# Constructions With Monoidal Categories

The Clowder Project Authors

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01UF This chapter contains some material on constructions with monoidal categories.

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# 01UG 13.1 Moduli Categories of Monoidal Structures

## 01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let $C$ be a category.

01UJ **Definition 13.1.1.1.1 ▶ The Moduli Category of Monoidal Structures on a Category**

The **moduli category of monoidal structures on** $C$ is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \overset{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \text{forget}} \\ \text{pt} & \xrightarrow[[C]]{} & \text{Cats}. \end{array}$$

01UK **Remark 13.1.1.1.2 ▶ Unwinding Definition 13.1.1.1.1, I**

In detail, **the moduli category of monoidal structures on** $C$ is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ whose underlying category is $C$.
- *Morphisms.* A morphism from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$ is a strong monoidal functor structure

$$\text{id}^{\otimes}_C : A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$
$$\text{id}^{\otimes}_{\mathbb{1}|C} : \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

  on the identity functor $\text{id}_C : C \to C$ of $C$.
- *Identities.* For each $M \overset{\text{def}}{=} \left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right) \in \text{Obj}\left(\mathcal{M}_{\mathbb{E}_1}(C)\right)$, the unit map

$$\mathbb{1}^{\mathcal{M}_{\mathbb{E}_1}(C)}_{M,M} : \text{pt} \to \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at $M$ is defined by

$$\mathrm{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \overset{\text{def}}{=} \left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right),$$

where $\left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right)$ is the identity monoidal functor of $C$ of **??**.

- *Composition.* For each $M, N, P \in \mathrm{Obj}\left(\mathcal{M}_{\mathbb{E}_1}(C)\right)$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} : \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N,P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,P)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at $(M,N,P)$ is defined by

$$\left(\mathrm{id}_C^{\otimes,\prime}, \mathrm{id}_{\mathbb{1}|C}^{\otimes,\prime}\right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right) \overset{\text{def}}{=} \left(\mathrm{id}_C^{\otimes,\prime} \circ \mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes,\prime} \circ \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right).$$

01UL **REMARK 13.1.1.1.3 ▶ UNWINDING DEFINITION 13.1.1.1.1, II**

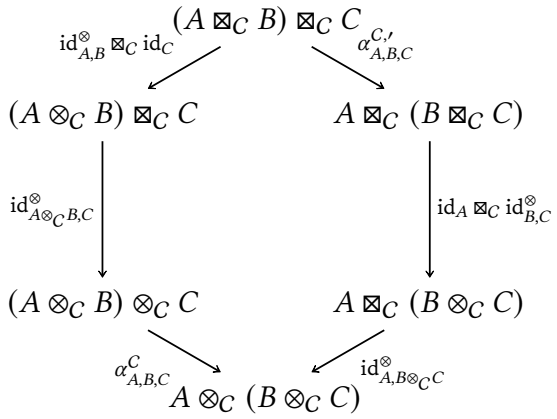
In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$ satisfies the following conditions:

01UM 1. *Naturality.* For each pair $f : A \to X$ and $g : B \to Y$ of morphisms of $C$, the diagram

$$\begin{array}{ccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \Big\downarrow{\scriptstyle \mathrm{id}_{A,B}^{\otimes}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{X,Y}^{\otimes}} \\ A \otimes_C B & \xrightarrow[f \otimes_C g]{} & X \otimes_C Y \end{array}$$

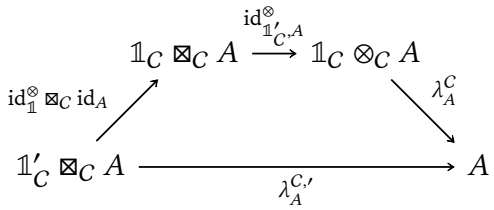
commutes.

01UN 2. *Monoidality.* For each $A, B, C \in \mathrm{Obj}(C)$, the diagram



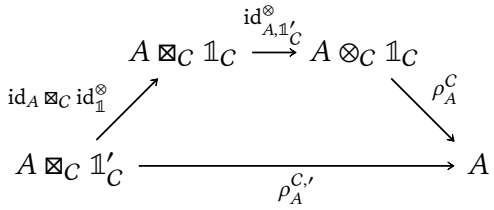

commutes.

01UP 3. *Left Monoidal Unity.* For each $A \in \mathrm{Obj}(C)$, the diagram




commutes.

01UQ 4. *Right Monoidal Unity.* For each $A \in \mathrm{Obj}(C)$, the diagram




commutes.

**Proposition 13.1.1.1.4 ▶ Properties of the Moduli Category of Monoidal Structures on a Category**

01UR

Let $C$ be a category.

01US 1. *Extra Monoidality Conditions.* Let $\left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

01UT (a) The diagram

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C} & (A \otimes_C B) \boxtimes_C C \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A \boxtimes_C B, C}} & & \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A \otimes_C B, C}} \\ (A \boxtimes_C B) \otimes_C C & \xrightarrow[\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C]{} & (A \otimes_C B) \otimes_C C \end{array}$$

commutes.

01UU (b) The diagram

$$\begin{array}{ccc} A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}} & A \boxtimes_C (B \otimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A, B \boxtimes_C C}} & & \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A, B \otimes_C C}} \\ A \otimes_C (B \boxtimes_C C) & \xrightarrow[\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}]{} & A \otimes_C (B \otimes_C C) \end{array}$$

commutes.

01WB 2. *Extra Monoidal Unity Constraints.* Let $\left(\mathrm{id}_C^{\otimes}, \mathrm{id}_{\mathbb{1}|C}^{\otimes}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

01WC (a) The diagram

$$\begin{array}{ccc} \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \\ \downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} & & \downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} \\ \mathbb{1}'_C & \xrightarrow[\mathrm{id}^{\otimes}_{\mathbb{1}}]{} & \mathbb{1}_C \end{array}$$

commutes.

01WD (b) The diagram

$$\begin{array}{ccc} \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\ \downarrow{\scriptstyle \rho^C_{\mathbb{1}'_C}} & & \downarrow{\scriptstyle \lambda^{C,\prime}_{\mathbb{1}_C}} \\ \mathbb{1}'_C & \xrightarrow[\mathrm{id}^{\otimes}_{\mathbb{1}}]{} & \mathbb{1}_C \end{array}$$

commutes.

01WE (c) The diagram

$$\begin{array}{ccc} \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\ \downarrow{\scriptstyle \lambda^{C,\prime}_{\mathbb{1}_C}} & & \downarrow{\scriptstyle \rho^C_{\mathbb{1}'_C}} \\ \mathbb{1}_C & \xrightarrow[\mathrm{id}^{\otimes,-1}_{\mathbb{1}}]{} & \mathbb{1}'_C \end{array}$$

commutes.

01WF (d) The diagram

$$\begin{array}{ccc} \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\ \downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} & & \downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} \\ \mathbb{1}_C & \xrightarrow[\mathrm{id}^{\otimes,-1}_{\mathbb{1}}]{} & \mathbb{1}'_C \end{array}$$

commutes.

01UV 3. *Mixed Associators.* Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on $C$ and let

$$\mathrm{id}^{\otimes}_{-_1,-_2} : -_1 \boxtimes_C -_2 \to -_1 \otimes_C -_2$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha^{\otimes}_{A,B,C} : (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha^{\otimes}_{A,B,C}} & A \otimes_C (B \boxtimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A\otimes_C B,C}} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}} \\ (A \otimes_C B) \otimes_C C & \xrightarrow[\alpha^{C}_{A,B,C}]{} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha^{C,\prime}_{A,B,C}} & A \boxtimes_C (B \boxtimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C} & & \downarrow{\scriptstyle \mathrm{id}_{A,B\boxtimes_C C}} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow[\alpha^{\otimes}_{A,B,C}]{} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UX (b) If there exists a natural transformation

$$\alpha^{\boxtimes}_{A,B,C} : (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha^{\boxtimes}_{A,B,C}} & A \boxtimes_C (B \otimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C} & & \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A,B\otimes_C C}} \\ (A \otimes_C B) \otimes_C C & \xrightarrow[\alpha^{C}_{A,B,C}]{} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha^{C,\prime}_{A,B,C}} & A \boxtimes_C (B \boxtimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A\boxtimes_C B,C}} & & \downarrow{\scriptstyle \mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}} \\ (A \boxtimes_C B) \otimes_C C & \xrightarrow[\alpha^{\boxtimes}_{A,B,C}]{} & A \boxtimes_C (B \otimes_C C) \end{array}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha^{\boxtimes,\otimes}_{A,B,C} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha^{\boxtimes,\otimes}_{A,B,C}} & A \otimes_C (B \boxtimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}} \\ (A \otimes_C B) \otimes_C C & \xrightarrow[\alpha^{C}_{A,B,C}]{} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha^{C,\prime}_{A,B,C}} & A \boxtimes_C (B \boxtimes_C C) \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A\boxtimes_C B,C}} & & \downarrow{\scriptstyle \mathrm{id}^{\otimes}_{A,B\boxtimes_C C}} \\ (A \boxtimes_C B) \otimes_C C & \xrightarrow[\alpha^{\boxtimes,\otimes}_{A,B,C}]{} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

**Proof 13.1.1.1.5 ▶ Proof of Proposition 13.1.1.1.4**

Item 1: Extra Monoidality Conditions

We claim that Items 1a and 1b are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}^{\otimes}_{A,B}$ and $\mathrm{id}_C$.

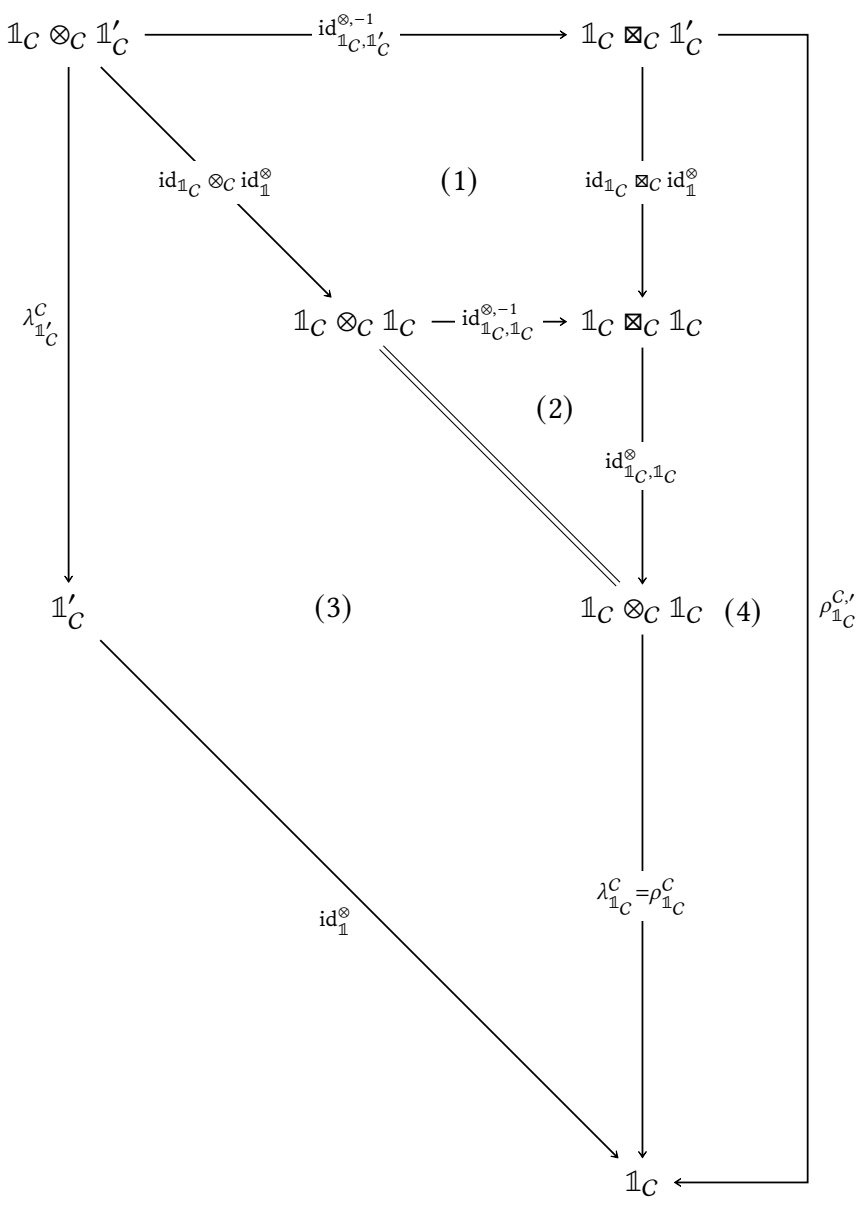
2. *Proof of Item 1b:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}_A$ and $\mathrm{id}^{\otimes}_{B,C}$.

This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:

1. *Proof of Item 1a:* Indeed, consider the diagram



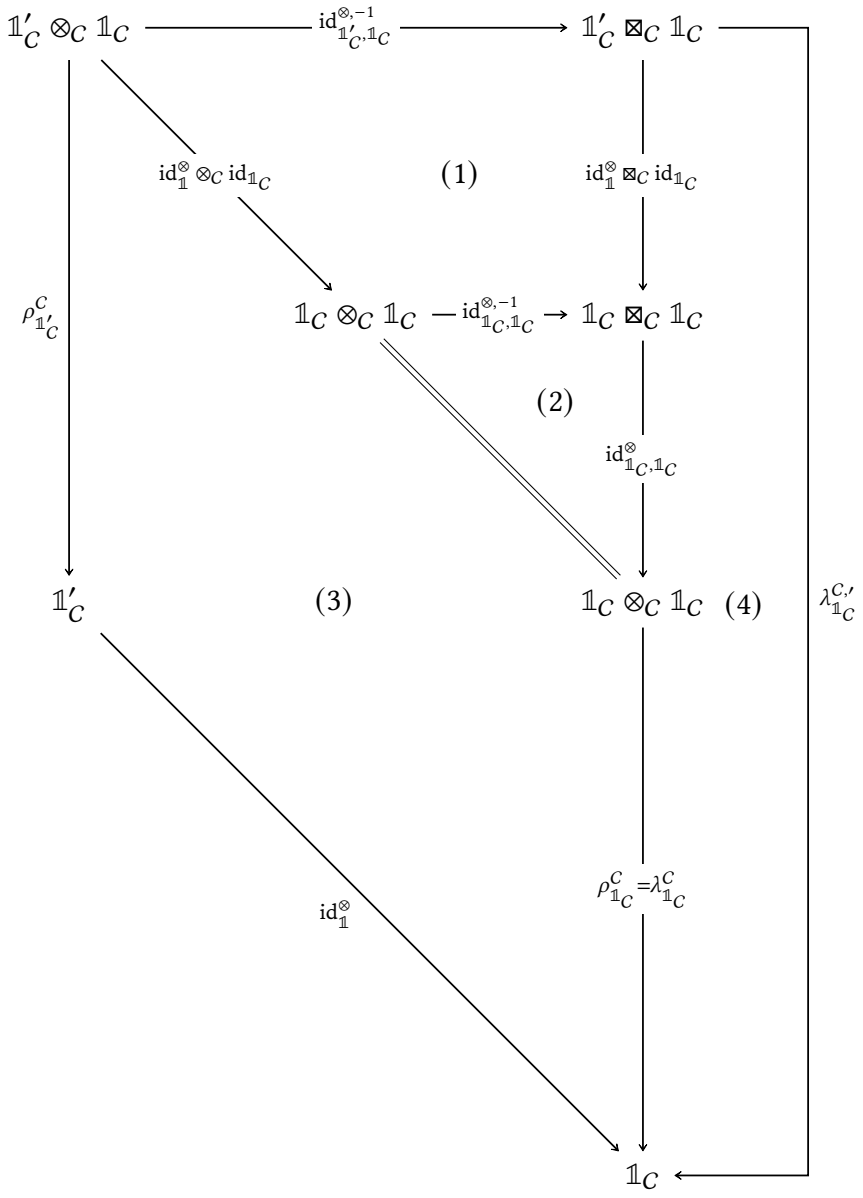

whose boundary diagram is the diagram whose commutativity

we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of $\lambda^C$, where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $\left(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}^{\otimes}_{C|\mathbb{1}}\right)$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram



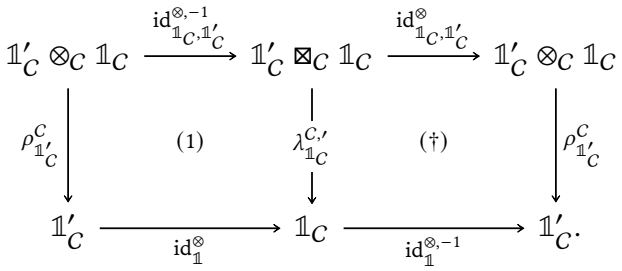

whose boundary diagram is the diagram whose commutativity

we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of $\rho^C$, where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $\left(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}^{\otimes}_{C|\mathbb{1}}\right)$;

so does the boundary diagram, and we are done.

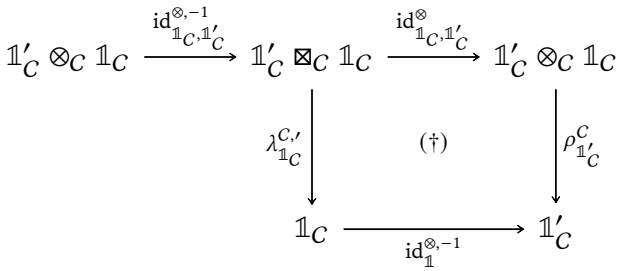
3. *Proof of Item 2c:* Indeed, consider the diagram




Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram



commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{array}{ccccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
\downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} & (1) & \downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} & (\dagger) & \downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} \\
\mathbb{1}'_C & \xrightarrow[\mathrm{id}^{\otimes}_{\mathbb{1}}]{} & \mathbb{1}_C & \xrightarrow[\mathrm{id}^{\otimes,-1}_{\mathbb{1}}]{} & \mathbb{1}_C
\end{array}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\begin{array}{ccccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 & & \downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} & (\dagger) & \downarrow{\scriptstyle \lambda^C_{\mathbb{1}'_C}} \\
 & & \mathbb{1}_C & \xrightarrow[\mathrm{id}^{\otimes,-1}_{\mathbb{1}}]{} & \mathbb{1}_C
\end{array}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

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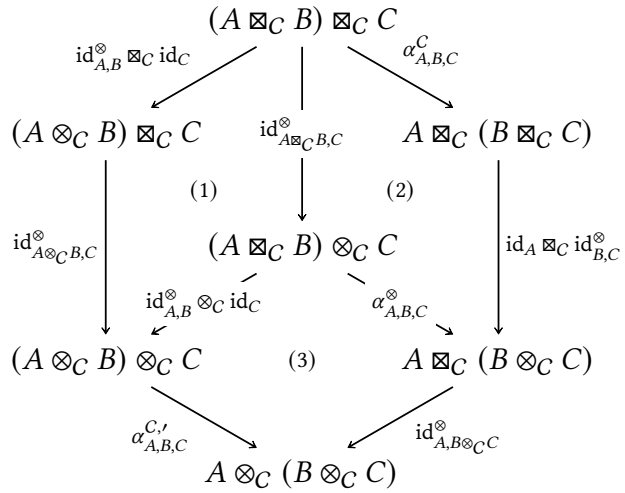
1. *Proof of Item 3a:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of Item 2 of Remark 13.1.1.1.3 as follows:




Since:

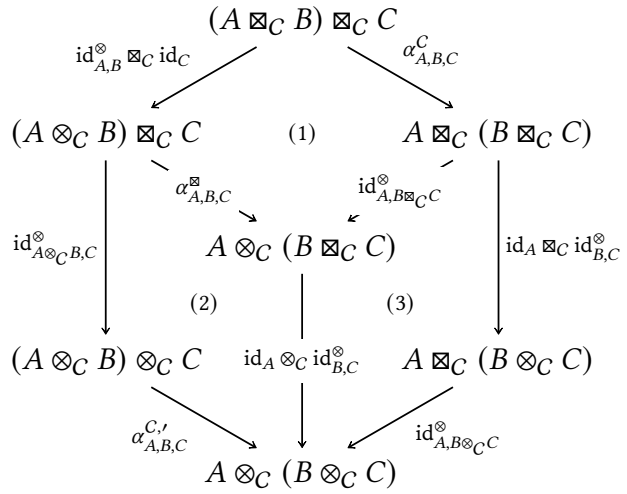
- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01V0

2. *Proof of Item 3b:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:



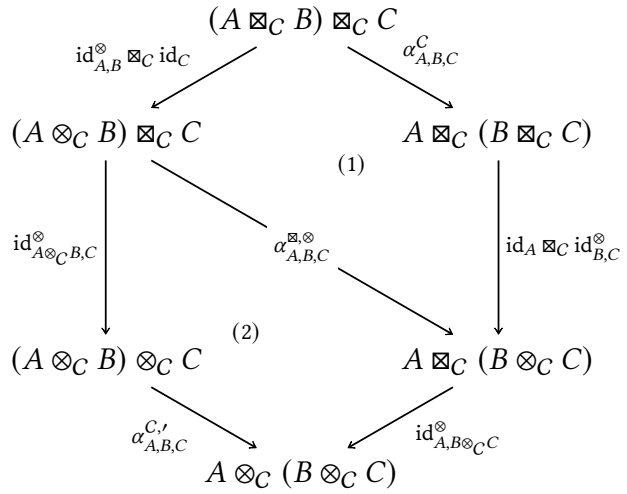

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01V1 3. *Proof of Item 3c:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:




Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof. ∎

01V2 **13.1.2 The Moduli Category of Braided Monoidal Structures on a Category**

01V3 **13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category**

01V4 **13.2 Moduli Categories of Closed Monoidal Structures**

01V5 **13.3 Moduli Categories of Refinements of Monoidal Structures**

01V6 **13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure**

# Appendices

## A Other Chapters

**Preliminaries**

1. Introduction
2. A Guide to the Literature

**Sets**

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6. Pointed Sets
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8. Relations
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