligned}$$

  where $\infty_Y$ is the top element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set $B$, the Kan extension $\mathrm{Ran}_{\chi_X}(f)$ is given by

$$\begin{aligned} [\mathrm{Ran}_{\chi_X}(f)](U) &= \bigwedge_{x \in U} f(x) \\ &= \bigcap_{x \in U} f(x) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

*Proof. Map I*: We define a map

$$\Phi_{X,Y} \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
*Map II*: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\mathrm{Ran}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[r, phantom, bend left, "\Downarrow"] & Y, \end{tikzcd}$$

for each $f \in \mathsf{Sets}(X, Y)$.
*Invertibility I*: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))}.$$

We have

$$\begin{aligned}[\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\overset{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\overset{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f \circ \chi_X)\end{aligned}$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned}\left[\mathrm{Ran}_{\chi_X}(f \circ \chi_X)\right](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\ &= f\left(\bigwedge_{x \in U} \chi_X(x)\right) \\ &= f\left(\bigcup_{x \in U} \{x\}\right) \\ &= f(U)\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that $f$ is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))}$ of $\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
*Invertibility II*: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

We have

$$\begin{aligned}[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\ &\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Ran}_{\chi_X}(f)) \\ &\overset{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f) \circ \chi_X\end{aligned}$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned}[\mathrm{Ran}_{\chi_X}(f) \circ \chi_X](x) &= \bigwedge_{y \in \{x\}} f(y) \\ &= f(x)\end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

*Naturality for* $\Phi$, *Part I*: We need to show that, given a function $f : X \to X'$, the diagram

$$\begin{tikzcd} \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_!(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\ \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y) \end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\ &\overset{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\ &\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\ &= \xi \circ (f_! \circ \chi_X)\end{aligned}$$

$$\begin{aligned}
&\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f)\\
&= (\xi \circ \chi_{X'}) \circ f\\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f\\
&\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi))\\
&\overset{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

*Naturality for* $\Phi$*, Part II*: We need to show that, given a cocontinuous morphism of posets

$$g\colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_!"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_!"]\\
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_!](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi))\\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X\\
&= g \circ (\xi \circ \chi_X)\\
&\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))\\
&\overset{\text{def}}{=} g_!(\Phi_{X,Y}(\xi))\\
&\overset{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

*Naturality for* $\Psi$: Since $\Phi$ is natural in each argument and $\Phi$ is a componentwise inverse to $\Psi$ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\Psi$ is also natural in each argument. □

01JS **Warning 4.4.6.1.3.** Although the assignment $X \mapsto \mathcal{P}(X)^{\mathsf{op}}$ is called the *free completion of* $X$, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}})^{\mathsf{op}} \neq \mathcal{P}(X)^{\mathsf{op}}$.

## 01JT 4.4.7 The Internal Hom of a Powerset

Let $X$ be a set and let $U, V \in \mathcal{P}(X)$.

01JU **Proposition 4.4.7.1.1.** The **internal Hom of** $\mathcal{P}(X)$ **from** $U$ **to** $V$ is the subset $[U, V]_X$[22] of $X$ given by

$$\begin{aligned}[U, V]_X &= U^{\mathsf{c}} \cup V \\ &= (U \setminus V)^{\mathsf{c}}\end{aligned}$$

where $U^{\mathsf{c}}$ is the complement of $U$ of Definition 4.3.11.1.1.

*Proof.* *Proof of the Equality $U^{\mathsf{c}} \cup V = (U \setminus V)^{\mathsf{c}}$*: We have

$$\begin{aligned}(U \setminus V)^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus (U \setminus V) \\ &= (X \cap V) \cup (X \setminus U) \\ &= V \cup (X \setminus U) \\ &\overset{\text{def}}{=} V \cup U^{\mathsf{c}} \\ &= U^{\mathsf{c}} \cup V,\end{aligned}$$

where we have used:

020S 1. Item 10 of Definition 4.3.10.1.2 for the second equality.

020T 2. Item 4 of Definition 4.3.9.1.2 for the third equality.

020U 3. Item 4 of Definition 4.3.8.1.2 for the last equality.

This finishes the proof.
*Proof that $U^{\mathsf{c}} \cup V$ Is Indeed the Internal Hom*: This follows from Item 2 of Definition 4.3.9.1.2. □

004C **Remark 4.4.7.1.2.** Henning Makholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from $U$ to $V$ ([MSE 267365]):

01JV 1. Since products in $\mathcal{P}(X)$ are given by binary intersections (Item 1 of Definition 4.4.1.1.4), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.

[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$.

01JW 2. Under the Curry–Howard correspondence (**??**), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.

01JX 3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.

01JY 4. The expression $\neg U \vee V$ then corresponds to the set $U^{\mathsf{c}} \cup V$ in $\mathcal{P}(X)$.

01JZ 5. The set $U^{\mathsf{c}} \vee V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

01K0 **Proposition 4.4.7.1.3.** Let $X$ be a set.

01K1 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$\begin{aligned}
[U, -]_X \colon & \quad (\mathcal{P}(X), \supset) && \to (\mathcal{P}(X), \subset),\\
[-, V]_X \colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset),\\
[-_1, -_2]_X \colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \to (\mathcal{P}(X), \subset).
\end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01K2 (a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.

01K3 (b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.

01K4 (c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

01K5 2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[U,-]_X}"] \end{tikzcd}$$

$$(- \cap V \dashv [V, -]_X) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[V,-]_X}"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X),\\
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X).
\end{aligned}$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

01K6 (a) The following conditions are equivalent:

01K7 i. We have $U \cap V \subset W$.

01K8 ii. We have $U \subset [V, W]_X$.

01K9 (b) The following conditions are equivalent:

01KA i. We have $U \cap V \subset W$.

01KB ii. We have $V \subset [U, W]_X$.

01KC 3. *Interaction With the Empty Set I.* We have

$$\begin{aligned} [U, \emptyset]_X &= U^{\mathsf{c}}, \\ [\emptyset, V]_X &= X, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

01KD 4. *Interaction With $X$.* We have

$$\begin{aligned} [U, X]_X &= X, \\ [X, V]_X &= V, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

01KE 5. *Interaction With the Empty Set II.* The functor

$$D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

defined by

$$\begin{aligned} D_X &\overset{\text{def}}{=} [-, \emptyset]_X \\ &= (-)^{\mathsf{c}} \end{aligned}$$

is an involutory isomorphism of categories, making $\emptyset$ into a dualising object for $(\mathcal{P}(X), \cap, X, [-,-]_X)$ in the sense of **??**. In particular:

01KF (a) The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "D_X"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}"'] & \mathcal{P}(X) \arrow[d, "D_X"] \\ & \mathcal{P}(X)^{\mathsf{op}} \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\overset{\text{def}}{=}[[U,\emptyset]_X,\emptyset]_X} = U$$

for each $U \in \mathcal{P}(X)$.

01KG (b) The diagram

$$\begin{tikzcd}
& \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] & \mathcal{P}(X)^{\mathsf{op}} \arrow[dr, "D_X"] & \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X"] \arrow[rrr, "{[-_1,-_2]_X}"'] & & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\text{def}}{=}[U\cap[V,\emptyset]_X,\emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

01KH 6. *Interaction With the Empty Set III.* Let $f : X \to Y$ be a function.

01KJ (a) *Interaction With Direct Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

01KK (b) *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "D_Y"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "D_X"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

01KL (c) *Interaction With Codirect Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

01KM 7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01KN 8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{tikzcd}
& \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcup^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[dl, "{[-,V]_X}"] \\
& \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01KP 9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V\right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01KQ 10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

*does not commute* in general, i.e. we may have

$$\bigcap_{W\in[\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U\in\mathcal{U}} U, \bigcap_{V\in\mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01KR 11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcap^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcup"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcap_{U\in\mathcal{U}} U, V\right]_X = \bigcup_{U\in\mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01KS 12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V\in\mathcal{V}} V\right]_X = \bigcap_{V\in\mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01KT 13. *Interaction With Binary Unions.* We have equalities of sets

$$\begin{aligned}[U \cap V, W]_X &= [U, W]_X \cup [V, W]_X, \\ [U, V \cap W]_X &= [U, V]_X \cap [U, W]_X\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01KU 14. *Interaction With Binary Intersections.* We have equalities of sets

$$\begin{aligned}[U \cup V, W]_X &= [U, W]_X \cap [V, W]_X, \\ [U, V \cup W]_X &= [U, V]_X \cup [U, W]_X\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01KV 15. *Interaction With Differences.* We have equalities of sets

$$\begin{aligned}[U \setminus V, W]_X &= [U, W]_X \cup [V^{\mathsf{c}}, W]_X \\ &= [U, W]_X \cup [U, V]_X, \\ [U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01KW 16. *Interaction With Complements.* We have equalities of sets

$$\begin{aligned}[U^{\mathsf{c}}, V]_X &= U \cup V, \\ [U, V^{\mathsf{c}}]_X &= U \cap V, \\ [U, V]^{\mathsf{c}}_X &= U \setminus V\end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

01KX 17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

01KY 18. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_!"] \arrow[d, "{[-_1,-_2]_X}"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

01KZ 19. *Interaction With Inverse Images.* Let $f: X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

01L0 20. *Interaction With Codirect Images.* Let $f: X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_*"] \arrow[d, "{[-_1,-_2]_X}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

*Proof.* *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove Items 1a to 1c.

020W 1. *Proof of Item 1a:* We have

$$\begin{aligned}[A, V]_X &\stackrel{\text{def}}{=} A^{\mathsf{c}} \cup V\\ &\subset U^{\mathsf{c}} \cup V\\ &\stackrel{\text{def}}{=} [U, V]_X,\end{aligned}$$

where we have used:

020X (a) Item 1 of Definition 4.3.11.1.2, which states that if $U \subset A$, then $A^{\mathsf{c}} \subset U^{\mathsf{c}}$.

020Y (b) Item 1a of Item 1 of Definition 4.3.11.1.2, which states that if $A^{\mathsf{c}} \subset U^{\mathsf{c}}$, then $A^{\mathsf{c}} \cup K \subset U^{\mathsf{c}} \cup K$ for any $K \in \mathcal{P}(X)$.

020Z 2. *Proof of Item 1b:* We have

$$\begin{aligned}[U, V]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup V\\ &\subset U^{\mathsf{c}} \cup B\\ &\stackrel{\text{def}}{=} [U, B]_X,\end{aligned}$$

where we have used Item 1b of Item 1 of Definition 4.3.11.1.2, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

0210 3. *Proof of Item 1c:* We have

$$\begin{aligned}[A, V]_X &\subset [U, V]_X\\ &\subset [U, B]_X,\end{aligned}$$

where we have used Items 1a and 1b.

This finishes the proof.
*Item 2, Adjointness*: This is a repetition of Item 2 of Definition 4.3.9.1.2 and is proved there.
*Item 3, Interaction With the Empty Set I*: We have

$$\begin{aligned}[U, \emptyset]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup \emptyset\\ &= U^{\mathsf{c}},\end{aligned}$$

where we have used Item 3 of Definition 4.3.8.1.2, and we have

$$[\emptyset, V]_X \stackrel{\text{def}}{=} \emptyset^{\mathsf{c}} \cup V$$

$$\begin{aligned} &\stackrel{\text{def}}{=} (X \setminus \varnothing) \cup V \\ &= X \cup V \\ &= X, \end{aligned}$$

where we have used:

0211 1. Item 12 of Definition 4.3.10.1.2 for the first equality.

0212 2. Item 5 of Definition 4.3.8.1.2 for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

*Item 4, Interaction With $X$*: We have

$$\begin{aligned} [U, X]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup X \\ &= X, \end{aligned}$$

where we have used Item 5 of Definition 4.3.8.1.2, and we have

$$\begin{aligned} [X, V]_X &\stackrel{\text{def}}{=} X^{\mathsf{c}} \cup V \\ &\stackrel{\text{def}}{=} (X \setminus X) \cup V \\ &= \varnothing \cup V \\ &= V, \end{aligned}$$

where we have used Item 3 of Definition 4.3.8.1.2 for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

*Item 5, Interaction With the Empty Set II*: We have

$$\begin{aligned} D_X(D_X(U)) &\stackrel{\text{def}}{=} [[U, \varnothing]_X, \varnothing]_X \\ &= [U^{\mathsf{c}}, \varnothing]_X \\ &= (U^{\mathsf{c}})^{\mathsf{c}} \\ &= U, \end{aligned}$$

where we have used:

0213 1. Item 3 for the second and third equalities.

0214 2. Item 3 of Definition 4.3.11.1.2 for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and thus we have

$$[[-, \emptyset]_X, \emptyset]_X \cong \mathrm{id}_{\mathcal{P}(X)}$$

This finishes the proof.

*Item 6, Interaction With the Empty Set III*: Since $D_X = (-)^{\mathsf{c}}$, this is essentially a repetition of the corresponding results for $(-)^{\mathsf{c}}$, namely Items 5 to 7 of Definition 4.3.11.1.2.

*Item 7, Interaction With Unions of Families of Subsets I*: By Item 3 of Definition 4.4.7.1.3, we have

$$\begin{aligned}[\mathcal{U}, \emptyset]_{\mathcal{P}(X)} &= \mathcal{U}^{\mathsf{c}},\\ [U, \emptyset]_X &= U^{\mathsf{c}}.\end{aligned}$$

With this, the counterexample given in the proof of Item 10 of Definition 4.3.6.1.2 then applies.

*Item 8, Interaction With Unions of Families of Subsets II*: We have

$$\begin{aligned}\left[\bigcup_{U\in\mathcal{U}} U, V\right]_X &\overset{\text{def}}{=} \left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathsf{c}} \cup V\\ &= \left(\bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}\right) \cup V\\ &= \bigcap_{U\in\mathcal{U}} (U^{\mathsf{c}} \cup V)\\ &\overset{\text{def}}{=} \bigcap_{U\in\mathcal{U}} [U, V]_X,\end{aligned}$$

where we have used:

0215 1. Item 11 of Definition 4.3.6.1.2 for the second equality.

0216 2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.

*Item 9, Interaction With Unions of Families of Subsets III*: We have

$$\bigcup_{V\in\mathcal{V}} [U, V]_X \overset{\text{def}}{=} \bigcup_{V\in\mathcal{V}} (U^{\mathsf{c}} \cup V)$$

$$
\begin{aligned}
&= U^{\mathsf{c}} \cup \left(\bigcup_{V\in\mathcal{V}} V\right)\\
&\stackrel{\text{def}}{=} \left[U, \bigcup_{V\in\mathcal{V}} V\right]_X.
\end{aligned}
$$

where we have used Item 6. This finishes the proof.

*Item 10, Interaction With Intersections of Families of Subsets I*: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$
\begin{aligned}
\bigcap_{W\in[\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W &= \bigcap_{W\in\mathcal{P}(X)} W\\
&= \{0,1\},
\end{aligned}
$$

whereas

$$
\begin{aligned}
\left[\bigcap_{U\in\mathcal{U}} U, \bigcap_{V\in\mathcal{V}} V\right]_X &= [\{0,1\},\{0\}]\\
&= \{0\},
\end{aligned}
$$

Thus we have

$$
\bigcap_{W\in[\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W = \{0,1\} \neq \{0\} = \left[\bigcap_{U\in\mathcal{U}} U, \bigcap_{V\in\mathcal{V}} V\right]_X.
$$

This finishes the proof.

*Item 11, Interaction With Intersections of Families of Subsets II*: We have

$$
\begin{aligned}
\left[\bigcap_{U\in\mathcal{U}} U, V\right]_X &\stackrel{\text{def}}{=} \left(\bigcap_{U\in\mathcal{U}} U\right)^{\mathsf{c}} \cup V\\
&= \left(\bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}\right) \cup V\\
&= \bigcup_{U\in\mathcal{U}} (U^{\mathsf{c}} \cup V)\\
&\stackrel{\text{def}}{=} \bigcup_{U\in\mathcal{U}} [U, V]_X,
\end{aligned}
$$

where we have used:

0217 1. Item 12 of Definition 4.3.6.1.2 for the second equality.

0218 2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.
*Item 12, Interaction With Intersections of Families of Subsets III*: We have

$$\begin{aligned}\bigcap_{V\in\mathcal{U}}[U,V]_X &\overset{\text{def}}{=} \bigcap_{V\in\mathcal{U}}(U^{\mathsf{c}}\cup V)\\ &= U^{\mathsf{c}}\cup\left(\bigcap_{V\in\mathcal{U}}V\right)\\ &\overset{\text{def}}{=} \left[U,\bigcap_{V\in\mathcal{U}}V\right]_X.\end{aligned}$$

where we have used Item 6. This finishes the proof.
*Item 13, Interaction With Binary Unions*: We have

$$\begin{aligned}[U\cap V,W]_X &\overset{\text{def}}{=} (U\cap V)^{\mathsf{c}}\cup W\\ &= (U^{\mathsf{c}}\cup V^{\mathsf{c}})\cup W\\ &= (U^{\mathsf{c}}\cup V^{\mathsf{c}})\cup(W\cup W)\\ &= (U^{\mathsf{c}}\cup W)\cup(V^{\mathsf{c}}\cup W)\\ &\overset{\text{def}}{=} [U,W]_X\cup[V,W]_X,\end{aligned}$$

where we have used:

0219 1. Item 2 of Definition 4.3.11.1.2 for the second equality.

021A 2. Item 8 of Definition 4.3.8.1.2 for the third equality.

021B 3. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the fourth equality.

For the second equality in the statement, we have

$$\begin{aligned}[U,V\cap W]_X &\overset{\text{def}}{=} U^{\mathsf{c}}\cup(V\cap W)\\ &= (U^{\mathsf{c}}\cup V)\cap(U^{\mathsf{c}}\cap W)\\ &\overset{\text{def}}{=} [U,V]_X\cap[U,W]_X,\end{aligned}$$

where we have used Item 6 of Definition 4.3.8.1.2 for the second equality.

*Item 14, Interaction With Binary Intersections*: We have

$$\begin{aligned}[U \cup V, W]_X &\overset{\text{def}}{=} (U \cup V)^{\mathsf{c}} \cup W\\ &= (U^{\mathsf{c}} \cap V^{\mathsf{c}}) \cup W\\ &= (U^{\mathsf{c}} \cup W) \cap (V^{\mathsf{c}} \cup W)\\ &\overset{\text{def}}{=} [U, W]_X \cap [V, W]_X,\end{aligned}$$

where we have used:

021C 1. Item 2 of Definition 4.3.11.1.2 for the second equality.

021D 2. Item 6 of Definition 4.3.8.1.2 for the third equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}[U, V \cup W]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup (V \cup W)\\ &= (U^{\mathsf{c}} \cup U^{\mathsf{c}}) \cup (V \cup W)\\ &= (U^{\mathsf{c}} \cup V) \cup (U^{\mathsf{c}} \cup W)\\ &\overset{\text{def}}{=} [U, V]_X \cup [U, W]_X,\end{aligned}$$

where we have used:

021E 1. Item 8 of Definition 4.3.8.1.2 for the second equality.

021F 2. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the third equality.

This finishes the proof.

*Item 15, Interaction With Differences*: We have

$$\begin{aligned}[U \setminus V, W]_X &\overset{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W\\ &\overset{\text{def}}{=} (X \setminus (U \setminus V)) \cup W\\ &= ((X \cap V) \cup (X \setminus U)) \cup W\\ &= (V \cup (X \setminus U)) \cup W\\ &\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W\\ &= (V \cup (U^{\mathsf{c}} \cup U^{\mathsf{c}})) \cup W\\ &= (U^{\mathsf{c}} \cup W) \cup (U^{\mathsf{c}} \cup V)\\ &\overset{\text{def}}{=} [U, W]_X \cup [U, V]_X,\end{aligned}$$

where we have used:

021G 1. Item 10 of Definition 4.3.10.1.2 for the third equality.

021H 2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.

021J 3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.

021K 4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

We also have

$$\begin{aligned}
[U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W \\
&\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\
&= (V \cup U^{\mathsf{c}}) \cup (W \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup (V \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup ((V^{\mathsf{c}})^{\mathsf{c}} \cup W) \\
&\stackrel{\text{def}}{=} [U, W]_X \cup [V^{\mathsf{c}}, W]_X,
\end{aligned}$$

where we have used:

021L 1. Item 10 of Definition 4.3.10.1.2 for the third equality.

021M 2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.

021N 3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.

021P 4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

021Q 5. Item 3 of Definition 4.3.11.1.2 for the eighth equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
[U, V \setminus W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup (V \setminus W) \\
&= (V \setminus W) \cup U^{\mathsf{c}} \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \setminus U^{\mathsf{c}})
\end{aligned}$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \setminus (W \setminus (X \setminus U))\\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup (W \setminus X))\\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup \emptyset)\\
&= (V \cup U^{\mathsf{c}}) \setminus (W \cap U)\\
&= (V \cup U^{\mathsf{c}}) \setminus (U \cap W)\\
&\stackrel{\text{def}}{=} [U, V]_X \setminus (U \cap W)
\end{aligned}
$$

where we have used:

021R 1. Item 4 of Definition 4.3.8.1.2 for the second equality.

021S 2. Item 4 of Definition 4.3.10.1.2 for the third equality.

021T 3. Item 10 of Definition 4.3.10.1.2 for the fifth equality.

021U 4. Item 13 of Definition 4.3.10.1.2 for the sixth equality.

021V 5. Item 3 of Definition 4.3.8.1.2 for the seventh equality.

021W 6. Item 5 of Definition 4.3.9.1.2 for the eighth equality.

This finishes the proof.
*Item 16, Interaction With Complements*: We have

$$
\begin{aligned}
[U^{\mathsf{c}}, V]_X &\stackrel{\text{def}}{=} (U^{\mathsf{c}})^{\mathsf{c}} \cup V,\\
&= U \cup V,
\end{aligned}
$$

where we have used Item 3 of Definition 4.3.11.1.2. We also have

$$
\begin{aligned}
[U, V^{\mathsf{c}}]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup V^{\mathsf{c}}\\
&= U \cap V
\end{aligned}
$$

where we have used Item 2 of Definition 4.3.11.1.2. Finally, we have

$$
\begin{aligned}
[U, V]_X^{\mathsf{c}} &= ((U \setminus V)^{\mathsf{c}})^{\mathsf{c}}\\
&= U \setminus V,
\end{aligned}
$$

where we have used Item 2 of Definition 4.3.11.1.2.

*Item 17, Interaction With Characteristic Functions*: We have

$$\begin{aligned}\chi_{[U,V]_{\mathcal{P}(X)}}(x) &\stackrel{\text{def}}{=} \chi_{U^{\mathsf{c}} \cup V}(x)\\ &= \max(\chi_{U^{\mathsf{c}}}, \chi_V)\\ &= \max(1 - \chi_U \pmod 2, \chi_V),\end{aligned}$$

where we have used:

021X 1. Item 10 of Definition 4.3.8.1.2 for the second equality.

021Y 2. Item 4 of Definition 4.3.11.1.2 for the third equality.

This finishes the proof.

*Item 18, Interaction With Direct Images*: This is a repetition of Item 10 of Definition 4.6.1.1.5 and is proved there.

*Item 19, Interaction With Inverse Images*: This is a repetition of Item 10 of Definition 4.6.2.1.3 and is proved there.

*Item 20, Interaction With Codirect Images*: This is a repetition of Item 9 of Definition 4.6.3.1.7 and is proved there. □

## 01L1 4.4.8 Isbell Duality for Sets

Let $X$ be a set.

01L2 **Definition 4.4.8.1.1.** The **Isbell function** of $X$ is the map

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

defined by

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [\![x \mapsto [U, \{x\}]_X]\!]$$

for each $U \in \mathcal{P}(X)$.

01L3 **Remark 4.4.8.1.2.** Recall from Definition 4.4.1.1.2 that we may view the powerset $\mathcal{P}(X)$ of a set $X$ as the decategorification of the category of presheaves $\mathsf{PSh}(\mathcal{C})$ of a category $\mathcal{C}$. Building upon this analogy, we want to mimic the definition of the Isbell $\mathsf{Spec}$ functor, which is given on objects by

$$\mathsf{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \mathrm{Nat}(\mathcal{F}, h_{(-)})$$

for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$. To this end, we could define

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [U, \chi_{(-)}]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ with the characteristic embedding $x \mapsto \chi_x$ of $X$ into $\mathcal{P}(X)$ of Definition 4.5.4.1.1.
- The internal Hom Nat of $\mathsf{PSh}(\mathcal{C})$ with the internal Hom $[-,-]_X$ of $\mathcal{P}(X)$ of Definition 4.4.7.1.1.

However, since $[U, \chi_x]_X$ is a subset of $U$ instead of a truth value, we get a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathcal{P}(X).$$

This makes some of the properties involving $\mathsf{I}$ a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $\mathsf{I}_! \circ \mathsf{I}$ evaluates to $\mathrm{id}_{\mathcal{P}(X)}$ in the sense of Definition 4.4.8.1.3.

01L4 **Proposition 4.4.8.1.3.** The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\mathsf{I}"] \arrow[dr, "\Delta_{\Delta_{\mathrm{id}_{\mathcal{P}(X)}}}"'] & \mathsf{Sets}(X, \mathcal{P}(X)) \arrow[d, "\mathsf{I}_!"] \\
 & \mathsf{Sets}(X, \mathsf{Sets}(X, \mathcal{P}(X)))
\end{tikzcd}$$

commutes, i.e. we have

$$\mathsf{I}_!(\mathsf{I}(U)) = [\![x \mapsto [\![y \mapsto U]\!]]\!]$$

for each $U \in \mathcal{P}(X)$.

*Proof.* We have

$$\begin{aligned}
\mathsf{I}_!(\mathsf{I}(U)) &\overset{\text{def}}{=} \mathsf{I}_!([\![x \mapsto U^{\mathsf{c}} \cup \{x\}]\!]) \\
&\overset{\text{def}}{=} [\![x \mapsto \mathsf{I}(U^{\mathsf{c}} \cup \{x\})]\!] \\
&\overset{\text{def}}{=} [\![x \mapsto [\![y \mapsto (U^{\mathsf{c}} \cup \{x\})^{\mathsf{c}} \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \setminus \{x\}) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto U]\!]]\!],
\end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2 for the fourth equality above. □

# 01L5 4.5 Characteristic Functions

## 005X 4.5.1 The Characteristic Function of a Subset

Let $X$ be a set and let $U \in \mathcal{P}(X)$.

005Z **Definition 4.5.1.1.1.** The **characteristic function of** $U$[23] is the function $\chi_U\colon X \to \{\mathsf{t}, \mathsf{f}\}$[24] defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

01L6 **Remark 4.5.1.1.2.** Under the analogy that $\{\mathsf{t}, \mathsf{f}\}$ should be the $(-1)$-categorical analogue of $\mathsf{Sets}$, we may view a function

$$f\colon X \to \{\mathsf{t}, \mathsf{f}\}$$

as a decategorification of presheaves and copresheaves

$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
$$F\colon \mathcal{C} \to \mathsf{Sets}.$$

The characteristic functions $\chi_U$ of the subsets of $X$ are then the primordial examples of such functions (and, in fact, all of them).

01L7 **Notation 4.5.1.1.3.** We will often employ the bijection $\{\mathsf{t}, \mathsf{f}\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when disucssing characteristic functions.

Examples of this include Items 4 to 11 of Definition 4.5.1.1.4 below.

0069 **Proposition 4.5.1.1.4.** Let $X$ be a set.

01L8 1. *Functionality.* The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}).$$

01L9 2. *Bijectivity.* The function $\chi_{(-)}$ from Item 1 is bijective.

[23] *Further Terminology:* Also called the **indicator function of** $U$.

[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

01LA 3. *Naturality.* The collection

$$\left\{\chi_{(-)}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})\right\}_{X \in \mathrm{Obj}(\mathsf{Sets})}$$

defines a natural isomorphism between $\mathcal{P}^{-1}$ and $\mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\})$. In particular, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\wr"', "\chi_{(-)}"] \\ \mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^{*}"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \end{tikzcd}$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

006B 4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

006C 5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

006D 6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

006E 7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

006F 8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

006G 9. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

006H 10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

01LB 11. *Interaction With Internal Homs.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

*Proof.* *Item 1*, *Functionality*: There is nothing to prove.
*Item 2*, *Bijectivity*: We proceed in three steps:

021Z 1. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi \colon \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \overset{\sim}{\dashrightarrow} \mathcal{P}(X),$$

defined by

$$\begin{aligned} \Phi(f) &\overset{\text{def}}{=} U_f \\ &\overset{\text{def}}{=} f^{-1}(\mathsf{true}) \\ &\overset{\text{def}}{=} \left\{ x \in X \,\middle|\, f(x) = \mathsf{true} \right\} \end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$.

0220 2. *Invertibility I.* We have

$$\begin{aligned}[\Phi \circ \chi_{(-)}](U) &\overset{\text{def}}{=} \Phi(\chi_U) \\ &\overset{\text{def}}{=} \chi_U^{-1}(\mathsf{true}) \\ &\overset{\text{def}}{=} \left\{x \in X \,\middle|\, \chi_U(x) = \mathsf{true}\right\} \\ &\overset{\text{def}}{=} \{x \in X \mid x \in U\} \\ &= U \\ &\overset{\text{def}}{=} [\mathrm{id}_{\mathcal{P}(X)}](U)\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus, we have

$$\Phi \circ \chi_{(-)} = \mathrm{id}_{\mathcal{P}(X)}.$$

0221 3. *Invertibility II.* We have

$$\begin{aligned}[\chi_{(-)} \circ \Phi](U) &\overset{\text{def}}{=} \chi_{\Phi(f)} \\ &\overset{\text{def}}{=} \chi_{f^{-1}(\mathsf{true})} \\ &\overset{\text{def}}{=} \left[\!\left[x \mapsto \begin{cases} \mathsf{true} & \text{if } x \in f^{-1}(\mathsf{true}) \\ \mathsf{false} & \text{otherwise} \end{cases}\right]\!\right] \\ &= [\![x \mapsto f(x)]\!] \\ &= f \\ &\overset{\text{def}}{=} [\mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})}](f)\end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$. Thus, we have

$$\chi_{(-)} \circ \Phi = \mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})}.$$

This finishes the proof.
*Item 3, Naturality*: We proceed in two steps:

0222 1. *Naturality of $\chi_{(-)}$.* We have

$$\begin{aligned}[\chi_V \circ f](v) &\overset{\text{def}}{=} \chi_V(f(v)) \\ &= \begin{cases} \mathsf{true} & \text{if } f(v) \in V, \\ \mathsf{false} & \text{otherwise} \end{cases}\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v)
\end{aligned}
$$

for each $v \in V$.

0223 2. *Naturality of* $\Phi$. Since $\chi_{(-)}$ is natural and a componentwise inverse to $\Phi$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\Phi$ is also natural in each argument.

This finishes the proof.

*Item 4, Interaction With Unions I*: This is a repetition of Item 10 of Definition 4.3.8.1.2 and is proved there.

*Item 5, Interaction With Unions II*: This is a repetition of Item 11 of Definition 4.3.8.1.2 and is proved there.

*Item 6, Interaction With Intersections I*: This is a repetition of Item 10 of Definition 4.3.9.1.2 and is proved there.

*Item 7, Interaction With Intersections II*: This is a repetition of Item 11 of Definition 4.3.9.1.2 and is proved there.

*Item 8, Interaction With Differences*: This is a repetition of Item 16 of Definition 4.3.10.1.2 and is proved there.

*Item 9, Interaction With Complements*: This is a repetition of Item 4 of Definition 4.3.11.1.2 and is proved there.

*Item 10, Interaction With Symmetric Differences*: This is a repetition of Item 15 of Definition 4.3.12.1.2 and is proved there.

*Item 11, Interaction With Internal Homs*: This is a repetition of Item 17 of Definition 4.4.7.1.3 and is proved there. □

0224 **Remark 4.5.1.1.5.** The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 2 of Definition 4.5.1.1.4, which

- Takes a subset $U \to X$ of $X$ and *straightens* it to a function $\chi_U \colon X \to \{\mathsf{true}, \mathsf{false}\}$;
- Takes a function $f \colon X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \to X$ of $X$;

may be viewed as the $(-1)$-categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\mathsf{FibSets}_X}_{\stackrel{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}_X}_{\stackrel{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}},\mathsf{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, **??**. Here we view:

- Subsets $U \hookrightarrow X$ as being analogous to $X$-fibred sets $\phi_X : A \to X$.
- Functions $f : X \to \{\mathsf{t}, \mathsf{f}\}$ as being analogous to $X$-indexed sets $A : X_{\mathsf{disc}} \to \mathsf{Sets}$.

## 01LC 4.5.2 The Characteristic Function of a Point

Let $X$ be a set and let $x \in X$.

0060 **Definition 4.5.2.1.1.** The **characteristic function of** $x$ is the function[25]

$$\chi_x : X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

01LD **Remark 4.5.2.1.2.** Expanding upon Definition 4.5.1.1.2, we may think of the characteristic function

$$\chi_x : X \to \{\mathsf{t}, \mathsf{f}\}$$

of an *element* $x$ of $X$ as a decategorification of the representable presheaf and of the representable copresheaf

$$h_X : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
$$h^X : \mathcal{C} \to \mathsf{Sets}$$

associated of an *object* $X$ of a category $\mathcal{C}$.

[25] *Further Notation:* Also written $\chi^x$, $\chi_X(x,-)$, or $\chi_X(-,x)$.

## 01LE 4.5.3 The Characteristic Relation of a Set

Let $X$ be a set.

0061 **Definition 4.5.3.1.1.** The **characteristic relation on** $X$[26] is the relation[27]

$$\chi_X(-_1,-_2)\colon X\times X\to\{\mathsf{t},\mathsf{f}\}$$

on $X$ defined by[28]

$$\chi_X(x,y)\overset{\mathrm{def}}{=}\begin{cases}\mathsf{true} & \text{if } x=y,\\ \mathsf{false} & \text{if } x\neq y\end{cases}$$

for each $x,y\in X$.

01LF **Remark 4.5.3.1.2.** Expanding upon Definitions 4.5.1.1.2 and 4.5.2.1.2, we may view the characteristic relation

$$\chi_X(-_1,-_2)\colon X\times X\to\{\mathsf{t},\mathsf{f}\}$$

of $X$ as a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)\colon \mathcal{C}^{\mathsf{op}}\times\mathcal{C}\to\mathsf{Sets}$$

of a category $\mathcal{C}$.

01LG **Proposition 4.5.3.1.3.** Let $f\colon X\to Y$ be a function.

006A 1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A\to B$ be a function. We have an inclusion[29]

$$\chi_B\circ(f\times f)\subset\chi_A,\qquad \begin{tikzcd} A\times A \arrow[rr, "f\times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B\times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t},\mathsf{f}\}. & \end{tikzcd}$$

*Proof.* *Item 1, The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a),f(b))\subset\chi_A(a,b)$ is equivalent to the statement "if $a=b$, then $f(a)=f(b)$", which is true. □

[26] *Further Terminology:* Also called the **identity relation on** $X$.

[27] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\mathrm{id}}$ in the context of relations.

[28] Under the bijection $\mathsf{Sets}(X\times X,\{\mathsf{t},\mathsf{f}\})\cong\mathcal{P}(X\times X)$ of Item 2 of Definition 4.5.1.1.4, the relation $\chi_X$ corresponds to the diagonal $\Delta_X\subset X\times X$ of $X$.

[29] *Note:* This is the 0-categorical version of Categories, Definition 11.5.4.1.1.

## 01LH 4.5.4 The Characteristic Embedding of a Set

Let $X$ be a set.

0062 **Definition 4.5.4.1.1.** The **characteristic embedding**[30] **of $X$ into $\mathcal{P}(X)$** is the function

$$\chi_{(-)}\colon X \to \mathcal{P}(X)$$

defined by[31]

$$\begin{aligned}\chi_{(-)}(x) &\overset{\text{def}}{=} \chi_x\\ &= \{x\}\end{aligned}$$

for each $x \in X$.

01LJ **Remark 4.5.4.1.2.** Expanding upon Definitions 4.5.1.1.2, 4.5.2.1.2 and 4.5.3.1.2, we may view the characteristic embedding

$$\chi_{(-)}\colon X \to \mathcal{P}(X)$$

of $X$ into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$よ\colon C^{\mathsf{op}} \to \mathsf{PSh}(C)$$

of a category $C$ into $\mathsf{PSh}(C)$.

01LK **Proposition 4.5.4.1.3.** Let $f\colon X \to Y$ be a map of sets.

01LL 1. *Interaction With Functions.* We have

$$f_! \circ \chi_X = \chi_Y \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & Y \arrow[d, "\chi_Y"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y). \end{tikzcd}$$

[30] The name "characteristic *embedding*" is justified by Definition 4.5.5.1.2, which gives an analogue of fully faithfulness for $\chi_{(-)}$.

[31] Here we are identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ as per Item 2 of Definition 4.5.1.1.4.

*Proof.* *Item 1*, *Interaction With Functions*: Indeed, we have

$$\begin{aligned}[f_! \circ \chi_X](x) &\stackrel{\text{def}}{=} f_!(\chi_X(x))\\ &\stackrel{\text{def}}{=} f_!(\{x\})\\ &= \{f(x)\}\\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x))\\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x),\end{aligned}$$

for each $x \in X$, showing the desired equality. □

006K **4.5.5 The Yoneda Lemma for Sets**

Let $X$ be a set and let $U \subset X$ be a subset of $X$.

006L **Proposition 4.5.5.1.1.** We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } U \subset V, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

*Proof.* We have

$$\begin{aligned}\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) &\stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } \{x\} \subset U, \\ \mathsf{false} & \text{otherwise} \end{cases}\\ &= \begin{cases} \mathsf{true} & \text{if } x \in U \\ \mathsf{false} & \text{otherwise} \end{cases}\\ &\stackrel{\text{def}}{=} \chi_U(x).\end{aligned}$$

This finishes the proof. □

006M **Corollary 4.5.5.1.2.** The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) \cong \chi_X(x, y)$$

for each $x, y \in X$.

*Proof.* We have

$$\begin{aligned}\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) &= \chi_y(x)\\ &\overset{\text{def}}{=} \begin{cases}\mathsf{true} & \text{if } x \in \{y\}\\ \mathsf{false} & \text{otherwise}\end{cases}\\ &= \begin{cases}\mathsf{true} & \text{if } x = y\\ \mathsf{false} & \text{otherwise}\end{cases}\\ &\overset{\text{def}}{=} \chi_X(x, y).\end{aligned}$$

where we have used Definition 4.5.5.1.1 for the first equality. □

## 01LM 4.6 The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

### 007F 4.6.1 Direct Images

Let $f : X \to Y$ be a function.

007G **Definition 4.6.1.1.1.** The **direct image function associated to** $f$ is the function[32]

$$f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[33]

$$\begin{aligned}f_!(U) &\overset{\text{def}}{=} \left\{ y \in Y \,\middle|\, \begin{array}{l}\text{there exists some } x \in U\\ \text{such that } y = f(x)\end{array} \right\}\\ &= \{ f(x) \in Y \mid x \in U \}\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

007H **Notation 4.6.1.1.2.** Sometimes one finds the notation

$$\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

[32] *Further Notation:* Also written simply $f : \mathcal{P}(X) \to \mathcal{P}(Y)$.
[33] *Further Terminology:* The set $f(U)$ is called the **direct image of** $U$ **by** $f$.

- We have $y \in \exists_f(U)$.
- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

0225 **Warning 4.6.1.1.3.** Notation for direct images between powersets is tricky:

0226 1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:

0227 (a) *Powersets.* Given a function $f: X \to Y$, we have an inverse image functor
$$f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X).$$
The *left* adjoint of this functor is the usual direct image, defined above in Definition 4.6.1.1.1.

0228 (b) *Presheaves.* Given a morphism of topological spaces $f: X \to Y$, we have an inverse image functor
$$f^{-1}: \mathsf{PSh}(Y) \to \mathsf{PSh}(X).$$
The *right* adjoint of this functor is the direct image functor of presheaves, defined in **??**.

0229 2. The presheaf direct image functor is denoted $f_*$, but the direct image functor for powersets is denoted $f_!$ (as it's a left adjoint).

022A 3. Adding to the confusion, it's somewhat common for $f_!: \mathcal{P}(X) \to \mathcal{P}(Y)$ to be denoted $f_*$.

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

| Situation | Adjoint String |
|---|---|
| Functoriality of Powersets | $(f_! \dashv f^{-1} \dashv f_*) \colon \mathcal{P}(X) \overset{\rightleftarrows}{\to} \mathcal{P}(Y)$ |
| Functoriality of Presheaf Categories | $(f_! \dashv f^{-1} \dashv f_*) \colon \mathsf{PSh}(X) \overset{\rightleftarrows}{\to} \mathsf{PSh}(Y)$ |
| Base Change | $(f_! \dashv f^* \dashv f_*) \colon \mathcal{C}_{/X} \overset{\rightleftarrows}{\to} \mathcal{C}_{/Y}$ |
| Kan Extensions | $(F_! \dashv F^* \dashv F_*) \colon \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \overset{\rightleftarrows}{\to} \mathsf{Fun}(\mathcal{D}, \mathcal{E})$ |

007J **Remark 4.6.1.1.4.** Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the direct image function associated to $f$ is equivalently the function

$$f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \mathrm{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\ &= \underset{\substack{x \in X \\ f(x) = -_1}}{\mathrm{colim}} (\chi_U(x)) \\ &= \bigvee_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)), \end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned} [f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $y \in Y$.

007K **Proposition 4.6.1.1.5.** Let $f : X \to Y$ be a function.

007L 1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

007M 2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left=45, "f_!"] \arrow[r, bend right=45, "f_*"'] & \mathcal{P}(Y), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by:

01LN (a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\to f^{-1} \circ f_!, & \mathrm{id}_{\mathcal{P}(Y)} &\to f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\to \mathrm{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\to \mathrm{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01LP (b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

01LQ i. The following conditions are equivalent:

01LR A. We have $f_!(U) \subset V$.

01LS B. We have $U \subset f^{-1}(V)$.

01LT ii. The following conditions are equivalent:

01LU A. We have $f^{-1}(U) \subset V$.

01LV B. We have $U \subset f_*(V)$.

01LW 3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\cup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01LX 4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01LY 5. *Interaction With Binary Unions.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cup"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

01LZ 6. *Interaction With Binary Intersections.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cap"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01M0 7. *Interaction With Differences.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathrm{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01M1 8. *Interaction With Complements.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

01M2 9. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_!(U) \triangle f_!(V) \subset f_!(U \triangle V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01M3 10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_!"] \arrow[d, "{[-_1,-_2]_X}"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

007N 11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f_!(U) \cup f_!(V) = f_!(U \cup V), \\ f_!(\emptyset) = \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

007P 12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} f_!(U \cap V) \subset f_!(U) \cap f_!(V), \\ f_!(X) \subset Y, \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

007Q 13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{gathered} f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \xrightarrow{=} f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

007R 14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} &: f_!(U \cap V) \to f_!(U) \cap f_!(V), \\ f_{!|\mathbb{1}}^{\otimes} &: f_!(X) \to Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

007S 15. *Interaction With Coproducts.* Let $f: X \to X'$ and $g: Y \to Y'$ be maps of sets. We have

$$(f \coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

007T 16. *Interaction With Products.* Let $f: X \to X'$ and $g: Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

007U 17. *Relation to Codirect Images.* We have

$$\begin{aligned} f_!(U) &= f_*(U^{\mathsf{c}})^{\mathsf{c}} \\ &\stackrel{\text{def}}{=} Y \setminus f_*(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

*Proof.* *Item 1, Functoriality*: Omitted.
*Item 2, Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
*Item 3, Interaction With Unions of Families of Subsets*: We have

$$\bigcup_{V \in f_!(\mathcal{U})} V = \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$

$$= \bigcup_{U \in \mathcal{U}} f_!(U).$$

This finishes the proof.

*Item 4, Interaction With Intersections of Families of Subsets*: We have

$$\begin{aligned}\bigcap_{V \in f_!(\mathcal{U})} V &= \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V\\ &= \bigcap_{U \in \mathcal{U}} f_!(U).\end{aligned}$$

This finishes the proof.

*Item 5, Interaction With Binary Unions*: See [Pro25p].

*Item 6, Interaction With Binary Intersections*: See [Pro25n].

*Item 7, Interaction With Differences*: See [Pro25o].

*Item 8, Interaction With Complements*: Applying Item 17 to $X \setminus U$, we have

$$\begin{aligned}f_!(U^{\mathsf{c}}) &= f_!(X \setminus U)\\ &= Y \setminus f_*(X \setminus (X \setminus U))\\ &= Y \setminus f_*(U)\\ &= f_*(U)^{\mathsf{c}}.\end{aligned}$$

This finishes the proof.

*Item 9, Interaction With Symmetric Differences*: We have

$$\begin{aligned}f_!(U) \triangle f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V))\\ &\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V))\\ &= (f_!(U \cup V)) \setminus (f_!(U \cap V))\\ &\subset f_!((U \cup V) \setminus (U \cap V))\\ &= f_!(U \triangle V),\end{aligned}$$

where we have used:

022C 1. Item 2 of Definition 4.3.12.1.2 for the first equality.

022D 2. Item 6 of this proposition together with Item 1 of Definition 4.3.10.1.2 for the first inclusion.

022E 3. Item 5 for the second equality.

022F 4. Item 7 for the second inclusion.

022G 5. Item 2 of Definition 4.3.12.1.2 for the tchird equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

*Item 10, Interaction With Internal Homs of Powersets*: We have

$$\begin{aligned} f_!([U, V]_X) &\stackrel{\text{def}}{=} f_!(U^{\mathsf{c}} \cup V) \\ &= f_!(U^{\mathsf{c}}) \cup f_!(V) \\ &= f_*(U)^{\mathsf{c}} \cup f_!(V) \\ &\stackrel{\text{def}}{=} [f_*(U), f_!(V)]_Y, \end{aligned}$$

where we have used:

022H 1. Item 5 for the second equality.

022J 2. Item 17 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

*Item 11, Preservation of Colimits*: This follows from Item 2 and **??**, **??** of **??**.[34]

*Item 12, Oplax Preservation of Limits*: The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25n] for the other inclusions.

*Item 13, Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 11.

*Item 14, Symmetric Oplax Monoidality With Respect to Intersections*: The inclusions in the statement follow from Item 12. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

*Item 15, Interaction With Coproducts*: Omitted.

*Item 16, Interaction With Products*: Omitted.

*Item 17, Relation to Codirect Images*: Applying Item 16 of Definition 4.6.3.1.7 to $X \setminus U$, we have

$$f_*(X \setminus U) = B \setminus f_!(X \setminus (X \setminus U))$$

[34] *Reference:* [Pro25p].

$$= B \setminus f_!(U).$$

Taking complements, we then obtain

$$\begin{aligned} f_!(U) &= B \setminus (B \setminus f_!(U)), \\ &= B \setminus f_*(X \setminus U), \end{aligned}$$

which finishes the proof. □

007V **Proposition 4.6.1.1.6.** Let $f : X \to Y$ be a function.

007W 1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} : \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

007X 2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} : \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

007Y 3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

007Z 4. *Interaction With Composition.* For each pair of composable functions $f : X \to Y$ and $g : Y \to Z$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(Y) \arrow[d, "g_!"] \\ & \mathcal{P}(Z). \end{tikzcd}$$

*Proof.* *Item 1*, *Functionality I*: There is nothing to prove.
*Item 2*, *Functionality II*: This follows from Item 1 of Definition 4.6.1.1.5.
*Item 3*, *Interaction With Identities*: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**.
*Item 4*, *Interaction With Composition*: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**. □

## 0080 4.6.2 Inverse Images

Let $f : X \to Y$ be a function.

0081 **Definition 4.6.2.1.1.** The **inverse image function associated to** $f$ is the function[35]

$$f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[36]

$$f^{-1}(V) \overset{\text{def}}{=} \left\{x \in X \;\middle|\; \text{we have } f(x) \in V\right\}$$

for each $V \in \mathcal{P}(Y)$.

0082 **Remark 4.6.2.1.2.** Identifying $\mathcal{P}(Y)$ with $\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the inverse image function associated to $f$ is equivalently the function

$$f^{*} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by

$$f^{*}(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in $\mathsf{Sets}$.

0083 **Proposition 4.6.2.1.3.** Let $f : X \to Y$ be a function.

0084 1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} \colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset).$$

In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

[35] *Further Notation:* Also written $f^{*} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$.
[36] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** $V$ **by** $f$.

0085 2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left=45, "f_!"] \arrow[r, bend right=45, "f_*"'] & \mathcal{P}(Y), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by:

01M4 (a) Units and counits of the form

$$\begin{gathered} \mathrm{id}_{\mathcal{P}(X)} \to f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \to f_* \circ f^{-1}, \\ f_! \circ f^{-1} \to \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \to \mathrm{id}_{\mathcal{P}(X)}, \end{gathered}$$

having components of the form

$$\begin{gathered} U \subset f^{-1}(f_!(U)), \qquad V \subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) \subset V, \qquad f^{-1}(f_*(U)) \subset U \end{gathered}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01M5 (b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

01M6 i. The following conditions are equivalent:

01M7 A. We have $f_!(U) \subset V$.

01M8 B. We have $U \subset f^{-1}(V)$.

01M9 ii. The following conditions are equivalent:

01MA A. We have $f^{-1}(U) \subset V$.

01MB B. We have $U \subset f_*(V)$.

01MC 3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V\in\mathcal{U}} f^{-1}(V) = \bigcup_{U\in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{U} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{U}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{U})$.

01MD 4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V\in\mathcal{U}} f^{-1}(V) = \bigcap_{U\in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{U} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{U}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{U})$.

01ME 5. *Interaction With Binary Unions.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1}\times f^{-1}"] \arrow[d, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01MF 6. *Interaction With Binary Intersections.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01MG 7. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\setminus"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\setminus"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

01MH 8. *Interaction With Complements.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

01MJ 9. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\triangle"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

i.e. we have

$$f^{-1}(U) \triangle f^{-1}(V) = f^{-1}(U \triangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

01MK 10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

0086 11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

0087 12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(Y) &= X, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

0088 13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(Y), \cup, \emptyset) \to (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} \colon f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f^{-1,\otimes}_{\mathbb{1}} \colon \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

0089 14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(Y), \cap, Y) \to (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} \colon f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cap V), \\ f^{-1,\otimes}_{\mathbb{1}} \colon X &\xrightarrow{=} f^{-1}(Y), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

008A 15. *Interaction With Coproducts.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \coprod g)^{-1}(U' \coprod V') = f^{-1}(U') \coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

008B 16. *Interaction With Products.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X'\times Y'} g)^{-1}(U' \boxtimes_{X'\times Y'} V') = f^{-1}(U') \boxtimes_{X\times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

*Proof. Item 1, Functoriality*: Omitted.
*Item 2, Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
*Item 3, Interaction With Unions of Families of Subsets*: We have

$$\begin{aligned}\bigcup_{U\in f^{-1}(\mathcal{U})} U &= \bigcup_{U\in\left\{f^{-1}(V)\in\mathcal{P}(X)\ \middle|\ V\in\mathcal{U}\right\}} U\\ &= \bigcup_{V\in\mathcal{U}} f^{-1}(V).\end{aligned}$$

This finishes the proof.
*Item 4, Interaction With Intersections of Families of Subsets*: We have

$$\begin{aligned}\bigcap_{U\in f^{-1}(\mathcal{U})} U &= \bigcap_{U\in\left\{f^{-1}(V)\in\mathcal{P}(X)\ \middle|\ V\in\mathcal{U}\right\}} U\\ &= \bigcap_{V\in\mathcal{U}} f^{-1}(V).\end{aligned}$$

This finishes the proof.
*Item 5, Interaction With Binary Unions*: See [Pro25y].
*Item 6, Interaction With Binary Intersections*: See [Pro25w].
*Item 7, Interaction With Differences*: See [Pro25x].
*Item 8, Interaction With Complements*: See [Pro25j].
*Item 9, Interaction With Symmetric Differences*: We have

$$f^{-1}(U \mathbin{\triangle} V) = f^{-1}((U \cup V) \setminus (U \cap V))$$

$$\begin{aligned}
&= f^{-1}(U \cup V) \setminus f^{-1}(U \cap V) \\
&= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U \cap V) \\
&= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U) \cap f^{-1}(V) \\
&= f^{-1}(U) \bigtriangleup f^{-1}(V),
\end{aligned}$$

where we have used:

022K 1. Item 2 of Definition 4.3.12.1.2 for the first equality.

022L 2. Item 7 for the second equality.

022M 3. Item 5 for the third equality.

022N 4. Item 6 for the fourth equality.

022P 5. Item 2 of Definition 4.3.12.1.2 for the fifth equality.

This finishes the proof.
*Item 10, Interaction With Internal Homs of Powersets*: We have

$$\begin{aligned}
f^{-1}([U, V]_Y) &\overset{\text{def}}{=} f^{-1}(U^{\mathsf{c}} \cup V) \\
&= f^{-1}(U^{\mathsf{c}}) \cup f^{-1}(V) \\
&= f^{-1}(U)^{\mathsf{c}} \cup f^{-1}(V) \\
&\overset{\text{def}}{=} [f^{-1}(U), f^{-1}(V)]_X,
\end{aligned}$$

where we have used:

022Q 1. Item 8 for the second equality.

022R 2. Item 5 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
*Item 11, Preservation of Colimits*: This follows from Item 2 and **??**, **??** of **??**.[37]
*Item 12, Preservation of Limits*: This follows from Item 2 and **??**, **??** of **??**.[38]
*Item 13, Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 11.

[37] *Reference:* [Pro25y].
[38] *Reference:* [Pro25w].

*Item 14, Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 12.
*Item 15, Interaction With Coproducts*: Omitted.
*Item 16, Interaction With Products*: Omitted. □

008C **Proposition 4.6.2.1.4.** Let $f: X \to Y$ be a function.

008D 1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{X,Y}: \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

008E 2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{X,Y}: \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

008F 3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_X^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

008G 4. *Interaction With Composition.* For each pair of composable functions $f: X \to Y$ and $g: Y \to Z$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(Z) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(Y) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

*Proof.* *Item 1, Functionality I*: There is nothing to prove.
*Item 2, Functionality II*: This follows from Item 1 of Definition 4.6.2.1.3.
*Item 3, Interaction With Identities*: This follows from Definition 4.6.2.1.2 and Categories, Item 5 of Definition 11.1.4.1.2.
*Item 4, Interaction With Composition*: This follows from Definition 4.6.2.1.2 and Categories, Item 2 of Definition 11.1.4.1.2. □

### 008H 4.6.3 Codirect Images

Let $f\colon X \to Y$ be a function.

008J **Definition 4.6.3.1.1.** The **codirect image function associated to** $f$ is the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[39,40]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right\} \\ &= \left\{ y \in Y \,\middle|\, \text{we have } f^{-1}(y) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

008K **Notation 4.6.3.1.2.** Sometimes one finds the notation

$$\forall_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.
- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

022V **Warning 4.6.3.1.3.** See Definition 4.6.1.1.3.

008L **Remark 4.6.3.1.4.** Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the codirect image function associated to $f$ is equivalently the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

[39] *Further Terminology:* The set $f_*(U)$ is called the **codirect image of** $U$ **by** $f$.

[40] We also have

$$\begin{aligned} f_*(U) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \\ &\overset{\text{def}}{=} Y \setminus f_!(X \setminus U); \end{aligned}$$

defined by

$$\begin{aligned} f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} X \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)) \\ &= \bigwedge_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)). \end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned} [f_*(\chi_U)](y) &= \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \mathsf{true} & \text{if, for each } x \in X \text{ such that} \\ & f(x) = y, \text{ we have } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } f^{-1}(y) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $y \in Y$.

---

see Item 16 of Definition 4.6.3.1.7.

008M **Definition 4.6.3.1.5.** Let $U$ be a subset of $X$.[41,42]

008N 1. The **image part of the codirect image** $f_*(U)$ **of** $U$ is the set $f_{*,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{*,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_*(U) \cap \mathrm{Im}(f) \\ &= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{array} \right\}. \end{aligned}$$

008P 2. The **complement part of the codirect image** $f_*(U)$ **of** $U$ is the set $f_{*,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{*,\mathrm{cp}}(U) &\overset{\mathrm{def}}{=} f_*(U) \cap (Y \setminus \mathrm{Im}(f)) \\ &= Y \setminus \mathrm{Im}(f) \\ &= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{array} \right\} \\ &= \left\{ y \in Y \,\middle|\, f^{-1}(y) = \emptyset \right\}. \end{aligned}$$

008Q **Example 4.6.3.1.6.** Here are some examples of codirect images.

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[41] Note that we have

$$f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_*(U) &= f_*(U) \cap Y \\ &= f_*(U) \cap (\mathrm{Im}(f) \cup (Y \setminus \mathrm{Im}(f))) \\ &= (f_*(U) \cap \mathrm{Im}(f)) \cup (f_*(U) \cap (Y \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U). \end{aligned}$$

[42] In terms of the meet computation of $f_*(U)$ of Definition 4.6.3.1.4, namely

$$f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),$$

we see that $f_{*,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

0231 1. *Multiplication by Two.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since $f$ is injective, we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &= f_!(U) \\ f_{*,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$. In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

0232 2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$\begin{aligned} f_{*,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{*,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{*,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{*,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

0233 3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$f_{*,\mathrm{im}}([-1,1]\times[-1,1]) = [0,1],$$
$$f_{*,\mathrm{im}}(([-1,1]\times[-1,1]) \setminus [-1,1]\times\{0\}) = \emptyset.$$

008R **Proposition 4.6.3.1.7.** Let $f\colon X \to Y$ be a function.

008S 1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(X),\subset) \to (\mathcal{P}(Y),\subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

008T 2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left=50, "f_!"] \arrow[r, bend right=50, "f_*"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{P}(Y), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by:

01ML (a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \to f^{-1}\circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \to f_*\circ f^{-1},$$
$$f_!\circ f^{-1} \to \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1}\circ f_* \to \mathrm{id}_{\mathcal{P}(X)},$$

having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*(f^{-1}(V)),$$
$$f_!(f^{-1}(V)) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01MM (b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

01MN i. The following conditions are equivalent:

01MP A. We have $f_!(U) \subset V$.

01MQ B. We have $U \subset f^{-1}(V)$.

01MR ii. The following conditions are equivalent:

01MS A. We have $f^{-1}(U) \subset V$.

01MT B. We have $U \subset f_*(V)$.

01MU 3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\cup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\cup"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01MV 4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01MW 5. *Interaction With Binary Unions.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cup"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01MX 6. *Interaction With Binary Intersections.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cap"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

01MY 7. *Interaction With Complements.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

01MZ 8. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_*"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U \triangle V) \subset f_*(U) \triangle f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01N0 9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_*"] \arrow[d, "{[-_1,-_2]_X}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

008U 10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} f_*(U_i) \subset f_*\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{gathered}
f_*(U) \cup f_*(V) \to f_*(U \cup V), \\
\emptyset \to f_*(\emptyset),
\end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

008V 11. *Preservation of Limits.* We have an equality of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_*(U) \cap f^{-1}(V), \\ f_*(X) &= Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008W 12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^{\otimes} &\colon f_*(U) \cup f_*(V) \to f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &\colon \emptyset \to f_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008X 13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &\colon f_*(U \cap V) \xrightarrow{=} f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} &\colon f_*(X) \xrightarrow{=} Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008Y 14. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \textstyle\coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

008Z 15. *Interaction With Products.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X\times Y} g)_*(U \boxtimes_{X\times Y} V) = f_*(U) \boxtimes_{X'\times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

0090 16. *Relation to Direct Images.* We have

$$\begin{aligned} f_*(U) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \\ &= Y \setminus f_!(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

0091 17. *Interaction With Injections.* If $f$ is injective, then we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &= f_!(U), \\ f_{*,\mathrm{cp}}(U) &= Y \setminus \mathrm{Im}(f), \end{aligned}$$

and so

$$\begin{aligned} f_*(U) &= f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U) \\ &= f_!(U) \cup (Y \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

0092 18. *Interaction With Surjections.* If $f$ is surjective, then we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &\subset f_!(U), \\ f_{*,\mathrm{cp}}(U) &= \emptyset, \end{aligned}$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

*Proof.* *Item 1, Functoriality*: Omitted.

*Item 2, Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
*Item 3, Interaction With Unions of Families of Subsets*: We have

$$\begin{aligned}\bigcup_{V\in f_*(\mathcal{U})} V &= \bigcup_{V\in\{f_*(U)\in\mathcal{P}(X)\mid U\in\mathcal{U}\}} V\\ &= \bigcup_{U\in\mathcal{U}} f_*(U).\end{aligned}$$

This finishes the proof.
*Item 4, Interaction With Intersections of Families of Subsets*: We have

$$\begin{aligned}\bigcap_{V\in f_*(\mathcal{U})} V &= \bigcap_{V\in\{f_*(U)\in\mathcal{P}(X)\mid U\in\mathcal{U}\}} V\\ &= \bigcap_{U\in\mathcal{U}} f_*(U).\end{aligned}$$

This finishes the proof.
*Item 5, Interaction With Binary Unions*: We have

$$\begin{aligned}f_*(U)\cup f_*(V) &= f_!(U^{\mathsf{c}})^{\mathsf{c}}\cup f_!(V^{\mathsf{c}})^{\mathsf{c}}\\ &= (f_!(U^{\mathsf{c}})\cap f_!(V^{\mathsf{c}}))^{\mathsf{c}}\\ &\subset (f_!(U^{\mathsf{c}}\cap V^{\mathsf{c}}))^{\mathsf{c}}\\ &= f_!((U\cup V)^{\mathsf{c}})^{\mathsf{c}}\\ &= f_*(U\cup V),\end{aligned}$$

where:

023X 1. We have used Item 16 for the first equality.

023Y 2. We have used Item 2 of Definition 4.3.11.1.2 for the second equality.

023Z 3. We have used Item 6 of Definition 4.6.1.1.5 for the third equality.

0240 4. We have used Item 2 of Definition 4.3.11.1.2 for the fourth equality.

0241 5. We have used Item 16 for the last equality.

This finishes the proof.
*Item 6, Interaction With Binary Intersections*: This follows from Item 11.
*Item 7, Interaction With Complements*: Omitted.
*Item 8, Interaction With Symmetric Differences*: Omitted.
*Item 9, Interaction With Internal Homs of Powersets*: We have

$$\begin{aligned}\left[f_!(U), f^!(V)\right]_X &\overset{\text{def}}{=} f_!(U)^{\mathsf{c}} \cup f_*(V) \\ &= f_*(U^{\mathsf{c}}) \cup f_*(V) \\ &\subset f_*(U^{\mathsf{c}} \cup V) \\ &\overset{\text{def}}{=} f_*([U, V]_X),\end{aligned}$$

where we have used:

0242 1. Item 7 of Definition 4.6.3.1.7 for the second equality.

0243 2. Item 5 of Definition 4.6.3.1.7 for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
*Item 10, Lax Preservation of Colimits*: Omitted.
*Item 11, Preservation of Limits*: This follows from Item 2 and **??**, **??** of **??**.
*Item 12, Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 10.
*Item 13, Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 11.
*Item 14, Interaction With Coproducts*: Omitted.
*Item 15, Interaction With Products*: Omitted.
*Item 16, Relation to Direct Images*: We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication.* We claim that

  $$f_*(U) \subset Y \setminus f_!(X \setminus U).$$

  Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.

  This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).

  Thus $y \in Y \setminus f_!(X \setminus U)$.

- *The Second Implication.* We claim that
$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$
Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.
Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.
Thus $y \in f_*(U)$.

This finishes the proof of Item 16.
*Item 17, Interaction With Injections*: Omitted.
*Item 18, Interaction With Surjections*: Omitted. □

0093 **Proposition 4.6.3.1.8.** Let $f : X \to B$ be a function.

0094 1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{!|X,Y} : \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

0095 2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{!|X,Y} : \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

0096 3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)} .$$

0097 4. *Interaction With Composition.* For each pair of composable functions $f : X \to Y$ and $g : Y \to Z$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(Y) \arrow[d, "g_*"] \\ & \mathcal{P}(Z). \end{tikzcd}$$

*Proof.* *Item 1, Functionality I*: There is nothing to prove.
*Item 2, Functionality II*: This follows from Item 1 of Definition 4.6.3.1.7.
*Item 3, Interaction With Identities*: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**.
*Item 4, Interaction With Composition*: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**. □

### 01N1 4.6.4 A Six-Functor Formalism for Sets

01N2 **Remark 4.6.4.1.1.** The assignment $X \mapsto \mathcal{P}(X)$ together with the functors $f_*$, $f^{-1}$, and $f_!$ of Item 1 of Definition 4.6.1.1.5, Item 1 of Definition 4.6.2.1.3, and Item 1 of Definition 4.6.3.1.7, and the functors

$$\begin{aligned} -_1 \cap -_2 &\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X), \\ [-_1, -_2]_X &\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

of Item 1 of Definition 4.3.9.1.2 and Item 1 of Definition 4.4.7.1.3 satisfy several properties reminiscent of a six functor formalism in the sense of **??**.

We collect these properties in Definition 4.6.4.1.2 below.[43]

01N3 **Proposition 4.6.4.1.2.** Let $X$ be a set.

01N4 1. *The Beck–Chevalley Condition.* Let

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a pullback diagram in $\mathsf{Sets}$. We have

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\mathrm{pr}_1^{-1}"] \arrow[d, "f_!"'] & \mathcal{P}(X \times_Z Y) \arrow[d, "(\mathrm{pr}_2)_!"] \\ \mathcal{P}(Z) \arrow[r, "g^{-1}"'] & \mathcal{P}(Y), \end{tikzcd} \qquad \begin{aligned} g^{-1} \circ f_! &= (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}, \\ f^{-1} \circ g_! &= (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}, \end{aligned} \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "\mathrm{pr}_2^{-1}"] \arrow[d, "g_!"'] & \mathcal{P}(X \times_Z Y) \arrow[d, "(\mathrm{pr}_1)_!"] \\ \mathcal{P}(Z) \arrow[r, "f^{-1}"'] & \mathcal{P}(Y). \end{tikzcd}$$

[43] See also [nLa25].

01N5 2. *The Projection Formula I.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"] & \\
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}"] \arrow[d, "f_! \times \mathrm{id}_{\mathcal{P}(Y)}"'] & & \mathcal{P}(X) \arrow[d, "f_!"] \\
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[rr, "\cap"'] & & \mathcal{P}(Y),
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U \cap f^{-1}(V)) = f_!(U) \cap V$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

01N6 3. *The Projection Formula II.* We have a natural transformation

$$\begin{tikzcd}
 & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"] & \\
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}"] \arrow[d, "f_* \times \mathrm{id}_{\mathcal{P}(Y)}"'] \arrow[rr, phantom, "\circlearrowright", shift right=4ex] & & \mathcal{P}(X) \arrow[d, "f_*"] \\
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[rr, "\cap"'] & & \mathcal{P}(Y),
\end{tikzcd}$$

with components

$$f_*(U) \cap V \subset f_*(U \cap f^{-1}(V))$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01N7 4. *Strong Closed Monoidality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

01N8 5. *The External Tensor Product.* We have an external tensor product

$$-_1 \boxtimes_{X\times Y} -_2 \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$$

given by

$$\begin{aligned} U \boxtimes_{X\times Y} V &\overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\ &= \{(u,v) \in X \times Y \mid u \in U \text{ and } v \in V\}. \end{aligned}$$

This is the same map as the one in Item 5 of Definition 4.4.1.1.4. Moreover, the following conditions are satisfied:

01N9 (a) *Interaction With Direct Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_! \times g_!"] \arrow[d, "\boxtimes_{X\times Y}"'] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, "\boxtimes_{X'\times Y'}"] \\ \mathcal{P}(X \times Y) \arrow[r, "f_! \times g_!"'] & \mathcal{P}(X' \times Y') \end{tikzcd}$$

commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X\times Y} V) = f_!(U) \boxtimes_{X'\times Y'} g_!(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

01NA (b) *Interaction With Inverse Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd} \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[r, "f^{-1} \times g^{-1}"] \arrow[d, "\boxtimes_{X'\times Y'}"'] & \mathcal{P}(X) \times \mathcal{P}(Y) \arrow[d, "\boxtimes_{X\times Y}"] \\ \mathcal{P}(X' \times Y') \arrow[r, "f^{-1} \times g^{-1}"'] & \mathcal{P}(X \times Y) \end{tikzcd}$$

commutes, i.e. we have

$$[f^{-1} \times g^{-1}](U \boxtimes_{X' \times Y'} V) = f^{-1}(U) \boxtimes_{X \times Y} g^{-1}(V)$$

for each $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$.

01NB (c) *Interaction With Codirect Images.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, "\boxtimes_{X \times Y}"'] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, "\boxtimes_{X' \times Y'}"] \\
\mathcal{P}(X \times Y) \arrow[r, "f_* \times g_*"'] & \mathcal{P}(X' \times Y')
\end{tikzcd}$$

commutes, i.e. we have

$$[f_* \times g_*](U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

01NC (d) *Interaction With Diagonals.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\boxtimes_{X \times X}"] \arrow[dr, "\cap"'] & \mathcal{P}(X \times X) \arrow[d, "\Delta_X^{-1}"] \\
 & \mathcal{P}(X),
\end{tikzcd}$$

i.e. we have

$$U \cap V = \Delta_X^{-1}(U \boxtimes_{X \times X} V)$$

for each $U, V \in \mathcal{P}(X)$.

01ND 6. *The Dualisation Functor.* We have a functor

$$D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

given by

$$\begin{aligned} D_X(U) &\overset{\text{def}}{=} [U, \emptyset]_X \\ &\overset{\text{def}}{=} U^{\mathsf{c}} \end{aligned}$$

for each $U \in \mathcal{P}(X)$, as in Item 5 of Definition 4.4.7.1.3, satisfying the following conditions:

01NE (a) *Duality.* We have

$$D_X(D_X(U)) = U, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "D_X"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \arrow[d, "D_X"] \\ & \mathcal{P}(X). \end{tikzcd}$$

01NF (b) *Duality.* The diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] & \mathcal{P}(X)^{\mathsf{op}} \arrow[dr, "D_X"] & \\ \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X"] \arrow[rrr, "{[-_1,-_2]_X}"'] & & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\mathrm{def}}{=}[U \cap [V,\emptyset]_X,\emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

01NG (c) *Interaction With Direct Images.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

01NH (d) *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathrm{op}} \arrow[r, "f^{-1,\mathrm{op}}"] \arrow[d, "D_Y"'] & \mathcal{P}(X)^{\mathrm{op}} \arrow[d, "D_X"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

01NJ (e) *Interaction With Codirect Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathrm{op}} \arrow[r, "f_!^{\mathrm{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathrm{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

*Proof.* *Item 1, The Beck–Chevalley Condition*: We have

$$\begin{aligned}
[g^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} g^{-1}(f_!(U)) \\
&\stackrel{\text{def}}{=} \left\{ y \in Y \,\middle|\, g(y) \in f_!(U) \right\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \,\middle|\, x \in U \right\} \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; x \in U \right\} \\ \text{such that } y = y \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; x \in U \right\} \\ \text{such that } \mathrm{pr}_2(x, y) = y \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; x \in U \right\}) \\
&= (\mathrm{pr}_2)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; \mathrm{pr}_1(x, y) \in U \right\}) \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\mathrm{pr}_1^{-1}(U)) \\
&\overset{\text{def}}{=} [(\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$
g^{-1} \circ f_! = (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}.
$$

For the second equality, we have

$$
\begin{aligned}
[f^{-1} \circ g_!](U) &\overset{\text{def}}{=} f^{-1}(g_!(U)) \\
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; f(x) \in g_!(V) \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\} \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\} \\ \text{such that } x = x \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\} \\ \text{such that } \mathrm{pr}_1(x, y) = x \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\}) \\
&= (\mathrm{pr}_1)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; \mathrm{pr}_2(x, y) \in V \right\}) \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\mathrm{pr}_2^{-1}(V))
\end{aligned}
$$

$$\stackrel{\text{def}}{=} [(\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}](V)$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$f^{-1} \circ g_! = (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}.$$

This finishes the proof.

*Item 2, The Projection Formula I*: We claim that

$$f_!(U) \cap V \subset f_!(U \cap f^{-1}(V)).$$

Indeed, we have

$$\begin{aligned} f_!(U) \cap V &\subset f_!(U) \cap f_!(f^{-1}(V)) \\ &= f_!(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

024B 1. Item 2 of Definition 4.6.1.1.5 for the inclusion.

024C 2. Item 6 of Definition 4.6.1.1.5 for the equality.

Conversely, we claim that

$$f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V.$$

Indeed:

024D 1. Let $y \in f_!(U \cap f^{-1}(V))$.

024E 2. Since $y \in f_!(U \cap f^{-1}(V))$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.

024F 3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.

024G 4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.

024H 5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.

024J 6. But $y = f(x)$, so $y \in f_!(U) \cap V$.

024K 7. Thus $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$.

This finishes the proof.
*Item 3, The Projection Formula II*: We have

$$\begin{aligned} f_*(U) \cap V &\subset f_*(U) \cap f_*(f^{-1}(V)) \\ &= f_*(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

024L 1. Item 2 of Definition 4.6.3.1.7 for the inclusion.

024M 2. Item 6 of Definition 4.6.3.1.7 for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
*Item 4, Strong Closed Monoidality*: This is a repetition of Item 19 of Definition 4.4.7.1.3 and is proved there.
*Item 5, The External Tensor Product*: We have

$$\begin{aligned} U \boxtimes_{X\times Y} V &\stackrel{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\ &\stackrel{\text{def}}{=} \left\{(x,y) \in X \times Y \,\middle|\, \mathrm{pr}_1(x,y) \in U\right\} \\ &\quad \cup \left\{(x,y) \in X \times Y \,\middle|\, \mathrm{pr}_2(x,y) \in V\right\} \\ &= \left\{(x,y) \in X \times Y \,\middle|\, x \in U\right\} \\ &\quad \cup \left\{(x,y) \in X \times Y \,\middle|\, y \in V\right\} \\ &= \left\{(x,y) \in X \times Y \,\middle|\, x \in U \text{ and } y \in V\right\} \\ &\stackrel{\text{def}}{=} U \times V. \end{aligned}$$

Next, we claim that Items 5a to 5d are indeed true:

024N 1. *Proof of Item 5a:* This is a repetition of Item 16 of Definition 4.6.1.1.5 and is proved there.

024P 2. *Proof of Item 5b:* This is a repetition of Item 16 of Definition 4.6.2.1.3 and is proved there.

024Q 3. *Proof of Item 5c:* This is a repetition of Item 15 of Definition 4.6.3.1.7 and is proved there.

024R 4. *Proof of Item 5d:* We have

$$\begin{aligned}\Delta_X^{-1}(U \boxtimes_{X\times X} V) &\stackrel{\text{def}}{=} \{x \in X \mid (x,x) \in U \boxtimes_{X\times X} V\}\\ &= \{x \in X \mid (x,x) \in \{(u,v) \in X \times X \mid u \in U \text{ and } v \in V\}\}\\ &= U \cap V.\end{aligned}$$

This finishes the proof.

*Item 6, The Dualisation Functor*: This is a repetition of Items 5 and 6 of Definition 4.4.7.1.3 and is proved there. □

02RG
## 4.7 Miscellany

02RH
### 4.7.1 Injective Functions

Let $A$ and $B$ be sets.

02RJ **Definition 4.7.1.1.1.** A function $f: A \to B$ is **injective** if it satisfies the following condition:

(★) For each $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

02RK **Proposition 4.7.1.1.2.** Let $f: A \to B$ be a function.

02RL 1. *Characterisations.* The following conditions are equivalent:[44]

02RM (a) The function $f$ is injective.

02RN (b) The function $f$ is a monomorphism in $\mathsf{Sets}$.

02RP (c) The direct image function

$$f_!: \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to $f$ is injective.

[44] Items 1c to 1f unwind respectively to the following statements:

- For each $U, V \in \mathcal{P}(A)$, if $f_!(U) = f_!(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_*(U) = f_*(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_!(U) \subset f_!(V)$, then $U \subset V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_*(U) \subset f_*(V)$, then $U \subset V$.

02RQ (d) The codirect image function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to $f$ is injective.

02RR (e) The direct image functor

$$f_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

associated to $f$ is full.

02RS (f) The codirect image function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to $f$ is full.

02RT (g) The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, hook, "\chi_A"'] & B \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A) \end{tikzcd}$$

commutes. That is, we have

$$f^{-1}(f(a)) = \{a\}$$

for each $a \in A$.

02RU (h) We have

$$f^{-1} \circ f_! = \mathrm{id}_{\mathcal{P}(A)} \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

In other words, we have

$$\left\{a \in A \,\middle|\, f(a) \in f(U)\right\} = U$$

for each $U \in \mathcal{P}(A)$.

02RV (i) We have

$$f^{-1} \circ f_* = \mathrm{id}_{\mathcal{P}(A)} \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

In other words, we have

$$\left\{a \in A \,\middle|\, f^{-1}(f(a)) \subset U\right\} = U$$

for each $U \in \mathcal{P}(A)$.

*Proof.* *Item 1, Characterisations*: We will proceed by showing:

- Step 1: Item 1a $\iff$ Item 1b.
- Step 2: Item 1a $\iff$ Item 1c.
- Step 3: Item 1a $\iff$ Item 1d.
- Step 4: Item 1c $\iff$ Item 1e.
- Step 5: Item 1e $\iff$ Item 1f.
- Step 6: Item 1a $\iff$ Item 1g.
- Step 7: Item 1g $\iff$ Item 1h.
- Step 8: Item 1a $\iff$ Item 1i.

**Step 1: Item 1a $\iff$ Item 1b.** We claim that Items 1a and 1b are equivalent:

- *Item 1a $\implies$ Item 1b:* We proceed in a few steps:
  - Proceeding by contrapositive, we claim that given a pair of maps $g, h \colon C \rightrightarrows A$ such that $g \neq h$, we have $f \circ g \neq f \circ h$.
  - Indeed, as $g$ and $h$ are different maps, there must exist at least one element $x \in C$ such that $g(x) \neq h(x)$.
  - But then we have $f(g(x)) \neq f(h(x))$, as $f$ is injective.

– Thus $f \circ g \neq f \circ h$, and we are done.

- *Item 1b* $\Longrightarrow$ *Item 1a:* We proceed in a few steps:

  – Consider the diagram

$$\text{pt} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{f} B,$$

  where $[x]$ and $[y]$ are the morphisms picking the elements $x$ and $y$ of $A$.

  – Note that we have $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$.

  – Since $f$ is assumed to be a monomorphism, if $f(x) = f(y)$, then $f \circ [x] = f \circ [y]$ and therefore $[x] = [y]$.

  – This shows that if $f(x) = f(y)$, then $x = y$, so $f$ is injective.

**Step 2: Item 1a $\Longleftrightarrow$ Item 1c.** We claim that Items 1a and 1c are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1c:* We proceed in a few steps:

  – Assume that $f$ is injective and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$. We wish to show that $U = V$.

  – To show that $U \subset V$, let $u \in U$.

  – By the definition of the direct image, we have $f(u) \in f_!(U)$.

  – Since $f_!(U) = f_!(V)$, it follows that $f(u) \in f_!(V)$.

  – Thus, there exists some $v \in V$ such that $f(v) = f(u)$.

  – Since $f$ is injective, the equality $f(v) = f(u)$ implies that $v = u$.

  – Thus $u \in V$ and $U \subset V$.

  – A symmetric argument shows that $V \subset U$.

  – Therefore $U = V$, showing $f_!$ to be injective.

- *Item 1c* $\Longrightarrow$ *Item 1a:* We proceed in a few steps:

  – Assume that the direct image function $f_!$ is injective and let $a, a' \in A$ such that $f(a) = f(a')$. We wish to show that $a = a'$.

– Since

$$\begin{aligned} f_!(\{a\}) &= \{f(a)\} \\ &= \{f(a')\} \\ &= f_!(\{a'\}), \end{aligned}$$

we must have $\{a\} = \{a'\}$, as $f_!$ is injective, so $a = a'$, showing $f$ to be injective.

**Step 3: Item 1c $\iff$ Item 1d.** This follows from Item 17 of Definition 4.6.1.1.5.
**Step 4: Item 1c $\iff$ Item 1e.** We claim that Items 1c and 1e are equivalent:

- *Item 1c $\implies$ Item 1e:* We proceed in a few steps:
  - Let $U, V \in \mathcal{P}(A)$ such that $f_!(U) \subset f_!(V)$, assume $f_!$ to be injective, and consider the set $U \cup V$.
  - Since $f_!(U) \subset f_!(V)$, we have

    $$\begin{aligned} f_!(U \cup V) &= f_!(U) \cup f_!(V) \\ &= f_!(V), \end{aligned}$$

    where we have used Item 5 of Definition 4.6.1.1.5 for the first equality.
  - Since $f_!$ is injective, this implies $U \cup V = V$.
  - Thus $U \subset V$, as we wished to show.
- *Item 1c $\implies$ Item 1e:* We proceed in a few steps:
  - Suppose Item 1e holds, and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$.
  - Since $f_!(U) = f_!(V)$, we have $f_!(U) \subset f_!(V)$ and $f_!(V) \subset f_!(U)$.
  - By assumption, this implies $U \subset V$ and $V \subset U$.
  - Thus $U = V$, showing $f_!$ to be injective.

**Step 5: Item 1e $\iff$ Item 1f.** This follows from Item 17 of Definition 4.6.1.1.5.
**Step 6: Item 1a $\iff$ Item 1g.** We have

$$f^{-1}(f(a)) = \left\{a' \in A \,\middle|\, f(a') = f(a)\right\}$$

so the condition $f^{-1}(f(a)) = \{a\}$ states precisely that if $f(a') = f(a)$, then $a' = a$.

**Step 7: Item 1g $\iff$ Item 1h.** We claim that Items 1g and 1h are indeed equivalent:

- *Item 1g $\implies$ Item 1h:* We have

$$\begin{aligned}[f^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} f^{-1}(f_!(U))\\ &= f^{-1}\left(f_!\left(\bigcup_{u \in U}\{u\}\right)\right)\\ &= f^{-1}\left(\bigcup_{u \in U} f_!(\{u\})\right)\\ &= \bigcup_{u \in U} f^{-1}(f_!(\{u\}))\\ &= \bigcup_{u \in U} f^{-1}(f_!(u))\\ &= \bigcup_{u \in U}\{u\}\\ &= U\end{aligned}$$

  for each $U \in \mathcal{P}(A)$, where we have used Item 5 of Definition 4.6.1.1.5 for the third equality and Item 5 of Definition 4.6.2.1.3 for the fourth equality.

- *Item 1h $\implies$ Item 1g:* Applying the condition $f^{-1} \circ f_! = \mathrm{id}_{\mathcal{P}(A)}$ to $U = \{a\}$ gives

$$f^{-1}(f_!(\{a\})) = \{a\}.$$

**Step 8: Item 1a $\iff$ Item 1i.** We claim that Items 1a and 1i are equivalent:

- *Item 1a $\implies$ Item 1i:* If $f$ is injective, then $f^{-1}(f(a)) = \{a\}$, so we have

$$\begin{aligned}f^{-1}(f_*(a)) &= \{a \in A \mid \{a\} \subset U\}\\ &= U.\end{aligned}$$

- *Item 1i $\implies$ Item 1a:* For $U = \{a\}$, the condition $f^{-1}(f_*(U)) = U$ becomes

$$\left\{a' \in A \,\middle|\, f^{-1}(f(a')) \subset \{a\}\right\} = \{a\}.$$

Since the set $f^{-1}(f(a'))$ is given by

$$\{a \in A \,|\, f(a) = f(a')\},$$

it follows that $f$ is injective.

This finishes the proof. □

02RW ## 4.7.2 Surjective Functions

Let $A$ and $B$ be sets.

02RX **Definition 4.7.2.1.1.** A function $f : A \to B$ is **surjective** if it satisfies the following condition:

($\star$) For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

02RY **Proposition 4.7.2.1.2.** Let $f : A \to B$ be a function.

02RZ 1. *Characterisations.* The following conditions are equivalent:

02S0 (a) The function $f$ is surjective.

02S1 (b) The function $f$ is an epimorphism in $\mathsf{Sets}$.

02S2 (c) The inverse image function

$$f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to $f$ is injective.

02S3 (d) The inverse image functor

$$f^{-1} : (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

associated to $f$ is full.

02S4 (e) The diagram

$$\begin{tikzcd} B \arrow[r, "f^{-1}"] \arrow[dr, hook, "\chi_B"'] & \mathcal{P}(A) \arrow[d, "f_!"] \\ & \mathcal{P}(B) \end{tikzcd}$$

commutes. That is, we have

$$f_!(f^{-1}(b)) = \{b\}$$

for each $b \in B$.

02S5 (f) We have

$$f_! \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "f^{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "f_!"] \\ & \mathcal{P}(B). \end{tikzcd}$$

In other words, we have

$$\left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in f^{-1}(U) \\ \text{such that } f(a) = b \end{array} \right\} = U$$

for each $U \in \mathcal{P}(A)$.

02S6 (g) We have

$$f_* \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "f^{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "f_*"] \\ & \mathcal{P}(B). \end{tikzcd}$$

In other words, we have

$$\left\{ b \in B \,\middle|\, f^{-1}(b) \subset f^{-1}(U) \right\} = U$$

for each $U \in \mathcal{P}(B)$.

*Proof. Item 1, Characterisations*: We will proceed by showing:

- Step 1: Item 1a $\iff$ Item 1b.
- Step 2: Item 1a $\iff$ Item 1c.

- Step 3: Item 1c $\iff$ Item 1d.
- Step 4: Item 1a $\iff$ Item 1e.
- Step 5: Item 1e $\iff$ Item 1f.
- Step 6: Item 1a $\iff$ Item 1g.

**Step 1: Item 1a $\iff$ Item 1b.** We claim Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* We proceed in a few steps:
  - Let $g, h \colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$.
  - For each $a \in A$, we have
  $$g(f(a)) = h(f(a)).$$
  - However, this implies that
  $$g(b) = h(b)$$
  for each $b \in B$, as $f$ is surjective.
  - Thus $g = h$ and $f$ is an epimorphism.
- *Item 1b $\implies$ Item 1a:* We proceed by contrapositive. Consider the diagram
  $$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$
  where $h$ is the map defined by $h(b) = 0$ for each $b \in B$ and $g$ is the map defined by
  $$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise}. \end{cases}$$
  Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have
  $$g(b) = 0 \neq 1 = h(b).$$
  Therefore $g \neq h$ and $f$ is not an epimorphism.

**Step 2: Item 1a $\iff$ Item 1c.** We claim Items 1a and 1c are indeed equivalent:

- *Item 1a* $\implies$ *Item 1c:* We proceed in a few steps:
  - Assume that $f$ is surjective. Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$. We wish to show that $U = V$.
  - To show that $U \subset V$, let $b \in U$.
  - Since $f$ is surjective, there must exist some $a \in A$ such that $f(a) = b$.
  - By the definition of the inverse image, since $f(a) = b$ and $b \in U$, we have $a \in f^{-1}(U)$.
  - By our initial assumption, $f^{-1}(U) = f^{-1}(V)$, so it follows that $a \in f^{-1}(V)$.
  - Again, by the definition of the inverse image, $a \in f^{-1}(V)$ means that $f(a) \in V$.
  - Since $f(a) = b$, we have shown that $b \in V$.
  - This establishes that $U \subset V$. A symmetric argument shows that $V \subset U$.
  - Thus $U = V$, proving that $f^{-1}$ is injective.
- *Item 1c* $\implies$ *Item 1a:* We proceed in a few steps:
  - Assume that the inverse image function $f^{-1}$ is injective. Suppose, for the sake of contradiction, that $f$ is not surjective.
  - The assumption that $f$ is not surjective means there exists some $b_0 \in B$ such that for all $a \in A$, we have $f(a) \neq b_0$.
  - By the definition of the inverse image, this is equivalent to stating that $f^{-1}(\{b_0\}) = \emptyset$.
  - Since $f^{-1}(\emptyset) = \emptyset$, we have $f^{-1}(\{b_0\}) = f^{-1}(\emptyset)$.
  - Since $f^{-1}$ is injective, this implies that $\{b_0\} = \emptyset$.
  - This is a contradiction, as the singleton set $\{b_0\}$ is non-empty.
  - Therefore, $f$ is surjective.

**Step 3: Item 1c $\iff$ Item 1d.** We claim that Items 1c and 1d are equivalent:

- *Item 1c* $\implies$ *Item 1d:* We proceed in a few steps:

- Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) \subset f^{-1}(V)$, assume $f^{-1}$ to be injective, and consider the set $U \cup V$.
- Since $f^{-1}(U) \subset f^{-1}(V)$, we have

$$\begin{aligned} f^{-1}(U \cup V) &= f^{-1}(U) \cup f^{-1}(V) \\ &= f^{-1}(V), \end{aligned}$$

  where we have used Item 5 of Definition 4.6.2.1.3 for the first equality.
- Since $f^{-1}$ is injective, this implies $U \cup V = V$.
- Thus $U \subset V$, as we wished to show.

• *Item 1d* $\implies$ *Item 1c:* We proceed in a few steps:

- Suppose Item 1d holds, and let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$.
- Since $f^{-1}(U) = f^{-1}(V)$, we have $f^{-1}(U) \subset f^{-1}(V)$ and $f^{-1}(V) \subset f^{-1}(U)$.
- By assumption, this implies $U \subset V$ and $V \subset U$.
- Thus $U = V$, showing $f^{-1}$ to be injective.

**Step 4: Item 1a $\iff$ Item 1e.** We have

$$f_!(f^{-1}(b)) = \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in f^{-1}(b) \\ \text{such that } f(a) = b \end{array} \right\},$$

so the condition $f_!(f^{-1}(b)) = \{b\}$ holds iff $f$ is surjective.
**Step 5: Item 1e $\iff$ Item 1f.** We claim that Items 1e and 1f are indeed equivalent:

• *Item 1e* $\implies$ *Item 1f:* We have

$$\begin{aligned} [f_! \circ f^{-1}](U) &\overset{\text{def}}{=} f_!(f^{-1}(U)) \\ &= f_!\left(f^{-1}\left(\bigcup_{u \in U} \{u\}\right)\right) \\ &= f_!\left(\bigcup_{u \in U} f^{-1}(\{u\})\right) \end{aligned}$$

$$\begin{aligned} &= \bigcup_{u\in U} f_!(f^{-1}(\{u\})) \\ &= \bigcup_{u\in U} f_!(f^{-1}(u)) \\ &= \bigcup_{u\in U} \{u\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(B)$, where we have used Item 5 of Definition 4.6.1.1.5 for the third equality and Item 5 of Definition 4.6.2.1.3 for the fourth equality.

- *Item 1f* $\implies$ *Item 1e:* Applying the condition $f_! \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)}$ to $U = \{b\}$ gives
$$f_!(f^{-1}(\{b\})) = \{b\}.$$

**Step 6: Item 1a $\iff$ Item 1g.** First, note that for the condition $f^{-1}(b) \subset f^{-1}(U)$ to hold, we must have $b \in U$ or $f^{-1}(b) = \emptyset$. Thus
$$f_*(f^{-1}(U)) = (U \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)).$$

We now claim that Items 1a and 1g are indeed equivalent:

- *Item 1a* $\implies$ *Item 1g:* If $f$ is surjective, we have
$$\begin{aligned} (U \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)) &= U \cup \emptyset \\ &= U, \end{aligned}$$
so $f_* \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)}$.
- *Item 1g* $\implies$ *Item 1a:* Taking $U = \emptyset$ gives
$$\begin{aligned} f_*(f^{-1}(\emptyset)) &= (\emptyset \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f), \end{aligned}$$
so the condition $f_*(f^{-1}(\emptyset)) = \emptyset$ implies $B \setminus \mathrm{Im}(f) = \emptyset$. Thus $\mathrm{Im}(f) = B$ and $f$ is surjective.

This finishes the proof. $\square$

# Appendices

# A Other Chapters



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