# Pointed Sets

# The Clowder Project Authors

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0098 This chapter contains some foundational material on pointed sets.

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009A	6.1	.1 Foundations	

- **Definition 6.1.1.1.1.** A **pointed set**<sup>1</sup> is equivalently:
  - An  $\mathbb{E}_0$ -monoid in  $(N_{\bullet}(Sets), pt)$ .
  - A pointed object in (Sets, pt).
- **Remark 6.1.1.1.2.** In detail, a **pointed set** is a pair  $(X, x_0)$  consisting of:
  - The Underlying Set. A set X, called the **underlying set of**  $(X, x_0)$ .
  - The Basepoint. A morphism

$$[x_0]: pt \to X$$

in Sets, determining an element  $x_0 \in X$ , called the **basepoint of** X.

- **Example 6.1.1.1.3.** The 0-sphere<sup>2</sup> is the pointed set  $(S^0, 0)^3$  consisting of:
  - *The Underlying Set.* The set  $S^0$  defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

• *The Basepoint*. The element 0 of  $S^0$ .

<sup>&</sup>lt;sup>1</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, pointed sets are viewed as  $\mathbb{F}_1$ -modules.

<sup>&</sup>lt;sup>2</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

<sup>&</sup>lt;sup>3</sup>Further Notation: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras,  $S^0$  is also denoted  $(\mathbb{F}_1, 0)$ .

**Example 6.1.1.1.4.** The **trivial pointed set** is the pointed set (pt, ★) consisting of:

- *The Underlying Set.* The punctual set pt  $\stackrel{\text{def}}{=} \{ \star \}$ .
- *The Basepoint*. The element ★ of pt.

01QB **Example 6.1.1.1.5.** The **standard pointed set with** n + 1 **elements** is the pointed set  $\langle n \rangle$  consisting of

• *The Underlying Set.* The set  $\langle n \rangle$  defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{ * \} \cup \{ 1, \ldots, n \}.$$

• *The Basepoint*. The element \* of  $\langle n \rangle$ .

## 009H 6.1.2 Morphisms of Pointed Sets

**Definition 6.1.2.1.1.** A morphism of pointed sets<sup>4,5</sup> is equivalently:

- A morphism of  $\mathbb{E}_0$ -monoids in  $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$ .
- A morphism of pointed objects in (Sets, pt).

**Remark 6.1.2.1.2.** In detail, a **morphism of pointed sets**  $f:(X,x_0) \to (Y,y_0)$  is a morphism of sets  $f:X\to Y$  such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & [y_0] \\
X & \xrightarrow{f} Y
\end{array}$$

commutes, i.e. such that

$$f(x_0)=y_0.$$

<sup>&</sup>lt;sup>4</sup>Further Terminology: Also called a **pointed function**.

<sup>&</sup>lt;sup>5</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, morphisms of pointed sets are also called **morphism of**  $\mathbb{F}_1$ -**modules**.

## 009L 6.1.3 The Category of Pointed Sets

- **Definition 6.1.3.1.1.** The **category of pointed sets** is the category Sets\* defined equivalently as:
  - The homotopy category of the ∞-category  $\mathsf{Mon}_{\mathbb{B}_0}(N_{\bullet}(\mathsf{Sets}),\mathsf{pt})$  of  $\ref{eq:sets}$ .
  - The category Sets\* of Constructions With Categories, ??.
- **Remark 6.1.3.1.2.** In detail, the **category of pointed sets** is the category Sets\* where:
  - Objects. The objects of Sets\* are pointed sets.
  - Morphisms. The morphisms of Sets\* are morphisms of pointed sets.
  - *Identities*. For each  $(X, x_0) \in Obj(Sets_*)$ , the unit map

$$\mathbb{1}_{(X,x_0)}^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets<sub>\*</sub> at  $(X, x_0)$  is defined by<sup>6</sup>

$$id_{(X,x_0)}^{Sets_*} \stackrel{\text{def}}{=} id_X$$
.

• *Composition*. For each  $(X, x_0), (Y, y_0), (Z, z_0) \in Obj(Sets_*)$ , the composition map

$$\circ_{(X,x_0),(Y,y_0),(Z,z_0)}^{\mathsf{Sets}_*} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

of Sets<sub>\*</sub> at  $((X, x_0), (Y, y_0), (Z, z_0))$  is defined by<sup>7</sup>

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

<sup>&</sup>lt;sup>6</sup>Note that  $id_X$  is indeed a morphism of pointed sets, as we have  $id_X(x_0) = x_0$ .

<sup>&</sup>lt;sup>7</sup>Note that the composition of two morphisms of pointed sets is indeed a morphism of

## **009P 6.1.4** Elementary Properties of Pointed Sets

- **Proposition 6.1.4.1.1.** Let  $(X, x_0)$  be a pointed set.
- Completeness. The category Sets\* of pointed sets and morphisms between them is complete, having in particular:
- 009S (a) Products, described as in Definition 6.2.3.1.1.
- 009T (b) Pullbacks, described as in Definition 6.2.4.1.1.
- 009U (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets\* of pointed sets and morphisms between them is cocomplete, having in particular:
- 009W (a) Coproducts, described as in Definition 6.3.3.1.1.
- 009X (b) Pushouts, described as in Definition 6.3.4.1.1;
- 009Y (c) Coequalisers, described as in Definition 6.3.5.1.1.
- 3. Failure To Be Cartesian Closed. The category Sets\* is not Cartesian closed.
- 4. Morphisms From the Monoidal Unit. We have a bijection of sets<sup>9</sup>

$$\mathsf{Sets}_*(S^0,X)\cong X,$$

pointed sets, as we have

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

<sup>8</sup>The category Sets<sub>\*</sub> does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

<sup>9</sup>In other words, the forgetful functor

defined on objects by sending a pointed set to its underlying set is corepresentable by  $S^0$ .

natural in  $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$ , internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0,X)\cong (X,x_0),$$

again natural in  $(X, x_0) \in Obj(Sets_*)$ .

00A1 5. Relation to Partial Functions. We have an equivalence of categories 10

$$\mathsf{Sets}_* \stackrel{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

024V i. A pointed set  $(X, x_0)$  to X.

024W ii. A pointed function

$$f\colon (X,x_0)\to (Y,y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on  $f^{-1}(Y \setminus y_0)$  and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in f^{-1}(Y \setminus y_0)$ .

024X (b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\mathsf{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

Warning: This is not an isomorphism of categories, only an equivalence.

024Y

- i. A set X is to the pointed set  $(X, \star)$  with  $\star$  an element that is not in X.
- 024Z
- ii. A partial function

$$f: X \to Y$$

defined on  $U \subset X$  to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each  $x \in X$ .

*Proof. Item 1, Completeness*: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

*Item 2, Cocompleteness*: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from  $S^0$  to a pointed set  $(X, x_0)$  sends  $0 \in S^0$  to  $x_0$  and then can send  $1 \in S^0$  to any element of X, we obtain a bijection between pointed maps  $S^0 \to X$  and the elements of X.

The isomorphism then

$$\mathsf{Sets}_*\big(S^0,X\big)\cong (X,x_0)$$

follows by noting that  $\Delta_{x_0} \colon S^0 \to X$ , the basepoint of  $\mathbf{Sets}_*(S^0, X)$ , corresponds to the pointed map  $S^0 \to X$  picking the element  $x_0$  of X, and thus we see that the bijection between pointed maps  $S^0 \to X$  and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

*Item 5*, *Relation to Partial Functions*: See [MSE 884460].

# 01QC 6.1.5 Active and Inert Morphisms of Pointed Sets

**Oldownian Definition 6.1.5.1.1.** Let  $f:(X,x_0)\to (Y,y_0)$  be a morphism of pointed sets.

01QE 1. The morphism f is **active** if  $f^{-1}(y_0) = x_0$ .

01QF 2. The morphism f is **inert** if, for each  $y \in Y$ , the set  $f^{-1}(y)$  has exactly one element.

**Notation 6.1.5.1.2.** We write Sets\*\* for the wide subcategory of Sets\* spanned by pointed sets and the active maps between them.

**O1QH** Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

**01QJ** 1. The map  $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$  given by



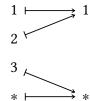
is active but not inert.

**01QK** 2. The map  $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$  given by

$$\begin{array}{cccc}
1 & \longrightarrow & 1 \\
2 & & & 2 \\
* & \longmapsto & *
\end{array}$$

is inert but not active.

01QL 3. The map  $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$  given by

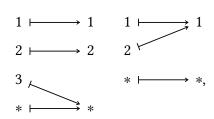


is neither inert nor active. However, it factors as  $f = a \circ i$ , where

$$i:\langle 3\rangle \rightarrow \langle 2\rangle,$$

$$a: \langle 2 \rangle \rightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by



with *i* being inert and *a* being active.

- **Olymorphisms** Proposition 6.1.5.1.4. Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.
- 01QN 1. Active-Inert Factorisation. Every morphism of pointed sets  $f:(X,x_0)\to (Y,y_0)$  factors uniquely as

$$f = a \circ i$$

where:

- 01QP (a) The map  $i: (X, x_0) \to (K, k_0)$  is an inert morphism of pointed sets
- 01QQ (b) The map  $a \colon (K, k_0) \to (Y, y_0)$  is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets<sub>\*</sub>.

*Proof. Item 1, Active-Inert Factorisation*: Let  $f: X \to Y$  be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\longrightarrow} K \stackrel{a}{\longrightarrow} Y$$
.

where:

• *K* is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$
  
=  $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$ 

•  $i: X \to K$  is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each  $x \in X$ ;

•  $a: K \to Y$  is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in K$ .

Next, let

$$X \xrightarrow{i} Y$$

$$f \downarrow \qquad \qquad \downarrow g$$

$$A \xrightarrow{g} B$$

be a commutative diagram in  $\mathsf{Sets}_*.$  Consider the morphism  $\phi\colon Y\to A$  given by

$$\phi(y) = f(i^{-1}(y))$$

for each  $y \in Y$  (which is well-defined since, as i is inert,  $i^{-1}(y)$  is a singleton for all  $y \in Y$ ). We claim that  $\phi$  is the unique diagonal filler in the diagram

$$X \xrightarrow{i} Y$$

$$f \downarrow \exists ! \qquad \downarrow g$$

$$A \xrightarrow{a} B.$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\stackrel{\text{def}}{=} \phi(i(x)) \\ &\stackrel{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each  $x \in X$  and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

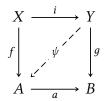
$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each  $y \in Y$ . Moreover, given another morphism  $\psi$  such that the diagram



commutes, it follows that we must have  $\psi = \phi$ , since, given  $y \in Y$ , there exists a unique  $x \in X$  such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

# **00A2 6.2** Limits of Pointed Sets

#### **00A3 6.2.1** The Terminal Pointed Set

- **Definition 6.2.1.1.1.** The **terminal pointed set** is the terminal object of Sets\* as in Limits and Colimits, ??.
- **Construction 6.2.1.1.2.** Concretely, the **terminal pointed set** is the pair  $(pt, \star), \{!_X\}_{(X,x_0) \in Obj(Sets_*)})$  consisting of:

- *The Limit*. The pointed set  $(pt, \star)$ .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathsf{pt}, \star)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each  $x \in X$  and each  $(X, x_0) \in Obj(Sets)$ .

*Proof.* We claim that  $(pt, \star)$  is the terminal object of Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt,  $\star$ )

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathsf{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists 1}} (\text{pt}, \star)$$

commute, namely  $!_X$ .

#### **00A5 6.2.2 Products of Families of Pointed Sets**

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

- **Definition 6.2.2.1.1.** The **product of**  $\{(X_i, x_0^i)\}_{i \in I}$  is the product of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- O251 Construction 6.2.2.1.2. Concretely, the **product of**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$  consisting of:
  - *The Limit.* The pointed set  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ .
  - The Cone. The collection

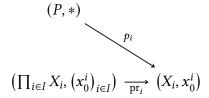
$$\left\{ \operatorname{pr}_{i} : \left( \prod_{i \in I} X_{i}, \left( x_{0}^{i} \right)_{i \in I} \right) \to \left( X_{i}, x_{0}^{i} \right) \right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_i\Big(\big(x_j\big)_{j\in I}\Big)\stackrel{\mathrm{def}}{=} x_i$$

for each  $(x_j)_{j \in I} \in \prod_{i \in I} X_i$  and each  $i \in I$ .

*Proof.* We claim that  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$  is the categorical product of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub>. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form



in Sets<sub>\*</sub>. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left( \prod_{i \in I} X_i, \left( x_0^i \right)_{i \in I} \right)$$

making the diagram

$$(P, *)$$

$$\downarrow \downarrow p_{i}$$

$$(\prod_{i \in I} X_{i}, (x_{0}^{i})_{i \in I}) \xrightarrow{\operatorname{pr}_{i}} (X_{i}, x_{0}^{i})$$

commute, being uniquely determined by the condition  $\operatorname{pr}_i \circ \phi = p_i$  for each  $i \in I$  via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= (x_0^i)_{i \in I},$$

where we have used that  $p_i$  is a morphism of pointed sets for each  $i \in I$ .  $\square$ 

**Proposition 6.2.2.1.3.** Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

6.2.3 Products

00A8 1. Functoriality. The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$  defines a functor

$$\prod_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

*Proof. Item* 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??. □

### **00A9 6.2.3 Products**

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

- **Definition 6.2.3.1.1.** The **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the product of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub> as in Limits and Colimits,??.
- **Construction 6.2.3.1.2.** Concretely, the **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pair consisting of:
  - *The Limit*. The pointed set  $(X \times Y, (x_0, y_0))$ .
  - The Cone. The morphisms of pointed sets

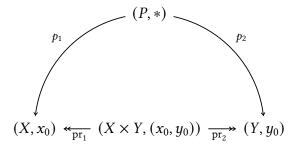
$$\operatorname{pr}_1 : (X \times Y, (x_0, y_0)) \to (X, x_0),$$
  
 $\operatorname{pr}_2 : (X \times Y, (x_0, y_0)) \to (Y, y_0)$ 

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$
  
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$ 

for each  $(x, y) \in X \times Y$ .

*Proof.* We claim that  $(X \times Y, (x_0, y_0))$  is the categorical product of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

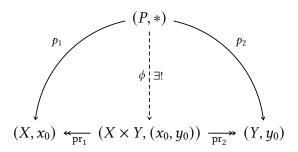


6.2.3 Products

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
  
 $\operatorname{pr}_2 \circ \phi = p_2$ 

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
  
=  $(x_0, y_0),$ 

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets.

**Proposition 6.2.3.1.3.** Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

00AC 1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{array}{ll} A\times -\colon & \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\ -\times B\colon & \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\ -_1\times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{array}$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

6.2.3 Products 16

01QR 2. Lack of Adjointness. The functors  $X \times -$  and  $- \times Y$  do not admit right adjoints.

OOAD 3. Associativity. We have an isomorphism of pointed sets

$$((X\times Y)\times Z,((x_0,y_0),z_0))\cong (X\times (Y\times Z),(x_0,(y_0,z_0)))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in Obj(Sets_*).$ 

**OOAE** 4. Unitality. We have isomorphisms of pointed sets

$$(pt, \star) \times (X, x_0) \cong (X, x_0),$$
  
 $(X, x_0) \times (pt, \star) \cong (X, x_0),$ 

natural in  $(X, x_0) \in Obj(Sets_*)$ .

00AF 5. Commutativity. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in  $(X, x_0), (Y, y_0) \in Obj(Sets_*)$ .

6. *Symmetric Monoidality*. The triple (Sets<sub>\*</sub>, ×, (pt, ★)) is a symmetric monoidal category.

*Proof. Item 1, Functoriality*: This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2, Lack of Adjointness: See [MSE 2855868].

*Item 3, Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.

*Item 4, Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.

*Item 5, Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.

*Item 6, Symmetric Monoidality*: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3. □

6.2.4 Pullbacks

### 00AH 6.2.4 Pullbacks

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (X, x_0) \to (Z, z_0)$  and  $g: (Y, y_0) \to (Z, z_0)$  be morphisms of pointed sets.

- OOAJ Definition 6.2.4.1.1. The pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) is the pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) in Sets<sub>\*</sub> as in Limits and Colimits,??.
- Construction 6.2.4.1.2. Concretely, the pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, q) is the pair consisting of:
  - *The Limit*. The pointed set  $(X \times_Z Y, (x_0, y_0))$ .
  - *The Cone*. The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$
  
 $\operatorname{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$ 

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$
  
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$ 

for each  $(x, y) \in X \times_Z Y$ .

*Proof.* We claim that  $X \times_Z Y$  is the categorical pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to (f, g) in Sets<sub>\*</sub>. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad (X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$

$$\downarrow g \qquad \qquad \downarrow g \qquad \qquad \downarrow g \qquad \qquad \downarrow g \qquad \qquad (X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given  $(x, y) \in X \times_Z Y$ , we have

$$[f \circ \operatorname{pr}_1](x, y) = f(\operatorname{pr}_1(x, y))$$
$$= f(x)$$

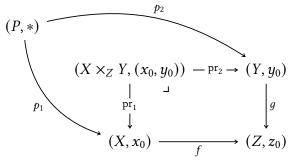
6.2.4 Pullbacks

$$= g(y)$$

$$= g(pr_2(x, y))$$

$$= [g \circ pr_2](x, y),$$

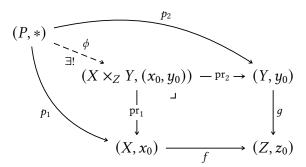
where f(x) = g(y) since  $(x, y) \in X \times_Z Y$ . Next, we prove that  $X \times_Z Y$  satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
  
 $\operatorname{pr}_2 \circ \phi = p_2$ 

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ , where we note that  $(p_1(x), p_2(x)) \in X \times Y$  indeed lies in  $X \times_Z Y$  by the condition

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$$f \circ p_1 = g \circ p_2$$
,

which gives

$$f(p_1(x)) = q(p_2(x))$$

for each  $x \in P$ , so that  $(p_1(x), p_2(x)) \in X \times_Z Y$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
  
=  $(x_0, y_0),$ 

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets.

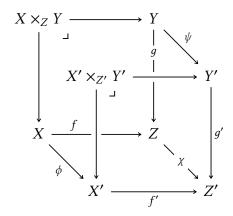
- **Proposition 6.2.4.1.3.** Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.
- 00AL 1. Functoriality. The assignment  $(X,Y,Z,f,g)\mapsto X\times_{f,Z,g}Y$  defines a functor

$$-_1 \times_{-_3} -_1 : \operatorname{\mathsf{Fun}}(\mathcal{P}, \operatorname{\mathsf{Sets}}_*) \to \operatorname{\mathsf{Sets}}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \times_{-3} -1$  is given by sending a morphism



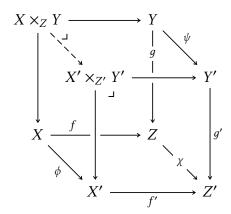
in  $Fun(\mathcal{P}, Sets_*)$  to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

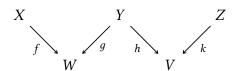
$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each  $(x, y) \in X \times_Z Y$ , which is the unique morphism of pointed sets making the diagram



commute.

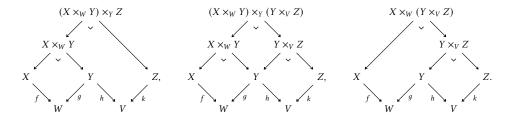
#### **00AM** 2. Associativity. Given a diagram



in Sets\*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



00AN 3. *Unitality*. We have isomorphisms of pointed sets

**OOAP** 4. Commutativity. We have an isomorphism of pointed sets

$$A \times_X B \longrightarrow B$$

$$\downarrow \qquad \qquad \downarrow g \qquad A \times_X B \cong B \times_X A \qquad \qquad \downarrow \qquad \downarrow f$$

$$A \xrightarrow{f} X, \qquad \qquad B \xrightarrow{g} X.$$

5. Interaction With Products. We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad \begin{array}{c} X \times Y \longrightarrow Y \\ & \downarrow \\ X \xrightarrow{!_{X}} \mathrm{pt.} \end{array}$$

6. Symmetric Monoidality. The triple (Sets<sub>\*</sub>,  $\times_X$ , X) is a symmetric monoidal category.

*Proof. Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pullback diagram.

*Item 2, Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.

*Item 3, Unitality*: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.

*Item 4, Commutativity*: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.

*Item 5, Interaction With Products*: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.

*Item 6, Symmetric Monoidality*: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5. □

## 00AS 6.2.5 Equalisers

Let  $f, g: (X, x_0) \Rightarrow (Y, y_0)$  be morphisms of pointed sets.

- **Definition 6.2.5.1.1.** The **equaliser of** (f, g) is the equaliser of f and g in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- **Construction 6.2.5.1.2.** Concretely, the **equaliser of** (f, g) is the pair consisting of:
  - *The Limit.* The pointed set  $(Eq(f, g), x_0)$ .
  - *The Cone*. The morphism of pointed sets

$$\operatorname{eq}(f,g) \colon (\operatorname{Eq}(f,g),x_0) \hookrightarrow (X,x_0)$$

given by the canonical inclusion  $\operatorname{eq}(f,g) \hookrightarrow \operatorname{Eq}(f,g) \hookrightarrow X$ .

*Proof.* We claim that  $(\text{Eq}(f,g),x_0)$  is the categorical equaliser of f and g in Sets\*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \operatorname{eq}(f, g) = g \circ \operatorname{eq}(f, g),$$

which indeed holds by the definition of the set  ${\rm Eq}(f,g)$ . Next, we prove that  ${\rm Eq}(f,g)$  satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$(E,*)$$

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\text{Eq}(f, g), x_0)$$

making the diagram

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow \downarrow \downarrow \downarrow e$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in A$  indeed lies in Eq(f,g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f,g)$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

**Proposition 6.2.5.1.3.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

00AV 1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \underbrace{\mathrm{Eq}(f,g,h)}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))} \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}$$

where Eq(f, q, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets\*, being explicitly given by

$$Eq(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW 2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,f)\cong X.$$

OOAX 3. Commutativity. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,g) \cong \operatorname{Eq}(g,f)$$
.

*Proof. Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.

*Item 2, Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.

*Item 3, Commutativity*: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3. □

# **00AY** 6.3 Colimits of Pointed Sets

- 00AZ 6.3.1 The Initial Pointed Set
- **Definition 6.3.1.1.1.** The **initial pointed set** is the initial object of Sets<sub>\*</sub> as in Limits and Colimits, ??.
- **Construction 6.3.1.1.2.** Concretely, the **initial pointed set** is the pair  $(pt, \star)$ ,  $\{\iota_X\}_{(X,x_0) \in Obj(Sets_*)}$  consisting of:
  - *The Limit*. The pointed set  $(pt, \star)$ .
  - *The Cone*. The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathsf{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

*Proof.* We claim that  $(pt, \star)$  is the initial object of Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
  $(X, x_0)$ 

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt},\star) \xrightarrow{-\frac{\phi}{\exists !}} (X,x_0)$$

commute, namely  $\iota_X$ .

## 00B1 6.3.2 Coproducts of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

- **Definition 6.3.2.1.1.** The **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}^{11}$  is the coproduct of  $\{(X_i, x_0^i)\}_{i \in I}^{11}$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- **Construction 6.3.2.1.2.** Concretely, the **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $((\bigvee_{i \in I} X_i, p_0), \{\inf_i\}_{i \in I})$  consisting of:
  - *The Colimit.* The pointed set  $(\bigvee_{i \in I} X_i, p_0)$  consisting of:
    - − *The Underlying Set.* The set  $\bigvee_{i \in I} X_i$  defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left( \prod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\coprod_{i \in I} X_i$  given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each  $i, j \in I$ .

<sup>&</sup>lt;sup>11</sup>Further Terminology: Also called the **wedge sum of the family**  $\{(X_i, x_0^i)\}_{i \in I}$ 

− *The Basepoint.* The element  $p_0$  of  $\bigvee_{i \in I} X_i$  defined by

$$p_0 \stackrel{\text{def}}{=} \left[ \left( i, x_0^i \right) \right] \\ = \left[ \left( j, x_0^j \right) \right]$$

for any  $i, j \in I$ .

• *The Cocone*. The collection

$$\left\{ \operatorname{inj}_i \colon \left( X_i, x_0^i \right) \to \left( \bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\operatorname{inj}_{i}(x) \stackrel{\text{def}}{=} (i, x)$$

for each  $x \in X_i$  and each  $i \in I$ .

*Proof.* We claim that  $(\bigvee_{i \in I} X_i, p_0)$  is the categorical coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub>. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

$$(X_i, x_0^i) \xrightarrow{\text{inj}_i} \left(\bigvee_{i \in I} X_i, p_0\right)$$

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i\in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$(X_i, x_0^i) \xrightarrow{\lim_{i \to \infty} l_i} \left( \bigvee_{i \in I} X_i, p_0 \right)$$

commute, being uniquely determined by the condition  $\phi \circ \operatorname{inj}_i = \iota_i$  for each  $i \in I$  via

$$\phi([(i,x)]) = \iota_i(x)$$

for each  $[(i,x)] \in \bigvee_{i \in I} X_i$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$
= \*,

as  $l_i$  is a morphism of pointed sets.

- OOB3 **Proposition 6.3.2.1.3.** Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.
- 00B4 1. Functoriality. The assignment  $\left\{\left(X_{i},x_{0}^{i}\right)\right\}_{i\in I}\mapsto\left(\bigvee_{i\in I}X_{i},p_{0}\right)$  defines a functor  $\bigvee_{i\in I}:\operatorname{Fun}(I_{\operatorname{disc}},\operatorname{Sets}_{*})\to\operatorname{Sets}_{*}.$

*Proof. Item* 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??. □

# 00B5 6.3.3 Coproducts

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

- **Definition 6.3.3.1.1.** The **coproduct of**  $(X, x_0)$  **and**  $(Y, y_0)^{12}$  is the coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub> as in Limits and Colimits,??.
- O257 Construction 6.3.3.1.2. Concretely, the coproduct of  $(X, x_0)$  and  $(Y, y_0)$ , also called their wedge sum, is the pair consisting of:
  - *The Colimit.* The pointed set  $(X \vee Y, p_0)$  consisting of:
    - *The Underlying Set.* The set *X* ∨ *Y* defined by

$$(X \lor Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \qquad X \lor Y \longleftarrow Y$$

$$\cong \left( X \coprod_{\text{pt}} Y, p_0 \right) \qquad \uparrow \qquad \uparrow \qquad \downarrow [y_0]$$

$$\cong (X \coprod Y/\sim, p_0), \qquad X \longleftarrow_{[x_0]} \text{pt,}$$

<sup>&</sup>lt;sup>12</sup> Further Terminology: Also called the **wedge sum of**  $(X, x_0)$  **and**  $(Y, y_0)$ .

where  $\sim$  is the equivalence relation on  $X \coprod Y$  obtained by declaring  $(0, x_0) \sim (1, y_0)$ .

- *The Basepoint.* The element  $p_0$  of  $X \vee Y$  defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$
  
=  $[(1, y_0)]$ 

• The Cocone. The morphisms of pointed sets

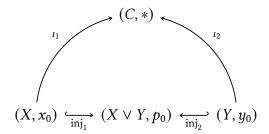
$$\operatorname{inj}_1 \colon (X, x_0) \to (X \vee Y, p_0),$$
  
 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \vee Y, p_0),$ 

given by

$$\operatorname{inj}_{1}(x) \stackrel{\text{def}}{=} [(0, x)],$$
  
 $\operatorname{inj}_{2}(y) \stackrel{\text{def}}{=} [(1, y)],$ 

for each  $x \in X$  and each  $y \in Y$ .

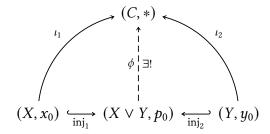
*Proof.* We claim that  $(X \vee Y, p_0)$  is the categorical coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$
  
$$\phi \circ \operatorname{inj}_V = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each  $z \in X \vee Y$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$
  
=  $\iota_Y([(1, y_0)])$   
= \*.

as  $\iota_X$  and  $\iota_Y$  are morphisms of pointed sets.

**Proposition 6.3.3.1.3.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00B8 1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$ 
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$ 

00B9 2. Associativity. We have an isomorphism of pointed sets

$$(X \lor Y) \lor Z \cong X \lor (Y \lor Z),$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$ .

00BA 3. *Unitality*. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$
  
 $(X, x_0) \lor (pt, *) \cong (X, x_0),$ 

natural in  $(X, x_0) \in \mathsf{Sets}_*$ .

00BB 4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$

natural in  $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$ .

OOBC 5. Symmetric Monoidality. The triple (Sets<sub>\*</sub>, ∨, pt) is a symmetric monoidal category.

6. The Fold Map. We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathsf{id}_{\mathsf{Sets}_*}, \qquad \begin{array}{c} \mathsf{Sets}_* \times \mathsf{Sets}_* \\ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} & \bigvee \\ \mathsf{V} & \bigvee \\ \mathsf{Sets}_* & \bigvee \\ \mathsf{Sets}_* & \bigvee \\ \mathsf{Sets}_*, \end{array}$$

called the fold map, whose component

$$\nabla_X : X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each  $p \in X \vee X$ .

*Proof.* Item 1, Functoriality: This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

Item 5, Symmetric Monoidality: Omitted.

*Item 6, The Fold Map*: Naturality for the transformation  $\nabla$  is the statement that, given a morphism of pointed sets  $f:(X,x_0)\to (Y,y_0)$ , we have

$$\nabla_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad \begin{array}{c} X \vee X \xrightarrow{\nabla_{X}} X \\ \downarrow_{f} \\ Y \vee Y \xrightarrow{\nabla_{Y}} Y. \end{array}$$

Indeed, we have

$$\begin{split} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{split}$$

for each  $[(i, x)] \in X \vee X$ , and thus  $\nabla$  is indeed a natural transformation.  $\square$ 

## 00BE 6.3.4 Pushouts

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (Z, z_0) \to (X, x_0)$  and  $g: (Z, z_0) \to (Y, y_0)$  be morphisms of pointed sets.

- OOBF Definition 6.3.4.1.1. The pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) is the pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) in Sets<sub>\*</sub> as in Limits and Colimits,??.
- O258 Construction 6.3.4.1.2. Concretely, the pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) is the pair consisting of:
  - *The Colimit*. The pointed set  $(X \coprod_{f,Z,g} Y, p_0)$ , where:
    - The set  $X \coprod_{f,Z,g} Y$  is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
    - We have  $p_0 = [x_0] = [y_0]$ .
  - *The Cocone*. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$
  
 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$ 

given by

$$\operatorname{inj}_{1}(x) \stackrel{\text{def}}{=} [(0, x)]$$
  
 $\operatorname{inj}_{2}(y) \stackrel{\text{def}}{=} [(1, y)]$ 

for each  $x \in X$  and each  $y \in Y$ .

*Proof.* Firstly, we note that indeed  $[x_0] = [y_0]$ , as we have

$$x_0 = f(z_0),$$
  
$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation  $\sim$  on  $X \coprod_Z Y$  then identifying  $x_0 = f(z_0) \sim g(z_0) = y_0$ .

We now claim that  $(X \coprod_Z Y, p_0)$  is the categorical pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to (f, g) in Sets<sub>\*</sub>. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \xleftarrow{\operatorname{inj}_{2}} (Y, y_{0})$$

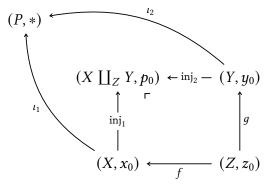
$$\operatorname{inj}_{1} \circ f = \operatorname{inj}_{2} \circ g, \qquad \operatorname{inj}_{1} \qquad \qquad \int_{g} g$$

$$(X, x_{0}) \xleftarrow{f} (Z, z_{0}).$$

Indeed, given  $z \in Z$ , we have

$$\begin{aligned} \left[ \inf_{1} \circ f \right](z) &= \inf_{1} (f(z)) \\ &= \left[ (0, f(z)) \right] \\ &= \left[ (1, g(z)) \right] \\ &= \inf_{2} (g(z)) \\ &= \left[ \inf_{2} \circ g \right](z), \end{aligned}$$

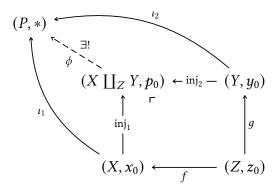
where [(0, f(z))] = [(1, g(z))] by the definition of the relation  $\sim$  on  $X \coprod Y$  (the coproduct of unpointed sets of X and Y). Next, we prove that  $X \coprod_Z Y$  satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1,$$
  
$$\phi \circ \operatorname{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each  $p \in X \coprod_Z Y$ , where the well-definedness of  $\phi$  is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$
  
=  $\iota_1(x_0)$   
= \*,

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$
  
=  $\iota_2(y_0)$   
= \*.

where we use that  $\iota_1$  (resp.  $\iota_2$ ) is a morphism of pointed sets.

**Proposition 6.3.4.1.3.** Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

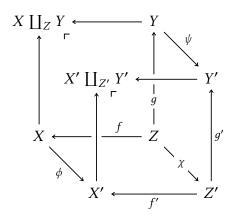
00BH 1. Functoriality. The assignment  $(X,Y,Z,f,g)\mapsto X\coprod_{f,Z,g}Y$  defines a functor

$$-_1 \coprod_{-_2} -_1 : \operatorname{\mathsf{Fun}}(\mathcal{P}, \operatorname{\mathsf{Sets}}) \to \operatorname{\mathsf{Sets}}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \coprod_{-3} -1$  is given by sending a morphism



in  $\operatorname{Fun}(\mathcal{P},\operatorname{Sets}_*)$  to the morphism of pointed sets

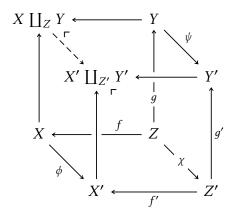
$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

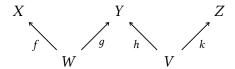
for each  $p \in X \coprod_Z Y$ , which is the unique morphism of pointed sets making the diagram

35



commute.

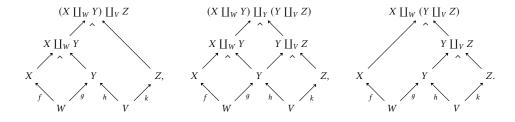
00BJ 2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

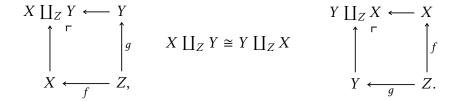
$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams



**00BK** 3. *Unitality*. We have isomorphisms of sets

**00BL** 4. *Commutativity*. We have an isomorphism of sets



**00BM** 5. *Interaction With Coproducts*. We have

$$X \coprod_{\mathrm{pt}} Y \cong X \vee Y,$$

$$X \bigvee_{\Gamma} \bigvee_{\Gamma} \bigvee_{[y_0]} [y_0]$$

$$X \longleftarrow_{[x_0]} \mathrm{pt}.$$

6. Symmetric Monoidality. The triple (Sets<sub>\*</sub>,  $\coprod_X$ ,  $(X, x_0)$ ) is a symmetric monoidal category.

*Proof. Item* 1, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pushout diagram.

*Item 2, Associativity*: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.

*Item 3, Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.

*Item 4, Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.

Item 5, Interaction With Coproducts: Omitted.

*Item 6*, *Symmetric Monoidality*: Omitted.

**00BP 6.3.5** Coequalisers

Let  $f, g: (X, x_0) \Rightarrow (Y, y_0)$  be morphisms of pointed sets.

- **Definition 6.3.5.1.1.** The **coequaliser of** (f, g) is the pointed set  $(CoEq(f, g), [y_0])$ .
- **Construction 6.3.5.1.2.** The **coequaliser of** (f, g) is the pair  $((CoEq(f, g), [y_0]), coeq(f, g))$  consisting of:
  - The Colimit. The pointed set  $(CoEq(f, g), [y_0])$ , where CoEq(f, g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
  - The Cocone. The map

$$coeq(f,g): Y \rightarrow (CoEq(f,g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

*Proof.* We claim that  $(CoEq(f,g),[y_0])$  is the categorical coequaliser of f and g in Sets $_*$ . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\operatorname{coeq}(f,g)\circ f=\operatorname{coeq}(f,g)\circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g) \circ g](x)$$

for each  $x \in X$ . Next, we prove that CoEq(f,g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each  $a \in A$ , it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map  $\phi \colon \operatorname{CoEq}(f,g) \xrightarrow{\exists !} C$  making the diagram

commute, where we note that  $\phi$  is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \text{coeq}(f, g)]([y_0])$$
  
=  $c([y_0])$   
= \*,

where we have used that c is a morphism of pointed sets.

**Proposition 6.3.5.1.3.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.

00BS 1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ g, \mathrm{coeq}(f,g) \circ h)} \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g, \mathrm{coeq}(g,h) \circ g, \mathrm{coeq}(g,h) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ h)}$$

where CoEq(f, q, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets\*.

00BT 2. Unitality. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

**OOBU** 3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f, g) \cong CoEq(g, f)$$
.

*Proof. Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.

*Item 2, Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.

*Item 3, Commutativity*: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5. □

## **00BV 6.4** Constructions With Pointed Sets

#### 00BW 6.4.1 Free Pointed Sets

Let *X* be a set.

- **Definition 6.4.1.1.1.** The **free pointed set on** X is the pointed set  $X^+$  consisting of:
  - The Underlying Set. The set  $X^+$  defined by <sup>13</sup>

$$X^+ \stackrel{\text{def}}{=} X \coprod \text{pt}$$
  
 $\stackrel{\text{def}}{=} X \coprod \{ \star \}.$ 

- *The Basepoint.* The element  $\star$  of  $X^+$ .
- **OOBY** Proposition 6.4.1.1.2. Let X be a set.
- 00BZ 1. Functoriality. The assignment  $X \mapsto X^+$  defines a functor

$$(-)^+$$
: Sets  $\rightarrow$  Sets<sub>\*</sub>,

where:

• *Action on Objects*. For each  $X \in Obj(Sets)$ , we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where  $X^+$  is the pointed set of Definition 6.4.1.1.

<sup>&</sup>lt;sup>13</sup> Further Notation: We sometimes write  $\star_X$  for the basepoint of  $X^+$  for clarity, specially when there are multiple free pointed sets involved in the current discussion.

• *Action on Morphisms*. For each morphism  $f: X \to Y$  of Sets, the image

$$f^+\colon X^+\to Y^+$$

of f by  $(-)^+$  is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

00C0 2. Adjointness. We have an adjunction

$$((-)^+$$
 ¬ 忘): Sets  $\underbrace{\bot}_{\stackrel{\leftarrow}{\Longrightarrow}}$  Sets<sub>\*</sub>,

witnessed by a bijection of sets

$$\operatorname{\mathsf{Sets}}_*((X^+, \star_X), (Y, y_0)) \cong \operatorname{\mathsf{Sets}}(X, Y),$$

natural in  $X \in \text{Obj}(\mathsf{Sets})$  and  $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+,(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathsf{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} : X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod} : \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in  $X, Y \in Obj(Sets)$ .

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)^+_{\parallel}): (\mathsf{Sets}, \times, \mathsf{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_{\sharp}^+ \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+.$$

natural in  $X, Y \in Obj(Sets)$ .

*Proof. Item 1, Functoriality:* We claim that  $(-)^+$  is indeed a functor:

• *Preservation of Identities.* Let  $X \in Obj(Sets)$ . We have

$$\operatorname{id}_{X}^{+}(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_{X} & \text{if } x = \star_{X}, \end{cases}$$

for each  $x \in X^+$ , so  $id_X^+ = id_{X^+}$ .

• Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$
,  $g: Y \to Z$ ,

we have

$$[g^+ \circ f^+](x) \stackrel{\text{def}}{=} g^+(f^+(x))$$

$$\stackrel{\text{def}}{=} g^+(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^+(x)$$

for each  $x \in X$  and

$$[g^{+} \circ f^{+}](\star_{X}) \stackrel{\text{def}}{=} g^{+}(f^{+}(\star_{X}))$$

$$\stackrel{\text{def}}{=} g^{+}(\star_{Y})$$

$$\stackrel{\text{def}}{=} \star_{Z}$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(\star_{X})$$

so 
$$(g \circ f)^+ = g^+ \circ f^+$$
.

This finishes the proof.

*Item 2, Adjointness*: We proceed in a few steps:

• Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_* (X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger} \colon X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X$ .

• Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function  $\xi \colon X \to Y$  to the morphism of pointed sets

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each  $x \in X^+$ .

• Invertibility I. Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\Phi_{X,Y}(\xi)\right) \\ &= \Psi_{X,Y} \left(\xi^{\dagger}\right) \\ &\stackrel{\text{def}}{=} \left[\!\!\left[x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\!\!\right] \\ &= \left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\!\!\right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[\text{id}_{\mathsf{Sets}_*(X^+,Y)}\right] (\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)} .$$

• *Invertibility II.* Given a map of sets  $\xi: X \to Y$ , we have

$$\begin{split} \left[ \Phi_{X,Y} \circ \Psi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big( \Psi_{X,Y} (\xi) \big) \\ &= \Phi_{X,Y} \bigg( \xi^{\dagger} \bigg) \\ &= \Phi_{X,Y} \bigg( \left[ x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \bigg) \\ &= \left[ x \mapsto \xi(x) \right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[ \text{id}_{\mathsf{Sets}(X,Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}$$
.

• Naturality for  $\Phi$ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\begin{aligned} \mathsf{Sets}_*(X'^{,+},Y) & \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}(X',Y) \\ f^* & & \downarrow f^* \\ \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X,Y) \end{aligned}$$

commutes. Indeed, given a morphism of pointed sets  $\xi\colon X'^{,+}\to Y$ , we have

$$\begin{aligned} \left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X',Y}(\xi) \circ f \\ &= f^*(\Phi_{X',Y}(\xi)) \\ &= f^*(\Phi_{X',Y}(\xi)) \end{aligned}$$

$$= [f^* \circ \Phi_{X',Y}](\xi).$$

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

• Naturality for  $\Phi$ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\begin{array}{ccc} \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X,Y) \\ & g_* & & \downarrow g_* \\ & & \downarrow g_* \\ & \mathsf{Sets}_*(X^+,Y'), \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}(X,Y') \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger} \colon X^+ \to Y,$$

we have

$$\begin{split} \left[ \Phi_{X,Y'} \circ g_* \right] (\xi) &= \Phi_{X,Y'} (g_*(\xi)) \\ &= \Phi_{X,Y'} (g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'} (\xi) \\ &= g_* \big( \Phi_{X,Y'} (\xi) \big) \\ &= \left[ g_* \circ \Phi_{X,Y'} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'} \circ q_* = q_* \circ \Phi_{X,Y'}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

• Naturality for  $\Psi$ . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\Psi$  is also natural in each argument.

This finishes the proof.

*Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums*: We construct the strong monoidal structure on  $(-)^+$  with respect to [] and  $\lor$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{+,\coprod}_{X,Y}:X^+\vee Y^+\stackrel{\sim}{\dashrightarrow}(X\coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each  $z \in X^+ \vee Y^+$ , with inverse

$$(-)_{X,Y}^{+,\coprod,-1}\colon (X\coprod Y)^+\stackrel{\sim}{\dashrightarrow} X^+\vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each  $z \in (X \coprod Y)^+$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} \colon \operatorname{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending  $\star_X$  to  $\star_{\emptyset}$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on  $(-)^+$  with respect to  $\times$  and  $\wedge$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each  $x \land y \in X^+ \land Y^+$ , with inverse

$$(-)_{XY}^{+,-1} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each  $z \in (X \times Y)^+$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \operatorname{pt}^+$$

is given by sending 0 to  $\star_{pt}$  and 1 to  $\star$ , where  $pt^+ = \{\star, \star_{pt}\}$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

### 01QS 6.4.2 Deleting Basepoints

Let  $(X, x_0)$  be a pointed set.

**Definition 6.4.2.1.1.** The **set with deleted basepoint associated to** X is the set  $X^-$  defined by

$$X^{-} \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

**Old Proposition 6.4.2.1.2.** Let  $(X, x_0)$  be a pointed set.

01QV 1. Functoriality. The assignment  $(X, x_0) \mapsto X^-$  defines a functor

$$X^-: \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets},$$

where:

• Action on Objects. For each  $X \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}}_*)$ , we have

$$[(-)^{-}](X) \stackrel{\text{def}}{=} X^{-},$$

where  $X^-$  is the set of Definition 6.4.2.1.1.

• Action on Morphisms. For each morphism  $f: X \to Y$  of  $\mathsf{Sets}^{\mathsf{actv}}_*$ , the image

$$f^-\colon X^-\to Y^-$$

of f by  $(-)^-$  is the map defined by

$$f^{-}(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in X^-$ .

**01QW** 2. *Adjoint Equivalence*. We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+)$$
: Sets<sup>actv</sup> $\underbrace{\stackrel{(-)^-}{\downarrow_{eq}}}$ Sets,

witnessed by a bijection of sets

$$Sets(X^-, Y) \cong Sets_*(X, Y^+),$$

natural in  $X \in \text{Obj}(\mathsf{Sets}_*)$  and  $Y \in \text{Obj}(\mathsf{Sets})$ , and by isomorphisms

$$(X^-)^+ \cong X,$$

$$(Y^+)^- \cong Y$$
,

once again natural in  $X \in \mathsf{Obj}(\mathsf{Sets}_*)$  and  $Y \in \mathsf{Obj}(\mathsf{Sets})$ .

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\vee},(-)_{1}^{-,\vee}\right)\colon \left(\mathsf{Sets}^{\mathsf{actv}}_{*},\vee,\mathsf{pt}\right), \to \left(\mathsf{Sets}, \coprod, \emptyset\right),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-,$$
$$(-)_{1}^{-,\vee} \colon \emptyset \xrightarrow{\sim} \mathsf{pt}^-,$$

natural in  $X, Y \in Obj(Sets)$ .

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)^{-,\times}_1): (\mathsf{Sets}^{\mathsf{actv}}_*, \wedge, S^0), \to (\mathsf{Sets}, \times, \mathsf{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^-: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-,$$
  
 $(-)_{\parallel}^-: \operatorname{pt} \xrightarrow{\sim} (S^0)^-,$ 

natural in  $X, Y \in Obj(Sets)$ .

*Proof.* Item 1, Functoriality: We claim that  $(-)^-$  is indeed a functor:

• Preservation of Identities. Let  $X \in \mathsf{Obj}(\mathsf{Sets})$ . We have

$$id_X^-(x) \stackrel{\text{def}}{=} x$$

for each  $x \in X^-$ , so  $\mathrm{id}_X^- = \mathrm{id}_{X^-}$ .

• Preservation of Composition. Given morphisms of pointed sets

$$f\colon (X,x_0)\to (Y,y_0),$$

$$q: (Y, y_0) \rightarrow (Z, z_0),$$

we have

$$[g^{-} \circ f^{-}](x) \stackrel{\text{def}}{=} g^{-}(f^{-}(x))$$

$$\stackrel{\text{def}}{=} g^{-}(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^{-}(x)$$

for each  $x \in X$ , so  $(g \circ f)^- = g^- \circ f^-$ .

This finishes the proof.

*Item 2, Adjoint Equivalence*: We proceed in a few steps:

025H 1. Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-, Y) \to \mathsf{Sets}^{\mathsf{actv}}_*(X, Y^+)$$

by sending a map  $\xi \colon X^- \to Y$  to the active morphism of pointed sets

$$\xi^{\dagger} \colon X \to Y^{+}$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each  $x \in X$ , where this morphism is indeed active since  $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$  for all  $x \in X^-$ .

025J 2. Map II. We define a map

$$\Psi_{XY} \colon \mathsf{Sets}^{\mathsf{actv}}_{\star}(X, Y^+) \to \mathsf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets  $\xi\colon X\to Y^+$  to the map

$$\xi^{\dagger} \colon X^{-} \to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X^-$ , which is indeed well-defined (in that  $\xi(x) \in Y$  for all  $x \in X^-$ ) since  $\xi$  is active.

**025K** 3. *Invertibility I.* Given a map of sets  $\xi: X^- \to Y$ , we have

$$\begin{split} \left[ \Psi_{X,Y} \circ \Phi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left( \Phi_{X,Y} (\xi) \right) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \left( \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^{-} \\ \star_{Y} & \text{if } x = x_{0} \end{cases} \right] \right) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \end{split}$$

$$= \left[ \mathrm{id}_{\mathsf{Sets}(X^-,Y)} \right] (\xi).$$

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}$$
.

**Q25L** 4. *Invertibility II*. Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big(\Psi_{X,Y}(\xi)\big) \\ &= \Phi_{X,Y} (\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= \left[ \text{id}_{\mathsf{Sets}^{\mathsf{actv}}(X,Y^+)} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} .$$

025M 5. Naturality for  $\Phi$ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\mathsf{Sets}(X^{',-},Y) \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X',Y^+)$$

$$f^* \downarrow \qquad \qquad \downarrow f^*$$

$$\mathsf{Sets}_*(X^-,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

commutes. Indeed, given a map of sets  $\xi \colon X' \to Y$ , we have

$$\big[\Phi_{X,Y}\circ f^*\big](\xi)=\Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \begin{bmatrix} x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{-} \\ \star_{Y} & \text{if } f(x) = x'_{0} \end{cases} \end{bmatrix}$$

$$= f^{*} \left( \begin{bmatrix} x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{-} \\ \star_{Y} & \text{if } x' = x'_{0} \end{cases} \end{bmatrix} \right)$$

$$= f^{*} \left( \Phi_{X',Y}(\xi) \right)$$

$$= \left[ f^{*} \circ \Phi_{X',Y} \right](\xi).$$

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

6. Naturality for  $\Phi$ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\mathsf{Sets}(X^-,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

$$g_* \qquad \qquad \downarrow g_*$$

$$\mathsf{Sets}(X^-,Y') \xrightarrow{\Phi_{X,Y'}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y'^{+})$$

commutes. Indeed, given a map of sets  $\xi \colon X^- \to Y$ , we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= \left[\!\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}\right]\!\!\right] \\ &= g_* \left(\!\!\left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases}\right]\!\!\right) \\ &= g_* \left(\Phi_{X,Y'}(\xi)\right) \end{split}$$

$$= [g_* \circ \Phi_{X,Y'}](\xi).$$

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

- 025P 7. Naturality for  $\Psi$ . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\Psi$  is also natural in each argument.
- 8. Fully Faithfulness of  $(-)^-$ . We aim to show that the assignment  $f \mapsto f^-$  sets up a bijection

$$(-)_{XY}^- \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y) \xrightarrow{\sim} \mathsf{Sets}(X^-,Y^-).$$

Indeed, the inverse map

$$(-)_{X,Y}^{-,-1} \colon \mathsf{Sets}(X^-,Y^-) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y)$$

is given by sending a map of sets  $f\colon X^-\to Y^-$  to the active morphism of pointed sets  $f^\dag\colon X\to Y$  defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each  $x \in X$ .

025R 9. Essential Surjectivity of  $(-)^-$ . We need to show that, given an object  $X \in Obj(Sets)$ , there exists some  $X' \in Obj(Sets^{actv}_*)$  such that  $(X')^- \cong X$ . Indeed, taking  $X' = X^+$ , we have

$$(X^{+})^{-} \stackrel{\text{def}}{=} (X \cup \{\star_{X}\})^{-}$$
$$\stackrel{\text{def}}{=} (X \cup \{\star_{X}\}) \setminus \{\star_{X}\}$$
$$= X,$$

and thus we have in fact an *equality*  $(X^+)^- = X$ , showing  $(-)^-$  to be essentially surjective.

025S 10. The Functor  $(-)^-$  Is an Equivalence. Since  $(-)^-$  is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.

*Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums*: We construct the strong monoidal structure on (-) with respect to  $\vee$  and [] as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{-,\vee}_{XY} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each  $z \in X^- \coprod Y^-$ , with inverse

$$(-)_{X,Y}^{-,\vee,-1}\colon (X\vee Y)^{-}\stackrel{\sim}{\dashrightarrow} X^{-}\coprod Y^{-}$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each  $z \in (X \vee Y)^-$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\vee,1} \colon \emptyset \xrightarrow{\sim} \mathsf{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^-$  into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on  $(-)^+$  with respect to  $\wedge$  and  $\times$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^-_{X,Y} \colon X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{X,Y}(x,y) = x \wedge y$$

for each  $(x, y) \in X^- \times Y^-$ , with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)_{XY}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x,y)$$

for each  $x \wedge y \in (X \wedge Y)^-$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{-,1} \colon \operatorname{pt} \xrightarrow{\sim} (S^0)^-$$

is given by sending  $\star$  to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

# **Appendices**

### **A** Other Chapters

#### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

3. Sets

- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

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#### **Relations**

**Categories** 

#### **Monoidal Categories**

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations
- 13. Constructions With Monoidal Categories

#### **Bicategories**

14. Types of Morphisms in Bicategories

11. Categories

# 12. Presheaves and the Yoneda

Lemma

15. Notes

Extra Part

### **References**

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