# Constructions With Monoidal Categories

The Clowder Project Authors

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This chapter contains some material on constructions with monoidal categories.

## Contents



## 13.1 Moduli Categories of Monoidal Structures

### 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let $\mathcal{C}$ be a category.

**Definition 13.1.1.1.1.** The **moduli category of monoidal structures on** $\mathcal{C}$ is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, \qquad \begin{tikzcd} \mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{MonCats} \arrow[d, "忘"] \\ \mathsf{pt} \arrow[r, "{[\mathcal{C}]}"'] & \mathsf{Cats}. \end{tikzcd}$$

**Remark 13.1.1.1.2.** In detail, **the moduli category of monoidal structures on $\mathcal{C}$** is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ are monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ whose underlying category is $\mathcal{C}$.
- *Morphisms.* A morphism from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ is a strong monoidal functor structure

$$\mathrm{id}^{\otimes}_{\mathcal{C}} \colon A \boxtimes_{\mathcal{C}} B \overset{\sim}{\dashrightarrow} A \otimes_{\mathcal{C}} B,$$
$$\mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}} \colon \mathbb{1}'_{\mathcal{C}} \overset{\sim}{\dashrightarrow} \mathbb{1}_{\mathcal{C}}$$

  on the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.
- *Identities.* For each $M \overset{\text{def}}{=} (\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}) \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,M} \colon \mathsf{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, M)$$

  of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at $M$ is defined by

$$\mathrm{id}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M} \overset{\text{def}}{=} (\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}),$$

  where $\left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right)$ is the identity monoidal functor of $\mathcal{C}$ of **??**.
- *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the composition map

$$\circ^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,N,P} \colon \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(N, P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, P)$$

  of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at $(M, N, P)$ is defined by

$$\left(\mathrm{id}^{\otimes,\prime}_{\mathcal{C}}, \mathrm{id}^{\otimes,\prime}_{\mathbb{1}|\mathcal{C}}\right) \circ^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,N,P} \left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right) \overset{\text{def}}{=} \left(\mathrm{id}^{\otimes,\prime}_{\mathcal{C}} \circ \mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes,\prime}_{\mathbb{1}|\mathcal{C}} \circ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right).$$

**Remark 13.1.1.1.3.** In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ satisfies the following conditions:

1. *Naturality.* For each pair $f\colon A \to X$ and $g\colon B \to Y$ of morphisms of $\mathcal{C}$, the diagram
$$\begin{tikzcd}
A \boxtimes_{\mathcal{C}} B \arrow[r, "f \boxtimes_{\mathcal{C}} g"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B}"'] & X \boxtimes_{\mathcal{C}} Y \arrow[d, "\mathrm{id}^{\otimes}_{X,Y}"] \\
A \otimes_{\mathcal{C}} B \arrow[r, "f \otimes_{\mathcal{C}} g"'] & X \otimes_{\mathcal{C}} Y
\end{tikzcd}$$
commutes.

2. *Monoidality.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
 & (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] \arrow[dr, "\alpha^{\mathcal{C},\prime}_{A,B,C}"] & \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"'] & & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[dr, "\alpha^{\mathcal{C}}_{A,B,C}"'] & & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[dl, "\mathrm{id}^{\otimes}_{A, B \otimes_{\mathcal{C}} C}"] \\
 & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) &
\end{tikzcd}$$
commutes.

3. *Left Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
 & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}'_{\mathcal{C}}, A}"] & \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A \arrow[dr, "\lambda^{\mathcal{C}}_A"] & \\
\mathbb{1}'_{\mathcal{C}} \boxtimes_{\mathcal{C}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_{\mathcal{C}} \mathrm{id}_A"] \arrow[rrr, "\lambda^{\mathcal{C},\prime}_A"'] & & & A
\end{tikzcd}$$
commutes.

4. *Right Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
 & A \boxtimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes}_{A, \mathbb{1}'_{\mathcal{C}}}"] & A \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[dr, "\rho^{\mathcal{C}}_A"] & \\
A \boxtimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[ur, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[rrr, "\rho^{\mathcal{C},\prime}_A"'] & & & A
\end{tikzcd}$$

commutes.

**Proposition 13.1.1.1.4.** Let $\mathcal{C}$ be a category.

1. *Extra Monoidality Conditions.* Let $\left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ to $\left(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime}\right)$.

   (a) The diagram

$$\begin{tikzcd}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"] \arrow[d, "\mathrm{id}^{\otimes}_{A \boxtimes_{\mathcal{C}} B, C}"'] & (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"] \\
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C"'] & (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C
\end{tikzcd}$$

   commutes.

   (b) The diagram

$$\begin{tikzcd}
A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[r, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A, B \boxtimes_{\mathcal{C}} C}"'] & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}^{\otimes}_{A, B \otimes_{\mathcal{C}} C}"] \\
A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[r, "\mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"'] & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{tikzcd}$$

   commutes.

2. *Extra Monoidal Unity Constraints.* Let $\left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ to $\left(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime}\right)$.

   (a) The diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathcal{C}},\mathbb{1}'_{\mathcal{C}}}"] \arrow[d, "\lambda^{\mathcal{C}}_{\mathbb{1}'_{\mathcal{C}}}"'] & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[d, "\rho^{\mathcal{C},\prime}_{\mathbb{1}_{\mathcal{C}}}"] \\
\mathbb{1}'_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_{\mathcal{C}}
\end{tikzcd}$$

   commutes.

(b) The diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^C_{\mathbb{1}'_C}"'] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

commutes.

(c) The diagram

$$\begin{tikzcd}
\mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"'] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^C_{\mathbb{1}'_C}"] \\
\mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes.

(d) The diagram

$$\begin{tikzcd}
\mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^C_{\mathbb{1}'_C}"] \\
\mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes.

3. *Mixed Associators.* Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ and $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ be monoidal structures on $\mathcal{C}$ and let

$$\mathrm{id}^{\otimes}_{-_1,-_2} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha^{\otimes}_{A,B,C} \colon (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$\begin{tikzcd}
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\otimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A\otimes_{\mathcal{C}}B,C}"'] & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C}}_{A,B,C}"'] & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C},\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_{A,B\boxtimes_{\mathcal{C}}C}"] \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\otimes}_{A,B,C}"'] & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha^{\boxtimes}_{A,B,C} \colon (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \longrightarrow A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

making the diagrams

$$\begin{tikzcd}
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\boxtimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C"'] & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}^{\otimes}_{A,B\otimes_{\mathcal{C}}C}"] \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C}}_{A,B,C}"'] & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C},\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A\boxtimes_{\mathcal{C}}B,C}"'] & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\boxtimes}_{A,B,C}"'] & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha^{\boxtimes,\otimes}_{A,B,C}\colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{tikzcd}
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes,\otimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C"'] & A \otimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[r, "\alpha^{C}_{A,B,C}"'] & A \otimes_C (B \otimes_C C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_C B) \boxtimes_C C \arrow[r, "\alpha^{C,\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A \boxtimes_C B,C}"'] & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}^{\otimes}_{A,B \boxtimes_C C}"] \\
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes,\otimes}_{A,B,C}"'] & A \otimes_C (B \boxtimes_C C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

*Proof. Item 1, Extra Monoidality Conditions*: We claim that Items 1a and 1b are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}^{\otimes}_{A,B}$ and $\mathrm{id}_C$.

2. *Proof of Item 1b:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}_A$ and $\mathrm{id}^{\otimes}_{B,C}$.

This finishes the proof.
*Item 2, Extra Monoidal Unity Constraints*: We claim that Items 2a and 2b are indeed true:

1. *Proof of Item 1a:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[dr, "\mathrm{id}_{\mathbb{1}_C} \otimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[dd, "\lambda^C_{\mathbb{1}'_C}"'] & & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[d, "\mathrm{id}_{\mathbb{1}_C} \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[ddd, bend left=60, "\rho^{C,\prime}_{\mathbb{1}_C}"] & \\
& \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dr, equals] \arrow[ur, phantom, "(1)"] & \mathbb{1}_C \boxtimes_C \mathbb{1}_C \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dl, phantom, "(2)"] & \\
\mathbb{1}'_C \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[r, phantom, "(3)"] & {} & \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[d, "\lambda^C_{\mathbb{1}_C} = \rho^C_{\mathbb{1}_C}"] \arrow[r, phantom, "(4)"] & {} \\
& & \mathbb{1}_C &
\end{tikzcd}$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_C$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of $\lambda^C$, where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $(\mathrm{id}_C, \mathrm{id}^{\otimes}_C, \mathrm{id}^{\otimes}_{C|\mathbb{1}})$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}'_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}'_{\mathcal{C}},\mathbb{1}_{\mathcal{C}}}"] \arrow[dr, "\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_{\mathcal{C}} \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}"] \arrow[dd, "\rho^{\mathcal{C}}_{\mathbb{1}'_{\mathcal{C}}}"'] & \arrow[d, phantom, "(1)"] & \mathbb{1}'_{\mathcal{C}} \boxtimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_{\mathcal{C}} \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}"] \arrow[ddd, bend left=60, "\lambda^{\mathcal{C},\prime}_{\mathbb{1}_{\mathcal{C}}}"] \\
& \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathcal{C}},\mathbb{1}_{\mathcal{C}}}"] \arrow[dr, equal] \arrow[d, phantom, "(3)"] & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}_{\mathcal{C}},\mathbb{1}_{\mathcal{C}}}"] \arrow[dl, phantom, "(2)"] \\
\mathbb{1}'_{\mathcal{C}} \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & & \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[d, "\rho^{\mathcal{C}}_{\mathbb{1}_{\mathcal{C}}}=\lambda^{\mathcal{C}}_{\mathbb{1}_{\mathcal{C}}}"] \arrow[r, phantom, "(4)"] & {} \\
& & \mathbb{1}_{\mathcal{C}}
\end{tikzcd}$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathcal{C}}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of $\rho^{\mathcal{C}}$, where the equality $\rho^{\mathcal{C}}_{\mathbb{1}_{\mathcal{C}}} = \lambda^{\mathcal{C}}_{\mathbb{1}_{\mathcal{C}}}$ comes from **??**;

- Subdiagram $(4)$ commutes by the left monoidal unity of $(\mathrm{id}_C, \mathrm{id}^{\otimes}_C, \mathrm{id}^{\otimes}_{C|\mathbb{1}})$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^C_{\mathbb{1}'_C}"'] \arrow[dr, phantom, "(1)"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^C_{\mathbb{1}'_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C.
\end{tikzcd}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram $(1)$ commutes by Item 1b;

it follows that the diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^C_{\mathbb{1}'_C}"] \\
& \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}$ is an isomorphism, it follows that the diagram $(\dagger)$ also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^C_{\mathbb{1}'_C}"'] \arrow[dr, phantom, "(1)"] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^C_{\mathbb{1}'_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathcal{C}},\mathbb{1}'_{\mathcal{C}}}"] & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathcal{C}},\mathbb{1}'_{\mathcal{C}}}"] \arrow[d, "\rho^{\mathcal{C},\prime}_{\mathbb{1}_{\mathcal{C}}}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[d, "\lambda^{\mathcal{C}}_{\mathbb{1}'_{\mathcal{C}}}"] \\
 & \mathbb{1}_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}_{\mathcal{C}}
\end{tikzcd}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}}$ is an isomorphism, it follows that the diagram $(\dagger)$ also commutes, and we are done.

This finishes the proof.
*Item 3, Mixed Associators*: We claim that Items 3a to 3c are indeed true:

1. *Proof of Item 3a:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of Item 2 of Definition 13.1.1.1.3 as follows:

$$\begin{tikzcd}
 & & (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dll, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] \arrow[drr, "\alpha^{\mathcal{C}}_{A,B,C}"] \arrow[dd, "\mathrm{id}^{\otimes}_{A \boxtimes_{\mathcal{C}} B, C}"] & & \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dd, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"'] & (1) & & (2) & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[dd, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
 & & (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[dll, "\mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C"'] \arrow[drr, "\alpha^{\otimes}_{A,B,C}"] & & \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[drr, "\alpha^{\mathcal{C},\prime}_{A,B,C}"'] & & (3) & & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[dll, "\mathrm{id}^{\otimes}_{A,B \otimes_{\mathcal{C}} C}"] \\
 & & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) & &
\end{tikzcd}$$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. *Proof of Item 3b:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of Item 2 of Definition 13.1.1.1.3 as follows:

$$\begin{tikzcd}
 & & (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dll, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] \arrow[drr, "\alpha^{\mathcal{C}}_{A,B,C}"] & & \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dd, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"'] \arrow[dr, "\alpha^{\boxtimes}_{A,B,C}"] & & (1) & & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[dl, "\mathrm{id}^{\otimes}_{A, B \boxtimes_{\mathcal{C}} C}"'] \arrow[dd, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
 & \phantom{X} \arrow[r, phantom, ""] & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[dd, "\mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] & & \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[drr, "\alpha^{\mathcal{C},\prime}_{A,B,C}"'] & (2) & & (3) & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[dll, "\mathrm{id}^{\otimes}_{A, B \otimes_{\mathcal{C}} C}"] \\
 & & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) & &
\end{tikzcd}$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item 3c:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of

Item 2 of Definition 13.1.1.1.3 as follows:

$$\begin{tikzcd}
 & (A \boxtimes_C B) \boxtimes_C C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[dr, "\alpha^{C}_{A,B,C}"] \arrow[dd, phantom, "(1)"] & \\
(A \otimes_C B) \boxtimes_C C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_C B,C}"'] \arrow[dr, phantom, "(2)"] \arrow[rr, phantom, ""] \arrow[drr, "\alpha^{\boxtimes,\otimes}_{A,B,C}" description] & & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[dr, "\alpha^{C,\prime}_{A,B,C}"'] & & A \boxtimes_C (B \otimes_C C) \arrow[dl, "\mathrm{id}^{\otimes}_{A,B \otimes_C C}"] \\
 & A \otimes_C (B \otimes_C C) &
\end{tikzcd}$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof. □

### 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category

### 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category

## 13.2 Moduli Categories of Closed Monoidal Structures

## 13.3 Moduli Categories of Refinements of Monoidal Structures

### 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# Appendices

# A Other Chapters