Pointed Sets

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This chapter contains some foundational material on pointed sets.

Contents

6.1 Pointed Sets

6.1.1 Foundations

Definition 6.1.1.1.1. A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 6.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 6.1.1.1.3. The 0**-sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

Example 6.1.1.1.4. The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 6.1.1.1.5. The **standard pointed set with** $n+1$ **elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \overset{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n \rangle$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$**-modules**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1. A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1. The **category of pointed sets** is the category Sets_* defined equivalently as:

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**.
- The category Sets_* of Constructions With Categories, **??**.

Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

  $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[6]

  $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\text{def}}{=} \mathrm{id}_X\,.$$

denoted $(\mathbb{F}_1, 0)$.

[4] *Further Terminology:* Also called a **pointed function**.

[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

[6] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

- *Composition.* For each $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

  of Sets_* at $((X,x_0),(Y,y_0),(Z,z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

6.1.4 Elementary Properties of Pointed Sets

Proposition 6.1.4.1.1. Let (X,x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:
   (a) Products, described as in Definition 6.2.3.1.1.
   (b) Pullbacks, described as in Definition 6.2.4.1.1.
   (c) Equalisers, described as in Definition 6.2.5.1.1.
2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:
   (a) Coproducts, described as in Definition 6.3.3.1.1.
   (b) Pushouts, described as in Definition 6.3.4.1.1;
   (c) Coequalisers, described as in Definition 6.3.5.1.1.
3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

[7] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned} \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

[8] The category Sets_* does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*\left(S^0, X\right) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

sends:

i. A pointed set (X, x_0) to X.

ii. A pointed function

$$f \colon (X, x_0) \longrightarrow (Y, y_0)$$

to the partial function

$$\xi_f \colon X \longrightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

[9] In other words, the forgetful functor

$$忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\mathrm{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.
ii. A partial function

$$f \colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, Completeness: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.
Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.
Item 3, Failure To Be Cartesian Closed: See [`MSE 2855868`].
Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.
Item 5, Relation to Partial Functions: See [`MSE 884460`]. □

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1. Let $f\colon (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2. We write $\mathsf{Sets}_*^{\mathrm{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

1. The map $\mu\colon \langle 2 \rangle \to \langle 1 \rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

   is active but not inert.

2. The map $f\colon \langle 2 \rangle \to \langle 2 \rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[dr, mapsto] & 2 \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

   is inert but not active.

3. The map $f\colon \langle 3 \rangle \to \langle 1 \rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & \\
3 \arrow[dr, mapsto] & \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

   is neither inert nor active. However, it factors as $f = a \circ i$, where

$$\begin{aligned}
i &\colon \langle 3 \rangle \to \langle 2 \rangle, \\
a &\colon \langle 2 \rangle \to \langle 1 \rangle
\end{aligned}$$

are the morphisms of pointed sets given by

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 1 \\ 2 \arrow[r, mapsto] & 2 \\ 3 \arrow[dr, mapsto] & \\ * \arrow[r, mapsto] & * \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, mapsto] & 1 \\ 2 \arrow[ur, mapsto] & \\ * \arrow[r, mapsto] & *, \end{tikzcd}$$

with i being inert and a being active.

Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

(a) The map $i\colon (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets

(b) The map $a\colon (K, k_0) \to (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_*.

Proof. Item 1, *Active-Inert Factorisation*: Let $f\colon X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= \left(X \setminus f^{-1}(y_0)\right) \cup \{x_0\}; \end{aligned}$$

- $i\colon X \to K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

- $a\colon K \to Y$ is the active morphism of pointed sets given by

  $$a(x) \overset{\text{def}}{=} f(x)$$

  for each $x \in K$.

Next, let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y \to A$ given by

$$\phi(y) = f\Big(i^{-1}(y)\Big)$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\exists!"', "\phi"] \\ A \arrow[r, "a"'] & B. \end{tikzcd}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\overset{\text{def}}{=} \phi(i(x)) \\ &\overset{\text{def}}{=} f\Big(i^{-1}(i(x))\Big) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [a \circ \phi](y) &\overset{\text{def}}{=} a(\phi(y)) \\ &\overset{\text{def}}{=} a\Big(f\Big(i^{-1}(y)\Big)\Big) \\ &\overset{\text{def}}{=} [a \circ f]\Big(i^{-1}(y)\Big) \\ &= [g \circ i]\Big(i^{-1}(y)\Big) \\ &\overset{\text{def}}{=} g\Big(i\Big(i^{-1}(y)\Big)\Big) \end{aligned}$$

$$\stackrel{\text{def}}{=} g(y)$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\
A \arrow[r, "a"'] & B
\end{tikzcd}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned}
\psi(y) &= \psi(i(x)) \\
&= f(x) \\
&= f\left(i^{-1}(y)\right) \\
&\stackrel{\text{def}}{=} \phi(y).
\end{aligned}$$

This finishes the proof. □

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1. The **terminal pointed set** is the terminal object of Sets_* as in Limits and Colimits, **??**.

Construction 6.2.1.1.2. Concretely, the **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{!_X : (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$!_X(x) \stackrel{\text{def}}{=} \star$$

  for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$\phi\colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\mathrm{pt}, \star)$$

commute, namely $!_X$. ∎

6.2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.2.2.1.1. The **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the product of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets$_*$ as in Limits and Colimits, **??**.

Construction 6.2.2.1.2. Concretely, the **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pair $\left(\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right), \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i\colon \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i\in I}$$

  of maps given by

$$\mathrm{pr}_i\left((x_j)_{j\in I}\right) \overset{\mathrm{def}}{=} x_i$$

  for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$ is the categorical product of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets$_*$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
(P, *) \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd}
(P,*) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i)
\end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_i(*))_{i\in I}\\ &= \left(x_0^i\right)_{i\in I},\end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

Proposition 6.2.2.1.3. Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.3.1.2. Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \to (X, x_0),\\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \to (Y, y_0)\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(x, y) &\overset{\mathrm{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\overset{\mathrm{def}}{=} y\end{aligned}$$

  for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\
(X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y, y_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
 & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
(X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y, y_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.3.1.3. Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

   $$\begin{aligned} A \times - \colon \quad & \mathsf{Sets}_* \quad \to \mathsf{Sets}_*, \\ - \times B \colon \quad & \mathsf{Sets}_* \quad \to \mathsf{Sets}_*, \\ -_1 \times -_2 \colon \quad & \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{aligned}$$

   defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

   $$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Unitality.* We have isomorphisms of pointed sets

   $$\begin{aligned}(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),\end{aligned}$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Commutativity.* We have an isomorphism of pointed sets

   $$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

   natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.
Item 2, Lack of Adjointness: See [MSE 2855868].
Item 3, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.
Item 4, Unitality: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.
Item 5, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.
Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3. □

6.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.4.1.2. Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{pr}_1 &: (X \times_Z Y, (x_0, y_0)) \to (X, x_0),\\ \mathrm{pr}_2 &: (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)\end{aligned}$$

  defined by

$$\begin{aligned}\mathrm{pr}_1(x, y) &\stackrel{\mathrm{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\stackrel{\mathrm{def}}{=} y\end{aligned}$$

  for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$
f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad
\begin{tikzcd}
(X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\
(X, x_0) \arrow[r, "f"'] & (Z, z_0).
\end{tikzcd}
$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$
\begin{aligned}
\big[f \circ \mathrm{pr}_1\big](x, y) &= f\big(\mathrm{pr}_1(x, y)\big) \\
&= f(x) \\
&= g(y) \\
&= g\big(\mathrm{pr}_2(x, y)\big) \\
&= \big[g \circ \mathrm{pr}_2\big](x, y),
\end{aligned}
$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$
\begin{tikzcd}
(P, *) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$
\phi \colon (P, *) \longrightarrow (X \times_Z Y, (x_0, y_0))
$$

making the diagram

$$
\begin{tikzcd}
(P, *) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists !"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.4.1.3. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

   $$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

   $$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

   $$\begin{tikzcd} X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\ & X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y' \arrow[dd, "g'"] \\ X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\ & X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z' \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi\colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \big(X' \times_{Z'} Y', \big(x'_0, y'_0\big)\big)$$

given by

$$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}
$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad A \times_X B \cong B \times_X A \qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}
$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.
Item 3, Unitality: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.
Item 4, Commutativity: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.
Item 5, Interaction With Products: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.
Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5. □

6.2.5 Equalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1. The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

  $$\mathrm{eq}(f, g)\colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists !"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g)\circ\phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= e(*)\\ &= x_0,\end{aligned}$$

where we have used that e is a morphism of pointed sets. □

Proposition 6.2.5.1.3. Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f\circ\mathrm{eq}(g,h), g\circ\mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f\circ\mathrm{eq}(f,g), h\circ\mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0) \end{tikzcd}$$

in Sets_*, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f,f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f,g) \cong \mathrm{Eq}(g,f).$$

Proof. *Item 1*, *Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.
Item 2, *Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.
Item 3, *Commutativity*: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3. □

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1. The **initial pointed set** is the initial object of Sets_* as in Limits and Colimits, **??**.

Construction 6.3.1.1.2. Concretely, the **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{\iota_X\colon (\mathrm{pt},\star) \to (X,x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$\iota_X(\star) \overset{\mathrm{def}}{=} x_0.$$

Proof. We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt},\star) \qquad (X,x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (\mathrm{pt},\star) \to (X,x_0)$$

making the diagram

$$(\mathrm{pt},\star) \underset{\exists!}{\overset{\phi}{\dashrightarrow}} (X,x_0)$$

commute, namely ι_X. □

6.3.2 Coproducts of Families of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.3.2.1.1. The **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$[11] is the coproduct of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_* as in Limits and Colimits, **??**.

Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ is the pair $\left(\left(\bigvee_{i\in I} X_i, p_0\right), \left\{\mathrm{inj}_i\right\}_{i\in I}\right)$ consisting of:

- *The Colimit.* The pointed set $\left(\bigvee_{i\in I} X_i, p_0\right)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by
  $$\bigvee_{i\in I} X_i \overset{\mathrm{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim,$$
  where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring
  $$\left(i, x_0^i\right) \sim \left(j, x_0^j\right)$$
  for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by
  $$\begin{aligned} p_0 &\overset{\mathrm{def}}{=} \left[\left(i, x_0^i\right)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$
  for any $i, j \in I$.
- *The Cocone.* The collection
$$\left\{\mathrm{inj}_i \colon \left(X_i, x_0^i\right) \to \left(\bigvee_{i\in I} X_i, p_0\right)\right\}_{i\in I}$$
of morphism of pointed sets given by
$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i, x)$$
for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $\left(\bigvee_{i\in I} X_i, p_0\right)$ is the categorical coproduct of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in

[11] *Further Terminology:* Also called the **wedge sum of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$.

Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon \left(\bigvee_{i\in I} X_i, p_0\right) \to (C,*)$$

making the diagram

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \arrow[u, dashed, "\phi"', "\exists!"] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i\left(\left[\left(i, x_0^i\right)\right]\right) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 6.3.2.1.3. Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality*. The assignment $\left\{(X_i, x_0^i)\right\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a functor

$$\bigvee_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0)[12] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}(X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \textstyle\coprod (Y, y_0)\\ &\cong \left(X \textstyle\coprod_{\mathrm{pt}} Y, p_0\right)\\ &\cong (X \textstyle\coprod Y/\sim, p_0),\end{aligned}$$

$$\begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\stackrel{\text{def}}{=} [(0, x_0)]\\ &= [(1, y_0)].\end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)],\\ \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0)

[12] *Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[r, hook, "\mathrm{inj}_1"'] \arrow[ur, bend left, "\iota_1"] & (X\vee Y,p_0) & (Y,y_0) \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"']
\end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X\vee Y,p_0)\longrightarrow(C,*)$$

making the diagram

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[r, hook, "\mathrm{inj}_1"'] \arrow[ur, bend left, "\iota_1"] & (X\vee Y,p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y,y_0) \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi\circ\mathrm{inj}_X &= \iota_X,\\
\phi\circ\mathrm{inj}_Y &= \iota_Y
\end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z=[(0,x)] \text{ with } x\in X,\\ \iota_Y(y) & \text{if } z=[(1,y)] \text{ with } y\in Y \end{cases}$$

for each $z\in X\vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \iota_X([(0,x_0)])\\
&= \iota_Y([(1,y_0)])\\
&= *,
\end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

Proposition 6.3.3.1.3. Let (X,x_0) and (Y,y_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

   define functors

$$\begin{aligned} X \vee - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \vee Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \vee -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0), \end{aligned}$$

   natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

   natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

   called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0,x)], \\ x & \text{if } p = [(1,x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, Associativity: Omitted.
Item 3, Unitality: Omitted.
Item 4, Commutativity: Omitted.
Item 5, Symmetric Monoidality: Omitted.
Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X,x_0) \to (Y,y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{aligned}$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. □

6.3.4 Pushouts

Let (X,x_0), (Y,y_0), and (Z,z_0) be pointed sets and let $f\colon (Z,z_0) \to (X,x_0)$ and $g\colon (Z,z_0) \to (Y,y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1. The **pushout of** (X,x_0) **and** (Y,y_0) **over** (Z,z_0) **along** (f,g) is the pushout of (X,x_0) and (Y,y_0) over (Z,z_0) along (f,g) in Sets_* as in Limits and Colimits, **??**.

Construction 6.3.4.1.2. Concretely, the **pushout of** (X,x_0) **and** (Y,y_0) **over** (Z,z_0) **along** (f,g) is the pair consisting of:

- *The Colimit*. The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:

  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)]\\ \mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)]\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}x_0 &= f(z_0),\\ y_0 &= g(z_0)\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}\left[\mathrm{inj}_1 \circ f\right](z) &= \mathrm{inj}_1(f(z))\\ &= [(0, f(z))]\\ &= [(1, g(z))]\\ &= \mathrm{inj}_2(g(z))\\ &= \left[\mathrm{inj}_2 \circ g\right](z),\end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, phantom] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner", very near end] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \coprod_Z Y, p_0) \longrightarrow (P, *)$$

making the diagram

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!"', "\phi"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner", very near end] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$

$$
\begin{aligned}
&= \iota_1(x_0) \\
&= *,
\end{aligned}
$$

or alternatively

$$
\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)]) \\
&= \iota_2(y_0) \\
&= *,
\end{aligned}
$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. □

Proposition 6.3.4.1.3. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

   $$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

   $$
   \begin{tikzcd}
    & \bullet \\
   \bullet & \bullet. \arrow[l] \arrow[u]
   \end{tikzcd}
   $$

   In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

   $$
   \begin{tikzcd}
   X \coprod_Z Y \arrow[dr, phantom, "\ulcorner", very near start] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
    & X' \coprod_{Z'} Y' \arrow[dr, phantom, "\ulcorner", very near start] & & Y' \arrow[ll] \\
   X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
    & X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
   \end{tikzcd}
   $$

   in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

   $$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} \left(X' \coprod_{Z'} Y', p_0'\right)$$

   given by

   $$
   \xi(p) \overset{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}
   $$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \textstyle\coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \textstyle\coprod_X A &\cong A, \\
A \textstyle\coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, equal] & X \arrow[l, "f"'] \arrow[d, equal] \\
X & X. \arrow[l, "f"]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} X \coprod_Z Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & Z, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad X \coprod_Z Y \cong Y \coprod_Z X \qquad \begin{tikzcd} Y \coprod_Z X \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ Y \arrow[u] & Z. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$X \coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *Associativity*: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.
Item 3, *Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.
Item 4, *Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.
Item 5, *Interaction With Coproducts*: Omitted.
Item 6, *Symmetric Monoidality*: Omitted. □

6.3.5 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.

- *The Cocone.* The map

$$\mathrm{coeq}(f,g)\colon Y \twoheadrightarrow (\mathrm{CoEq}(f,g),[y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(\mathrm{CoEq}(f,g),[y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g)\circ f=\mathrm{coeq}(f,g)\circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g)\circ f](x) &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](f(x))\\
&\overset{\text{def}}{=} [f(x)]\\
&= [g(x)]\\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](g(x))\\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g)\circ g](x)
\end{aligned}$$

for each $x\in X$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \\
& & (C,*)
\end{tikzcd}$$

in Sets. Then, since $c(f(a))=c(g(a))$ for each $a\in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\phi\colon\ \mathrm{CoEq}(f,g)\xrightarrow{\exists!}C$ making the diagram

$$\begin{tikzcd}
(X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\
& & (C,*)
\end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0])=[\phi\circ\mathrm{coeq}(f,g)]([y_0])$$

$$= c([y_0])$$
$$= *,$$

where we have used that c is a morphism of pointed sets. □

Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ f, \mathrm{coeq}(f,g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0) \end{tikzcd}$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

Proof. *Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.
Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.
Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5. □

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let X be a set.

Definition 6.4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[13]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 6.4.1.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \overset{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^+ \colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[13] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+},(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon(\mathsf{Sets},\coprod,\emptyset)\to(\mathsf{Sets}_{*},\vee,\mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned}(-)^{+,\coprod}_{X,Y}&\colon X^{+}\vee Y^{+}\overset{\sim}{\dashrightarrow}\left(X\coprod Y\right)^{+},\\(-)^{+,\coprod}_{\mathbb{1}}&\colon\mathrm{pt}\overset{\sim}{\dashrightarrow}\emptyset^{+},\end{aligned}$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+},(-)^{+},(-)^{+}_{\mathbb{1}}\right)\colon(\mathsf{Sets},\times,\mathrm{pt})\to\left(\mathsf{Sets}_{*},\wedge,S^{0}\right),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned}(-)^{+}_{X,Y}&\colon X^{+}\wedge Y^{+}\overset{\sim}{\dashrightarrow}(X\times Y)^{+},\\(-)^{+}_{\mathbb{1}}&\colon S^{0}\overset{\sim}{\dashrightarrow}\mathrm{pt}^{+},\end{aligned}$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: We claim that $(-)^{+}$ is indeed a functor:

- *Preservation of Identities.* Let $X\in\mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}^{+}_{X}(x)\overset{\mathrm{def}}{=}\begin{cases}x & \text{if } x\in X,\\ \star_{X} & \text{if } x=\star_{X},\end{cases}$$

for each $x\in X^{+}$, so $\mathrm{id}^{+}_{X}=\mathrm{id}_{X^{+}}$.

- *Preservation of Composition.* Given morphisms of sets

$$\begin{aligned}f&\colon X\to Y,\\g&\colon Y\to Z,\end{aligned}$$

we have

$$\begin{aligned}\left[g^{+}\circ f^{+}\right](x)&\overset{\mathrm{def}}{=}g^{+}\left(f^{+}(x)\right)\\&\overset{\mathrm{def}}{=}g^{+}(f(x))\end{aligned}$$

$$\begin{aligned} &\overset{\text{def}}{=} g(f(x)) \\ &\overset{\text{def}}{=} [g \circ f]^+(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [g^+ \circ f^+](\star_X) &\overset{\text{def}}{=} g^+(f^+(\star_X)) \\ &\overset{\text{def}}{=} g^+(\star_Y) \\ &\overset{\text{def}}{=} \star_Z \\ &\overset{\text{def}}{=} [g \circ f]^+(\star_X), \end{aligned}$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.
Item 2, *Adjointness*: We proceed in a few steps:

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

  by sending a morphism of pointed sets

  $$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

  to the function

  $$\xi^\dagger \colon X \to Y$$

  given by

  $$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

  for each $x \in X$.

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*(X^+, Y)$$

  given by sending a function $\xi \colon X \to Y$ to the morphism of pointed sets

  $$\xi^\dagger \colon (X^+, \star_X) \to (Y, y_0)$$

  defined by

  $$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

  for each $x \in X^+$.

- *Invertibility I.* Given a morphism of pointed sets

$$\xi\colon (X^{+},\star_X) \to (Y,y_0),$$

we have

$$\begin{aligned}
\left[\Psi_{X,Y}\circ\Phi_{X,Y}\right](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi))\\
&= \Psi_{X,Y}\left(\xi^{\dagger}\right)\\
&\overset{\text{def}}{=} \left[\!\left[x\mapsto \begin{cases}\xi^{\dagger}(x) & \text{if } x\in X\\ y_0 & \text{if } x=\star_X\end{cases}\right]\!\right]\\
&= \left[\!\left[x\mapsto \begin{cases}\xi(x) & \text{if } x\in X\\ y_0 & \text{if } x=\star_X\end{cases}\right]\!\right]\\
&= \xi\\
&\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}_*(X^{+},Y)}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Psi_{X,Y}\circ\Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^{+},Y)}\,.$$

- *Invertibility II.* Given a map of sets $\xi\colon X\to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y}\circ\Psi_{X,Y}\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi))\\
&= \Phi_{X,Y}\left(\xi^{\dagger}\right)\\
&= \Phi_{X,Y}\left(\left[\!\left[x\mapsto \begin{cases}\xi(x) & \text{if } x\in X\\ y_0 & \text{if } x=\star_X\end{cases}\right]\!\right]\right)\\
&= [\![x\mapsto \xi(x)]\!]\\
&= \xi\\
&\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,Y)}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y}\circ\Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}\,.$$

- *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0)\to (X',x_0'),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(X'^{,+}, Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+} \to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g\colon (Y, y_0) \to \left(Y', y'_0\right),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*(X^+, Y'), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger\colon X^+ \to Y,$$

we have

$$\left[\Phi_{X,Y'} \circ g_*\right](\xi) = \Phi_{X,Y'}(g_*(\xi))$$

$$
\begin{aligned}
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*\big(\Phi_{X,Y'}(\xi)\big) \\
&= \big[g_* \circ \Phi_{X,Y'}\big](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \textstyle\coprod Y)^+$$

is given by

$$
(-)^{+,\coprod}_{X,Y}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}
$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)^{+,\coprod,-1}_{X,Y} \colon (X \textstyle\coprod Y)^+ \overset{\sim}{\dashrightarrow} X^+ \vee Y^+$$

given by

$$
(-)^{+,\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}
$$

for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

  $$(-)^{+,\coprod,\mathbb{1}}_{X,Y}\colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \emptyset^{+}$$

  is given by sending $\star_X$ to $\star_\emptyset$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\times$ and $\wedge$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

  $$(-)^{+}_{X,Y}\colon X^{+} \wedge Y^{+} \overset{\sim}{\dashrightarrow} (X \times Y)^{+}$$

  is given by

  $$(-)^{+}_{X,Y}(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X\times Y} & \text{otherwise} \end{cases}$$

  for each $x \wedge y \in X^{+} \wedge Y^{+}$, with inverse

  $$(-)^{+,-1}_{X,Y}\colon (X \times Y)^{+} \overset{\sim}{\dashrightarrow} X^{+} \wedge Y^{+}$$

  given by

  $$(-)^{+,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X\times Y}, \end{cases}$$

  for each $z \in (X \times Y)^{+}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

  $$(-)^{+,\mathbb{1}}_{X,Y}\colon S^{0} \overset{\sim}{\dashrightarrow} \mathrm{pt}^{+}$$

  is given by sending 0 to $\star_{\mathrm{pt}}$ and 1 to $\star$, where $\mathrm{pt}^{+} = \{\star, \star_{\mathrm{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted. □

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1. The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2. Let (X, x_0) be a pointed set.

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- : \mathsf{Sets}_*^{\mathrm{actv}} \longrightarrow \mathsf{Sets},$$

   where:

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*^{\mathrm{actv}})$, we have

   $$[(-)^-](X) \overset{\text{def}}{=} X^-,$$

     where X^- is the set of Definition 6.4.2.1.1.
   - *Action on Morphisms.* For each morphism $f \colon X \longrightarrow Y$ of $\mathsf{Sets}_*^{\mathrm{actv}}$, the image

   $$f^- : X^- \longrightarrow Y^-$$

     of f by $(-)^-$ is the map defined by

   $$f^-(x) \overset{\text{def}}{=} f(x)$$

     for each $x \in X^-$.

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$\big((-)^- \dashv (-)^+\big)\colon \quad \begin{tikzcd} \mathsf{Sets}_*^{\mathrm{actv}} \arrow[r, bend left, "(-)^-"] \arrow[r, phantom, "\perp_{\mathrm{eq}}"] & \mathsf{Sets}, \arrow[l, bend left, "(-)^+"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*\big(X, Y^+\big),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms

$$\begin{aligned} (X^-)^+ &\cong X, \\ \big(Y^+\big)^- &\cong Y, \end{aligned}$$

   once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\vee},(-)^{-,\vee}_{\mathbb{1}}\right)\colon\left(\mathsf{Sets}^{\mathrm{actv}}_{*},\vee,\mathrm{pt}\right),\to(\mathsf{Sets},\textstyle\coprod,\emptyset),$$

being equipped with isomorphisms of pointed sets

$$(-)^{-,\vee}_{X,Y}\colon X^{-}\textstyle\coprod Y^{-}\overset{\sim}{\dashrightarrow}(X\vee Y)^{-},$$
$$(-)^{-,\vee}_{\mathbb{1}}\colon\emptyset\overset{\sim}{\dashrightarrow}\mathrm{pt}^{-},$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\times},(-)^{-,\times}_{\mathbb{1}}\right)\colon\left(\mathsf{Sets}^{\mathrm{actv}}_{*},\wedge,S^{0}\right),\to(\mathsf{Sets},\times,\mathrm{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)^{-}_{X,Y}\colon X^{-}\times Y^{-}\overset{\sim}{\dashrightarrow}(X\wedge Y)^{-},$$
$$(-)^{-}_{\mathbb{1}}\colon\mathrm{pt}\overset{\sim}{\dashrightarrow}\left(S^{0}\right)^{-},$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: We claim that $(-)^{-}$ is indeed a functor:

- *Preservation of Identities.* Let $X\in\mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}^{-}_{X}(x)\overset{\mathrm{def}}{=}x$$

for each $x\in X^{-}$, so $\mathrm{id}^{-}_{X}=\mathrm{id}_{X^{-}}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f\colon(X,x_0)\to(Y,y_0),$$
$$g\colon(Y,y_0)\to(Z,z_0),$$

we have

$$\begin{aligned}[g^{-}\circ f^{-}](x)&\overset{\mathrm{def}}{=}g^{-}(f^{-}(x))\\&\overset{\mathrm{def}}{=}g^{-}(f(x))\\&\overset{\mathrm{def}}{=}g(f(x))\\&\overset{\mathrm{def}}{=}[g\circ f]^{-}(x)\end{aligned}$$

for each $x\in X$, so $(g\circ f)^{-}=g^{-}\circ f^{-}$.

This finishes the proof.
Item 2, *Adjoint Equivalence*: We proceed in a few steps:

1. *Map I.* We define a map
$$\Phi_{X,Y}\colon \mathsf{Sets}(X^-,Y) \to \mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)$$
by sending a map $\xi\colon X^- \to Y$ to the active morphism of pointed sets
$$\xi^\dagger\colon X \to Y^+$$
given by
$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$
for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. *Map II.* We define a map
$$\Psi_{X,Y}\colon \mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$
given by sending an active morphism of pointed sets $\xi\colon X \to Y^+$ to the map
$$\xi^\dagger\colon X^- \to Y$$
defined by
$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \xi(x)$$
for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi\colon X^- \to Y$, we have
$$\begin{aligned}
\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\overset{\mathrm{def}}{=} \Psi_{X,Y}\left(\Phi_{X,Y}(\xi)\right) \\
&\overset{\mathrm{def}}{=} \Psi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]\right) \\
&= \left[\!\left[x \mapsto \xi(x)\right]\!\right] \\
&= \xi \\
&= \left[\mathrm{id}_{\mathsf{Sets}(X^-,Y)}\right](\xi).
\end{aligned}$$
Therefore we have
$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}.$$

4. *Invertibility II.* Given a morphism of pointed sets

$$\xi\colon (X,x_0) \to \left(Y^+,\star_Y\right),$$

we have

$$\begin{aligned}
\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right) \\
&= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\
&= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\
&= \xi \\
&= \left[\mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\,.$$

5. *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0) \to \left(X',x_0'\right),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}\left(X^{\prime,-},Y\right) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X',Y^+) \arrow[d, "f^*"] \\
\mathsf{Sets}_*(X^-,Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)
\end{tikzcd}$$

commutes. Indeed, given a map of sets $\xi\colon X' \to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \llbracket x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X^{\prime,-} \\ \star_Y & \text{if } f(x) = x_0' \end{cases} \rrbracket \\
&= f^*\left(\llbracket x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X^{\prime,-} \\ \star_Y & \text{if } x' = x_0' \end{cases} \rrbracket\right) \\
&= f^*\left(\Phi_{X',Y}(\xi)\right)
\end{aligned}$$

$$= \left[f^* \circ \Phi_{X',Y}\right](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g\colon (Y, y_0) \to \left(Y', y_0'\right),$$

the diagram

$$\begin{tikzcd} \mathsf{Sets}(X^-, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+) \arrow[d, "g_*"] \\ \mathsf{Sets}(X^-, Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X, Y'^{,+}) \end{tikzcd}$$

commutes. Indeed, given a map of sets $\xi\colon X^- \to Y$, we have

$$\begin{aligned} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= \left[\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}\right]\!\right] \\ &= g_*\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases}\right]\!\right]\right) \\ &= g_*\left(\Phi_{X,Y'}(\xi)\right) \\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

7. *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

8. *Fully Faithfulness of* $(-)^{-}$. We aim to show that the assignment $f \mapsto f^{-}$ sets up a bijection

$$(-)^{-}_{X,Y}\colon \mathsf{Sets}^{\mathrm{actv}}_{*}(X,Y) \xrightarrow{\sim} \mathsf{Sets}(X^{-},Y^{-}).$$

   Indeed, the inverse map

$$(-)^{-,-1}_{X,Y}\colon \mathsf{Sets}(X^{-},Y^{-}) \xrightarrow{\sim} \mathsf{Sets}^{\mathrm{actv}}_{*}(X,Y)$$

   is given by sending a map of sets $f\colon X^{-} \to Y^{-}$ to the active morphism of pointed sets $f^{\dagger}\colon X \to Y$ defined by

$$f^{\dagger}(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_0 & \text{if } x = x_0 \end{cases}$$

   for each $x \in X$.

9. *Essential Surjectivity of* $(-)^{-}$. We need to show that, given an object $X \in \mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}\left(\mathsf{Sets}^{\mathrm{actv}}_{*}\right)$ such that $(X')^{-} \cong X$. Indeed, taking $X' = X^{+}$, we have

$$\begin{aligned} \left(X^{+}\right)^{-} &\overset{\mathrm{def}}{=} (X \cup \{\star_X\})^{-} \\ &\overset{\mathrm{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

   and thus we have in fact an *equality* $(X^{+})^{-} = X$, showing $(-)^{-}$ to be essentially surjective.

10. *The Functor* $(-)^{-}$ *Is an Equivalence.* Since $(-)^{-}$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.
Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^{-}$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-,\vee}_{X,Y}\colon X^{-} \coprod Y^{-} \xrightarrow{\sim} (X \vee Y)^{-}$$

  is given by

$$(-)^{-,\vee}_{X,Y}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^{-} \coprod Y^{-}$, with inverse

$$(-)^{-,\vee,-1}_{X,Y} \colon (X \vee Y)^{-} \overset{\sim}{\dashrightarrow} X^{-} \coprod Y^{-}$$

given by

$$(-)^{-,\vee,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\vee,\mathbb{1}}_{X,Y} \colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^{-}$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{-}$ into a symmetric strong monoidal functor is omitted.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \overset{\sim}{\dashrightarrow} (X \wedge Y)^{-}$$

is given by

$$(-)^{-}_{X,Y}(x,y) = x \wedge y$$

for each $(x,y) \in X^{-} \times Y^{-}$, with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \overset{\sim}{\dashrightarrow} X^{-} \times Y^{-}$$

given by

$$(-)^{-,-1}_{X,Y}(x \wedge y) \overset{\text{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{-,\mathbb{1}}_{X,Y} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^{0}\right)^{-}$$

is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted. □

Appendices

A Other Chapters

References

[MSE 2855868] Qiaochu Yuan. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: `https://math.stackexchange.com/q/2855868` (cit. on pp. 6, 15).

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