

Constructions With Sets

The Clowder Project Authors

July 29, 2025

000J This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

01YT 1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see [Definitions 4.2.4.1.1](#) and [4.2.5.1.1](#) and [Remarks 4.2.4.1.4](#) and [4.2.5.1.4](#)).

01YU 2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:

01YV (a) A discussion of the internal Hom of a powerset ([Section 4.4.7](#)).

01YW (b) A 0-categorical version of the Yoneda lemma ([Presheaves and the Yoneda Lemma, Theorem 12.1.5.1.1](#)), which we term the *Yoneda lemma for sets* ([Proposition 4.5.5.1.1](#)).

01YX (c) A characterisation of powersets as free cocompletions ([Section 4.4.5](#)), mimicking the corresponding statement for categories of presheaves (??).

01YY (d) A characterisation of powersets as free completions ([Section 4.4.6](#)), mimicking the corresponding statement for categories of copresheaves (??).

01YZ (e) A (-1) -categorical version of un/straightening ([Item 2](#) of [Proposition 4.5.1.1.4](#) and [Remark 4.5.1.1.6](#)).

01Z0 (f) A 0-categorical form of Isbell duality internal to powersets ([Section 4.4.8](#)).

01Z1 3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* : \mathcal{P}(A) \xrightarrow{\cong} \mathcal{P}(B)$$

of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f : A \rightarrow B$, including in particular:

- 01Z2 (a) How f^{-1} can be described as a precomposition while $f_!$ and f_* can be described as Kan extensions (Remarks 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).
- 01Z3 (b) An extensive list of the properties of $f_!$, f^{-1} , and f_* (Propositions 4.6.1.1.5, 4.6.1.1.7, 4.6.2.1.3, 4.6.2.1.5, 4.6.3.1.7 and 4.6.3.1.9).
- 01Z4 (c) How the functors $f_!$, f^{-1} , f_* , along with the functors

$$\begin{aligned} -_1 \cap -_2 &: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X), \\ [-_1, -_2]_X &: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X) \end{aligned}$$

may be viewed as a six-functor formalism with the empty set \emptyset as the dualising object (Section 4.6.4).

Contents

4.1 Limits of Sets	4
4.1.1 The Terminal Set	4
4.1.2 Products of Families of Sets	4
4.1.3 Binary Products of Sets	7
4.1.4 Pullbacks	20
4.1.5 Equalisers	31
4.1.6 Inverse Limits	36
4.2 Colimits of Sets	40
4.2.1 The Initial Set	40
4.2.2 Coproducts of Families of Sets	41
4.2.3 Binary Coproducts	43
4.2.4 Pushouts	47
4.2.5 Coequalisers	57
4.2.6 Direct Colimits	61
4.3 Operations With Sets	67
4.3.1 The Empty Set	67
4.3.2 Singleton Sets	67
4.3.3 Pairings of Sets	68
4.3.4 Ordered Pairs	68
4.3.5 Sets of Maps	69
4.3.6 Unions of Families of Subsets	72
4.3.7 Intersections of Families of Subsets	88
4.3.8 Binary Unions	102

4.3.9	Binary Intersections.....	108
4.3.10	Differences.....	115
4.3.11	Complements.....	120
4.3.12	Symmetric Differences.....	125
4.4	Powersets.....	134
4.4.1	Foundations.....	134
4.4.2	Functoriality of Powersets.....	137
4.4.3	Adjointness of Powersets I.....	139
4.4.4	Adjointness of Powersets II.....	140
4.4.5	Powersets as Free Cocompletions.....	141
4.4.6	Powersets as Free Completions.....	148
4.4.7	The Internal Hom of a Powerset.....	155
4.4.8	Isbell Duality for Sets.....	173
4.5	Characteristic Functions.....	175
4.5.1	The Characteristic Function of a Subset.....	175
4.5.2	The Characteristic Function of a Point.....	180
4.5.3	The Characteristic Relation of a Set.....	182
4.5.4	The Characteristic Embedding of a Set.....	184
4.5.5	The Yoneda Lemma for Sets.....	185
4.6	The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$.....	186
4.6.1	Direct Images.....	186
4.6.2	Inverse Images.....	200
4.6.3	Codirect Images.....	212
4.6.4	A Six-Functor Formalism for Sets.....	226
4.7	Miscellany.....	235
4.7.1	Injective Functions.....	235
4.7.2	Surjective Functions.....	241
A	Other Chapters.....	248

000K 4.1 Limits of Sets

000L 4.1.1 The Terminal Set

000M DEFINITION 4.1.1.1 ► THE TERMINAL SET

The **terminal set** is the terminal object of Sets as in Limits and Colimits, ??.

01DB CONSTRUCTION 4.1.1.1.2 ► CONSTRUCTION OF THE TERMINAL SET

Concretely, the terminal set is the pair $(\text{pt}, \{!_A\}_{A \in \text{Obj}(\text{Sets})})$ consisting of:

01DC 1. *The Limit.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.

01DD 2. *The Cone.* The collection of maps

$$\{!_A : A \rightarrow \text{pt}\}_{A \in \text{Obj}(\text{Sets})}$$

defined by

$$!_A(a) \stackrel{\text{def}}{=} \star$$

for each $a \in A$ and each $A \in \text{Obj}(\text{Sets})$.

PROOF 4.1.1.1.3 ► PROOF OF CONSTRUCTION 4.1.1.1.2

We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \quad \text{pt}$$

in Sets. Then there exists a unique map $\phi : A \rightarrow \text{pt}$ making the diagram

$$A \xrightarrow[\exists!]{\phi} \text{pt}$$

commute, namely $!_A$.



000N 4.1.2 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

000P

DEFINITION 4.1.2.1.1 ► THE PRODUCT OF A FAMILY OF SETS

The **product**¹ of $\{A_i\}_{i \in I}$ is the product of $\{A_i\}_{i \in I}$ in Sets as in Limits and Colimits, ??.

¹Further Terminology: Also called the **Cartesian product** of $\{A_i\}_{i \in I}$.

01DE

CONSTRUCTION 4.1.2.1.2 ► CONSTRUCTION OF THE PRODUCT OF A FAMILY OF SETS

Concretely, the product of $\{A_i\}_{i \in I}$ is the pair $(\prod_{i \in I} A_i, \{\text{pr}_i\}_{i \in I})$ consisting of:

01DF

1. *The Limit.* The set $\prod_{i \in I} A_i$ defined by

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \text{Sets} \left(I, \bigcup_{i \in I} A_i \right) \mid \begin{array}{l} \text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

01DG

2. *The Cone.* The collection

$$\left\{ \text{pr}_i : \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$

for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

PROOF 4.1.2.1.3 ► PROOF OF CONSTRUCTION 4.1.2.1.2

We claim that $\prod_{i \in I} A_i$ is the categorical product of $\{A_i\}_{i \in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form


$$\begin{array}{ccc} P & & \\ & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

in Sets. Then there exists a unique map $\phi: P \rightarrow \prod_{i \in I} A_i$ making the diagram

$$\begin{array}{ccc} P & & \\ \downarrow \phi \exists! & \searrow p_i & \\ \prod_{i \in I} A_i & \xrightarrow{\text{pr}_i} & A_i \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. 

01DH

REMARK 4.1.2.1.4 ► UNWINDING CONSTRUCTION 4.1.2.1.2

Less formally, we may think of Cartesian products and projection maps as follows:

01DJ

1. We think of $\prod_{i \in I} A_i$ as the set whose elements are I -indexed collections $(a_i)_{i \in I}$ with $a_i \in A_i$ for each $i \in I$.

01DK

2. We view the projection maps

$$\left\{ \text{pr}_i: \prod_{i \in I} A_i \rightarrow A_i \right\}_{i \in I}$$

as being given by

$$\text{pr}_i((a_j)_{j \in I}) \stackrel{\text{def}}{=} a_i$$

for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

000Q

PROPOSITION 4.1.2.1.5 ► PROPERTIES OF PRODUCTS OF FAMILIES OF SETS

Let $\{A_i\}_{i \in I}$ be a family of sets.

000R

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}) \rightarrow \text{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, we have

$$\left[\prod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \text{Sets} \left(\prod_{i \in I} A_i, \prod_{i \in I} B_i \right)$$

of $\prod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i : A_i \rightarrow B_i\}_{i \in I}$$

in $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\prod_{i \in I} f_i : \prod_{i \in I} A_i \rightarrow \prod_{i \in I} B_i$$

defined by

$$\left[\prod_{i \in I} f_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i \in I}$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} A_i$.

PROOF 4.1.2.1.6 ► PROOF OF PROPOSITION 4.1.2.1.5

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.



000S 4.1.3 Binary Products of Sets

Let A and B be sets.

000T DEFINITION 4.1.3.1.1 ► BINARY PRODUCTS OF SETS

The **product of A and B** ¹ is the product of A and B in Sets as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **Cartesian product of A and B** .

01DL

CONSTRUCTION 4.1.3.1.2 ► CONSTRUCTION OF BINARY PRODUCTS OF SETS

Concretely, the product of A and B is the pair $(A \times B, \{\text{pr}_1, \text{pr}_2\})$ consisting of:

01DM

1. *The Limit.* The set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \text{Sets}(\{0, 1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\ &\cong \left\{ \begin{array}{l} \text{ordered pairs } (a, b) \text{ with} \\ a \in A \text{ and } b \in B \end{array} \right\}. \end{aligned}$$

01DN

2. *The Cone.* The maps

$$\begin{aligned} \text{pr}_1 &: A \times B \rightarrow A, \\ \text{pr}_2 &: A \times B \rightarrow B \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(a, b) &\stackrel{\text{def}}{=} a, \\ \text{pr}_2(a, b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each $(a, b) \in A \times B$.

PROOF 4.1.3.1.3 ► PROOF OF CONSTRUCTION 4.1.3.1.2

We claim that $A \times B$ is the categorical product of A and B in the category of sets. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & P & \\ p_1 \swarrow & & \searrow p_2 \\ A & \xleftarrow{\text{pr}_1} A \times B \xrightarrow{\text{pr}_2} & B \end{array}$$

in Sets. Then there exists a unique map $\phi: P \rightarrow A \times B$ making the diagram

$$\begin{array}{ccc} & P & \\ p_1 \swarrow & \downarrow \phi \exists! & \searrow p_2 \\ A & \xleftarrow{\text{pr}_1} A \times B \xrightarrow{\text{pr}_2} & B \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. 

000U

PROPOSITION 4.1.3.1.4 ► PROPERTIES OF PRODUCTS OF SETS

Let A , B , C , and X be sets.

000V

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -: \quad \text{Sets} &\rightarrow \text{Sets}, \\ - \times B: \quad \text{Sets} &\rightarrow \text{Sets}, \\ -_1 \times -_2: \text{Sets} \times \text{Sets} &\rightarrow \text{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} : \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \times B, X \times Y)$$

of \times at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g : A \times B \rightarrow X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \text{Obj}(\text{Sets})$.

000W

2. *Adjointness I.* We have adjunctions

$$\begin{aligned} (A \times - \dashv \text{Sets}(A, -)) : \text{Sets} &\begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets}, \\ (- \times B \dashv \text{Sets}(B, -)) : \text{Sets} &\begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets}, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

01Z5

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\text{Sets}} \dashv -_1 \times -_2) : \text{Sets} \begin{array}{c} \xrightarrow{\Delta_{\text{Sets}}} \\ \perp \\ \xleftarrow{-_1 \times -_2} \end{array} \text{Sets} \times \text{Sets},$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets} \times \text{Sets}}((A, A), (B, C)) \cong \text{Sets}(A, B \times C),$$

natural in $A \in \text{Obj}(\text{Sets})$ and in $(B, C) \in \text{Obj}(\text{Sets} \times \text{Sets})$.

000X

4. *Associativity*. We have an isomorphism of sets

$$\alpha_{A,B,C}^{\text{Sets}} : (A \times B) \times C \xrightarrow{\sim} A \times (B \times C),$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

000Y

5. *Unitality*. We have isomorphisms of sets

$$\lambda_A^{\text{Sets}} : \text{pt} \times A \xrightarrow{\sim} A,$$

$$\rho_A^{\text{Sets}} : A \times \text{pt} \xrightarrow{\sim} A,$$

natural in $A \in \text{Obj}(\text{Sets})$.

000Z

6. *Commutativity*. We have an isomorphism of sets

$$\sigma_{A,B}^{\text{Sets}} : A \times B \xrightarrow{\sim} B \times A,$$

natural in $A, B \in \text{Obj}(\text{Sets})$.

01DP

7. *Distributivity Over Coproducts*. We have isomorphisms of sets

$$\delta_\ell^{\text{Sets}} : A \times (B \amalg C) \xrightarrow{\sim} (A \times B) \amalg (A \times C),$$

$$\delta_r^{\text{Sets}} : (A \amalg B) \times C \xrightarrow{\sim} (A \times C) \amalg (B \times C),$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

0010

8. *Annihilation With the Empty Set*. We have isomorphisms of sets

$$\zeta_\ell^{\text{Sets}} : \emptyset \times A \xrightarrow{\sim} \emptyset,$$

$$\zeta_r^{\text{Sets}} : A \times \emptyset \xrightarrow{\sim} \emptyset,$$

natural in $A \in \text{Obj}(\text{Sets})$.

0011

9. *Distributivity Over Unions*. Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cup W) = (U \times V) \cup (U \times W),$$

$$(U \cup V) \times W = (U \times W) \cup (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

- 0012 10. *Distributivity Over Intersections.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned} U \times (V \cap W) &= (U \times V) \cap (U \times W), \\ (U \cap V) \times W &= (U \times W) \cap (V \times W) \end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

- 0014 11. *Distributivity Over Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned} U \times (V \setminus W) &= (U \times V) \setminus (U \times W), \\ (U \setminus V) \times W &= (U \times W) \setminus (V \times W) \end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

- 0015 12. *Distributivity Over Symmetric Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned} U \times (V \triangle W) &= (U \times V) \triangle (U \times W), \\ (U \triangle V) \times W &= (U \times W) \triangle (V \times W) \end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

- 0013 13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$\begin{array}{ccc} (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \xrightarrow{\cap \times \cap} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \mathcal{P}_{X,X}^\times \times \mathcal{P}_{X,X}^\times \downarrow & & \downarrow \mathcal{P}_{X,X}^\times \\ \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) & \xrightarrow{\cap} & \mathcal{P}(X \times X) \end{array}$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).$$

for each $U, V, W, T \in \mathcal{P}(X)$.

- 0016 14. *Symmetric Monoidality.* The 8-tuple $(\mathbf{Sets}, \times, \text{pt}, \mathbf{Sets}(-1, -2), \alpha^{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$ is a closed symmetric monoidal category.

0017

15. *Symmetric Bimonoidality*. The 18-tuple

$$\left(\text{Sets}, \coprod, \times, \emptyset, \text{pt}, \text{Sets}(-1, -2), \alpha^{\text{Sets}}, \lambda^{\text{Sets}}, \rho^{\text{Sets}}, \sigma^{\text{Sets}}, \right. \\ \left. \alpha^{\text{Sets}, \coprod}, \lambda^{\text{Sets}, \coprod}, \rho^{\text{Sets}, \coprod}, \sigma^{\text{Sets}, \coprod}, \delta_\ell^{\text{Sets}}, \delta_r^{\text{Sets}}, \zeta_\ell^{\text{Sets}}, \zeta_r^{\text{Sets}} \right),$$

is a symmetric closed bimonoidal category, where $\alpha^{\text{Sets}, \coprod}$, $\lambda^{\text{Sets}, \coprod}$, $\rho^{\text{Sets}, \coprod}$, and $\sigma^{\text{Sets}, \coprod}$ are the natural transformations from **Items 3 to 5** of **Proposition 4.2.3.1.4**.

PROOF 4.1.3.1.5 ► PROOF OF PROPOSITION 4.1.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

Item 2: Adjointness

We prove only that there's an adjunction $- \times B \dashv \text{Sets}(B, -)$, witnessed by a bijection

$$\text{Sets}(A \times B, C) \cong \text{Sets}(A, \text{Sets}(B, C)),$$

natural in $B, C \in \text{Obj}(\text{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \text{Sets}(A, -)$ follows almost exactly in the same way.

0126

- *Map I.* We define a map

$$\Phi_{B,C} : \text{Sets}(A \times B, C) \rightarrow \text{Sets}(A, \text{Sets}(B, C)),$$

by sending a function

$$\xi : A \times B \rightarrow C$$

to the function

$$\begin{aligned} \xi^\dagger : A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a^\dagger : B \rightarrow C), \end{aligned}$$

where we define

$$\xi_a^\dagger(b) \stackrel{\text{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $\llbracket a \mapsto f(a) \rrbracket$ notation of **Sets, Notation 3.1.1.1.2**, we have

$$\xi^\dagger \stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket.$$

01Z7

- *Map II.* We define a map

$$\Psi_{B,C} : \text{Sets}(A, \text{Sets}(B, C)), \rightarrow \text{Sets}(A \times B, C)$$

given by sending a function

$$\begin{aligned} \xi : A &\longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a : B \rightarrow C), \end{aligned}$$

to the function

$$\xi^\dagger : A \times B \rightarrow C$$

defined by

$$\begin{aligned} \xi^\dagger(a, b) &\stackrel{\text{def}}{=} \text{ev}_b(\text{ev}_a(\xi)) \\ &\stackrel{\text{def}}{=} \text{ev}_b(\xi_a) \\ &\stackrel{\text{def}}{=} \xi_a(b) \end{aligned}$$

for each $(a, b) \in A \times B$.

01Z8

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \text{id}_{\text{Sets}(A \times B, C)}.$$

Indeed, given a function $\xi : A \times B \rightarrow C$, we have

$$\begin{aligned} [\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\ &= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\ &= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\ &= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket) \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\text{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket (a, b) \mapsto \text{ev}_b(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\ &= \llbracket (a, b) \mapsto \xi(a, b) \rrbracket \\ &= \xi. \end{aligned}$$

01Z9

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \text{id}_{\text{Sets}(A, \text{Sets}(B, C))}.$$

Indeed, given a function

$$\begin{aligned} \xi &: A \longrightarrow \text{Sets}(B, C), \\ a &\mapsto (\xi_a: B \rightarrow C), \end{aligned}$$

we have

$$\begin{aligned} [\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\stackrel{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\ &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a, b) \mapsto \xi_a(b) \rrbracket) \\ &\stackrel{\text{def}}{=} \Phi_{A,B}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \text{ev}_{(a,b)}(\llbracket (a', b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\ &\stackrel{\text{def}}{=} \xi. \end{aligned}$$

01ZA

- *Naturality for Φ , Part I.* We need to show that, given a function $g: B \rightarrow B'$, the diagram

$$\begin{array}{ccc} \text{Sets}(A \times B', C) & \xrightarrow{\Phi_{B',C}} & \text{Sets}(A, \text{Sets}(B', C)), \\ \text{id}_A \times g^* \downarrow & & \downarrow (g^*)! \\ \text{Sets}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Sets}(A, \text{Sets}(B, C)) \end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B' \rightarrow C,$$

we have

$$\begin{aligned} [\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\ &= \Phi_{B,C}(\xi(-_1, g(-_2))) \end{aligned}$$

$$\begin{aligned}
&= [\xi(-1, g(-2))]^\dagger \\
&= \xi_{-1}^\dagger(g(-2)) \\
&= (g^*)_!(\xi^\dagger) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

Alternatively, using the $\llbracket a \mapsto f(a) \rrbracket$ notation of **Sets**, **Notation 3.1.1.1.2**, we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\text{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\text{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}([\text{id}_A \times g^*](\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\
&= \Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, g(b)) \rrbracket) \\
&= \llbracket a \mapsto \llbracket b \mapsto \xi(a, g(b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto g^*(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\
&= (g^*)_!(\llbracket a \mapsto \llbracket b' \mapsto \xi(a, b') \rrbracket \rrbracket) \\
&= (g^*)_!(\Phi_{B',C}(\llbracket (a, b') \mapsto \xi(a, b') \rrbracket)) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

01ZB

- *Naturality for Φ , Part II.* We need to show that, given a function $h: C \rightarrow C'$, the diagram

$$\begin{array}{ccc}
\text{Sets}(A \times B, C) & \xrightarrow{\Phi_{B,C}} & \text{Sets}(A, \text{Sets}(B, C)), \\
\downarrow h! & & \downarrow (h!)_! \\
\text{Sets}(A \times B, C') & \xrightarrow{\Phi_{B,C'}} & \text{Sets}(A, \text{Sets}(B, C'))
\end{array}$$

commutes. Indeed, given a function

$$\xi: A \times B \rightarrow C,$$

we have

$$[\Phi_{B,C} \circ h!](\xi) = \Phi_{B,C}(h!(\xi))$$

$$\begin{aligned}
&= \Phi_{B,C}(h_!(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\
&= \Phi_{B,C}(\llbracket (a, b) \mapsto h(\xi(a, b)) \rrbracket) \\
&= \llbracket a \mapsto \llbracket b \mapsto h(\xi(a, b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto h_!(\llbracket b \mapsto \xi(a, b) \rrbracket) \rrbracket \\
&= (h_!)_!(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\
&= (h_!)_!(\Phi_{B,C}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\
&= (h_!)_!(\Phi_{B,C}(\xi)) \\
&= [(h_!)_! \circ \Phi_{B,C}](\xi).
\end{aligned}$$

01ZC

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that Ψ is also natural in each argument.

This finishes the proof.

Item 3: Adjointness II

This follows from the universal property of the product.

Item 4: Associativity

This is proved in the proof of [Monoidal Structures on the Category of Sets, Definition 5.1.4.1.1](#).

Item 5: Unitality

This is proved in the proof of [Monoidal Structures on the Category of Sets, Definitions 5.1.5.1.1 and 5.1.6.1.1](#).

Item 6: Commutativity

This is proved in the proof of [Monoidal Structures on the Category of Sets, Definition 5.1.7.1.1](#).

Item 7: Distributivity Over Coproducts

This is proved in the proof of [Monoidal Structures on the Category of Sets, Definitions 5.3.1.1.1 and 5.3.2.1.1](#).

Item 8: Annihilation With the Empty Set

This is proved in the proof of [Monoidal Structures on the Category of Sets, Definitions 5.3.3.1.1 and 5.3.4.1.1](#).

Item 9: Distributivity Over Unions

See [Pro25c].

Item 10: Distributivity Over Intersections

See [Pro25d, Corollary 1].

Item 11: Distributivity Over Differences

See [Pro25a].

Item 12: Distributivity Over Symmetric Differences

See [Pro25b].


Item 13: Middle-Four Exchange With Respect to Intersections

See [Pro25d, Corollary 1].

Item 14: Symmetric Monoidality

This is a repetition of **Monoidal Structures on the Category of Sets**, Proposition 5.1.9.1.1, and is proved there.

Item 15: Symmetric Bimonoidality

This is a repetition of **Monoidal Structures on the Category of Sets**, Proposition 5.3.5.1.1, and is proved there. 

01DR

REMARK 4.1.3.1.6 ► THE CARTESIAN PRODUCT OF SETS AS AN $(\mathbb{E}_k, \mathbb{E}_\ell)$ -TENSOR PRODUCT

As shown in **Item 1** of Proposition 4.1.3.1.4, the Cartesian product of sets defines a functor

$$-_1 \times -_2 : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}.$$

This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k, \ell} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_\ell}(\mathbf{Sets}) \rightarrow \mathbf{Mon}_{\mathbb{E}_{k+\ell}}(\mathbf{Sets})$$

of tensor products of \mathbb{E}_k -monoid objects on Sets with \mathbb{E}_ℓ -monoid objects on Sets; see ??.

01DS

REMARK 4.1.3.1.7 ► DIAGRAMS FOR ITEMS 9 TO 12 OF PROPOSITION 4.1.3.1.4

We may state the equalities in **Items 9 to 12** of **Proposition 4.1.3.1.4** as the commutativity of the following diagrams:



0018 4.1.4 Pullbacks

Let A , B , and C be sets and let $f: A \rightarrow C$ and $g: B \rightarrow C$ be functions.

0019 DEFINITION 4.1.4.1.1 ► PULLBACKS OF SETS

The **pullback of A and B over C along f and g** ¹ is the pullback of A and B over C along f and g in **Sets** as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **fibre product of A and B over C along f and g** .

01DT CONSTRUCTION 4.1.4.1.2 ► CONSTRUCTION OF PULLBACKS OF SETS

Concretely, the pullback of A and B over C along f and g is the pair $(A \times_C B, \{pr_1, pr_2\})$ consisting of:

01DU 1. *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

01DV 2. *The Cone.* The maps¹

$$pr_1: A \times_C B \rightarrow A,$$

$$pr_2: A \times_C B \rightarrow B$$

defined by

$$pr_1(a, b) \stackrel{\text{def}}{=} a,$$

$$pr_2(a, b) \stackrel{\text{def}}{=} b$$

for each $(a, b) \in A \times_C B$.

¹*Further Notation:* Also written $pr_1^{A \times_C B}$ and $pr_2^{A \times_C B}$.

PROOF 4.1.4.1.3 ► PROOF OF CONSTRUCTION 4.1.4.1.2

We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in **Sets**. First we need to check that the relevant pullback

diagram commutes, i.e. that we have

$$f \circ \text{pr}_1 = g \circ \text{pr}_2,$$

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\text{pr}_2} & B \\ \text{pr}_1 \downarrow & & \downarrow g \\ A & \xrightarrow{f} & C. \end{array}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned} [f \circ \text{pr}_1](a, b) &= f(\text{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\text{pr}_2(a, b)) \\ &= [g \circ \text{pr}_2](a, b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{array}{ccccc} P & & \xrightarrow{p_2} & & B \\ & \searrow p_1 & & \searrow \text{pr}_2 & \\ & & A \times_C B & \xrightarrow{\text{pr}_2} & B \\ & & \downarrow \text{pr}_1 & \lrcorner & \downarrow g \\ & & A & \xrightarrow{f} & C \end{array}$$

in Sets. Then there exists a unique map $\phi: P \rightarrow A \times_C B$ making the diagram

$$\begin{array}{ccccc} P & & \xrightarrow{p_2} & & B \\ & \searrow \phi & & \searrow \text{pr}_2 & \\ & & A \times_C B & \xrightarrow{\text{pr}_2} & B \\ & & \downarrow \text{pr}_1 & \lrcorner & \downarrow g \\ & & A & \xrightarrow{f} & C \end{array}$$

$\exists!$

commute, being uniquely determined by the conditions

$$\begin{aligned}\text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2\end{aligned}$$

via


$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. 

01DW

REMARK 4.1.4.1.4 ► PULLBACKS OF SETS DEPEND ON THE MAPS

It is common practice to write $A \times_C B$ for the pullback of A and B over C along f and g , omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \times_C B$ depends very much on the maps f and g , and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

001A

EXAMPLE 4.1.4.1.5 ► EXAMPLES OF PULLBACKS OF SETS

Here are some examples of pullbacks of sets.

001B

1. *Unions via Intersections.* Let X be a set. We have

$$A \cap B \cong A \times_{A \cup B} B,$$

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow \lrcorner & & \downarrow \iota_B \\ A & \xrightarrow{\iota_A} & A \cup B \end{array}$$


for each $A, B \in \mathcal{P}(X)$.

PROOF 4.1.4.1.6 ► PROOF OF EXAMPLE 4.1.4.1.5

Item 1: Unions via Intersections

Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. 

001C

PROPOSITION 4.1.4.1.7 ► PROPERTIES OF PULLBACKS OF SETS

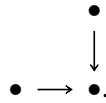
Let A, B, C , and X be sets.

001D

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f, C, g} B$ defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of ${}_1 \times_{-3} {}_1$ is given by sending a morphism

$$\begin{array}{ccccc} A \times_C B & \xrightarrow{\quad} & B & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & A' \times_{C'} B' & \xrightarrow{\quad} & B' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ A & \xrightarrow{f} & C & & \\ \searrow \phi & & \searrow \chi & & \\ & A' & \xrightarrow{f'} & C' & \end{array}$$

in $\text{Fun}(\mathcal{P}, \text{Sets})$ to the map $\xi: A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{array}{ccccc}
 A \times_C B & \xrightarrow{\quad} & B & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & A' \times_{C'} B' & \xrightarrow{\quad} & B' & \\
 & \downarrow \lrcorner & \downarrow & \downarrow g' & \\
 A & \xrightarrow{f} & C & & \\
 \downarrow \phi & & \downarrow \chi & & \\
 A' & \xrightarrow{f'} & C' & &
 \end{array}$$

commute.

01DX

2. *Adjointness I.* We have adjunctions

$$\begin{aligned}
 (A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)) : \quad & \mathbf{Sets}_{/X} \begin{array}{c} \xrightarrow{A \times_X -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_{/X}(A, -)} \end{array} \mathbf{Sets}_{/X}, \\
 (- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)) : \quad & \mathbf{Sets}_{/X} \begin{array}{c} \xrightarrow{- \times_X B} \\ \perp \\ \xleftarrow{\mathbf{Sets}_{/X}(B, -)} \end{array} \mathbf{Sets}_{/X},
 \end{aligned}$$

witnessed by bijections

$$\begin{aligned}
 \mathbf{Sets}_{/X}(A \times_X B, C) &\cong \mathbf{Sets}_{/X}(A, \mathbf{Sets}_{/X}(B, C)), \\
 \mathbf{Sets}_{/X}(A \times_X B, C) &\cong \mathbf{Sets}_{/X}(B, \mathbf{Sets}_{/X}(A, C)),
 \end{aligned}$$

natural in $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \mathbf{Obj}(\mathbf{Sets}_{/X})$, where $\mathbf{Sets}_{/X}(A, B)$ is the object of $\mathbf{Sets}_{/X}$ consisting of (see Fibred Sets, ??):

- *The Set.* The set $\mathbf{Sets}_{/X}(A, B)$ defined by

$$\mathbf{Sets}_{/X}(A, B) \stackrel{\text{def}}{=} \coprod_{x \in X} \mathbf{Sets}(\phi_A^{-1}(x), \phi_B^{-1}(x))$$

- *The Map to X.* The map

$$\phi_{\mathbf{Sets}_{/X}(A, B)} : \mathbf{Sets}_{/X}(A, B) \rightarrow X$$

defined by

$$\phi_{\mathbf{Sets}_{/X}(A, B)}(x, f) \stackrel{\text{def}}{=} x$$

for each $(x, f) \in \mathbf{Sets}_{/X}(A, B)$.

01ZD

3. *Adjointness II.* We have an adjunction

$$\left(\Delta_{\mathbf{Sets}/X} \dashv -1 \times -2 \right): \mathbf{Sets}/X \xrightleftharpoons[-1 \times -2]{\Delta_{\mathbf{Sets}/X}} \mathbf{Sets}/X \times \mathbf{Sets}/X,$$

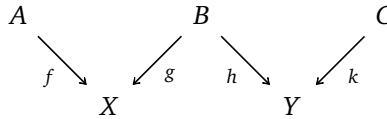
witnessed by a bijection

$$\mathrm{Hom}_{\mathbf{Sets}/X \times \mathbf{Sets}/X}((A, A), (B, C)) \cong \mathbf{Sets}/X(A, B \times_X C),$$

natural in $A \in \mathrm{Obj}(\mathbf{Sets}/X)$ and in $(B, C) \in \mathrm{Obj}(\mathbf{Sets}/X \times \mathbf{Sets}/X)$.

001E

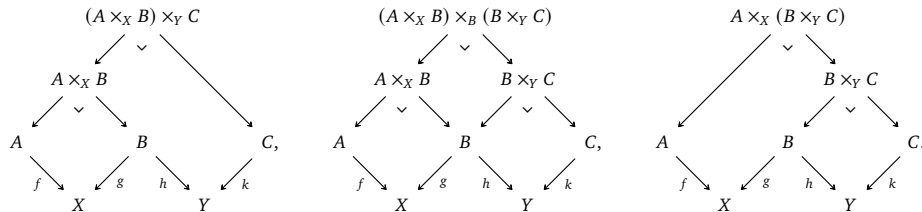
4. *Associativity.* Given a diagram



in \mathbf{Sets} , we have isomorphisms of sets

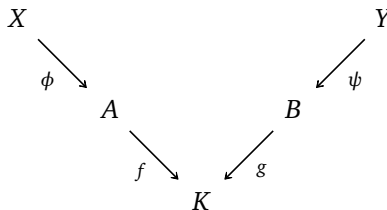
$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams



01DY

5. *Interaction With Composition.* Given a diagram



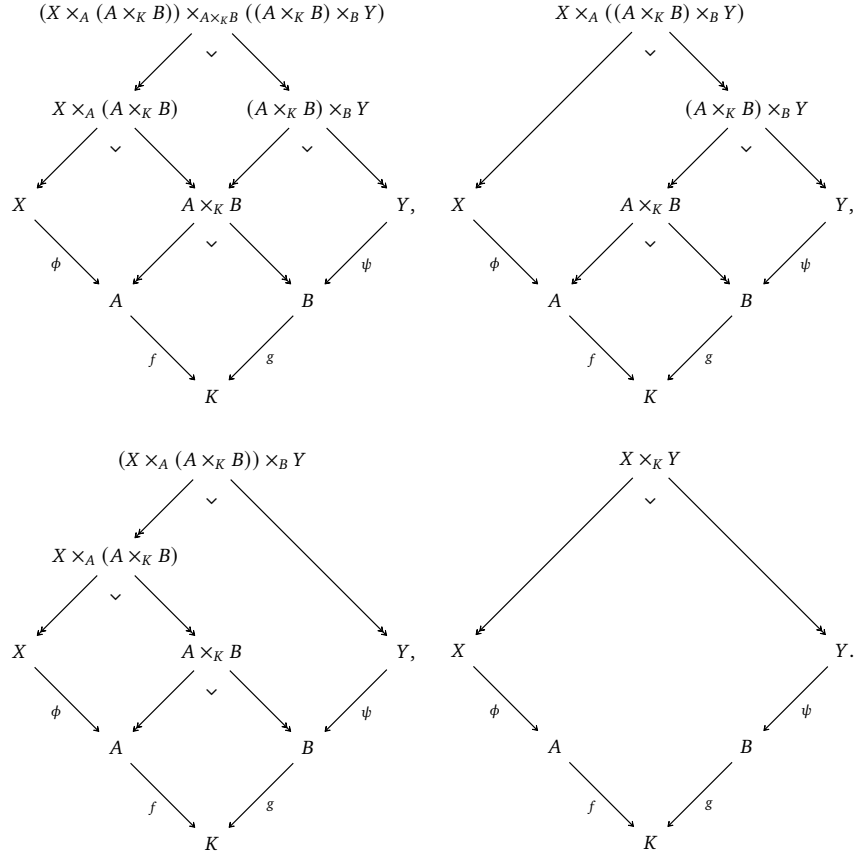
in Sets, we have isomorphisms of sets

$$\begin{aligned}
 X \times_K^{f \circ \phi, g \circ \psi} Y &\cong (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \times_{A \times_K^{f, g} B}^{p_2, p_1} ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y) \\
 &\cong X \times_A^{\phi, p} ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y) \\
 &\cong (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \times_B^{q, \psi} Y
 \end{aligned}$$

where

$$\begin{aligned}
 q_1 &= \text{pr}_1^{A \times_K^{f, g} B}, & q_2 &= \text{pr}_2^{A \times_K^{f, g} B}, \\
 p_1 &= \text{pr}_1^{(A \times_K^{f, g} B) \times_Y^{q_2, \psi}}, & p_2 &= \text{pr}_2^{X \times_{A \times_K^{f, g} B}^{q_1, \phi} (A \times_K^{f, g} B)}, \\
 p &= q_1 \circ \text{pr}_1^{(A \times_K^{f, g} B) \times_B^{q_2, \psi} Y}, & q &= q_2 \circ \text{pr}_2^{X \times_A^{\phi, q_1} (A \times_K^{f, g} B)},
 \end{aligned}$$

and where these pullbacks are built as in the following diagrams:



001F

6. *Unitality*. We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 f \downarrow \lrcorner & & \downarrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{c}
 \lambda_A^{\text{Sets}/X} : X \times_X A \xrightarrow{\sim} A, \\
 \rho_A^{\text{Sets}/X} : A \times_X X \xrightarrow{\sim} A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \parallel \lrcorner & & \parallel \\
 X & \xrightarrow{f} & X,
 \end{array}$$

natural in $(A, f) \in \text{Obj}(\text{Sets}/X)$.

001G

7. *Commutativity*. We have an isomorphism of sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 \sigma_{A,B}^{\text{Sets}/X} : A \times_X B \xrightarrow{\sim} B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X,
 \end{array}$$

natural in $(A, f), (B, g) \in \text{Obj}(\text{Sets}/X)$.

01DZ

8. *Distributivity Over Coproducts*. Let A, B , and C be sets and let $\phi_A : A \rightarrow X$, $\phi_B : B \rightarrow X$, and $\phi_C : C \rightarrow X$ be morphisms of sets. We have isomorphisms of sets

$$\begin{aligned}
 \delta_\ell^{\text{Sets}/X} : A \times_X (B \amalg C) &\xrightarrow{\sim} (A \times_X B) \amalg (A \times_X C), \\
 \delta_r^{\text{Sets}/X} : (A \amalg B) \times_X C &\xrightarrow{\sim} (A \times_X C) \amalg (B \times_X C),
 \end{aligned}$$

as in the diagrams

$$\begin{array}{ccc}
 (A \times_X B) \amalg (A \times_X C) & \longrightarrow & B \amalg C \\
 \downarrow \lrcorner & & \downarrow \phi_B \amalg \phi_C \\
 A & \xrightarrow{\phi_A} & X
 \end{array}
 \quad
 \begin{array}{ccc}
 (A \times_X C) \amalg (B \times_X C) & \longrightarrow & C \\
 \downarrow \lrcorner & & \downarrow \phi_C \\
 A \amalg B & \xrightarrow{\phi_A \amalg \phi_B} & X
 \end{array}$$

natural in $A, B, C \in \text{Obj}(\text{Sets}/X)$.

001H

9. *Annihilation With the Empty Set*. We have isomorphisms of sets

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & \emptyset \\
 \downarrow \lrcorner & & \downarrow \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 \begin{array}{c}
 \zeta_\ell^{\text{Sets}/X} : A \times_X \emptyset \xrightarrow{\sim} \emptyset, \\
 \zeta_r^{\text{Sets}/X} : \emptyset \times_X A \xrightarrow{\sim} \emptyset,
 \end{array}
 \quad
 \begin{array}{ccc}
 \emptyset & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 \emptyset & \longrightarrow & X,
 \end{array}$$

natural in $(A, f) \in \text{Obj}(\text{Sets}_{/X})$.

001J

10. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\text{pt}} B \cong A \times B,$$

$$\begin{array}{ccc} A \times B & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow !_B \\ A & \xrightarrow{!_A} & \text{pt.} \end{array}$$

001K

11. *Symmetric Monoidality.* The 8-tuple $(\text{Sets}_{/X}, \times_X, X, \mathbf{Sets}_{/X}, \alpha^{\text{Sets}_{/X}}, \lambda^{\text{Sets}_{/X}}, \rho^{\text{Sets}_{/X}}, \sigma^{\text{Sets}_{/X}})$ is a symmetric closed monoidal category.

PROOF 4.1.4.1.8 ► PROOF OF PROPOSITION 4.1.4.1.7

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Adjointness I

This is a repetition of Fibred Sets, ?? of ??, and is proved there.

Item 3: Adjointness II

This follows from the universal property of the product (pullbacks are products in $\text{Sets}_{/X}$).

Item 4: Associativity

We have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{(a, b), c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{(a, b), c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a, (b, c)) \in A \times (B \times_Y C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a, (b, c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$(A \times_X B) \times_B (B \times_Y C) \cong \{(a, b), (b', c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\}$$

$$\begin{aligned}
&\cong \left\{ ((a, b), (b', c)) \in (A \times B) \times (B \times C) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\
&\cong \left\{ (a, (b, (b', c))) \in A \times (B \times (B \times C)) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\
&\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times B) \times C) \mid \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\
&\cong \left\{ (a, ((b, b'), c)) \in A \times ((B \times_B B) \times C) \mid \begin{array}{l} f(a) = g(b) \text{ and } \\ h(b') = k(c) \end{array} \right\} \\
&\cong \{ (a, (b, c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c) \} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}$$

where we have used **Item 6** for the isomorphism $B \times_B B \cong B$.

Item 5: Interaction With Composition

By **Item 4**, it suffices to construct only the isomorphism

$$X \times_K^{f \circ \phi, g \circ \psi} Y \cong (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \times_{A \times_K^{f, g} B}^{p_2, p_1} ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y).$$

We have

$$\begin{aligned}
(X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) &\stackrel{\text{def}}{=} \{ (x, (a, b)) \in X \times (A \times_K^{f, g} B) \mid \phi(x) = q_1(a, b) \} \\
&\stackrel{\text{def}}{=} \{ (x, (a, b)) \in X \times (A \times_K^{f, g} B) \mid \phi(x) = a \} \\
&\cong \{ (x, (a, b)) \in X \times (A \times B) \mid \phi(x) = a \text{ and } f(a) = g(b) \}, \\
((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y) &\stackrel{\text{def}}{=} \{ ((a, b), y) \in (A \times_K^{f, g} B) \times Y \mid q_2(a, b) = \psi(y) \} \\
&\stackrel{\text{def}}{=} \{ ((a, b), y) \in (A \times_K^{f, g} B) \times Y \mid b = \psi(y) \} \\
&\cong \{ ((a, b), y) \in (A \times B) \times Y \mid b = \psi(y) \text{ and } f(a) = g(b) \},
\end{aligned}$$

so writing

$$\begin{aligned}
S &= (X \times_A^{\phi, q_1} (A \times_K^{f, g} B)) \\
S' &= ((A \times_K^{f, g} B) \times_B^{q_2, \psi} Y),
\end{aligned}$$

we have

$$\begin{aligned}
S \times_{A \times_K^{f, g} B}^{p_2, p_1} S' &\stackrel{\text{def}}{=} \{ ((x, (a, b)), ((a', b'), y)) \in S \times S' \mid p_1(x, (a, b)) = p_2((a', b'), y) \} \\
&\stackrel{\text{def}}{=} \{ ((x, (a, b)), ((a', b'), y)) \in S \times S' \mid (a, b) = (a', b') \} \\
&\cong \{ ((x, a, b, y)) \in X \times A \times B \times Y \mid \phi(x) = a, \psi(y) = b, \text{ and } f(a) = g(b) \}
\end{aligned}$$

$$\begin{aligned} &\cong \{(x, a, b, y) \in X \times A \times B \times Y \mid f(\phi(x)) = g(\psi(y))\} \\ &\stackrel{\text{def}}{=} X \times_K Y. \end{aligned}$$

This finishes the proof.

Item 6: Unitality

We have

$$\begin{aligned} X \times_X A &\cong \{(x, a) \in X \times A \mid f(a) = x\}, \\ A \times_X X &\cong \{(a, x) \in X \times A \mid f(a) = x\}, \end{aligned}$$

which are isomorphic to A via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$. The proof of the naturality of $\lambda^{\text{Sets}/X}$ and $\rho^{\text{Sets}/X}$ is omitted.

Item 7: Commutativity

We have

$$\begin{aligned} A \times_C B &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\} \\ &= \{(a, b) \in A \times B \mid g(b) = f(a)\} \\ &\cong \{(b, a) \in B \times A \mid g(b) = f(a)\} \\ &\stackrel{\text{def}}{=} B \times_C A. \end{aligned}$$

The proof of the naturality of $\sigma^{\text{Sets}/X}$ is omitted.

Item 8: Distributivity Over Coproducts

We have

$$\begin{aligned} A \times_X (B \amalg C) &\stackrel{\text{def}}{=} \{(a, z) \in A \times (B \amalg C) \mid \phi_A(a) = \phi_{B \amalg C}(z)\} \\ &= \{(a, z) \in A \times (B \amalg C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, z) \in A \times (B \amalg C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c)\} \\ &= \{(a, z) \in A \times (B \amalg C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, z) \in A \times (B \amalg C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c)\} \\ &\cong \{(a, b) \in A \times B \mid \phi_A(a) = \phi_B(b)\} \\ &\quad \cup \{(a, c) \in A \times C \mid \phi_A(a) = \phi_C(c)\} \\ &\stackrel{\text{def}}{=} (A \times_X B) \cup (A \times_X C) \\ &\cong (A \times_X B) \amalg (A \times_X C), \end{aligned}$$

with the construction of the isomorphism

$$\delta_r^{\text{Sets}/X} : (A \amalg B) \times_X C \xrightarrow{\sim} (A \times_X C) \amalg (B \times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\text{Sets}/X}$ and $\delta_r^{\text{Sets}/X}$ is omitted.

Item 9: Annihilation With the Empty Set

We have

$$\begin{aligned} A \times_X \emptyset &\stackrel{\text{def}}{=} \{(a, b) \in A \times \emptyset \mid f(a) = g(b)\} \\ &= \{k \in \emptyset \mid f(a) = g(b)\} \\ &= \emptyset, \end{aligned}$$

and similarly for $\emptyset \times_X A$, where we have used **Item 8** of **Proposition 4.1.3.1.4**. The proof of the naturality of $\zeta_\ell^{\text{Sets}/X}$ and $\zeta_r^{\text{Sets}/X}$ is omitted.

Item 10: Interaction With Products

We have

$$\begin{aligned} A \times_{\text{pt}} B &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid !_A(a) = !_B(b)\} \\ &\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid \star = \star\} \\ &= \{(a, b) \in A \times B\} \\ &= A \times B. \end{aligned}$$

Item 11: Symmetric Monoidality

Omitted. 

001L 4.1.5 Equalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

001M DEFINITION 4.1.5.1.1 ► EQUALISERS OF SETS

The **equaliser** of f and g is the equaliser of f and g in **Sets** as in **Limits and Colimits**, ??.

01E0 CONSTRUCTION 4.1.5.1.2 ► CONSTRUCTION OF EQUALISERS OF SETS

Concretely, the equaliser of f and g is the pair $(\text{Eq}(f, g), \text{eq}(f, g))$ consisting of:

01E1

1. *The Limit.* The set $\text{Eq}(f, g)$ defined by

$$\text{Eq}(f, g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

01E2

2. *The Cone.* The inclusion map

$$\text{eq}(f, g): \text{Eq}(f, g) \hookrightarrow A.$$

PROOF 4.1.5.1.3 ► PROOF OF CONSTRUCTION 4.1.5.1.2

We claim that $\text{Eq}(f, g)$ is the categorical equaliser of f and g in **Sets**. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A \xrightleftharpoons[f]{f} B \\ & \nearrow e & \\ E & & \end{array}$$

in **Sets**. Then there exists a unique map $\phi: E \rightarrow \text{Eq}(f, g)$ making the diagram

$$\begin{array}{ccc} \text{Eq}(f, g) & \xrightarrow{\text{eq}(f, g)} & A \xrightleftharpoons[f]{f} B \\ \uparrow \phi \exists! & \nearrow e & \\ E & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via


$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. 

001N

PROPOSITION 4.1.5.1.4 ► PROPERTIES OF EQUALISERS OF SETS

Let A , B , and C be sets.

001P

1. *Associativity.* We have isomorphisms of sets¹

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$\begin{array}{ccc} & f & \\ A & \xrightarrow{\quad} & B \\ & \underset{h}{\xleftarrow{\quad}} & \\ & g & \end{array}$$

in Sets, being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

001Q

2. *Unitality.* We have an isomorphism of sets

$$\text{Eq}(f, f) \cong A.$$

001R

3. *Commutativity.* We have an isomorphism of sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

001S

4. *Interaction With Composition.* Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have an inclusion of sets

$$\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) \subset \text{Eq}(h \circ f, k \circ g),$$

where $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ is the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{g} \end{matrix} B \begin{matrix} \xrightarrow{h} \\ \xrightarrow[k]{k} \end{matrix} C.$$

¹That is, the following three ways of forming “the” equaliser of (f, g, h) agree:

01ZE

- (a) Take the equaliser of (f, g, h) , i.e. the limit of the diagram

$$A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{g} \\ \xrightarrow[h]{h} \end{matrix} B$$

in Sets.

01ZF

- (b) First take the equaliser of f and g , forming a diagram

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{g} \end{matrix} B$$

and then take the equaliser of the composition

$$\text{Eq}(f, g) \xrightarrow{\text{eq}(f, g)} A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[h]{h} \end{matrix} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) = \text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))$$

of $\text{Eq}(f, g)$.

01ZG

- (c) First take the equaliser of g and h , forming a diagram

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{matrix} \xrightarrow{g} \\ \xrightarrow[h]{h} \end{matrix} B$$

and then take the equaliser of the composition

$$\text{Eq}(g, h) \xrightarrow{\text{eq}(g, h)} A \begin{matrix} \xrightarrow{f} \\ \xrightarrow[g]{g} \end{matrix} B,$$

obtaining a subset

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) = \text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))$$

of $\text{Eq}(g, h)$.

PROOF 4.1.5.1.5 ► PROOF OF PROPOSITION 4.1.5.1.4

Item 1: Associativity

We first prove that $\text{Eq}(f, g, h)$ is indeed given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & & f & \\ & & & \rightarrow & \\ \text{Eq}(f, g, h) & \xrightarrow{\text{eq}(f, g, h)} & A & \xrightarrow[\begin{smallmatrix} -g \\ h \end{smallmatrix}]{f} & B \\ & \nearrow e & & & \\ E & & & & \end{array}$$

in Sets. Then there exists a unique map $\phi: E \rightarrow \text{Eq}(f, g, h)$, uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g, h)$.

We now check the equalities

$$\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) \cong \text{Eq}(f, g, h) \cong \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)).$$

Indeed, we have

$$\begin{aligned} \text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h)) &\cong \{x \in \text{Eq}(g, h) \mid [f \circ \text{eq}(g, h)](a) = [g \circ \text{eq}(g, h)](a)\} \\ &\cong \{x \in \text{Eq}(g, h) \mid f(a) = g(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\ &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\ &\cong \text{Eq}(f, g, h). \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g)) &\cong \{x \in \text{Eq}(f, g) \mid [f \circ \text{eq}(f, g)](a) = [h \circ \text{eq}(f, g)](a)\} \\
 &\cong \{x \in \text{Eq}(f, g) \mid f(a) = h(a)\} \\
 &\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
 &\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
 &\cong \text{Eq}(f, g, h).
 \end{aligned}$$

Item 2: Unitality

Indeed, we have

$$\begin{aligned}
 \text{Eq}(f, f) &\stackrel{\text{def}}{=} \{a \in A \mid f(a) = f(a)\} \\
 &= A.
 \end{aligned}$$

Item 3: Commutativity

Indeed, we have

$$\begin{aligned}
 \text{Eq}(f, g) &\stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\} \\
 &= \{a \in A \mid g(a) = f(a)\} \\
 &\stackrel{\text{def}}{=} \text{Eq}(g, f).
 \end{aligned}$$


Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}
 \text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g)) &\cong \{a \in \text{Eq}(f, g) \mid h(f(a)) = k(g(a))\} \\
 &\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.
 \end{aligned}$$

and

$$\text{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\text{Eq}(h \circ f \circ \text{eq}(f, g), k \circ g \circ \text{eq}(f, g))$ to $\text{Eq}(h \circ f, k \circ g)$. 

01E3 4.1.6 Inverse Limits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}: (I, \preceq) \rightarrow \text{Sets}$ be an inverse system of sets.

01E4

DEFINITION 4.1.6.1.1 ► INVERSE LIMITS OF SETS

The **inverse limit** of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ in Sets as in Limits and Colimits, ??.

01E5

CONSTRUCTION 4.1.6.1.2 ► CONSTRUCTION OF INVERSE LIMITS OF SETS

Concretely, the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the pair $\left(\varprojlim_{\alpha \in I} (X_\alpha), \{\text{pr}_\alpha\}_{\alpha \in I} \right)$ consisting of:

01E6

1. *The Limit.* The set $\varprojlim_{\alpha \in I} (X_\alpha)$ defined by

$$\varprojlim_{\alpha \in I} (X_\alpha) \stackrel{\text{def}}{=} \left\{ (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \mid \begin{array}{l} \text{for each } \alpha, \beta \in I, \text{ if } \alpha \preceq \beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta) \end{array} \right\}.$$

01E7

2. *The Cone.* The collection

$$\left\{ \text{pr}_\gamma : \varprojlim_{\alpha \in I} (X_\alpha) \rightarrow X_\gamma \right\}_{\gamma \in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{ \text{pr}_\gamma : \prod_{\alpha \in I} X_\alpha \rightarrow X_\gamma \right\}_{\gamma \in I}$$

of **Item 2** of **Construction 4.1.2.1.2** to $\varprojlim_{\alpha \in I} (X_\alpha)$ and hence given by

$$\text{pr}_\gamma((x_\alpha)_{\alpha \in I}) \stackrel{\text{def}}{=} x_\gamma$$

for each $\gamma \in I$ and each $(x_\alpha)_{\alpha \in I} \in \varprojlim_{\alpha \in I} (X_\alpha)$.

PROOF 4.1.6.1.3 ► PROOF OF CONSTRUCTION 4.1.6.1.2

We claim that $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ is the limit of the inverse system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$. First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta} \circ \text{pr}_\alpha = \text{pr}_\beta,$$

$$\begin{array}{ccc} & \lim_{\leftarrow \alpha \in I} (X_\alpha) & \\ \text{pr}_\alpha \swarrow & & \searrow \text{pr}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

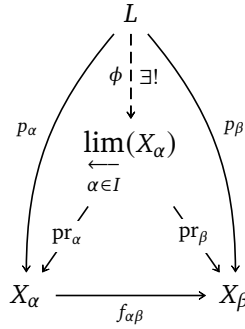
for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma \in I} \in \lim_{\leftarrow \gamma \in I} (X_\gamma)$, we have

$$\begin{aligned} [f_{\alpha\beta} \circ \text{pr}_\alpha]((x_\gamma)_{\gamma \in I}) &\stackrel{\text{def}}{=} f_{\alpha\beta}(\text{pr}_\alpha((x_\gamma)_{\gamma \in I})) \\ &\stackrel{\text{def}}{=} f_{\alpha\beta}(x_\alpha) \\ &= x_\beta \\ &\stackrel{\text{def}}{=} \text{pr}_\beta((x_\gamma)_{\gamma \in I}), \end{aligned}$$

where the third equality comes from the definition of $\lim_{\leftarrow \alpha \in I} (X_\alpha)$. Next, we prove that $\lim_{\leftarrow \alpha \in I} (X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{array}{ccc} & L & \\ p_\alpha \swarrow & & \searrow p_\beta \\ & \lim_{\leftarrow \alpha \in I} (X_\alpha) & \\ \text{pr}_\alpha \swarrow & & \searrow \text{pr}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

in Sets. Then there indeed exists a unique map $\phi: L \xrightarrow{\exists!} \varprojlim_{\alpha \in I} (X_\alpha)$ making the diagram



commute, being uniquely determined by the family of conditions


$$\{p_\alpha = \text{pr}_\alpha \circ \phi\}_{\alpha \in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha \in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$ indeed lies in $\varprojlim_{\alpha \in I} (X_\alpha)$, as we have

$$\begin{aligned} f_{\alpha\beta}(p_\alpha(\ell)) &\stackrel{\text{def}}{=} [f_{\alpha\beta} \circ p_\alpha](\ell) \\ &\stackrel{\text{def}}{=} p_\beta(\ell) \end{aligned}$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $(L, \{p_\alpha\}_{\alpha \in I})$. 

01E8

EXAMPLE 4.1.6.1.4 ► EXAMPLES OF INVERSE LIMITS OF SETS

Here are some examples of inverse limits of sets.

01E9

1. *The p-Adic Integers.* The ring of p -adic integers \mathbb{Z}_p of ?? is the inverse limit

$$\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N}} (\mathbb{Z}/p^n);$$

see ??.

01EA

2. *Rings of Formal Power Series.* The ring $R[[t]]$ of formal power series in a variable t is the inverse limit

$$R[[t]] \cong \varprojlim_{n \in \mathbb{N}} (R[t]/t^n R[t]);$$

see ??.

01EB

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see ??.

001T 4.2 Colimits of Sets

001U 4.2.1 The Initial Set

001V

DEFINITION 4.2.1.1.1 ► THE INITIAL SET

The **initial set** is the initial object of Sets as in Limits and Colimits, ??.

01EC

CONSTRUCTION 4.2.1.1.2 ► CONSTRUCTION OF THE INITIAL SET

Concretely, the initial set is the pair $(\emptyset, \{\iota_A\}_{A \in \text{Obj}(\text{Sets})})$ consisting of:

01ED

1. *The Colimit.* The empty set \emptyset of **Definition 4.3.1.1.1**.

01EE

2. *The Cocone.* The collection of maps

$$\{\iota_A : \emptyset \rightarrow A\}_{A \in \text{Obj}(\text{Sets})}$$

given by the inclusion maps from \emptyset to A .

PROOF 4.2.1.1.3 ► PROOF OF CONSTRUCTION 4.2.1.1.2

We claim that \emptyset is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \quad A$$

in Sets. Then there exists a unique map $\phi: \emptyset \rightarrow A$ making the diagram

$$\emptyset \xrightarrow[\exists!]{\phi} A$$

commute, namely the inclusion map ι_A .



001W 4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

001X DEFINITION 4.2.2.1.1 ► THE COPRODUCT OF A FAMILY OF SETS

The **coproduct** of $\{A_i\}_{i \in I}$ ¹ is the coproduct of $\{A_i\}_{i \in I}$ in Sets as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i \in I}$.

01EF CONSTRUCTION 4.2.2.1.2 ► CONSTRUCTION OF THE COPRODUCT OF A FAMILY OF SETS

Concretely, the disjoint union of $\{A_i\}_{i \in I}$ is the pair $(\coprod_{i \in I} A_i, \{\text{inj}_i\}_{i \in I})$ consisting of:

- 01EG 1. *The Colimit.* The set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ (i, x) \in I \times \left(\bigcup_{i \in I} A_i \right) \mid x \in A_i \right\}.$$

- 01EH 2. *The Cocone.* The collection

$$\left\{ \text{inj}_i: A_i \rightarrow \coprod_{i \in I} A_i \right\}_{i \in I}$$

of maps given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in A_i$ and each $i \in I$.

PROOF 4.2.2.1.3 ► PROOF OF CONSTRUCTION 4.2.2.1.2

We claim that $\coprod_{i \in I} A_i$ is the categorical coproduct of $\{A_i\}_{i \in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form


$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

in Sets. Then there exists a unique map $\phi: \coprod_{i \in I} A_i \rightarrow C$ making the diagram

$$\begin{array}{ccc} & & C \\ & \nearrow \iota_i & \uparrow \hat{\phi} \exists! \\ A_i & \xrightarrow{\text{inj}_i} & \coprod_{i \in I} A_i \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i, x)) = \iota_i(x)$$

for each $(i, x) \in \coprod_{i \in I} A_i$. 

001Y

PROPOSITION 4.2.2.1.4 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF SETS

Let $\{A_i\}_{i \in I}$ be a family of sets.

001Z

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \coprod_{i \in I} A_i$ defines a functor

$$\coprod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}) \rightarrow \text{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, we have

$$\left[\coprod_{i \in I} \right] ((A_i)_{i \in I}) \stackrel{\text{def}}{=} \coprod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \text{Obj}(\text{Fun}(I_{\text{disc}}, \text{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} : \text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \rightarrow \text{Sets} \left(\coprod_{i \in I} A_i, \coprod_{i \in I} B_i \right)$$

of $\coprod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i : A_i \rightarrow B_i\}_{i \in I}$$

in $\text{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\coprod_{i \in I} f_i : \coprod_{i \in I} A_i \rightarrow \coprod_{i \in I} B_i$$

defined by

$$\left[\coprod_{i \in I} f_i \right] (i, a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i \in I} A_i$.

PROOF 4.2.2.1.5 ► PROOF OF PROPOSITION 4.2.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.



0020 4.2.3 Binary Coproducts

Let A and B be sets.

0021 DEFINITION 4.2.3.1.1 ► COPRODUCTS OF SETS

The **coproduct of A and B** ¹ is the coproduct of A and B in Sets as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **disjoint union of A and B** .

01EJ

CONSTRUCTION 4.2.3.1.2 ► CONSTRUCTION OF COPRODUCTS OF SETS

Concretely, the coproduct of A and B is the pair $(A \amalg B, \{\text{inj}_1, \text{inj}_2\})$ consisting of:

01EK

1. *The Colimit.* The set $A \amalg B$ defined by

$$\begin{aligned} A \amalg B &\stackrel{\text{def}}{=} \coprod_{z \in \{A, B\}} z \\ &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in A\} \cup \{(1, b) \in S \mid b \in B\}, \end{aligned}$$

where $S = \{0, 1\} \times (A \cup B)$.

01EL

2. *The Cocone.* The maps

$$\text{inj}_1: A \rightarrow A \amalg B,$$

$$\text{inj}_2: B \rightarrow A \amalg B,$$

given by

$$\text{inj}_1(a) \stackrel{\text{def}}{=} (0, a),$$

$$\text{inj}_2(b) \stackrel{\text{def}}{=} (1, b),$$

for each $a \in A$ and each $b \in B$.

PROOF 4.2.3.1.3 ► PROOF OF CONSTRUCTION 4.2.3.1.2

We claim that $A \amalg B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & C & \\ \iota_1 \nearrow & & \nwarrow \iota_2 \\ A \xrightarrow{\text{inj}_1} & A \amalg B & \xleftarrow{\text{inj}_2} B \end{array}$$

in Sets. Then there exists a unique map $\phi: A \amalg B \rightarrow C$ making the diagram

$$\begin{array}{ccccc}
 & & C & & \\
 & \swarrow \iota_1 & \uparrow \phi \mid \exists! & \nwarrow \iota_2 & \\
 A & \xrightarrow{\text{inj}_1} & A \amalg B & \xleftarrow{\text{inj}_2} & B
 \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
 \phi \circ \text{inj}_A &= \iota_A, \\
 \phi \circ \text{inj}_B &= \iota_B
 \end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \amalg B$.



0022

PROPOSITION 4.2.3.1.4 ► PROPERTIES OF COPRODUCTS OF SETS

Let A, B, C , and X be sets.

0023

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \amalg B$ defines functors

$$\begin{aligned}
 A \amalg -: \quad \text{Sets} &\rightarrow \text{Sets}, \\
 - \amalg B: \quad \text{Sets} &\rightarrow \text{Sets}, \\
 -_1 \amalg -_2: \text{Sets} \times \text{Sets} &\rightarrow \text{Sets},
 \end{aligned}$$

where $-_1 \amalg -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$, we have

$$[-_1 \amalg -_2](A, B) \stackrel{\text{def}}{=} A \amalg B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \text{Obj}(\text{Sets})$, the action on Hom-sets

$$\amalg_{(A,B),(X,Y)}: \text{Sets}(A, X) \times \text{Sets}(B, Y) \rightarrow \text{Sets}(A \amalg B, X \amalg Y)$$

of \amalg at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \amalg g: A \amalg B \rightarrow X \amalg Y$$

defined by

$$[f \amalg g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \amalg B$.

and where $A \amalg -$ and $- \amalg B$ are the partial functors of $-_1 \amalg -_2$ at $A, B \in \text{Obj}(\text{Sets})$.

01ZH

2. *Adjointness.* We have an adjunction

$$(-_1 \amalg -_2 \dashv \Delta_{\text{Sets}}): \text{Sets} \times \text{Sets} \begin{array}{c} \xrightarrow{-_1 \amalg -_2} \\ \perp \\ \xleftarrow{\Delta_{\text{Sets}}} \end{array} \text{Sets},$$

witnessed by a bijection

$$\text{Sets}(A \amalg B, C) \cong \text{Hom}_{\text{Sets} \times \text{Sets}}((A, B), (C, C))$$

natural in $(A, B) \in \text{Obj}(\text{Sets} \times \text{Sets})$ and in $C \in \text{Obj}(\text{Sets})$.

0024

3. *Associativity.* We have an isomorphism of sets

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg}: (X \amalg Y) \amalg Z \xrightarrow{\sim} X \amalg (Y \amalg Z),$$

natural in $X, Y, Z \in \text{Obj}(\text{Sets})$.

0025

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \lambda_X^{\text{Sets}, \amalg}: \emptyset \amalg X &\xrightarrow{\sim} X, \\ \rho_X^{\text{Sets}, \amalg}: X \amalg \emptyset &\xrightarrow{\sim} X, \end{aligned}$$

natural in $X \in \text{Obj}(\text{Sets})$.

0026

5. *Commutativity.* We have an isomorphism of sets

$$\sigma_{X,Y}^{\text{Sets}, \amalg}: X \amalg Y \xrightarrow{\sim} Y \amalg X,$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

0027

6. *Symmetric Monoidality.* The 7-tuple $(\text{Sets}, \amalg, \emptyset, \alpha_{\amalg}^{\text{Sets}}, \lambda_{\amalg}^{\text{Sets}}, \rho_{\amalg}^{\text{Sets}}, \sigma^{\text{Sets}})$ is a symmetric monoidal category.

PROOF 4.2.3.1.5 ► PROOF OF PROPOSITION 4.2.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

Item 2: Adjointness

This follows from the universal property of the coproduct.

Item 3: Associativity

This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.2.3.1.1.

Item 4: Unitality

This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definitions 5.2.4.1.1 and 5.2.5.1.1.

Item 5: Commutativity

This is proved in the proof of **Monoidal Structures on the Category of Sets**, Definition 5.2.6.1.1.

Item 6: Symmetric Monoidality

This is a repetition of **Monoidal Structures on the Category of Sets**, Proposition 5.2.7.1.1, and is proved there. 

0028 4.2.4 Pushouts

Let A , B , and C be sets and let $f: C \rightarrow A$ and $g: C \rightarrow B$ be functions.

0029 DEFINITION 4.2.4.1.1 ► PUSHOUTS OF SETS

The **pushout of A and B over C along f and g** ¹ is the pushout of A and B over C along f and g in Sets as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g** .

01EM CONSTRUCTION 4.2.4.1.2 ► CONSTRUCTION OF PUSHOUTS OF SETS

Concretely, the pushout of A and B over C along f and g is the pair $(A \amalg_C B, \{\text{inj}_1, \text{inj}_2\})$ consisting of:

01EN

1. *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B / \sim_C,$$

where \sim_C is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

01EP

2. *The Cocone.* The maps

$$\begin{aligned} \text{inj}_1 &: A \rightarrow A \coprod_C B, \\ \text{inj}_2 &: B \rightarrow A \coprod_C B \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(a) &\stackrel{\text{def}}{=} [(0, a)] \\ \text{inj}_2(b) &\stackrel{\text{def}}{=} [(1, b)] \end{aligned}$$

for each $a \in A$ and each $b \in B$.

PROOF 4.2.4.1.3 ► PROOF OF CONSTRUCTION 4.2.4.1.2

We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

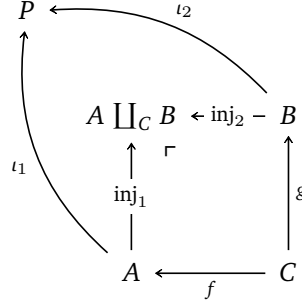
$$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$$

$$\begin{array}{ccc} A \coprod_C B & \xleftarrow{\text{inj}_2} & B \\ \text{inj}_1 \uparrow & & \uparrow g \\ A & \xleftarrow{f} & C. \end{array}$$

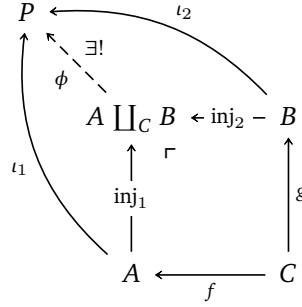
Indeed, given $c \in C$, we have

$$\begin{aligned} [\text{inj}_1 \circ f](c) &= \text{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \text{inj}_2(g(c)) \\ &= [\text{inj}_2 \circ g](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation \sim on $A \amalg B$. Next, we prove that $A \amalg_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets. Then there exists a unique map $\phi: A \amalg_C B \rightarrow P$ making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_1 &= \iota_1, \\ \phi \circ \text{inj}_2 &= \iota_2\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \amalg_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation \sim on $A \amalg B$ as follows:

01EQ

1. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by [Remark 4.2.4.1.4](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

01ER

2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by [Remark 4.2.4.1.4](#), we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

01ES

3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by [Remark 4.2.4.1.4](#), we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives


$$\begin{aligned} \phi([x]) &= \phi([(0, f(c))]) \\ &\stackrel{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]), \end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned} (0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b) \end{aligned}$$

gives

$$\begin{aligned} \phi([(0, a)]) &= \phi([(0, a')]), \\ \phi([(1, b)]) &= \phi([(1, b')]), \\ \phi([(0, a)]) &= \phi([(1, b)]), \end{aligned}$$

showing ϕ to be well-defined. 

002A

REMARK 4.2.4.1.4 ► UNWINDING DEFINITION 4.2.4.1.1

In detail, by **Conditions on Relations**, **Construction 10.5.2.1.2**, the relation \sim of **Definition 4.2.4.1.1** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

01ET

1. We have $a, b \in A$ and $a = b$.

01EU

2. We have $a, b \in B$ and $a = b$.

01EV

3. There exist $x_1, \dots, x_n \in A \amalg B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

01EW

(a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.

01EX

(b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

In other words, there exist $x_1, \dots, x_n \in A \amalg B$ satisfying the following conditions:

01EY

(c) There exists $c_0 \in C$ satisfying one of the following conditions:

01ZJ

i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.

01ZK

ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.

01EZ

(d) For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:

01ZL

i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.

01ZM

ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.

01F0

(e) There exists $c_n \in C$ satisfying one of the following conditions:

01F1

i. We have $x_n = f(c_n)$ and $b = g(c_n)$.

01F2

ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

01ZN

REMARK 4.2.4.1.5 ► PUSHOUTS OF SETS DEPEND ON THE MAPS

It is common practice to write $A \amalg_C B$ for the pushout of A and B over C along f and g , omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \amalg_C B$ depends very much on the maps f and g , and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \amalg_{f,C,g} B$ or $A \amalg_C^{f,g} B$ for $A \amalg_C B$.

002B EXAMPLE 4.2.4.1.6 ► EXAMPLES OF PUSHOUTS OF SETS

Here are some examples of pushouts of sets.

002C 1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of **Pointed Sets**, Definition 6.3.3.1.1 is an example of a pushout of sets.

002D 2. *Intersections via Unions.* Let X be a set. We have

$$A \cup B \cong A \amalg_{A \cap B} B,$$

$$\begin{array}{ccc} A \cup B & \longleftarrow & B \\ \uparrow \lrcorner & & \uparrow \\ A & \longleftarrow & A \cap B \end{array}$$


for each $A, B \in \mathcal{P}(X)$.

PROOF 4.2.4.1.7 ► PROOF OF EXAMPLE 4.2.4.1.6

Item 1: Wedge Sums of Pointed Sets

This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.

Item 2: Intersections via Unions

Indeed, $A \amalg_{A \cap B} B$ is the quotient of $A \amalg B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. 

002E PROPOSITION 4.2.4.1.8 ► PROPERTIES OF PUSHOUTS OF SETS

Let A, B, C , and X be sets.

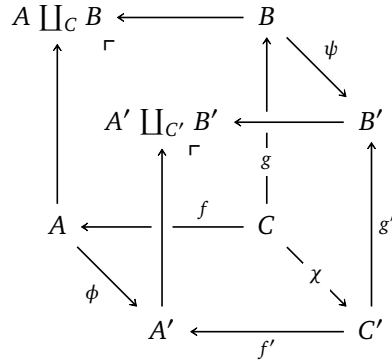
002F 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \amalg_{f, C, g} B$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets},$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \uparrow & \\ \bullet & \longleftarrow & \bullet \end{array}$$

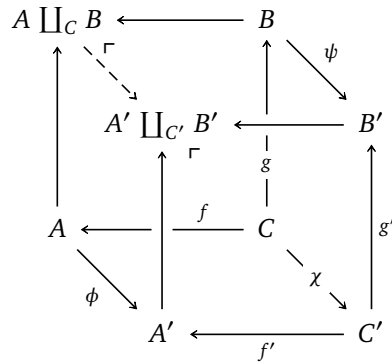
In particular, the action on morphisms of $-_1 \coprod_{-3} -_1$ is given by sending a morphism



in $\text{Fun}(\mathcal{P}, \text{Sets})$ to the map $\xi: A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram



commute.

01ZP

2. *Adjointness.* We have an adjunction

$$\left(-_1 \coprod_{X^{-2}} \dashv \Delta_{\text{Sets}_{X/}} \right): \text{Sets}_{X/} \times \text{Sets}_{X/} \begin{matrix} \xrightarrow{-_1 \coprod_{X^{-2}}} \\ \perp \\ \xleftarrow{\Delta_{\text{Sets}_{X/}}} \end{matrix} \text{Sets}_{X/},$$

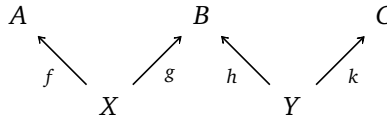
witnessed by a bijection

$$\text{Sets}_{X/}(A \amalg_X B, C) \cong \text{Hom}_{\text{Sets}_{X/} \times \text{Sets}_{X/}}((A, B), (C, C))$$

natural in $(A, B) \in \text{Obj}(\text{Sets}_{X/} \times \text{Sets}_{X/})$ and in $C \in \text{Obj}(\text{Sets}_{X/})$.

002G

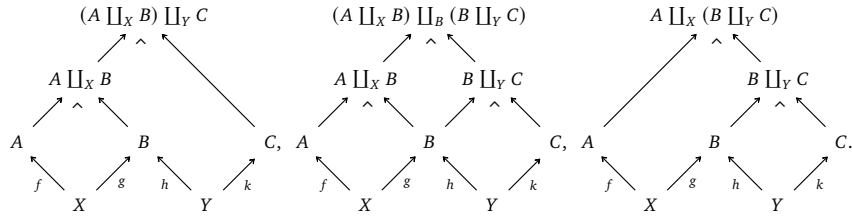
3. *Associativity.* Given a diagram



in Sets , we have isomorphisms of sets

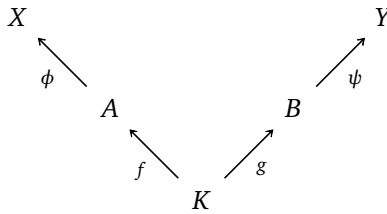
$$(A \amalg_X B) \amalg_Y C \cong (A \amalg_X B) \amalg_B (B \amalg_Y C) \cong A \amalg_X (B \amalg_Y C)$$

where these pullbacks are built as in the diagrams



01F4

4. *Interaction With Composition.* Given a diagram



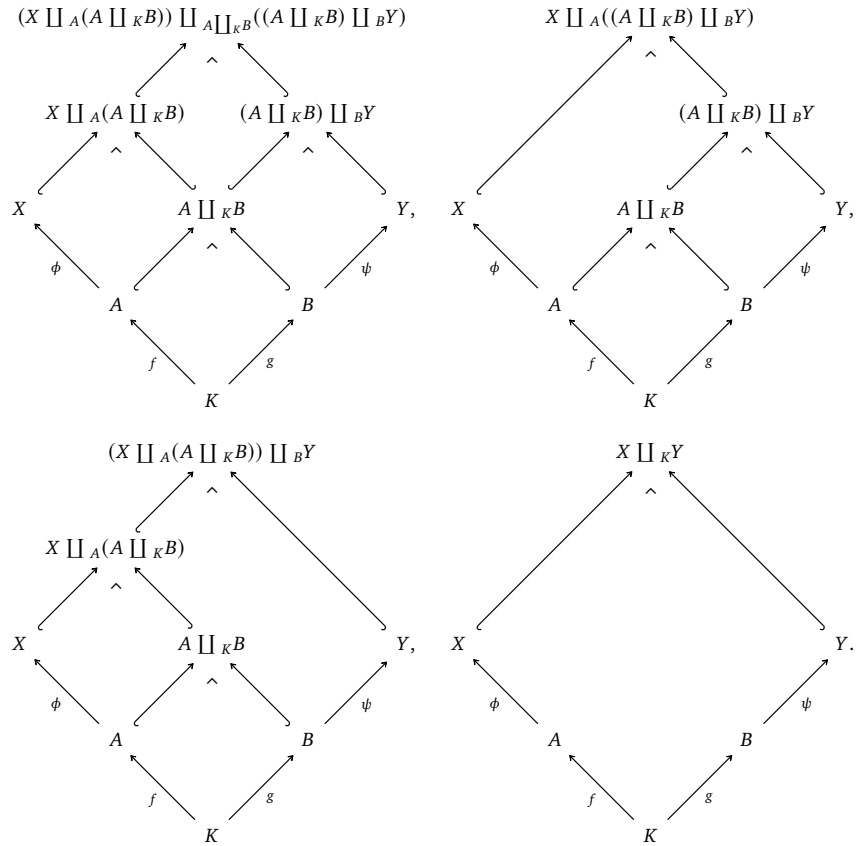
in Sets , we have isomorphisms of sets

$$\begin{aligned} X \amalg_K^{\phi \circ f, \psi \circ g} Y &\cong (X \amalg_A^{\phi, j_1} (A \amalg_K^{f, g} B)) \amalg_{A \amalg_K^{f, g} B}^{i_2, i_1} ((A \amalg_K^{f, g} B) \amalg_B^{j_2, \psi} Y) \\ &\cong X \amalg_A^{\phi, i} ((A \amalg_K^{f, g} B) \amalg_B^{j_2, \psi} Y) \\ &\cong (X \amalg_A^{\phi, i_1} (A \amalg_K^{f, g} B)) \amalg_B^{j, \psi} Y \end{aligned}$$

where

$$\begin{aligned}
 j_1 &= \text{inj}_1^{A \times_K^{f,g} B}, & j_2 &= \text{inj}_2^{A \times_K^{f,g} B}, \\
 i_1 &= \text{inj}_1^{(A \times_K^{f,g} B) \times_Y^{q_2, \psi}}, & i_2 &= \text{inj}_2^{X \times_{A \times_K^{f,g} B}^{q_1, \phi} (A \times_K^{f,g} B)}, \\
 i &= j_1 \circ \text{inj}_1^{(A \times_K^{f,g} B) \times_B^{q_2, \psi} Y}, & j &= j_2 \circ \text{inj}_2^{X \times_A^{q_1, \phi} (A \times_K^{f,g} B)},
 \end{aligned}$$

and where these pullbacks are built as in the diagrams



002H

5. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \uparrow f & \lrcorner & \uparrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 \lambda_A^{\text{Sets}_X/} : X \amalg_X A \xrightarrow{\sim} A, \\
 \rho_A^{\text{Sets}_X/} : A \amalg_X X \xrightarrow{\sim} A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xleftarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xleftarrow{f} & X,
 \end{array}$$

natural in $(A, f) \in \text{Obj}(\text{Sets}_{X/})$.

002J

6. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc} A \amalg_X B & \longleftarrow & B \\ \uparrow \ulcorner & & \uparrow g \\ A & \xleftarrow{f} & X, \end{array} \quad \sigma_A^{\text{Sets}_{X/}} : A \amalg_X B \xrightarrow{\sim} B \amalg_X A \quad \begin{array}{ccc} B \amalg_X A & \longleftarrow & A \\ \uparrow \ulcorner & & \uparrow f \\ B & \xleftarrow{g} & X. \end{array}$$

natural in $(A, f), (B, g) \in \text{Obj}(\text{Sets}_{X/})$.

002K

7. *Interaction With Coproducts.* We have

$$A \amalg_{\emptyset} B \cong A \amalg B, \quad \begin{array}{ccc} A \amalg B & \longleftarrow & B \\ \uparrow \ulcorner & & \uparrow \iota_B \\ A & \xleftarrow{\iota_A} & \emptyset. \end{array}$$

002L

8. *Symmetric Monoidality.* The triple $(\text{Sets}_{X/}, \amalg_X, X)$ is a symmetric monoidal category.

PROOF 4.2.4.1.9 ► PROOF OF PROPOSITION 4.2.4.1.8

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: : Adjointness

This follows from the universal property of the coproduct (pushouts are coproducts in $\text{Sets}_{X/}$).

Item 3: Associativity

Omitted.

Item 4: Interaction With Composition

Omitted.

Item 5: Unitality

Omitted.

Item 6: Commutativity

Omitted.

Item 7: Interaction With Coproducts

Omitted.

Item 8: Symmetric Monoidality

Omitted.



002M 4.2.5 Coequalisers

Let A and B be sets and let $f, g: A \rightrightarrows B$ be functions.

002N DEFINITION 4.2.5.1.1 ► COEQUALISERS OF SETS

The **coequaliser of f and g** is the coequaliser of f and g in Sets as in Limits and Colimits, ??.

01F5 CONSTRUCTION 4.2.5.1.2 ► CONSTRUCTION OF COEQUALISERS OF SETS

Concretely, the coequaliser of f and g is the pair $(\text{CoEq}(f, g), \text{coeq}(f, g))$ consisting of:

01F6 1. *The Colimit.* The set $\text{CoEq}(f, g)$ defined by

$$\text{CoEq}(f, g) \stackrel{\text{def}}{=} B/\sim,$$

where \sim is the equivalence relation on B generated by $f(a) \sim g(a)$.

01F7 2. *The Cocone.* The map

$$\text{coeq}(f, g): B \twoheadrightarrow \text{CoEq}(f, g)$$

given by the quotient map $\pi: B \rightarrow B/\sim$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

PROOF 4.2.5.1.3 ► PROOF OF CONSTRUCTION 4.2.5.1.2

We claim that $\text{CoEq}(f, g)$ is the categorical coequaliser of f and g in **Sets**. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have


$$\begin{aligned} [\text{coeq}(f, g) \circ f](a) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(a)) \\ &\stackrel{\text{def}}{=} [f(a)] \\ &= [g(a)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(a)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](a) \end{aligned}$$

for each $a \in A$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow[f]{g} B & \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & \searrow c \\ & & C \end{array}$$

in **Sets**. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **Conditions on Relations**, **Items 4** and **5** of **Proposition 10.6.2.1.3** that there exists a unique map $\text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow[f]{g} B & \xrightarrow{\text{coeq}(f, g)} \text{CoEq}(f, g) \\ & & \searrow c \quad \downarrow \exists! \\ & & C \end{array}$$

commute. 

002P

REMARK 4.2.5.1.4 ► UNWINDING DEFINITION 4.2.5.1.1

In detail, by **Conditions on Relations**, **Construction 10.5.2.1.2**, the relation \sim of **Definition 4.2.5.1.1** is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

01ZQ

1. We have $a = b$;

01ZR

2. There exist $x_1, \dots, x_n \in B$ such that $a \sim' x_1 \sim' \dots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

01ZS

(a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.

01ZT

(b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

In other words, there exist $x_1, \dots, x_n \in B$ satisfying the following conditions:

01ZU

(a) There exists $z_0 \in A$ satisfying one of the following conditions:

01ZV

i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.

01ZW

ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.

01ZX

(b) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:

01ZY

i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.

01ZZ

ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.

0200

(c) There exists $z_n \in A$ satisfying one of the following conditions:

0201

i. We have $x_n = f(z_n)$ and $b = g(z_n)$.

0202

ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

002Q

EXAMPLE 4.2.5.1.5 ► EXAMPLES OF COEQUALISERS OF SETS

Here are some examples of coequalisers of sets.

002R

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X . We have a bijection of sets

$$X/\sim_R \cong \text{CoEq}(R \hookrightarrow X \times X \begin{matrix} \xrightarrow{\text{pr}_1} \\ \xrightarrow{\text{pr}_2} \end{matrix} X).$$

PROOF 4.2.5.1.6 ► PROOF OF EXAMPLE 4.2.5.1.5

Item 1: Quotients by Equivalence Relations

See [Pro25z].



002S

PROPOSITION 4.2.5.1.7 ► PROPERTIES OF COEQUALISERS OF SETS

Let A , B , and C be sets.

002T

1. *Associativity.* We have isomorphisms of sets¹

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g-} \\ \xrightarrow{h} \end{array} B$$

in Sets.

002U

2. *Unitality.* We have an isomorphism of sets

$$\text{CoEq}(f, f) \cong B.$$

002V

3. *Commutativity.* We have an isomorphism of sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

002W

4. *Interaction With Composition.* Let

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{k} \end{array} C$$

be functions. We have a surjection

$$\text{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$$

exhibiting $\text{CoEq}(\text{coeq}(h, k) \circ h \circ f, \text{coeq}(h, k) \circ k \circ g)$ as a quotient of $\text{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

¹That is, the following three ways of forming “the” coequaliser of (f, g, h) agree:

0203

(a) Take the coequaliser of (f, g, h) , i.e. the colimit of the diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{-g-} \\ \xrightarrow{h} \end{array} B$$

in Sets.

0204

(b) First take the coequaliser of f and g , forming a diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{\text{coeq}(f,g)} \text{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{h} \end{array} B \xrightarrow{\text{coeq}(f,g)} \text{CoEq}(f,g),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(f,g) \circ f, \text{coeq}(f,g) \circ h) = \text{CoEq}(\text{coeq}(f,g) \circ g, \text{coeq}(f,g) \circ h)$$

of $\text{CoEq}(f,g)$

0205

(c) First take the coequaliser of g and h , forming a diagram

$$A \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} B \xrightarrow{\text{coeq}(g,h)} \text{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \xrightarrow{\text{coeq}(g,h)} \text{CoEq}(g,h),$$

obtaining a quotient

$$\text{CoEq}(\text{coeq}(g,h) \circ f, \text{coeq}(g,h) \circ g) = \text{CoEq}(\text{coeq}(g,h) \circ f, \text{coeq}(g,h) \circ h)$$

of $\text{CoEq}(g,h)$.**PROOF 4.2.5.1.8 ► PROOF OF PROPOSITION 4.2.5.1.7**

Item 1: Associativity

Omitted.

Item 2: Unitality

Omitted.

Item 3: Commutativity

Omitted.

Item 4: Interaction With Composition

Omitted.

**01F8 4.2.6 Direct Colimits**Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} : (I, \preceq) \rightarrow \Pi$ be a direct system of sets.

01F9

DEFINITION 4.2.6.1.1 ► DIRECT COLIMITS OF SETS

The **direct colimit** of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ in Sets as in Limits and Colimits, ??.

01FA

CONSTRUCTION 4.2.6.1.2 ► CONSTRUCTION OF DIRECT COLIMITS OF SETS

Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$ is the pair $\left(\underset{\alpha \in I}{\operatorname{colim}}(X_\alpha), \{\operatorname{inj}_\alpha\}_{\alpha \in I} \right)$ consisting of:

01FB

1. *The Colimit.* The set $\underset{\alpha \in I}{\operatorname{colim}}(X_\alpha)$ defined by

$$\underset{\alpha \in I}{\operatorname{colim}}(X_\alpha) \stackrel{\text{def}}{=} \left(\coprod_{\alpha \in I} X_\alpha \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{\alpha \in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

01FC

- (a) We have $\alpha \preceq \gamma$.

01FD

- (b) We have $\beta \preceq \gamma$.

01FE

- (c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

01FF

2. *The Cocone.* The collection


$$\left\{ \operatorname{inj}_\gamma : X_\gamma \rightarrow \underset{\alpha \in I}{\operatorname{colim}}(X_\alpha) \right\}_{\gamma \in I}$$

of maps of sets defined by

$$\operatorname{inj}_\gamma(x) \stackrel{\text{def}}{=} [(\gamma, x)]$$

for each $\gamma \in I$ and each $x \in X_\gamma$.

PROOF 4.2.6.1.3 ► PROOF OF CONSTRUCTION 4.2.6.1.2

We will prove **Construction 4.2.6.1.2** below in a bit, but first we need a lemma (which is interesting in its own right). 

01FG

LEMMA 4.2.6.1.4 ► IDENTIFICATION OF x WITH $f_{\alpha\beta}(x)$ IN DIRECT COLIMITS

For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have


$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in $\operatorname{colim}_{\alpha \in I}(X_\alpha)$.

PROOF 4.2.6.1.5 ► PROOF OF LEMMA 4.2.6.1.4

Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \stackrel{\text{def}}{=} \operatorname{id}_{X_\beta}$, and we have

$$\begin{aligned} f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} \operatorname{id}_{X_\beta}(f_{\alpha\beta}(x)), \\ &= f_{\alpha\beta}(x). \end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, **Items 1a to 1c of Construction 4.2.6.1.2** are met. Thus we have $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$. 

We can now prove **Construction 4.2.6.1.2**:

PROOF 4.2.6.1.6 ► PROOF OF CONSTRUCTION 4.2.6.1.2

We claim that $\operatorname{colim}_{\alpha \in I}(X_\alpha)$ is the colimit of the direct system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in I}$.

Commutativity of the Colimit Diagram

First, we need to check that the colimit diagram defined by $\operatorname{colim}_{\alpha \in I}(X_\alpha)$

commutes, i.e. that we have

$$\text{inj}_\alpha = \text{inj}_\beta \circ f_{\alpha\beta}, \quad \begin{array}{ccc} & \text{colim}(X_\alpha) & \\ \text{inj}_\alpha \nearrow & \xrightarrow[\alpha \in I]{} & \nwarrow \text{inj}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$\begin{aligned} [\text{inj}_\beta \circ f_{\alpha\beta}](x) &\stackrel{\text{def}}{=} \text{inj}_\beta(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\ &= [(\alpha, x)] \\ &\stackrel{\text{def}}{=} \text{inj}_\alpha(x), \end{aligned}$$

where we have used [Lemma 4.2.6.1.4](#) for the third equality.

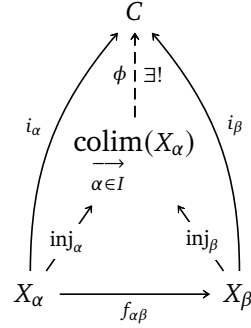
Proof of the Universal Property of the Colimit

Next, we prove that $\text{colim}_{\alpha \in I} (X_\alpha)$ as constructed in [Construction 4.2.6.1.2](#) satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{array}{ccc} & C & \\ i_\alpha \nearrow & & \nwarrow i_\beta \\ & \text{colim}(X_\alpha) & \\ \text{inj}_\alpha \nearrow & \xrightarrow[\alpha \in I]{} & \nwarrow \text{inj}_\beta \\ X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta \end{array}$$

in Sets. We claim that there exists a unique map $\phi: \text{colim}(X_\alpha) \xrightarrow[\alpha \in I]{\exists!} C$ making

the diagram



commute. To this end, first consider the diagram

$$\begin{array}{ccc} \coprod_{\alpha \in I} X_{\alpha} & \xrightarrow{\text{pr}} & \text{colim}(X_{\alpha}) \\ & \searrow & \downarrow \\ & & \coprod_{\alpha \in I} i_{\alpha} \\ & & C. \end{array}$$

Lemma. If $(\alpha, x) \sim (\beta, y)$, then we have

$$\left[\coprod_{\alpha \in I} i_{\alpha} \right] (x) = \left[\coprod_{\alpha \in I} i_{\alpha} \right] (y).$$

Proof. Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

- 0206 1. We have $\alpha \preceq \gamma$.
- 0207 2. We have $\beta \preceq \gamma$.
- 0208 3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$\begin{aligned} \left[\coprod_{\alpha \in I} i_{\alpha} \right] (x) &\stackrel{\text{def}}{=} i_{\alpha}(x) \\ &\stackrel{\text{def}}{=} [i_{\gamma} \circ f_{\alpha\gamma}](x) \end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} i_\gamma(f_{\alpha\gamma}(x)) \\
&= i_\gamma(f_{\beta\gamma}(x)) \\
&\stackrel{\text{def}}{=} [i_\gamma \circ f_{\beta\gamma}](x) \\
&= i_\beta(y) \\
&\stackrel{\text{def}}{=} \left[\coprod_{\alpha \in I} i_\alpha \right](y).
\end{aligned}$$

This finishes the proof of the lemma. Continuing, by **Conditions on Relations**, ?? of **Proposition 10.6.2.1.3**, there then exists a map $\phi: \text{colim}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc}
\coprod_{\alpha \in I} X_\alpha & \xrightarrow{\text{pr}} & \text{colim}(X_\alpha) \\
& \searrow & \downarrow \phi \\
& \coprod_{\alpha \in I} i_\alpha & C
\end{array}$$

commute. In particular, this implies that the diagram

$$\begin{array}{ccc}
X_\alpha & \xrightarrow{\text{inj}_\alpha} & \text{colim}(X_\alpha) \\
& \searrow i_\alpha & \downarrow \phi \\
& & C
\end{array}$$

also commutes, and thus so does the diagram

$$\begin{array}{ccc}
& & C \\
& \nearrow i_\alpha & \nwarrow i_\beta \\
& \text{colim}(X_\alpha) & \\
& \nwarrow \text{inj}_\alpha & \nearrow \text{inj}_\beta \\
X_\alpha & \xrightarrow{f_{\alpha\beta}} & X_\beta
\end{array}$$

This finishes the proof.¹



¹Incidentally, the conditions

$$\{i_\alpha = \phi \circ \text{inj}_\alpha\}_{\alpha \in I}$$

show that ϕ must be given by

$$\phi([\langle \alpha, x \rangle]) = (i_\alpha(x))_{\alpha \in I}$$

for each $[\langle \alpha, x \rangle] \in \varinjlim_{\alpha \in I} (X_\alpha)$, although we would need to show that this assignment is well-defined were we to prove [Construction 4.2.6.1.2](#) in this way. Instead, invoking [Conditions on Relations](#), ?? of [Proposition 10.6.2.1.3](#) gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

01FH

EXAMPLE 4.2.6.1.7 ► EXAMPLES OF DIRECT COLIMITS OF SETS

Here are some examples of direct colimits of sets.

01FJ

1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \stackrel{\text{def}}{=} \varinjlim_{n \in \mathbb{N}} (\mathbb{Z}/p^n);$$

see ??.

002X 4.3 Operations With Sets

002Y 4.3.1 The Empty Set

002Z

DEFINITION 4.3.1.1.1 ► THE EMPTY SET

The **empty set** is the set \emptyset defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where X is the set in the set existence axiom, ?? of ??.

0030 4.3.2 Singleton Sets

Let X be a set.

0031 DEFINITION 4.3.2.1.1 ► SINGLETON SETS

The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself of **Definition 4.3.3.1.1**.

0032 4.3.3 Pairings of Sets

Let X and Y be sets.

0033 DEFINITION 4.3.3.1.1 ► PAIRINGS OF SETS

The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, ?? of ??.

0034 4.3.4 Ordered Pairs

Let A and B be sets.

0035 DEFINITION 4.3.4.1.1 ► ORDERED PAIRS

The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

0036 PROPOSITION 4.3.4.1.2 ► PROPERTIES OF ORDERED PAIRS

Let A and B be sets.


0037 1. *Uniqueness.* Let A , B , C , and D be sets. The following conditions are equivalent:

0038 (a) We have $(A, B) = (C, D)$.

0039 (b) We have $A = C$ and $B = D$.

PROOF 4.3.4.1.3 ► PROOF OF PROPOSITION 4.3.4.1.2

Item 1: Uniqueness

See [Cie97, Theorem 1.2.3]. 

003A 4.3.5 Sets of Maps

Let A and B be sets.

003B DEFINITION 4.3.5.1.1 ► SETS OF MAPS

The **set of maps from A to B** ¹ is the set $\text{Sets}(A, B)$ ² whose elements are the functions from A to B .

¹*Further Terminology:* Also called the **Hom set from A to B** .²*Further Notation:* Also written $\text{Hom}_{\text{Sets}}(A, B)$.

003C PROPOSITION 4.3.5.1.2 ► PROPERTIES OF SETS OF MAPS

Let A and B be sets.003D 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \text{Hom}_{\text{Sets}}(X, Y)$ define functors

$$\begin{aligned} \text{Sets}(X, -) &: \text{Sets} \rightarrow \text{Sets}, \\ \text{Sets}(-, Y) &: \text{Sets}^{\text{op}} \rightarrow \text{Sets}, \\ \text{Sets}(-_1, -_2) &: \text{Sets}^{\text{op}} \times \text{Sets} \rightarrow \text{Sets}. \end{aligned}$$

01FK 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (A \times - \dashv \text{Sets}(A, -)) &: \text{Sets} \begin{array}{c} \xrightarrow{A \times -} \\ \perp \\ \xleftarrow{\text{Sets}(A, -)} \end{array} \text{Sets}, \\ (- \times B \dashv \text{Sets}(B, -)) &: \text{Sets} \begin{array}{c} \xrightarrow{- \times B} \\ \perp \\ \xleftarrow{\text{Sets}(B, -)} \end{array} \text{Sets}, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Sets}(A \times B, C) &\cong \text{Sets}(A, \text{Sets}(B, C)), \\ \text{Sets}(A \times B, C) &\cong \text{Sets}(B, \text{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \text{Obj}(\text{Sets})$.

01FL

3. *Maps From the Punctual Set.* We have a bijection

$$\text{Sets}(\text{pt}, A) \cong A,$$

natural in $A \in \text{Obj}(\text{Sets})$.

01FM

4. *Maps to the Punctual Set.* We have a bijection

$$\text{Sets}(A, \text{pt}) \cong \text{pt},$$

natural in $A \in \text{Obj}(\text{Sets})$.

PROOF 4.3.5.1.3 ► PROOF OF PROPOSITION 4.3.5.1.2

Item 1: Functoriality

This follows from [Categories](#), [Items 2](#) and [5](#) of [Proposition 11.1.4.1.2](#).

Item 2: Adjointness

This is a repetition of [Item 2](#) of [Proposition 4.1.3.1.4](#) and is proved there.

Item 3: Maps From the Punctual Set

The bijection

$$\Phi_A : \text{Sets}(\text{pt}, A) \xrightarrow{\sim} A$$

is given by

$$\Phi_A(f) \stackrel{\text{def}}{=} f(\star)$$

for each $f \in \text{Sets}(\text{pt}, A)$, admitting an inverse

$$\Phi_A^{-1} : A \xrightarrow{\sim} \text{Sets}(\text{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \stackrel{\text{def}}{=} [\star \mapsto a]$$

for each $a \in A$. Indeed, we have

$$\begin{aligned} [\Phi_A^{-1} \circ \Phi_A](f) &\stackrel{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\ &\stackrel{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\ &\stackrel{\text{def}}{=} [\star \mapsto f(\star)] \\ &\stackrel{\text{def}}{=} f \end{aligned}$$

$$\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(\text{pt}, A)}](f)$$

for each $f \in \text{Sets}(\text{pt}, A)$ and

$$\begin{aligned} [\Phi_A \circ \Phi_A^{-1}](a) &\stackrel{\text{def}}{=} \Phi_A(\Phi_A^{-1}(a)) \\ &\stackrel{\text{def}}{=} \Phi_A(\llbracket \star \mapsto a \rrbracket) \\ &\stackrel{\text{def}}{=} \text{ev}_\star(\llbracket \star \mapsto a \rrbracket) \\ &\stackrel{\text{def}}{=} a \\ &\stackrel{\text{def}}{=} [\text{id}_A](a) \end{aligned}$$

for each $a \in A$, and thus we have

$$\begin{aligned} \Phi_A^{-1} \circ \Phi_A &= \text{id}_{\text{Sets}(\text{pt}, A)} \\ \Phi_A \circ \Phi_A^{-1} &= \text{id}_A. \end{aligned}$$

To prove naturality, we need to show that the diagram

$$\begin{array}{ccc} \text{Sets}(\text{pt}, A) & \xrightarrow{f_!} & \text{Sets}(\text{pt}, B) \\ \Phi_A \downarrow \wr & & \downarrow \wr \Phi_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes. Indeed, we have

$$\begin{aligned} [f \circ \Phi_A](\phi) &\stackrel{\text{def}}{=} f(\Phi_A(\phi)) \\ &\stackrel{\text{def}}{=} f(\phi(\star)) \\ &\stackrel{\text{def}}{=} [f \circ \phi](\star) \\ &\stackrel{\text{def}}{=} \Phi_B(f \circ \phi) \\ &\stackrel{\text{def}}{=} \Phi_B(f_!(\phi)) \\ &\stackrel{\text{def}}{=} [\Phi_B \circ f_!](\phi) \end{aligned}$$

for each $\phi \in \text{Sets}(\text{pt}, A)$. This finishes the proof.

Item 4: Maps to the Punctual Set

This follows from the universal property of pt as the terminal set, [Definition 4.1.1.1.1](#). 

003E **4.3.6 Unions of Families of Subsets**

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

003F **DEFINITION 4.3.6.1.1 ► UNIONS OF FAMILIES OF SUBSETS**

The **union** of \mathcal{U} is the set $\bigcup_{U \in \mathcal{U}} U$ defined by

$$\bigcup_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}.$$

01FN **PROPOSITION 4.3.6.1.2 ► PROPERTIES OF UNIONS OF FAMILIES OF SUBSETS**

Let X be a set.

01FP 1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$ defines a functor

$$\bigcup: (\mathcal{P}(\mathcal{P}(X)), \subset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

$$(\star) \text{ If } \mathcal{U} \subset \mathcal{V}, \text{ then } \bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

01FQ 2. *Associativity.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \bigcup} & \mathcal{P}(\mathcal{P}(X)) \\ \bigcup \star \text{id}_{\mathcal{P}(X)} \downarrow & & \downarrow \bigcup \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

01FR

3. *Left Unitality*. The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) & & \\
 \chi_{\mathcal{P}(X)} \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \{U\}} V = U$$

for each $U \in \mathcal{P}(X)$.

01FS

4. *Right Unitality*. The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) & & \\
 \mathcal{P}(\chi_X) \downarrow & \searrow \text{id}_{\mathcal{P}(X)} & \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\bigcup_{\{u\} \in \chi_X(U)} \{u\} = U$$

for each $U \in \mathcal{P}(X)$.

01FT

5. *Interaction With Unions I*. The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcup} & \mathcal{P}(\mathcal{P}(X)) \\
 \bigcup \times \bigcup \downarrow & & \downarrow \bigcup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\bigcup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01FU

6. *Interaction With Unions II.* The diagrams

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)} & \mathcal{P}(\mathcal{P}(X)) \\
\downarrow \cup & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{U \cup -} & \mathcal{P}(X)
\end{array}
\quad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)} & \mathcal{P}(\mathcal{P}(X)) \\
\downarrow \cup & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{- \cup V} & \mathcal{P}(X)
\end{array}$$

commute, i.e. we have

$$\begin{aligned}
U \cup \left(\bigcup_{V \in \mathcal{V}} V \right) &= \bigcup_{V \in \mathcal{V}} (U \cup V), \\
\left(\bigcup_{U \in \mathcal{U}} U \right) \cup V &= \bigcup_{U \in \mathcal{U}} (U \cup V)
\end{aligned}$$

for each nonempty $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

01FV

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\
\downarrow \cup \times \cup & \wr & \downarrow \cup \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X),
\end{array}$$

with components

$$\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \subset \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01FW

8. *Interaction With Intersections II.* The diagrams

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\
\downarrow \cup & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X)
\end{array}
\quad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\
\downarrow \cup & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X)
\end{array}$$

commute, i.e. we have

$$U \cap \left(\bigcup_{V \in \mathcal{V}} V \right) = \bigcup_{V \in \mathcal{V}} (U \cap V),$$

$$\left(\bigcup_{U \in \mathcal{U}} U \right) \cap V = \bigcup_{U \in \mathcal{U}} (U \cap V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

01FX

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\setminus} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \mathcal{U} \times \mathcal{U} & \text{X} & \downarrow \mathcal{U} \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\setminus} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01FY

10. *Interaction With Complements I.* The diagram

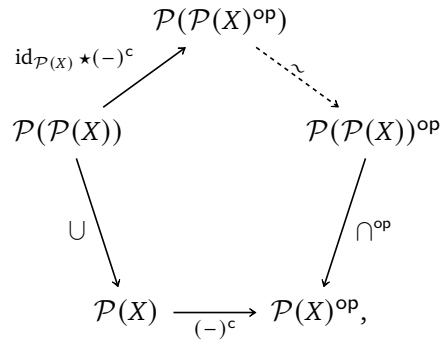
$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(\mathcal{P}(X)) \\ \downarrow \mathcal{U}^{\text{op}} & \text{X} & \downarrow \mathcal{U} \\ \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{U \in \mathcal{U}^c} U \neq \bigcup_{U \in \mathcal{U}} U^c$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01FZ

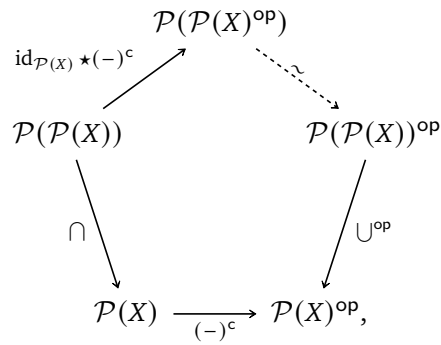
11. *Interaction With Complements II.* The diagram

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U \right)^c = \bigcap_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01G0

12. *Interaction With Complements III.* The diagram

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U \right)^c = \bigcup_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01G1

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup \times \cup & \text{X} & \downarrow \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U \right) \Delta \left(\bigcup_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01G2

14. *Interaction With Internal Homs I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup^{\text{op}} \times \cup^{\text{op}} & \text{X} & \downarrow \cup \\
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01G3

15. *Interaction With Internal Homs II.* The diagram

$$\begin{array}{ccccc}
 & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\
 & \nearrow \sim & & \searrow \cup^{\text{op}} & \\
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & & & \mathcal{P}(X)^{\text{op}} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & & & \downarrow [-, V]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01G4

16. *Interaction With Internal Homs III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \\ \text{id}_{\mathcal{P}(X)} \star [U, -]_X \downarrow & & \downarrow [U, -]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01G5

17. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01G6

18. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

01G7

19. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01G8

20. *Interaction With Intersections of Families I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(x)) \\ \downarrow \cup \star \text{id}_{\mathcal{P}(X)} & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{\cap} & X \end{array}$$

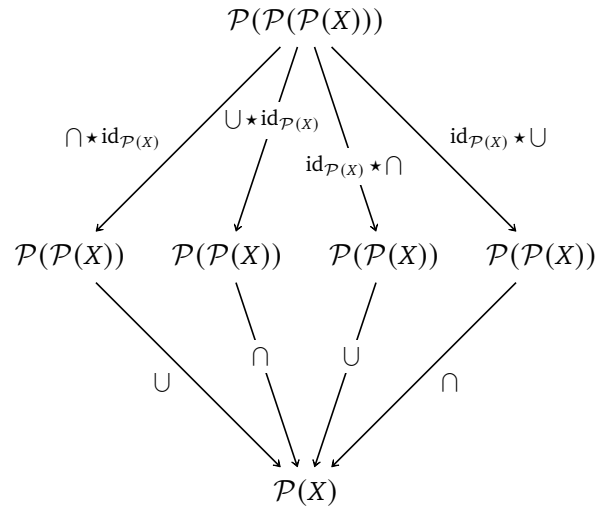
commutes, i.e. we have

$$\bigcap_{A \in \mathcal{A}} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

01G9

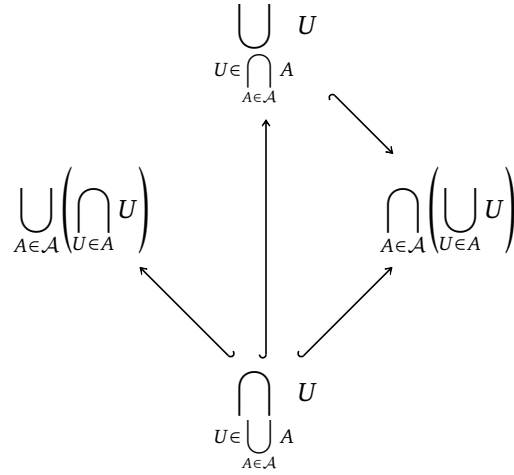
21. *Interaction With Intersections of Families II.* Let X be a set and consider the compositions



given by

$$\begin{aligned} \mathcal{A} &\mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, & \mathcal{A} &\mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U, \\ \mathcal{A} &\mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), & \mathcal{A} &\mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \end{aligned}$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:



All other possible inclusions fail to hold in general.

PROOF 4.3.6.1.3 ► PROOF OF PROPOSITION 4.3.6.1.2

Item 1: Functoriality

Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition (\star) . So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

Item 2: Associativity

We have

$$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\}$$

$$\begin{aligned}
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right).
\end{aligned}$$

This finishes the proof.

Item 3: Left Unitality

We have

$$\begin{aligned}
\bigcup_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in V \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

Item 4: Right Unitality

We have

$$\begin{aligned}
\bigcup_{\{u\} \in \chi_X(U)} \{u\} &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

Item 5: Interaction With Unions I

We have

$$\begin{aligned}
 \bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \\ \text{such that we have } x \in W \end{array} \right\} \\
 &\quad \cup \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \left(\bigcup_{W \in \mathcal{U}} W \right) \cup \left(\bigcup_{W \in \mathcal{V}} W \right) \\
 &= \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right).
 \end{aligned}$$

This finishes the proof.

Item 6: Interaction With Unions II

Assume \mathcal{V} is nonempty. We have

$$\begin{aligned}
 U \cup \bigcup_{V \in \mathcal{V}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid x \in U \text{ or } x \in \bigcup_{V \in \mathcal{V}} V \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} x \in U \text{ or there exists some} \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ or } x \in V \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cup V \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cup V.
 \end{aligned}$$

This concludes the proof of the first statement. For the second statement, use **Item 4** of **Proposition 4.3.8.1.2** to rewrite

$$\begin{aligned} \left(\bigcup_{U \in \mathcal{U}} U \right) \cup V &= V \cup \left(\bigcup_{U \in \mathcal{U}} U \right), \\ \bigcup_{U \in \mathcal{U}} (U \cup V) &= \bigcup_{U \in \mathcal{U}} (V \cup U). \end{aligned}$$

But these two sets are equal by the first statement.

Item 7: Interaction With Intersections I

We have

$$\begin{aligned} \bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cap \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\ &\subset \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \text{ and some } V \in \mathcal{V} \\ \text{such that we have } x \in U \text{ and } x \in V \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\} \\ &\quad \cup \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that we have } x \in V \end{array} \right\} \\ &\stackrel{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right). \end{aligned}$$

This finishes the proof.

Item 8: Interaction With Intersections II

We have

$$\begin{aligned} U \cap \bigcup_{V \in \mathcal{V}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} x \in U \text{ and } x \in \bigcup_{V \in \mathcal{V}} V \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} x \in U \text{ and there exists some } \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ and } x \in V \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ x \in X \mid \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cap V \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cap V.
\end{aligned}$$

This concludes the proof of the first statement. For the second statement, use [Item 5](#) of [Proposition 4.3.9.1.2](#) to rewrite

$$\begin{aligned}
\left(\bigcup_{U \in \mathcal{U}} U \right) \cap V &= V \cap \left(\bigcup_{U \in \mathcal{U}} U \right), \\
\bigcup_{U \in \mathcal{U}} (U \cap V) &= \bigcup_{U \in \mathcal{U}} (V \cap U).
\end{aligned}$$

But these two sets are equal by the first statement.

Item 9: Interaction With Differences

Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned}
\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W &= \bigcup_{W \in \{\{0, 1\}\}} W \\
&= \{0, 1\},
\end{aligned}$$

whereas

$$\begin{aligned}
\left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right) &= \{0, 1\} \setminus \{0\} \\
&= \{1\}.
\end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \{1\} = \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 10: Interaction With Complements I

Let $X = \{0, 1\}$ and let $\mathcal{U} = \{0\}$. We have

$$\bigcup_{U \in \mathcal{U}^c} U = \bigcup_{U \in \{\emptyset, \{1\}, \{0, 1\}\}} U$$

$$= \{0, 1\},$$

whereas

$$\begin{aligned} \bigcup_{U \in \mathcal{U}} U^c &= \{0\}^c \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcup_{U \in \mathcal{U}^c} U = \{0, 1\} \neq \{1\} = \bigcup_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

Item 11: Interaction With Complements II

We have

$$\begin{aligned} \left(\bigcup_{U \in \mathcal{U}} U \right)^c &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{there exists no } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{for all } U \in \mathcal{U} \\ \text{we have } x \notin U \end{array} \right\} \\ &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for all } U \in \mathcal{U} \\ \text{we have } x \in U^c \end{array} \right\} \\ &\stackrel{\text{def}}{=} \bigcap_{U \in \mathcal{U}} U^c. \end{aligned}$$

Item 12: Interaction With Complements III

By **Item 11 Item 3** of **Proposition 4.3.11.1.2**, we have

$$\begin{aligned} \left(\bigcap_{U \in \mathcal{U}} U \right)^c &= \left(\bigcap_{U \in \mathcal{U}} (U^c)^c \right)^c \\ &= \left(\left(\bigcup_{U \in \mathcal{U}} U^c \right)^c \right)^c \\ &= \bigcup_{U \in \mathcal{U}} U^c. \end{aligned}$$

Item 13: Interaction With Symmetric Differences

Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned} \bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W &= \bigcup_{W \in \{\{0\}\}} W \\ &= \{0\}, \end{aligned}$$

whereas

$$\begin{aligned} \left(\bigcup_{U \in \mathcal{U}} U \right) \Delta \left(\bigcup_{V \in \mathcal{V}} V \right) &= \{0, 1\} \Delta \{0, 1\} \\ &= \emptyset, \end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \Delta \mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U \in \mathcal{U}} U \right) \Delta \left(\bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 14: Interaction With Internal Homs I

This is a repetition of **Item 7** of **Proposition 4.4.7.1.4** and is proved there.

Item 15: Interaction With Internal Homs II

This is a repetition of **Item 8** of **Proposition 4.4.7.1.4** and is proved there.

Item 16: Interaction With Internal Homs III

This is a repetition of **Item 9** of **Proposition 4.4.7.1.4** and is proved there.

Item 17: Interaction With Direct Images

This is a repetition of **Item 3** of **Proposition 4.6.1.1.5** and is proved there.

Item 18: Interaction With Inverse Images

This is a repetition of **Item 3** of **Proposition 4.6.2.1.3** and is proved there.

Item 19: Interaction With Codirect Images

This is a repetition of **Item 3** of **Proposition 4.6.3.1.7** and is proved there.

Item 20: Interaction With Intersections of Families I

We have

$$\begin{aligned}
 \bigcap_{A \in \mathcal{A}} U &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
 &= \left\{ x \in X \mid \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\} \\
 &\stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right).
 \end{aligned}$$

This finishes the proof.

Item 21: Interaction With Intersections of Families II

Omitted.



003V 4.3.7 Intersections of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

003W

DEFINITION 4.3.7.1.1 ► INTERSECTIONS OF FAMILIES OF SUBSETS

The **intersection** of \mathcal{U} is the set $\bigcap_{U \in \mathcal{U}} U$ defined by

$$\bigcap_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\}.$$

01GA

PROPOSITION 4.3.7.1.2 ► PROPERTIES OF INTERSECTIONS OF FAMILIES OF SUBSETS

Let X be a set.

01GB

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcap_{U \in \mathcal{U}} U$ defines a functor

$$\bigcap: (\mathcal{P}(\mathcal{P}(X)), \supset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

$$(\star) \text{ If } \mathcal{U} \subset \mathcal{V}, \text{ then } \bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

01GC

2. *Oplax Associativity.* We have a natural transformation

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap \star \text{id}_{\mathcal{P}(X)} \downarrow & \wr & \downarrow \cap \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

with components

$$\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \subset \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

01GD

3. *Left Unitality.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) & & \\
 \downarrow \chi_{\mathcal{P}(X)} & \searrow \text{id}_{\mathcal{P}(X)} & \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \{U\}} V = U.$$

for each $U \in \mathcal{P}(X)$.

01GE

4. *Oplax Right Unitality.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) & & \\
 \downarrow \mathcal{P}(\chi_X) & \searrow \text{id}_{\mathcal{P}(X)} & \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

✗

does not commute in general, i.e. we may have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \neq U$$

in general, where $U \in \mathcal{P}(X)$. However, when U is nonempty, we have

$$\bigcap_{\{x\} \in \chi_X(U)} \{x\} \subset U.$$

01GF

5. *Interaction With Unions I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GG

6. *Interaction With Unions II.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{U \cup -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{- \cup V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) &= \bigcap_{V \in \mathcal{V}} (U \cup V), \\ \left(\bigcap_{U \in \mathcal{U}} U \right) \cup V &= \bigcap_{U \in \mathcal{U}} (U \cup V) \end{aligned}$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

01GH

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \wr & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X), \end{array}$$

with components

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) \subset \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GJ

8. *Interaction With Intersections II.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) &= \bigcap_{V \in \mathcal{V}} (U \cup V), \\ \left(\bigcap_{U \in \mathcal{U}} U \right) \cup V &= \bigcap_{U \in \mathcal{U}} (U \cup V) \end{aligned}$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

01GK

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\setminus} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \times \cap \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\setminus} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GL

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(\mathcal{P}(X)) \\
 \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\
 \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U}^c} W \neq \bigcap_{U \in \mathcal{U}} U^c$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GM

11. *Interaction With Complements II.* The diagram

$$\begin{array}{ccccc}
 & & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \text{---} \sim \text{---} \searrow & & \\
 \mathcal{P}(\mathcal{P}(X)) & & & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cap & & & & \downarrow \cup^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}}, & &
 \end{array}$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U \right)^c = \bigcup_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GN

12. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\text{op}}) & \\
 \text{id}_{\mathcal{P}(X)} \star (-)^c \nearrow & & \searrow \sim \\
 \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} \\
 \downarrow \cup & & \downarrow \cap^{\text{op}} \\
 \mathcal{P}(X) & \xrightarrow{(-)^c} & \mathcal{P}(X)^{\text{op}},
 \end{array}$$

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U \right)^c = \bigcap_{U \in \mathcal{U}} U^c$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GP

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cap \times \cap & \text{X} & \downarrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \Delta \left(\bigcap_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GQ

14. *Interaction With Internal Homs I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cap^{\text{op}} \times \cap^{\text{op}} & \text{X} & \downarrow \cap \\
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01GR

15. *Interaction With Internal Homs II.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & \\ \swarrow \sim & & \searrow \cap^{\text{op}} \\ \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \mathcal{P}(X)^{\text{op}} \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & \downarrow [-, V]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01GS

16. *Interaction With Internal Homs III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01GT

17. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01GU

18. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{V})} U$$

for each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(Y))$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

01GV

19. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a map of sets. The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01GW

20. *Interaction With Unions of Families I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(x)) \\
 \downarrow \cup \star \text{id}_{\mathcal{P}(X)} & & \downarrow \cap \\
 \mathcal{P}(X) & \xrightarrow{\cap} & X
 \end{array}$$

commutes, i.e. we have

$$\bigcup_{A \in \mathcal{A}} \bigcap_{U \in A} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

01GX

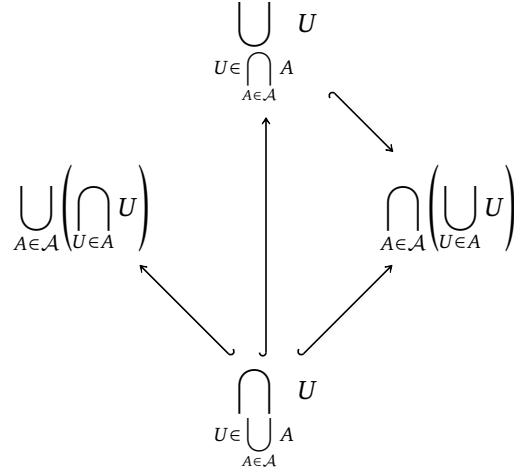
21. *Interaction With Unions of Families II.* Let X be a set and consider the compositions

$$\begin{array}{c}
 \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \\
 \swarrow \cap \star \text{id}_{\mathcal{P}(X)} \quad \searrow \cup \star \text{id}_{\mathcal{P}(X)} \quad \downarrow \text{id}_{\mathcal{P}(X)} \star \cap \quad \searrow \text{id}_{\mathcal{P}(X)} \star \cup \\
 \mathcal{P}(\mathcal{P}(X)) \quad \mathcal{P}(\mathcal{P}(X)) \quad \mathcal{P}(\mathcal{P}(X)) \quad \mathcal{P}(\mathcal{P}(X)) \\
 \swarrow \cup \quad \searrow \cap \quad \swarrow \cup \quad \searrow \cap \\
 \mathcal{P}(X)
 \end{array}$$

given by

$$\begin{aligned}
 \mathcal{A} &\mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, & \mathcal{A} &\mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U, \\
 \mathcal{A} &\mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), & \mathcal{A} &\mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)
 \end{aligned}$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:



All other possible inclusions fail to hold in general.

PROOF 4.3.7.1.3 ► PROOF OF PROPOSITION 4.3.6.1.2

Item 1: Functoriality

Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition (\star) . So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion.

Item 2: Oplax Associativity

We have

$$\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \bigcap_{U \in A} U \end{array} \right\}$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
&\subset \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.
\end{aligned}$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic ([Categories, Item 4 of Proposition 11.2.7.1.2](#)). This finishes the proof.

Item 3: Left Unitality

We have

$$\begin{aligned}
\bigcap_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in V \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}$$

This finishes the proof.

Item 4: Oplax Right Unitality

If $U = \emptyset$, then we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \bigcap_{\{u\} \in \emptyset} \{u\} \\
&= X,
\end{aligned}$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$. When U is nonempty, we have two cases:

1. If U is a singleton, say $U = \{u\}$, we have

$$\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \{u\} \\
&\stackrel{\text{def}}{=} U.
\end{aligned}$$

020B

020C

2. If U contains at least two elements, we have

$$\bigcap_{\{u\} \in \chi_X(U)} \{u\} = \emptyset \\ \subset U.$$

This finishes the proof.

Item 5: Interaction With Unions I

We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\ &= \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \text{ and each} \\ W \in \mathcal{V}, \text{ we have } x \in W \end{array} \right\} \\ &\stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U}, \\ \text{we have } x \in W \end{array} \right\} \\ &\quad \cap \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\ &\stackrel{\text{def}}{=} \left(\bigcap_{W \in \mathcal{U}} W \right) \cap \left(\bigcap_{W \in \mathcal{V}} W \right) \\ &= \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right). \end{aligned}$$

This finishes the proof.

Item 6: Interaction With Unions II

Omitted.

Item 7: Interaction With Intersections I

We have

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) \stackrel{\text{def}}{=} \left\{ x \in X \mid \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\}$$

$$\begin{aligned}
& \cup \left\{ x \in X \mid \begin{array}{l} \text{for each } V \in \mathcal{V}, \\ \text{we have } x \in V \end{array} \right\} \\
&= \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \cap \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
&\subset \left\{ x \in X \mid \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W.
\end{aligned}$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic ([Categories](#), [Item 4 of Proposition 11.2.7.1.2](#)). This finishes the proof.

Item 8: Interaction With Intersections II

Omitted.

Item 9: Interaction With Differences

Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0\}, \{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned}
\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W &= \bigcap_{W \in \{\{0, 1\}\}} W \\
&= \{0, 1\},
\end{aligned}$$

whereas

$$\begin{aligned}
\left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right) &= \{0\} \setminus \{0\} \\
&= \emptyset.
\end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \emptyset = \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 10: Interaction With Complements I

Let $X = \{0, 1\}$ and let $\mathcal{U} = \{\{0\}\}$. We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U}^c} W &= \bigcap_{W \in \{\emptyset, \{1\}, \{0, 1\}\}} W \\ &= \emptyset, \end{aligned}$$

whereas

$$\begin{aligned} \bigcap_{U \in \mathcal{U}} U^c &= \{0\}^c \\ &= \{1\}. \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U}^c} W = \emptyset \neq \{1\} = \bigcap_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

Item 11: Interaction With Complements II

This is a repetition of **Item 12** of **Proposition 4.3.6.1.2** and is proved there.

Item 12: Interaction With Complements III

This is a repetition of **Item 11** of **Proposition 4.3.6.1.2** and is proved there.

Item 13: Interaction With Symmetric Differences

Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned} \bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W &= \bigcap_{W \in \{\{0\}\}} W \\ &= \{0\}, \end{aligned}$$

whereas

$$\begin{aligned} \left(\bigcap_{U \in \mathcal{U}} U \right) \Delta \left(\bigcap_{V \in \mathcal{V}} V \right) &= \{0, 1\} \Delta \{0\} \\ &= \emptyset, \end{aligned}$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \Delta \mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcap_{U \in \mathcal{U}} U \right) \Delta \left(\bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 14: Interaction With Internal Homs I

This is a repetition of **Item 10** of **Proposition 4.4.7.1.4** and is proved there.

Item 15: Interaction With Internal Homs II

This is a repetition of **Item 11** of **Proposition 4.4.7.1.4** and is proved there.

Item 16: Interaction With Internal Homs III

This is a repetition of **Item 12** of **Proposition 4.4.7.1.4** and is proved there.

Item 17: Interaction With Direct Images

This is a repetition of **Item 4** of **Proposition 4.6.1.1.5** and is proved there.

Item 18: Interaction With Inverse Images

This is a repetition of **Item 4** of **Proposition 4.6.2.1.3** and is proved there.

Item 19: Interaction With Codirect Images

This is a repetition of **Item 4** of **Proposition 4.6.3.1.7** and is proved there.

Item 20: Interaction With Unions of Families I

This is a repetition of **Item 20** of **Proposition 4.3.6.1.2** and is proved there.

Item 21: Interaction With Unions of Families II

This is a repetition of **Item 21** of **Proposition 4.3.6.1.2** and is proved there.



003G 4.3.8 Binary Unions

Let X be a set and let $U, V \in \mathcal{P}(X)$.

003H DEFINITION 4.3.8.1.1 ► BINARY UNIONS

The **union** of U and V is the set $U \cup V$ defined by

$$\begin{aligned} U \cup V &\stackrel{\text{def}}{=} \bigcup_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

003J PROPOSITION 4.3.8.1.2 ► PROPERTIES OF BINARY UNIONS

Let X be a set.

003K

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \cup V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cup -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01GY

(a) If $U \subset A$, then $U \cup V \subset A \cup V$.

01GZ

(b) If $V \subset B$, then $U \cup V \subset U \cup B$.

01H0

(c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

003M

2. *Associativity.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\ \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \swarrow \sim & & \searrow \text{id}_{\mathcal{P}(X)} \times \cup \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cup \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \cup \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X), \end{array}$$

commutes, i.e. we have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

003N

3. *Unitality.* The diagrams

$$\begin{array}{ccc} \text{pt} \times \mathcal{P}(X) & \xrightarrow{[\emptyset] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \lambda_{\mathcal{P}(X)}^{\text{Sets}} \searrow \sim & & \downarrow \cup \\ & & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [\emptyset]} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \rho_{\mathcal{P}(X)}^{\text{Sets}} \searrow \sim & & \downarrow \cup \\ & & \mathcal{P}(X) \end{array}$$

commute, i.e. we have equalities of sets

$$\emptyset \cup U = U,$$

$$U \cup \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

003P

4. *Commutativity.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 & \searrow \cup & \downarrow \cup \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cup V = V \cup U$$

for each $U, V \in \mathcal{P}(X)$.

01H1

5. *Annihilation With X .* The diagrams

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \\
 \text{id}_{\text{pt}} \times \epsilon_{\mathcal{P}(X)}^{\text{Sets}} \nearrow & & \nwarrow \mu_{4|\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \text{pt} \times \mathcal{P}(X) & & \text{pt} \\
 [X] \times \text{id}_{\mathcal{P}(X)} \searrow & & \nearrow [X] \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$U \cup X = X,$$

$$X \cup V = X$$

for each $U, V \in \mathcal{P}(X)$.

003R

6. *Distributivity of Unions Over Intersections.* The diagrams

$$\begin{array}{ccc}
 & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \Delta_{\mathcal{P}(X)} \times (\text{id}_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)}) \nearrow & & \nwarrow \mu_{4|\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
 \text{id}_{\mathcal{P}(X)} \times \cap \searrow & & \nearrow \cup \times \cup \\
 \mathcal{P}(X) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \cup \searrow & & \nearrow \cap \\
 & \mathcal{P}(X) &
 \end{array}$$

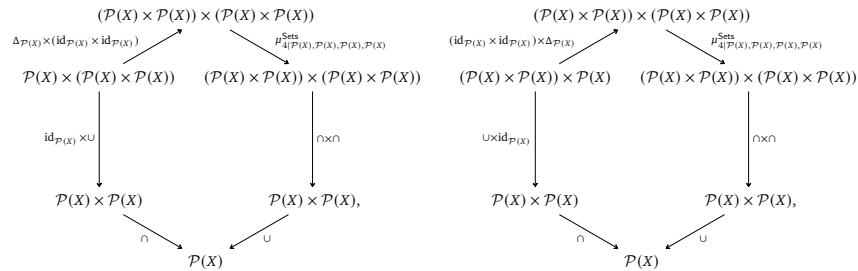
commute, i.e. we have equalities of sets

$$\begin{aligned} U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01H2

7. *Distributivity of Intersections Over Unions.* The diagrams



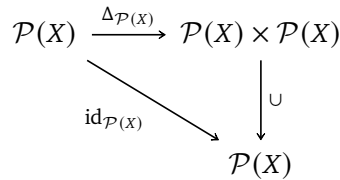
commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

003Q

8. *Idempotency.* The diagram



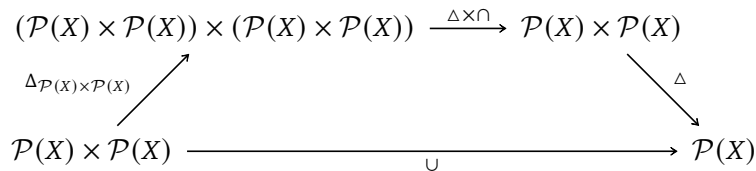
commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each $U \in \mathcal{P}(X)$.

003L

9. *Via Intersections and Symmetric Differences.* The diagram



commutes, i.e. we have an equality of sets

$$U \cup V = (U \Delta V) \Delta (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

003S

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

003T

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

01H3

12. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

01H4

13. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01H5

14. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cup \downarrow & \subset & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

003U

15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

PROOF 4.3.8.1.3 ► PROOF OF PROPOSITION 4.3.8.1.2

Item 1: Functoriality

See [Pro25an].

Item 2: Associativity

See [Pro25ba].

Item 3: Unitality

This follows from [Pro25bd] and Item 4.

Item 4: Commutativity

See [Pro25bb].

Item 5: Annihilation With X

We have

$$\begin{aligned} U \cup X &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\ &= \{x \in X \mid x \in X\}, \\ &= X \end{aligned}$$

and

$$\begin{aligned} X \cup V &\stackrel{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\ &= \{x \in X \mid x \in X\} \\ &= X. \end{aligned}$$

This finishes the proof.

Item 6: Distributivity of Unions Over Intersections

See [Pro25az].

Item 7: Distributivity of Intersections Over Unions

See [Pro25aj].

Item 8: Idempotency

See [Pro25am].

Item 9: Via Intersections and Symmetric Differences

See [Pro25ay].

Item 10: Interaction With Characteristic Functions I

See [Pro25h].

Item 11: Interaction With Characteristic Functions II

See [Pro25h].

Item 12: Interaction With Direct Images

See [Pro25p].


Item 13: Interaction With Inverse Images

See [Pro25y].

Item 14: Interaction With Codirect Images

This is a repetition of **Item 5** of **Proposition 4.6.3.1.7** and is proved there.

Item 15: Interaction With Powersets and Semirings

This follows from **Items 2 to 4** and **8** of this proposition and **Items 3 to 6** and **8** of **Proposition 4.3.9.1.2**. 

003X 4.3.9 Binary Intersections

Let X be a set and let $U, V \in \mathcal{P}(X)$.

003Y

DEFINITION 4.3.9.1.1 ► BINARY INTERSECTIONS

The **intersection of U and V** is the set $U \cap V$ defined by

$$\begin{aligned} U \cap V &\stackrel{\text{def}}{=} \bigcap_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

003Z

PROPOSITION 4.3.9.1.2 ► PROPERTIES OF BINARY INTERSECTIONS

Let X be a set.

0040

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \cap V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \cap -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01H6

(a) If $U \subset A$, then $U \cap V \subset A \cap V$.

01H7

(b) If $V \subset B$, then $U \cap V \subset U \cap B$.

01H8

(c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

0041

2. *Adjointness.* We have adjunctions

$$\begin{aligned} (U \cap - \dashv [U, -]_X): \mathcal{P}(X) &\overset{U \cap -}{\underset{[U, -]_X}{\rightleftarrows}} \mathcal{P}(X), \\ (- \cap V \dashv [V, -]_X): \mathcal{P}(X) &\overset{- \cap V}{\underset{[V, -]_X}{\rightleftarrows}} \mathcal{P}(X), \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(V, [U, W]_X), \end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is the bifunctor of [Section 4.4.7](#). In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

01H9

(a) The following conditions are equivalent:

01HA

i. We have $U \cap V \subset W$.

01HB

ii. We have $U \subset [V, W]_X$.

01HC

(b) The following conditions are equivalent:

01HD

i. We have $U \cap V \subset W$.

01HE

ii. We have $V \subset [U, W]_X$.

0042

3. *Associativity.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
 \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \swarrow \sim & & \searrow \text{id}_{\mathcal{P}(X)} \times \cap \\
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \cap \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

0043

4. *Unitality.* The diagrams

$$\begin{array}{ccc}
 \text{pt} \times \mathcal{P}(X) & \xrightarrow{[X] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \lambda_{\mathcal{P}(X)}^{\text{Sets}} \searrow \sim & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [X]} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \rho_{\mathcal{P}(X)}^{\text{Sets}} \searrow \sim & & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$X \cap U = U,$$

$$U \cap X = U$$

for each $U \in \mathcal{P}(X)$.

0044

5. *Commutativity.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 & \searrow \cap & \downarrow \cap \\
 & & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each $U, V \in \mathcal{P}(X)$.

0047

6. *Annihilation With the Empty Set.* The diagrams

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \\
 \text{id}_{\text{pt}} \times \epsilon_{\mathcal{P}(X)}^{\text{Sets}} \nearrow & & \nwarrow \mu_{4|\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \text{pt} \times \mathcal{P}(X) & & \text{pt} \\
 [\emptyset] \times \text{id}_{\mathcal{P}(X)} \searrow & & \nearrow [\emptyset] \\
 \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
 \emptyset \cap X &= \emptyset, \\
 X \cap \emptyset &= \emptyset
 \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

01HF

7. *Distributivity of Unions Over Intersections.* The diagrams

$$\begin{array}{ccc}
 (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
 \Delta_{\mathcal{P}(X)} \times (\text{id}_{\mathcal{P}(X)} \times \text{id}_{\mathcal{P}(X)}) \nearrow & & \nwarrow \mu_{4|\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \\
 \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
 \text{id}_{\mathcal{P}(X)} \times \cap \searrow & & \nearrow \cap \times \text{id}_{\mathcal{P}(X)} \\
 \mathcal{P}(X) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
 \cup \nearrow & & \nwarrow \cup \\
 & \mathcal{P}(X) &
 \end{array}$$

commute, i.e. we have equalities of sets

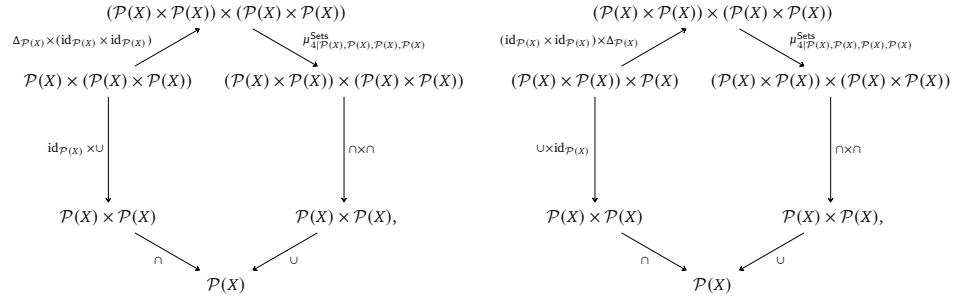
$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$

$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

0046

8. *Distributivity of Intersections Over Unions.* The diagrams



commute, i.e. we have equalities of sets

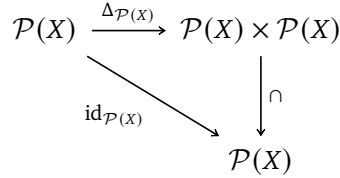
$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$

$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

0045

9. *Idempotency.* The diagram



commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

0048

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

0049

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

01HG

12. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & \subset & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01HH

13. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01HJ

14. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

004A 15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

004B 16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

PROOF 4.3.9.1.3 ► PROOF OF PROPOSITION 4.3.9.1.2

Item 1: Functoriality

See [Pro25al].

Item 2: Adjointness

See [MSE 267469].

Item 3: Associativity

See [Pro25r].

Item 4: Unitality

This follows from [Pro25v] and Item 5.

Item 5: Commutativity

See [Pro25s].

Item 6: Annihilation With the Empty Set

This follows from [Pro25t] and Item 5.

Item 7: Distributivity of Unions Over Intersections

See [Pro25az].

Item 8: Distributivity of Intersections Over Unions

See [Pro25aj].

Item 9: Idempotency

See [Pro25ak].

Item 10: Interaction With Characteristic Functions I

See [Pro25e].

Item 11: Interaction With Characteristic Functions II

See [Pro25e].

Item 12: Interaction With Direct Images

See [Pro25n].

Item 13: Interaction With Inverse Images

See [Pro25w].


Item 14: Interaction With Codirect Images

This is a repetition of **Item 6** of **Proposition 4.6.3.1.7** and is proved there.

Item 15: Interaction With Powersets and Monoids With Zero

This follows from **Items 3** to **6**.

Item 16: Interaction With Powersets and Semirings

This follows from **Items 2** to **4** and **8** and **Items 3** to **6** and **8** of **Proposition 4.3.9.1.2**. 

004D 4.3.10 Differences

Let X and Y be sets.

004E DEFINITION 4.3.10.1.1 ► DIFFERENCES

The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

004F PROPOSITION 4.3.10.1.2 ► PROPERTIES OF DIFFERENCES

Let X be a set.

004G 1. *Functoriality*. The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus - &: (\mathcal{P}(X), \supset) && \rightarrow (\mathcal{P}(X), \subset), \\ - \setminus V &: (\mathcal{P}(X), \subset) && \rightarrow (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

01HK (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.

01HL

(b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.

01HM

(c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

004H

2. *De Morgan's Laws.* We have equalities of sets

$$X \setminus (U \cup V) = (X \setminus U) \cap (X \setminus V),$$

$$X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$$

for each $U, V \in \mathcal{P}(X)$.

004J

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

004K

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

004L

5. *Interaction With Unions III.* We have equalities of sets

$$U \setminus (V \cup W) = (U \cup W) \setminus (V \cup W)$$

$$= (U \setminus V) \setminus W$$

$$= (U \setminus W) \setminus V$$

for each $U, V, W \in \mathcal{P}(X)$.

004M

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

004N

7. *Interaction With Intersections.* We have equalities of sets

$$(U \setminus V) \cap W = (U \cap W) \setminus V$$

$$= U \cap (W \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

004P

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^c$$

for each $U, V \in \mathcal{P}(X)$.

004Q

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \Delta (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

004R

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

004S

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

004T

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

01HN

13. *Right Annihilation.* We have

$$U \setminus X = \emptyset$$

for each $U \in \mathcal{P}(X)$.

004U

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

004V

15. *Interaction With Containment.* The following conditions are equivalent:

004W

(a) We have $V \setminus U \subset W$.

004X (b) We have $V \setminus W \subset U$.

004Y 16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

01HP 17. *Interaction With Direct Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \setminus & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01HQ 18. *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow \setminus & & \downarrow \setminus \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

01HR 19. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \setminus & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

PROOF 4.3.10.1.3 ► PROOF OF PROPOSITION 4.3.10.1.2

Item 1: Functoriality

See [Pro25ad] and [Pro25ah].

Item 2: De Morgan's Laws

See [Pro25k].

Item 3: Interaction With Unions I

See [Pro25l].

Item 4: Interaction With Unions II

We have

$$\begin{aligned} (U \setminus V) \cup W &\stackrel{\text{def}}{=} \{x \in X \mid (x \in U \text{ and } x \notin V) \text{ or } x \in W\} \\ &= \{x \in X \mid (x \in U \text{ or } x \in W) \text{ and } (x \notin V \text{ or } x \in W)\} \\ &= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \text{ and } x \notin W)\} \\ &= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \setminus W)\} \\ &= \{x \in X \mid (x \in (U \cup W) \setminus (V \setminus W))\} \\ &= (U \cup W) \setminus (V \setminus W). \end{aligned}$$

Item 5: Interaction With Unions III

See [Pro25ai].

Item 6: Interaction With Unions IV

See [Pro25ac].

Item 7: Interaction With Intersections

See [Pro25u].

Item 8: Interaction With Complements

See [Pro25aa].

Item 9: Interaction With Symmetric Differences

See [Pro25ab].

Item 10: Triple Differences

See [Pro25ag].

Item 11: Left Annihilation

The direction $\emptyset \subset \emptyset \setminus U$ always holds. Now assume $x \in \emptyset \setminus U$. Then, $x \in \emptyset$ and $x \notin U$. Hence $\emptyset \setminus U \subset \emptyset$ must hold and the sets are equal.

Item 12: Right Unitality

See [Pro25ae].

Item 13: Right Annihilation

It suffices to show that no $x \in X$ can be an element of $U \setminus X$. Assume $x \in U \setminus X$. Then $x \notin X$, contradicting $x \in X$. This completes the proof.

Item 14: Invertibility

See [Pro25af].

Item 15: Interaction With Containment

The conditions are symmetric in U, W , hence it suffices to show that $V \setminus U \subset W$ implies $V \setminus W \subset U$. So assume $V \setminus U \subset W, x \in V \setminus W$. Then $x \in V, x \notin W$. So by contraposition, $x \notin V \setminus U$. But $x \in V$, so we must have $x \in U$, completing the proof.

Item 16: Interaction With Characteristic Functions

See [Pro25f].

Item 17: Interaction With Direct Images

See [Pro25o].

Item 18: Interaction With Inverse Images

See [Pro25x].



004Z 4.3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

0050 DEFINITION 4.3.11.1 ► COMPLEMENTS

The **complement of U** is the set U^c defined by

$$\begin{aligned} U^c &\stackrel{\text{def}}{=} X \setminus U \\ &\stackrel{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

0051

Let X be a set.

0052

1. *Functoriality.* The assignment $U \mapsto U^c$ defines a functor

$$(-)^c: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X).$$

In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

(★) If $U \subset V$, then $V^c \subset U^c$.

0053

2. *De Morgan's Laws.* The diagrams

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cup^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X) \end{array} \quad \begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X) \end{array}$$

commute, i.e. we have equalities of sets

$$(U \cup V)^c = U^c \cap V^c,$$

$$(U \cap V)^c = U^c \cup V^c$$

for each $U, V \in \mathcal{P}(X)$.

0054

3. *Involutority.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{(-)^c} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)^{\text{op}}} & \downarrow (-)^{c, \text{op}} \\ & & \mathcal{P}(X)^{\text{op}} \end{array}$$

commutes, i.e. we have

$$(U^c)^c = U$$

for each $U \in \mathcal{P}(X)$.

0055

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

01HS

5. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U^c) = f_*(U)^c$$

for each $U \in \mathcal{P}(X)$.

01HT

6. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U^c) = f^{-1}(U)^c$$

for each $U \in \mathcal{P}(X)$.

01HU

7. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U^c) = f_!(U)^c$$

for each $U \in \mathcal{P}(X)$.

|

|

Item 1: Functoriality

This follows from **Item 1** of **Proposition 4.3.10.1.2**.

Item 2: De Morgan's Laws

See [**Pro25k**].

Item 3: Involutivity

See [**Pro25i**].

Item 4: Interaction With Characteristic Functions

We consider the two cases $x \in U, x \notin U$.

1. If $x \in U$, then $x \notin U^c$. So $\chi_U(x) = 1$ and

$$\begin{aligned}\chi_{U^c}(x) &= 0 \\ &= 1 - \chi_U(x).\end{aligned}$$

2. If $x \notin U$, then $x \in U^c$. So $\chi_U(x) = 0$ and

$$\begin{aligned}\chi_{U^c}(x) &= 1 \\ &= 1 - \chi_U(x).\end{aligned}$$

Hence, the equation holds for all $x \in X$.

Item 5: Interaction With Direct Images

This is a repetition of **Item 8** of **Proposition 4.6.1.1.5** and is proved there.

Item 6: Interaction With Inverse Images

This is a repetition of **Item 8** of **Proposition 4.6.2.1.3** and is proved there.

Item 7: Interaction With Codirect Images

This is a repetition of **Item 7** of **Proposition 4.6.3.1.7** and is proved there. 

0056 4.3.12 Symmetric Differences

Let X be a set and let $U, V \in \mathcal{P}(X)$.

0057 DEFINITION 4.3.12.1.1 ► SYMMETRIC DIFFERENCES

The **symmetric difference of U and V** is the set $U \triangle V$ defined by¹

$$U \triangle V \stackrel{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

¹Illustration:

0058

PROPOSITION 4.3.12.1.2 ► PROPERTIES OF SYMMETRIC DIFFERENCESLet X be a set.

0059

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \Delta V$ **does not** in general define functors

$$\begin{aligned} U \Delta -: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ - \Delta V: (\mathcal{P}(X), \subset) &\rightarrow (\mathcal{P}(X), \subset), \\ -_1 \Delta -_2: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

005A

2. *Via Unions and Intersections.* We have

$$U \Delta V = (U \cup V) \setminus (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram

01HV

3. *Symmetric Differences of Disjoint Sets.* If U and V are disjoint, then we have

$$U \Delta V = U \cup V.$$

005B

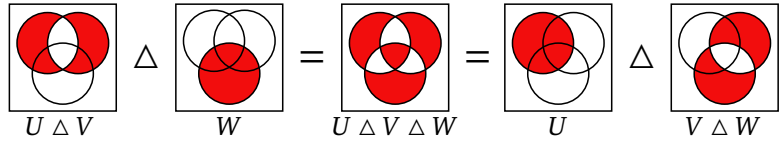
4. *Associativity.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\ \alpha_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}} \swarrow & \searrow \text{id}_{\mathcal{P}(X)} \times \Delta & \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\ \Delta \times \text{id}_{\mathcal{P}(X)} \searrow & & \searrow \Delta \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X), \end{array}$$

commutes, i.e. we have

$$(U \Delta V) \Delta W = U \Delta (V \Delta W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram



005D

5. *Unitality.* The diagrams

$$\begin{array}{ccc} \text{pt} \times \mathcal{P}(X) & \xrightarrow{[\emptyset] \times \text{id}_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \searrow \lambda_{\mathcal{P}(X)}^{\text{Sets}} \sim & \downarrow \Delta & \\ & \mathcal{P}(X) & \end{array} \quad \begin{array}{ccc} \mathcal{P}(X) \times \text{pt} & \xrightarrow{\text{id}_{\mathcal{P}(X)} \times [\emptyset]} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \searrow \rho_{\mathcal{P}(X)}^{\text{Sets}} \sim & \downarrow \Delta & \\ & \mathcal{P}(X) & \end{array}$$

commute, i.e. we have

$$\begin{aligned} U \Delta \emptyset &= U, \\ \emptyset \Delta U &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

005C

6. *Commutativity.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma_{\mathcal{P}(X), \mathcal{P}(X)}^{\text{Sets}}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \searrow \Delta & & \downarrow \Delta \\ & & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$U \Delta V = V \Delta U$$

for each $U, V \in \mathcal{P}(X)$.

005E

7. *Invertibility.* We have

$$U \Delta U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

005F 8. *Interaction With Unions.* We have

$$(U \Delta V) \cup (V \Delta T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

005G 9. *Interaction With Complements I.* We have

$$U \Delta U^c = X$$

for each $U \in \mathcal{P}(X)$.

005H 10. *Interaction With Complements II.* We have

$$U \Delta X = U^c,$$

$$X \Delta U = U^c$$

for each $U \in \mathcal{P}(X)$.

005J 11. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X) \\ (-)^c \times (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\Delta} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$U^c \Delta V^c = U \Delta V$$

for each $U, V \in \mathcal{P}(X)$.

005K 12. *“Transitivity”.* We have

$$(U \Delta V) \Delta (V \Delta W) = U \Delta W$$

for each $U, V, W \in \mathcal{P}(X)$.

005L 13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \Delta W \subset U \Delta V \cup V \Delta W$$

for each $U, V, W \in \mathcal{P}(X)$.

005M

14. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

005N

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

005P

16. *Bijectivity.* Given $U, V \in \mathcal{P}(X)$, the maps

$$\begin{aligned} U \triangle -: \mathcal{P}(X) &\rightarrow \mathcal{P}(X), \\ - \triangle V: \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \end{aligned}$$

are self-inverse bijections. Moreover, the map

$$\begin{aligned} \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ C &\longmapsto C \triangle (U \triangle V) \end{aligned}$$

is a bijection of $\mathcal{P}(X)$ onto itself sending U to V and V to U .

005Q

17. *Interaction With Powersets and Groups.* Let X be a set.

005R

(a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \text{id}_{\mathcal{P}(X)})$ is an abelian group.¹

005S

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to \triangle , and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

005T

18. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

- The group $\mathcal{P}(X)$ of **Item 17**;
- The map $\alpha_{\mathcal{P}(X)}: \mathbb{F}_2 \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

is an \mathbb{F}_2 -vector space.

005U

19. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

020L

(a) The set of singletons sets on the elements of X forms a basis for the \mathbb{F}_2 -vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of [Item 18](#).

020M

(b) We have

$$\dim(\mathcal{P}(X)) = \#X.$$

005V

20. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \Delta, \cap, \emptyset, X)$ is a commutative ring.²

01HW

21. *Interaction With Direct Images.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \wr & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \Delta f_!(V) \subset f_!(U \Delta V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01HX

22. *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \Delta \downarrow & & \downarrow \Delta \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

i.e. we have

$$f^{-1}(U) \Delta f^{-1}(V) = f^{-1}(U \Delta V)$$

for each $U, V \in \mathcal{P}(Y)$.

01HY

23. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{array}{ccc}
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\
 \Delta \downarrow & \wr & \downarrow \Delta \\
 \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
 \end{array}$$

with components

$$f_*(U \Delta V) \subset f_*(U) \Delta f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.¹Here are some examples:

020H

i. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \Delta, \emptyset, \text{id}_{\mathcal{P}(\emptyset)}) \cong \text{pt.}$$

020J


ii. When $X = \text{pt}$, we have an isomorphism of groups between $\mathcal{P}(\text{pt})$ and $\mathbb{Z}/2$:

$$(\mathcal{P}(\text{pt}), \Delta, \emptyset, \text{id}_{\mathcal{P}(\text{pt})}) \cong \mathbb{Z}/2.$$

020K

iii. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$:

$$(\mathcal{P}(\{0, 1\}), \Delta, \emptyset, \text{id}_{\mathcal{P}(\{0, 1\})}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

²  **Warning:** The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \Delta, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25aw] for a proof.

PROOF 4.3.12.1.3 ► PROOF OF PROPOSITION 4.3.12.1.2

Item 1: Lack of Functoriality

Let $X = \{0, 1\}$, $U = \{0\}$. Then $\emptyset \subset U$, but $U \Delta \emptyset = U \not\subset \emptyset = U \Delta U$ from **Item 5** and **Item 7**. This gives a counterexample to the first statement. By using **Item 6**, we can adapt it to the second and third statement.

Item 2: Via Unions and Intersections

See [Pro25m].

Item 3: Symmetric Differences of Disjoint Sets

Since U and V are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$\begin{aligned} U \Delta V &= (U \cup V) \setminus (U \cap V) \\ &= (U \cup V) \setminus \emptyset \\ &= U \cup V, \end{aligned}$$

where we've used [Item 2](#) and [Item 12](#) of [Proposition 4.3.10.1.2](#).

Item 4: Associativity

See [[Pro25ao](#)].

Item 5: Unitality

This follows from [Item 6](#) and [[Pro25at](#)].

Item 6: Commutativity

See [[Pro25ap](#)].

Item 7: Invertibility

See [[Pro25av](#)].

Item 8: Interaction With Unions

See [[Pro25bc](#)].

Item 9: Interaction With Complements I

See [[Pro25as](#)].

Item 10: Interaction With Complements II

This follows from [Item 6](#) and [[Pro25ax](#)].

Item 11: Interaction With Complements III

See [[Pro25aq](#)].

Item 12: "Transitivity"

We have

$$\begin{aligned} (U \Delta V) \Delta (V \Delta W) &= U \Delta (V \Delta (V \Delta W)) && \text{(by Item 4)} \\ &= U \Delta ((V \Delta V) \Delta W) && \text{(by Item 4)} \\ &= U \Delta (\emptyset \Delta W) && \text{(by Item 7)} \\ &= U \Delta W. && \text{(by Item 5)} \end{aligned}$$

This finishes the proof.

Item 13: The Triangle Inequality for Symmetric Differences

This follows from [Items 2](#) and [12](#).

Item 14: Distributivity Over Intersections

See [[Pro25q](#)].

Item 15: Interaction With Characteristic Functions

See [[Pro25g](#)].

Item 16: Bijectivity

- We show that

$$(U \triangle -): \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is self-inverse.

Let $W \in \mathcal{P}(X)$. Then,

$$\begin{aligned} U \triangle (U \triangle W) &= (U \triangle U) \triangle W && \text{(by [Item 4](#))} \\ &= \emptyset \triangle W && \text{(by [Item 7](#))} \\ &= W. && \text{(by [Item 5](#))} \end{aligned}$$

- By [Item 6](#), $(- \triangle V) = (V \triangle -)$, hence the former is also self-inverse by the first point.
- The map $- \triangle (U \triangle V)$ is a bijection as a special case of the second point. From the first two points and [Item 6](#), we get

$$U \triangle (U \triangle V) = V, \quad V \triangle (U \triangle V) = V \triangle (V \triangle U) = U.$$

Hence the function maps U to V and V to U .

Item 17: Interaction With Powersets and Groups

[Item 17a](#) follows from [Items 4](#) to [7](#), while [Item 17b](#) follows from [Item 7](#).¹

Item 18: Interaction With Powersets and Vector Spaces I

See [[MSE 2719059](#)].

Item 19: Interaction With Powersets and Vector Spaces II

See [[MSE 2719059](#)].

Item 20: Interaction With Powersets and Rings

This follows from **Items 6 and 15** of **Proposition 4.3.9.1.2** and **Items 14 and 17**.²

Item 21: Interaction With Direct Images

This is a repetition of **Item 9** of **Proposition 4.6.1.1.5** and is proved there.

Item 22: Interaction With Inverse Images

This is a repetition of **Item 9** of **Proposition 4.6.2.1.3** and is proved there.

Item 23: Interaction With Codirect Images

This is a repetition of **Item 8** of **Proposition 4.6.3.1.7** and is proved there. 

¹Reference: [Pro25ar].

²Reference: [Pro25au].

005W 4.4 Powersets

01HZ 4.4.1 Foundations

Let X be a set.

006P DEFINITION 4.4.1.1.1 ► POWERSETS

The **powerset** of X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, ?? of ??.

006Q REMARK 4.4.1.1.2 ► POWERSETS AS DECATEGORIFICATIONS OF CO/PRESHEAF CATEGORIES

Under the analogy that $\{t, f\}$ should be the (-1) -categorical analogue of Sets, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set X is equivalently (**Item 2** of **Proposition 4.5.1.1.4**) the set

$$\text{Sets}(X, \{t, f\})$$

of functions from X to the set $\{t, f\}$ of classical truth values.

- The category of presheaves on a category C is the category

$$\text{Fun}(C^{\text{op}}, \text{Sets})$$

of functors from C^{op} to the category Sets of sets.

01J0

NOTATION 4.4.1.1.3 ► FURTHER NOTATION FOR POWERSETS

Let X be a set.

01J1

1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of X .

01J2

2. We write $\mathcal{P}_{\text{fin}}(X)$ for the set of finite subsets of X .

01J3

PROPOSITION 4.4.1.1.4 ► ELEMENTARY PROPERTIES OF POWERSETS

Let X be a set.

01J4

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

020P

(a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

020Q

(b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

020R

(c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

01J5

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.

01J6

3. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \text{Rel}(\text{pt}, X),$$

$$\mathcal{P}(X) \cong \text{Rel}(X, \text{pt}),$$

natural in $X \in \text{Obj}(\text{Sets})$.

01J7

4. *Interaction With Products I.* The map

$$\mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \amalg Y)$$

$$(U, V) \longmapsto U \cup V$$

is an isomorphism of sets, natural in $X, Y \in \text{Obj}(\text{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* on \mathcal{P} of **Proposition 4.4.2.1.1**. Moreover, this makes each of $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* into a symmetric monoidal functor.

01J8

5. *Interaction With Products II.* The map

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \amalg Y) \\ (U, V) &\longmapsto U \boxtimes_{X \times Y} V,\end{aligned}$$

where¹

$$U \boxtimes_{X \times Y} V \stackrel{\text{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

is an inclusion of sets, natural in $X, Y \in \text{Obj}(\text{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* on \mathcal{P} of [Proposition 4.4.2.1.1](#). Moreover, this makes each of $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* into a symmetric monoidal functor.

01J9

6. *Interaction With Products III.* We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in $X, Y \in \text{Obj}(\text{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* on \mathcal{P} of [Proposition 4.4.2.1.1](#), where \otimes denotes the tensor product of suplattices of \mathcal{P} . Moreover, this makes each of $\mathcal{P}_!$, \mathcal{P}^{-1} , and \mathcal{P}_* into a symmetric monoidal functor.

¹The set $U \boxtimes_{X \times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see also [Section 4.6.4](#)).

PROOF 4.4.1.1.5 ► PROOF OF PROPOSITION 4.4.1.1.4

Item 1: Co/Completeness

Omitted.

Item 2: Cartesian Closedness

See [Section 4.4.7](#).

Item 3: Powersets as Sets of Relations

Indeed, we have

$$\begin{aligned}\text{Rel}(\text{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\text{pt} \times X) \\ &\cong \mathcal{P}(X)\end{aligned}$$

and

$$\begin{aligned}\text{Rel}(X, \text{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \text{pt}) \\ &\cong \mathcal{P}(X),\end{aligned}$$

where we have used **Item 5** of **Proposition 4.1.3.1.4**.

Item 4: Interaction With Products I

The inverse of the map in the statement is the map

$$\Phi: \mathcal{P}(X \amalg Y) \rightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \stackrel{\text{def}}{=} (S_X, S_Y)$$

for each $S \in \mathcal{P}(X \amalg Y)$, where

$$\begin{aligned}S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\} \\ S_Y &\stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}.\end{aligned}$$

The rest of the proof is omitted.

Item 5: Interaction With Products II

Omitted.

Item 6: Interaction With Products III

Omitted.



01JA 4.4.2 Functoriality of Powersets

01JB PROPOSITION 4.4.2.1.1 ► FUNCTORIALITY OF POWERSSETS

Let X be a set.

01JC 1. *Functoriality I*. The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_!: \text{Sets} \rightarrow \text{Sets},$$

where

- *Action on Objects*. For each $A \in \text{Obj}(\text{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} : \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by sending a map of sets $f : A \rightarrow B$ to the map

$$\mathcal{P}_!(f) : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in **Definition 4.6.1.1.1**.

01JD

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} : \text{Sets}^{\text{op}} \rightarrow \text{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on morphisms

$$\mathcal{P}_{A,B}^{-1} : \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of \mathcal{P}^{-1} at (A, B) is the map defined by sending a map of sets $f : A \rightarrow B$ to the map

$$\mathcal{P}^{-1}(f) : \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in **Definition 4.6.2.1.1**.

01JE

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* : \text{Sets} \rightarrow \text{Sets},$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\text{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \text{Obj}(\text{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B}: \text{Sets}(A, B) \rightarrow \text{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of \mathcal{P}_* at (A, B) is the map defined by sending a map of sets $f: A \rightarrow B$ to the map

$$\mathcal{P}_*(f): \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in [Definition 4.6.3.1.1](#).

PROOF 4.4.2.1.2 ► PROOF OF PROPOSITION 4.4.2.1.1


Item 1: Functoriality I

This follows from [Items 3 and 4 of Proposition 4.6.1.1.7](#).

Item 2: Functoriality II

This follows from [Items 3 and 4 of Proposition 4.6.2.1.5](#).

Item 3: Functoriality III

This follows from [Items 3 and 4 of Proposition 4.6.3.1.9](#). 

01JF 4.4.3 Adjointness of Powersets I

01JG PROPOSITION 4.4.3.1.1 ► ADJOINTNESS OF POWERSETS I

We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1, \text{op}} \right): \text{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{P}^{-1}} \\ \perp \\ \xleftarrow{\mathcal{P}^{-1, \text{op}}} \end{array} \text{Sets},$$

witnessed by a bijection

$$\underbrace{\text{Sets}^{\text{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \text{Sets}(Y, \mathcal{P}(X))} \cong \text{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $Y \in \text{Obj}(\text{Sets}^{\text{op}})$.

PROOF 4.4.3.1.2 ► PROOF OF PROPOSITION 4.4.3.1.1

We have

$$\begin{aligned} \text{Sets}^{\text{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \text{Sets}(B, \mathcal{P}(A)) \\ &\cong \text{Sets}(B, \text{Sets}(A, \{t, f\})) && \text{(by Item 2 of Proposition 4.5.1.1.4)} \\ &\cong \text{Sets}(A \times B, \{t, f\}) && \text{(by Item 2 of Proposition 4.1.3.1.4)} \\ &\cong \text{Sets}(A, \text{Sets}(B, \{t, f\})) && \text{(by Item 2 of Proposition 4.1.3.1.4)} \\ &\cong \text{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Proposition 4.5.1.1.4)} \end{aligned}$$

where all bijections are natural in A and B .¹



¹Here we are using Item 3 of Proposition 4.5.1.1.4.

01JH 4.4.4 Adjointness of Powersets II

01JJ PROPOSITION 4.4.4.1.1 ► ADJOINTNESS OF POWERSSETS II

We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_!): \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_!} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(X), Y) \cong \text{Sets}(X, \mathcal{P}(Y))$$

natural in $X \in \text{Obj}(\text{Sets})$ and $Y \in \text{Obj}(\text{Rel})$, where Gr is the graph functor of Relations, Item 1 of Proposition 8.2.2.1.2 and $\mathcal{P}_!$ is the functor of Relations, Proposition 8.7.5.1.1.

PROOF 4.4.4.1.2 ► PROOF OF PROPOSITION 4.4.4.1.1


We have

$$\begin{aligned}
 \text{Rel}(\text{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
 &\cong \text{Sets}(A \times B, \{t, f\}) && \text{(by Item 2 of Proposition 4.5.1.1.4)} \\
 &\cong \text{Sets}(A, \text{Sets}(B, \{t, f\})) && \text{(by Item 2 of Proposition 4.1.3.1.4)} \\
 &\cong \text{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Proposition 4.5.1.1.4)}
 \end{aligned}$$

where all bijections are natural in A , (where we are using [Item 3 of Proposition 4.5.1.1.4](#)). Explicitly, this isomorphism is given by sending a relation $R: \text{Gr}(A) \rightarrowtail B$ to the map $R^\dagger: A \rightarrow \mathcal{P}(B)$ sending a to the subset $R(a)$ of B , as in [Relations, Definition 8.1.1.1.1](#).

Naturality in B is then the statement that given a relation $R: B \rightarrowtail B'$, the diagram

$$\begin{array}{ccc}
 \text{Rel}(\text{Gr}(A), B) & \xrightarrow{R \circ -} & \text{Rel}(\text{Gr}(A), B') \\
 \downarrow \wr & & \downarrow \wr \\
 \text{Sets}(A, \mathcal{P}(B)) & \xrightarrow{R_!} & \text{Sets}(A, \mathcal{P}(B'))
 \end{array}$$

commutes, which follows from [Relations, Remark 8.7.1.1.3](#). 

01JK 4.4.5 Powersets as Free Cocompletions

Let X be a set.

01JL

PROPOSITION 4.4.5.1.1 ► POWERSETS AS FREE COCOMPLETIONS: UNIVERSAL PROPERTY

The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of X of [Definition 4.4.1.1.1](#);
- The characteristic embedding $\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of [Definition 4.5.4.1.1](#);

satisfies the following universal property:

- (★) Given another pair (Y, f) consisting of
- A suplattice (Y, \preceq) ;
 - A function $f: X \rightarrow Y$;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc} & \mathcal{P}(X) & \\ \chi_X \nearrow & & \downarrow \exists! \\ X & \xrightarrow{f} & Y \end{array}$$

commute.

PROOF 4.4.5.1.2 ► PROOF OF PROPOSITION 4.4.5.1.1

This is a rephrasing of [Proposition 4.4.5.1.3](#), which we prove below.¹ 

¹Here we only remark that the unique morphism of suplattices in the statement is given by the left Kan extension $\text{Lan}_{\chi_X}(f)$ of f along χ_X .

01JM

PROPOSITION 4.4.5.1.3 ► POWERSETS AS FREE COCOMPLETIONS: ADJOINTNESS

We have an adjunction

$$(\mathcal{P} \dashv \overline{\omega}): \text{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\overline{\omega}} \end{array} \text{SupLat},$$

witnessed by a bijection

$$\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, \preceq) \in \text{Obj}(\text{SupLat})$, where:

- The category SupLat is the category of suplattices of ??.
- The map

$$\chi_X^*: \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of suplattices $f: \mathcal{P}(X) \rightarrow Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

$$\text{Lan}_{\chi_X}: \text{Sets}(X, Y) \rightarrow \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function $f: X \rightarrow Y$ to its left Kan extension along χ_X ,

$$\text{Lan}_{\chi_X}(f): \mathcal{P}(X) \rightarrow Y,$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \text{Sets}(X, \{t, f\})$ of [Item 2 of Proposition 4.5.1.1.4](#), $\text{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned} [\text{Lan}_{\chi_X}(f)](U) &= \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_X, U) \odot f(x) \\ &= \int^{x \in X} \chi_U(x) \odot f(x) \\ &= \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \\ &= \left(\bigvee_{x \in U} (\chi_U(x) \odot f(x)) \right) \vee \left(\bigvee_{x \in U^c} (\chi_U(x) \odot f(x)) \right) \\ &= \left(\bigvee_{x \in U} f(x) \right) \vee \left(\bigvee_{x \in U^c} \emptyset_Y \right) \\ &= \bigvee_{x \in U} f(x) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- We have used ?? for the first equality.
- We have used [Proposition 4.5.5.1.1](#) for the second equality.
- We have used ?? for the third equality.
- The symbol \vee denotes the join in (Y, \preceq) .

- The symbol \odot denotes the tensor of an element of Y by a truth value as in ???. In particular, we have

$$\begin{aligned}\text{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \odot f(x) &\stackrel{\text{def}}{=} \emptyset_Y,\end{aligned}$$

where \emptyset_Y is the bottom element of (Y, \preceq) .

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B , the Kan extension $\text{Lan}_{\chi_X}(f)$ is given by

$$\begin{aligned}[\text{Lan}_{\chi_X}(f)](U) &= \bigvee_{x \in U} f(x) \\ &= \bigcup_{x \in U} f(x)\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

PROOF 4.4.5.1.4 ► PROOF OF PROPOSITION 4.4.5.1.3

Map I

We define a map

$$\Phi_{X,Y}: \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Map II

We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f),$$

for each $f \in \text{Sets}(X, Y)$.

Invertibility I

We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))}.$$

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. We now claim that

$$\text{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned} [\text{Lan}_{\chi_X}(f \circ \chi_X)](U) &= \bigvee_{x \in U} f(\chi_X(x)) \\ &= f\left(\bigvee_{x \in U} \chi_X(x)\right) \\ &= f\left(\bigcup_{x \in U} \{x\}\right) \\ &= f(U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))}$ of $\text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Invertibility II

We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)}.$$

We have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Lan}_{\chi_X}(f)) \\ &\stackrel{\text{def}}{=} \text{Lan}_{\chi_X}(f) \circ \chi_X \end{aligned}$$

for each $f \in \text{Sets}(X, Y)$. We now claim that

$$\text{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \text{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned} [\text{Lan}_{\chi_X}(f) \circ \chi_X](x) &= \bigvee_{y \in \{x\}} f(y) \\ &= f(x) \end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \text{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{Sets}(X,Y)}$ of $\text{Sets}(X, Y)$.

Naturality for Φ , Part I

We need to show that, given a function $f: X \rightarrow X'$, the diagram

$$\begin{array}{ccc} \text{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ \mathcal{P}_!(f)^* \downarrow & & \downarrow f^* \\ \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, we have

$$\begin{aligned}
 [\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
 &\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
 &\stackrel{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
 &= \xi \circ (f_! \circ \chi_X) \\
 &\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
 &= (\xi \circ \chi_{X'}) \circ f \\
 &\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
 &\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
 &\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
 \end{aligned}$$

for each $\xi \in \text{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$, where we have used **Item 1** of **Proposition 4.5.4.1.3** for the fifth equality above.

Naturality for Φ , Part II

We need to show that, given a morphism of suplattices

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc}
 \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\
 g_! \downarrow & & \downarrow g_! \\
 \text{SupLat}((\mathcal{P}(X), \subset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y')
 \end{array}$$


commutes. Indeed, we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_!](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi)) \\
 &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
 &\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
 &= g \circ (\xi \circ \chi_X) \\
 &\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
 &\stackrel{\text{def}}{=} g_!(\Phi_{X,Y}(\xi))
 \end{aligned}$$

$$\stackrel{\text{def}}{=} [g! \circ \Phi_{X,Y}](\xi).$$

for each $\xi \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Naturality for Ψ

Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from **Categories, Item 2 of Proposition 11.9.7.1.2** that Ψ is also natural in each argument. 

WARNING 4.4.5.1.5 ► FREE COCOMPLETION IS NOT AN IDEMPOTENT OPERATION

Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion* of X , it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.



01JP 4.4.6 Powersets as Free Completions

Let X be a set.

01JQ

PROPOSITION 4.4.6.1.1 ► POWERSETS AS FREE COMPLETIONS: UNIVERSAL PROPERTY

The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset of X together with reverse inclusion $\mathcal{P}(X)^{\text{op}} = (\mathcal{P}(X), \supset)$ of **Definition 4.4.1.1.1**;
- The characteristic embedding $\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of **Definition 4.5.4.1.1**;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

- An inflattice (Y, \preceq) ;
- A function $f: X \rightarrow Y$;

there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X)^{\text{op}} & \\
 \chi_X \nearrow & \downarrow \exists! & \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commute.

PROOF 4.4.6.1.2 ► PROOF OF PROPOSITION 4.4.6.1.1

This is a rephrasing of **Proposition 4.4.6.1.3**, which we prove below.¹ 

¹Here we only remark that the unique morphism of inflattices in the statement is given by the right Kan extension $\text{Ran}_{\chi_X}(f)$ of f along χ_X .

01JR

PROPOSITION 4.4.6.1.3 ► POWERSETS AS FREE COMPLETIONS: ADJOINTNESS

We have an adjunction

$$(\mathcal{P} \dashv \text{忘}): \text{Sets} \begin{array}{c} \xrightarrow{\mathcal{P}} \\ \perp \\ \xleftarrow{\text{忘}} \end{array} \text{InfLat},$$

witnessed by a bijection

$$\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, \preceq) \in \text{Obj}(\text{InfLat})$, where:

- The category InfLat is the category of inflattices of ??.
- The map

$$\chi_X^*: \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of inflattices $f: \mathcal{P}(X)^{\text{op}} \rightarrow Y$ to the composition

$$X \xrightarrow{\chi_X} \mathcal{P}(X)^{\text{op}} \xrightarrow{f} Y.$$

- The map

$$\text{Ran}_{\chi_X} : \text{Sets}(X, Y) \rightarrow \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function $f : X \rightarrow Y$ to its right Kan extension along χ_X ,

$$\text{Ran}_{\chi_X}(f) : \mathcal{P}(X)^{\text{op}} \rightarrow Y,$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \text{Sets}(X, \{t, f\})$ of [Item 2 of Proposition 4.5.1.1.4](#), $\text{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned} [\text{Ran}_{\chi_X}(f)](U) &= \int_{x \in X} \chi_{\mathcal{P}(X)^{\text{op}}}(\chi_x, U) \multimap f(x) \\ &= \int_{x \in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \multimap f(x) \\ &= \int_{x \in X} \chi_U(x) \multimap f(x) \\ &= \bigwedge_{x \in X} \chi_U(x) \multimap f(x) \\ &= \left(\bigwedge_{x \in U} \chi_U(x) \multimap f(x) \right) \wedge \left(\bigwedge_{x \in U^c} \chi_U(x) \multimap f(x) \right) \\ &= \left(\bigwedge_{x \in U} f(x) \right) \wedge \left(\bigwedge_{x \in U^c} \infty_Y \right) \\ &= \left(\bigwedge_{x \in U} f(x) \right) \wedge \infty_Y \\ &= \bigwedge_{x \in U} f(x) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- We have used ?? for the first equality.
- We have used [Proposition 4.5.5.1.1](#) for the second equality.
- We have used ?? for the third equality.

- The symbol \wedge denotes the meet in (Y, \preceq) .
- The symbol \pitchfork denotes the cotensor of an element of Y by a truth value as in ???. In particular, we have

$$\begin{aligned}\text{true} \pitchfork f(x) &\stackrel{\text{def}}{=} f(x), \\ \text{false} \pitchfork f(x) &\stackrel{\text{def}}{=} \infty_Y,\end{aligned}$$

where ∞_Y is the top element of (Y, \preceq) .

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B , the Kan extension $\text{Ran}_{\chi_X}(f)$ is given by

$$\begin{aligned}[\text{Ran}_{\chi_X}(f)](U) &= \bigwedge_{x \in U} f(x) \\ &= \bigcap_{x \in U} f(x)\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

PROOF 4.4.6.1.4 ► PROOF OF PROPOSITION 4.4.5.1.3

Map I

We define a map

$$\Phi_{X,Y}: \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \rightarrow \text{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Map II

We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f),$$

for each $f \in \text{Sets}(X, Y)$.

Invertibility I

We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}.$$

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\text{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned} [\text{Ran}_{\chi_X}(f \circ \chi_X)](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\ &= f\left(\bigwedge_{x \in U} \chi_X(x)\right) \\ &= f\left(\bigcup_{x \in U} \{x\}\right) \\ &= f(U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}$ of $\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Invertibility II

We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)}.$$

We have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Phi_{X,Y}(\text{Ran}_{\chi_X}(f)) \\ &\stackrel{\text{def}}{=} \text{Ran}_{\chi_X}(f) \circ \chi_X \end{aligned}$$

for each $f \in \text{Sets}(X, Y)$. We now claim that

$$\text{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \text{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned} [\text{Ran}_{\chi_X}(f) \circ \chi_X](x) &= \bigwedge_{y \in \{x\}} f(y) \\ &= f(x) \end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \text{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\text{id}_{\text{Sets}(X,Y)}$ of $\text{Sets}(X, Y)$.

Naturality for Φ , Part I

We need to show that, given a function $f: X \rightarrow X'$, the diagram

$$\begin{array}{ccc} \text{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\ \mathcal{P}_!(f)^* \downarrow & & \downarrow f^* \\ \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \end{array}$$

commutes. Indeed, we have

$$\begin{aligned}
 [\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
 &\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
 &\stackrel{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
 &= \xi \circ (f_! \circ \chi_X) \\
 &\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
 &= (\xi \circ \chi_{X'}) \circ f \\
 &\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
 &\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
 &\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
 \end{aligned}$$

for each $\xi \in \text{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used **Item 1** of **Proposition 4.5.4.1.3** for the fifth equality above.

Naturality for Φ , Part II

We need to show that, given a cocontinuous morphism of posets

$$g: (Y, \preceq_Y) \rightarrow (Y', \preceq_{Y'}),$$

the diagram

$$\begin{array}{ccc}
 \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\
 g_! \downarrow & & \downarrow g_! \\
 \text{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y')
 \end{array}$$


commutes. Indeed, we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_!](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi)) \\
 &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
 &\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
 &= g \circ (\xi \circ \chi_X) \\
 &\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
 &\stackrel{\text{def}}{=} g_!(\Phi_{X,Y}(\xi))
 \end{aligned}$$

$$\stackrel{\text{def}}{=} [g! \circ \Phi_{X,Y}](\xi).$$

for each $\xi \in \text{InfLat}((\mathcal{P}(X), \supset), (Y, \leq))$.

Naturality for Ψ

Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from **Categories, Item 2 of Proposition 11.9.7.1.2** that Ψ is also natural in each argument. 

WARNING 4.4.6.1.5 ► FREE COMPLETION IS NOT AN IDEMPOTENT OPERATION

Although the assignment $X \mapsto \mathcal{P}(X)^{\text{op}}$ is called the *free completion* of X , it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\text{op}})^{\text{op}} \neq \mathcal{P}(X)^{\text{op}}$.



01JT 4.4.7 The Internal Hom of a Powerset

Let X be a set and let $U, V \in \mathcal{P}(X)$.

01JU PROPOSITION 4.4.7.1.1 ► THE INTERNAL HOM OF A POWERSET

The **internal Hom of $\mathcal{P}(X)$ from U to V** is the subset $[U, V]_X^1$ of X given by

$$\begin{aligned} [U, V]_X &= U^c \cup V \\ &= (U \setminus V)^c \end{aligned}$$

where U^c is the complement of U of **Definition 4.3.11.1.1**.

¹Further Notation: Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$.

PROOF 4.4.7.1.2 ► PROOF OF PROPOSITION 4.4.7.1.1

Proof of the Equality $U^c \cup V = (U \setminus V)^c$

We have

$$\begin{aligned} (U \setminus V)^c &\stackrel{\text{def}}{=} X \setminus (U \setminus V) \\ &= (X \cap V) \cup (X \setminus U) \\ &= V \cup (X \setminus U) \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{def}}{=} V \cup U^c \\ &= U^c \cup V, \end{aligned}$$

where we have used:

- 020S 1. **Item 10** of **Proposition 4.3.10.1.2** for the second equality.
- 020T 2. **Item 4** of **Proposition 4.3.9.1.2** for the third equality.
- 020U 3. **Item 4** of **Proposition 4.3.8.1.2** for the last equality.

This finishes the proof.

Proof that $U^c \cup V$ Is Indeed the Internal Hom

This follows from **Item 2** of **Proposition 4.3.9.1.2**. 

004C REMARK 4.4.7.1.3 ► INTUITION FOR THE INTERNAL HOM OF $\mathcal{P}(X)$

Henning Makholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from U to V ([MSE 267365]):

- 01JV 1. Since products in $\mathcal{P}(X)$ are given by binary intersections (**Item 1** of **Proposition 4.4.1.1.4**), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.
- 01JW 2. Under the Curry–Howard correspondence (??), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.
- 01JX 3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.
- 01JY 4. The expression $\neg U \vee V$ then corresponds to the set $U^c \cup V$ in $\mathcal{P}(X)$.
- 01JZ 5. The set $U^c \cup V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

01K0 PROPOSITION 4.4.7.1.4 ► PROPERTIES OF INTERNAL HOMS OF POWERSETS

Let X be a set.

- 01K1 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$\begin{aligned} [U, -]_X &: (\mathcal{P}(X), \supset) && \rightarrow (\mathcal{P}(X), \subset), \\ [-, V]_X &: (\mathcal{P}(X), \subset) && \rightarrow (\mathcal{P}(X), \subset), \\ [-_1, -_2]_X &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \rightarrow (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

- 01K2 (a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.
- 01K3 (b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.
- 01K4 (c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

01K5 2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X): \mathcal{P}(X) \begin{array}{c} \xrightarrow{U \cap -} \\ \perp \\ \xleftarrow{[U, -]_X} \end{array} \mathcal{P}(X),$$

$$(- \cap V \dashv [V, -]_X): \mathcal{P}(X) \begin{array}{c} \xrightarrow{- \cap V} \\ \perp \\ \xleftarrow{[V, -]_X} \end{array} \mathcal{P}(X),$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \text{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \text{Hom}_{\mathcal{P}(X)}(V, [U, W]_X). \end{aligned}$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

- 01K6 (a) The following conditions are equivalent:
- 01K7 i. We have $U \cap V \subset W$.
- 01K8 ii. We have $U \subset [V, W]_X$.
- 01K9 (b) The following conditions are equivalent:
- 01KA i. We have $U \cap V \subset W$.
- 01KB ii. We have $V \subset [U, W]_X$.

01KC 3. *Interaction With the Empty Set I.* We have

$$\begin{aligned} [U, \emptyset]_X &= U^c, \\ [\emptyset, V]_X &= X, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

01KD 4. *Interaction With X.* We have

$$\begin{aligned} [U, X]_X &= X, \\ [X, V]_X &= V, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

01KE

5. *Interaction With the Empty Set II.* The functor

$$D_X: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X)$$

defined by

$$\begin{aligned} D_X &\stackrel{\text{def}}{=} [-, \emptyset]_X \\ &= (-)^c \end{aligned}$$

is an involutory isomorphism of categories, making \emptyset into a dualising object for $(\mathcal{P}(X), \cap, X, [-, -]_X)$ in the sense of ?? . In particular:

01KF

(a) The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)^{\text{op}}} & \downarrow D_X \\ & & \mathcal{P}(X)^{\text{op}} \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X} = U$$

for each $U \in \mathcal{P}(X)$.

01KG

(b) The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} & \mathcal{P}(X)^{\text{op}} \\ & \nearrow \text{id}_{\mathcal{P}(X)^{\text{op}}} \times D_X & & & \searrow D_X \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{\quad \quad \quad} & [-_1, -_2]_X & \xrightarrow{\quad \quad \quad} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

01KH

6. *Interaction With the Empty Set III.* Let $f: X \rightarrow Y$ be a function.

01KJ

(a) *Interaction With Direct Images.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\
 D_X \downarrow & & \downarrow D_Y \\
 \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
 \end{array}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

01KK

(b) *Interaction With Inverse Images.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\
 D_Y \downarrow & & \downarrow D_X \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

01KL

(c) *Interaction With Codirect Images.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\
 D_X \downarrow & & \downarrow D_Y \\
 \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
 \end{array}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

01KM

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
 \downarrow \cup^{\text{op}} \times \cup^{\text{op}} & \text{X} & \downarrow \cup \\
 \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X),
 \end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01KN

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{array}{ccccc}
 & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\
 & \nearrow \sim & & \searrow \cup^{\text{op}} & \\
 \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & & & \mathcal{P}(X)^{\text{op}} \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & & & \downarrow [-, V]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) & &
 \end{array}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01KP

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X) \\
 \downarrow \text{id}_{\mathcal{P}(X)} \star [U, -]_X & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01KQ

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\text{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1, -2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap^{\text{op}} \times \cap^{\text{op}} \downarrow & \text{X} & \downarrow \cap \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

01KR

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$\begin{array}{ccccc} & & \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & \\ & \nearrow \text{dashed} & & \searrow \cap^{\text{op}} & \\ \mathcal{P}(\mathcal{P}(X))^{\text{op}} & & & & \mathcal{P}(X)^{\text{op}} \\ \downarrow \text{id}_{\mathcal{P}(X)} \star [-, V]_X & & & & \downarrow [-, V]_X \\ \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cup} & & & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

01KS

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X) \\
 \text{id}_{\mathcal{P}(X)} \star [U, -]_X \downarrow & & \downarrow [U, -]_X \\
 \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

01KT

13. *Interaction With Binary Unions.* We have equalities of sets

$$\begin{aligned}
 [U \cap V, W]_X &= [U, W]_X \cup [V, W]_X, \\
 [U, V \cap W]_X &= [U, V]_X \cap [U, W]_X
 \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01KU

14. *Interaction With Binary Intersections.* We have equalities of sets

$$\begin{aligned}
 [U \cup V, W]_X &= [U, W]_X \cap [V, W]_X, \\
 [U, V \cup W]_X &= [U, V]_X \cup [U, W]_X
 \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01KV

15. *Interaction With Differences.* We have equalities of sets

$$\begin{aligned}
 [U \setminus V, W]_X &= [U, W]_X \cup [V^c, W]_X \\
 &= [U, W]_X \cup [U, V]_X, \\
 [U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)
 \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

01KW

16. *Interaction With Complements.* We have equalities of sets

$$\begin{aligned}
 [U^c, V]_X &= U \cup V, \\
 [U, V^c]_X &= U \cap V, \\
 [U, V]_X^c &= U \setminus V
 \end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

01KX

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod{2}, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

01KY

18. *Interaction With Direct Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

01KZ

19. *Interaction With Inverse Images.* Let $f: X \rightarrow Y$ be a function. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

01L0

20. *Interaction With Codirect Images.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & \wr & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

PROOF 4.4.7.1.5 ► PROOF OF PROPOSITION 4.4.7.1.4

Item 1: Functoriality

Since $\mathcal{P}(X)$ is posetal, it suffices to prove **Items 1a** to **1c**.

020W

1. *Proof of Item 1a:* We have

$$\begin{aligned} [A, V]_X &\stackrel{\text{def}}{=} A^c \cup V \\ &\subset U^c \cup V \\ &\stackrel{\text{def}}{=} [U, V]_X, \end{aligned}$$

where we have used:

020X

(a) **Item 1** of **Proposition 4.3.11.1.2**, which states that if $U \subset A$, then $A^c \subset U^c$.

020Y

(b) **Item 1a** of **Item 1** of **Proposition 4.3.11.1.2**, which states that if $A^c \subset U^c$, then $A^c \cup K \subset U^c \cup K$ for any $K \in \mathcal{P}(X)$.

020Z

2. *Proof of Item 1b:* We have

$$\begin{aligned} [U, V]_X &\stackrel{\text{def}}{=} U^c \cup V \\ &\subset U^c \cup B \\ &\stackrel{\text{def}}{=} [U, B]_X, \end{aligned}$$

where we have used **Item 1b** of **Item 1** of **Proposition 4.3.11.1.2**, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

0210

3. *Proof of Item 1c:* We have

$$\begin{aligned} [A, V]_X &\subset [U, V]_X \\ &\subset [U, B]_X, \end{aligned}$$

where we have used **Items 1a** and **1b**.

This finishes the proof.

Item 2: Adjointness

This is a repetition of **Item 2** of **Proposition 4.3.9.1.2** and is proved there.

Item 3: Interaction With the Empty Set I

We have

$$\begin{aligned}[U, \emptyset]_X &\stackrel{\text{def}}{=} U^c \cup \emptyset \\ &= U^c,\end{aligned}$$

where we have used **Item 3** of **Proposition 4.3.8.1.2**, and we have

$$\begin{aligned}[\emptyset, V]_X &\stackrel{\text{def}}{=} \emptyset^c \cup V \\ &\stackrel{\text{def}}{=} (X \setminus \emptyset) \cup V \\ &= X \cup V \\ &= X,\end{aligned}$$

where we have used:

- 0211 1. **Item 12** of **Proposition 4.3.10.1.2** for the first equality.
- 0212 2. **Item 5** of **Proposition 4.3.8.1.2** for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Proposition 11.2.7.1.2**).

Item 4: Interaction With X

We have

$$\begin{aligned}[U, X]_X &\stackrel{\text{def}}{=} U^c \cup X \\ &= X,\end{aligned}$$

where we have used **Item 5** of **Proposition 4.3.8.1.2**, and we have

$$\begin{aligned}[X, V]_X &\stackrel{\text{def}}{=} X^c \cup V \\ &\stackrel{\text{def}}{=} (X \setminus X) \cup V \\ &= \emptyset \cup V \\ &= V,\end{aligned}$$

where we have used **Item 3** of **Proposition 4.3.8.1.2** for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Proposition 11.2.7.1.2**).

Item 5: Interaction With the Empty Set II

We have

$$\begin{aligned} D_X(D_X(U)) &\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X \\ &= [U^c, \emptyset]_X \\ &= (U^c)^c \\ &= U, \end{aligned}$$

where we have used:

- 0213 1. **Item 3** for the second and third equalities.
- 0214 2. **Item 3** of **Proposition 4.3.11.1.2** for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Proposition 11.2.7.1.2**), and thus we have

$$[[-, \emptyset]_X, \emptyset]_X \cong \text{id}_{\mathcal{P}(X)}$$

This finishes the proof.

Item 6: Interaction With the Empty Set III

Since $D_X = (-)^c$, this is essentially a repetition of the corresponding results for $(-)^c$, namely **Items 5** to **7** of **Proposition 4.3.11.1.2**.

Item 7: Interaction With Unions of Families of Subsets I

By **Item 3** of **Proposition 4.4.7.1.4**, we have

$$\begin{aligned} [\mathcal{U}, \emptyset]_{\mathcal{P}(X)} &= \mathcal{U}^c, \\ [U, \emptyset]_X &= U^c. \end{aligned}$$

With this, the counterexample given in the proof of **Item 10** of **Proposition 4.3.6.1.2** then applies.

Item 8: Interaction With Unions of Families of Subsets II

We have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X \stackrel{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right)^c \cup V$$

$$\begin{aligned}
&= \left(\bigcap_{U \in \mathcal{U}} U^c \right) \cup V \\
&= \bigcap_{U \in \mathcal{U}} (U^c \cup V) \\
&\stackrel{\text{def}}{=} \bigcap_{U \in \mathcal{U}} [U, V]_X,
\end{aligned}$$

where we have used:

- 0215 1. **Item 11** of **Proposition 4.3.6.1.2** for the second equality.
- 0216 2. **Item 6** of **Proposition 4.3.7.1.2** for the third equality.

This finishes the proof.

Item 9: Interaction With Unions of Families of Subsets III

We have

$$\begin{aligned}
\bigcup_{V \in \mathcal{V}} [U, V]_X &\stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} (U^c \cup V) \\
&= U^c \cup \left(\bigcup_{V \in \mathcal{V}} V \right) \\
&\stackrel{\text{def}}{=} \left[U, \bigcup_{V \in \mathcal{V}} V \right]_X.
\end{aligned}$$

where we have used **Item 6**. This finishes the proof.

Item 10: Interaction With Intersections of Families of Subsets I

Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned}
\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W &= \bigcap_{W \in \mathcal{P}(X)} W \\
&= \{0, 1\},
\end{aligned}$$

whereas

$$\left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X = [\{0, 1\}, \{0\}]$$

$$= \{0\},$$

Thus we have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X.$$

This finishes the proof.

Item 11: Interaction With Intersections of Families of Subsets II

We have

$$\begin{aligned} \left[\bigcap_{U \in \mathcal{U}} U, V \right]_X &\stackrel{\text{def}}{=} \left(\bigcap_{U \in \mathcal{U}} U \right)^c \cup V \\ &= \left(\bigcup_{U \in \mathcal{U}} U^c \right) \cup V \\ &= \bigcup_{U \in \mathcal{U}} (U^c \cup V) \\ &\stackrel{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X, \end{aligned}$$

where we have used:

- 0217 1. Item 12 of Proposition 4.3.6.1.2 for the second equality.
- 0218 2. Item 6 of Proposition 4.3.7.1.2 for the third equality.

This finishes the proof.

Item 12: Interaction With Intersections of Families of Subsets III

We have

$$\begin{aligned} \bigcap_{V \in \mathcal{V}} [U, V]_X &\stackrel{\text{def}}{=} \bigcap_{V \in \mathcal{V}} (U^c \cup V) \\ &= U^c \cup \left(\bigcap_{V \in \mathcal{V}} V \right) \\ &\stackrel{\text{def}}{=} \left[U, \bigcap_{V \in \mathcal{V}} V \right]_X. \end{aligned}$$

where we have used **Item 6**. This finishes the proof.

Item 13: Interaction With Binary Unions

We have

$$\begin{aligned}
 [U \cap V, W]_X &\stackrel{\text{def}}{=} (U \cap V)^c \cup W \\
 &= (U^c \cup V^c) \cup W \\
 &= (U^c \cup V^c) \cup (W \cup W) \\
 &= (U^c \cup W) \cup (V^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cup [V, W]_X,
 \end{aligned}$$

where we have used:

- 0219 1. **Item 2** of **Proposition 4.3.11.1.2** for the second equality.
- 021A 2. **Item 8** of **Proposition 4.3.8.1.2** for the third equality.
- 021B 3. Several applications of **Items 2** and **4** of **Proposition 4.3.8.1.2** and for the fourth equality.

For the second equality in the statement, we have

$$\begin{aligned}
 [U, V \cap W]_X &\stackrel{\text{def}}{=} U^c \cup (V \cap W) \\
 &= (U^c \cup V) \cap (U^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, V]_X \cap [U, W]_X,
 \end{aligned}$$

where we have used **Item 6** of **Proposition 4.3.8.1.2** for the second equality.

Item 14: Interaction With Binary Intersections

We have

$$\begin{aligned}
 [U \cup V, W]_X &\stackrel{\text{def}}{=} (U \cup V)^c \cup W \\
 &= (U^c \cap V^c) \cup W \\
 &= (U^c \cup W) \cap (V^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cap [V, W]_X,
 \end{aligned}$$

where we have used:

- 021C 1. **Item 2** of **Proposition 4.3.11.1.2** for the second equality.

021D 2. Item 6 of Proposition 4.3.8.1.2 for the third equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
 [U, V \cup W]_X &\stackrel{\text{def}}{=} U^c \cup (V \cup W) \\
 &= (U^c \cup U^c) \cup (V \cup W) \\
 &= (U^c \cup V) \cup (U^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, V]_X \cup [U, W]_X,
 \end{aligned}$$

where we have used:

021E 1. Item 8 of Proposition 4.3.8.1.2 for the second equality.

021F 2. Several applications of Items 2 and 4 of Proposition 4.3.8.1.2 and for the third equality.

This finishes the proof.

Item 15: Interaction With Differences

We have

$$\begin{aligned}
 [U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^c \cup W \\
 &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
 &= ((X \cap V) \cup (X \setminus U)) \cup W \\
 &= (V \cup (X \setminus U)) \cup W \\
 &\stackrel{\text{def}}{=} (V \cup U^c) \cup W \\
 &= (V \cup (U^c \cup U^c)) \cup W \\
 &= (U^c \cup W) \cup (U^c \cup V) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cup [U, V]_X,
 \end{aligned}$$

where we have used:

021G 1. Item 10 of Proposition 4.3.10.1.2 for the third equality.

021H 2. Item 4 of Proposition 4.3.9.1.2 for the fourth equality.

021J 3. Item 8 of Proposition 4.3.8.1.2 for the sixth equality.

021K 4. Several applications of Items 2 and 4 of Proposition 4.3.8.1.2 and for the seventh equality.

We also have

$$\begin{aligned}
 [U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^c \cup W \\
 &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
 &= ((X \cap V) \cup (X \setminus U)) \cup W \\
 &= (V \cup (X \setminus U)) \cup W \\
 &\stackrel{\text{def}}{=} (V \cup U^c) \cup W \\
 &= (V \cup U^c) \cup (W \cup W) \\
 &= (U^c \cup W) \cup (V \cup W) \\
 &= (U^c \cup W) \cup ((V^c)^c \cup W) \\
 &\stackrel{\text{def}}{=} [U, W]_X \cup [V^c, W]_X,
 \end{aligned}$$

where we have used:

- 021L 1. **Item 10** of **Proposition 4.3.10.1.2** for the third equality.
- 021M 2. **Item 4** of **Proposition 4.3.9.1.2** for the fourth equality.
- 021N 3. **Item 8** of **Proposition 4.3.8.1.2** for the sixth equality.
- 021P 4. Several applications of **Items 2** and **4** of **Proposition 4.3.8.1.2** and for the seventh equality.
- 021Q 5. **Item 3** of **Proposition 4.3.11.1.2** for the eighth equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
 [U, V \setminus W]_X &\stackrel{\text{def}}{=} U^c \cup (V \setminus W) \\
 &= (V \setminus W) \cup U^c \\
 &= (V \cup U^c) \setminus (W \setminus U^c) \\
 &\stackrel{\text{def}}{=} (V \cup U^c) \setminus (W \setminus (X \setminus U)) \\
 &= (V \cup U^c) \setminus ((W \cap U) \cup (W \setminus X)) \\
 &= (V \cup U^c) \setminus ((W \cap U) \cup \emptyset) \\
 &= (V \cup U^c) \setminus (W \cap U) \\
 &= (V \cup U^c) \setminus (U \cap W) \\
 &\stackrel{\text{def}}{=} [U, V]_X \setminus (U \cap W)
 \end{aligned}$$

where we have used:

- 021R 1. Item 4 of Proposition 4.3.8.1.2 for the second equality.
- 021S 2. Item 4 of Proposition 4.3.10.1.2 for the third equality.
- 021T 3. Item 10 of Proposition 4.3.10.1.2 for the fifth equality.
- 021U 4. Item 13 of Proposition 4.3.10.1.2 for the sixth equality.
- 021V 5. Item 3 of Proposition 4.3.8.1.2 for the seventh equality.
- 021W 6. Item 5 of Proposition 4.3.9.1.2 for the eighth equality.

This finishes the proof.

Item 16: Interaction With Complements

We have

$$\begin{aligned} [U^c, V]_X &\stackrel{\text{def}}{=} (U^c)^c \cup V, \\ &= U \cup V, \end{aligned}$$

where we have used Item 3 of Proposition 4.3.11.1.2. We also have

$$\begin{aligned} [U, V^c]_X &\stackrel{\text{def}}{=} U^c \cup V^c \\ &= U \cap V \end{aligned}$$

where we have used Item 2 of Proposition 4.3.11.1.2. Finally, we have

$$\begin{aligned} [U, V]_X^c &= ((U \setminus V)^c)^c \\ &= U \setminus V, \end{aligned}$$

where we have used Item 2 of Proposition 4.3.11.1.2.

Item 17: Interaction With Characteristic Functions

We have

$$\begin{aligned} \chi_{[U, V]_{\mathcal{P}(X)}}(x) &\stackrel{\text{def}}{=} \chi_{U^c \cup V}(x) \\ &= \max(\chi_{U^c}, \chi_V) \\ &= \max(1 - \chi_U \pmod{2}, \chi_V), \end{aligned}$$

where we have used:

021X 1. **Item 10** of **Proposition 4.3.8.1.2** for the second equality.

021Y 2. **Item 4** of **Proposition 4.3.11.1.2** for the third equality.

This finishes the proof.

Item 18: Interaction With Direct Images

This is a repetition of **Item 10** of **Proposition 4.6.1.1.5** and is proved there.

Item 19: Interaction With Inverse Images

This is a repetition of **Item 10** of **Proposition 4.6.2.1.3** and is proved there.

Item 20: Interaction With Codirect Images

This is a repetition of **Item 9** of **Proposition 4.6.3.1.7** and is proved there. 

01L1 4.4.8 Isbell Duality for Sets

Let X be a set.

01L2 DEFINITION 4.4.8.1.1 ► THE ISBELL FUNCTION

The **Isbell function** of X is the map

$$I: \mathcal{P}(X) \rightarrow \text{Sets}(X, \mathcal{P}(X))$$

defined by

$$I(U) \stackrel{\text{def}}{=} \llbracket x \mapsto [U, \{x\}]_X \rrbracket$$

for each $U \in \mathcal{P}(X)$.

01L3 REMARK 4.4.8.1.2 ► MOTIVATION FOR THE ISBELL FUNCTION

Recall from **Remark 4.4.1.1.2** that we may view the powerset $\mathcal{P}(X)$ of a set X as the decategorification of the category of presheaves $\text{PSh}(C)$ of a category C . Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\text{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \text{Nat}(\mathcal{F}, h_{(-)})$$

for each $\mathcal{F} \in \text{Obj}(\text{PSh}(C))$. To this end, we could define

$$I(U) \stackrel{\text{def}}{=} [U, \chi_{(-)}]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of C into $\mathbf{PSh}(C)$ with the characteristic embedding $x \mapsto \chi_x$ of X into $\mathcal{P}(X)$ of [Definition 4.5.4.1.1](#).
- The internal Hom Nat of $\mathbf{PSh}(C)$ with the internal Hom $[-, -]_X$ of $\mathcal{P}(X)$ of [Proposition 4.4.7.1.1](#).

However, since $[U, \chi_x]_X$ is a subset of U instead of a truth value, we get a function

$$l: \mathcal{P}(X) \rightarrow \mathbf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$l: \mathcal{P}(X) \rightarrow \mathcal{P}(X).$$

This makes some of the properties involving l a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $l_! \circ l$ evaluates to $\text{id}_{\mathcal{P}(X)}$ in the sense of [Proposition 4.4.8.1.3](#).

01L4

PROPOSITION 4.4.8.1.3 ► ISBELL DUALITY FOR SETS

The diagram

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{l} & \mathbf{Sets}(X, \mathcal{P}(X)) \\ & \searrow \Delta_{\text{id}_{\mathcal{P}(X)}} & \downarrow l_! \\ & & \mathbf{Sets}(X, \mathbf{Sets}(X, \mathcal{P}(X))) \end{array}$$

commutes, i.e. we have

$$l_!(l(U)) = \llbracket x \mapsto \llbracket y \mapsto U \rrbracket \rrbracket$$

for each $U \in \mathcal{P}(X)$.

PROOF 4.4.8.1.4 ► PROOF OF PROPOSITION 4.4.8.1.3

We have

$$\begin{aligned} l_!(l(U)) &\stackrel{\text{def}}{=} l_!(\llbracket x \mapsto U^c \cup \{x\} \rrbracket) \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto l(U^c \cup \{x\}) \rrbracket \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \llbracket y \mapsto (U^c \cup \{x\})^c \cup \{x\} \rrbracket \rrbracket \end{aligned}$$

$$\begin{aligned}
&= \llbracket x \mapsto \llbracket y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\} \rrbracket \rrbracket \\
&= \llbracket x \mapsto \llbracket y \mapsto (U \setminus \{x\}) \cup \{x\} \rrbracket \rrbracket \\
&= \llbracket x \mapsto \llbracket y \mapsto U \rrbracket \rrbracket,
\end{aligned}$$

where we have used **Item 2** of **Proposition 4.3.11.1.2** for the fourth equality above. 

01L5 4.5 Characteristic Functions

005X 4.5.1 The Characteristic Function of a Subset

Let X be a set and let $U \in \mathcal{P}(X)$.

005Z DEFINITION 4.5.1.1.1 ► THE CHARACTERISTIC FUNCTION OF A SUBSET

The **characteristic function of U** ¹ is the function $\chi_U: X \rightarrow \{t, f\}$ ² defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

¹*Further Terminology:* Also called the **indicator function of U** .

²*Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

01L6

REMARK 4.5.1.1.2 ► CHARACTERISTIC FUNCTIONS OF SUBSETS AS DECATEGORYIFICATIONS OF PRESHEAVES

Under the analogy that $\{t, f\}$ should be the (-1) -categorical analogue of Sets, we may view a function

$$f: X \rightarrow \{t, f\}$$

as a decategorification of presheaves and copresheaves

$$\mathcal{F}: C^{\text{op}} \rightarrow \text{Sets},$$

$$F: C \rightarrow \text{Sets}.$$

The characteristic functions χ_U of the subsets of X are then the primordial examples of such functions (and, in fact, all of them).

01L7

NOTATION 4.5.1.1.3 ► FURTHER NOTATION FOR CHARACTERISTIC FUNCTIONS

We will often employ the bijection $\{t, f\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when discussing characteristic functions. Examples of this include **Items 4 to 11** of **Proposition 4.5.1.1.4** below.

0069

PROPOSITION 4.5.1.1.4 ► PROPERTIES OF CHARACTERISTIC FUNCTIONS OF SUBSETS

Let X be a set.

01L8

1. *Functionality*. The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)} : \mathcal{P}(X) \rightarrow \text{Sets}(X, \{t, f\}).$$

01L9

2. *Bijectivity*. The function $\chi_{(-)}$ from **Item 1** is bijective.

01LA

3. *Naturality*. The collection

$$\{\chi_{(-)} : \mathcal{P}(X) \rightarrow \text{Sets}(X, \{t, f\})\}_{X \in \text{Obj}(\text{Sets})}$$

defines a natural isomorphism between \mathcal{P}^{-1} and $\text{Sets}(-, \{t, f\})$. In particular, given a function $f : X \rightarrow Y$, the diagram

$$\begin{array}{ccc} \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \\ \chi_{(-)} \downarrow \wr & & \wr \downarrow \chi_{(-)} \\ \text{Sets}(Y, \{t, f\}) & \xrightarrow{f^*} & \text{Sets}(X, \{t, f\}) \end{array}$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

006B

4. *Interaction With Unions I*. We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

006C

5. *Interaction With Unions II*. We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

006D 6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

006E 7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

006F 8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

006G 9. *Interaction With Complements.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

006H 10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \Delta V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \Delta V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

01LB 11. *Interaction With Internal Homs.* We have

$$\chi_{[U, V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod{2}, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

PROOF 4.5.1.1.5 ► PROOF OF PROPOSITION 4.5.1.1.4

Item 1: Functionality

There is nothing to prove.

Item 2: Bijectivity

We proceed in three steps:

021Z 1. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi: \text{Sets}(X, \{t, f\}) \xrightarrow{\sim} \mathcal{P}(X),$$

defined by

$$\begin{aligned} \Phi(f) &\stackrel{\text{def}}{=} U_f \\ &\stackrel{\text{def}}{=} f^{-1}(\text{true}) \\ &\stackrel{\text{def}}{=} \{x \in X \mid f(x) = \text{true}\} \end{aligned}$$

for each $f \in \text{Sets}(X, \{t, f\})$.

0220 2. *Invertibility I.* We have

$$\begin{aligned} [\Phi \circ \chi_{(-)}](U) &\stackrel{\text{def}}{=} \Phi(\chi_U) \\ &\stackrel{\text{def}}{=} \chi_U^{-1}(\text{true}) \\ &\stackrel{\text{def}}{=} \{x \in X \mid \chi_U(x) = \text{true}\} \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U\} \\ &= U \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathcal{P}(X)}](U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus, we have

$$\Phi \circ \chi_{(-)} = \text{id}_{\mathcal{P}(X)}.$$

0221 3. *Invertibility II.* We have

$$\begin{aligned} [\chi_{(-)} \circ \Phi](U) &\stackrel{\text{def}}{=} \chi_{\Phi(f)} \\ &\stackrel{\text{def}}{=} \chi_{f^{-1}(\text{true})} \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \text{true} & \text{if } x \in f^{-1}(\text{true}) \\ \text{false} & \text{otherwise} \end{cases} \rrbracket \end{aligned}$$

$$\begin{aligned}
&= \llbracket x \mapsto f(x) \rrbracket \\
&= f \\
&\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(X, \{t, f\})}](f)
\end{aligned}$$

for each $f \in \text{Sets}(X, \{t, f\})$. Thus, we have

$$\chi_{(-)} \circ \Phi = \text{id}_{\text{Sets}(X, \{t, f\})}.$$

This finishes the proof.

Item 3: Naturality

We proceed in two steps:

0222

1. *Naturality of $\chi_{(-)}$.* We have

$$\begin{aligned}
[\chi_V \circ f](v) &\stackrel{\text{def}}{=} \chi_V(f(v)) \\
&= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases} \\
&\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v)
\end{aligned}$$

for each $v \in V$.

0223

2. *Naturality of Φ .* Since $\chi_{(-)}$ is natural and a componentwise inverse to Φ , it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that Φ is also natural in each argument.

This finishes the proof.

Item 4: Interaction With Unions I

This is a repetition of **Item 10** of **Proposition 4.3.8.1.2** and is proved there.

Item 5: Interaction With Unions II

This is a repetition of **Item 11** of **Proposition 4.3.8.1.2** and is proved there.

Item 6: Interaction With Intersections I

This is a repetition of **Item 10** of **Proposition 4.3.9.1.2** and is proved there.

Item 7: Interaction With Intersections II

This is a repetition of **Item 11** of **Proposition 4.3.9.1.2** and is proved there.

Item 8: Interaction With Differences

This is a repetition of **Item 16** of **Proposition 4.3.10.1.2** and is proved there.

Item 9: Interaction With Complements

This is a repetition of **Item 4** of **Proposition 4.3.11.1.2** and is proved there.

Item 10: Interaction With Symmetric Differences

This is a repetition of **Item 15** of **Proposition 4.3.12.1.2** and is proved there.

Item 11: Interaction With Internal Homs

This is a repetition of **Item 17** of **Proposition 4.4.7.1.4** and is proved there.



0224

REMARK 4.5.1.1.6 ► POWERSETS AS SETS OF FUNCTIONS AND UN/STRAIGHTENING

The bijection

$$\mathcal{P}(X) \cong \text{Sets}(X, \{t, f\})$$

of **Item 2** of **Proposition 4.5.1.1.4**, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U: X \rightarrow \{\text{true}, \text{false}\}$;
- Takes a function $f: X \rightarrow \{\text{true}, \text{false}\}$ and *unstraightens* it to a subset $f^{-1}(\text{true}) \hookrightarrow X$ of X ;

may be viewed as the (-1) -categorical version of the 0-categorical un/s-traightening isomorphism between indexed and fibred sets

$$\underbrace{\text{FibSets}_X}_{\text{def} \equiv \text{Sets}_{/X}} \cong \underbrace{\text{ISets}_X}_{\text{def} \equiv \text{Fun}(X_{\text{disc}}, \text{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, ???. Here we view:

- Subsets $U \hookrightarrow X$ as being analogous to X -fibred sets $\phi_X: A \rightarrow X$.
- Functions $f: X \rightarrow \{t, f\}$ as being analogous to X -indexed sets $A: X_{\text{disc}} \rightarrow \text{Sets}$.

01LC 4.5.2 The Characteristic Function of a Point

Let X be a set and let $x \in X$.

0060

The **characteristic function** of x is the function¹

$$\chi_x: X \rightarrow \{t, f\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

¹Further Notation: Also written χ^x , $\chi_X(x, -)$, or $\chi_X(-, x)$.

01LD

REMARK 4.5.2.1.2 ► CHARACTERISTIC FUNCTIONS OF POINTS AS DECATEGORIFICATIONS OF REPRESENTABLE PRESHEAVES

Expanding upon **Remark 4.5.1.1.2**, we may think of the characteristic function

$$\chi_x: X \rightarrow \{t, f\}$$

of an *element* x of X as a decategorification of the representable presheaf and of the representable copresheaf

$$h_X: C^{\text{op}} \rightarrow \text{Sets},$$

$$h^X: C \rightarrow \text{Sets}$$

associated of an *object* X of a category C .

01LE 4.5.3 The Characteristic Relation of a Set

Let X be a set.

0061

DEFINITION 4.5.3.1.1 ► THE CHARACTERISTIC RELATION OF A SET

The **characteristic relation** on X ¹ is the relation²

$$\chi_X(-, -): X \times X \rightarrow \{t, f\}$$

on X defined by³

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

¹*Further Terminology:* Also called the **identity relation** on X .

²*Further Notation:* Also written χ_X^{-1} , or \sim_{id} in the context of relations.

³Under the bijection $\text{Sets}(X \times X, \{\text{t}, \text{f}\}) \cong \mathcal{P}(X \times X)$ of **Item 2** of **Proposition 4.5.1.1.4**, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X .

01LF

REMARK 4.5.3.1.2 ► THE CHARACTERISTIC RELATION OF A SET AS A DECATEGORIFICATION OF THE HOM PROFUNCTOR

Expanding upon **Remarks 4.5.1.1.2** and **4.5.2.1.2**, we may view the characteristic relation

$$\chi_X(-1, -2): X \times X \rightarrow \{\text{t}, \text{f}\}$$

of X as a decategorification of the Hom profunctor

$$\text{Hom}_C(-1, -2): C^{\text{op}} \times C \rightarrow \text{Sets}$$

of a category C .

01LG

PROPOSITION 4.5.3.1.3 ► PROPERTIES OF CHARACTERISTIC RELATIONS

Let $f: X \rightarrow Y$ be a function.

006A

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f: A \rightarrow B$ be a function. We have an inclusion¹


$$\chi_B \circ (f \times f) \subset \chi_A,$$

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ & \searrow \chi_A \quad \hookrightarrow \quad \swarrow \chi_B & \\ & \{ \text{t}, \text{f} \}. & \end{array}$$

¹*Note:* This is the 0-categorical version of **Categories, Definition 11.5.4.1.1**.

PROOF 4.5.3.1.4 ► PROOF OF PROPOSITION 4.5.3.1.3

Item 1: The Inclusion of Characteristic Relations Associated to a Function

The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement “if $a = b$, then $f(a) = f(b)$ ”, which is true. 

01LH 4.5.4 The Characteristic Embedding of a Set

Let X be a set.

0062 DEFINITION 4.5.4.1.1 ► THE CHARACTERISTIC EMBEDDING OF A SET

The **characteristic embedding**¹ of X into $\mathcal{P}(X)$ is the function

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

defined by²

$$\begin{aligned} \chi_{(-)}(x) &\stackrel{\text{def}}{=} \chi_x \\ &= \{x\} \end{aligned}$$

for each $x \in X$.

¹The name “characteristic embedding” is justified by [Corollary 4.5.5.1.3](#), which gives an analogue of fully faithfulness for $\chi_{(-)}$.

²Here we are identifying $\mathcal{P}(X)$ with $\text{Sets}(X, \{t, f\})$ as per [Item 2 of Proposition 4.5.1.1.4](#).

01LJ

REMARK 4.5.4.1.2 ► THE CHARACTERISTIC EMBEDDING OF A SET AS A DECATEGORIFICATION OF THE YONEDA EMBEDDING

Expanding upon [Remarks 4.5.1.1.2](#), [4.5.2.1.2](#) and [4.5.3.1.2](#), we may view the characteristic embedding

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$\mathfrak{Y} : C^{\text{op}} \hookrightarrow \text{PSh}(C)$$

of a category C into $\text{PSh}(C)$.

01LK

PROPOSITION 4.5.4.1.3 ► PROPERTIES OF CHARACTERISTIC EMBEDDINGS

Let $f: X \rightarrow Y$ be a map of sets.

01LL

1. *Interaction With Functions.* We have


$$f_! \circ \chi_X = \chi_Y \circ f,$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \chi_X \downarrow & & \downarrow \chi_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y). \end{array}$$

PROOF 4.5.4.1.4 ► PROOF OF PROPOSITION 4.5.4.1.3**Item 1: Interaction With Functions**

Indeed, we have

$$\begin{aligned} [f_! \circ \chi_X](x) &\stackrel{\text{def}}{=} f_!(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f_!(\{x\}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x), \end{aligned}$$

for each $x \in X$, showing the desired equality. 

006K 4.5.5 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X .

006L

PROPOSITION 4.5.5.1.1 ► THE YONEDA LEMMA FOR SETS

We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi(-), \chi_U) = \chi_U,$$


where

$$\chi_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } U \subset V, \\ \text{false} & \text{otherwise.} \end{cases}$$

PROOF 4.5.5.1.2 ► PROOF OF PROPOSITION 4.5.5.1.1

We have

$$\begin{aligned}\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) &\stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } \{x\} \subset U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } x \in U \\ \text{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_U(x).\end{aligned}$$

This finishes the proof. 

006M

COROLLARY 4.5.5.1.3 ► THE CHARACTERISTIC EMBEDDING IS FULLY FAITHFUL

The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) \cong \chi_X(x, y)$$

for each $x, y \in X$.

PROOF 4.5.5.1.4 ► PROOF OF COROLLARY 4.5.5.1.3

We have

$$\begin{aligned}\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) &= \chi_y(x) \\ &\stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in \{y\} \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } x = y \\ \text{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_X(x, y).\end{aligned}$$

where we have used **Proposition 4.5.5.1.1** for the first equality. 

01LM **4.6 The Adjoint Triple** $f_! \dashv f^{-1} \dashv f_*$ 007F **4.6.1 Direct Images**

Let $f: X \rightarrow Y$ be a function.

DEFINITION 4.6.1.1 (Direct Images)

007G

The **direct image function** associated to f is the function¹

$$f_! : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by²

$$\begin{aligned} f_!(U) &\stackrel{\text{def}}{=} \left\{ y \in Y \mid \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\} \\ &= \{f(x) \in Y \mid x \in U\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

¹*Further Notation:* Also written simply $f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$.

²*Further Terminology:* The set $f(U)$ is called the **direct image of U by f** .

007H

NOTATION 4.6.1.1.2 ► FURTHER NOTATION FOR DIRECT IMAGES

Sometimes one finds the notation

$$\exists_f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \exists_f(U)$.
- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

0225

WARNING 4.6.1.1.3 ► NOTATION FOR DIRECT IMAGES IS CONFUSING

Notation for direct images between powersets is tricky:

- 0226 1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:

- 0227 (a) *Powersets*. Given a function $f: X \rightarrow Y$, we have an inverse image functor

$$f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X).$$

The *left* adjoint of this functor is the usual direct image, defined above in **Definition 4.6.1.1.1**.

- 0228 (b) *Presheaves*. Given a morphism of topological spaces $f: X \rightarrow Y$, we have an inverse image functor

$$f^{-1}: \text{PSh}(Y) \rightarrow \text{PSh}(X).$$

The *right* adjoint of this functor is the direct image functor of presheaves, defined in ??.



- 0229 2. The presheaf direct image functor is denoted f_* , but the direct image functor for powersets is denoted $f_!$ (as it's a left adjoint).
- 022A 3. Adding to the confusion, it's somewhat common for $f_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ to be denoted f_* .

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

SITUATION	ADJOINT STRING
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*): \mathcal{P}(X) \xrightarrow{\cong} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*): \text{PSh}(X) \xrightarrow{\cong} \text{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*): C_X \xrightarrow{\cong} C_Y$
Kan Extensions	$(F_! \dashv F^* \dashv F_*): \text{Fun}(C, \mathcal{E}) \xrightarrow{\cong} \text{Fun}(D, \mathcal{E})$

007J

REMARK 4.6.1.1.4 ► UNWINDING DEFINITION 4.6.1.1.1

Identifying $\mathcal{P}(X)$ with $\text{Sets}(X, \{t, f\})$ via **Item 2** of **Proposition 4.5.1.1.4**, we see that the direct image function associated to f is equivalently the function

$$f_! : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \text{Lan}_f(\chi_U) \\ &= \text{colim}((f \times \vec{(-1)}) \xrightarrow{\text{pr}} A \xrightarrow{\chi_U} \{t, f\}) \\ &= \text{colim}_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)) \\ &= \bigvee_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)), \end{aligned}$$

where we have used ?? for the second equality. In other words, we have

$$\begin{aligned} [f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \text{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $y \in Y$.

007K

PROPOSITION 4.6.1.1.5 ► PROPERTIES OF DIRECT IMAGES I

Let $f: X \rightarrow Y$ be a function.

007L

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! : (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f_!(U) \subset f_!(V).$$

007M

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \mathcal{P}(X) \begin{matrix} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{matrix} \mathcal{P}(Y),$$

witnessed by:

01LN

(a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01LP

(b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

01LQ

i. The following conditions are equivalent:

01LR

A. We have $f_!(U) \subset V$.

01LS

B. We have $U \subset f^{-1}(V)$.

01LT

ii. The following conditions are equivalent:

01LU

A. We have $f^{-1}(U) \subset V$.

01LV

B. We have $U \subset f_*(V)$.

01LW

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)!} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01LX

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

01LY

5. *Interaction With Binary Unions.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

01LZ

6. *Interaction With Binary Intersections.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & \subset & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01M0

7. *Interaction With Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \setminus & \supset & \downarrow \setminus \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01M1

8. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(U^c) = f_*(U)^c$$

for each $U \in \mathcal{P}(X)$.

01M2

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow \Delta & \supset & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

with components

$$f_!(U) \Delta f_!(V) \subset f_!(U \Delta V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01M3

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_!} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \downarrow [-1, -2]_X & & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

007N

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_!(U) \cup f_!(V) &= f_!(U \cup V), \\ f_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

007P

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U \cap V) &\subset f_!(U) \cap f_!(V), \\ f_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

007Q

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_!, f_!^\otimes, f_{!|\mathbb{1}}^\otimes): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$f_{!|U,V}^\otimes: f_!(U) \cup f_!(V) \xrightarrow{=} f_!(U \cup V),$$

$$f_{!|\mathbb{1}}^\otimes: \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(X)$.

007R

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(f_!, f_!^\otimes, f_{!|\mathbb{1}}^\otimes): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$f_{!|U,V}^\otimes: f_!(U \cap V) \hookrightarrow f_!(U) \cap f_!(V),$$

$$f_{!|\mathbb{1}}^\otimes: f_!(X) \hookrightarrow Y,$$

natural in $U, V \in \mathcal{P}(X)$.

007S

15. *Interaction With Coproducts.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \amalg g)_!(U \amalg V) = f_!(U) \amalg g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

007T

16. *Interaction With Products.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

007U

17. *Relation to Codirect Images.* We have

$$\begin{aligned} f_!(U) &= f_*(U^c)^c \\ &\stackrel{\text{def}}{=} Y \setminus f_*(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

PROOF 4.6.1.1.6 ► PROOF OF PROPOSITION 4.6.1.5

Item 1: Functoriality

Omitted.

Item 2: Triple Adjointness

This follows from [Remark 4.6.1.1.4](#), [Remark 4.6.2.1.2](#), [Remark 4.6.3.1.4](#), and Kan Extensions, ?? of ??.

Item 3: Interaction With Unions of Families of Subsets

We have

$$\begin{aligned} \bigcup_{V \in f_!(\mathcal{U})} V &= \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.

Item 4: Interaction With Intersections of Families of Subsets

We have

$$\begin{aligned} \bigcap_{V \in f_!(\mathcal{U})} V &= \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.

Item 5: Interaction With Binary Unions

See [\[Pro25p\]](#).

Item 6: Interaction With Binary Intersections

See [\[Pro25n\]](#).

Item 7: Interaction With Differences

See [\[Pro25o\]](#).

Item 8: Interaction With Complements

Applying [Item 17](#) to $X \setminus U$, we have

$$f_!(U^c) = f_!(X \setminus U)$$

$$\begin{aligned}
&= Y \setminus f_*(X \setminus (X \setminus U)) \\
&= Y \setminus f_*(U) \\
&= f_*(U)^c.
\end{aligned}$$

This finishes the proof.

Item 9: Interaction With Symmetric Differences

We have

$$\begin{aligned}
f_!(U) \triangle f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\
&\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\
&= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\
&\subset f_!((U \cup V) \setminus (U \cap V)) \\
&= f_!(U \triangle V),
\end{aligned}$$

where we have used:

- 022C 1. [Item 2](#) of [Proposition 4.3.12.1.2](#) for the first equality.
- 022D 2. [Item 6](#) of this proposition together with [Item 1](#) of [Proposition 4.3.10.1.2](#) for the first inclusion.
- 022E 3. [Item 5](#) for the second equality.
- 022F 4. [Item 7](#) for the second inclusion.
- 022G 5. [Item 2](#) of [Proposition 4.3.12.1.2](#) for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Proposition 11.2.7.1.2](#)). This finishes the proof.

Item 10: Interaction With Internal Homs of Powersets

We have

$$\begin{aligned}
f_!([U, V]_X) &\stackrel{\text{def}}{=} f_!(U^c \cup V) \\
&= f_!(U^c) \cup f_!(V) \\
&= f_*(U)^c \cup f_!(V) \\
&\stackrel{\text{def}}{=} [f_*(U), f_!(V)]_Y,
\end{aligned}$$

where we have used:

022H

1. **Item 5** for the second equality.

022J

2. **Item 17** for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**Categories**, **Item 4** of **Proposition 11.2.7.1.2**). This finishes the proof.

Item 11: Preservation of Colimits

This follows from **Item 2** and ??, ?? of ??.¹

Item 12: Oplax Preservation of Limits

The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25n] for the other inclusions.

Item 13: Symmetric Strict Monoidality With Respect to Unions

This follows from **Item 11**.

Item 14: Symmetric Oplax Monoidality With Respect to Intersections

The inclusions in the statement follow from **Item 12**. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (**Categories**, **Item 4** of **Proposition 11.2.7.1.2**).

Item 15: Interaction With Coproducts

Omitted.

Item 16: Interaction With Products

Omitted.


Item 17: Relation to Codirect Images

Applying **Item 16** of **Proposition 4.6.3.1.7** to $X \setminus U$, we have

$$\begin{aligned} f_*(X \setminus U) &= B \setminus f_!(X \setminus (X \setminus U)) \\ &= B \setminus f_!(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_!(U) &= B \setminus (B \setminus f_!(U)), \\ &= B \setminus f_*(X \setminus U), \end{aligned}$$

which finishes the proof. 

¹Reference: [Pro25p].

007V PROPOSITION 4.6.1.1.7 ► PROPERTIES OF DIRECT IMAGES II

Let $f: X \rightarrow Y$ be a function.

007W 1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

007X 2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

007Y 3. *Interaction With Identities.* For each $X \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_X)_! = \text{id}_{\mathcal{P}(X)}.$$

007Z 4. *Interaction With Composition.* For each pair of composable functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \\ & \searrow (g \circ f)_! & \downarrow g_! \\ & & \mathcal{P}(Z). \end{array}$$

PROOF 4.6.1.1.8 ► PROOF OF PROPOSITION 4.6.1.1.7

Item 1: Functionality I

There is nothing to prove.

Item 2: Functionality II

This follows from **Item 1** of **Proposition 4.6.1.1.5**.

Item 3: Interaction With Identities

This follows from **Remark 4.6.1.1.4** and Kan Extensions, ?? of ??.

Item 4: Interaction With Composition

This follows from **Remark 4.6.1.1.4** and Kan Extensions, ?? of ??.



0080 4.6.2 Inverse Images

Let $f: X \rightarrow Y$ be a function.

0081

The **inverse image function** associated to f is the function¹

$$f^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by²

$$f^{-1}(V) \stackrel{\text{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each $V \in \mathcal{P}(Y)$.

¹Further Notation: Also written $f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$.

²Further Terminology: The set $f^{-1}(V)$ is called the **inverse image of V by f** .

0082

REMARK 4.6.2.1.2 ► UNWINDING DEFINITION 4.6.2.1.1

Identifying $\mathcal{P}(Y)$ with $\text{Sets}(Y, \{\text{t}, \text{f}\})$ via **Item 2** of **Proposition 4.5.1.1.4**, we see that the inverse image function associated to f is equivalently the function

$$f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\text{true}, \text{false}\}$$

in Sets .

0083

PROPOSITION 4.6.2.1.3 ► PROPERTIES OF INVERSE IMAGES I

Let $f: X \rightarrow Y$ be a function.

0084

1. *Functoriality*. The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}: (\mathcal{P}(Y), \subset) \rightarrow (\mathcal{P}(X), \subset).$$

In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

$$(\star) \text{ If } U \subset V, \text{ then } f^{-1}(U) \subset f^{-1}(V).$$

0085

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*): \mathcal{P}(X) \begin{array}{c} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{array} \mathcal{P}(Y),$$

witnessed by:

01M4

(a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01M5

(b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

01M6

i. The following conditions are equivalent:

01M7

A. We have $f_!(U) \subset V$.

01M8

B. We have $U \subset f^{-1}(V)$.

01M9

ii. The following conditions are equivalent:

01MA

A. We have $f^{-1}(U) \subset V$.

01MB

B. We have $U \subset f_*(V)$.

01MC

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

01MD

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

01ME

5. *Interaction With Binary Unions.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cup \downarrow & & \downarrow \cup \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01MF

6. *Interaction With Binary Intersections.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

01MG

7. *Interaction With Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow \setminus & & \downarrow \setminus \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

01MH

8. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(U^c) = f^{-1}(U)^c$$

for each $U \in \mathcal{P}(X)$.

01MJ

9. *Interaction With Symmetric Differences.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\text{op}, -1} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \Delta \downarrow & & \downarrow \Delta \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

i.e. we have

$$f^{-1}(U) \triangle f^{-1}(V) = f^{-1}(U \triangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

01MK

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\ \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

0086

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

0087

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(Y) &= X, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

0088

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{1}}^{-1, \otimes}): (\mathcal{P}(Y), \cup, \emptyset) \rightarrow (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1, \otimes}: f^{-1}(U) \cup f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cup V), \\ f_{\mathbb{1}}^{-1, \otimes}: \emptyset &\xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

0089

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of **Item 1** has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1, \otimes}, f_{\mathbb{1}}^{-1, \otimes}): (\mathcal{P}(Y), \cap, Y) \rightarrow (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$\begin{aligned} f_{U,V}^{-1, \otimes}: f^{-1}(U) \cap f^{-1}(V) &\xrightarrow{=} f^{-1}(U \cap V), \\ f_{\mathbb{1}}^{-1, \otimes}: X &\xrightarrow{=} f^{-1}(Y), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

008A

15. *Interaction With Coproducts.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \amalg g)^{-1}(U' \amalg V') = f^{-1}(U') \amalg g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

008B

16. *Interaction With Products.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

PROOF 4.6.2.1.4 ► PROOF OF PROPOSITION 4.6.2.1.3

Item 1: Functoriality

Omitted.

Item 2: Triple Adjointness

This follows from [Remark 4.6.1.1.4](#), [Remark 4.6.2.1.2](#), [Remark 4.6.3.1.4](#), and Kan Extensions, ?? of ??.

Item 3: Interaction With Unions of Families of Subsets

We have

$$\begin{aligned} \bigcup_{U \in f^{-1}(\mathcal{V})} U &= \bigcup_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcup_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.

Item 4: Interaction With Intersections of Families of Subsets

We have

$$\begin{aligned} \bigcap_{U \in f^{-1}(\mathcal{V})} U &= \bigcap_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcap_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.

Item 5: Interaction With Binary Unions

See [\[Pro25y\]](#).

Item 6: Interaction With Binary Intersections

See [\[Pro25w\]](#).

Item 7: Interaction With Differences

See [\[Pro25x\]](#).

Item 8: Interaction With Complements

See [\[Pro25j\]](#).

Item 9: Interaction With Symmetric Differences

We have

$$\begin{aligned}
 f^{-1}(U \Delta V) &= f^{-1}((U \cup V) \setminus (U \cap V)) \\
 &= f^{-1}(U \cup V) \setminus f^{-1}(U \cap V) \\
 &= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U \cap V) \\
 &= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U) \cap f^{-1}(V) \\
 &= f^{-1}(U) \Delta f^{-1}(V),
 \end{aligned}$$

where we have used:

- 022K 1. [Item 2](#) of [Proposition 4.3.12.1.2](#) for the first equality.
- 022L 2. [Item 7](#) for the second equality.
- 022M 3. [Item 5](#) for the third equality.
- 022N 4. [Item 6](#) for the fourth equality.
- 022P 5. [Item 2](#) of [Proposition 4.3.12.1.2](#) for the fifth equality.

This finishes the proof.

Item 10: Interaction With Internal Homs of Powersets

We have

$$\begin{aligned}
 f^{-1}([U, V]_Y) &\stackrel{\text{def}}{=} f^{-1}(U^c \cup V) \\
 &= f^{-1}(U^c) \cup f^{-1}(V) \\
 &= f^{-1}(U)^c \cup f^{-1}(V) \\
 &\stackrel{\text{def}}{=} [f^{-1}(U), f^{-1}(V)]_X,
 \end{aligned}$$

where we have used:

- 022Q 1. [Item 8](#) for the second equality.
- 022R 2. [Item 5](#) for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Proposition 11.2.7.1.2](#)). This finishes the proof.

Item 11: Preservation of Colimits

This follows from **Item 2** and ??, ?? of ??.¹

Item 12: Preservation of Limits

This follows from **Item 2** and ??, ?? of ??.²

Item 13: Symmetric Strict Monoidality With Respect to Unions

This follows from **Item 11**.

Item 14: Symmetric Strict Monoidality With Respect to Intersections

This follows from **Item 12**.

Item 15: Interaction With Coproducts

Omitted.

Item 16: Interaction With Products

Omitted.

¹Reference: [Pro25y].

²Reference: [Pro25w].

008C

PROPOSITION 4.6.2.1.5 ► PROPERTIES OF INVERSE IMAGES II

Let $f: X \rightarrow Y$ be a function.

008D

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{X,Y}^{-1}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

008E

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{X,Y}^{-1}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

008F

3. *Interaction With Identities.* For each $X \in \text{Obj}(\text{Sets})$, we have

$$\text{id}_X^{-1} = \text{id}_{\mathcal{P}(X)}.$$

008G

4. *Interaction With Composition.* For each pair of composable functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y) \\ & \searrow (g \circ f)^{-1} & \downarrow f^{-1} \\ & & \mathcal{P}(X). \end{array}$$

|

|

Item 1: Functionality I

There is nothing to prove.


Item 2: Functionality II

This follows from **Item 1** of **Proposition 4.6.2.1.3**.

Item 3: Interaction With Identities

This follows from **Remark 4.6.2.1.2** and **Categories, Item 5** of **Proposition 11.1.4.1.2**.

Item 4: Interaction With Composition

This follows from **Remark 4.6.2.1.2** and **Categories, Item 2** of **Proposition 11.1.4.1.2**. 

008H 4.6.3 Codirect Images

Let $f: X \rightarrow Y$ be a function.

008J DEFINITION 4.6.3.1.1 ► CODIRECT IMAGES

The **codirect image function associated to f** is the function

$$f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by^{1,2}

$$\begin{aligned} f_*(U) &\stackrel{\text{def}}{=} \left\{ y \in Y \mid \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right\} \\ &= \{ y \in Y \mid \text{we have } f^{-1}(y) \subset U \} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

¹*Further Terminology:* The set $f_*(U)$ is called the **codirect image of U by f** .

²We also have

$$\begin{aligned} f_*(U) &= f_!(U^c)^c \\ &\stackrel{\text{def}}{=} Y \setminus f_!(X \setminus U); \end{aligned}$$

see **Item 16** of **Proposition 4.6.3.1.7**.

008K

NOTATION 4.6.3.1.2 ► FURTHER NOTATION FOR CODIRECT IMAGES

Sometimes one finds the notation

$$\forall_f: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.
- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

WARNING 4.6.3.1.3 ► NOTATION FOR CODIRECT IMAGES IS CONFUSING



022V

See [Warning 4.6.1.1.3](#).

008L

REMARK 4.6.3.1.4 ► UNWINDING DEFINITION 4.6.3.1.1

Identifying $\mathcal{P}(X)$ with $\text{Sets}(X, \{t, f\})$ via [Item 2 of Proposition 4.5.1.1.4](#), we see that the codirect image function associated to f is equivalently the function

$$f_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \text{Ran}_f(\chi_U) \\ &= \lim_{\substack{x \in X \\ f(x) = -1}} ((-1) \times f) \xrightarrow{\text{pr}} X \xrightarrow{\chi_U} \{\text{true}, \text{false}\} \\ &= \lim_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)) \\ &= \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)). \end{aligned}$$

where we have used ?? for the second equality. In other words, we have

$$[f_*(\chi_U)](y) = \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x))$$

$$\begin{aligned}
&= \begin{cases} \text{true} & \text{if, for each } x \in X \text{ such that} \\ & f(x) = y, \text{ we have } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } f^{-1}(y) \subset U \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $y \in Y$.

008M

DEFINITION 4.6.3.1.5 ► THE IMAGE AND COMPLEMENT PARTS OF f_*

Let U be a subset of X .^{1,2}

008N

1. The **image part of the codirect image** $f_*(U)$ of U is the set $f_{*,\text{im}}(U)$ defined by

$$\begin{aligned}
f_{*,\text{im}}(U) &\stackrel{\text{def}}{=} f_*(U) \cap \text{Im}(f) \\
&= \left\{ y \in Y \mid \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{array} \right\}.
\end{aligned}$$

008P

2. The **complement part of the codirect image** $f_*(U)$ of U is the set $f_{*,\text{cp}}(U)$ defined by

$$\begin{aligned}
f_{*,\text{cp}}(U) &\stackrel{\text{def}}{=} f_*(U) \cap (Y \setminus \text{Im}(f)) \\
&= Y \setminus \text{Im}(f) \\
&= \left\{ y \in Y \mid \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{array} \right\} \\
&= \{ y \in Y \mid f^{-1}(y) = \emptyset \}.
\end{aligned}$$

¹Note that we have

$$f_*(U) = f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U),$$

as

$$\begin{aligned}
f_*(U) &= f_*(U) \cap Y \\
&= f_*(U) \cap (\text{Im}(f) \cup (Y \setminus \text{Im}(f))) \\
&= (f_*(U) \cap \text{Im}(f)) \cup (f_*(U) \cap (Y \setminus \text{Im}(f))) \\
&\stackrel{\text{def}}{=} f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U).
\end{aligned}$$

²In terms of the meet computation of $f_*(U)$ of Remark 4.6.3.1.4, namely

$$f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),$$

we see that $f_{*,\text{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\text{cp}}$ corresponds to meets indexed over the empty set.

008Q

EXAMPLE 4.6.3.1.6 ► EXAMPLES OF CODIRECT IMAGES

Here are some examples of codirect images.

0231

1. *Multiplication by Two.* Consider the function $f: \mathbb{N} \rightarrow \mathbb{N}$ given by

$$f(n) \stackrel{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$f_{*,\text{im}}(U) = f_!(U)$$

$$f_{*,\text{cp}}(U) = \{\text{odd natural numbers}\}$$

for any $U \subset \mathbb{N}$. In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

0232

2. *Parabolas.* Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) \stackrel{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{*,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$f_{*,\text{im}}([0, 1]) = \{0\},$$

$$f_{*,\text{im}}([-1, 1]) = [0, 1],$$

$$f_{*,\text{im}}([1, 2]) = \emptyset,$$

$$f_{*,\text{im}}([-2, -1] \cup [1, 2]) = [1, 4].$$

0233

3. *Circles.* Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x, y) \stackrel{\text{def}}{=} x^2 + y^2$$

for each $(x, y) \in \mathbb{R}^2$. We have

$$f_{*,\text{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0, 0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{*,\text{im}}([-1, 1] \times [-1, 1]) &= [0, 1], \\ f_{*,\text{im}}([-1, 1] \times [-1, 1] \setminus [-1, 1] \times \{0\}) &= \emptyset. \end{aligned}$$

008R

PROPOSITION 4.6.3.1.7 ► PROPERTIES OF CODIRECT IMAGES I

Let $f: X \rightarrow Y$ be a function.

008S

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

(★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

008T

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_* \right): \mathcal{P}(X) \begin{matrix} \xrightarrow{f_!} \\ \perp \\ \xleftarrow{f^{-1}} \\ \perp \\ \xrightarrow{f_*} \end{matrix} \mathcal{P}(Y),$$

witnessed by:

01ML

- (a) Units and counits of the form

$$\begin{aligned} \text{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \text{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \text{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \text{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01MM

(b) Bijections of sets

$$\begin{aligned} \text{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \text{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \text{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \text{Hom}_{\mathcal{P}(Y)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

01MN

i. The following conditions are equivalent:

01MP

A. We have $f_!(U) \subset V$.

01MQ

B. We have $U \subset f^{-1}(V)$.

01MR

ii. The following conditions are equivalent:

01MS

A. We have $f^{-1}(U) \subset V$.

01MT

B. We have $U \subset f_*(V)$.

01MU

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cup & & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01MV

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\ \downarrow \cap & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

01MW

5. *Interaction With Binary Unions.* Let $f: X \rightarrow Y$ be a function. We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cup \downarrow & \subset & \downarrow \cup \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01MX

6. *Interaction With Binary Intersections.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\ \cap \downarrow & & \downarrow \cap \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

01MY

7. *Interaction With Complements.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ (-)^c \downarrow & & \downarrow (-)^c \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(U^c) = f_!(U)^c$$

for each $U \in \mathcal{P}(X)$.

01MZ

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ \Delta \downarrow & \wr & \downarrow \Delta \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$f_*(U \Delta V) \subset f_*(U) \Delta f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

01N0

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\text{op}} \times f_*} & \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) \\ [-1, -2]_X \downarrow & \wr & \downarrow [-1, -2]_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

008U

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_*(U) \cup f_*(V) &\hookrightarrow f_*(U \cup V), \\ \emptyset &\hookrightarrow f_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008V

11. *Preservation of Limits.* We have an equality of sets

$$f_* \left(\bigcap_{i \in I} U_i \right) = \bigcap_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_*(U) \cap f^{-1}(V), \\ f_*(X) &= Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008W

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of **Item 1** has a symmetric lax monoidal structure

$$(f_*, f_*^\otimes, f_{*|\mathbb{1}}^\otimes): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U) \cup f_*(V) &\hookrightarrow f_*(U \cup V), \\ f_{*|\mathbb{1}}^\otimes: \emptyset &\hookrightarrow f_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008X

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(f_*, f_*^\otimes, f_{*|\mathbb{1}}^\otimes): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^\otimes: f_*(U \cap V) &\xrightarrow{=} f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^\otimes: f_*(X) &\xrightarrow{=} Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

008Y

14. *Interaction With Coproducts.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \amalg g)_*(U \amalg V) = f_*(U) \amalg g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

- 008Z 15. *Interaction With Products.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_*(U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

- 0090 16. *Relation to Direct Images.* We have

$$\begin{aligned} f_*(U) &= f_!(U^c)^c \\ &= Y \setminus f_!(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

- 0091 17. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{*,\text{im}}(U) &= f_!(U), \\ f_{*,\text{cp}}(U) &= Y \setminus \text{Im}(f), \end{aligned}$$

and so

$$\begin{aligned} f_*(U) &= f_{*,\text{im}}(U) \cup f_{*,\text{cp}}(U) \\ &= f_!(U) \cup (Y \setminus \text{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

- 0092 18. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{*,\text{im}}(U) &\subset f_!(U), \\ f_{*,\text{cp}}(U) &= \emptyset, \end{aligned}$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

PROOF 4.6.3.1.8 ► PROOF OF PROPOSITION 4.6.3.1.7

Item 1: Functoriality

Omitted.

Item 2: Triple Adjointness

This follows from [Remark 4.6.1.1.4](#), [Remark 4.6.2.1.2](#), [Remark 4.6.3.1.4](#), and Kan Extensions, ?? of ??.

Item 3: Interaction With Unions of Families of Subsets

We have

$$\begin{aligned} \bigcup_{V \in f_*(\mathcal{U})} V &= \bigcup_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.

Item 4: Interaction With Intersections of Families of Subsets

We have

$$\begin{aligned} \bigcap_{V \in f_*(\mathcal{U})} V &= \bigcap_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.

Item 5: Interaction With Binary Unions

We have

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_!(U^c)^c \cup f_!(V^c)^c \\ &= (f_!(U^c) \cap f_!(V^c))^c \\ &\subset (f_!(U^c \cap V^c))^c \\ &= f_!((U \cup V)^c)^c \\ &= f_*(U \cup V), \end{aligned}$$

where:

023X

1. We have used [Item 16](#) for the first equality.

- 023Y 2. We have used [Item 2](#) of [Proposition 4.3.11.1.2](#) for the second equality.
- 023Z 3. We have used [Item 6](#) of [Proposition 4.6.1.1.5](#) for the third equality.
- 0240 4. We have used [Item 2](#) of [Proposition 4.3.11.1.2](#) for the fourth equality.
- 0241 5. We have used [Item 16](#) for the last equality.

This finishes the proof.

Item 6: Interaction With Binary Intersections

This follows from [Item 11](#).

Item 7: Interaction With Complements

Omitted.

Item 8: Interaction With Symmetric Differences

Omitted.

Item 9: Interaction With Internal Homs of Powersets

We have

$$\begin{aligned}
 [f_!(U), f^!(V)]_X &\stackrel{\text{def}}{=} f_!(U)^c \cup f_*(V) \\
 &= f_*(U^c) \cup f_*(V) \\
 &\subset f_*(U^c \cup V) \\
 &\stackrel{\text{def}}{=} f_*([U, V]_X),
 \end{aligned}$$

where we have used:

- 0242 1. [Item 7](#) of [Proposition 4.6.3.1.7](#) for the second equality.
- 0243 2. [Item 5](#) of [Proposition 4.6.3.1.7](#) for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic ([Categories](#), [Item 4](#) of [Proposition 11.2.7.1.2](#)). This finishes the proof.

Item 10: Lax Preservation of Colimits

Omitted.

Item 11: Preservation of Limits

This follows from [Item 2](#) and [??](#), [??](#) of [??](#).

Item 12: Symmetric Lax Monoidality With Respect to Unions

This follows from **Item 10**.

Item 13: Symmetric Strict Monoidality With Respect to Intersections

This follows from **Item 11**.

Item 14: Interaction With Coproducts

Omitted.

Item 15: Interaction With Products

Omitted.

Item 16: Relation to Direct Images

We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication.* We claim that

$$f_*(U) \subset Y \setminus f_!(X \setminus U).$$

Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.

This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).

Thus $y \in Y \setminus f_!(X \setminus U)$.

- *The Second Implication.* We claim that

$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$

Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.

Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.

Thus $y \in f_*(U)$.

This finishes the proof of **Item 16**.

Item 17: Interaction With Injections

Omitted.

Item 18: Interaction With Surjections

Omitted.



0093 PROPOSITION 4.6.3.1.9 ► PROPERTIES OF CODIRECT IMAGES II

Let $f: X \rightarrow B$ be a function.

0094 1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

0095 2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

0096 3. *Interaction With Identities.* For each $X \in \text{Obj}(\text{Sets})$, we have

$$(\text{id}_X)_* = \text{id}_{\mathcal{P}(X)}.$$

0097 4. *Interaction With Composition.* For each pair of composable functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$(g \circ f)_* = g_* \circ f_*,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \\ & \searrow (g \circ f)_* & \downarrow g_* \\ & & \mathcal{P}(Z). \end{array}$$

PROOF 4.6.3.1.10 ► PROOF OF PROPOSITION 4.6.3.1.9

Item 1: Functionality I

There is nothing to prove.

Item 2: Functionality II

This follows from **Item 1** of **Proposition 4.6.3.1.7**.

Item 3: Interaction With Identities

This follows from **Remark 4.6.3.1.4** and Kan Extensions, ?? of ??.

Item 4: Interaction With Composition

This follows from **Remark 4.6.3.1.4** and Kan Extensions, ?? of ??.



01N1 4.6.4 A Six-Functor Formalism for Sets

01N2 REMARK 4.6.4.1.1 ► A SIX-FUNCTOR FORMALISM FOR SETS

The assignment $X \mapsto \mathcal{P}(X)$ together with the functors f_* , f^{-1} , and $f_!$ of [Item 1 of Proposition 4.6.1.1.5](#), [Item 1 of Proposition 4.6.2.1.3](#), and [Item 1 of Proposition 4.6.3.1.7](#), and the functors

$$\begin{aligned} -_1 \cap -_2 &: \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X), \\ [-_1, -_2]_X &: \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \rightarrow \mathcal{P}(X) \end{aligned}$$

of [Item 1 of Proposition 4.3.9.1.2](#) and [Item 1 of Proposition 4.4.7.1.4](#) satisfy several properties reminiscent of a six functor formalism in the sense of [??](#). We collect these properties in [Proposition 4.6.4.1.2](#) below.¹

¹See also [\[nLa25\]](#).

01N3 PROPOSITION 4.6.4.1.2 ► A SIX-FUNCTOR FORMALISM FOR SETS

Let X be a set.

01N4 1. The Beck–Chevalley Condition. Let

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\text{pr}_2} & Y \\ \text{pr}_1 \downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

be a pullback diagram in Sets. We have

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{pr}_1^{-1}} & \mathcal{P}(X \times_Z Y) \\ f_! \downarrow & & \downarrow (\text{pr}_2)_! \\ \mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y), \end{array} \quad \begin{array}{l} g^{-1} \circ f_! = (\text{pr}_2)_! \circ \text{pr}_1^{-1}, \\ f^{-1} \circ g_! = (\text{pr}_1)_! \circ \text{pr}_2^{-1}, \end{array} \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{\text{pr}_2^{-1}} & \mathcal{P}(X \times_Z Y) \\ g_! \downarrow & & \downarrow (\text{pr}_1)_! \\ \mathcal{P}(Z) & \xrightarrow{f^{-1}} & \mathcal{P}(Y). \end{array}$$

01N5

2. *The Projection Formula I.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times \mathcal{P}(X) & \\
 \text{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
 f_! \times \text{id}_{\mathcal{P}(Y)} \searrow & & \downarrow f_! \\
 \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\cap} & \mathcal{P}(Y),
 \end{array}$$

commutes, i.e. we have

$$f_!(U \cap f^{-1}(V)) = f_!(U) \cap V$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

01N6

3. *The Projection Formula II.* We have a natural transformation

$$\begin{array}{ccc}
 & \mathcal{P}(X) \times \mathcal{P}(X) & \\
 \text{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap \\
 \mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
 f_* \times \text{id}_{\mathcal{P}(Y)} \searrow & \cup & \downarrow f_* \\
 \mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\cap} & \mathcal{P}(Y),
 \end{array}$$

with components

$$f_*(U) \cap V \subset f_*(U \cap f^{-1}(V))$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

01N7

4. *Strong Closed Monoidality.* The diagram

$$\begin{array}{ccc}
 \mathcal{P}(Y)^{\text{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1, \text{op}} \times f^{-1}} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) \\
 \downarrow [-1, -2]_Y & & \downarrow [-1, -2]_X \\
 \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
 \end{array}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

01N8

5. *The External Tensor Product.* We have an external tensor product

$$-_1 \boxtimes_{X \times Y} -_2 : \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathcal{P}(X \times Y)$$

given by

$$\begin{aligned} U \boxtimes_{X \times Y} V &\stackrel{\text{def}}{=} \text{pr}_1^{-1}(U) \cap \text{pr}_2^{-1}(V) \\ &= \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}. \end{aligned}$$

This is the same map as the one in **Item 5** of **Proposition 4.4.1.1.4**. Moreover, the following conditions are satisfied:

01N9

(a) *Interaction With Direct Images.* Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_! \times g_!} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \boxtimes_{X \times Y} \downarrow & & \downarrow \boxtimes_{X' \times Y'} \\ \mathcal{P}(X \times Y) & \xrightarrow{f_! \times g_!} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

01NA

(b) *Interaction With Inverse Images.* Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X') \times \mathcal{P}(Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X) \times \mathcal{P}(Y) \\ \boxtimes_{X' \times Y'} \downarrow & & \downarrow \boxtimes_{X \times Y} \\ \mathcal{P}(X' \times Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X \times Y) \end{array}$$

commutes, i.e. we have

$$[f^{-1} \times g^{-1}](U \boxtimes_{X' \times Y'} V) = f^{-1}(U) \boxtimes_{X \times Y} g^{-1}(V)$$

for each $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$.

01NB

(c) *Interaction With Codirect Images.* Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be functions. The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\ \boxtimes_{X \times Y} \downarrow & & \downarrow \boxtimes_{X' \times Y'} \\ \mathcal{P}(X \times Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X' \times Y') \end{array}$$

commutes, i.e. we have

$$[f_* \times g_*](U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

01NC

(d) *Interaction With Diagonals.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\boxtimes_{X \times X}} & \mathcal{P}(X \times X) \\ & \searrow \cap & \downarrow \Delta_X^{-1} \\ & & \mathcal{P}(X), \end{array}$$

i.e. we have

$$U \cap V = \Delta_X^{-1}(U \boxtimes_{X \times X} V)$$

for each $U, V \in \mathcal{P}(X)$.

01ND

6. *The Dualisation Functor.* We have a functor

$$D_X: \mathcal{P}(X)^{\text{op}} \rightarrow \mathcal{P}(X)$$

given by

$$D_X(U) \stackrel{\text{def}}{=} [U, \emptyset]_X \\ \stackrel{\text{def}}{=} U^c$$

for each $U \in \mathcal{P}(X)$, as in [Item 5](#) of [Proposition 4.4.7.1.4](#), satisfying the following conditions:

01NE

(a) *Duality*. We have

$$D_X(D_X(U)) = U,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \searrow \text{id}_{\mathcal{P}(X)} & \downarrow D_X \\ & & \mathcal{P}(X). \end{array}$$

01NF

(b) *Duality*. The diagram

$$\begin{array}{ccc} & \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X)^{\text{op}} & \xrightarrow{\cap^{\text{op}}} \mathcal{P}(X)^{\text{op}} \\ \text{id}_{\mathcal{P}(X)^{\text{op}}} \times D_X \nearrow & & \searrow D_X \\ \mathcal{P}(X)^{\text{op}} \times \mathcal{P}(X) & \xrightarrow{[-1, -2]_X} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

01NG

(c) *Interaction With Direct Images*. The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_*^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

01NH

(d) *Interaction With Inverse Images*. The diagram

$$\begin{array}{ccc} \mathcal{P}(Y)^{\text{op}} & \xrightarrow{f^{-1, \text{op}}} & \mathcal{P}(X)^{\text{op}} \\ D_Y \downarrow & & \downarrow D_X \\ \mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X) \end{array}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

01NJ

(e) *Interaction With Codirect Images.* The diagram

$$\begin{array}{ccc} \mathcal{P}(X)^{\text{op}} & \xrightarrow{f_!^{\text{op}}} & \mathcal{P}(Y)^{\text{op}} \\ D_X \downarrow & & \downarrow D_Y \\ \mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \end{array}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

PROOF 4.6.4.1.3 ► PROOF OF PROPOSITION 4.6.4.1.2

Item 1: The Beck–Chevalley Condition

We have

$$\begin{aligned} [g^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} g^{-1}(f_!(U)) \\ &\stackrel{\text{def}}{=} \{y \in Y \mid g(y) \in f_!(U)\} \\ &= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right. \right\} \\ &= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \end{array} \right. \right\} \\ &= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } y = y \end{array} \right. \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\{ y \in Y \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } \text{pr}_2(x, y) = y \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} (\text{pr}_2)_! (\{(x, y) \in X \times_Z Y \mid x \in U\}) \\
&= (\text{pr}_2)_! (\{(x, y) \in X \times_Z Y \mid \text{pr}_1(x, y) \in U\}) \\
&\stackrel{\text{def}}{=} (\text{pr}_2)_! (\text{pr}_1^{-1}(U)) \\
&\stackrel{\text{def}}{=} [(\text{pr}_2)_! \circ \text{pr}_1^{-1}](U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$g^{-1} \circ f_! = (\text{pr}_2)_! \circ \text{pr}_1^{-1}.$$

For the second equality, we have

$$\begin{aligned}
[f^{-1} \circ g_!](U) &\stackrel{\text{def}}{=} f^{-1}(g_!(U)) \\
&\stackrel{\text{def}}{=} \{x \in X \mid f(x) \in g_!(V)\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } x = x \end{array} \right. \right\} \\
&= \left\{ x \in X \left| \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } \text{pr}_1(x, y) = x \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} (\text{pr}_1)_! (\{(x, y) \in X \times_Z Y \mid y \in V\}) \\
&= (\text{pr}_1)_! (\{(x, y) \in X \times_Z Y \mid \text{pr}_2(x, y) \in V\}) \\
&\stackrel{\text{def}}{=} (\text{pr}_1)_! (\text{pr}_2^{-1}(V)) \\
&\stackrel{\text{def}}{=} [(\text{pr}_1)_! \circ \text{pr}_2^{-1}](V)
\end{aligned}$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$f^{-1} \circ g_! = (\text{pr}_1)_! \circ \text{pr}_2^{-1}.$$

This finishes the proof.

Item 2: The Projection Formula I

We claim that

$$f_!(U) \cap V \subset f_!(U \cap f^{-1}(V)).$$

Indeed, we have

$$\begin{aligned} f_!(U) \cap V &\subset f_!(U) \cap f_!(f^{-1}(V)) \\ &= f_!(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

- 024B 1. Item 2 of Proposition 4.6.1.1.5 for the inclusion.
- 024C 2. Item 6 of Proposition 4.6.1.1.5 for the equality.

Conversely, we claim that

$$f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V.$$

Indeed:

- 024D 1. Let $y \in f_!(U \cap f^{-1}(V))$.
- 024E 2. Since $y \in f_!(U \cap f^{-1}(V))$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.
- 024F 3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.
- 024G 4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.
- 024H 5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.
- 024J 6. But $y = f(x)$, so $y \in f_!(U) \cap V$.
- 024K 7. Thus $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$.

This finishes the proof.

Item 3: The Projection Formula II

We have

$$\begin{aligned} f_*(U) \cap V &\subset f_*(U) \cap f_*(f^{-1}(V)) \\ &= f_*(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

024L 1. *Item 2* of *Proposition 4.6.3.1.7* for the inclusion.

024M 2. *Item 6* of *Proposition 4.6.3.1.7* for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (*Categories*, *Item 4* of *Proposition 11.2.7.1.2*).

Item 4: Strong Closed Monoidality

This is a repetition of *Item 19* of *Proposition 4.4.7.1.4* and is proved there.

Item 5: The External Tensor Product

We have

$$\begin{aligned}
 U \boxtimes_{X \times Y} V &\stackrel{\text{def}}{=} \text{pr}_1^{-1}(U) \cap \text{pr}_2^{-1}(V) \\
 &\stackrel{\text{def}}{=} \{(x, y) \in X \times Y \mid \text{pr}_1(x, y) \in U\} \\
 &\quad \cup \{(x, y) \in X \times Y \mid \text{pr}_2(x, y) \in V\} \\
 &= \{(x, y) \in X \times Y \mid x \in U\} \\
 &\quad \cup \{(x, y) \in X \times Y \mid y \in V\} \\
 &= \{(x, y) \in X \times Y \mid x \in U \text{ and } y \in V\} \\
 &\stackrel{\text{def}}{=} U \times V.
 \end{aligned}$$

Next, we claim that *Items 5a* to *5d* are indeed true:

024N 1. *Proof of Item 5a*: This is a repetition of *Item 16* of *Proposition 4.6.1.1.5* and is proved there.

024P 2. *Proof of Item 5b*: This is a repetition of *Item 16* of *Proposition 4.6.2.1.3* and is proved there.


024Q 3. *Proof of Item 5c*: This is a repetition of *Item 15* of *Proposition 4.6.3.1.7* and is proved there.

024R 4. *Proof of Item 5d*: We have

$$\begin{aligned}
 \Delta_X^{-1}(U \boxtimes_{X \times X} V) &\stackrel{\text{def}}{=} \{x \in X \mid (x, x) \in U \boxtimes_{X \times X} V\} \\
 &= \{x \in X \mid (x, x) \in \{(u, v) \in X \times X \mid u \in U \text{ and } v \in V\}\} \\
 &= U \cap V.
 \end{aligned}$$

This finishes the proof.

Item 6: The Dualisation Functor

This is a repetition of **Items 5** and **6** of **Proposition 4.4.7.1.4** and is proved there. 

02RG 4.7 Miscellany

02RH 4.7.1 Injective Functions

Let A and B be sets.

02RJ DEFINITION 4.7.1.1.1 ► INJECTIVE FUNCTIONS

A function $f: A \rightarrow B$ is **injective** if it satisfies the following condition:

(★) For each $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

02RK PROPOSITION 4.7.1.1.2 ► PROPERTIES OF INJECTIVE FUNCTIONS

Let $f: A \rightarrow B$ be a function.

02RL 1. *Characterisations.* The following conditions are equivalent:¹

02RM (a) The function f is injective.

02RN (b) The function f is a monomorphism in Sets.

02RP (c) The direct image function

$$f_!: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to f is injective.

02RQ (d) The codirect image function

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to f is injective.

02RR (e) The direct image functor

$$f_!: (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

associated to f is full.

02RS

(f) The codirect image function

$$f_*: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to f is full.

02RT

(g) The diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \chi_A & \downarrow f^{-1} \\ & & \mathcal{P}(A) \end{array}$$

commutes. That is, we have

$$f^{-1}(f(a)) = \{a\}$$

for each $a \in A$.

02RU

(h) We have

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{f_!} & \mathcal{P}(B) \\ & \searrow & \downarrow f^{-1} \\ & & \mathcal{P}(A). \end{array}$$

$f^{-1} \circ f_! = \text{id}_{\mathcal{P}(A)}$

In other words, we have

$$\{a \in A \mid f(a) \in f(U)\} = U$$

for each $U \in \mathcal{P}(A)$.

02RV

(i) We have

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{f_*} & \mathcal{P}(B) \\ & \searrow & \downarrow f^{-1} \\ & & \mathcal{P}(A). \end{array}$$

$f^{-1} \circ f_* = \text{id}_{\mathcal{P}(A)}$

In other words, we have

$$\{a \in A \mid f^{-1}(f(a)) \subset U\} = U$$

for each $U \in \mathcal{P}(A)$.

¹Items 1c to 1f unwind respectively to the following statements:

- For each $U, V \in \mathcal{P}(A)$, if $f_!(U) = f_!(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_*(U) = f_*(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_!(U) \subset f_!(V)$, then $U \subset V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_*(U) \subset f_*(V)$, then $U \subset V$.

PROOF 4.7.1.1.3 ► PROOF OF PROPOSITION 4.7.1.1.2

Item 1: Characterisations

We will proceed by showing:

- Step 1: Item 1a \iff Item 1b.
- Step 2: Item 1a \iff Item 1c.
- Step 3: Item 1a \iff Item 1d.
- Step 4: Item 1c \iff Item 1e.
- Step 5: Item 1e \iff Item 1f.
- Step 6: Item 1a \iff Item 1g.
- Step 7: Item 1g \iff Item 1h.
- Step 8: Item 1a \iff Item 1i.

Step 1: Item 1a \iff Item 1b

We claim that Items 1a and 1b are equivalent:

- Item 1a \implies Item 1b: We proceed in a few steps:
 - Proceeding by contrapositive, we claim that given a pair of maps $g, h: C \rightrightarrows A$ such that $g \neq h$, we have $f \circ g \neq f \circ h$.
 - Indeed, as g and h are different maps, there must exist at least one element $x \in C$ such that $g(x) \neq h(x)$.
 - But then we have $f(g(x)) \neq f(h(x))$, as f is injective.
 - Thus $f \circ g \neq f \circ h$, and we are done.

- *Item 1b* \implies *Item 1a*: We proceed in a few steps:

- Consider the diagram

$$\text{pt} \begin{array}{c} \xrightarrow{[x]} \\ \xrightarrow{[y]} \end{array} A \xrightarrow{f} B,$$

where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A .

- Note that we have $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$.
- Since f is assumed to be a monomorphism, if $f(x) = f(y)$, then $f \circ [x] = f \circ [y]$ and therefore $[x] = [y]$.
- This shows that if $f(x) = f(y)$, then $x = y$, so f is injective.

Step 2: Item 1a \iff Item 1c

We claim that *Items 1a* and *1c* are indeed equivalent:

- *Item 1a* \implies *Item 1c*: We proceed in a few steps:
 - Assume that f is injective and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$. We wish to show that $U = V$.
 - To show that $U \subset V$, let $u \in U$.
 - By the definition of the direct image, we have $f(u) \in f_!(U)$.
 - Since $f_!(U) = f_!(V)$, it follows that $f(u) \in f_!(V)$.
 - Thus, there exists some $v \in V$ such that $f(v) = f(u)$.
 - Since f is injective, the equality $f(v) = f(u)$ implies that $v = u$.
 - Thus $u \in V$ and $U \subset V$.
 - A symmetric argument shows that $V \subset U$.
 - Therefore $U = V$, showing $f_!$ to be injective.
- *Item 1c* \implies *Item 1a*: We proceed in a few steps:
 - Assume that the direct image function $f_!$ is injective and let $a, a' \in A$ such that $f(a) = f(a')$. We wish to show that $a = a'$.

– Since

$$\begin{aligned} f_!(\{a\}) &= \{f(a)\} \\ &= \{f(a')\} \\ &= f_!(\{a'\}), \end{aligned}$$

we must have $\{a\} = \{a'\}$, as $f_!$ is injective, so $a = a'$, showing f to be injective.

Step 3: Item 1c \iff Item 1d

This follows from **Item 17** of **Proposition 4.6.1.1.5**.

Step 4: Item 1c \iff Item 1e

We claim that **Items 1c** and **1e** are equivalent:

- **Item 1c \implies Item 1e:** We proceed in a few steps:
 - Let $U, V \in \mathcal{P}(A)$ such that $f_!(U) \subset f_!(V)$, assume $f_!$ to be injective, and consider the set $U \cup V$.
 - Since $f_!(U) \subset f_!(V)$, we have

$$\begin{aligned} f_!(U \cup V) &= f_!(U) \cup f_!(V) \\ &= f_!(V), \end{aligned}$$
 where we have used **Item 5** of **Proposition 4.6.1.1.5** for the first equality.
 - Since $f_!$ is injective, this implies $U \cup V = V$.
 - Thus $U \subset V$, as we wished to show.
- **Item 1e \implies Item 1c:** We proceed in a few steps:
 - Suppose **Item 1e** holds, and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$.
 - Since $f_!(U) = f_!(V)$, we have $f_!(U) \subset f_!(V)$ and $f_!(V) \subset f_!(U)$.
 - By assumption, this implies $U \subset V$ and $V \subset U$.

– Thus $U = V$, showing $f_!$ to be injective.

Step 5: Item 1e \iff Item 1f

This follows from **Item 17** of **Proposition 4.6.1.1.5**.

Step 6: Item 1a \iff Item 1g

We have

$$f^{-1}(f(a)) = \{a' \in A \mid f(a') = f(a)\}$$

so the condition $f^{-1}(f(a)) = \{a\}$ states precisely that if $f(a') = f(a)$, then $a' = a$.

Step 7: Item 1g \iff Item 1h

We claim that **Items 1g** and **1h** are indeed equivalent:

- **Item 1g \implies Item 1h:** We have

$$\begin{aligned} [f^{-1} \circ f_!](U) &\stackrel{\text{def}}{=} f^{-1}(f_!(U)) \\ &= f^{-1}\left(f_!\left(\bigcup_{u \in U} \{u\}\right)\right) \\ &= f^{-1}\left(\bigcup_{u \in U} f_!(\{u\})\right) \\ &= \bigcup_{u \in U} f^{-1}(f_!(\{u\})) \\ &= \bigcup_{u \in U} f^{-1}(f_!(u)) \\ &= \bigcup_{u \in U} \{u\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we have used **Item 5** of **Proposition 4.6.1.1.5** for the third equality and **Item 5** of **Proposition 4.6.2.1.3** for the fourth equality.

- **Item 1h** \implies **Item 1g**: Applying the condition $f^{-1} \circ f_! = \text{id}_{\mathcal{P}(A)}$ to $U = \{a\}$ gives

$$f^{-1}(f_!(\{a\})) = \{a\}.$$

Step 8: Item 1a \iff Item 1i

We claim that **Items 1a** and **1i** are equivalent:

- **Item 1a** \implies **Item 1i**: If f is injective, then $f^{-1}(f(a)) = \{a\}$, so we have

$$\begin{aligned} f^{-1}(f_*(a)) &= \{a \in A \mid \{a\} \subset U\} \\ &= U. \end{aligned}$$


- **Item 1i** \implies **Item 1a**: For $U = \{a\}$, the condition $f^{-1}(f_*(U)) = U$ becomes

$$\{a' \in A \mid f^{-1}(f(a')) \subset \{a\}\} = \{a\}.$$

Since the set $f^{-1}(f(a'))$ is given by

$$\{a \in A \mid f(a) = f(a')\},$$

it follows that f is injective.

This finishes the proof. 

02RW 4.7.2 Surjective Functions

Let A and B be sets.

02RX DEFINITION 4.7.2.1.1 ► SURJECTIVE FUNCTIONS

A function $f: A \rightarrow B$ is **surjective** if it satisfies the following condition:

- (★) For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

02RY PROPOSITION 4.7.2.1.2 ► PROPERTIES OF SURJECTIVE FUNCTIONS

Let $f: A \rightarrow B$ be a function.

1. *Characterisations*. The following conditions are equivalent:

- (a) The function f is surjective.

02RZ

02S0

02S1 (b) The function f is an epimorphism in Sets.

02S2 (c) The inverse image function

$$f^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

associated to f is injective.

02S3 (d) The inverse image functor

$$f^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

associated to f is full.

02S4 (e) The diagram

$$\begin{array}{ccc} B & \xrightarrow{f^{-1}} & \mathcal{P}(A) \\ & \searrow \chi_B & \downarrow f_! \\ & & \mathcal{P}(B) \end{array}$$

commutes. That is, we have

$$f_!(f^{-1}(b)) = \{b\}$$

for each $b \in B$.

02S5 (f) We have

$$f_! \circ f^{-1} = \text{id}_{\mathcal{P}(B)} \quad \begin{array}{ccc} \mathcal{P}(B) & \xrightarrow{f^{-1}} & \mathcal{P}(A) \\ & \searrow & \downarrow f_! \\ & & \mathcal{P}(B). \end{array}$$

In other words, we have

$$\left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in f^{-1}(U) \\ \text{such that } f(a) = b \end{array} \right. \right\} = U$$

for each $U \in \mathcal{P}(A)$.

02S6

(g) We have

$$f_* \circ f^{-1} = \text{id}_{\mathcal{P}(B)}$$

$$\begin{array}{ccc} \mathcal{P}(B) & \xrightarrow{f^{-1}} & \mathcal{P}(A) \\ & \searrow & \downarrow f_* \\ & & \mathcal{P}(B). \end{array}$$

In other words, we have

$$\{b \in B \mid f^{-1}(b) \subset f^{-1}(U)\} = U$$

for each $U \in \mathcal{P}(B)$.

PROOF 4.7.2.1.3 ► PROOF OF PROPOSITION 4.7.2.1.2

Item 1: Characterisations

We will proceed by showing:

- Step 1: **Item 1a** \iff **Item 1b**.
- Step 2: **Item 1a** \iff **Item 1c**.
- Step 3: **Item 1c** \iff **Item 1d**.
- Step 4: **Item 1a** \iff **Item 1e**.
- Step 5: **Item 1e** \iff **Item 1f**.
- Step 6: **Item 1a** \iff **Item 1g**.

Step 1: **Item 1a** \iff **Item 1b**We claim **Items 1a** and **1b** are indeed equivalent:

- **Item 1a** \implies **Item 1b**: We proceed in a few steps:
 - Let $g, h: B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$.
 - For each $a \in A$, we have

$$g(f(a)) = h(f(a)).$$

- However, this implies that

$$g(b) = h(b)$$

for each $b \in B$, as f is surjective.

- Thus $g = h$ and f is an epimorphism.

- **Item 1b \implies Item 1a:** We proceed by contrapositive. Consider the diagram

$$A \xrightarrow{f} B \begin{matrix} \xrightarrow{g} \\ \xrightarrow{h} \end{matrix} C,$$

where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by

$$g(b) = \begin{cases} 1 & \text{if } b \in \text{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \text{Im}(f)$, we have

$$g(b) = 0 \neq 1 = h(b).$$

Therefore $g \neq h$ and f is not an epimorphism.

Step 2: Item 1a \iff Item 1c

We claim **Items 1a** and **1c** are indeed equivalent:

- **Item 1a \implies Item 1c:** We proceed in a few steps:
 - Assume that f is surjective. Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$. We wish to show that $U = V$.
 - To show that $U \subset V$, let $b \in U$.
 - Since f is surjective, there must exist some $a \in A$ such that $f(a) = b$.
 - By the definition of the inverse image, since $f(a) = b$ and $b \in U$, we have $a \in f^{-1}(U)$.

- By our initial assumption, $f^{-1}(U) = f^{-1}(V)$, so it follows that $a \in f^{-1}(V)$.
- Again, by the definition of the inverse image, $a \in f^{-1}(V)$ means that $f(a) \in V$.
- Since $f(a) = b$, we have shown that $b \in V$.
- This establishes that $U \subset V$. A symmetric argument shows that $V \subset U$.
- Thus $U = V$, proving that f^{-1} is injective.
- *Item 1c \implies Item 1a:* We proceed in a few steps:
 - Assume that the inverse image function f^{-1} is injective. Suppose, for the sake of contradiction, that f is not surjective.
 - The assumption that f is not surjective means there exists some $b_0 \in B$ such that for all $a \in A$, we have $f(a) \neq b_0$.
 - By the definition of the inverse image, this is equivalent to stating that $f^{-1}(\{b_0\}) = \emptyset$.
 - Since $f^{-1}(\emptyset) = \emptyset$, we have $f^{-1}(\{b_0\}) = f^{-1}(\emptyset)$.
 - Since f^{-1} is injective, this implies that $\{b_0\} = \emptyset$.
 - This is a contradiction, as the singleton set $\{b_0\}$ is non-empty.
 - Therefore, f is surjective.

Step 3: Item 1c \iff Item 1d

We claim that *Items 1c* and *1d* are equivalent:

- *Item 1c \implies Item 1d:* We proceed in a few steps:
 - Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) \subset f^{-1}(V)$, assume f^{-1} to be injective, and consider the set $U \cup V$.
 - Since $f^{-1}(U) \subset f^{-1}(V)$, we have

$$\begin{aligned} f^{-1}(U \cup V) &= f^{-1}(U) \cup f^{-1}(V) \\ &= f^{-1}(V), \end{aligned}$$

where we have used *Item 5* of *Proposition 4.6.2.1.3* for the first equality.

- Since f^{-1} is injective, this implies $U \cup V = V$.
- Thus $U \subset V$, as we wished to show.
- **Item 1d \implies Item 1c:** We proceed in a few steps:
 - Suppose **Item 1d** holds, and let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$.
 - Since $f^{-1}(U) = f^{-1}(V)$, we have $f^{-1}(U) \subset f^{-1}(V)$ and $f^{-1}(V) \subset f^{-1}(U)$.
 - By assumption, this implies $U \subset V$ and $V \subset U$.
 - Thus $U = V$, showing f^{-1} to be injective.

Step 4: Item 1a \iff Item 1e

We have

$$f_!(f^{-1}(b)) = \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in f^{-1}(b) \\ \text{such that } f(a) = b \end{array} \right\},$$

so the condition $f_!(f^{-1}(b)) = \{b\}$ holds iff f is surjective.

Step 5: Item 1e \iff Item 1f

We claim that **Items 1e** and **1f** are indeed equivalent:

- **Item 1e \implies Item 1f:** We have

$$\begin{aligned}
 [f_! \circ f^{-1}](U) &\stackrel{\text{def}}{=} f_!(f^{-1}(U)) \\
 &= f_!\left(f^{-1}\left(\bigcup_{u \in U} \{u\}\right)\right) \\
 &= f_!\left(\bigcup_{u \in U} f^{-1}(\{u\})\right) \\
 &= \bigcup_{u \in U} f_!(f^{-1}(\{u\})) \\
 &= \bigcup_{u \in U} f_!(f^{-1}(u))
 \end{aligned}$$

$$\begin{aligned}
&= \bigcup_{u \in U} \{u\} \\
&= U
\end{aligned}$$

for each $U \in \mathcal{P}(B)$, where we have used **Item 5 of Proposition 4.6.1.1.5** for the third equality and **Item 5 of Proposition 4.6.2.1.3** for the fourth equality.

- **Item 1f** \implies **Item 1e**: Applying the condition $f_! \circ f^{-1} = \text{id}_{\mathcal{P}(B)}$ to $U = \{b\}$ gives

$$f_!(f^{-1}(\{b\})) = \{b\}.$$

Step 6: Item 1a \iff Item 1g

First, note that for the condition $f^{-1}(b) \subset f^{-1}(U)$ to hold, we must have $b \in U$ or $f^{-1}(b) = \emptyset$. Thus

$$f_*(f^{-1}(U)) = (U \cap \text{Im}(f)) \cup (B \setminus \text{Im}(f)).$$

We now claim that **Items 1a** and **1g** are indeed equivalent:

- **Item 1a** \implies **Item 1g**: If f is surjective, we have


$$\begin{aligned}
(U \cap \text{Im}(f)) \cup (B \setminus \text{Im}(f)) &= U \cup \emptyset \\
&= U,
\end{aligned}$$

$$\text{so } f_* \circ f^{-1} = \text{id}_{\mathcal{P}(B)}.$$

- **Item 1g** \implies **Item 1a**: Taking $U = \emptyset$ gives

$$\begin{aligned}
f_*(f^{-1}(\emptyset)) &= (\emptyset \cap \text{Im}(f)) \cup (B \setminus \text{Im}(f)) \\
&= B \setminus \text{Im}(f),
\end{aligned}$$

so the condition $f_*(f^{-1}(\emptyset)) = \emptyset$ implies $B \setminus \text{Im}(f) = \emptyset$. Thus $\text{Im}(f) = B$ and f is surjective.

This finishes the proof. 

Appendices

A Other Chapters

Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

Relations

8. [Relations](#)
9. [Constructions With Relations](#)

10. [Conditions on Relations](#)

Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

Monoidal Categories

13. [Constructions With Monoidal Categories](#)

Bicategories

14. [Types of Morphisms in Bicategories](#)

Extra Part

15. [Notes](#)

References

- [MSE 267365] [J. B.](#) *Show that the powerset partial order is a cartesian closed category.* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/267365> (cit. on p. 156).
- [MSE 267469] [Zhen Lin.](#) *Show that the powerset partial order is a cartesian closed category.* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/267469> (cit. on p. 114).
- [MSE 2719059] [Vinny Chase.](#) *$\mathcal{P}(X)$ with symmetric difference as addition as a vector space over \mathbb{Z}_2 .* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2719059> (cit. on p. 133).
- [Cie97] Krzysztof Ciesielski. *Set Theory for the Working Mathematician.* Vol. 39. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. xii+236. ISBN: 0-521-59441-3; 0-521-59465-0. DOI: [10.1017/CB09781139173131](https://doi.org/10.1017/CB09781139173131).

- URL: <https://doi.org/10.1017/CB09781139173131> (cit. on p. 69).
- [nLa25] nLab Authors. *Interactions of Images and Pre-images with Unions and Intersections*. <https://ncatlab.org/nlab/show/interactions+of+images+and+pre-images+with+unions+and+intersections>. Oct. 2025 (cit. on p. 226).
- [Pro25a] Proof Wiki Contributors. *Cartesian Product Distributes Over Set Difference — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Set_Difference (cit. on p. 18).
- [Pro25b] Proof Wiki Contributors. *Cartesian Product Distributes Over Symmetric Difference — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Symmetric_Difference (cit. on p. 18).
- [Pro25c] Proof Wiki Contributors. *Cartesian Product Distributes Over Union — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Union (cit. on p. 18).
- [Pro25d] Proof Wiki Contributors. *Cartesian Product of Intersections — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Cartesian_Product_of_Intersections (cit. on p. 18).
- [Pro25e] Proof Wiki Contributors. *Characteristic Function of Intersection — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Intersection (cit. on p. 115).
- [Pro25f] Proof Wiki Contributors. *Characteristic Function of Set Difference — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Set_Difference (cit. on p. 120).
- [Pro25g] Proof Wiki Contributors. *Characteristic Function of Symmetric Difference — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Symmetric_Difference (cit. on p. 133).
- [Pro25h] Proof Wiki Contributors. *Characteristic Function of Union — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Union (cit. on p. 108).
- [Pro25i] Proof Wiki Contributors. *Complement of Complement — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Complement_of_Complement (cit. on p. 125).

- [Pro25j] Proof Wiki Contributors. *Complement of Preimage equals Preimage of Complement* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Complement_of_Preimage_equals_Preimage_of_Complement (cit. on p. 208).
- [Pro25k] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_\(Set_Theory\)](https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory)) (cit. on pp. 119, 125).
- [Pro25l] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)/Set Difference/Difference with Union* — Proof Wiki. 2025. URL: [https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_\(Set_Theory\)/Set_Difference/Difference_with_Union](https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory)/Set_Difference/Difference_with_Union) (cit. on p. 119).
- [Pro25m] Proof Wiki Contributors. *Equivalence of Definitions of Symmetric Difference* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Equivalence_of_Definitions_of_Symmetric_Difference (cit. on p. 131).
- [Pro25n] Proof Wiki Contributors. *Image of Intersection Under Mapping* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Image_of_Intersection_under_Mapping (cit. on pp. 115, 197, 199).
- [Pro25o] Proof Wiki Contributors. *Image of Set Difference Under Mapping* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Image_of_Set_Difference_under_Mapping (cit. on pp. 120, 197).
- [Pro25p] Proof Wiki Contributors. *Image of Union Under Mapping* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Image_of_Union_under_Mapping (cit. on pp. 108, 197, 199).
- [Pro25q] Proof Wiki Contributors. *Intersection Distributes Over Symmetric Difference* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Intersection_Distributes_over_Symmetric_Difference (cit. on p. 133).
- [Pro25r] Proof Wiki Contributors. *Intersection Is Associative* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Intersection_is_Associative (cit. on p. 114).
- [Pro25s] Proof Wiki Contributors. *Intersection Is Commutative* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Intersection_is_Commutative (cit. on p. 114).

- [Pro25t] Proof Wiki Contributors. *Intersection With Empty Set — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Intersection_with_Empty_Set (cit. on p. 114).
- [Pro25u] Proof Wiki Contributors. *Intersection With Set Difference Is Set Difference With Intersection — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Intersection_with_Set_Difference_is_Set_Difference_with_Intersection (cit. on p. 119).
- [Pro25v] Proof Wiki Contributors. *Intersection With Subset Is Subset — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Intersection_with_Subset_is_Subset (cit. on p. 114).
- [Pro25w] Proof Wiki Contributors. *Preimage of Intersection Under Mapping — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Preimage_of_Intersection_under_Mapping (cit. on pp. 115, 208, 210).
- [Pro25x] Proof Wiki Contributors. *Preimage of Set Difference Under Mapping — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Preimage_of_Set_Difference_under_Mapping (cit. on pp. 120, 208).
- [Pro25y] Proof Wiki Contributors. *Preimage of Union Under Mapping — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Preimage_of_Union_under_Mapping (cit. on pp. 108, 208, 210).
- [Pro25z] Proof Wiki Contributors. *Quotient Mapping Is Coequalizer — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Quotient_Mapping_is_Coequalizer (cit. on p. 59).
- [Pro25aa] Proof Wiki Contributors. *Set Difference as Intersection With Complement — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Set_Difference_as_Intersection_with_Complement (cit. on p. 119).
- [Pro25ab] Proof Wiki Contributors. *Set Difference as Symmetric Difference With Intersection — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Set_Difference_as_Symmetric_Difference_with_Intersection (cit. on p. 120).
- [Pro25ac] Proof Wiki Contributors. *Set Difference Is Right Distributive Over Union — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Set_Difference_is_Right_Distributive_over_Union (cit. on p. 119).

- [Pro25ad] Proof Wiki Contributors. *Set Difference Over Subset* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Difference_over_Subset (cit. on p. 119).
- [Pro25ae] Proof Wiki Contributors. *Set Difference With Empty Set Is Self* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Difference_with_Empty_Set_is_Self (cit. on p. 120).
- [Pro25af] Proof Wiki Contributors. *Set Difference With Self Is Empty Set* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Difference_with_Self_is_Empty_Set (cit. on p. 120).
- [Pro25ag] Proof Wiki Contributors. *Set Difference With Set Difference Is Union of Set Difference With Intersection* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Difference_with_Set_Difference_is_Union_of_Set_Difference_with_Intersection (cit. on p. 120).
- [Pro25ah] Proof Wiki Contributors. *Set Difference With Subset Is Superset of Set Difference* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Difference_with_Subset_is_Superset_of_Set_Difference (cit. on p. 119).
- [Pro25ai] Proof Wiki Contributors. *Set Difference With Union* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Difference_with_Union (cit. on p. 119).
- [Pro25aj] Proof Wiki Contributors. *Set Intersection Distributes Over Union* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Intersection_Distributes_over_Union (cit. on pp. 108, 114).
- [Pro25ak] Proof Wiki Contributors. *Set Intersection Is Idempotent* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Intersection_is_Idempotent (cit. on p. 114).
- [Pro25al] Proof Wiki Contributors. *Set Intersection Preserves Subsets* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Intersection_Preserves_Subsets (cit. on p. 114).
- [Pro25am] Proof Wiki Contributors. *Set Union Is Idempotent* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Union_is_Idempotent (cit. on p. 108).
- [Pro25an] Proof Wiki Contributors. *Set Union Preserves Subsets* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Set_Union_Preserves_Subsets (cit. on p. 107).

- [Pro25ao] Proof Wiki Contributors. *Symmetric Difference Is Associative* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_is_Associative (cit. on p. 132).
- [Pro25ap] Proof Wiki Contributors. *Symmetric Difference Is Commutative* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_is_Commutative (cit. on p. 132).
- [Pro25aq] Proof Wiki Contributors. *Symmetric Difference of Complements* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_of_Complements (cit. on p. 132).
- [Pro25ar] Proof Wiki Contributors. *Symmetric Difference on Power Set Forms Abelian Group* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_on_Power_Set_forms_Abelian_Group (cit. on p. 134).
- [Pro25as] Proof Wiki Contributors. *Symmetric Difference With Complement* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Complement (cit. on p. 132).
- [Pro25at] Proof Wiki Contributors. *Symmetric Difference With Empty Set* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Empty_Set (cit. on p. 132).
- [Pro25au] Proof Wiki Contributors. *Symmetric Difference With Intersection Forms Ring* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Intersection_forms_Ring (cit. on p. 134).
- [Pro25av] Proof Wiki Contributors. *Symmetric Difference With Self Is Empty Set* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Self_is_Empty_Set (cit. on p. 132).
- [Pro25aw] Proof Wiki Contributors. *Symmetric Difference With Union Does Not Form Ring* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Union_does_not_form_Ring (cit. on p. 131).
- [Pro25ax] Proof Wiki Contributors. *Symmetric Difference With Universe* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Universe (cit. on p. 132).
- [Pro25ay] Proof Wiki Contributors. *Union as Symmetric Difference With Intersection* — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Union_as_Symmetric_Difference_with_Intersection (cit. on p. 108).

- [Pro25az] Proof Wiki Contributors. *Union Distributes Over Intersection* — *Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Union_Distributes_over_Intersection (cit. on pp. 108, 114).
- [Pro25ba] Proof Wiki Contributors. *Union Is Associative* — *Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Union_is_Associative (cit. on p. 107).
- [Pro25bb] Proof Wiki Contributors. *Union Is Commutative* — *Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Union_is_Commutative (cit. on p. 107).
- [Pro25bc] Proof Wiki Contributors. *Union of Symmetric Differences* — *Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Union_of_Symmetric_Differences (cit. on p. 132).
- [Pro25bd] Proof Wiki Contributors. *Union With Empty Set* — *Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Union_with_Empty_Set (cit. on p. 107).