## Constructions With Monoidal Categories

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01UF This chapter contains some material on constructions with monoidal categories.

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### 01UG 13.1 Moduli Categories of Monoidal Structures

# 01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let *C* be a category.

**Definition 13.1.1.1.1.** The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathrm{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, egin{pmatrix} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \mathsf{MonCats}, \\ & & \downarrow & & \downarrow & & \downarrow \\ & \mathsf{pt} & & & \mathsf{Cats}. \end{pmatrix}$$

- 01UK Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:
  - Objects. The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is C.
  - *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{1|C}^{\otimes} \colon \mathbb{1}_{C}' \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C : C \to C$  of C.

• *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{\mathcal{M},\mathcal{M}}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(\mathcal{M},\mathcal{M})$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \stackrel{\operatorname{def}}{=} (\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes}),$$

where  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  is the identity monoidal functor of C of ??.

• Composition. For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{E_1}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{E_1}(C)}(M,P)$$
 of  $\mathcal{M}_{E_1}(C)$  at  $(M,N,P)$  is defined by 
$$\left( \operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{1|C}^{\otimes,\prime} \right) \circ_{M,N,P}^{\mathcal{M}_{E_1}(C)} \left( \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes} \right) \stackrel{\operatorname{def}}{=} \left( \operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes,\prime} \circ \operatorname{id}_{1|C}^{\otimes} \right).$$

- **Q1UL Remark 13.1.1.1.3.** In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:
- 01UM I. Naturality. For each pair  $f:A\to X$  and  $g:B\to Y$  of morphisms of C, the diagram

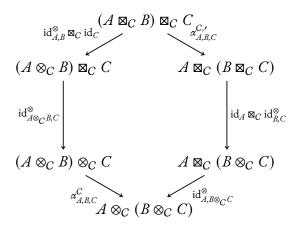
$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

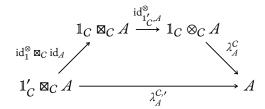
commutes.

**O1UN** 2. *Monoidality*. For each  $A, B, C \in Obj(C)$ , the diagram



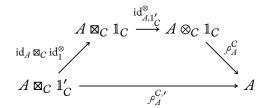
commutes.

01UP 3. Left Monoidal Unity. For each  $A \in \text{Obj}(C)$ , the diagram



commutes.

**01UQ** 4. *Right Monoidal Unity*. For each  $A \in \text{Obj}(C)$ , the diagram



commutes.

**Olur** Proposition 13.1.1.4. Let C be a category.

01US

1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{I}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{I}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .

01UT (a) The diagram

commutes.

01UU (b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B\boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

01WB 2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .

01WC (a) The diagram

commutes.

01WD (b) The diagram

commutes.

01WE (c) The diagram

commutes.

01WF (d) The diagram

commutes.

01UV 3. Mixed Associators. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} : -_1 \boxtimes_C -_2 \longrightarrow -_1 \otimes_C -_2$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes} : (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$id_{A \otimes_{C} B,C} \downarrow \qquad \qquad \downarrow id_{A} \otimes_{C} id_{B,C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

commute, then the natural transformation id<sup>®</sup> satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UX (b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

and

$$\begin{array}{c|c} (A\boxtimes_{C}B)\boxtimes_{C}C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_{C}(B\boxtimes_{C}C) \\ \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} & \operatorname{id}_{A\boxtimes_{C}}\operatorname{id}_{B,C}^{\otimes} \\ (A\boxtimes_{C}B)\otimes_{C}C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A\boxtimes_{C}(B\otimes_{C}C) \end{array}$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha_{A.B.C}^{\boxtimes,\otimes} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

and

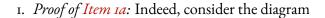
commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

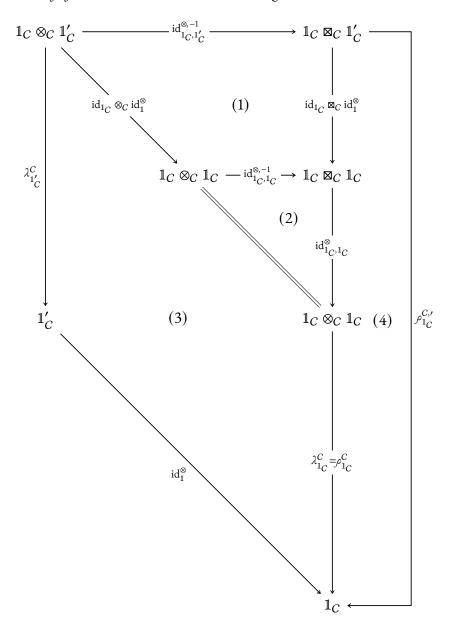
*Proof. Item 1*, *Extra Monoidality Conditions*: We claim that Items 1a and 1b are indeed true:

- I. *Proof of Item 1a:* This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A,B}^{\otimes}$  and  $id_{C}$ .
- 2. *Proof of Item 1b:* This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A}$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

*Item 2, Extra Monoidal Unity Constraints*: We claim that Items 2a and 2b are indeed true:



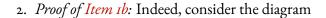


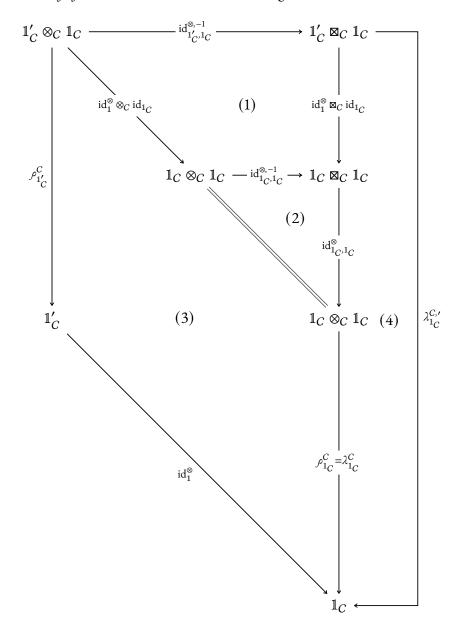
whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{1_C}^C=\lambda_{1_C}^C$  comes from  $\ref{eq:compare}$ ;
- Subdiagram (4) commutes by the right monoidal unity of (id $_C$ , id $_C^{\otimes}$ , id $_{C|1}^{\otimes}$ );

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from  $\ref{eq:compare}$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

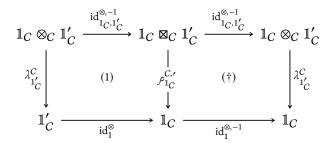
Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $\mathrm{id}_{1_C,1_C'}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram



Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$1_{C} \otimes_{C} 1_{C}' \xrightarrow{\operatorname{id}_{1_{C},1_{C}'}^{\otimes,-1}} 1_{C} \boxtimes_{C} 1_{C}' \xrightarrow{\operatorname{id}_{1_{C},1_{C}'}^{\otimes,-1}} 1_{C} \otimes_{C} 1_{C}'$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{1_{C}'}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{1_{C}'}^{C}$$

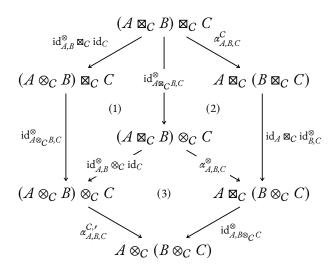
commutes. But since  $id_1^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

*Item 3, Mixed Associators*: We claim that Items 3a to 3c are indeed true:

01UZ I. Proof of Item 3a: We may partition the monoidality diagram for  $id^{\otimes}$  of

#### Item 2 of Definition 13.1.1.1.3 as follows:



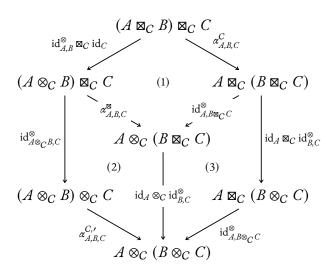
Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01V0 2. *Proof of Item 3b*: We may partition the monoidality diagram for  $id^{\otimes}$  of



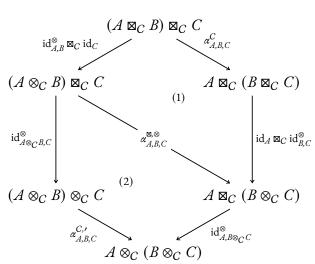


#### Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01V1 3. Proof of Item 3c: We may partition the monoidality diagram for  $id^{\otimes}$  of



Item 2 of Definition 13.1.1.1.3 as follows:

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id<sup>®</sup> satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- 01V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# **Appendices**

## A Other Chapters

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- 2. A Guide to the Literature

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- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

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- 8. Relations
- 9. Constructions With Relations

#### 10. Conditions on Relations

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- 11. Categories
- 12. Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

13. Constructions With Monoidal Categories

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14. Types of Morphisms in Bicategories

#### Extra Part

15. Notes