

# Constructions With Relations

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This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (??).
2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, converse relations, composition of relations, and collages ([Section 9.2](#)).

This chapter is under revision. TODO:

1. Rename range to image
2. Co/limits in **Rel**.

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## 9.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

## 9.2 More Constructions With Relations

### 9.2.1 The Domain and Range of a Relation

Let  $A$  and  $B$  be sets.

#### DEFINITION 9.2.1.1.1 ► THE DOMAIN AND RANGE OF A RELATION

Let  $R: A \rightarrowtail B$  be a relation.<sup>1,2</sup>

1. The **domain** of  $R$  is the subset  $\text{dom}(R)$  of  $A$  defined by

$$\text{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \mid \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range** of  $R$  is the subset  $\text{range}(R)$  of  $B$  defined by

$$\text{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \mid \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

<sup>1</sup>Following ??, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\text{dom}(R)}(a) &\cong \text{colim}_{b \in B} (R_a^b) & (a \in A) \\ &\cong \bigvee_{b \in B} R_a^b, \\ \chi_{\text{range}(R)}(b) &\cong \text{colim}_{a \in A} (R_a^b) & (b \in B) \\ &\cong \bigvee_{a \in A} R_a^b, \end{aligned}$$

where the join  $\bigvee$  is taken in the poset  $(\{\text{true}, \text{false}\}, \preceq)$  of **Constructions With Sets**, Definition 3.2.2.1.3.

<sup>2</sup>Viewing  $R$  as a function  $R: A \rightarrow \mathcal{P}(B)$ , we have

$$\begin{aligned} \text{dom}(R) &\cong \text{colim}_{y \in Y} (R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \text{range}(R) &\cong \text{colim}_{x \in X} (R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

## 9.2.2 Binary Unions of Relations

Let  $A$  and  $B$  be sets and let  $R$  and  $S$  be relations from  $A$  to  $B$ .

### DEFINITION 9.2.2.1.1 ► BINARY UNIONS OF RELATIONS

The **union of  $R$  and  $S$** <sup>1</sup> is the relation  $R \cup S$  from  $A$  to  $B$  defined as follows:

- Viewing relations from  $A$  to  $B$  as subsets of  $A \times B$ , we define<sup>2</sup>

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from  $A$  to  $B$  as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each  $a \in A$ .

<sup>1</sup>*Further Terminology:* Also called the **binary union of  $R$  and  $S$** , for emphasis.

<sup>2</sup>This is the same as the union of  $R$  and  $S$  as subsets of  $A \times B$ .

### PROPOSITION 9.2.2.1.2 ► PROPERTIES OF BINARY UNIONS OF RELATIONS

Let  $R, S, R_1$ , and  $R_2$  be relations from  $A$  to  $B$ , and let  $S_1$  and  $S_2$  be relations from  $B$  to  $C$ .

1. *Interaction With Converses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

### PROOF 9.2.2.1.3 ► PROOF OF PROPOSITION 9.2.2.1.2

Item 1: Interaction With Converses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- The condition for  $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$  is:

- There exists some  $b \in B$  such that:

$$* a \sim_{R_1} b \text{ and } b \sim_{S_1} c;$$

or

$$* a \sim_{R_2} b \text{ and } b \sim_{S_2} c;$$


- The condition for  $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$  is:

- There exists some  $b \in B$  such that:

$$* a \sim_{R_1} b \text{ or } a \sim_{R_2} b;$$

and

$$* b \sim_{S_1} c \text{ or } b \sim_{S_2} c.$$

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on  $A \times C$  may differ. 

### 9.2.3 Unions of Families of Relations

Let  $A$  and  $B$  be sets and let  $\{R_i\}_{i \in I}$  be a family of relations from  $A$  to  $B$ .

#### DEFINITION 9.2.3.1.1 ► THE UNION OF A FAMILY OF RELATIONS

The **union of the family**  $\{R_i\}_{i \in I}$  is the relation  $\bigcup_{i \in I} R_i$  from  $A$  to  $B$  defined as follows:

- Viewing relations from  $A$  to  $B$  as subsets of  $A \times B$ , we define<sup>1</sup>

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \mid \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from  $A$  to  $B$  as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$\left[ \bigcup_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

for each  $a \in A$ .

<sup>1</sup>This is the same as the union of  $\{R_i\}_{i \in I}$  as a collection of subsets of  $A \times B$ .

**PROPOSITION 9.2.3.1.2 ► PROPERTIES OF UNIONS OF FAMILIES OF RELATIONS**

Let  $A$  and  $B$  be sets and let  $\{R_i\}_{i \in I}$  be a family of relations from  $A$  to  $B$ .

1. *Interaction With Converses.* We have

$$\left( \bigcup_{i \in I} R_i \right)^\dagger = \bigcup_{i \in I} R_i^\dagger.$$

**PROOF 9.2.3.1.3 ► PROOF OF PROPOSITION 9.2.3.1.2**

Item 1: Interaction With Converses

Clear.

**9.2.4 Binary Intersections of Relations**

Let  $A$  and  $B$  be sets and let  $R$  and  $S$  be relations from  $A$  to  $B$ .

**DEFINITION 9.2.4.1.1 ► BINARY INTERSECTIONS OF RELATIONS**

The **intersection of  $R$  and  $S$** <sup>1</sup> is the relation  $R \cap S$  from  $A$  to  $B$  defined as follows:

- Viewing relations from  $A$  to  $B$  as subsets of  $A \times B$ , we define<sup>2</sup>

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from  $A$  to  $B$  as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each  $a \in A$ .

<sup>1</sup>*Further Terminology:* Also called the **binary intersection of  $R$  and  $S$** , for emphasis.

<sup>2</sup>This is the same as the intersection of  $R$  and  $S$  as subsets of  $A \times B$ .

**PROPOSITION 9.2.4.1.2 ► PROPERTIES OF BINARY INTERSECTIONS OF RELATIONS**

Let  $R, S, R_1$ , and  $R_2$  be relations from  $A$  to  $B$ , and let  $S_1$  and  $S_2$  be relations from  $B$  to  $C$ .

1. *Interaction With Converses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

PROOF 9.2.4.1.3 ► PROOF OF PROPOSITION 9.2.4.1.2


Item 1: Interaction With Converses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- The condition for  $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$  is:
  - There exists some  $b \in B$  such that:
    - \*  $a \sim_{R_1} b$  and  $b \sim_{S_1} c$ ;
 and
    - \*  $a \sim_{R_2} b$  and  $b \sim_{S_2} c$ ;
- The condition for  $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$  is:
  - There exists some  $b \in B$  such that:
    - \*  $a \sim_{R_1} b$  and  $a \sim_{R_2} b$ ;
 and
    - \*  $b \sim_{S_1} c$  and  $b \sim_{S_2} c$ .

These two conditions agree, and thus so do the two resulting relations on  $A \times C$ . 

## 9.2.5 Intersections of Families of Relations

Let  $A$  and  $B$  be sets and let  $\{R_i\}_{i \in I}$  be a family of relations from  $A$  to  $B$ .

DEFINITION 9.2.5.1.1 ► THE INTERSECTION OF A FAMILY OF RELATIONS

The **intersection of the family**  $\{R_i\}_{i \in I}$  is the relation  $\bigcup_{i \in I} R_i$  defined as follows:

- Viewing relations from  $A$  to  $B$  as subsets of  $A \times B$ , we define<sup>1</sup>

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \mid \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from  $A$  to  $B$  as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$\left[ \bigcap_{i \in I} R_i \right](a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each  $a \in A$ .

<sup>1</sup>This is the same as the intersection of  $\{R_i\}_{i \in I}$  as a collection of subsets of  $A \times B$ .

#### PROPOSITION 9.2.5.1.2 ► PROPERTIES OF INTERSECTIONS OF FAMILIES OF RELATIONS

Let  $A$  and  $B$  be sets and let  $\{R_i\}_{i \in I}$  be a family of relations from  $A$  to  $B$ .

1. *Interaction With Converses.* We have

$$\left( \bigcap_{i \in I} R_i \right)^\dagger = \bigcap_{i \in I} R_i^\dagger.$$

#### PROOF 9.2.5.1.3 ► PROOF OF PROPOSITION 9.2.5.1.2

Item 1: Interaction With Converses

Clear.



## 9.2.6 Binary Products of Relations

Let  $A$ ,  $B$ ,  $X$ , and  $Y$  be sets, let  $R: A \rightarrow B$  be a relation from  $A$  to  $B$ , and let  $S: X \rightarrow Y$  be a relation from  $X$  to  $Y$ .

#### DEFINITION 9.2.6.1.1 ► BINARY PRODUCTS OF RELATIONS

The **product of  $R$  and  $S$** <sup>1</sup> is the relation  $R \times S$  from  $A \times X$  to  $B \times Y$  defined as follows:

- Viewing relations from  $A \times X$  to  $B \times Y$  as subsets of  $(A \times X) \times (B \times Y)$ , we define  $R \times S$  as the Cartesian product of  $R$  and  $S$  as subsets of  $A \times X$  and  $B \times Y$ .<sup>2</sup>

- Viewing relations from  $A \times X$  to  $B \times Y$  as functions  $A \times X \rightarrow \mathcal{P}(B \times Y)$ , we define  $R \times S$  as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \xrightarrow{\mathcal{P}_{B,Y}^{\otimes}} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each  $(a, x) \in A \times X$ .

<sup>1</sup>Further Terminology: Also called the **binary product of  $R$  and  $S$** , for emphasis.  
<sup>2</sup>That is,  $R \times S$  is the relation given by declaring  $(a, x) \sim_{R \times S} (b, y)$  iff  $a \sim_R b$  and  $x \sim_S y$ .

#### PROPOSITION 9.2.6.1.2 ► PROPERTIES OF BINARY PRODUCTS OF RELATIONS

Let  $A, B, X$ , and  $Y$  be sets.

1. *Interaction With Converses.* Let

$$R: A \rightarrowtail B,$$

$$S: X \rightarrowtail Y$$

We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$R_1: A \rightarrowtail B,$$

$$S_1: B \rightarrowtail C,$$

$$R_2: X \rightarrowtail Y,$$

$$S_2: Y \rightarrowtail Z$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$



## PROOF 9.2.6.1.3 ► PROOF OF PROPOSITION 9.2.6.1.2

## Item 1: Interaction With Converses

Unwinding the definitions, we see that:


- We have  $(a, x) \sim_{(R \times S)^\dagger} (b, y)$  iff:
  - We have  $(b, y) \sim_{R \times S} (a, x)$ , i.e. iff:
    - \* We have  $b \sim_R a$ ;
    - \* We have  $y \sim_S x$ ;
- We have  $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$  iff:
  - We have  $a \sim_{R^\dagger} b$  and  $x \sim_{S^\dagger} y$ , i.e. iff:
    - \* We have  $b \sim_R a$ ;
    - \* We have  $y \sim_S x$ .

These two conditions agree, and thus the two resulting relations on  $A \times X$  are equal.

## Item 2: Interaction With Composition

Unwinding the definitions, we see that:

- We have  $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$  iff:
  - We have  $a \sim_{S_1 \diamond R_1} c$  and  $x \sim_{S_2 \diamond R_2} z$ , i.e. iff:
    - \* There exists some  $b \in B$  such that  $a \sim_{R_1} b$  and  $b \sim_{S_1} c$ ;
    - \* There exists some  $y \in Y$  such that  $x \sim_{R_2} y$  and  $y \sim_{S_2} z$ ;
- We have  $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$  iff:
  - There exists some  $(b, y) \in B \times Y$  such that  $(a, x) \sim_{R_1 \times R_2} (b, y)$  and  $(b, y) \sim_{S_1 \times S_2} (c, z)$ , i.e. such that:
    - \* We have  $a \sim_{R_1} b$  and  $x \sim_{R_2} y$ ;
    - \* We have  $b \sim_{S_1} c$  and  $y \sim_{S_2} z$ .

These two conditions agree, and thus the two resulting relations from  $A \times X$  to  $C \times Z$  are equal. 

### 9.2.7 Products of Families of Relations

Let  $\{A_i\}_{i \in I}$  and  $\{B_i\}_{i \in I}$  be families of sets, and let  $\{R_i: A_i \rightarrow B_i\}_{i \in I}$  be a family of relations.

#### DEFINITION 9.2.7.1.1 ► THE PRODUCT OF A FAMILY OF RELATIONS

The **product of the family**  $\{R_i\}_{i \in I}$  is the relation  $\prod_{i \in I} R_i$  from  $\prod_{i \in I} A_i$  to  $\prod_{i \in I} B_i$  defined as follows:

- Viewing relations as subsets, we define  $\prod_{i \in I} R_i$  as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \mid \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[ \prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

for each  $(a_i)_{i \in I} \in \prod_{i \in I} R_i$ .

### 9.2.8 The Collage of a Relation

Let  $A$  and  $B$  be sets and let  $R: A \rightarrow B$  be a relation from  $A$  to  $B$ .

#### DEFINITION 9.2.8.1.1 ► THE COLLAGE OF A RELATION

The **collage of  $R$** <sup>1</sup> is the poset  $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \preceq_{\mathbf{Coll}(R)})$  consisting of:

- *The Underlying Set.* The set  $\mathbf{Coll}(R)$  defined by

$$\mathbf{Coll}(R) \stackrel{\text{def}}{=} A \amalg B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)}: \mathbf{Coll}(R) \times \mathbf{Coll}(R) \rightarrow \{\text{true}, \text{false}\}$$

on  $\mathbf{Coll}(R)$  defined by

$$\preceq(a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

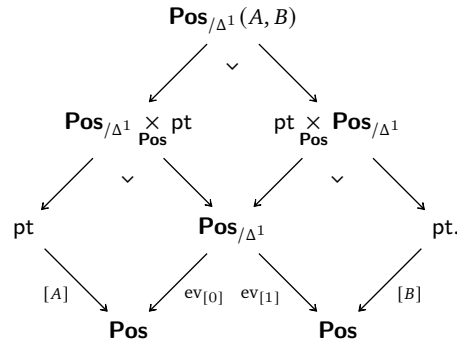
<sup>1</sup>Further Terminology: Also called the **cograph** of  $R$ .

NOTATION 9.2.8.1.2 ► NOTATION:  $\text{Pos}_{/\Delta^1}(A, B)$ 

We write  $\text{Pos}_{/\Delta^1}(A, B)$  for the category defined as the pullback

$$\text{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \text{pt} \times_{[A], \text{Pos}, \text{ev}_0} \text{Pos}_{/\Delta^1} \times_{\text{ev}_1, \text{Pos}, [B]} \text{pt},$$

as in the diagram



## REMARK 9.2.8.1.3 ► UNWINDING NOTATION 9.2.8.1.2

In detail,  $\text{Pos}_{/\Delta^1}(A, B)$  is the category where:

- *Objects.* An object of  $\text{Pos}_{/\Delta^1}(A, B)$  is a pair  $(X, \phi_X)$  consisting of
  - A poset  $X$ ;
  - A morphism  $\phi_X: X \rightarrow \Delta^1$ ;

such that we have

$$\begin{aligned} \phi_X^{-1}(0) &= A, \\ \phi_X^{-1}(1) &= B. \end{aligned}$$

- *Morphisms.* A morphism of  $\text{Pos}_{/\Delta^1}(A, B)$  from  $(X, \phi_X)$  to  $(Y, \phi_Y)$  is a morphism of posets  $f: X \rightarrow Y$  making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi_X \searrow & & \swarrow \phi_Y \\ & \Delta^1 & \end{array}$$

commute.



Let  $A$  and  $B$  be sets and let  $R: A \rightarrowtail B$  be a relation from  $A$  to  $B$ .

1. *Functoriality.* The assignment  $R \mapsto \mathbf{Coll}(R)$  defines a functor

$$\mathbf{Coll}: \mathbf{Rel}(A, B) \rightarrow \mathbf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each  $R \in \mathbf{Obj}(\mathbf{Rel}(A, B))$ , we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each  $R \in \mathbf{Rel}(A, B)$ , where

- The poset  $\mathbf{Coll}(R)$  is the collage of  $R$  of [Definition 9.2.8.1.1](#).
- The morphism  $\phi_R: \mathbf{Coll}(R) \rightarrow \Delta^1$  is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each  $x \in \mathbf{Coll}(R)$ .

- *Action on Morphisms.* For each  $R, S \in \mathbf{Obj}(\mathbf{Rel}(A, B))$ , the action on Hom-sets

$$\mathbf{Coll}_{R,S}: \mathbf{Hom}_{\mathbf{Rel}(A,B)}(R, S) \rightarrow \mathbf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of  $\mathbf{Coll}$  at  $(R, S)$  is given by sending an inclusion

$$\iota: R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota): \mathbf{Coll}(R) \rightarrow \mathbf{Coll}(S)$$

of posets over  $\Delta^1$  defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each  $x \in \mathbf{Coll}(R)$ .<sup>1</sup>

2. *Equivalence.* The functor of [Item 1](#) is an equivalence of categories.

<sup>1</sup>Note that this is indeed a morphism of posets: if  $x \preceq_{\mathbf{Coll}(R)} y$ , then  $x = y$  or  $x \sim_R y$ , so we have either  $x = y$  or  $x \sim_S y$  (as  $R \subset S$ ), and thus  $x \preceq_{\mathbf{Coll}(S)} y$ .

|

|

Item 1: Functoriality

Clear.

Item 2: Equivalence

Omitted.



# Appendices

## A Other Chapters

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