Pointed Sets

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This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

Definition 6.1.1.1.1. A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_{\bullet}(Sets), pt)$.
- A pointed object in (Sets, pt).

Remark 6.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- The Underlying Set. A set X, called the **underlying set of** (X, x_0) .
- The Basepoint. A morphism

$$[x_0]: pt \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 6.1.1.1.3. The 0-sphere² is the pointed set $(S^0, 0)^3$ consisting of:

• The Underlying Set. The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

• *The Basepoint*. The element 0 of S^0 .

Example 6.1.1.1.4. The **trivial pointed set** is the pointed set (pt, \star) consisting of:

¹Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -modules.

²Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

³ Further Notation: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also

- *The Underlying Set.* The punctual set pt $\stackrel{\text{def}}{=} \{ \star \}$.
- *The Basepoint.* The element \star of pt.

Example 6.1.1.1.5. The **standard pointed set with** n + 1 **elements** is the pointed set $\langle n \rangle$ consisting of

• The Underlying Set. The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

• The Basepoint. The element * of $\langle n \rangle$.

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1. A **morphism of pointed sets**^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(N_{\bullet}(Sets), pt)$.
- A morphism of pointed objects in (Sets, pt).

Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f: (X, x_0) \to (Y, y_0)$ is a morphism of sets $f: X \to Y$ such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & & [y_0] \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes, i.e. such that

$$f(x_0)=y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1. The **category of pointed sets** is the category Sets_{*} defined equivalently as:

denoted $(\mathbb{F}_1, 0)$.

⁴ Further Terminology: Also called a **pointed function**.

⁵ Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of** \mathbb{F}_1 -**modules**.

- The homotopy category of the ∞ -category $\mathsf{Mon}_{\mathbb{E}_n}(\mathsf{N}_{\bullet}(\mathsf{Sets}),\mathsf{pt})$ of ??,??.
- The category Sets* of Constructions With Categories, ??.

Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category $Sets_*$ where:

- Objects. The objects of Sets* are pointed sets.
- *Morphisms*. The morphisms of Sets* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in Obj(Sets_*)$, the unit map

$$\mathbb{1}_{(X,x_0)}^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets_{*} at (X, x_0) is defined by⁶

$$id_{(X,x_0)}^{Sets_*} \stackrel{\text{def}}{=} id_X$$
.

• *Composition*. For each (X, x_0) , (Y, y_0) , $(Z, z_0) \in Obj(Sets_*)$, the composition map

6.1.4 Elementary Properties of Pointed Sets

Proposition 6.1.4.1.1. Let (X, x_0) be a pointed set.

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

⁶Note that id_X is indeed a morphism of pointed sets, as we have $id_X(x_0) = x_0$.

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

- 1. *Completeness*. The category Sets_{*} of pointed sets and morphisms between them is complete, having in particular:
 - (a) Products, described as in Definition 6.2.3.1.1.
 - (b) Pullbacks, described as in Definition 6.2.4.1.1.
 - (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets_{*} of pointed sets and morphisms between them is cocomplete, having in particular:
 - (a) Coproducts, described as in Definition 6.3.3.1.1.
 - (b) Pushouts, described as in Definition 6.3.4.1.1;
 - (c) Coequalisers, described as in Definition 6.3.5.1.1.
- 3. Failure To Be Cartesian Closed. The category Sets* is not Cartesian closed.8
- 4. Morphisms From the Monoidal Unit. We have a bijection of sets⁹

$$\mathsf{Sets}_*(S^0,X)\cong X,$$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\big(S^0,X\big)\cong (X,x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

5. Relation to Partial Functions. We have an equivalence of categories 10

$$\mathsf{Sets}_* \stackrel{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

Warning: This is not an isomorphism of categories, only an equivalence.

⁸The category Sets_{*} does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

⁹In other words, the forgetful functor

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

- i. A pointed set (X, x_0) to X.
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\mathsf{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

- i. A set *X* is to the pointed set (X, \star) with \star an element that is not in *X*.
- ii. A partial function

$$f: X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, *Completeness*: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, *Morphisms From the Monoidal Unit*: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*\big(S^0,X\big)\cong (X,x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of **Sets**_{*}(S^0, X), corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets. *Item* 5, *Relation to Partial Functions*: See [MSE 884460].

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1. Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a morphism of pointed sets.

- 1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
- 2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2. We write $Sets^{actv}_*$ for the wide subcategory of $Sets_*$ spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

1. The map $\mu: \langle 2 \rangle \to \langle 1 \rangle$ given by

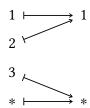
$$\begin{array}{ccc}
1 & & & \\
2 & & & \\
* & & & & \\
\end{array}$$

is active but not inert.

2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by

is inert but not active.

3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by



is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i: \langle 3 \rangle \to \langle 2 \rangle,$$

 $a: \langle 2 \rangle \to \langle 1 \rangle$

are the morphisms of pointed sets given by

with *i* being inert and *a* being active.

Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. Active-Inert Factorisation. Every morphism of pointed sets $f:(X,x_0) \to (Y,y_0)$ factors uniquely as

$$f = a \circ i$$
,

where:

- (a) The map $i: (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets
- (b) The map $a: (K, k_0) \rightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets*.

Proof. Item 1, *Active-Inert Factorisation*: Let $f: X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\longrightarrow} K \stackrel{a}{\longrightarrow} Y$$
.

where:

• *K* is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$

= $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$

• $i: X \to K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

• $a: K \to Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$X \xrightarrow{i} Y$$

$$f \downarrow \qquad \qquad \downarrow g$$

$$A \xrightarrow{g} B$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y\to A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$X \xrightarrow{i} Y$$

$$f \downarrow \exists ! \nearrow \phi \downarrow g$$

$$A \xrightarrow{g} B.$$

Indeed, this diagram commutes, as we have

$$[\phi \circ i](x) \stackrel{\text{def}}{=} \phi(i(x))$$
$$\stackrel{\text{def}}{=} f(i^{-1}(i(x)))$$
$$= f(x)$$

for each $x \in X$ and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & \downarrow & \downarrow g \\
A & \xrightarrow{a} & B
\end{array}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1. The **terminal pointed set** is the terminal object of Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.1.1.2. Concretely, the **terminal pointed set** is the pair (pt, \star) , $\{!_X\}_{(X,x_0) \in Obj(Sets_*)}$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- The Cone. The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\operatorname{pt}, \star)\}_{(X, x_0) \in \operatorname{Obj}(\operatorname{\mathsf{Sets}})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathsf{Sets})$.

Proof. We claim that (pt, \star) is the terminal object of Sets_{*}. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt, \star)

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\operatorname{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists 1}} (pt, \star)$$

commute, namely $!_X$.

6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 6.2.2.1.1. The **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.2.1.2. Concretely, the **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $(\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$ consisting of:

- The Limit. The pointed set $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$.
- The Cone. The collection

$$\left\{ \operatorname{pr}_{i} : \left(\prod_{i \in I} X_{i}, \left(x_{0}^{i} \right)_{i \in I} \right) \to \left(X_{i}, x_{0}^{i} \right) \right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_i\Big(\big(x_j\big)_{j\in I}\Big)\stackrel{\mathrm{def}}{=} x_i$$

for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i\in I$.

Proof. We claim that $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets*. Indeed, suppose we have, for each $i\in I$, a diagram of the form

$$(P, *) \xrightarrow{p_i} (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \xrightarrow{\text{DI}_i} (X_i, x_0^i)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left(\prod_{i \in I} X_i, \left(x_0^i \right)_{i \in I} \right)$$

making the diagram

$$(P, *)$$

$$\phi \mid \exists !$$

$$(\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \xrightarrow{\operatorname{pr}_i} (X_i, x_0^i)$$

commute, being uniquely determined by the condition $\operatorname{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

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for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= (x_0^i)_{i \in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$.

Proposition 6.2.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I} : \operatorname{Fun}(I_{\operatorname{disc}}, \operatorname{Sets}_*) \to \operatorname{Sets}_*.$$

Proof. Item 1, Functoriality: This follows from Limits and Colimits, ?? of ??. \Box

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.3.1.2. Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- The Limit. The pointed set $(X \times Y, (x_0, y_0))$.
- The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 \colon (X \times Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$

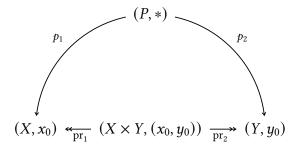
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and

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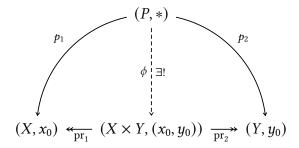
 (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$pr_1 \circ \phi = p_1,$$

$$pr_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 6.2.3.1.3. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

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1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$A \times -:$$
 Sets_{*} \rightarrow Sets_{*},
 $- \times B:$ Sets_{*} \rightarrow Sets_{*},
 $-_1 \times -_2:$ Sets_{*} \times Sets_{*} \rightarrow Sets_{*},

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

- 2. *Lack of Adjointness.* The functors $X \times -$ and $\times Y$ do not admit right adjoints.
- 3. Associativity. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(Sets_*).$

4. *Unitality*. We have isomorphisms of pointed sets

$$(pt, \star) \times (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \times (pt, \star) \cong (X, x_0),$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

5. Commutativity. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\mathsf{Sets}_*).$

6. *Symmetric Monoidality.* The triple (Sets $_*$, \times , (pt, \star)) is a symmetric monoidal category.

Proof. Item **1**, *Functoriality*: This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.

Item 2, Lack of Adjointness: See [MSE 2855868].

Item 3, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.

Item 4, *Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.

Item 5, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.

Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3.

6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.4.1.2. Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, q) is the pair consisting of:

- The Limit. The pointed set $(X \times_Z Y, (x_0, y_0))$.
- The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad \operatorname{pr}_{1} \bigg| g$$

$$(X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$

$$\operatorname{pr}_{1} \bigg| g$$

$$(X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$[f \circ \operatorname{pr}_1](x, y) = f(\operatorname{pr}_1(x, y))$$

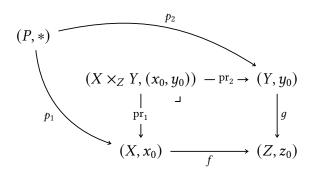
$$= f(x)$$

$$= g(y)$$

$$= g(\operatorname{pr}_2(x, y))$$

$$= [g \circ \operatorname{pr}_2](x, y),$$

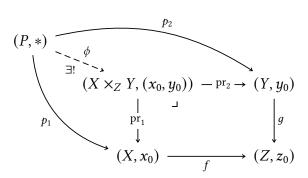
where f(x) = g(y) since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$

 $\operatorname{pr}_2 \circ \phi = p_2$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2$$
,

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets. \Box

Proposition 6.2.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

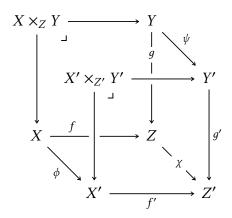
1. Functoriality. The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 : \operatorname{Fun}(\mathcal{P}, \operatorname{Sets}_*) \to \operatorname{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \times_{-3} -1$ is given by sending a morphism



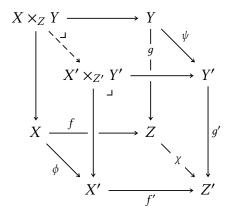
in $\operatorname{Fun}(\mathcal{P},\operatorname{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

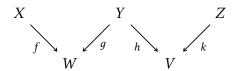
$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

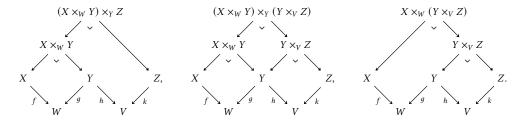
2. Associativity. Given a diagram



in Sets*, we have isomorphisms of pointed sets

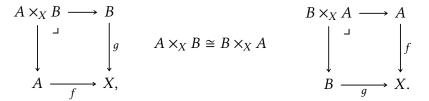
$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



3. *Unitality*. We have isomorphisms of pointed sets

4. *Commutativity*. We have an isomorphism of pointed sets



5. *Interaction With Products*. We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y,$$

$$X \times_{\mathrm{pt}} Y \cong X \times Y,$$

$$X \xrightarrow{!_{Y}} pt.$$

6. *Symmetric Monoidality.* The triple (Sets_{*}, \times_X , X) is a symmetric monoidal category.

Proof. Item **1**, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.

Item 3, Unitality: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.

Item 4, Commutativity: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.

Item 5, Interaction With Products: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.

Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5.

6.2.5 Equalisers

Let $f, g: (X, x_0) \Rightarrow (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1. The **equaliser of** (f, g) is the equaliser of f and g in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(Eq(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\operatorname{eq}(f,g) \colon (\operatorname{Eq}(f,g),x_0) \hookrightarrow (X,x_0)$$

given by the canonical inclusion $eq(f, g) \hookrightarrow Eq(f, g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f,g),x_0)$ is the categorical equaliser of f and g in Sets_{*}. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ eq(f, g) = g \circ eq(f, g),$$

which indeed holds by the definition of the set Eq(f,g). Next, we prove that Eq(f,g) satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$(E,*)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\text{Eq}(f, g), x_0)$$

making the diagram

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow^{\phi} |\exists! \qquad e$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f, q) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in Eq(f, g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

Proposition 6.2.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \underbrace{\mathrm{Eq}(f,g,h)}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))} = \underbrace{\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}$$

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets*, being explicitly given by

$$Eq(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, f) \cong X$$
.

3. *Commutativity*. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, g) \cong \operatorname{Eq}(g, f).$$

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3.

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1. The **initial pointed set** is the initial object of Sets_{*} as in Limits and Colimits, **??**.

Construction 6.3.1.1.2. Concretely, the **initial pointed set** is the pair (pt, \star) , $\{\iota_X\}_{(X,x_0) \in Obj(Sets_*)}$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- The Cone. The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathrm{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_{*}. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
 (X, x_0)

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt},\star) \xrightarrow{-\frac{\phi}{\exists !}} (X,x_0)$$

commute, namely ι_X .

6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 6.3.2.1.1. The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \{\text{inj}_i\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
 - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i \in I} X_i \right) / \sim,$$

where ~ is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \stackrel{\text{def}}{=} \left[(i, x_0^i) \right] \\ = \left[\left(j, x_0^j \right) \right]$$

for any $i, j \in I$.

• *The Cocone*. The collection

$$\left\{\operatorname{inj}_i\colon \left(X_i,x_0^i\right)\to \left(\bigvee_{i\in I}X_i,p_0\right)\right\}_{i\in I}$$

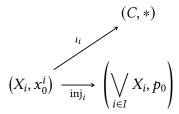
of morphism of pointed sets given by

$$\operatorname{inj}_{i}(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

 $^{^{11}}$ Further Terminology: Also called the **wedge sum of the family** $\left\{\left(X_{i},x_{0}^{i}\right)\right\}_{i\in I}$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets*. Indeed, suppose we have, for each $i \in I$, a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi : \left(\bigvee_{i \in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$(X_i, x_0^i) \xrightarrow[\inf_i]{l_i} (X_i, p_0)$$

commute, being uniquely determined by the condition $\phi \circ \operatorname{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$

$$= *.$$

as ι_i is a morphism of pointed sets.

Proposition 6.3.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \operatorname{Fun}(I_{\operatorname{disc}}, \operatorname{Sets}_*) \to \operatorname{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** $(Y, y_0)^{12}$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
 - The Underlying Set. The set $X \vee Y$ defined by

$$(X \lor Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \qquad X \lor Y \longleftarrow Y$$

$$\cong \left(X \coprod_{\text{pt}} Y, p_0 \right) \qquad \uparrow \qquad \downarrow [y_0]$$

$$\cong (X \coprod Y/\sim, p_0), \qquad X \longleftarrow_{[x_0]} \text{pt,}$$

where \sim is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$

= $[(1, y_0)].$

• *The Cocone*. The morphisms of pointed sets

$$inj_1: (X, x_0) \to (X \lor Y, p_0),
inj_2: (Y, y_0) \to (X \lor Y, p_0),$$

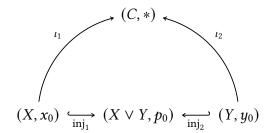
given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)],
inj_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

The state of (X, y_0) and (Y, y_0) .

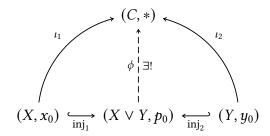
Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$

$$\phi \circ \operatorname{inj}_V = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

= $\iota_Y([(1, y_0)])$
= *.

as ι_X and ι_Y are morphisms of pointed sets.

Proposition 6.3.3.1.3. Let (X, x_0) and (Y, y_0) be pointed sets.

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$

2. Associativity. We have an isomorphism of pointed sets

$$(X \lor Y) \lor Z \cong X \lor (Y \lor Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in Sets_*$.

3. *Unitality*. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \lor (pt, *) \cong (X, x_0),$

natural in $(X, x_0) \in Sets_*$.

4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$

natural in $(X, x_0), (Y, y_0) \in Sets_*$.

- 5. *Symmetric Monoidality*. The triple (Sets_∗, ∨, pt) is a symmetric monoidal category.
- 6. *The Fold Map.* We have a natural transformation

$$\nabla\colon \vee\circ\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}\Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{array}{c} \mathsf{Sets}_*\times\mathsf{Sets}_*\\ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}& \bigvee\\ \mathsf{Sets}_*& \bigvee\\ \mathsf{Sets}_*& \bigvee\\ \mathsf{id}_{\mathsf{Sets}_*}& \mathsf{Sets}_*, \end{array}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at *X* is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

Item 5, Symmetric Monoidality: Omitted.

Item 6, *The Fold Map*: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \to (Y, y_0)$, we have

$$X \vee X \xrightarrow{\nabla_X} X$$

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \quad f \vee f \downarrow \qquad \qquad \downarrow f$$

$$Y \vee Y \xrightarrow{\nabla_Y} Y.$$

Indeed, we have

$$[\nabla_Y \circ (f \vee f)]([(i,x)]) = \nabla_Y([(i,f(x))])$$

$$= f(x)$$

$$= f(\nabla_X([(i,x)]))$$

$$= [f \circ \nabla_X]([(i,x)])$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. \square

6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.4.1.2. Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, q) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
 - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
 - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone*. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$

given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)]
inj_2(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \stackrel{\text{inj}_{2}}{\longleftarrow} (Y, y_{0})$$

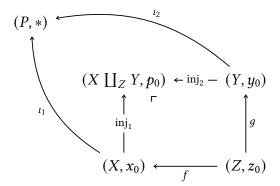
$$\text{inj}_{1} \circ f = \text{inj}_{2} \circ g, \qquad \text{inj}_{1} \qquad \qquad \int_{g} g$$

$$(X, x_{0}) \longleftarrow_{f} (Z, z_{0}).$$

Indeed, given $z \in Z$, we have

$$\begin{aligned} \left[\operatorname{inj}_1 \circ f \right](z) &= \operatorname{inj}_1(f(z)) \\ &= \left[(0, f(z)) \right] \\ &= \left[(1, g(z)) \right] \\ &= \operatorname{inj}_2(g(z)) \\ &= \left[\operatorname{inj}_2 \circ g \right](z), \end{aligned}$$

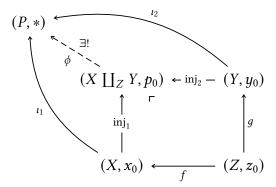
where [(0, f(z))] = [(1, g(z))] by the definition of the relation \sim on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1$$
,

$$\phi \circ \text{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$

= $\iota_1(x_0)$
= *,

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$

= $\iota_2(y_0)$
= *,

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

Proposition 6.3.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

1. Functoriality. The assignment $(X,Y,Z,f,g)\mapsto X\coprod_{f,Z,g}Y$ defines a functor

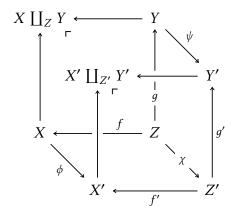
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P},\mathsf{Sets}) \to \mathsf{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \coprod_{-3} -1$ is given by sending

a morphism



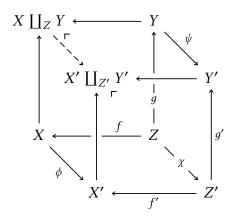
in $Fun(\mathcal{P}, Sets_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

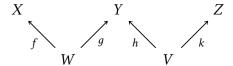
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

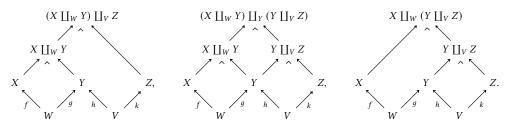
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

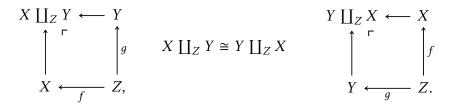
where these pullbacks are built as in the diagrams



3. *Unitality*. We have isomorphisms of sets



4. Commutativity. We have an isomorphism of sets



5. Interaction With Coproducts. We have

6. *Symmetric Monoidality.* The triple (Sets_{*}, \coprod_X , (X, x_0)) is a symmetric monoidal category.

Proof. Item **1**, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.

Item 3, Unitality: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.

Item 4, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.

Item 5, *Interaction With Coproducts*: Omitted.

Item 6, *Symmetric Monoidality*: Omitted.

6.3.5 Coequalisers

Let $f, g: (X, x_0) \Rightarrow (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(CoEq(f, g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f, g) is the pair $((CoEq(f, g), [y_0]), coeq(f, g))$ consisting of:

- The Colimit. The pointed set $(CoEq(f, g), [y_0])$, where CoEq(f, g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
- *The Cocone*. The map

$$coeq(f,g): Y \rightarrow (CoEq(f,g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(CoEq(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_{*}. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f, g) \circ f = coeq(f, g) \circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\text{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\text{coeq}(f,g) \circ g](x)$$

for each $x \in X$. Next, we prove that CoEq(f, g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\phi \colon \operatorname{CoEq}(f,g) \xrightarrow{\exists !} C$ making the diagram

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ coeq(f, g)]([y_0])$$
= $c([y_0])$
= *.

where we have used that *c* is a morphism of pointed sets.

Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)}\cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)}\cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets*.

2. Unitality. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f, g) \cong CoEq(g, f)$$
.

Proof. Item 1, *Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.

Item 2, *Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5.

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let *X* be a set.

Definition 6.4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

• The Underlying Set. The set X^+ defined by ¹³

$$X^{+} \stackrel{\text{def}}{=} X \coprod \text{pt}$$
$$\stackrel{\text{def}}{=} X \coprod \{ \star \}.$$

• *The Basepoint.* The element \star of X^+ .

¹³ Further Notation: We sometimes write \star_X for the basepoint of X^+ for clarity, specially when

Proposition 6.4.1.1.2. Let *X* be a set.

1. Functoriality. The assignment $X \mapsto X^+$ defines a functor

$$(-)^+$$
: Sets \rightarrow Sets_{*},

where:

• Action on Objects. For each $X \in \text{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

• Action on Morphisms. For each morphism $f: X \to Y$ of Sets, the image

$$f^+\colon X^+\to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. Adjointness. We have an adjunction

$$((-)^+ \dashv \stackrel{\leftarrow}{\sim}): \operatorname{Sets}_{\stackrel{\leftarrow}{\sim}}^{(-)^+} \operatorname{Sets}_*,$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+,(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathsf{pt}),$$

there are multiple free pointed sets involved in the current discussion.

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} : X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod} : \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)^+_{\parallel}) : (\operatorname{Sets}, \times, \operatorname{pt}) \to (\operatorname{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_{1}^+ \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+,$$

natural in $X, Y \in Obj(Sets)$.

Proof. Item 1, *Functoriality*: We claim that $(-)^+$ is indeed a functor:

• Preservation of Identities. Let $X \in \text{Obj}(\mathsf{Sets})$. We have

$$\operatorname{id}_{X}^{+}(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_{X} & \text{if } x = \star_{X}, \end{cases}$$

for each $x \in X^+$, so $id_X^+ = id_{X^+}$.

• Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$
, $g: Y \to Z$,

we have

$$[g^{+} \circ f^{+}](x) \stackrel{\text{def}}{=} g^{+}(f^{+}(x))$$

$$\stackrel{\text{def}}{=} g^{+}(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(x)$$

for each $x \in X$ and

$$[g^{+} \circ f^{+}](\star_{X}) \stackrel{\text{def}}{=} g^{+}(f^{+}(\star_{X}))$$

$$\stackrel{\text{def}}{=} g^{+}(\star_{Y})$$

$$\stackrel{\text{def}}{=} \star_{Z}$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(\star_{X}),$$

so
$$(g \circ f)^+ = g^+ \circ f^+$$
.

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

• Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_* \big(X^+, Y \big) \to \mathsf{Sets}(X,Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger} \colon X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

• Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function $\xi \colon X \to Y$ to the morphism of pointed sets

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

• Invertibility I. Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\Phi_{X,Y}(\xi)\right) \\ &= \Psi_{X,Y} \left(\xi^{\dagger}\right) \\ &\stackrel{\text{def}}{=} \left[\!\!\left[x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\!\!\right] \\ &= \left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\!\!\right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[\text{id}_{\mathsf{Sets}_*(X^+,Y)}\right](\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)} .$$

• *Invertibility II.* Given a map of sets $\xi \colon X \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big(\Psi_{X,Y} (\xi) \big) \\ &= \Phi_{X,Y} \Big(\left[\!\! \left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \right) \\ &= \left[\!\! \left[x \mapsto \xi(x) \right] \!\! \right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[\text{id}_{\mathsf{Sets}(X,Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}$$
.

• Naturality for Φ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\begin{aligned} \mathsf{Sets}_*(X'^{+},Y) & \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}(X',Y) \\ f^* & & \downarrow f^* \\ \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X,Y) \end{aligned}$$

commutes. Indeed, given a morphism of pointed sets $\xi \colon X'^{+} \to Y$, we have

$$[\Phi_{X,Y} \circ f^*](\xi) = \Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \xi \circ f$$

$$= \Phi_{X',Y}(\xi) \circ f$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= [f^* \circ \Phi_{X',Y}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

• Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g\colon (Y,y_0)\to (Y',y_0'),$$

the diagram

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger} \colon X^{+} \to Y,$$

we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_* \left(\Phi_{X,Y'}(\xi)\right) \\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

• Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to [] and \lor as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{X,Y}^{+,\coprod}: X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)^{+,\coprod,-1}_{X,Y}\colon (X\coprod Y)^+\stackrel{\sim}{\dashrightarrow} X^+\vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} : \operatorname{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_{\emptyset} .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \land y \in X^+ \land Y^+$, with inverse

$$(-)^{+,-1}_{XY} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \operatorname{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $pt^+ = \{\star, \star_{pt}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1. The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^{-} \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2. Let (X, x_0) be a pointed set.

1. Functoriality. The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^-: \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets},$$

where:

• Action on Objects. For each $X \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}}_*)$, we have

$$[(-)^{-}](X) \stackrel{\text{def}}{=} X^{-},$$

where X^- is the set of Definition 6.4.2.1.1.

• Action on Morphisms. For each morphism $f\colon X\to Y$ of $\mathsf{Sets}^{\mathsf{actv}}_*$, the image

$$f^-\colon X^-\to Y^-$$

of f by $(-)^-$ is the map defined by

$$f^{-}(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

2. Adjoint Equivalence. We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+)$$
: Sets* $(-)^-$ Sets,

witnessed by a bijection of sets

$$Sets(X^-, Y) \cong Sets_*(X, Y^+),$$

natural in $X \in \text{Obj}(\mathsf{Sets}_*)$ and $Y \in \text{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^{-})^{+} \cong X,$$
$$(Y^{+})^{-} \cong Y,$$

once again natural in $X \in \text{Obj}(\mathsf{Sets}_*)$ and $Y \in \text{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\vee},(-)_{\scriptscriptstyle 1}^{-,\vee}\right)\colon \left(\mathsf{Sets}^{\mathsf{actv}}_*,\vee,\mathsf{pt}\right),\to \left(\mathsf{Sets},\mathrel{\textstyle\coprod},\varnothing\right),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-,$$
$$(-)_{1}^{-,\vee} \colon \emptyset \xrightarrow{\sim} \mathsf{pt}^-,$$

natural in $X, Y \in Obj(Sets)$.

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)^{-,\times}): (\operatorname{Sets}^{\operatorname{actv}}_*, \wedge, S^0), \to (\operatorname{Sets}, \times, \operatorname{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-},$$
$$(-)_{1}^{-} \colon \operatorname{pt} \xrightarrow{\sim} (S^{0})^{-},$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

Proof. Item 1, *Functoriality*: We claim that $(-)^-$ is indeed a functor:

• Preservation of Identities. Let $X \in \text{Obj}(\mathsf{Sets})$. We have

$$id_X^-(x) \stackrel{\text{def}}{=} x$$

for each $x \in X^-$, so $id_X^- = id_{X^-}$.

• Preservation of Composition. Given morphisms of pointed sets

$$f: (X, x_0) \to (Y, y_0),$$

 $g: (Y, y_0) \to (Z, z_0),$

we have

$$[g^{-} \circ f^{-}](x) \stackrel{\text{def}}{=} g^{-}(f^{-}(x))$$

$$\stackrel{\text{def}}{=} g^{-}(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^{-}(x)$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2, *Adjoint Equivalence*: We proceed in a few steps:

1. *Map I*. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-, Y) \to \mathsf{Sets}^{\mathsf{actv}}_*(X, Y^+)$$

by sending a map $\xi \colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^{\dagger} \colon X \to Y^{+}$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$

given by sending an active morphism of pointed sets $\xi\colon X\to Y^+$ to the map

$$\xi^{\dagger}\colon X^{-}\to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi \colon X^- \to Y$, we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\Phi_{X,Y}(\xi)\right) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\begin{bmatrix} x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^{-} \\ \star_{Y} & \text{if } x = x_{0} \end{cases} \right] \right) \\ &= \begin{bmatrix} x \mapsto \xi(x) \end{bmatrix} \\ &= \xi \\ &= \left[\text{id}_{\mathsf{Sets}(X^{-},Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}$$
.

4. Invertibility II. Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big(\Psi_{X,Y}(\xi)\big) \\ &= \Phi_{X,Y} \big(\llbracket x \mapsto \xi(x) \rrbracket \big) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \end{bmatrix} \\ &= \xi \\ &= \left[\text{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathrm{actv}}_{\circ}(X,Y^+)}$$
.

5. Naturality for Φ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\operatorname{Sets}(X^{',-},Y) \xrightarrow{\Phi_{X',Y}} \operatorname{Sets}^{\operatorname{actv}}_*(X',Y^+)$$

$$f^* \downarrow \qquad \qquad \downarrow f^*$$

$$\operatorname{Sets}_*(X^-,Y) \xrightarrow{\Phi_{X,Y}} \operatorname{Sets}^{\operatorname{actv}}_*(X,Y^+)$$

commutes. Indeed, given a map of sets $\xi \colon X' \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \left[\!\left[x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases}\right]\!\right] \\ &= f^* \left(\!\left[\!\left[x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x'_0 \end{cases}\right]\!\right) \end{split}$$

$$= f^* (\Phi_{X',Y}(\xi))$$

= $[f^* \circ \Phi_{X',Y}](\xi).$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

commutes. Indeed, given a map of sets $\xi \colon X^- \to Y$, we have

$$\begin{aligned} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \end{aligned}$$

$$= \left[\!\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \right]\!\!\right]$$

$$= g_* \left(\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases} \right]\!\!\right)$$

$$= g_* \left(\Phi_{X,Y'}(\xi)\right)$$

$$= \left[g_* \circ \Phi_{X,Y'}\right](\xi).$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for $\boldsymbol{\Phi}$ above indeed commutes.

- 7. Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.
- 8. Fully Faithfulness of $(-)^-$. We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{X,Y}^{-} \colon \mathsf{Sets}^{\mathsf{actv}}_{*}(X,Y) \xrightarrow{\sim} \mathsf{Sets}(X^{-},Y^{-}).$$

Indeed, the inverse map

$$(-)_{X,Y}^{-,-1} \colon \mathsf{Sets}(X^-,Y^-) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y)$$

is given by sending a map of sets $f\colon X^-\to Y^-$ to the active morphism of pointed sets $f^\dagger\colon X\to Y$ defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_{0} & \text{if } x = x_{0} \end{cases}$$

for each $x \in X$.

9. Essential Surjectivity of $(-)^-$. We need to show that, given an object $X \in \text{Obj}(\mathsf{Sets})$, there exists some $X' \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$(X^{+})^{-} \stackrel{\text{def}}{=} (X \cup \{\star_{X}\})^{-}$$
$$\stackrel{\text{def}}{=} (X \cup \{\star_{X}\}) \setminus \{\star_{X}\}$$
$$= X,$$

and thus we have in fact an equality $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. The Functor $(-)^-$ Is an Equivalence. Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on (-) with respect to \vee and [] as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{-,\vee}_{X,Y} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \coprod Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1} \colon (X \vee Y)^{-} \xrightarrow{\sim} X^{-} \coprod Y^{-}$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\vee,1} \colon \overset{\sim}{\mathcal{Q}} \xrightarrow{\sim} \operatorname{pt}^{-}$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^-_{YY}: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{X,Y}(x,y)=x\wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\mathrm{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{-,1} \colon \operatorname{pt} \xrightarrow{\sim} (S^0)^-$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

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