

# Relations

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- 00HD This chapter contains some material about relations. Notably, we discuss and explore:
- 028A 1. The definition of relations ([Section 8.1.1](#)).
  - 028B 2. How relations may be viewed as decategorification of profunctors ([Section 8.1.2](#)).
  - 02KS 3. The various kinds of categories that relations form, namely:
    - 02KT (a) A category ([Section 8.3.2](#)).
    - 02KU (b) A monoidal category ([Section 8.3.3](#)).
    - 02KV (c) A 2-category ([Section 8.3.4](#)).
    - 02KW (d) A double category ([Section 8.3.5](#)).
  - 028H 4. The various categorical properties of the 2-category of relations, including:
    - 028J (a) The self-duality of  $\mathbf{Rel}$  and  $\mathbf{Rel}$  ([Definition 8.5.1.1.1](#)).
    - 028K (b) Identifications of equivalences and isomorphisms in  $\mathbf{Rel}$  with bijections ([Definition 8.5.2.1.2](#)).
    - 028L (c) Identifications of adjunctions in  $\mathbf{Rel}$  with functions ([Definition 8.5.3.1.1](#)).
    - 028M (d) Identifications of monads in  $\mathbf{Rel}$  with preorders (??).
    - 028N (e) Identifications of comonads in  $\mathbf{Rel}$  with subsets (??).
    - 028P (f) A description of the monoids and comonoids in  $\mathbf{Rel}$  with respect to the Cartesian product ([Definition 8.5.9.1.1](#)).
    - 028Q (g) Characterisations of monomorphisms in  $\mathbf{Rel}$  ([Definition 8.5.10.1.1](#)).
    - 028R (h) Characterisations of 2-categorical notions of monomorphisms in  $\mathbf{Rel}$  ([Definition 8.5.11.1.1](#)).
    - 028S (i) Characterisations of epimorphisms in  $\mathbf{Rel}$  ([Definition 8.5.12.1.1](#)).

- 028T (j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Definition 8.5.13.1.1).
- 028U (k) The partial co/completeness of **Rel** (Definition 8.5.14.1.1).
- 028V (l) The existence or non-existence of Kan extensions and Kan lifts in **Rel** (??).
- 028W (m) The closedness of **Rel** (Definition 8.5.19.1.1).
- 028X (n) The identification of **Rel** with the category of free algebras of the powerset monad on **Sets** (Definition 8.5.20.1.1).
- 02KX 5. The adjoint pairs

$$R_! \dashv R_{-1} : \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$

$$R^{-1} \dashv R_* : \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between  $\mathcal{P}(A)$  and  $\mathcal{P}(B)$  induced by a relation  $R : A \multimap B$ , as well as the properties of  $R_!$ ,  $R_{-1}$ ,  $R^{-1}$ , and  $R_*$  (Section 8.7).

Of particular note are the following points:

- 02KY (a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple  $f_! \dashv f^{-1} \dashv f_*$  induced by a function  $f : A \rightarrow B$  studied in **Constructions With Sets**, Section 4.6.
- 02KZ (b) We have  $R_{-1} = R^{-1}$  iff  $R$  is total and functional (Item 8 of Definition 8.7.2.1.3).
- 02L0 (c) As a consequence of the previous item, when  $R$  comes from a function  $f$ , the pair of adjunctions

$$R_! \dashv R_{-1} = R^{-1} \dashv R_*$$

reduces to the triple adjunction

$$f_! \dashv f^{-1} \dashv f_*$$

from **Constructions With Sets**, Section 4.6.

- 02L1 (d) The pairs  $R_! \dashv R_{-1}$  and  $R^{-1} \dashv R_*$  turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (Topological Spaces, ??).

- 028Y 6. A description of two notions of “skew composition” on  $\mathbf{Rel}(A, B)$ , giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on  $\mathbf{Fun}(C, \mathcal{D})$  appearing in the definition of a relative monad (Sections 8.8 and 8.9).

This chapter is under revision. TODO:

1. Replicate [Section 8.5](#) for apartness composition
2. Revise [Section 8.7](#)
3. Add subsection “A Six Functor Formalism for Sets, Part 2”, now with relations, building upon [Section 8.7](#).
4. Replicate [Section 8.7](#) for apartness composition
5. Revise sections on skew monoidal structures on  $\mathbf{Rel}(A, B)$
6. Replicate the sections on skew monoidal structures on  $\mathbf{Rel}(A, B)$  for apartness composition.
7. Explore relative co/monads in  $\mathbf{Rel}$ , defined to be co/monoids in  $\mathbf{Rel}(A, B)$  with its left/right skew monoidal structures of [Relations](#), [Sections 8.8](#) and [8.9](#)
8. functional total relations defined with “satisfying the following equivalent conditions:”

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## 00HE 8.1 Relations

### 00HF 8.1.1 Foundations

Let  $A$  and  $B$  be sets.

00HG **Definition 8.1.1.1.1.** A relation  $R: A \rightarrow B$  from  $A$  to  $B$ <sup>1,2</sup> is equivalently:

- 00HR 1. A subset  $R$  of  $A \times B$ .
- 00HS 2. A function from  $A \times B$  to  $\{\text{true}, \text{false}\}$ .
- 00HT 3. A function from  $A$  to  $\mathcal{P}(B)$ .
- 00HU 4. A function from  $B$  to  $\mathcal{P}(A)$ .
- 00HV 5. A cocontinuous morphism of posets from  $(\mathcal{P}(A), \subset)$  to  $(\mathcal{P}(B), \subset)$ .
- 02C6 6. A continuous morphism of posets from  $(\mathcal{P}(B), \supset)$  to  $(\mathcal{P}(A), \supset)$ .

*Proof.* (We will prove that **Items 1 to 6** are indeed equivalent in a bit.)  $\square$

01QZ **Remark 8.1.1.1.2.** We may think of a relation  $R: A \rightarrow B$  as a function from  $A$  to  $B$  that is *multivalued*, assigning to each element  $a$  in  $A$  a set  $R(a)$  of elements of  $B$ , thought of as the *set of values of  $R$  at  $a$* .

Note that this includes also the possibility of  $R$  having no value at all on a given  $a \in A$  when  $R(a) = \emptyset$ .

01R0 **Remark 8.1.1.1.3.** Another way of stating the equivalence between **Items 1 to 5** of **Definition 8.1.1.1.1** is by saying that we have bijections of sets

$$\begin{aligned}
 \{\text{relations from } A \text{ to } B\} &\cong \mathcal{P}(A \times B) \\
 &\cong \text{Sets}(A \times B, \{\text{true}, \text{false}\}) \\
 &\cong \text{Sets}(A, \mathcal{P}(B)) \\
 &\cong \text{Sets}(B, \mathcal{P}(A)) \\
 &\cong \text{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B)) \\
 &\cong \text{Pos}^{\mathcal{C}}(\mathcal{P}(B), \mathcal{P}(A))
 \end{aligned}$$

natural in  $A, B \in \text{Obj}(\text{Sets})$ , where  $\mathcal{P}(A)$  and  $\mathcal{P}(B)$  are endowed with the poset structure given by inclusion.

*Proof.* We claim that **Items 1 to 5** are indeed equivalent:

<sup>1</sup>*Further Terminology:* Also called a **multivalued function from  $A$  to  $B$** .

<sup>2</sup>*Further Terminology:* When  $A = B$ , we also call  $R \subset A \times A$  a **relation on  $A$** .

- *Item 1*  $\iff$  *Item 2*: This is a special case of **Constructions With Sets, Items 2 and 3 of Definition 4.5.1.1.4.**

- *Item 2*  $\iff$  *Item 3*: This follows from the bijections

$$\begin{aligned} \text{Sets}(A \times B, \{\text{true}, \text{false}\}) &\cong \text{Sets}(A, \text{Sets}(B, \{\text{true}, \text{false}\})) \\ &\cong \text{Sets}(A, \mathcal{P}(B)), \end{aligned}$$

where the last bijection is from **Constructions With Sets, Items 2 and 3 of Definition 4.5.1.1.4.**

- *Item 2*  $\iff$  *Item 4*: This follows from the bijections

$$\begin{aligned} \text{Sets}(A \times B, \{\text{true}, \text{false}\}) &\cong \text{Sets}(B, \text{Sets}(A, \{\text{true}, \text{false}\})) \\ &\cong \text{Sets}(B, \mathcal{P}(A)), \end{aligned}$$

where again the last bijection is from **Constructions With Sets, Items 2 and 3 of Definition 4.5.1.1.4.**

- *Item 2*  $\iff$  *Item 5*: This follows from the universal property of the powerset  $\mathcal{P}(X)$  of a set  $X$  as the free cocompletion of  $X$  via the characteristic embedding

$$\chi_X: X \hookrightarrow \mathcal{P}(X)$$

of  $X$  into  $\mathcal{P}(X)$ , as in **Constructions With Sets, Definition 4.4.5.1.1.** In particular, the bijection

$$\text{Sets}(A, \mathcal{P}(B)) \cong \text{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B))$$

is given by extending each  $f: A \rightarrow \mathcal{P}(B)$  in  $\text{Sets}(A, \mathcal{P}(B))$  from  $A$  to all of  $\mathcal{P}(A)$  by taking its left Kan extension along  $\chi_X$ , recovering the direct image function  $f_!: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$  of  $f$  of **Constructions With Sets, Definition 4.6.1.1.1.**

- *Item 5*  $\iff$  *Item 6*: Omitted.

This finishes the proof.  $\square$

**00HH Notation 8.1.1.1.4.** Let  $A$  and  $B$  be sets and let  $R: \dashv B$  be a relation from  $A$  to  $B$ .

**00HM** 1. We write  $\text{Rel}(A, B)$  for the set of relations from  $A$  to  $B$ .

**00HN** 2. We write  $\mathbf{Rel}(A, B)$  for the sub-poset of  $(\mathcal{P}(A \times B), \subset)$  spanned by the relations from  $A$  to  $B$ .

00HJ 3. Given  $a \in A$  and  $b \in B$ , we write  $a \sim_R b$  to mean  $(a, b) \in R$ .

00HK 4. When viewing  $R$  as a function

$$R: A \times B \rightarrow \{t, f\},$$

we write  $R_a^b$  for the value of  $R$  at  $(a, b)$ .<sup>3</sup>

00HW **Proposition 8.1.1.1.5.** Let  $A$  and  $B$  be sets and let  $R, S: A \rightarrow B$  be relations.

00HX 1. *End Formula for the Set of Inclusions of Relations.* We have

$$\text{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a \in A} \int_{b \in B} \text{Hom}_{\{t,f\}}(R_a^b, S_a^b).$$

*Proof.* **Item 1, End Formula for the Set of Inclusions of Relations:** Unwinding the expression inside the end on the right hand side, we have

$$\int_{a \in A} \int_{b \in B} \text{Hom}_{\{t,f\}}(R_a^b, S_a^b) \cong \begin{cases} \text{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{we have } \text{Hom}_{\{t,f\}}(R_a^b, S_a^b) \cong \text{pt} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since we have  $\text{Hom}_{\{t,f\}}(R_a^b, S_a^b) = \{\text{true}\} \cong \text{pt}$  exactly when  $R_a^b = \text{false}$  or  $R_a^b = S_a^b = \text{true}$ , we get

$$\int_{a \in A} \int_{b \in B} \text{Hom}_{\{t,f\}}(R_a^b, S_a^b) \cong \begin{cases} \text{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\text{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \begin{cases} \text{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since  $(a \sim_R b \implies a \sim_S b)$  iff  $R \subset S$ , the two sets above are isomorphic. This finishes the proof.  $\square$

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<sup>3</sup>The choice to write  $R_a^b$  in place of  $R_b^a$  is to keep the notation consistent with the notation we will later employ for profunctors in ??.



### 00HY 8.1.2 Relations as Decategorifications of Profunctors

00HZ **Remark 8.1.2.1.1.** The notion of a relation is a decategorification of that of a profunctor:

028Z 1. A profunctor from a category  $C$  to a category  $\mathcal{D}$  is a functor

$$\mathfrak{p}: \mathcal{D}^{\text{op}} \times C \rightarrow \text{Sets}.$$

0290 2. A relation on sets  $A$  and  $B$  is a function

$$R: A \times B \rightarrow \{\text{true}, \text{false}\}.$$

Here we notice that:

- The opposite  $X^{\text{op}}$  of a set  $X$  is itself, as  $(-)^{\text{op}}: \text{Cats} \rightarrow \text{Cats}$  restricts to the identity endofunctor on  $\text{Sets}$ .
- The values that profunctors and relations take are analogous:
  - A category is enriched over the category

$$\text{Sets} \stackrel{\text{def}}{=} \text{Cats}_0$$

of sets, with profunctors taking values on it.

- A set is enriched over the set

$$\{\text{true}, \text{false}\} \stackrel{\text{def}}{=} \text{Cats}_{-1}$$

of classical truth values, with relations taking values on it.

00J0 **Remark 8.1.2.1.2.** Extending **Definition 8.1.2.1.1**, the equivalent definitions of relations in **Definition 8.1.1.1.1** are also related to the corresponding ones for profunctors (??), which state that a profunctor  $\mathfrak{p}: C \rightarrow \mathcal{D}$  is equivalently:

00J1 1. A functor  $\mathfrak{p}: \mathcal{D}^{\text{op}} \times C \rightarrow \text{Sets}$ .

00J2 2. A functor  $\mathfrak{p}: C \rightarrow \text{PSh}(\mathcal{D})$ .

00J3 3. A functor  $\mathfrak{p}: \mathcal{D}^{\text{op}} \rightarrow \text{CoPSh}(C)$ .

00J4 4. A colimit-preserving functor  $\mathfrak{p}: \text{PSh}(C) \rightarrow \text{PSh}(\mathcal{D})$ .

02C7 5. A limit-preserving functor  $\mathfrak{p}: \text{CoPSh}(\mathcal{D})^{\text{op}} \rightarrow \text{CoPSh}(C)^{\text{op}}$ .

Indeed:

- The equivalence between **Items 1** and **2** (and also that between **Items 1** and **3**, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\text{Sets}(A \times B, \{\text{true}, \text{false}\}) &\cong \text{Sets}(A, \text{Sets}(B, \{\text{true}, \text{false}\})) \\ &\cong \text{Sets}(A, \mathcal{P}(B)),\end{aligned}$$

and

$$\begin{aligned}\text{Fun}(\mathcal{D}^{\text{op}} \times \mathcal{D}, \text{Sets}) &\cong \text{Fun}(C, \text{Fun}(\mathcal{D}^{\text{op}}, \text{Sets})) \\ &\cong \text{Fun}(C, \text{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between **Items 2** and **4** follows from the universal properties of:

- The powerset  $\mathcal{P}(X)$  of a set  $X$  as the free cocompletion of  $X$  via the characteristic embedding

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

of  $X$  into  $\mathcal{P}(X)$ , as stated and proved in **Constructions With Sets, Definition 4.4.5.1.1**.

- The category  $\text{PSh}(C)$  of presheaves on a category  $C$  as the free cocompletion of  $C$  via the Yoneda embedding

$$\mathcal{Y} : C \hookrightarrow \text{PSh}(C)$$

of  $C$  into  $\text{PSh}(C)$ , as stated and proved in **Presheaves and the Yoneda Lemma, ?? of Definition 12.1.4.1.3**.

- The equivalence between **Items 3** and **5** follows from the universal properties of:

- The powerset  $\mathcal{P}(X)$  of a set  $X$  as the free completion of  $X$  via the characteristic embedding

$$\chi_{(-)} : X \hookrightarrow \mathcal{P}(X)$$

of  $X$  into  $\mathcal{P}(X)$ , as stated and proved in **Constructions With Sets, Definition 4.4.6.1.1**.

- The category  $\text{CoPSh}(\mathcal{D})^{\text{op}}$  of copresheaves on a category  $\mathcal{D}$  as the free completion of  $\mathcal{D}$  via the dual Yoneda embedding

$$\mathcal{Y} : \mathcal{D} \hookrightarrow \text{CoPSh}(\mathcal{D})^{\text{op}}$$

of  $\mathcal{D}$  into  $\text{CoPSh}(\mathcal{D})^{\text{op}}$ , as stated and proved in **Presheaves and the Yoneda Lemma, ?? of Definition 12.1.4.1.3**.

### 00QT 8.1.3 Composition of Relations

Let  $A$ ,  $B$ , and  $C$  be sets and let  $R: A \rightarrow B$  and  $S: B \rightarrow C$  be relations.

00QU **Definition 8.1.3.1.1.** The **composition of  $R$  and  $S$**  is the relation  $S \diamond R$  defined as follows:

02D1 1. Viewing relations from  $A$  to  $C$  as subsets of  $A \times C$ , we define

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

02D2 2. Viewing relations as functions  $A \times B \rightarrow \{\text{true}, \text{false}\}$ , we define

$$\begin{aligned} (S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{b \in B} S_b^{-1} \times R_{-2}^b \\ &= \bigvee_{b \in B} S_b^{-1} \times R_{-2}^b, \end{aligned}$$

where the join  $\bigvee$  is taken in the poset  $(\{\text{true}, \text{false}\}, \preceq)$  of **Sets, Definition 3.2.2.1.3**.

02D3 3. Viewing relations as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$S \diamond R \stackrel{\text{def}}{=} \text{Lan}_{\chi_B}(S) \circ R,$$

where  $\text{Lan}_{\chi_B}(S)$  is computed by the formula

$$\begin{aligned} [\text{Lan}_{\chi_B}(S)](V) &\cong \int^{b \in B} \chi_{\mathcal{P}(B)}(\chi_b, V) \odot S(b) \\ &\cong \int^{b \in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b \in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b \in V} S(b) \end{aligned}$$

for each  $V \in \mathcal{P}(B)$ , so we have<sup>4</sup>

$$\begin{aligned} [S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} S(b). \end{aligned}$$

for each  $a \in A$ .

**02D4 Remark 8.1.3.1.2.** You might wonder what happens if we instead define an alternative composition of relations  $\diamond'$  via right Kan extensions. In this case, we would take the right Kan extension of  $S$  along the dual characteristic embedding  $B \hookrightarrow \mathcal{P}(B)^{\text{op}}$ :

$$S \diamond' R \stackrel{\text{def}}{=} \text{Ran}_{\chi_B}(S) \circ R,$$

The diagram illustrates the right Kan extension of  $S$  along the dual characteristic embedding  $\chi_B: B \rightarrow \mathcal{P}(B)^{\text{op}}$ . It shows a commutative square with  $B$  at the top-left,  $\mathcal{P}(C)$  at the top-right,  $A$  at the bottom-left, and  $\mathcal{P}(B)^{\text{op}}$  at the bottom-right. The horizontal arrow is  $R: A \rightarrow \mathcal{P}(B)^{\text{op}}$ , the vertical arrow is  $\chi_B: B \rightarrow \mathcal{P}(B)^{\text{op}}$ , the top arrow is  $S: B \rightarrow \mathcal{P}(C)$ , and the diagonal arrow is  $\text{Ran}_{\chi'_B}(S): \mathcal{P}(B)^{\text{op}} \rightarrow \mathcal{P}(C)$ .

In this case, we would have<sup>5</sup>

$$[S \diamond' R](a) \stackrel{\text{def}}{=} \bigcap_{b \in R(a)} S(b).$$

This alternative composition turns out to actually be a different kind of structure: it's an internal right Kan extension in **Rel**, namely  $\text{Ran}_{R^\dagger}(S)$  — see [Section 8.5.17](#).

**00QV Example 8.1.3.1.3.** Here are some examples of composition of relations.

**02AX 1.** *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let  $A = B = C = \mathbb{R}$ . We have

$$\begin{aligned} \leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}. \end{aligned}$$

<sup>4</sup>That is: the relation  $R$  may send  $a \in A$  to a number of elements  $\{b_i\}_{i \in I}$  in  $B$ , and then the relation  $S$  may send the image of each of the  $b_i$ 's to a number of elements  $\{S(b_i)\}_{i \in I} = \left\{ \{c_{ji}\}_{j_i \in J_i} \right\}_{i \in I}$  in  $C$ .

<sup>5</sup>If we replace  $R(a)$  with  $B \setminus R(a)$ , defining

$$S \square R \stackrel{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b),$$

we instead obtain the apartness composition of relations; see [Section 8.1.4](#).

- 02AY 2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let  $A = B = C = \mathbb{R}$ . We have

$$\begin{aligned}\leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq.\end{aligned}$$

00QW **Proposition 8.1.3.1.4.** Let  $R: A \rightarrowtail B$ ,  $S: B \rightarrowtail C$ , and  $T: C \rightarrowtail D$  be relations.

- 02D5 1. *Functoriality.* The assignments  $R, S, (R, S) \mapsto S \diamond R$  define functors

$$\begin{aligned}S \diamond -: \quad & \mathbf{Rel}(A, B) && \rightarrow \mathbf{Rel}(A, C), \\ - \diamond R: \quad & \mathbf{Rel}(B, C) && \rightarrow \mathbf{Rel}(A, C), \\ -_1 \diamond -_2: & \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) && \rightarrow \mathbf{Rel}(A, C).\end{aligned}$$

In particular, given relations

$$A \begin{array}{c} \xrightarrow{R_1} \\ \xleftrightarrow{\quad} \\ \xrightarrow{R_2} \end{array} B \begin{array}{c} \xrightarrow{S_1} \\ \xleftrightarrow{\quad} \\ \xrightarrow{S_2} \end{array} C,$$

if  $R_1 \subset R_2$  and  $S_1 \subset S_2$ , then  $S_1 \diamond R_1 \subset S_2 \diamond R_2$ .

- 00QY 2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

That is, we have

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)$$

for each  $a \in A$ .

- 00QZ 3. *Unitality.* We have

$$\begin{aligned}\Delta_B \diamond R &= R, \\ R \diamond \Delta_A &= R.\end{aligned}$$

That is, we have

$$\begin{aligned}\bigcup_{b \in R(a)} \{b\} &= R(a), \\ \bigcup_{a \in \{a\}} R(a) &= R(a)\end{aligned}$$

for each  $a \in A$ .

02D6 4. *Relation to Apartness Composition of Relations.* We have

$$\begin{aligned}(S \diamond R)^c &= S^c \sqcap R^c, \\ (S \sqcap R)^c &= S^c \diamond R^c,\end{aligned}$$

where  $(-)^c$  is the complement functor of [Constructions With Sets, Section 4.3.11](#). In particular,  $\diamond$  is a special case of apartness composition of relations, as we have

$$S \diamond R = (S^c \sqcap R^c)^c.$$

This is also compatible with units, as we have  $\Delta_A^c = \nabla_A$ .

02D7 5. *Linear Distributivity.* We have inclusions of relations

$$\begin{aligned}T \diamond (S \sqcap R) &\subset (T \diamond S) \sqcap R, \\ (T \sqcap S) \diamond R &\subset T \sqcap (S \diamond R).\end{aligned}$$

That is, we have

$$\begin{aligned}T\left(\bigcap_{b \in B \setminus R(a)} S(b)\right) &\subset \bigcap_{b \in B \setminus R(a)} T(S(b)) \\ \bigcup_{b \in R(a)} \bigcap_{c \in C \setminus S(b)} T(c) &\subset \bigcap_{c \in C \setminus S(R(a))} T(c)\end{aligned}$$

or, unwinding the expression for  $S(R(a))$ , we have

$$\begin{aligned}\bigcup_{c \in \bigcap_{b \in B \setminus R(a)} S(b)} T(c) &\subset \bigcap_{b \in B \setminus R(a)} \bigcup_{c \in S(b)} T(c) \\ \bigcup_{b \in R(a)} \bigcap_{c \in C \setminus S(b)} T(c) &\subset \bigcap_{c \in C \setminus \bigcup_{b \in R(a)} S(b)} T(c)\end{aligned}$$

for each  $a \in A$ .

00R0 6. *Interaction With Converses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

00QX 7. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\text{dom}(S \diamond R) &\subset \text{dom}(R), \\ \text{range}(S \diamond R) &\subset \text{range}(S).\end{aligned}$$

*Proof. Item 1, Functoriality:* We have

$$\begin{aligned} S_1 \diamond R_1 &\stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{there exists some } b \in B, \text{ such} \\ \text{that } a \sim_{R_1} b \text{ or } b \sim_{S_1} c \end{array} \right. \right\} \\ &\subset \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{there exists some } b \in B, \text{ such} \\ \text{that } a \sim_{R_2} b \text{ or } b \sim_{S_2} c \end{array} \right. \right\} \\ &\stackrel{\text{def}}{=} S_2 \diamond R_2. \end{aligned}$$

This finishes the proof.

*Item 2, Associativity, Proof I:* Indeed, we have

$$\begin{aligned} (T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left( \int^{c \in C} T_c^{-1} \times S_{-2}^c \right) \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in B} \left( \int^{c \in C} T_c^{-1} \times S_b^c \right) \times R_{-2}^b \\ &= \int^{b \in B} \int^{c \in C} (T_c^{-1} \times S_b^c) \times R_{-2}^b \\ &= \int^{c \in C} \int^{b \in B} (T_c^{-1} \times S_b^c) \times R_{-2}^b \\ &= \int^{c \in C} \int^{b \in B} T_c^{-1} \times (S_b^c \times R_{-2}^b) \\ &= \int^{c \in C} T_c^{-1} \times \left( \int^{b \in B} S_b^c \times R_{-2}^b \right) \\ &\stackrel{\text{def}}{=} \int^{c \in C} T_c^{-1} \times (S \diamond R)_{-2}^c \\ &\stackrel{\text{def}}{=} T \diamond (S \diamond R). \end{aligned}$$

In the language of relations, given  $a \in A$  and  $d \in D$ , the stated equality witnesses the equivalence of the following two statements:

- 02D8    1. We have  $a \sim_{(T \diamond S) \diamond R} d$ , i.e. there exists some  $b \in B$  such that:
- We have  $a \sim_R b$ ;
  - We have  $b \sim_{T \diamond S} d$ , i.e. there exists some  $c \in C$  such that:
    - We have  $b \sim_S c$ ;
    - We have  $c \sim_T d$ ;
- 02D9    2. We have  $a \sim_{T \diamond (S \diamond R)} d$ , i.e. there exists some  $c \in C$  such that:

- We have  $a \sim_{S \diamond R} c$ , i.e. there exists some  $b \in B$  such that:
  - We have  $a \sim_R b$ ;
  - We have  $b \sim_S c$ ;
- We have  $c \sim_T d$ ;

both of which are equivalent to the statement

(★) There exist  $b \in B$  and  $c \in C$  such that  $a \sim_R b \sim_S c \sim_T d$ .

*Item 2, Associativity, Proof II:* Using **Item 3** of **Definition 8.1.3.1.1**, we have

$$\begin{aligned} [(T \diamond S) \diamond R](a) &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} (T \diamond S)(b) \\ &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) \end{aligned}$$

on the one hand and

$$\begin{aligned} [T \diamond (S \diamond R)](a) &\stackrel{\text{def}}{=} \bigcup_{c \in [S \diamond R](a)} T(c) \\ &\stackrel{\text{def}}{=} \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c) \end{aligned}$$

on the other, so we want to prove an equality of the form

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c).$$

This then follows from an application of **Constructions With Sets, Item 2** of **Definition 4.3.6.1.2** in which we consider  $X = D$ , consider  $\mathcal{P}(\mathcal{P}(\mathcal{P}(D)))$ , take  $U = U_c = T(c)$ , take  $A$  to be

$$A_b \stackrel{\text{def}}{=} \{T(c) \in \mathcal{P}(D) \mid c \in S(b)\},$$

and then finally take

$$\begin{aligned} \mathcal{A} &\stackrel{\text{def}}{=} \{A_b \in \mathcal{P}(\mathcal{P}(D)) \mid b \in R(a)\} \\ &\stackrel{\text{def}}{=} \{\{T(c) \in \mathcal{P}(D) \mid c \in S(b)\} \mid b \in R(a)\}. \end{aligned}$$

Indeed, we have

$$\bigcup_{A \in \mathcal{A}} \left( \bigcup_{U \in A} U \right) = \bigcup_{A_b \in \mathcal{A}} \left( \bigcup_{c \in S(b)} T(c) \right)$$



$$= \bigcup_{b \in R(a)} \left( \bigcup_{c \in S(b)} T(c) \right)$$

and

$$\begin{aligned} \bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U &= \bigcup_{U_c \in \bigcup_{b \in R(a)} A_b} U_c \\ &= \bigcup_{T(c) \in \bigcup_{b \in R(a)} A_b} T(c) \\ &= \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c). \end{aligned}$$

This finishes the proof.

*Item 3, Unitality:* Indeed, we have

$$\begin{aligned} \Delta_B \diamond R &\stackrel{\text{def}}{=} \int^{b \in B} (\Delta_B)_b^{-1} \times R_{-2}^b \\ &= \bigvee_{b \in B} (\Delta_B)_b^{-1} \times R_{-2}^b \\ &= \bigvee_{\substack{b \in B \\ b = -1}} R_{-2}^b \\ &= R_{-2}^{-1}, \end{aligned}$$

and

$$\begin{aligned} R \diamond \Delta_A &\stackrel{\text{def}}{=} \int^{a \in A} R_a^{-1} \times (\Delta_A)_{-2}^a \\ &= \bigvee_{a \in B} R_a^{-1} \times (\Delta_A)_{-2}^a \\ &= \bigvee_{\substack{a \in B \\ a = -2}} R_a^{-1} \\ &= R_{-2}^{-1}. \end{aligned}$$

In the language of relations, given  $a \in A$  and  $b \in B$ :

- The equality

$$\Delta_B \diamond R = R$$

witnesses the equivalence of the following two statements:

- We have  $a \sim_b B$ .

- There exists some  $b' \in B$  such that:
  - \* We have  $a \sim_R b'$
  - \* We have  $b' \sim_{\Delta_B} b$ , i.e.  $b' = b$ .
- The equality

$$R \diamond \Delta_A = R$$

witnesses the equivalence of the following two statements:

- There exists some  $a' \in A$  such that:
  - \* We have  $a \sim_{\Delta_B} a'$ , i.e.  $a = a'$ .
  - \* We have  $a' \sim_R b$
- We have  $a \sim_b B$ .

*Item 4, Relation to Apartness Composition of Relations:* This is a repetition of *Item 4* of [Definition 8.1.4.1.3](#) and is proved there.

*Item 5, Linear Distributivity:* This is a repetition of *Item 5* of [Definition 8.1.4.1.3](#) and is proved there.

*Item 6, Interaction With Converses:* This is a repetition of *Item 3* of [Definition 8.1.5.1.3](#) and is proved there.

*Item 7, Interaction With Ranges and Domains:* We have

$$\begin{aligned}
 \text{dom}(S \diamond R) &\stackrel{\text{def}}{=} \{a \in A \mid a \sim_{S \diamond R} c \text{ for some } c \in C\}, \\
 &= \left\{ a \in A \mid \begin{array}{l} \text{there exists some } b \in B \text{ and } c \in C \\ \text{such that } a \sim_R b \text{ and } b \sim_R c \end{array} \right\}, \\
 &\subset \left\{ a \in A \mid \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}, \\
 &\stackrel{\text{def}}{=} \text{dom}(R)
 \end{aligned}$$

and

$$\begin{aligned}
 \text{range}(S \diamond R) &\stackrel{\text{def}}{=} \{c \in C \mid a \sim_{S \diamond R} c \text{ for some } a \in A\}, \\
 &= \left\{ c \in C \mid \begin{array}{l} \text{there exists some } a \in A \text{ and } b \in B \\ \text{such that } a \sim_R b \text{ and } b \sim_R c \end{array} \right\}, \\
 &\subset \left\{ c \in C \mid \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } b \sim_S c \end{array} \right\}, \\
 &\stackrel{\text{def}}{=} \text{range}(S).
 \end{aligned}$$

This finishes the proof. □

### 02CY 8.1.4 Apartness Composition of Relations

Let  $A$ ,  $B$ , and  $C$  be sets and let  $R: A \rightarrowtail B$  and  $S: B \rightarrowtail C$  be relations.

02DA **Definition 8.1.4.1.1.** The **apartness composition of  $R$  and  $S$**  is the relation  $S \sqcap R$  defined as follows:

- Viewing relations as subsets of  $A \times C$ , we define

$$S \sqcap R \stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \mid \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions  $A \times C \rightarrow \{\text{true}, \text{false}\}$ , we define

$$\begin{aligned} (S \sqcap R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int_{b \in B} S_b^{-1} \amalg R_{-2}^b \\ &= \bigwedge_{b \in B} S_b^{-1} \amalg R_{-2}^b, \end{aligned}$$

where the meet  $\bigwedge$  is taken in the poset  $(\{\text{true}, \text{false}\}, \preceq)$  of **Sets, Definition 3.2.2.1.3**.

- Viewing relations as functions  $A \rightarrow \mathcal{P}(C)$ , we define

$$[S \sqcap R](a) \stackrel{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b)$$

for each  $a \in A$ .

02DB **Example 8.1.4.1.2.** Here are some examples of apartness composition of relations.

02DC 1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let  $A = B = C = \mathbb{R}$ . We have

$$\begin{aligned} \leq \sqcap \geq &= \emptyset, \\ \geq \sqcap \leq &= \emptyset. \end{aligned}$$

02DD 2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let  $A = B = C = \mathbb{R}$ . We have

$$\begin{aligned} \leq \sqcap \leq &= \emptyset, \\ \geq \sqcap \geq &= \emptyset. \end{aligned}$$

02DE 3. *Equality and Inequality.* Let  $A = B = C = \mathbb{Z}$ . We have

$$\begin{aligned} &= \square \neq =, \\ &\neq \square = =. \end{aligned}$$

02DF 4. *Subset Inclusion.* Let  $X$  be a set with at least three elements and consider the relations  $\subset$  and  $\supset$  in  $\mathcal{P}(X)$ . We have

$$\supset \square \subset = \{(U, V) \in \mathcal{P}(X) \mid U = \emptyset \text{ or } V = \emptyset\}.$$

02DG **Proposition 8.1.4.1.3.** Let  $R: A \rightarrowtail B$ ,  $S: B \rightarrowtail C$ , and  $T: C \rightarrowtail D$  be relations.

02DH 1. *Functoriality.* The assignments  $R, S, (R, S) \mapsto S \square R$  define functors

$$\begin{aligned} S \square -: \quad & \mathbf{Rel}(A, B) && \rightarrow \mathbf{Rel}(A, C), \\ - \square R: \quad & \mathbf{Rel}(B, C) && \rightarrow \mathbf{Rel}(A, C), \\ -_1 \square -_2: & \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) && \rightarrow \mathbf{Rel}(A, C). \end{aligned}$$

In particular, given relations

$$A \begin{array}{c} \xrightarrow{R_1} \\ \xleftrightarrow{\quad} \\ \xrightarrow{R_2} \end{array} B \begin{array}{c} \xrightarrow{S_1} \\ \xleftrightarrow{\quad} \\ \xrightarrow{S_2} \end{array} C,$$

if  $R_1 \subset R_2$  and  $S_1 \subset S_2$ , then  $S_1 \square R_1 \subset S_2 \square R_2$ .

02DJ 2. *Associativity.* We have

$$(T \square S) \square R = T \square (S \square R).$$

02DK 3. *Unitality.* We have

$$\begin{aligned} \nabla_B \square R &= R, \\ R \square \nabla_A &= R. \end{aligned}$$

02DL 4. *Relation to Composition of Relations.* We have

$$\begin{aligned} (S \square R)^c &= S^c \diamond R^c, \\ (S \diamond R)^c &= S^c \square R^c, \end{aligned}$$

where  $(-)^c$  is the complement functor of **Constructions With Sets, Section 4.3.11**. In particular,  $\square$  is a special case of composition of relations, as we have

$$S \square R = (S^c \diamond R^c)^c.$$

This is also compatible with units, as we have  $\nabla_A^c = \Delta_A$ .

02DM 5. *Linear Distributivity*. We have inclusions of relations

$$\begin{aligned} T \diamond (S \sqcap R) &\subset (T \diamond S) \sqcap R, \\ (T \sqcap S) \diamond R &\subset T \sqcap (S \diamond R). \end{aligned}$$

02DN 6. *Interaction With Converses*. We have

$$(S \sqcap R)^\dagger = R^\dagger \sqcap S^\dagger.$$

*Proof.* *Item 1, Functoriality*: We have

$$\begin{aligned} S_1 \sqcap R_1 &\stackrel{\text{def}}{=} \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_{R_1} b \text{ or } b \sim_{S_1} c \end{array} \right. \right\} \\ &\subset \left\{ (a, c) \in A \times C \left| \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_{R_2} b \text{ or } b \sim_{S_2} c \end{array} \right. \right\} \\ &\stackrel{\text{def}}{=} S_2 \sqcap R_2. \end{aligned}$$

This finishes the proof.

*Item 2, Associativity*: Indeed, we have

$$\begin{aligned} (T \sqcap S) \sqcap R &\stackrel{\text{def}}{=} \left( \int_{c \in C} T_c^{-1} \sqcup S_{-2}^c \right) \sqcap R \\ &\stackrel{\text{def}}{=} \int_{b \in B} \left( \int_{c \in C} T_c^{-1} \sqcup S_b^c \right) \sqcup R_{-2}^b \\ &= \int_{b \in B} \int_{c \in C} (T_c^{-1} \sqcup S_b^c) \sqcup R_{-2}^b \\ &= \int_{c \in C} \int_{b \in B} (T_c^{-1} \sqcup S_b^c) \sqcup R_{-2}^b \\ &= \int_{c \in C} \int_{b \in B} T_c^{-1} \sqcup (S_b^c \sqcup R_{-2}^b) \\ &= \int_{c \in C} T_c^{-1} \sqcup \left( \int_{b \in B} S_b^c \sqcup R_{-2}^b \right) \\ &\stackrel{\text{def}}{=} \int_{c \in C} T_c^{-1} \sqcup (S \sqcap R)_{-2}^c \\ &\stackrel{\text{def}}{=} T \sqcap (S \sqcap R). \end{aligned}$$

In the language of relations, given  $a \in A$  and  $d \in D$ , the stated equality witnesses the equivalence of the following two statements:

- We have  $a \sim_{(T \sqcap S) \sqcap R} d$ , i.e. there exists some  $b \in B$  such that:

- We have  $a \sim_R b$ ;
- We have  $b \sim_{T \sqcap S} d$ , i.e. there exists some  $c \in C$  such that:
  - \* We have  $b \sim_S c$ ;
  - \* We have  $c \sim_T d$ ;
- We have  $a \sim_{T \sqcap (S \sqcap R)} d$ , i.e. there exists some  $c \in C$  such that:
  - We have  $a \sim_{S \sqcap R} c$ , i.e. there exists some  $b \in B$  such that:
    - \* We have  $a \sim_R b$ ;
    - \* We have  $b \sim_S c$ ;
  - We have  $c \sim_T d$ ;

both of which are equivalent to the statement

- There exist  $b \in B$  and  $c \in C$  such that  $a \sim_R b \sim_S c \sim_T d$ .

*Item 3, Unitality:* Indeed, we have

$$\begin{aligned}
 \nabla_B \sqcap R &\stackrel{\text{def}}{=} \int_{b \in B} (\nabla_B)_b^{-1} \amalg R_{-2}^b \\
 &= \bigwedge_{b \in B} (\nabla_B)_b^{-1} \amalg R_{-2}^b \\
 &= \left( \bigwedge_{\substack{b \in B \\ b = -1}} (\nabla_B)_b^{-1} \amalg R_{-2}^b \right) \wedge \left( \bigwedge_{\substack{b \in B \\ b \neq -1}} (\nabla_B)_b^{-1} \amalg R_{-2}^b \right) \\
 &= ((\nabla_B)_{-1}^{-1} \amalg R_{-2}^{-1}) \wedge \left( \bigwedge_{\substack{b \in B \\ b \neq -1}} \mathbf{t} \amalg R_{-2}^b \right) \\
 &= (\mathbf{f} \amalg R_{-2}^{-1}) \wedge \left( \bigwedge_{\substack{b \in B \\ b \neq -1}} \mathbf{t} \right) \\
 &= R_{-2}^{-1} \wedge \mathbf{t} \\
 &= R_{-2}^{-1},
 \end{aligned}$$

and

$$\begin{aligned}
 R \sqcap \nabla_A &\stackrel{\text{def}}{=} \int_{a \in A} R_a^{-1} \amalg (\nabla_A)_a \\
 &= \bigwedge_{a \in A} R_a^{-1} \amalg (\nabla_A)_a
 \end{aligned}$$

$$\begin{aligned}
&= \left( \bigwedge_{\substack{a \in A \\ a \neq -2}} R_a^{-1} \sqcup (\nabla_A)_{-2}^a \right) \wedge \left( \bigwedge_{\substack{a \in A \\ a \neq -2}} R_a^{-1} \sqcup (\nabla_A)_{-2}^a \right) \\
&= (R_{-2}^{-1} \sqcup (\nabla_A)_{-2}^{-2}) \wedge \left( \bigwedge_{\substack{a \in A \\ a \neq -2}} R_a^{-1} \sqcup \mathbf{t} \right) \\
&= (R_{-2}^{-1} \sqcup \mathbf{f}) \wedge \left( \bigwedge_{\substack{a \in A \\ a \neq -2}} \mathbf{t} \right) \\
&= R_{-2}^{-1} \wedge \mathbf{t} \\
&= R_{-2}^{-1},
\end{aligned}$$

This finishes the proof.

**Item 4, Relation to Composition of Relations:** We proceed in a few steps.

- We have  $a \sim_{(S \sqcap R)^c} b$  iff  $a \not\sim_{S \sqcap R} b$ .
- We have  $a \not\sim_{S \sqcap R} b$  iff the assertion “for each  $b \in B$ , we have  $a \sim_R b$  or  $b \sim_S c$ ” is false.
- That happens iff there exists some  $b \in B$  such that  $a \not\sim_R b$  and  $b \not\sim_S c$ .
- That happens iff there exists some  $b \in B$  such that  $a \sim_{R^c} b$  and  $b \sim_{S^c} c$ .

The second equality then follows from the first one by **Constructions With Sets, Item 3** of **Definition 4.3.11.1.2**.

**Item 5, Linear Distributivity:** We have

$$\begin{aligned}
T \diamond (S \sqcap R) &\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{there exists some } c \in C \text{ such} \\ \text{that } a \sim_{S \sqcap R} c \text{ and } c \sim_T d \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{there exists some } c \in C \text{ such that} \\ c \sim_T d \text{ and, for each } b \in B, \\ \text{we have } a \sim_R b \text{ or } b \sim_S c \end{array} \right. \right\} \\
&= \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{the following conditions are satisfied:} \\ 1. \text{ For each } b \in B, \text{ we have } a \sim_R b \text{ or } b \sim_S c. \\ 2. \text{ There exists } c \in C \text{ such that } c \sim_T d. \end{array} \right. \right\} \\
&\subset \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{for each } b \in B, \text{ at least one of the} \\ \text{following conditions is satisfied:} \\ 1. \text{ We have } a \sim_R b. \\ 2. \text{ There exists } c \in C \text{ such that } b \sim_S c \\ \text{and } c \sim_T d. \end{array} \right. \right\}
\end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or there exists some } c \in C \\ \text{such that } b \sim_S c \text{ and } c \sim_T d \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_{T \circ S} d \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} (T \diamond S) \square R
\end{aligned}$$

and

$$\begin{aligned}
(T \square S) \diamond R &\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_{T \square S} d \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and, for each } c \in C, \\ \text{we have } b \sim_S c \text{ or } c \sim_T d \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{there exists some } b \in B \text{ satisfying} \\ \text{the following conditions:} \\ \quad 1. \text{ We have } a \sim_R b. \\ \quad 2. \text{ For each } c \in C, \text{ we have } b \sim_S c \\ \quad \text{or } c \sim_T d. \end{array} \right. \right\} \\
&\subset \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{for each } c \in C, \text{ at least one of the} \\ \text{following conditions is satisfied:} \\ \quad 1. \text{ We have } c \sim_T d. \\ \quad 2. \text{ There exists some } b \in B \text{ such that} \\ \quad \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{for each } c \in C, \text{ we have } c \sim_T d \\ \text{or there exists some } b \in B, \text{ such that} \\ \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right. \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d, a) \in D \times A \left| \begin{array}{l} \text{for each } c \in C, \text{ we have} \\ a \sim_{S \diamond R} c \text{ or } c \sim_T d \end{array} \right. \right\} \\
&\subset T \square (S \diamond R).
\end{aligned}$$

This finishes the proof.

*Item 6, Interaction With Converses:* This is a repetition of *Item 4* of *Definition 8.1.5.1.3* and is proved there.  $\square$

### 00QE 8.1.5 The Converse of a Relation

Let  $A$ ,  $B$ , and  $C$  be sets and let  $R \subset A \times B$  be a relation.



**00QF Definition 8.1.5.1.1.** The **converse** of  $R^6$  is the relation  $R^\dagger$  defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } a \sim_R b\}.$$

- Viewing relations as functions  $A \times B \rightarrow \{\text{true}, \text{false}\}$ , we define

$$[R^\dagger]_b^a \stackrel{\text{def}}{=} R_a^b$$

for each  $(b, a) \in B \times A$ .

- Viewing relations as functions  $A \rightarrow \mathcal{P}(B)$ , we define<sup>7</sup>

$$R^\dagger(b) \stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\}$$

for each  $b \in B$ .

**00QG Example 8.1.5.1.2.** Here are some examples of converses of relations.

**00QH** 1. *Less Than Equal Signs.* We have  $(\leq)^\dagger = \geq$ .

**00QJ** 2. *Greater Than Equal Signs.* Dually to **Item 1**, we have  $(\geq)^\dagger = \leq$ .

**00QK** 3. *Functions.* Let  $f: A \rightarrow B$  be a function. We have

$$\begin{aligned} \text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f), \end{aligned}$$

where  $\text{Gr}(f)$  and  $f^{-1}$  are the relations of **Sections 8.2.2** and **8.2.3**.

**00QL Proposition 8.1.5.1.3.** Let  $R: A \rightarrowtail B$  and  $S: B \rightarrowtail C$  be relations.

**00QM** 1. *Functoriality.* The assignment  $R \mapsto R^\dagger$  defines a functor (i.e. morphism of posets)

$$(-)^\dagger: \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(B, A).$$

In other words, given relations  $R, S: A \rightarrowtail B$ , we have:

(★) If  $R \subset S$ , then  $R^\dagger \subset S^\dagger$ .

<sup>6</sup>*Further Terminology:* Also called the **opposite** of  $R$  or the **transpose** of  $R$ .

<sup>7</sup>Note that  $R^\dagger(b) = R^{-1}(\{b\})$ .

00QN 2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\text{dom}(R^\dagger) &= \text{range}(R), \\ \text{range}(R^\dagger) &= \text{dom}(R).\end{aligned}$$

00QP 3. *Interaction With Composition.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

02DP 4. *Interaction With Apartness Composition.* We have

$$(S \square R)^\dagger = R^\dagger \square S^\dagger.$$

00QR 5. *Invertibility.* We have

$$(R^\dagger)^\dagger = R.$$

00QS 6. *Identity I.* We have

$$\Delta_A^\dagger = \Delta_A.$$

02DQ 7. *Identity II.* We have

$$\nabla_A^\dagger = \nabla_A.$$

*Proof.* **Item 1, Functoriality:** We have

$$\begin{aligned}R^\dagger &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\} \\ &\subset \{a \in A \mid b \in S(a)\} \\ &\stackrel{\text{def}}{=} S^\dagger.\end{aligned}$$

This finishes the proof.

**Item 2, Interaction With Ranges and Domains:** We have

$$\begin{aligned}\text{dom}(R^\dagger) &\stackrel{\text{def}}{=} \{b \in B \mid b \sim_{R^\dagger} a \text{ for some } a \in A\} \\ &= \{b \in B \mid a \sim_R b \text{ for some } a \in A\} \\ &\stackrel{\text{def}}{=} \text{range}(R)\end{aligned}$$

and

$$\text{range}(R^\dagger) \stackrel{\text{def}}{=} \{a \in A \mid b \sim_{R^\dagger} a \text{ for some } b \in B\}$$

$$\begin{aligned}
&= \{a \in A \mid a \sim_R b \text{ for some } b \in B\} \\
&\stackrel{\text{def}}{=} \text{dom}(R).
\end{aligned}$$

This finishes the proof.

*Item 3, Interaction With Composition:* We have

$$\begin{aligned}
(S \diamond R)^\dagger &\stackrel{\text{def}}{=} \left\{ (c, a) \in C \times A \mid c \sim_{(S \diamond R)^\dagger} a \right\} \\
&= \{ (c, a) \in C \times A \mid a \sim_{S \diamond R} c \} \\
&= \left\{ (c, a) \in C \times A \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&= \left\{ (c, a) \in C \times A \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } b \sim_{R^\dagger} a \text{ and } c \sim_{S^\dagger} b \end{array} \right\} \\
&= \left\{ (c, a) \in C \times A \mid \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } c \sim_{S^\dagger} b \text{ and } b \sim_{R^\dagger} a \end{array} \right\} \\
&\stackrel{\text{def}}{=} R^\dagger \diamond S^\dagger.
\end{aligned}$$

This finishes the proof.

*Item 4, Interaction With Apartness Composition:* We have

$$\begin{aligned}
(S \square R)^\dagger &\stackrel{\text{def}}{=} \left\{ (c, a) \in C \times A \mid c \sim_{(S \square R)^\dagger} a \right\} \\
&= \{ (c, a) \in C \times A \mid a \sim_{S \square R} c \} \\
&= \left\{ (c, a) \in C \times A \mid \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_S c \end{array} \right\} \\
&= \left\{ (c, a) \in C \times A \mid \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ b \sim_{R^\dagger} a \text{ or } c \sim_{S^\dagger} b \end{array} \right\} \\
&= \left\{ (c, a) \in C \times A \mid \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ c \sim_{S^\dagger} b \text{ or } b \sim_{R^\dagger} a \end{array} \right\} \\
&\stackrel{\text{def}}{=} R^\dagger \square S^\dagger.
\end{aligned}$$

This finishes the proof.

*Item 5, Invertibility:* We have

$$\begin{aligned}
(R^\dagger)^\dagger &\stackrel{\text{def}}{=} \{ (a, b) \in A \times B \mid b \sim_{R^\dagger} a \} \\
&= \{ (a, b) \in A \times B \mid a \sim_R b \} \\
&\stackrel{\text{def}}{=} R.
\end{aligned}$$

This finishes the proof.

*Item 6, Identity I:* We have

$$\begin{aligned}\Delta_A^\dagger &\stackrel{\text{def}}{=} \{(a, b) \in A \times A \mid a \sim_{\Delta_A} b\} \\ &= \{(a, b) \in A \times A \mid a = b\} \\ &= \Delta_A.\end{aligned}$$

This finishes the proof.

*Item 7, Identity II:* We have

$$\begin{aligned}\nabla_A^\dagger &\stackrel{\text{def}}{=} \{(a, b) \in A \times A \mid a \sim_{\nabla_A} b\} \\ &= \{(a, b) \in A \times A \mid a \neq b\} \\ &= \nabla_A.\end{aligned}$$

This finishes the proof. □

## 02CZ 8.2 Examples of Relations

### 00J5 8.2.1 Elementary Examples of Relations

00J6 **Example 8.2.1.1.1.** The **trivial relation on  $A$  and  $B$**  is the relation  $\sim_{\text{triv}}$  defined equivalently as follows:

0291 1. As a subset of  $A \times B$ , we have

$$\sim_{\text{triv}} \stackrel{\text{def}}{=} A \times B.$$

0292 2. As a function from  $A \times B$  to  $\{\text{true}, \text{false}\}$ , the relation  $\sim_{\text{triv}}$  is the constant function

$$\Delta_{\text{true}} : A \times B \rightarrow \{\text{true}, \text{false}\}$$

from  $A \times B$  to  $\{\text{true}, \text{false}\}$  taking the value true.

0293 3. As a function from  $A$  to  $\mathcal{P}(B)$ , the relation  $\sim_{\text{triv}}$  is the function

$$\Delta_{\text{true}} : A \rightarrow \mathcal{P}(B)$$

defined by

$$\Delta_{\text{true}}(a) \stackrel{\text{def}}{=} B$$

for each  $a \in A$ .

0294 4. Lastly, it is the unique relation  $R$  on  $A$  and  $B$  such that we have  $a \sim_R b$  for each  $a \in A$  and each  $b \in B$ .

**00J7 Example 8.2.1.1.2.** The **cotrivial relation on  $A$  and  $B$**  is the relation  $\sim_{\text{cotriv}}$  defined equivalently as follows:

**0295** 1. As a subset of  $A \times B$ , we have

$$\sim_{\text{cotriv}} \stackrel{\text{def}}{=} \emptyset.$$

**0296** 2. As a function from  $A \times B$  to  $\{\text{true}, \text{false}\}$ , the relation  $\sim_{\text{cotriv}}$  is the constant function

$$\Delta_{\text{false}}: A \times B \rightarrow \{\text{true}, \text{false}\}$$

from  $A \times B$  to  $\{\text{true}, \text{false}\}$  taking the value false.

**0297** 3. As a function from  $A$  to  $\mathcal{P}(B)$ , the relation  $\sim_{\text{cotriv}}$  is the function

$$\Delta_{\text{false}}: A \rightarrow \mathcal{P}(B)$$

defined by

$$\Delta_{\text{false}}(a) \stackrel{\text{def}}{=} \emptyset$$

for each  $a \in A$ .

**0298** 4. Lastly, it is the unique relation  $R$  on  $A$  and  $B$  such that we have  $a \not\sim_R b$  for each  $a \in A$  and each  $b \in B$ .

**00J8 Example 8.2.1.1.3.** The characteristic relation  $\chi_X$  on  $X$  of **Constructions With Sets, Definition 4.5.3.1.1**:

**02DR** 1. As a subset of  $X \times X$ , we have

$$\begin{aligned} \sim_{\chi_X} &\stackrel{\text{def}}{=} \Delta_X \\ &\stackrel{\text{def}}{=} \{(x, x) \in X \times X\}. \end{aligned}$$

**02DS** 2. As a function from  $X \times X$  to  $\{\text{true}, \text{false}\}$ , we have

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each  $x, y \in X$ .

**02DT** 3. As a function from  $X$  to  $\mathcal{P}(X)$ , we have

$$\chi_X(x) \stackrel{\text{def}}{=} \{x\}$$

for each  $x \in X$ .

**02DU Example 8.2.1.1.4.** The **antidiagonal relation on  $X$**  is the relation  $\nabla_X$  defined equivalently as follows:

**02DV** 1. As a subset of  $X \times X$ , we have

$$\begin{aligned}\sim_{\nabla_X} &\stackrel{\text{def}}{=} \nabla_X \\ &\stackrel{\text{def}}{=} X \setminus \Delta_X \\ &= \{(x, y) \in X \times X \mid x \neq y\}.\end{aligned}$$

**02DW** 2. As a function from  $X \times X$  to  $\{\text{true}, \text{false}\}$ , we have

$$\nabla_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a \neq b, \\ \text{false} & \text{if } a = b \end{cases}$$

for each  $x, y \in X$ .

**02DX** 3. As a function from  $X$  to  $\mathcal{P}(X)$ , we have

$$\nabla_X(x) \stackrel{\text{def}}{=} X \setminus \{x\}$$

for each  $x \in X$ .

**02DY Example 8.2.1.1.5.** Partial functions may be viewed (or defined) as being exactly those relations which are functional; see **Conditions on Relations, Section 10.1.1**.

**00J9 Example 8.2.1.1.6.** Square roots are examples of relations:

**0299** 1. *Square Roots in  $\mathbb{R}$ .* The assignment  $x \mapsto \sqrt{x}$  defines a relation

$$\sqrt{\phantom{x}}: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$$

from  $\mathbb{R}$  to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \{-\sqrt{|x|}, \sqrt{|x|}\} & \text{if } x \neq 0. \end{cases}$$

**029A** 2. *Square Roots in  $\mathbb{Q}$ .* Square roots in  $\mathbb{Q}$  are similar to square roots in  $\mathbb{R}$ , though now additionally it may also occur that  $\sqrt{\phantom{x}}: \mathbb{Q} \rightarrow \mathcal{P}(\mathbb{Q})$  sends a rational number  $x$  (e.g. 2) to the empty set (since  $\sqrt{2} \notin \mathbb{Q}$ ).

**00JA Example 8.2.1.1.7.** The complex logarithm defines a relation

$$\log: \mathbb{C} \rightarrow \mathcal{P}(\mathbb{C})$$

from  $\mathbb{C}$  to itself, where we have

$$\log(a + bi) \stackrel{\text{def}}{=} \left\{ \log\left(\sqrt{a^2 + b^2}\right) + i \arg(a + bi) + (2\pi i)k \mid k \in \mathbb{Z} \right\}$$

for each  $a + bi \in \mathbb{C}$ .

**00JB Example 8.2.1.1.8.** See [Wik25] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

### 00P0 8.2.2 The Graph of a Function

Let  $f: A \rightarrow B$  be a function.

**00P1 Definition 8.2.2.1.1.** The **graph of  $f$**  is the relation  $\text{Gr}(f): A \rightarrow B$  defined as follows:<sup>8</sup>

- Viewing relations from  $A$  to  $B$  as subsets of  $A \times B$ , we define

$$\text{Gr}(f) \stackrel{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from  $A$  to  $B$  as functions  $A \times B \rightarrow \{\text{true}, \text{false}\}$ , we define

$$\text{Gr}(f)_a^b \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

for each  $(a, b) \in A \times B$ .

- Viewing relations from  $A$  to  $B$  as functions  $A \rightarrow \mathcal{P}(B)$ , we define

$$[\text{Gr}(f)](a) \stackrel{\text{def}}{=} \{f(a)\}$$

for each  $a \in A$ , i.e. we define  $\text{Gr}(f)$  as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

**00P2 Proposition 8.2.2.1.2.** Let  $f: A \rightarrow B$  be a function.

**00P3 1. Functoriality.** The assignment  $A \mapsto \text{Gr}(A)$  defines a functor

$$\text{Gr}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each  $A \in \text{Obj}(\text{Sets})$ , we have

$$\text{Gr}(A) \stackrel{\text{def}}{=} A.$$

---

<sup>8</sup>*Further Terminology and Notation:* When  $f = \text{id}_A$ , we write  $\text{Gr}(A)$  for  $\text{Gr}(\text{id}_A)$ , calling it the **graph of  $A$** .

- *Action on Morphisms.* For each  $A, B \in \text{Obj}(\text{Sets})$ , the action on Hom-sets

$$\text{Gr}_{A,B}: \text{Sets}(A, B) \rightarrow \underbrace{\text{Rel}(\text{Gr}(A), \text{Gr}(B))}_{\stackrel{\text{def}}{=} \text{Rel}(A, B)}$$

of  $\text{Gr}$  at  $(A, B)$  is defined by

$$\text{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \text{Gr}(f),$$

where  $\text{Gr}(f)$  is the graph of  $f$  as in **Definition 8.2.2.1.1**.

In particular, the following statements are true:

- *Preservation of Identities.* We have

$$\text{Gr}(\text{id}_A) = \chi_A$$

for each  $A \in \text{Obj}(\text{Sets})$ .

- *Preservation of Composition.* We have

$$\text{Gr}(g \circ f) = \text{Gr}(g) \diamond \text{Gr}(f)$$

for each pair of functions  $f: A \rightarrow B$  and  $g: B \rightarrow C$ .

**00P5** 2. *Adjointness.* We have an adjunction

$$(\text{Gr} \dashv \mathcal{P}_!): \text{Sets} \begin{array}{c} \xrightarrow{\text{Gr}} \\ \perp \\ \xleftarrow{\mathcal{P}_!} \end{array} \text{Rel},$$

witnessed by a bijection of sets

$$\text{Rel}(\text{Gr}(A), B) \cong \text{Sets}(A, \mathcal{P}(B))$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $B \in \text{Obj}(\text{Rel})$ .

**00P7** 3. *Cocontinuity.* The functor  $\text{Gr}: \text{Sets} \rightarrow \text{Rel}$  of **Item 1** preserves colimits.

**00P4** 4. *Adjointness Inside Rel.* We have an internal adjunction

$$\left( \text{Gr}(f) \dashv f^{-1} \right): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \perp \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**, where  $f^{-1}$  is the inverse of  $f$  of **Definition 8.2.3.1.1**.



00P6 5. *Interaction With Converses.* We have

$$\begin{aligned}\text{Gr}(f)^\dagger &= f^{-1}, \\ (f^{-1})^\dagger &= \text{Gr}(f).\end{aligned}$$

00P8 6. *Characterisations.* Let  $R: A \dashv B$  be a relation. The following conditions are equivalent:

- 00P9 (a) There exists a function  $f: A \rightarrow B$  such that  $R = \text{Gr}(f)$ .
- 00PA (b) The relation  $R$  is total and functional.
- 00PB (c) The weak and strong inverse images of  $R$  agree, i.e. we have  $R^{-1} = R_{-1}$ .
- 00PC (d) The relation  $R$  has a right adjoint  $R^\dagger$  in  $\mathbf{Rel}$ .

*Proof.* *Item 1, Functoriality:* Omitted.

*Item 2, Adjointness:* This is a repetition of **Constructions With Sets, Definition 4.4.4.1.1**, and is proved there.

*Item 3, Cocontinuity:* This follows from **Item 2** and ??.

*Item 4, Adjointness Inside Rel:* We need to check that there are inclusions

$$\begin{aligned}\chi_A &\subset f^{-1} \diamond \text{Gr}(f), \\ \text{Gr}(f) \diamond f^{-1} &\subset \chi_B.\end{aligned}$$

These correspond respectively to the following conditions:

- 02AR 1. For each  $a \in A$ , there exists some  $b \in B$  such that  $a \sim_{\text{Gr}(f)} b$  and  $b \sim_{f^{-1}} a$ .
- 02AS 2. For each  $a, b \in A$ , if  $a \sim_{\text{Gr}(f)} b$  and  $b \sim_{f^{-1}} a$ , then  $a = b$ .

In other words, the first condition states that the image of any  $a \in A$  by  $f$  is nonempty, whereas the second condition states that  $f$  is not multivalued. As  $f$  is a function, both of these statements are true, and we are done.

*Item 5, Interaction With Converses:* Omitted.

*Item 6, Characterisations:* We claim that **Items 6a to 6d** are indeed equivalent:

- **Item 6a**  $\iff$  **Item 6b**. This is shown in the proof of **Definition 8.5.2.1.2**.
- **Item 6b**  $\implies$  **Item 6c**. If  $R$  is total and functional, then, for each  $a \in A$ , the set  $R(a)$  is a singleton. Since the conditions
  - $R(a) \cap V \neq \emptyset$ ;
  - $R(a) \subset V$ ;

are equivalent when  $R(a)$  is a singleton, it follows that the sets

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

are equal for all  $V \in \mathcal{P}(B)$ .

- *Item 6c*  $\implies$  *Item 6b*. We claim that  $R$  is indeed total and functional:

- *Totality*. We proceed in a few steps:

- \* If we had  $R(a) = \emptyset$  for some  $a \in A$ , then we would have  $a \in R_{-1}(\emptyset)$ , so that  $R_{-1}(\emptyset) \neq \emptyset$ .
- \* But since  $R^{-1}(\emptyset) = \emptyset$ , this would imply  $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$ , a contradiction.
- \* Thus  $R(a) \neq \emptyset$  for all  $a \in A$  and  $R$  is total.

- *Functionality*. If  $R^{-1} = R_{-1}$ , then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

for each  $b \in R(a)$  and each  $a \in A$ , and thus  $R(a) \subset \{b\}$ . But since  $R$  is total, we must have  $R(a) = \{b\}$ , so  $R$  is functional.

- *Item 6a*  $\iff$  *Item 6d*. This follows from [Relations, Definition 8.5.3.1.1](#).

This finishes the proof. □

### 00PD 8.2.3 The Inverse of a Function

Let  $f: A \rightarrow B$  be a function.

**00PE Definition 8.2.3.1.1.** The **inverse of  $f$**  is the relation  $f^{-1}: B \rightarrowtail A$  defined as follows:

- Viewing relations from  $B$  to  $A$  as subsets of  $B \times A$ , we define

$$f^{-1} \stackrel{\text{def}}{=} \left\{ (b, f^{-1}(b)) \in B \times A \mid a \in A \right\},$$

where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each  $b \in B$ .

- Viewing relations from  $B$  to  $A$  as functions  $B \times A \rightarrow \{\text{true}, \text{false}\}$ , we define

$$[f^{-1}]_a^b \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \text{false} & \text{otherwise} \end{cases}$$

for each  $(b, a) \in B \times A$ .

- Viewing relations from  $B$  to  $A$  as functions  $B \rightarrow \mathcal{P}(A)$ , we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each  $b \in B$ .

**00PF Proposition 8.2.3.1.2.** Let  $f: A \rightarrow B$  be a function.

**00PG 1. Functoriality.** The assignment  $A \mapsto A, f \mapsto f^{-1}$  defines a functor

$$(-)^{-1}: \text{Sets} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each  $A \in \text{Obj}(\text{Sets})$ , we have

$$[(-)^{-1}](A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each  $A, B \in \text{Obj}(\text{Sets})$ , the action on Hom-sets

$$(-)^{-1}_{A,B}: \text{Sets}(A, B) \rightarrow \text{Rel}(A, B)$$

of  $(-)^{-1}$  at  $(A, B)$  is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} [(-)^{-1}](f),$$

where  $f^{-1}$  is the inverse of  $f$  as in **Definition 8.2.3.1.1**.

In particular, the following statements are true:

- *Preservation of Identities.* We have

$$\text{id}_A^{-1} = \chi_A$$

for each  $A \in \text{Obj}(\text{Sets})$ .

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions  $f: A \rightarrow B$  and  $g: B \rightarrow C$ .

00PH 2. *Adjointness Inside Rel.* We have an adjunction

$$\left( \text{Gr}(f) \dashv f^{-1} \right): A \begin{array}{c} \xrightarrow{\text{Gr}(f)} \\ \vdash \\ \xleftarrow{f^{-1}} \end{array} B$$

in **Rel**.

00PJ 3. *Interaction With Converses of Relations.* We have

$$\begin{aligned} \left( f^{-1} \right)^{\dagger} &= \text{Gr}(f), \\ \text{Gr}(f)^{\dagger} &= f^{-1}. \end{aligned}$$

*Proof.* *Item 1, Functoriality:* Omitted.

*Item 2, Adjointness Inside Rel:* This is a repetition of *Item 4* of *Definition 8.2.2.1.2* and is proved there.

*Item 3, Interaction With Converses of Relations:* This is a repetition of *Item 5* of *Definition 8.2.2.1.2* and is proved there.  $\square$

## 00PK 8.2.4 Representable Relations

Let  $A$  and  $B$  be sets.

00PL **Definition 8.2.4.1.1.** Let  $f: A \rightarrow B$  and  $g: B \rightarrow A$  be functions.<sup>9</sup>

02AT 1. The **representable relation associated to  $f$**  is the relation  $\chi_f: A \rightarrowtail B$  defined as the composition

$$A \times B \xrightarrow{f \times \text{id}_B} B \times B \xrightarrow{\chi_B} \{\text{true}, \text{false}\},$$

i.e. given by declaring  $a \sim_{\chi_f} b$  iff  $f(a) = b$ .

---

<sup>9</sup>More generally, given functions

$$\begin{aligned} f: A &\rightarrow C, \\ g: B &\rightarrow D \end{aligned}$$

and a relation  $B \rightarrowtail D$ , we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\text{true}, \text{false}\},$$

for which we have  $a \sim_{R \circ (f \times g)} b$  iff  $f(a) \sim_R g(b)$ .

- 02AU 2. The **corepresentable relation associated to  $g$**  is the relation  $\chi^g: B \rightarrowtail A$  defined as the composition

$$B \times A \xrightarrow{g \times \text{id}_A} A \times A \xrightarrow{\chi_A} \{\text{true}, \text{false}\},$$

i.e. given by declaring  $b \sim_{\chi^g} a$  iff  $g(b) = a$ .

## 00JQ 8.3 Categories of Relations

### 01R1 8.3.1 The Category of Relations Between Two Sets

- 01R2 **Definition 8.3.1.1.1.** The **category of relations from  $A$  to  $B$**  is the category  $\mathbf{Rel}(A, B)$  defined by<sup>10</sup>

$$\mathbf{Rel}(A, B) \stackrel{\text{def}}{=} \mathbf{Rel}(A, B)_{\text{pos}},$$

where  $\mathbf{Rel}(A, B)_{\text{pos}}$  is the posetal category associated to the poset  $\mathbf{Rel}(A, B)$  of [Item 2 of Definition 8.1.1.1.4](#) and [Categories, Definition 11.2.7.1.1](#).

### 00JR 8.3.2 The Category of Relations

- 00JS **Definition 8.3.2.1.1.** The **category of relations** is the category  $\mathbf{Rel}$  where

- *Objects.* The objects of  $\mathbf{Rel}$  are sets.
- *Morphisms.* For each  $A, B \in \text{Obj}(\mathbf{Sets})$ , we have

$$\mathbf{Rel}(A, B) \stackrel{\text{def}}{=} \mathbf{Rel}(A, B).$$

- *Identities.* For each  $A \in \text{Obj}(\mathbf{Rel})$ , the unit map

$$\mathbb{1}_A^{\mathbf{Rel}}: \text{pt} \rightarrow \mathbf{Rel}(A, A)$$

of  $\mathbf{Rel}$  at  $A$  is defined by

$$\text{id}_A^{\mathbf{Rel}} \stackrel{\text{def}}{=} \chi_A(-1, -2),$$

where  $\chi_A(-1, -2)$  is the characteristic relation of  $A$  of [Definition 8.2.1.1.3](#).

- *Composition.* For each  $A, B, C \in \text{Obj}(\mathbf{Rel})$ , the composition map

$$\circ_{A,B,C}^{\mathbf{Rel}}: \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, C)$$

<sup>10</sup>Here we choose to abuse notation by writing  $\mathbf{Rel}(A, B)$  instead of  $\mathbf{Rel}(A, B)_{\text{pos}}$  for the posetal

of  $\text{Rel}$  at  $(A, B, C)$  is defined by

$$S \circ_{A,B,C}^{\text{Rel}} R \stackrel{\text{def}}{=} S \diamond R$$

for each  $(S, R) \in \text{Rel}(B, C) \times \text{Rel}(A, B)$ , where  $S \diamond R$  is the composition of  $S$  and  $R$  of [Definition 8.1.3.1.1](#).

### 00JT 8.3.3 The Closed Symmetric Monoidal Category of Relations

#### 00JU 8.3.3.1 The Monoidal Product

00JV **Definition 8.3.3.1.1.** The **monoidal product** of  $\text{Rel}$  is the functor

$$\times: \text{Rel} \times \text{Rel} \rightarrow \text{Rel}$$

where

- *Action on Objects.* For each  $A, B \in \text{Obj}(\text{Rel})$ , we have

$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

where  $A \times B$  is the Cartesian product of sets of [Constructions With Sets, Definition 4.1.3.1.1](#).

- *Action on Morphisms.* For each  $(A, C), (B, D) \in \text{Obj}(\text{Rel} \times \text{Rel})$ , the action on morphisms

$$\times_{(A,C),(B,D)}: \text{Rel}(A, B) \times \text{Rel}(C, D) \rightarrow \text{Rel}(A \times C, B \times D)$$

of  $\times$  is given by sending a pair of morphisms  $(R, S)$  of the form

$$\begin{aligned} R &: A \rightarrowtail B, \\ S &: C \rightarrowtail D \end{aligned}$$

to the relation

$$R \times S: A \times C \rightarrowtail B \times D$$

of [Constructions With Relations, Definition 9.2.6.1.1](#).

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category of relations from  $A$  to  $B$ , even though the same notation is used for the poset of relations from  $A$  to  $B$ .

**00JW 8.3.3.2 The Monoidal Unit****00JX Definition 8.3.3.2.1.** The **monoidal unit** of  $\text{Rel}$  is the functor

$$\mathbb{1}^{\text{Rel}} : \text{pt} \rightarrow \text{Rel}$$

picking the set

$$\mathbb{1}_{\text{Rel}} \stackrel{\text{def}}{=} \text{pt}$$

of  $\text{Rel}$ .**00JY 8.3.3.3 The Associator****00JZ Definition 8.3.3.3.1.** The **associator** of  $\text{Rel}$  is the natural isomorphism

$$\alpha^{\text{Rel}} : \times \circ ((\times) \times \text{id}) \xrightarrow{\sim} \times \circ (\text{id} \times (\times)) \circ \alpha_{\text{Rel}, \text{Rel}, \text{Rel}}^{\text{Cats}},$$

as in the diagram

$$\begin{array}{ccc}
 & \text{Rel} \times (\text{Rel} \times \text{Rel}) & \\
 \alpha_{\text{Rel}, \text{Rel}, \text{Rel}}^{\text{Cats}} \swarrow & & \searrow \text{id} \times (\times) \\
 (\text{Rel} \times \text{Rel}) \times \text{Rel} & & \text{Rel} \times \text{Rel} \\
 (\times) \times \text{id} \searrow & \nearrow \alpha^{\text{Rel}} & \searrow \times \\
 \text{Rel} \times \text{Rel} & \xrightarrow[\times]{} & \text{Rel}
 \end{array}$$

whose component

$$\alpha_{A,B,C}^{\text{Rel}} : (A \times B) \times C \rightarrowtail A \times (B \times C)$$

at  $A, B, C \in \text{Obj}(\text{Rel})$  is the relation defined by declaring

$$((a, b), c) \sim_{\alpha_{A,B,C}^{\text{Rel}}} (a', (b', c'))$$

iff  $a = a'$ ,  $b = b'$ , and  $c = c'$ .

## 00K0 8.3.3.4 The Left Unitor

00K1 Definition 8.3.3.4.1. The **left unitor** of Rel is the natural isomorphism

$$\lambda^{\text{Rel}} : \times \circ (\mathbb{1}^{\text{Rel}} \times \text{id}) \xRightarrow{\sim} \lambda_{\text{Rel}}^{\text{Cats}_2},$$

whose component

$$\lambda_A^{\text{Rel}} : \mathbb{1}_{\text{Rel}} \times A \rightarrowtail A$$

at  $A$  is defined by declaring

$$(\star, a) \sim_{\lambda_A^{\text{Rel}}} b$$

iff  $a = b$ .

## 00K2 8.3.3.5 The Right Unitor

00K3 Definition 8.3.3.5.1. The **right unitor** of Rel is the natural isomorphism

$$\rho^{\text{Rel}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Rel}}) \xRightarrow{\sim} \rho_{\text{Rel}}^{\text{Cats}_2},$$

whose component

$$\rho_A^{\text{Rel}} : A \times \mathbb{1}_{\text{Rel}} \rightarrowtail A$$

at  $A$  is defined by declaring

$$(a, \star) \sim_{\rho_A^{\text{Rel}}} b$$

iff  $a = b$ .



## 00K4 8.3.3.6 The Symmetry

00K5 **Definition 8.3.3.6.1.** The **symmetry of Rel** is the natural isomorphism

$$\sigma^{\text{Rel}} : \times \Rightarrow \times \circ \sigma_{\text{Rel}, \text{Rel}}^{\text{Cats}_2}$$

whose component

$$\sigma_{A,B}^{\text{Rel}} : A \times B \rightarrow B \times A$$

at  $(A, B)$  is defined by declaring

$$(a, b) \sim_{\sigma_{A,B}^{\text{Rel}}} (b', a')$$

iff  $a = a'$  and  $b = b'$ .

## 00K6 8.3.3.7 The Internal Hom

00K7 **Definition 8.3.3.7.1.** The **internal Hom of Rel** is the functor

$$\text{Rel} : \text{Rel}^{\text{op}} \times \text{Rel} \rightarrow \text{Rel}$$

defined

- On objects by sending  $A, B \in \text{Obj}(\text{Rel})$  to the set  $\text{Rel}(A, B)$  of ?? of ??.
- On morphisms by pre/post-composition defined as in [Definition 8.1.3.1.1](#).

00K8 **Proposition 8.3.3.7.2.** Let  $A, B, C \in \text{Obj}(\text{Rel})$ .00K9 1. *Adjointness.* We have adjunctions

$$(A \times - \dashv \text{Rel}(A, -)) : \text{Rel} \rightleftarrows \text{Rel},$$

$$(- \times B \dashv \text{Rel}(B, -)) : \text{Rel} \rightleftarrows \text{Rel},$$

witnessed by bijections

$$\text{Rel}(A \times B, C) \cong \text{Rel}(A, \text{Rel}(B, C)),$$

$$\text{Rel}(A \times B, C) \cong \text{Rel}(B, \text{Rel}(A, C)),$$

natural in  $A, B, C \in \text{Obj}(\text{Rel})$ .

*Proof.* **Item 1, Adjointness:** Indeed, we have

$$\begin{aligned} \text{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \text{Sets}(A \times B \times C, \{\text{true}, \text{false}\}) \\ &\stackrel{\text{def}}{=} \text{Rel}(A, B \times C) \\ &\stackrel{\text{def}}{=} \text{Rel}(A, \text{Rel}(B, C)), \end{aligned}$$

and similarly for the bijection  $\text{Rel}(A \times B, C) \cong \text{Rel}(B, \text{Rel}(A, C))$ .  $\square$

### 00KA 8.3.3.8 The Closed Symmetric Monoidal Category of Relations

00KB **Proposition 8.3.3.8.1.** The category  $\text{Rel}$  admits a closed symmetric monoidal category structure consisting of<sup>11</sup>

- *The Underlying Category.* The category  $\text{Rel}$  of sets and relations of **Definition 8.3.2.1.1**.
- *The Monoidal Product.* The functor

$$\times: \text{Rel} \times \text{Rel} \rightarrow \text{Rel}$$

of **Definition 8.3.3.1.1**.

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Rel}: \text{Rel}^{\text{op}} \times \text{Rel} \rightarrow \text{Rel}$$

of **Definition 8.3.3.7.1**.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\text{Rel}}: \text{pt} \rightarrow \text{Rel}$$

of **Definition 8.3.3.2.1**.

- *The Associators.* The natural isomorphism

$$\alpha^{\text{Rel}}: \times \circ (\times \times \text{id}_{\text{Rel}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Rel}} \times \times) \circ \alpha_{\text{Rel}, \text{Rel}, \text{Rel}}^{\text{Cats}}$$

of **Definition 8.3.3.3.1**.

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<sup>11</sup>  **Warning:** This is not a Cartesian monoidal structure, as the product on  $\text{Rel}$  is in fact given by the disjoint union of sets; see **Constructions With Relations**, ??.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Rel}}: \times \circ (\mathbb{1}^{\mathbf{Rel}} \times \mathrm{id}_{\mathbf{Rel}}) \xRightarrow{\sim} \lambda_{\mathbf{Rel}}^{\mathbf{Cats}_2}$$

of [Definition 8.3.3.4.1](#).

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathbf{Rel}}: \times \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}}) \xRightarrow{\sim} \rho_{\mathbf{Rel}}^{\mathbf{Cats}_2}$$

of [Definition 8.3.3.5.1](#).

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathbf{Rel}}: \times \xRightarrow{\sim} \times \circ \sigma_{\mathbf{Rel}, \mathbf{Rel}}^{\mathbf{Cats}_2}$$

of [Definition 8.3.3.6.1](#).

*Proof.* Omitted. □

## 00KC 8.3.4 The 2-Category of Relations

00KD **Definition 8.3.4.1.1.** The **2-category of relations** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets.
- *Hom-Objects.* For each  $A, B \in \mathrm{Obj}(\mathbf{Sets})$ , we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Rel}}(A, B) &\stackrel{\mathrm{def}}{=} \mathbf{Rel}(A, B) \\ &\stackrel{\mathrm{def}}{=} (\mathbf{Rel}(A, B), \subset). \end{aligned}$$

- *Identities.* For each  $A \in \mathrm{Obj}(\mathbf{Rel})$ , the unit map

$$\mathbb{1}_A^{\mathbf{Rel}}: \mathrm{pt} \rightarrow \mathbf{Rel}(A, A)$$

of **Rel** at  $A$  is defined by

$$\mathrm{id}_A^{\mathbf{Rel}} \stackrel{\mathrm{def}}{=} \chi_A(-1, -2),$$

where  $\chi_A(-1, -2)$  is the characteristic relation of  $A$  of [Definition 8.2.1.1.3](#).

- *Composition.* For each  $A, B, C \in \text{Obj}(\mathbf{Rel})$ , the composition map<sup>12</sup>

$$\circ_{A,B,C}^{\mathbf{Rel}} : \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, C)$$

of  $\mathbf{Rel}$  at  $(A, B, C)$  is defined by

$$S \circ_{A,B,C}^{\mathbf{Rel}} R \stackrel{\text{def}}{=} S \diamond R$$

for each  $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$ , where  $S \diamond R$  is the composition of  $S$  and  $R$  of [Definition 8.1.3.1.1](#).

### 00KE 8.3.5 The Double Category of Relations

#### 00KF 8.3.5.1 The Double Category of Relations

00KG **Definition 8.3.5.1.1.** The **double category of relations** is the locally posetal double category  $\mathbf{Rel}^{\text{dbl}}$  where

- *Objects.* The objects of  $\mathbf{Rel}^{\text{dbl}}$  are sets.
- *Vertical Morphisms.* The vertical morphisms of  $\mathbf{Rel}^{\text{dbl}}$  are maps of sets  $f: A \rightarrow B$ .
- *Horizontal Morphisms.* The horizontal morphisms of  $\mathbf{Rel}^{\text{dbl}}$  are relations  $R: A \dashv\vdash X$ .
- *2-Morphisms.* A 2-cell

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ X & \xrightarrow{S} & Y \end{array}$$

of  $\mathbf{Rel}^{\text{dbl}}$  is either non-existent or an inclusion of relations of the form

$$\begin{array}{ccccc} A \times B & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ X \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\}. \end{array}$$

<sup>12</sup>That this is indeed a morphism of posets is proven in ?? of [Definition 8.1.3.1.4](#).

- *Horizontal Identities.* The horizontal unit functor of  $\text{Rel}^{\text{dbl}}$  is the functor of [Definition 8.3.5.2.1](#).
- *Vertical Identities.* For each  $A \in \text{Obj}(\text{Rel}^{\text{dbl}})$ , we have

$$\text{id}_A^{\text{Rel}^{\text{dbl}}} \stackrel{\text{def}}{=} \text{id}_A.$$

- *Identity 2-Morphisms.* For each horizontal morphism  $R: A \rightarrowtail B$  of  $\text{Rel}^{\text{dbl}}$ , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{R} & B \\ \text{id}_A \downarrow & \parallel & \downarrow \text{id}_B \\ A & \xrightarrow{R} & B \end{array}$$

of  $R$  is the identity inclusion

$$\begin{array}{ccc} B \times A & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ \text{id}_B \times \text{id}_A \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times A & \xrightarrow{R} & \{\text{true}, \text{false}\}. \end{array}$$

$R \subset R,$

- *Horizontal Composition.* The horizontal composition functor of  $\text{Rel}^{\text{dbl}}$  is the functor of [Definition 8.3.5.3.1](#).
- *Vertical Composition of 1-Morphisms.* For each composable pair  $A \xrightarrow{F} B \xrightarrow{G} C$  of vertical morphisms of  $\text{Rel}^{\text{dbl}}$ , i.e. maps of sets, we have

$$g \circ^{\text{Rel}^{\text{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in  $\text{Rel}^{\text{dbl}}$  is defined as in [Definition 8.3.5.4.1](#).
- *Associators.* The associators of  $\text{Rel}^{\text{dbl}}$  are defined as in [Definition 8.3.5.5.1](#).
- *Left Unitors.* The left unitors of  $\text{Rel}^{\text{dbl}}$  are defined as in [Definition 8.3.5.6.1](#).
- *Right Unitors.* The right unitors of  $\text{Rel}^{\text{dbl}}$  are defined as in [Definition 8.3.5.7.1](#).

### 00KH 8.3.5.2 Horizontal Identities

00KJ **Definition 8.3.5.2.1.** The **horizontal unit functor** of  $\text{Rel}^{\text{dbl}}$  is the functor

$$\mathbb{1}^{\text{Rel}^{\text{dbl}}} : \text{Rel}_0^{\text{dbl}} \rightarrow \text{Rel}_1^{\text{dbl}}$$

of  $\text{Rel}^{\text{dbl}}$  is the functor where

- *Action on Objects.* For each  $A \in \text{Obj}(\text{Rel}_0^{\text{dbl}})$ , we have

$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-1, -2).$$

- *Action on Morphisms.* For each vertical morphism  $f : A \rightarrow B$  of  $\text{Rel}^{\text{dbl}}$ , i.e. each map of sets  $f$  from  $A$  to  $B$ , the identity 2-morphism

$$\begin{array}{ccc} A & \xrightarrow{\mathbb{1}_A} & A \\ f \downarrow & \Downarrow \mathbb{1}_f & \downarrow f \\ B & \xrightarrow{\mathbb{1}_B} & B \end{array}$$

of  $f$  is the inclusion

$$\begin{array}{ccccc} A \times A & \xrightarrow{\chi_A(-1, -2)} & \{ \text{true}, \text{false} \} \\ f \times f \downarrow & \subset & \downarrow \text{id}_{\{ \text{true}, \text{false} \}} \\ B \times B & \xrightarrow{\chi_B(-1, -2)} & \{ \text{true}, \text{false} \} \end{array}$$

of **Constructions With Sets, Item 1** of **Definition 4.5.3.1.3**.

### 00KK 8.3.5.3 Horizontal Composition

00KL **Definition 8.3.5.3.1.** The **horizontal composition functor** of  $\text{Rel}^{\text{dbl}}$  is the functor

$$\odot^{\text{Rel}^{\text{dbl}}} : \text{Rel}_1^{\text{dbl}} \times_{\text{Rel}_0^{\text{dbl}}} \text{Rel}_1^{\text{dbl}} \rightarrow \text{Rel}_1^{\text{dbl}}$$

of  $\text{Rel}^{\text{dbl}}$  is the functor where

- *Action on Objects.* For each composable pair  $A \xrightarrow{R} B \xrightarrow{S} C$  of horizontal morphisms of  $\text{Rel}^{\text{dbl}}$ , we have

$$S \odot R \stackrel{\text{def}}{=} S \diamond R,$$

where  $S \diamond R$  is the composition of  $R$  and  $S$  of **Definition 8.1.3.1.1**.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{array}{ccc}
 A & \xrightarrow{R} & B \\
 f \downarrow & \Downarrow \alpha & \downarrow g \\
 X & \xrightarrow{T} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 B & \xrightarrow{S} & C \\
 g \downarrow & \Downarrow \beta & \downarrow h \\
 Y & \xrightarrow{U} & Z
 \end{array}$$

of 2-morphisms of  $\text{Rel}^{\text{dbl}}$ , i.e. for each pair

$$\begin{array}{ccc}
 A \times B & \xrightarrow{R} & \{\text{true}, \text{false}\} \\
 f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\
 X \times Y & \xrightarrow{T} & \{\text{true}, \text{false}\}
 \end{array}
 \quad
 \begin{array}{ccc}
 B \times C & \xrightarrow{S} & \{\text{true}, \text{false}\} \\
 g \times h \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\
 Y \times Z & \xrightarrow{U} & \{\text{true}, \text{false}\}
 \end{array}$$

of inclusions of relations, the horizontal composition

$$\begin{array}{ccc}
 A & \xrightarrow{S \diamond R} & C \\
 f \downarrow & \Downarrow \beta \diamond \alpha & \downarrow h \\
 X & \xrightarrow{U \diamond T} & Z
 \end{array}$$

of  $\alpha$  and  $\beta$  is the inclusion of relations

$$\begin{array}{ccc}
 A \times C & \xrightarrow{S \diamond R} & \{\text{true}, \text{false}\} \\
 f \times h \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\
 X \times Z & \xrightarrow{U \diamond T} & \{\text{true}, \text{false}\}.
 \end{array}$$

*Proof.* The inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R)$$

follows from the fact that the statement

- We have  $a \sim_{(U \diamond T) \circ (f \times h)} c$ , i.e.  $f(a) \sim_{U \diamond T} h(c)$ , i.e. there exists some  $y \in Y$  such that:
  - We have  $f(a) \sim_T y$ .

- We have  $y \sim_U h(c)$ .

is implied by the statement

- We have  $a \sim_{S \circ R} c$ , i.e. there exists some  $b \in B$  such that:
  - We have  $a \sim_R b$ .
  - We have  $b \sim_S c$ .

since:

- If  $a \sim_R b$ , then  $f(a) \sim_T g(b)$ , as  $T \circ (f \times g) \subset R$ ;
- If  $b \sim_S c$ , then  $g(b) \sim_U h(c)$ , as  $U \circ (g \times h) \subset S$ .

This finishes the proof.  $\square$

#### 00KM 8.3.5.4 Vertical Composition of 2-Morphisms

00KN **Definition 8.3.5.4.1.** The **vertical composition** in  $\text{Rel}^{\text{dbl}}$  is defined as follows: for each vertically composable pair

$$\begin{array}{ccc} A & \xrightarrow{R} & X \\ f \downarrow & \Downarrow \alpha & \downarrow g \\ B & \xrightarrow{S} & Y \end{array} \quad \begin{array}{ccc} B & \xrightarrow{S} & Y \\ h \downarrow & \Downarrow \beta & \downarrow k \\ C & \xrightarrow{T} & Z \end{array}$$

of 2-morphisms of  $\text{Rel}^{\text{dbl}}$ , i.e. for each pair

$$\begin{array}{ccc} A \times X & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\} \end{array} \quad \begin{array}{ccc} B \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\} \\ h \times k \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ C \times Z & \xrightarrow{T} & \{\text{true}, \text{false}\} \end{array}$$

of inclusions of relations, we define the vertical composition

$$\begin{array}{ccc} A & \xrightarrow{R} & X \\ h \circ f \downarrow & \Downarrow \beta \circ \alpha & \downarrow k \circ g \\ C & \xrightarrow{T} & Z \end{array}$$



of  $\alpha$  and  $\beta$  as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \quad \begin{array}{ccc} A \times X & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ (h \circ f) \times (k \circ g) \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ C \times Z & \xrightarrow{T} & \{\text{true}, \text{false}\} \end{array}$$

given by the pasting of inclusions

$$\begin{array}{ccc} A \times X & \xrightarrow{R} & \{\text{true}, \text{false}\} \\ f \times g \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ B \times Y & \xrightarrow{S} & \{\text{true}, \text{false}\} \\ h \times k \downarrow & \subset & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ C \times Z & \xrightarrow{T} & \{\text{true}, \text{false}\}. \end{array}$$

*Proof.* The inclusion

$$T \circ [(h \circ f) \times (k \circ g)] \subset R$$

follows from the fact that, given  $(a, x) \in A \times X$ , the statement

- We have  $h(f(a)) \sim_T k(g(x))$ ;

is implied by the statement

- We have  $a \sim_R x$ ;

since

- If  $a \sim_R x$ , then  $f(a) \sim_S g(x)$ , as  $S \circ (f \times g) \subset R$ ;
- If  $b \sim_S y$ , then  $h(b) \sim_T k(y)$ , as  $T \circ (h \times k) \subset S$ , and thus, in particular:
  - If  $f(a) \sim_S g(x)$ , then  $h(f(a)) \sim_T k(g(x))$ .

This finishes the proof. □

**00KP 8.3.5.5 The Associators****00KQ Definition 8.3.5.5.1.** For each composable triple

$$A \xrightarrow{R} B \xrightarrow{S} C \xrightarrow{T} D$$

of horizontal morphisms of  $\text{Rel}^{\text{dbl}}$ , the component

$$\alpha_{T,S,R}^{\text{Rel}^{\text{dbl}}} : (T \odot S) \odot R \xrightarrow{\sim} T \odot (S \odot R),$$

of the associator of  $\text{Rel}^{\text{dbl}}$  at  $(R, S, T)$  is the identity inclusion<sup>13</sup>

$$(T \odot S) \odot R = T \odot (S \odot R)$$

**00KR 8.3.5.6 The Left Unitors****00KS Definition 8.3.5.6.1.** For each horizontal morphism  $R: A \rightarrow B$  of  $\text{Rel}^{\text{dbl}}$ , the component

$$\lambda_R^{\text{Rel}^{\text{dbl}}} : \mathbb{1}_B \odot R \xrightarrow{\sim} R,$$

of the left unitor of  $\text{Rel}^{\text{dbl}}$  at  $R$  is the identity inclusion<sup>14</sup>

$$R = \chi_B \odot R,$$

<sup>13</sup>As proved in Item 2 of Definition 8.1.3.1.4.<sup>14</sup>As proved in Item 3 of Definition 8.1.3.1.4.

### 00KT 8.3.5.7 The Right Unitors

00KU **Definition 8.3.5.7.1.** For each horizontal morphism  $R: A \rightarrowtail B$  of  $\mathbf{Rel}^{\text{dbl}}$ , the component

$$\rho_R^{\mathbf{Rel}^{\text{dbl}}}: R \odot \mathbb{1}_A \xRightarrow{\sim} R,$$

$$\begin{array}{ccccc} A & \xrightarrow{\mathbb{1}_A} & A & \xrightarrow{R} & B \\ \text{id}_A \downarrow & & \rho_R^{\mathbf{Rel}^{\text{dbl}}} \Downarrow & & \downarrow \text{id}_B \\ A & \xrightarrow{\quad} & A & \xrightarrow{R} & B \end{array}$$

of the right unitor of  $\mathbf{Rel}^{\text{dbl}}$  at  $R$  is the identity inclusion<sup>15</sup>

$$R = R \diamond \chi_A,$$

$$\begin{array}{ccc} A \times B & \xrightarrow{R \diamond \chi_A} & \{\text{true}, \text{false}\} \\ \parallel & \cong & \downarrow \text{id}_{\{\text{true}, \text{false}\}} \\ A \times B & \xrightarrow{R} & \{\text{true}, \text{false}\}. \end{array}$$

## 02DZ 8.4 Categories of Relations With Apartness Composition

### 02E0 8.4.1 The Category of Relations With Apartness Composition

02E1 **Definition 8.4.1.1.1.** The **category of relations with apartness composition** is the category  $\mathbf{Rel}^\square$  where

- *Objects.* The objects of  $\mathbf{Rel}^\square$  are sets.
- *Morphisms.* For each  $A, B \in \text{Obj}(\mathbf{Sets})$ , we have

$$\mathbf{Rel}^\square(A, B) \stackrel{\text{def}}{=} \mathbf{Rel}(A, B).$$

- *Identities.* For each  $A \in \text{Obj}(\mathbf{Rel}^\square)$ , the unit map

$$\mathbb{1}_A^{\mathbf{Rel}^\square}: \text{pt} \rightarrow \mathbf{Rel}(A, A)$$

of  $\mathbf{Rel}^\square$  at  $A$  is defined by

$$\text{id}_A^{\mathbf{Rel}^\square} \stackrel{\text{def}}{=} \nabla_A(-_1, -_2),$$

where  $\nabla_A(-_1, -_2)$  is the antidiagonal relation of  $A$  of [Definition 8.2.1.1.4](#).

<sup>15</sup>As proved in [Item 3](#) of [Definition 8.1.3.1.4](#).

- *Composition.* For each  $A, B, C \in \text{Obj}(\text{Rel}^\square)$ , the composition map

$$\circ_{A,B,C}^{\text{Rel}^\square} : \text{Rel}(B, C) \times \text{Rel}(A, B) \rightarrow \text{Rel}(A, C)$$

of  $\text{Rel}^\square$  at  $(A, B, C)$  is defined by

$$S \circ_{A,B,C}^{\text{Rel}^\square} R \stackrel{\text{def}}{=} S \sqcap R$$

for each  $(S, R) \in \text{Rel}(B, C) \times \text{Rel}(A, B)$ , where  $S \diamond R$  is the composition of  $S$  and  $R$  of [Definition 8.1.4.1.1](#).

**02E2 Proposition 8.4.1.1.2.** The functor

$$(-)^c : \text{Rel} \rightarrow \text{Rel}^\square$$

given by the identity on objects and by  $R \mapsto R^c$  on morphisms is an isomorphism of categories.

*Proof.* By [Item 4](#) of [Definition 8.1.4.1.3](#), we see that  $(-)^c$  is indeed a functor.

By [Categories, Item 1](#) of [Definition 11.6.8.1.3](#), it suffices to show that  $(-)^c$  is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^c : \text{Rel}(A, B) \rightarrow \text{Rel}(A, B)$$

defined by the assignment  $R \mapsto R^c$  is a bijection by [Constructions With Sets, Item 3](#) of [Definition 4.3.11.1.2](#). Thus  $(-)^c$  is an isomorphism of categories.  $\square$

## **02E3 8.4.2 The 2-Category of Relations With Apartness Composition**

**02E4 Definition 8.4.2.1.1.** The 2-category of relations with apartness composition is the locally posetal 2-category  $\mathbf{Rel}$  where

- *Objects.* The objects of  $\mathbf{Rel}$  are sets.
- *Hom-Objects.* For each  $A, B \in \text{Obj}(\text{Sets})$ , we have

$$\begin{aligned} \text{Hom}_{\mathbf{Rel}}(A, B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B) \\ &\stackrel{\text{def}}{=} (\text{Rel}(A, B), \subset). \end{aligned}$$

- *Identities.* For each  $A \in \text{Obj}(\mathbf{Rel})$ , the unit map

$$\mathbb{1}_A^{\mathbf{Rel}} : \text{pt} \rightarrow \mathbf{Rel}(A, A)$$

of  $\mathbf{Rel}$  at  $A$  is defined by

$$\text{id}_A^{\mathbf{Rel}} \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

where  $\chi_A(-_1, -_2)$  is the characteristic relation of  $A$  of [Definition 8.2.1.1.3](#).

- *Composition.* For each  $A, B, C \in \text{Obj}(\mathbf{Rel})$ , the composition map<sup>16</sup>

$$\circ_{A,B,C}^{\mathbf{Rel}}: \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, C)$$

of  $\mathbf{Rel}$  at  $(A, B, C)$  is defined by

$$S \circ_{A,B,C}^{\mathbf{Rel}} R \stackrel{\text{def}}{=} S \diamond R$$

for each  $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$ , where  $S \diamond R$  is the composition of  $S$  and  $R$  of [Definition 8.1.3.1.1](#).

**02E5 Proposition 8.4.2.1.2.** The functor

$$(-)^c: \mathbf{Rel} \rightarrow \mathbf{Rel}^{\square, \text{co}}$$

given by the identity on objects and by  $R \mapsto R^c$  on 1-morphisms is a 2-isomorphism of 2-categories.

*Proof.* By [Item 4 of Definition 8.1.4.1.3](#), we see that  $(-)^c$  is indeed a functor. By [Constructions With Sets, Item 1 of Definition 4.3.11.1.2](#), it is also a 2-functor.

By ??, it suffices to show that  $(-)^c$  is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in [Definition 8.4.1.1.1](#).
- Bijective on 2-morphisms, which follows from [Constructions With Sets, Item 1 of Definition 4.3.11.1.2](#).

Thus  $(-)^c$  is indeed a 2-isomorphism of categories.  $\square$

### 02E6 8.4.3 The Linear Bicategory of Relations

**02E7 Definition 8.4.3.1.1.** The **linear bicategory of relations** is the linear bicategory consisting of:

- *The Underlying Bicategory I.* The bicategory  $\mathbf{Rel}$  of [Definition 8.3.4.1.1](#).
- *The Underlying Bicategory II.* The bicategory  $\mathbf{Rel}$  of [Definition 8.4.2.1.1](#).
- *Linear Distributors.* The inclusions

$$\begin{aligned} \delta_{R,S,T}^{\ell}: T \diamond (S \square R) &\hookrightarrow (T \diamond S) \square R, \\ \delta_{R,S,T}^r: (T \square S) \diamond R &\hookrightarrow T \square (S \diamond R) \end{aligned}$$

of [Item 5 of Definition 8.1.4.1.3](#).

*Proof.* Since  $\mathbf{Rel}$  and  $\mathbf{Rel}^{\square}$  are locally posetal, the commutativity of the coherence conditions for linear bicategories follows automatically ([Categories, Item 4 of Definition 11.2.7.1.2](#)).  $\square$

<sup>16</sup>That this is indeed a morphism of posets is proven in ?? of [Definition 8.1.4.1.3](#).

## 02E8 8.4.4 Other Categorical Structures With Apartness Composition

02E9 **Remark 8.4.4.1.1.** It seems apartness composition fails to form the following categorical structures:

- 02EA • *Monoidal Category With Products.* Products don't seem to endow  $\text{Rel}^\square$  with a monoidal structure.
- 02EB • *Monoidal Category With Coproducts.* Coproducts also don't seem to endow  $\text{Rel}^\square$  with a monoidal structure.
- 02EC • *Double Categorical Structure.* It seems the apartness composition of relations doesn't form a double category in a natural<sup>17</sup> way.

## 00KV 8.5 Properties of the 2-Category of Relations

### 00KW 8.5.1 Self-Duality

00KX **Proposition 8.5.1.1.1.** The 2-/category of relations is self-dual:

- 00KY 1. *Self-Duality I.* We have an isomorphism

$$\text{Rel}^{\text{op}} \cong \text{Rel}$$

of categories.

- 00KZ 2. *Self-Duality II.* We have a 2-isomorphism

$$\mathbf{Rel}^{\text{op}} \cong \mathbf{Rel}$$

of 2-categories.

*Proof.* **Item 1, Self-Duality I:** We claim that the functor

$$(-)^\dagger: \text{Rel}^{\text{op}} \rightarrow \text{Rel}$$

given by the identity on objects and by  $R \mapsto R^\dagger$  on morphisms is an isomorphism of categories. Note that this is indeed a functor by **Items 3** and **6** of **Definition 8.1.5.1.3**.

By **Categories, Item 1** of **Definition 11.6.8.1.3**, it suffices to show that  $(-)^\dagger$  is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^\dagger: \text{Rel}(A, B) \rightarrow \text{Rel}(B, A)$$

<sup>17</sup>I.e. such that the composition of vertical morphisms is the usual composition of functions, as

defined by the assignment  $R \mapsto R^\dagger$  is a bijection by [Item 5 of Definition 8.1.5.1.3](#), showing  $(-)^{\dagger}$  to be fully faithful.

*Item 2, Self-Duality II:* We claim that the 2-functor

$$(-)^{\dagger}: \text{Rel}^{\text{op}} \rightarrow \text{Rel}$$

given by the identity on objects, by  $R \mapsto R^\dagger$  on morphisms, and by preserving inclusions on 2-morphisms via [Item 1 of Definition 8.1.5.1.3](#), is an isomorphism of categories.

By ??, it suffices to show that  $(-)^{\dagger}$  is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in [Item 1](#).
- Bijective on 2-morphisms, which follows from [Item 1 of Definition 8.1.5.1.3](#).

Thus  $(-)^{\dagger}$  is indeed a 2-isomorphism of categories.  $\square$

## 00L0 8.5.2 Isomorphisms and Equivalences

Let  $R: A \rightarrowtail B$  be a relation from  $A$  to  $B$ .

**02ED Lemma 8.5.2.1.1.** The conditions below are row-wise equivalent:

CONDITION	INCLUSION
$R$ is functional	$R \diamond R^\dagger \subset \Delta_B$
$R$ is total	$\Delta_A \subset R^\dagger \diamond R$
$R$ is injective	$R^\dagger \diamond R \subset \Delta_A$
$R$ is surjective	$\Delta_B \subset R \diamond R^\dagger$

*Proof. Functionality Is Equivalent to  $R \diamond R^\dagger \subset \Delta_B$ :* The condition  $R \diamond R^\dagger \subset \Delta_B$  unwinds to

- (★) For each  $b, b' \in B$ , if there exists some  $a \in A$  such that  $b \sim_{R^\dagger} a$  and  $a \sim_R b'$ , then  $b = b'$ .

Since  $b \sim_{R^\dagger} a$  is the same as  $a \sim_R b$ , the condition says that  $a \sim_R b$  and  $a \sim_R b'$  imply  $b = b'$ . This is precisely the condition for  $R$  to be functional.

*Totality Is Equivalent to  $\Delta_A \subset R^\dagger \diamond R$ :* The condition  $\Delta_A \subset R^\dagger \diamond R$  unwinds to

- (★) For each  $a, a' \in A$ , if  $a = a'$ , then there exists some  $b \in B$  such that  $a \sim_R b$  and  $b \sim_{R^\dagger} a'$ .
-

Since  $b \sim_{R^\dagger} a'$  is the same as  $a' \sim_R b$ , the condition says that for each  $a \in A$ , there is some  $b \in B$  with  $b \in R(a)$ , so  $R(a) \neq \emptyset$ . This is precisely the condition for  $R$  to be total.

*Injectivity Is Equivalent to  $R^\dagger \diamond R \subset \Delta_A$ :* The condition  $R^\dagger \diamond R \subset \Delta_A$  unwinds to

- (★) For each  $a, a' \in A$ , if there exists some  $b \in B$  such that  $a \sim_R b$  and  $b \sim_{R^\dagger} a'$ , then  $a = a'$ .

Since  $b \sim_{R^\dagger} a'$  is the same as  $a' \sim_R b$ , the condition says that for each  $b \in B$ , if  $a \sim_R b$  and  $a' \sim_R b$ , then  $a = a'$ . This is precisely the condition for  $R$  to be injective.

*Surjectivity Is Equivalent to  $\Delta_B \subset R \diamond R^\dagger$ :* The condition  $\Delta_B \subset R \diamond R^\dagger$  unwinds to

- (★) For each  $b, b' \in B$ , if  $b = b'$ , then there exists some  $a \in A$  such that  $b \sim_{R^\dagger} a$  and  $a \sim_R b'$ .

Since  $b \sim_{R^\dagger} a$  is the same as  $a \sim_R b$ , the condition says that for each  $b \in B$ , there is some  $a \in A$  with  $b \in R(a)$ , so  $R^{-1}(b) \neq \emptyset$ . This is precisely the condition for  $R$  to be surjective.  $\square$

**00L1 Proposition 8.5.2.1.2.** The following conditions are equivalent:

**00L2** 1. The relation  $R: A \rightarrowtail B$  is an equivalence in **Rel**, i.e.:

- (★) There exists a relation  $R^{-1}: B \rightarrowtail A$  from  $B$  to  $A$  together with isomorphisms

$$\begin{aligned} R^{-1} \diamond R &\cong \Delta_A, \\ R \diamond R^{-1} &\cong \Delta_B. \end{aligned}$$

**00L3** 2. The relation  $R: A \rightarrowtail B$  is an isomorphism in **Rel**, i.e.:

- (★) There exists a relation  $R^{-1}: B \rightarrowtail A$  from  $B$  to  $A$  such that we have

$$\begin{aligned} R^{-1} \diamond R &= \Delta_A, \\ R \diamond R^{-1} &= \Delta_B. \end{aligned}$$

**00L4** 3. There exists a bijection  $f: A \xrightarrow{\sim} B$  with  $R = \text{Gr}(f)$ .

*Proof.* We claim that **Items 1 to 3** are indeed equivalent:

- **Item 1**  $\iff$  **Item 2**: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.



- *Item 2*  $\implies$  *Item 3*: We proceed in a few steps:
  - First, note that the equalities in *Item 2* imply  $R \dashv R^{-1}$  and thus, by *Definition 8.5.3.1.1*, there exists a function  $f_R: A \rightarrow B$  associated to  $R$ .
  - By *Definition 8.5.2.1.1*,  $f_R$  is a bijection.
- *Item 3*  $\implies$  *Item 2*: By *Item 4* of *Definition 8.2.2.1.2*, we have an adjunction  $\text{Gr}(f) \dashv f^{-1}$ , giving inclusions

$$\begin{aligned}\Delta_A &\subset f^{-1} \diamond \text{Gr}(f), \\ \text{Gr}(f) \diamond f^{-1} &\subset \Delta_B.\end{aligned}$$

If  $f$  is bijective, then the reverse inclusions are also true by *Definition 8.5.2.1.1*.

This finishes the proof.  $\square$

### 00L5 8.5.3 Internal Adjunctions

Let  $A$  and  $B$  be sets.

00L6 **Proposition 8.5.3.1.1.** We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Functions} \\ \text{from } A \text{ to } B \end{array} \right\},$$

with every adjunction in  $\mathbf{Rel}$  being of the form  $\text{Gr}(f) \dashv f^{-1}$  for some function  $f$ .

*Proof.* We proceed step by step:

- 029B 1. *From Adjunctions in Rel to Functions.* An adjunction in  $\mathbf{Rel}$  from  $A$  to  $B$  consists of a pair of relations

$$\begin{aligned}R &: A \rightarrowtail B, \\ S &: B \rightarrowtail A,\end{aligned}$$

together with inclusions

$$\begin{aligned}\Delta_A &\subset S \diamond R, \\ R \diamond S &\subset \Delta_B.\end{aligned}$$

---

in Sets.

By **Definition 8.5.2.1.1**,  $R$  is total and functional. In particular,  $R(a)$  is a singleton for all  $a \in A$ . Defining  $f_R$  such that  $f_R(a)$  is the unique element of  $R(a)$  then gives us our desired function, forming a map

$$\left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Functions} \\ \text{from } A \text{ to } B \end{array} \right\}.$$

Moreover, by uniqueness of adjoints (??), this implies also that  $S = f^{-1}$ .

- 029K** 2. *From Functions to Adjunctions in Rel.* By **Item 4** of **Definition 8.2.2.1.2**, every function  $f: A \rightarrow B$  gives rise to an adjunction  $\text{Gr}(f) \dashv f^{-1}$  in **Rel**, giving a map

$$\left\{ \begin{array}{c} \text{Functions} \\ \text{from } A \text{ to } B \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B \end{array} \right\}.$$

- 029L** 3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function  $f: A \rightarrow B$ , passing to  $\text{Gr}(f) \dashv f^{-1}$ , and then passing again to a function gives  $f$  again. This follows from the fact that we have  $a \sim_{\text{Gr}(f)} b$  iff  $f(a) = b$ .
- 029M** 4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction  $R \dashv S$  in **Rel** giving rise to a function  $f_{R,S}: A \rightarrow B$ , we have

$$\begin{aligned} \text{Gr}(f_{R,S}) &= R, \\ f_{R,S}^{-1} &= S. \end{aligned}$$

We check these explicitly:

- $\text{Gr}(f_{R,S}) = R$ . We have

$$\begin{aligned} \text{Gr}(f_{R,S}) &\stackrel{\text{def}}{=} \{(a, f_{R,S}(a)) \in A \times B \mid a \in A\} \\ &\stackrel{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

- $f_{R,S}^{-1} = S$ . We first claim that, given  $a \in A$  and  $b \in B$ , the following conditions are equivalent:
  - We have  $a \sim_R b$ .
  - We have  $b \sim_S a$ .

Indeed:

- If  $a \sim_R b$ , then  $b \sim_S a$ : We proceed in a few steps.
  - \* Since  $\Delta_A \subset S \diamond R$ , there exists  $k \in B$  such that  $a \sim_R k$  and  $k \sim_S a$ .
  - \* Since  $a \sim_R b$  and  $R$  is functional, we have  $k = b$ .
  - \* Thus  $b \sim_S a$ .
- If  $b \sim_S a$ , then  $a \sim_R b$ : We proceed in a few steps.
  - \* First note that, since  $R$  is total, we have  $a \sim_R b'$  for some  $b' \in B$ .
  - \* Since  $R \diamond S \subset \Delta_B$ ,  $b \sim_S a$ , and  $a \sim_R b'$ , we have  $b = b'$ .
  - \* Thus  $a \sim_R b$ .

Having show this, we now have

$$\begin{aligned}
 f_{R,S}^{-1}(b) &\stackrel{\text{def}}{=} \{a \in A \mid f_{R,S}(a) = b\} \\
 &\stackrel{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\
 &= \{a \in A \mid b \sim_S a\} \\
 &\stackrel{\text{def}}{=} S(b).
 \end{aligned}$$

for each  $b \in B$ , and thus  $f_{R,S}^{-1} = S$ .

This finishes the proof. □

### 00L7 8.5.4 Internal Monads

Let  $X$  be a set.

00L8 **Proposition 8.5.4.1.1.** We have a natural identification<sup>18</sup>

$$\left\{ \begin{array}{l} \text{Monads in} \\ \mathbf{Rel} \text{ on } X \end{array} \right\} \cong \{\text{Preorders on } X\}.$$

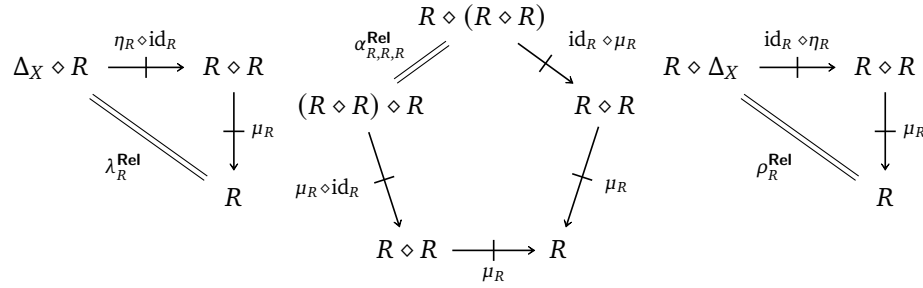
*Proof.* A monad in **Rel** on  $X$  consists of a relation  $R: X \rightarrowtail X$  together with maps

$$\begin{aligned}
 \mu_R: R \diamond R &\subset R, \\
 \eta_R: \Delta_X &\subset R
 \end{aligned}$$

---

<sup>18</sup>See also ?? for an extension of this correspondence to “relative monads in **Rel**”.

making the diagrams



commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (**Categories**, **Item 4 of Definition 11.2.7.1.2**), and hence all that is left is the data of the two maps  $\mu_R$  and  $\eta_R$ , which correspond respectively to the following conditions:

- 029N 1. For each  $x, z \in X$ , if there exists some  $y \in Y$  such that  $x \sim_R y$  and  $y \sim_R z$ , then  $x \sim_R z$ .
- 029P 2. For each  $x \in X$ , we have  $x \sim_R x$ .

These are exactly the requirements for  $R$  to be a preorder (??). Conversely, any preorder  $\preceq$  gives rise to a pair of maps  $\mu_{\preceq}$  and  $\eta_{\preceq}$ , forming a monad on  $X$ .  $\square$

02EE **Example 8.5.4.1.2.** Let  $R: A \rightarrowtail B$  be a relation.

- 02EF 1. The codensity monad  $\text{Ran}_R(R): B \rightarrowtail B$  is given by

$$[\text{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b)$$

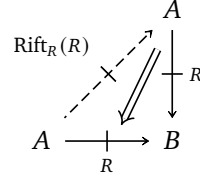
for each  $b \in B$ . Thus, it corresponds to the preorder

$$\preceq_{\text{Ran}_R(R)}: B \times B \rightarrow \{t, f\}$$

on  $B$  obtained by declaring  $b \preceq_{\text{Ran}_R(R)} b'$  iff the following equivalent conditions are satisfied:

- 02EG (a) For each  $a \in A$ , if  $a \sim_R b$ , then  $a \sim_R b'$ .
- 02EH (b) We have  $R^{-1}(b) \subset R^{-1}(b')$ .

02EJ 2. The dual codensity monad  $\text{Rift}_R(R) : A \rightarrow A$  is given by

$$[\text{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\}$$


for each  $a \in A$ . Thus, it corresponds to the preorder

$$\preceq_{\text{Rift}_R(R)} : A \times A \rightarrow \{t, f\}$$

on  $A$  obtained by declaring  $a \preceq_{\text{Rift}_R(R)} a'$  iff the following equivalent conditions are satisfied:

02EK (a) For each  $a \in A$ , if  $a \sim_R b$ , then  $a' \sim_R b$ .

02EL (b) We have  $R(a') \subset R(a)$ .

### 00L9 8.5.5 Internal Comonads

Let  $X$  be a set.

00LA **Proposition 8.5.5.1.1.** We have a natural identification

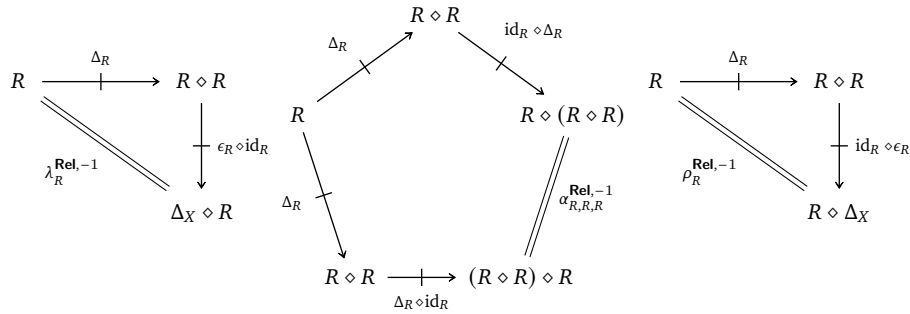
$$\left\{ \begin{array}{l} \text{Comonads in} \\ \mathbf{Rel} \text{ on } X \end{array} \right\} \cong \{\text{Subsets of } X\}.$$

*Proof.* A comonad in  $\mathbf{Rel}$  on  $X$  consists of a relation  $R : X \rightarrow X$  together with maps

$$\Delta_R : R \subset R \diamond R,$$

$$\epsilon_R : R \subset \Delta_X$$

making the diagrams



commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**), and hence all that is left is the data of the two maps  $\Delta_R$  and  $\epsilon_R$ , which correspond respectively to the following conditions:

- 029Q** 1. For each  $x, y \in X$ , if  $x \sim_R y$ , then there exists some  $k \in X$  such that  $x \sim_R k$  and  $k \sim_R y$ .
- 029R** 2. For each  $x, y \in X$ , if  $x \sim_R y$ , then  $x = y$ .

The second condition implies that  $R \subset \Delta_X$ , so  $R$  must be a subset of  $X$ . Taking  $k = y$  in the first condition above then shows it to be trivially satisfied. Conversely, any subset  $U$  of  $X$  satisfies  $U \subset \Delta_X$ , defining a comonad as above.  $\square$

**02EM** **Example 8.5.5.1.2.** Let  $f: A \rightarrow B$  be a function.

- 02EN** 1. The density comonad  $\text{Lan}_f(f): B \rightarrow B$  is given by

$$[\text{Lan}_f(f)](b) = \bigcup_{a \in f^{-1}(b)} f(a)$$

for each  $b \in B$ . Thus, it corresponds to the image  $\text{Im}(f)$  of  $f$  as a subset of  $B$ .

- 02EP** 2. The dual density comonad  $\text{Lift}_{f^\dagger}(f^\dagger): A \rightarrow A$  is given by

$$[\text{Lift}_{f^\dagger}(f^\dagger)](b) = \bigcup_{a \in f^{-1}(b)} f(a)$$

for each  $b \in B$ . Thus, it also corresponds to the image  $\text{Im}(f)$  of  $f$  as a subset of  $B$ .

## **02EQ** 8.5.6 Modules Over Internal Monads

Let  $A$  be a set.

- 02ER** **Proposition 8.5.6.1.1.** Let  $\preceq_A$  be a preorder on  $A$ , viewed also as an internal monad on  $A$  via **Definition 8.5.4.1.1**.

02ES 1. *Left Modules.* We have a natural identification

$$\{\text{Left modules over } \preceq_A\} \cong \left\{ \begin{array}{l} \text{Relations } R: B \rightarrowtail A \text{ such that,} \\ \text{for each } b \in B, \text{ the set } R(b) \text{ is} \\ \text{upward-closed in } A \end{array} \right\}.$$

02ET 2. *Right Modules.* We have a natural identification

$$\{\text{Right modules over } \preceq_A\} \cong \left\{ \begin{array}{l} \text{Relations } R: A \rightarrowtail B \text{ such that,} \\ \text{for each } b \in B, \text{ the set } R^{-1}(b) \text{ is} \\ \text{downward-closed in } A \end{array} \right\}.$$

02EU 3. *Bimodules.* We have a natural identification

$$\{\text{Bimodules over } \preceq_A\} \cong \left\{ \begin{array}{l} \text{Quadruples } (B, C, R, S) \text{ such that:} \\ 1. \text{ For each } b \in B, \text{ the set } R(b) \text{ is} \\ \quad \text{upward-closed in } A. \\ 2. \text{ For each } c \in C, \text{ the set } S^{-1}(c) \text{ is} \\ \quad \text{downward-closed in } A. \end{array} \right\}.$$

*Proof.* **Item 1, Left Modules:** A left module over  $\preceq_A$  in **Rel** consists of a relation  $R: B \rightarrowtail A$  together with an inclusion

$$\alpha_B: \preceq_A \diamond R \subset R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (**Categories, Item 4** of **Definition 11.2.7.1.2**), and hence all that is left is the data of the inclusion  $\alpha_B$ . This corresponds to the following condition:

- (★) For each  $a, a' \in A$ , if there exists some  $b \in B$  such that  $b \sim_R a$  and  $a \preceq_a a'$ , then  $b \sim_R a'$ .

This condition is equivalent to  $R(b)$  being downward-closed for all  $b \in B$ .

**Item 2, Right Modules:** The proof is dual to **Item 1**, and is therefore omitted.

**Item 3, Bimodules:** Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (**Categories, Item 4** of **Definition 11.2.7.1.2**), and hence a bimodule is just a left module along with a right module.  $\square$

### 02EV 8.5.7 Comodules Over Internal Comonads

Let  $A$  be a set.

02EW **Proposition 8.5.7.1.1.** Let  $U$  be a subset of  $A$ , viewed also as an internal comonad on  $A$  via [Definition 8.5.5.1.1](#).

02EX 1. *Left Comodules.* We have a natural identification

$$\{\text{Left comodules over } U\} \cong \left\{ \begin{array}{l} \text{Relations } R: B \rightarrowtail A \text{ such that,} \\ \text{for each } b \in B, \text{ we have } R(b) \subset U \end{array} \right\}.$$

02EY 2. *Right Comodules.* We have a natural identification

$$\{\text{Right comodules over } U\} \cong \left\{ \begin{array}{l} \text{Relations } R: A \rightarrowtail B \text{ such that,} \\ \text{for each } b \in B, \text{ we have } R^{-1}(b) \subset U \end{array} \right\}.$$

02EZ 3. *Bicomodules.* We have a natural identification

$$\{\text{Bicomodules over } U\} \cong \left\{ \begin{array}{l} \text{Quadruples } (B, C, R, S) \text{ such that:} \\ 1. \text{ For each } b \in B, \text{ we have } R(b) \subset U \\ 2. \text{ For each } c \in C, \text{ we have } S^{-1}(c) \subset U \end{array} \right\}.$$

*Proof.* **Item 1, Left Comodules:** A left comodule over  $U$  in **Rel** consists of a relation  $R: B \rightarrowtail A$  together with an inclusion

$$R \subset U \diamond R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic ([Categories, Item 4](#) of [Definition 11.2.7.1.2](#)), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

- (★) For each  $b \in B$ , if  $b \sim_R a$ , then there exists some  $a' \in A$  such that  $b \sim_R a'$  and  $a' \sim_U a$ .

Since  $a' \sim_U a$  is true if  $a = a'$  and  $a \in U$ , this condition ends up being equivalent to  $R(b) \subset U$ .

**Item 2, Right Comodules:** A right comodule over  $U$  in **Rel** consists of a relation  $R: A \rightarrowtail B$  together with an inclusion

$$R \subset R \diamond U$$



making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

- (★) For each  $a \in A$ , if  $a \sim_R b$ , then there exists some  $x \in A$  such that  $a \sim_U x$  and  $x \sim_R b$ .

Since  $a \sim_U x$  is true if  $a = x$  and  $a \in U$ , this condition ends up being equivalent to  $R^{-1}(b) \subset U$ .

**Item 3, Bicomodules:** Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (**Categories**, **Item 4** of **Definition 11.2.7.1.2**), and hence a bicomodule is just a left comodule along with a right comodule.  $\square$

## 02F0 8.5.8 Eilenberg–Moore and Kleisli Objects

Let  $X$  be a set.

02F1 **Proposition 8.5.8.1.1.** Let  $R$  be a preorder on  $X$ , viewed as an internal monad on  $X$  via **Definition 8.5.4.1.1**.

02F2 1. *Eilenberg–Moore Objects in Rel.* The Eilenberg–Moore object for  $R$  exists iff it is an equivalence relation, in which case it is the quotient  $X/\sim_R$  of  $X$  by  $R$ .

02F3 2. *Kleisli Objects in Rel.* [...]

*Proof.* Omitted.  $\square$

## 00LB 8.5.9 Co/Monoids

00LC **Remark 8.5.9.1.1.** The monoids in **Rel** with respect to the Cartesian monoidal structure of **Definition 8.3.3.8.1** are called *hypermonoids*, and their theory is explored in ???. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in ???.

## 00LD 8.5.10 Monomorphisms

In this section we characterise the epimorphisms in the category **Rel**, following ??, ??.

00LE **Proposition 8.5.10.1.1.** Let  $R: A \rightarrowtail B$  be a relation. The following conditions are equivalent:

00LF 1. The relation  $R$  is a monomorphism in  $\text{Rel}$ .

00LG 2. The direct image function

$$R_! : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to  $R$  is injective.

00LH 3. The codirect image function

$$R_* : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$$

associated to  $R$  is injective.

Moreover, if  $R$  is a monomorphism, then it satisfies the following condition, and the converse holds if  $R$  is total:

(★) For each  $a, a' \in A$ , if there exists some  $b \in B$  such that

$$a \sim_R b,$$

$$a' \sim_R b,$$

then  $a = a'$ .

*Proof. First Proof of the Equivalence of Items 1 to 3:* Firstly note that Items 2 and 3 are equivalent by Item 7 of Definition 8.7.1.1.4. We then claim that Items 1 and 2 are also equivalent:

- *Item 1  $\implies$  Item 2:* Let  $U, V \in \mathcal{P}(A)$  and consider the diagram

$$\text{pt} \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} A \xrightarrow{R} B.$$

By Definition 8.7.1.1.3, we have

$$R_!(U) = R \diamond U,$$

$$R_!(V) = R \diamond V.$$

Now, if  $R \diamond U = R \diamond V$ , i.e.  $R_!(U) = R_!(V)$ , then  $U = V$  since  $R$  is assumed to be a monomorphism, showing  $R_!$  to be injective.

- *Item 2  $\implies$  Item 1:* Conversely, suppose that  $R_!$  is injective, consider the diagram

$$X \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

and suppose that  $R \diamond S = R \diamond T$ . Note that, since  $R_!$  is injective, given a diagram of the form

$$\text{pt} \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} A \xrightarrow{R} B,$$

if  $R_!(U) = R \diamond U = R \diamond V = R_!(V)$ , then  $U = V$ . In particular, for each  $x \in X$ , we may consider the diagram

$$\text{pt} \xrightarrow{[x]} X \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

where we have  $R \diamond S \diamond [x] = R \diamond T \diamond [x]$ , implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each  $x \in X$ . Thus  $S = T$  and  $R$  is a monomorphism.

*Second Proof of the Equivalence of Items 1 to 3:* A more abstract proof can also be given, following [MSE 350788]:

- **Item 1  $\implies$  Item 2:** Assume that  $R$  is a monomorphism.
  - We first notice that the functor  $\text{Rel}(\text{pt}, -) : \text{Rel} \rightarrow \text{Sets}$  maps  $R$  to  $R_!$  by **Definition 8.7.1.1.3**.
  - Since  $\text{Rel}(\text{pt}, -)$  preserves all limits by Limits and Colimits, ?? of ??, it follows by ??, ?? of ?? that  $\text{Rel}(\text{pt}, -)$  also preserves monomorphisms.
  - Since  $R$  is a monomorphism and  $\text{Rel}(\text{pt}, -)$  maps  $R$  to  $R_!$ , it follows that  $R_!$  is also a monomorphism.
  - Since the monomorphisms in **Sets** are precisely the injections (??, ?? of ??), it follows that  $R_!$  is injective.
- **Item 2  $\implies$  Item 1:** Assume that  $R_!$  is injective.
  - We first notice that the functor  $\text{Rel}(\text{pt}, -) : \text{Rel} \rightarrow \text{Sets}$  maps  $R$  to  $R_!$  by **Definition 8.7.1.1.3**.
  - Since the monomorphisms in **Sets** are precisely the injections (??, ?? of ??), it follows that  $R_!$  is a monomorphism.
  - Since  $\text{Rel}(\text{pt}, -)$  is faithful, it follows by ??, ?? of ?? that  $\text{Rel}(\text{pt}, -)$  reflects monomorphisms.
  - Since  $R_!$  is a monomorphism and  $\text{Rel}(\text{pt}, -)$  maps  $R$  to  $R_!$ , it follows that  $R$  is also a monomorphism.

*Proof of the Second Half of Definition 8.5.10.1.1:* Finally, we prove the second part of the statement. Assume that  $R$  is a monomorphism, let  $a, a' \in A$  such that  $a \sim_R b$  and  $a' \sim_R b$  for some  $b \in B$ , and consider the diagram

$$\text{pt} \begin{array}{c} \xrightarrow{[a]} \\ \xrightarrow{[a']} \end{array} A \xrightarrow{R} B.$$

Then:

- Since  $\star \sim_{[a]} a$  and  $a \sim_R b$ , we have  $\star \sim_{R \diamond [a]} b$ .
- Similarly,  $\star \sim_{R \diamond [a']} b$ .
- Thus  $R \diamond [a] = R \diamond [a']$ .
- Since  $R$  is a monomorphism, we have  $[a] = [a']$ , so  $a = a'$ .

Conversely, assume the condition

( $\star$ ) For each  $a, a' \in A$ , if there exists some  $b \in B$  such that

$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$

then  $a = a'$ .

consider the diagram

$$X \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

and let  $x \in X$  and  $a \in A$  such that  $x \sim_S a$ .

- Since  $R$  is total and  $a \in A$ , there exists some  $b \in B$  such that  $a \sim_R b$ .
- In this case, we have  $x \sim_{R \diamond S} b$ , and since  $R \diamond S = R \diamond T$ , we have also  $x \sim_{R \diamond T} b$ .
- Thus there must exist some  $a' \in A$  such that  $x \sim_T a'$  and  $a' \sim_R b$ .
- However, since  $a \sim_R b$  and  $a' \sim_R b$ , we must have  $a = a'$ .
- Thus  $x \sim_T a$  as well.
- A similar argument shows that if  $x \sim_T a$ , then  $x \sim_S a$ .
- Thus  $S = T$  and it follows that  $R$  is a monomorphism.

This finishes the proof.  $\square$

### 00LJ 8.5.11 2-Categorical Monomorphisms

In this section we characterise (for now, some of) the 2-categorical monomorphisms in **Rel**, following [Types of Morphisms in Bicategories, Section 14.1](#).

00LK **Proposition 8.5.11.1.1.** Let  $R: A \rightarrowtail B$  be a relation.

00LL 1. *Representably Faithful Morphisms in Rel.* Every morphism of **Rel** is a representably faithful morphism.

00LM 2. *Representably Full Morphisms in Rel.* The following conditions are equivalent:

00LN (a) The morphism  $R: A \rightarrowtail B$  is a representably full morphism.

00LP (b) For each pair of relations  $S, T: X \rightarrowtail A$ , the following condition is satisfied:

(★) If  $R \diamond S \subset R \diamond T$ , then  $S \subset T$ .

00LQ (c) The direct image functor

$$R_! : (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

of [Item 2](#) of [Definition 8.7.1.1.5](#) is full.

00LR (d) The codirect image functor

$$R_* : (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

of [Item 2](#) of [Definition 8.7.4.1.4](#) is full.

00LS (e) For each  $U, V \in \mathcal{P}(A)$ , if  $R_!(U) \subset R_!(V)$ , then  $U \subset V$ .

00LT (f) For each  $U, V \in \mathcal{P}(A)$ , if  $R_*(U) \subset R_*(V)$ , then  $U \subset V$ .

00LU 3. *Representably Fully Faithful Morphisms in Rel.* Every representably full morphism in **Rel** is a representably fully faithful morphism.

*Proof.* [Item 1](#), *Representably Faithful Morphisms in Rel:* The relation  $R$  is a representably faithful morphism in **Rel** iff, for each  $X \in \text{Obj}(\mathbf{Rel})$ , the functor

$$R_! : \mathbf{Rel}(X, A) \rightarrow \mathbf{Rel}(X, B)$$

is faithful, i.e. iff the morphism

$$R_{*|S,T} : \text{Hom}_{\mathbf{Rel}(X,A)}(S, T) \rightarrow \text{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each  $S, T \in \text{Obj}(\mathbf{Rel}(X, A))$ . However,  $\text{Hom}_{\mathbf{Rel}(X,A)}(S, T)$  is either empty or a singleton, in either case of which the map  $R_{*|S,T}$  is necessarily injective.

[Item 2](#), *Representably Full Morphisms in Rel:* We claim [Items 2a](#) to [2f](#) are indeed equivalent:

- *Item 2a*  $\iff$  *Item 2b*: This is simply a matter of unwinding definitions: The relation  $R$  is a representably full morphism in  $\mathbf{Rel}$  iff, for each  $X \in \text{Obj}(\mathbf{Rel})$ , the functor

$$R_! : \mathbf{Rel}(X, A) \rightarrow \mathbf{Rel}(X, B)$$

is full, i.e. iff the morphism

$$R_{*|S,T} : \text{Hom}_{\mathbf{Rel}(X,A)}(S, T) \rightarrow \text{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is surjective for each  $S, T \in \text{Obj}(\mathbf{Rel}(X, A))$ , i.e. iff, whenever  $R \diamond S \subset R \diamond T$ , we also have  $S \subset T$ .

- *Item 2c*  $\iff$  *Item 2e*: This is also simply a matter of unwinding definitions: The functor

$$R_! : (\mathcal{P}(A), \subset) \rightarrow (\mathcal{P}(B), \subset)$$

is full iff, for each  $U, V \in \mathcal{P}(A)$ , the morphism

$$R_{*|U,V} : \text{Hom}_{\mathcal{P}(A)}(U, V) \rightarrow \text{Hom}_{\mathcal{P}(B)}(R_!(U), R_!(V))$$

is surjective, i.e. iff whenever  $R_!(U) \subset R_!(V)$ , we also necessarily have  $U \subset V$ .

- *Item 2d*  $\iff$  *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between *Items 2c* and *2e* given above.
- *Item 2e*  $\implies$  *Item 2f*: Suppose that the following condition is true:

( $\star$ ) For each  $U, V \in \mathcal{P}(A)$ , if  $R_!(U) \subset R_!(V)$ , then  $U \subset V$ .

We need to show that the condition

( $\star$ ) For each  $U, V \in \mathcal{P}(A)$ , if  $R_*(U) \subset R_*(V)$ , then  $U \subset V$ .

is also true. We proceed step by step:

- Suppose we have  $U, V \in \mathcal{P}(A)$  with  $R_*(U) \subset R_*(V)$ .
- By **Constructions With Relations**, ?? of ??, we have

$$R_*(U) = B \setminus R_!(A \setminus U),$$

$$R_*(V) = B \setminus R_!(A \setminus V).$$

- By **Constructions With Sets**, *Item 1* of **Definition 4.3.10.1.2** we have  $R_!(A \setminus V) \subset R_!(A \setminus U)$ .

- By assumption, we then have  $A \setminus V \subset A \setminus U$ .
- By **Constructions With Sets, Item 1** of **Definition 4.3.10.1.2** again, we have  $U \subset V$ .

• **Item 2f**  $\implies$  **Item 2e**: Suppose that the following condition is true:

- ( $\star$ ) For each  $U, V \in \mathcal{P}(A)$ , if  $R_*(U) \subset R_*(V)$ , then  $U \subset V$ .

We need to show that the condition

- ( $\star$ ) For each  $U, V \in \mathcal{P}(A)$ , if  $R_!(U) \subset R_!(V)$ , then  $U \subset V$ .

is also true. We proceed step by step:

- Suppose we have  $U, V \in \mathcal{P}(A)$  with  $R_!(U) \subset R_!(V)$ .
- By **Constructions With Relations, ??** of **??**, we have

$$\begin{aligned} R_!(U) &= B \setminus R_*(A \setminus U), \\ R_!(V) &= B \setminus R_*(A \setminus V). \end{aligned}$$

- By **Constructions With Sets, Item 1** of **Definition 4.3.10.1.2** we have  $R_*(A \setminus V) \subset R_*(A \setminus U)$ .
- By assumption, we then have  $A \setminus V \subset A \setminus U$ .
- By **Constructions With Sets, Item 1** of **Definition 4.3.10.1.2** again, we have  $U \subset V$ .

• **Item 2b**  $\implies$  **Item 2e**: Consider the diagram

$$X \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

and suppose that  $R \diamond S \subset R \diamond T$ . Note that, by assumption, given a diagram of the form

$$\text{pt} \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} A \xrightarrow{R} B,$$

if  $R_!(U) = R \diamond U \subset R \diamond V = R_!(V)$ , then  $U \subset V$ . In particular, for each  $x \in X$ , we may consider the diagram

$$\text{pt} \xrightarrow{[x]} X \begin{array}{c} \xrightarrow{S} \\ \text{---} \\ \xrightarrow{T} \end{array} A \xrightarrow{R} B,$$

for which we have  $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$ , implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each  $x \in X$ , implying  $S \subset T$ .

- *Item 2e*  $\implies$  *Item 2b*: Let  $U, V \in \mathcal{P}(A)$  and consider the diagram

$$\text{pt} \begin{array}{c} \xrightarrow{U} \\ \text{---} \\ \xrightarrow{V} \end{array} A \xrightarrow{R} B.$$

By **Definition 8.7.1.1.3**, we have

$$R_!(U) = R \diamond U,$$

$$R_!(V) = R \diamond V.$$

Now, if  $R_!(U) \subset R_!(V)$ , i.e.  $R \diamond U \subset R \diamond V$ , then  $U \subset V$  by assumption.

*Item 3, Representably Fully Faithful Morphisms in Rel*: This follows from **Items 1 and 2**.  $\square$

**00LV Question 8.5.11.1.2.** *Item 2 of Definition 8.5.11.1.1* gives a characterisation of the representably full morphisms in **Rel**.

Are there other nice characterisations of these?

This question also appears as [MO 467527].

## **00LW 8.5.12 Epimorphisms**

In this section we characterise the epimorphisms in the category **Rel**, following ??, ??.

**00LX Proposition 8.5.12.1.1.** Let  $R: A \rightarrowtail B$  be a relation. The following conditions are equivalent:

**00LY** 1. The relation  $R$  is an epimorphism in **Rel**.

**00LZ** 2. The weak inverse image function

$$R^{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

associated to  $R$  is injective.

**00M0** 3. The strong inverse image function

$$R_{-1}: \mathcal{P}(B) \rightarrow \mathcal{P}(A)$$

associated to  $R$  is injective.

**00M1** 4. The function  $R: A \rightarrow \mathcal{P}(B)$  is “surjective on singletons”:

( $\star$ ) For each  $b \in B$ , there exists some  $a \in A$  such that  $R(a) = \{b\}$ .



Moreover, if  $R$  is total and an epimorphism, then it satisfies the following equivalent conditions:

- 029S 1. For each  $b \in B$ , there exists some  $a \in A$  such that  $a \sim_R b$ .
- 029T 2. We have  $\text{Im}(R) = B$ .

*Proof. First Proof of the Equivalence of Items 1 to 3:* Firstly note that Items 2 and 3 are equivalent by Item 7 of Definition 8.7.2.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1  $\implies$  Item 2:* Let  $U, V \in \mathcal{P}(A)$  and consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{U} \\ \xrightarrow{V} \end{array} \text{pt.}$$

By **Constructions With Relations**, ??, we have

$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

Now, if  $U \diamond R = V \diamond R$ , i.e.  $R^{-1}(U) = R^{-1}(V)$ , then  $U = V$  since  $R$  is assumed to be an epimorphism, showing  $R^{-1}$  to be injective.

- *Item 2  $\implies$  Item 1:* Conversely, suppose that  $R^{-1}$  is injective, consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \end{array} X,$$

and suppose that  $S \diamond R = T \diamond R$ . Note that, since  $R^{-1}$  is injective, given a diagram of the form

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{U} \\ \xrightarrow{V} \end{array} \text{pt},$$

if  $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$ , then  $U = V$ . In particular, for each  $x \in X$ , we may consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \end{array} X \xrightarrow{[x]} \text{pt},$$

for which we have  $[x] \diamond S \diamond R = [x] \diamond T \diamond R$ , implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

for each  $x \in X$ . Thus  $S = T$  and  $R$  is an epimorphism.

*Second Proof of the Equivalence of Items 1 to 3:* A more abstract proof can also be given, following [MSE 350788]:

- *Item 1*  $\implies$  *Item 2*: Assume that  $R$  is an epimorphism.
  - We first notice that the functor  $\text{Rel}(-, \text{pt}): \text{Rel}^{\text{op}} \rightarrow \text{Sets}$  maps  $R$  to  $R^{-1}$  by [Constructions With Relations](#), ??.
  - Since  $\text{Rel}(-, \text{pt})$  preserves limits by Limits and Colimits, ?? of ??, it follows by ??, ?? of ?? that  $\text{Rel}(-, \text{pt})$  also preserves monomorphisms.
  - That is:  $\text{Rel}(-, \text{pt})$  sends monomorphisms in  $\text{Rel}^{\text{op}}$  to monomorphisms in Sets.
  - The monomorphisms in  $\text{Rel}^{\text{op}}$  are precisely the epimorphisms in Rel by ??, ?? of ??.
  - Since  $R$  is an epimorphism and  $\text{Rel}(-, \text{pt})$  maps  $R$  to  $R^{-1}$ , it follows that  $R^{-1}$  is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (??, ?? of ??), it follows that  $R^{-1}$  is injective.
- *Item 2*  $\implies$  *Item 1*: Assume that  $R^{-1}$  is injective.
  - We first notice that the functor  $\text{Rel}(-, \text{pt}): \text{Rel}^{\text{op}} \rightarrow \text{Sets}$  maps  $R$  to  $R^{-1}$  by [Constructions With Relations](#), ??.
  - Since the monomorphisms in Sets are precisely the injections (??, ?? of ??), it follows that  $R^{-1}$  is a monomorphism.
  - Since  $\text{Rel}(-, \text{pt})$  is faithful, it follows by ??, ?? of ?? that  $\text{Rel}(-, \text{pt})$  reflects monomorphisms.
  - That is:  $\text{Rel}(-, \text{pt})$  reflects monomorphisms in Sets to monomorphisms in  $\text{Rel}^{\text{op}}$ .
  - The monomorphisms in  $\text{Rel}^{\text{op}}$  are precisely the epimorphisms in Rel by ??, ?? of ??.
  - Since  $R^{-1}$  is a monomorphism and  $\text{Rel}(-, \text{pt})$  maps  $R$  to  $R^{-1}$ , it follows that  $R$  is an epimorphism.

*Proof of the Equivalence of Items 2 and 4:* We claim that [Items 2](#) and [4](#) are equivalent, following [MO 350788]:

- *Item 2*  $\implies$  *Item 4*: We proceed in two steps.
  - Since  $B \setminus \{b\} \subset B$  and  $R^{-1}$  is injective, we have  $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$ .
  - Taking some  $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ , we obtain an element of  $A$  such that  $R(a) = \{b\}$ .

- *Item 4*  $\implies$  *Item 2*: We proceed in a few steps.
  - Let  $U, V \subset B$  with  $U \neq V$ .
  - Without loss of generality, we can assume  $U \setminus V \neq \emptyset$ ; otherwise just swap  $U$  and  $V$ .
  - Let then  $b \in U \setminus V$ .
  - By assumption, there exists an  $a \in A$  with  $R(a) = \{b\}$ .
  - Then  $a \in R^{-1}(U)$  but  $a \notin R^{-1}(V)$ , so  $R^{-1}(U) \neq R^{-1}(V)$ , showing  $R^{-1}$  to be injective.

*Proof of the Second Half of Definition 8.5.12.1.1*: Finally, we prove the second part of the statement. Assume  $R$  is a total epimorphism in **Rel** and consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \dashrightarrow \\ \xrightarrow{T} \end{array} \{0, 1\},$$

where  $b \sim_S 0$  for each  $b \in B$  and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \text{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each  $b \in B$ .

- Since  $R$  is total, we have  $a \sim_{S \diamond R} 0$  and  $a \sim_{T \diamond R} 0$  for all  $a \in A$ , and no element of  $A$  is related to 1 by  $S \diamond R$  or  $T \diamond R$ .
- Thus  $S \diamond R = T \diamond R$ .
- Since  $R$  is an epimorphism, we have  $S = T$ .
- But by the definition of  $T$ , this implies  $\text{Im}(R) = B$ .

This finishes the proof. □

## 00M2 8.5.13 2-Categorical Epimorphisms

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following [Types of Morphisms in Bicategories, Section 14.2](#).

00M3 **Proposition 8.5.13.1.1.** Let  $R: A \dashrightarrow B$  be a relation.

- 00M4 1. *Corepresentably Faithful Morphisms in Rel.* Every morphism of **Rel** is a corepresentably faithful morphism.

00M5 2. *Corepresentably Full Morphisms in Rel*. The following conditions are equivalent:

00M6 (a) The morphism  $R: A \rightarrowtail B$  is a corepresentably full morphism.

00M7 (b) For each pair of relations  $S, T: X \rightarrowtail A$ , the following condition is satisfied:

(★) If  $S \diamond R \subset T \diamond R$ , then  $S \subset T$ .

00M8 (c) The functor

$$R^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

is full.

00M9 (d) For each  $U, V \in \mathcal{P}(B)$ , if  $R^{-1}(U) \subset R^{-1}(V)$ , then  $U \subset V$ .

00MA (e) The functor

$$R_{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

is full.

00MB (f) For each  $U, V \in \mathcal{P}(B)$ , if  $R_{-1}(U) \subset R_{-1}(V)$ , then  $U \subset V$ .

00MC 3. *Corepresentably Fully Faithful Morphisms in Rel*. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

*Proof.* *Item 1, Corepresentably Faithful Morphisms in Rel*: The relation  $R$  is a corepresentably faithful morphism in **Rel** iff, for each  $X \in \text{Obj}(\mathbf{Rel})$ , the functor

$$R^*: \mathbf{Rel}(B, X) \rightarrow \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R_{S,T}^*: \text{Hom}_{\mathbf{Rel}(B,X)}(S, T) \rightarrow \text{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each  $S, T \in \text{Obj}(\mathbf{Rel}(B, X))$ . However,  $\text{Hom}_{\mathbf{Rel}(B,X)}(S, T)$  is either empty or a singleton, in either case of which the map  $R_{S,T}^*$  is necessarily injective.

*Item 2, Corepresentably Full Morphisms in Rel*: We claim *Items 2a to 2f* are indeed equivalent:

- *Item 2a*  $\iff$  *Item 2b*: This is simply a matter of unwinding definitions: The relation  $R$  is a corepresentably full morphism in **Rel** iff, for each  $X \in \text{Obj}(\mathbf{Rel})$ , the functor

$$R^*: \mathbf{Rel}(B, X) \rightarrow \mathbf{Rel}(A, X)$$

is full, i.e. iff the morphism

$$R_{S,T}^*: \text{Hom}_{\mathbf{Rel}(B,X)}(S, T) \rightarrow \text{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is surjective for each  $S, T \in \text{Obj}(\mathbf{Rel}(B, X))$ , i.e. iff, whenever  $S \diamond R \subset T \diamond R$ , we also have  $S \subset T$ .

- *Item 2c*  $\iff$  *Item 2d*: This is also simply a matter of unwinding definitions: The functor

$$R^{-1}: (\mathcal{P}(B), \subset) \rightarrow (\mathcal{P}(A), \subset)$$

is full iff, for each  $U, V \in \mathcal{P}(A)$ , the morphism

$$R_{U,V}^{-1}: \text{Hom}_{\mathcal{P}(B)}(U, V) \rightarrow \text{Hom}_{\mathcal{P}(A)}(R^{-1}(U), R^{-1}(V))$$

is surjective, i.e. iff whenever  $R^{-1}(U) \subset R^{-1}(V)$ , we also necessarily have  $U \subset V$ .

- *Item 2e*  $\iff$  *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between *Items 2c* and *2d* given above.
- *Item 2d*  $\implies$  *Item 2f*: Suppose that the following condition is true:

( $\star$ ) For each  $U, V \in \mathcal{P}(B)$ , if  $R^{-1}(U) \subset R^{-1}(V)$ , then  $U \subset V$ .

We need to show that the condition

( $\star$ ) For each  $U, V \in \mathcal{P}(B)$ , if  $R_{-1}(U) \subset R_{-1}(V)$ , then  $U \subset V$ .

is also true. We proceed step by step:

- Suppose we have  $U, V \in \mathcal{P}(B)$  with  $R_{-1}(U) \subset R_{-1}(V)$ .
- By **Constructions With Relations**, ?? of ??, we have

$$\begin{aligned} R_{-1}(U) &= B \setminus R^{-1}(A \setminus U), \\ R_{-1}(V) &= B \setminus R^{-1}(A \setminus V). \end{aligned}$$

- By **Constructions With Sets**, *Item 1* of **Definition 4.3.10.1.2** we have  $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$ .
- By assumption, we then have  $A \setminus V \subset A \setminus U$ .
- By **Constructions With Sets**, *Item 1* of **Definition 4.3.10.1.2** again, we have  $U \subset V$ .

- *Item 2f*  $\implies$  *Item 2d*: Suppose that the following condition is true:

( $\star$ ) For each  $U, V \in \mathcal{P}(B)$ , if  $R_{-1}(U) \subset R_{-1}(V)$ , then  $U \subset V$ .

We need to show that the condition

( $\star$ ) For each  $U, V \in \mathcal{P}(B)$ , if  $R^{-1}(U) \subset R^{-1}(V)$ , then  $U \subset V$ .

is also true. We proceed step by step:

- Suppose we have  $U, V \in \mathcal{P}(B)$  with  $R^{-1}(U) \subset R^{-1}(V)$ .
- By **Constructions With Relations**, ?? of ??, we have

$$\begin{aligned} R^{-1}(U) &= B \setminus R_{-1}(A \setminus U), \\ R^{-1}(V) &= B \setminus R_{-1}(A \setminus V). \end{aligned}$$

- By **Constructions With Sets**, **Item 1** of **Definition 4.3.10.1.2** we have  $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$ .
- By assumption, we then have  $A \setminus V \subset A \setminus U$ .
- By **Constructions With Sets**, **Item 1** of **Definition 4.3.10.1.2** again, we have  $U \subset V$ .

- *Item 2b*  $\implies$  *Item 2d*: Consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \end{array} X,$$

and suppose that  $S \diamond R \subset T \diamond R$ . Note that, by assumption, given a diagram of the form

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{U} \\ \xrightarrow{V} \end{array} \text{pt},$$

if  $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$ , then  $U \subset V$ . In particular, for each  $x \in X$ , we may consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{S} \\ \xrightarrow{T} \end{array} X \xrightarrow{[x]} \text{pt},$$

for which we have  $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$ , implying that we have

$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

for each  $x \in X$ , implying  $S \subset T$ .

- *Item 2e*  $\implies$  *Item 2b*: Let  $U, V \in \mathcal{P}(B)$  and consider the diagram

$$A \xrightarrow{R} B \begin{array}{c} \xrightarrow{U} \\ \xrightarrow{V} \end{array} \text{pt.}$$

By **Definition 8.7.1.1.3**, we have

$$R^{-1}(U) = U \diamond R,$$

$$R^{-1}(V) = V \diamond R.$$

Now, if  $R^{-1}(U) \subset R^{-1}(V)$ , i.e.  $U \diamond R \subset V \diamond R$ , then  $U \subset V$  by assumption.

*Item 3, Corepresentably Fully Faithful Morphisms in Rel*: This follows from **Items 1** and **2**.  $\square$

**00MD Question 8.5.13.1.2.** *Item 2* of **Definition 8.5.13.1.1** gives a characterisation of the corepresentably full morphisms in **Rel**.

Are there other nice characterisations of these?

This question also appears as [MO 467527].

## **00ME 8.5.14 Co/Limits**

**00MF Proposition 8.5.14.1.1.** This will be properly written later on.

*Proof.* Omitted.  $\square$

## **00NH 8.5.15 Internal Left Kan Extensions**

**00NJ Proposition 8.5.15.1.1.** Let  $R: A \rightarrowtail B$  be a relation.

**00NK 1.** *Non-Existence of All Internal Left Kan Extensions in Rel.* Not all relations in **Rel** admit left Kan extensions.

**00NL 2.** *Characterisation of Relations Admitting Internal Left Kan Extensions Along Them.* The following conditions are equivalent:

**02A1 (a)** The left Kan extension

$$\text{Lan}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along  $R$  exists.

**02A2 (b)** The relation  $R$  admits a left adjoint in **Rel**.

**02A3 (c)** The relation  $R$  is of the form  $\text{Gr}(f)$  (as in **Definition 8.2.2.1.1**) for some function  $f$ .

*Proof.* **Item 1**, Non-Existence of All Internal Left Kan Extensions in **Rel**: By **Item 2**, it suffices to take a relation that doesn't have a left adjoint.

**Item 2**, Characterisation of Relations Admitting Left Kan Extensions Along Them: This proof is mostly due to Tim Campion, via [MO 460693].

- We may view precomposition

$$-\diamond R: \mathbf{Rel}(B, C) \rightarrow \mathbf{Rel}(A, C)$$

with  $R: A \rightarrowtail B$  as a cocontinuous functor from  $\mathcal{P}(B \times C)$  to  $\mathcal{P}(A \times C)$  (via **Item 5** of **Definition 8.1.1.1.1**).

- By the adjoint functor theorem (??), this map has a left adjoint iff it preserves limits.
- If  $C = \emptyset$ , this holds trivially.
- Otherwise,  $C$  admits  $\text{pt}$  as a retract, and we reduce to the case  $C = \text{pt}$  via ??.
- For the case  $C = \text{pt}$ , a relation  $T: B \rightarrowtail \text{pt}$  is the same as a subset of  $B$ , and  $-\diamond R$  becomes the weak inverse image functor  $R^{-1}$  of **Section 8.7.3**.
- Now, again by the adjoint functor theorem,  $R^{-1}$  preserves limits exactly when it has a left adjoint.
- Finally  $R^{-1}$  has a left adjoint precisely when  $R = \text{Gr}(f)$  for  $f$  a function (**Item 8** of **Definition 8.7.3.1.3**).

This finishes the proof. □

**02F7 Example 8.5.15.1.2.** Given a function  $f: A \rightarrow B$ , the left Kan extension

$$\text{Lan}_f: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

along  $f$  exists by **Item 2** of **Definition 8.5.15.1.1**. Explicitly, given a relation  $R: A \rightarrowtail X$ , the left Kan extension

$$\text{Lan}_f(R): B \rightarrowtail X,$$

may be described as follows:



02F8 1. We declare  $b \sim_{\text{Lan}_f(R)} x$  iff there exists some  $a \in R$  such that  $b = f(a)$  and  $a \sim_R x$ .

02F9 2. We have<sup>19</sup>

$$[\text{Lan}_f(R)](b) = \bigcup_{a \in f^{-1}(b)} R(a)$$

for each  $b \in B$ .

02FA **Remark 8.5.15.1.3.** Following [Definition 8.5.15.1.2](#), given a relation  $R: A \rightarrowtail B$  and a relation  $F: A \rightarrowtail X$ , we could perhaps try to define an “honorary” left Kan extension

$$\text{Lan}'_R(F): B \rightarrowtail X$$

by

$$[\text{Lan}'_R(F)](b) \stackrel{\text{def}}{=} \bigcup_{a \in R^{-1}(b)} F(a)$$

for each  $b \in B$ .

The failure of  $\text{Lan}'_R(F)$  to be a Kan extension can then be seen as follows. Let  $G: B \rightarrowtail X$  be a relation. If  $\text{Lan}'_R(F)$  were a left Kan extension, then the following conditions **would be** equivalent:

02FB 1. For each  $b \in B$ , we have  $\bigcup_{a \in R^{-1}(b)} F(a) \subset G(b)$ .

02FC 2. For each  $a \in A$ , we have  $F(a) \subset \bigcup_{b \in R(a)} G(b)$ .

The issue is two-fold:

- *Totality.* If  $R$  isn't total, then the implication [Item 1](#)  $\Rightarrow$  [Item 2](#) fails.
- *Functionality.* If  $R$  isn't functional, then the implication [Item 2](#)  $\Rightarrow$  [Item 1](#) fails.

00NM **Question 8.5.15.1.4.** Given relations  $S: A \rightarrowtail X$  and  $R: A \rightarrowtail B$ , is there a characterisation of when the left Kan extension<sup>20</sup>

$$\text{Lan}_S(R): B \rightarrowtail X$$

exists in terms of properties of  $R$  and  $S$ ?

This question also appears as [\[MO 461592\]](#).

<sup>19</sup>Cf. [Item 3](#) of [Definition 8.5.17.1.2](#).

<sup>20</sup>Specifically for  $R$  and  $S$ , not  $\text{Lan}_S$  the functor.

### 00NP 8.5.16 Internal Left Kan Lifts

00NQ **Proposition 8.5.16.1.1.** Let  $R: A \rightarrowtail B$  be a relation.

00NR 1. *Non-Existence of All Internal Left Kan Lifts in **Rel**.* Not all relations in **Rel** admit left Kan lifts.

00NS 2. *Characterisation of Relations Admitting Internal Left Kan Lifts Along Them.* The following conditions are equivalent:

02A4 (a) The left Kan lift

$$\text{Lift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$

along  $R$  exists.

02A5 (b) The relation  $R$  admits a right adjoint in **Rel**.

02A6 (c) The relation  $R$  is of the form  $f^{-1}$  (as in **Definition 8.2.3.1.1**) for some function  $f$ .

*Proof. Item 1, Non-Existence of All Internal Left Kan Lifts in **Rel**:* By **Item 2**, it suffices to take a relation that doesn't have a right adjoint.

*Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them:* This proof is dual to that of **Item 2** of **Definition 8.5.15.1.1**, and is therefore omitted.  $\square$

02FG **Example 8.5.16.1.2.** Given a function  $f: A \rightarrow B$ , the left Kan lift

$$\text{Lift}_{f^\dagger}: \mathbf{Rel}(X, A) \rightarrow \mathbf{Rel}(X, B)$$

along  $f^\dagger$  exists by **Item 2** of **Definition 8.5.16.1.1**. Explicitly, given a relation  $R: X \rightarrowtail A$ , the left Kan lift

$$\text{Lift}_{f^\dagger}(R): X \rightarrowtail B,$$

is given by

$$\begin{aligned} [\text{Lift}_f(R)](x) &= [\text{Gr}(f) \diamond R](a) \\ &= \bigcup_{a \in R(x)} f(a) \end{aligned}$$

for each  $x \in X$ .

**00NT Question 8.5.16.1.3.** Given relations  $S: A \rightarrowtail X$  and  $R: A \rightarrowtail B$ , is there a characterisation of when the left Kan lift<sup>21</sup>

$$\mathrm{Lift}_S(R): X \rightarrowtail A$$

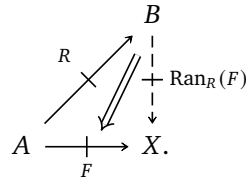
exists in terms of properties of  $R$  and  $S$ ?

This question also appears as [MO 461592].

### 00NV 8.5.17 Internal Right Kan Extensions

Let  $A$ ,  $B$ , and  $X$  be sets and let  $R: A \rightarrowtail B$  and  $F: A \rightarrowtail X$  be relations.

**02FH Motivation 8.5.17.1.1.** We want to understand internal right Kan extensions in  $\mathbf{Rel}$ , which look like this:



Note in particular here that  $F: A \rightarrowtail X$  is a relation from  $A$  to  $X$ . These will form a functor

$$\mathrm{Ran}_R: \mathbf{Rel}(A, X) \rightarrow \mathbf{Rel}(B, X)$$

that is right adjoint to the precomposition by  $R$  functor

$$R^*: \mathbf{Rel}(B, X) \rightarrow \mathbf{Rel}(A, X).$$

**00NW Proposition 8.5.17.1.2.** The internal right Kan extension of  $F$  along  $R$  is the relation  $\mathrm{Ran}_R(F)$  described as follows:

**02A7** 1. Viewing relations from  $B$  to  $X$  as subsets of  $B \times X$ , we have

$$\mathrm{Ran}_R(F) = \left\{ (b, x) \in B \times X \left| \begin{array}{l} \text{for each } a \in A, \text{ if } a \sim_R b, \\ \text{then we have } a \sim_F x \end{array} \right. \right\}.$$

**02A8** 2. Viewing relations as functions  $B \times X \rightarrow \{\text{true}, \text{false}\}$ , we have

$$(\mathrm{Ran}_R(F))_{-2}^{-1} = \int_{a \in A} \mathbf{Hom}_{\{\text{t}, \text{f}\}}(R_a^{-2}, F_a^{-1})$$

<sup>21</sup>Specifically for  $R$  and  $S$ , not  $\mathrm{Lift}_S$  the functor.

$$= \bigwedge_{a \in A} \mathbf{Hom}_{\{t,f\}}(R_a^{-2}, F_a^{-1}),$$

where the meet  $\bigwedge$  is taken in the poset  $(\{\text{true}, \text{false}\}, \preceq)$  of **Sets, Definition 3.2.2.1.3**.

**02FL** 3. Viewing relations as functions  $B \rightarrow \mathcal{P}(X)$ , we have

$$\text{Ran}_R(F) = \text{Ran}_{\chi'_A}(F) \circ R^{-1},$$

$$\begin{array}{ccc} A & \xrightarrow{F} & \mathcal{P}(X), \\ \chi_A \downarrow & \swarrow & \nearrow \text{Ran}_{\chi_A}(F) \\ B & \xrightarrow{R^{-1}} & \mathcal{P}(A)^{\text{op}} \end{array}$$

where  $\text{Ran}_{\chi'_A}(F)$  is computed by the formula

$$\begin{aligned} [\text{Ran}_{\chi'_A}(F)](V) &\cong \int_{a \in A} \chi_{\mathcal{P}(A)^{\text{op}}}(V, \chi_a) \upharpoonright F(a) \\ &\cong \int_{a \in A} \chi_{\mathcal{P}(A)}(\chi_a, V) \upharpoonright F(a) \\ &\cong \int_{a \in A} \chi_V(a) \upharpoonright F(a) \\ &\cong \bigcap_{a \in A} \chi_V(a) \upharpoonright F(a) \\ &\cong \bigcap_{a \in V} F(a) \end{aligned}$$

for each  $V \in \mathcal{P}(B)$ , so we have

$$[\text{Ran}_R(F)](b) = \bigcap_{a \in R^{-1}(b)} F(a)$$

for each  $b \in B$ .

*Proof.* We have

$$\begin{aligned} \text{Hom}_{\mathbf{Rel}(A,X)}(F \diamond R, T) &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}((F \diamond R)_a^x, T_a^x) \\ &\cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{t,f\}}\left(\left(\int_{b \in B} F_b^x \times R_a^b\right), T_a^x\right) \\ &\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(F_b^x \times R_a^b, T_a^x) \\ &\cong \int_{a \in A} \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t,f\}}(F_b^x, \mathbf{Hom}_{\{t,f\}}(R_a^b, T_a^x)) \end{aligned}$$

$$\begin{aligned}
&\cong \int_{b \in B} \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t, f\}} \left( F_b^x, \mathbf{Hom}_{\{t, f\}} \left( R_a^b, T_a^x \right) \right) \\
&\cong \int_{b \in B} \int_{x \in X} \mathbf{Hom}_{\{t, f\}} \left( F_b^x, \int_{a \in A} \mathbf{Hom}_{\{t, f\}} \left( R_a^b, T_a^x \right) \right) \\
&\cong \mathbf{Hom}_{\mathbf{Rel}(B, X)} \left( F, \int_{a \in A} \mathbf{Hom}_{\{t, f\}} \left( R_a^{-2}, T_a^{-1} \right) \right)
\end{aligned}$$

naturally in each  $F \in \mathbf{Rel}(B, X)$  and each  $T \in \mathbf{Rel}(A, X)$ , showing that

$$\int_{a \in A} \mathbf{Hom}_{\{t, f\}} (R_a^{-2}, T_a^{-1})$$

is right adjoint to the precomposition functor  $- \diamond R$ , being thus the right Kan extension along  $R$ . Here we have used the following results, respectively (i.e. for each  $\cong$  sign):

- 02A9 1. **Relations, Item 1** of **Definition 8.1.1.1.5**.
- 02AA 2. **Definition 8.1.3.1.1**.
- 02AB 3. Ends and Coends, ?? of ??.
- 02AC 4. **Sets, Definition 3.2.2.1.5**.
- 02AD 5. Ends and Coends, ?? of ??.
- 02AE 6. Ends and Coends, ?? of ??.
- 02AF 7. **Relations, Item 1** of **Definition 8.1.1.1.5**.

This finishes the proof. □

**02FU Example 8.5.17.1.3.** Here are some examples of internal right Kan extensions of relations.

- 02FV 1. *Orthogonal Complements.* Let  $A = B = X = \mathcal{V}$  be an inner product space, and let  $R = F = \perp$  be the orthogonality relation, so that we have

$$\begin{aligned}
R(v) &= v^\perp \\
F(u) &= u^\perp,
\end{aligned}$$

for each  $u, v \in \mathcal{V}$ , where

$$v^\perp \stackrel{\text{def}}{=} \{u \in V \mid v \perp u\}$$

is the orthogonal complement of  $v$ . The right Kan extension  $\text{Ran}_R(F)$  is then given by

$$\begin{aligned} [\text{Ran}_R(F)](v) &= \bigcap_{u \in R^{-1}(v)} F(u) \\ &= \bigcap_{\substack{u \in V \\ u \perp v}} u^\perp \\ &= (v^\perp)^\perp, \end{aligned}$$

the double orthogonal complement. In particular:

- If  $\mathcal{V}$  is finite-dimensional, then  $[\text{Ran}_R(F)](v) = \text{Span}(v)$ .
- If  $\mathcal{V}$  is a Hilbert space, then  $[\text{Ran}_R(F)](v) = \overline{\text{Span}(v)}$ .

**02FW** 2. *Galois Connections and Closure Operators.* Let:

- $B = X = (P, \preceq_P)$  and  $A = (Q, \preceq_Q)$  be posets;
- $(f, g)$  be a Galois connection (adjunction) between  $P$  and  $Q$ ;
- $R, F: Q \rightrightarrows P$  be the relations defined by

$$\begin{aligned} R(q) &\stackrel{\text{def}}{=} \{p \in P \mid q \preceq_Q f(p)\}, \\ F(q) &\stackrel{\text{def}}{=} \{p \in P \mid p \preceq_P g(q)\} \end{aligned}$$

for each  $q \in Q$ .

We have

$$\begin{aligned} [\text{Ran}_R(F)](p) &= \bigcap_{q \in R^{-1}(p)} F(q) \\ &= \bigcap_{\substack{q \in Q \\ q \preceq_Q f(p)}} \{p \in P \mid p \preceq_P g(q)\} \\ &= \{p \in P \mid p \preceq_P g(f(p))\} \\ &= \downarrow g(f(p)), \end{aligned}$$

the down set of  $g(f(p))$ . In other words,  $\text{Ran}_R(F)$  is the closure operator on  $P$  associated with the Galois connection  $(f, g)$ .

**02FX** **Proposition 8.5.17.1.4.** Let  $A, B, C$  and  $X$  be sets and let  $R: A \rightrightarrows B, S: B \rightrightarrows C$ , and  $F: A \rightrightarrows X$  be relations.

02FY 1. *Functoriality.* The assignments  $R, F, (R, F) \mapsto \text{Ran}_R(F)$  define functors

$$\begin{aligned} \text{Ran}_{(-)}(F) : \quad & \mathbf{Rel}(A, B)^{\text{op}} && \rightarrow \mathbf{Rel}(B, X), \\ \text{Ran}_R : \quad & \mathbf{Rel}(A, X) && \rightarrow \mathbf{Rel}(B, X), \\ \text{Ran}_{(-)}(-_2) : \quad & \mathbf{Rel}(A, X) \times \mathbf{Rel}(A, B)^{\text{op}} && \rightarrow \mathbf{Rel}(B, X). \end{aligned}$$

In other words, given relations

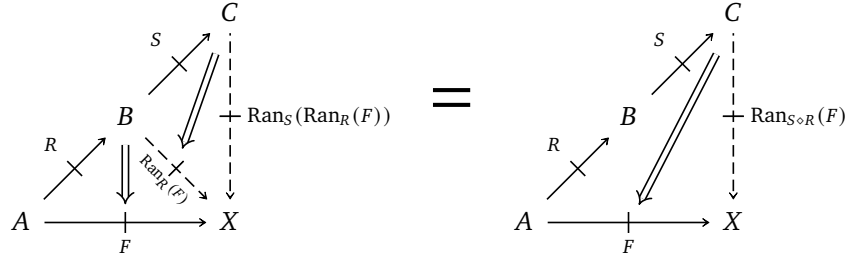
$$A \begin{array}{c} \xrightarrow{R_1} \\ \xleftarrow{R_2} \end{array} B \quad A \begin{array}{c} \xrightarrow{F_1} \\ \xleftarrow{F_2} \end{array} X,$$

if  $R_1 \subset R_2$  and  $F_1 \subset F_2$ , then  $\text{Ran}_{R_2}(F_1) \subset \text{Ran}_{R_1}(F_2)$ .

02FZ 2. *Interaction With Composition.* We have

$$\text{Ran}_{S \circ R}(F) = \text{Ran}_S(\text{Ran}_R(F))$$

and an equality



of pasting diagrams in **Rel**.

02G0 3. *Interaction With Converses.* We have

$$\text{Ran}_R(F)^\dagger = \text{Rift}_{R^\dagger}(F^\dagger).$$

02G1 4. *Interaction With Weak Inverse Images.* We have

$$[\text{Ran}_R(F)]^{-1}(x) = \{b \in B \mid R^{-1}(b) \subset F^{-1}(x)\}$$

for each  $x \in X$ .

*Proof.* **Item 1, Functoriality:** We have

$$[\text{Ran}_{R_2}(F_1)](b) = \bigcap_{a \in R_2^{-1}(b)} F_1(a)$$

$$\begin{aligned}
&\subset \bigcap_{a \in R_1^{-1}(b)} F_1(a) \\
&\subset \bigcap_{a \in R_1^{-1}(b)} F_2(a) \\
&= [\text{Ran}_{R_1}(F_2)](b)
\end{aligned}$$

for each  $b \in B$ , so we therefore have  $\text{Ran}_{R_2}(F_1) \subset \text{Ran}_{R_1}(F_2)$ .

**Item 2, Interaction With Composition:** This holds in a general bicategory with the necessary right Kan extensions, being therefore a special case of ??.

**Item 3, Interaction With Converses:** We have

$$\begin{aligned}
[\text{Rift}_{R^\dagger}(F^\dagger)](x) &= \{b \in B \mid R^\dagger(b) \subset F^\dagger(x)\} \\
&= \{b \in B \mid R^{-1}(b) \subset F^{-1}(x)\} \\
&= \text{Ran}_R(F)^{-1}(x) \\
&= \text{Ran}_R(F)^\dagger(x)
\end{aligned}$$

where we have used **Definition 8.5.18.1.2** and **Item 4**.

**Item 4, Interaction With Weak Inverse Images:** We proceed in a few steps.

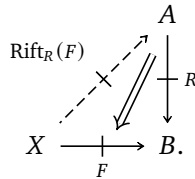
- We have  $b \in [\text{Ran}_R(F)]^{-1}(x)$  iff, for each  $a \in R^{-1}(b)$ , we have  $b \in F(a)$ .
- This holds iff, for each  $a \in R^{-1}(b)$ , we have  $a \in F^{-1}(b)$ .
- This holds iff  $R^{-1}(b) \subset F^{-1}(b)$ .

This finishes the proof. □

## 00NX 8.5.18 Internal Right Kan Lifts

Let  $A$ ,  $B$ , and  $X$  be sets and let  $R: A \rightarrowtail B$  and  $F: X \rightarrowtail B$  be relations.

**02G2 Motivation 8.5.18.1.1.** We want to understand internal right Kan lifts in **Rel**, which look like this:



Note in particular here that  $F: B \rightarrowtail X$  is a relation from  $B$  to  $X$ . These will form a functor

$$\text{Rift}_R: \mathbf{Rel}(X, B) \rightarrow \mathbf{Rel}(X, A)$$



that is right adjoint to the postcomposition by  $R$  functor

$$R_* : \mathbf{Rel}(X, A) \rightarrow \mathbf{Rel}(X, B).$$

**00NY Proposition 8.5.18.1.2.** The internal right Kan lift of  $F$  along  $R$  is the relation  $\text{Rift}_R(F)$  described as follows:

**02AG** 1. Viewing relations from  $X$  to  $A$  as subsets of  $X \times A$ , we have

$$\text{Rift}_R(F) = \left\{ (x, a) \in X \times A \mid \begin{array}{l} \text{for each } b \in B, \text{ if } a \sim_R b, \\ \text{then we have } x \sim_F b \end{array} \right\}.$$

**02AH** 2. Viewing relations as functions  $X \times A \rightarrow \{\text{true}, \text{false}\}$ , we have

$$\begin{aligned} (\text{Rift}_R(F))_{-2}^{-1} &= \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, F_{-2}^b) \\ &= \bigwedge_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, F_{-2}^b), \end{aligned}$$

where the meet  $\bigwedge$  is taken in the poset  $(\{\text{true}, \text{false}\}, \preceq)$  of **Sets, Definition 3.2.2.1.3**.

**02G3** 3. Viewing relations as functions  $X \rightarrow \mathcal{P}(A)$ , we have

$$[\text{Rift}_R(F)](x) = \{a \in A \mid R(a) \subset F(x)\}$$

for each  $a \in A$ .

*Proof.* We have

$$\begin{aligned} \mathbf{Hom}_{\mathbf{Rel}(X, B)}(R \diamond F, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}((R \diamond F)_x^b, T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}\left(\left(\int_{a \in A} R_a^b \times F_x^a\right), T_x^b\right) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(R_a^b \times F_x^a, T_x^b) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}(F_x^a, \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(F_x^a, \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)) \\ &\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{t, f\}}\left(F_x^a, \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_a^b, T_x^b)\right) \\ &\cong \mathbf{Hom}_{\mathbf{Rel}(X, A)}\left(F, \int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, T_{-2}^b)\right) \end{aligned}$$

naturally in each  $F \in \mathbf{Rel}(X, A)$  and each  $T \in \mathbf{Rel}(X, B)$ , showing that

$$\int_{b \in B} \mathbf{Hom}_{\{t, f\}}(R_{-1}^b, F_{-2}^b)$$

is right adjoint to the postcomposition functor  $R \diamond -$ , being thus the right Kan lift along  $R$ . Here we have used the following results, respectively (i.e. for each  $\cong$  sign):

- 02AJ 1. Relations, Item 1 of Definition 8.1.1.1.5.
- 02AK 2. Definition 8.1.3.1.1.
- 02AL 3. Ends and Coends, ?? of ??.
- 02AM 4. Sets, Definition 3.2.2.1.5.
- 02AN 5. Ends and Coends, ?? of ??.
- 02AP 6. Ends and Coends, ?? of ??.
- 02AQ 7. Relations, Item 1 of Definition 8.1.1.1.5.

This finishes the proof. □

02G4 **Example 8.5.18.1.3.** Here are some examples of internal right Kan lifts of relations.

02G5 1. *Pullbacks.* Let  $p: A \rightarrow B$  and  $f: X \rightarrow B$  be functions. We have

$$\begin{aligned} [\mathbf{Rift}_{\mathbf{Gr}(p)}(\mathbf{Gr}(f))](x) &= \{a \in A \mid [\mathbf{Gr}(p)](a) \subset [\mathbf{Gr}(f)](x)\} \\ &= \{a \in A \mid p(a) = f(x)\}. \end{aligned}$$

Thus, as a subset of  $X \times A$ , the right Kan lift  $\mathbf{Rift}_{\mathbf{Gr}(p)}(\mathbf{Gr}(f))$  corresponds precisely to the pullback  $X \times_B A$  of  $X$  and  $A$  along  $p$  and  $f$  of **Constructions With Sets, Section 4.1.4**.

02G6 **Proposition 8.5.18.1.4.** Let  $A, B, C$  and  $X$  be sets and let  $R: A \rightarrowtail B$ ,  $S: B \rightarrowtail C$ , and  $F: X \rightarrowtail B$  be relations.

02G7 1. *Functoriality.* The assignments  $R, F, (R, F) \mapsto \mathbf{Rift}_R(F)$  define functors

$$\begin{aligned} \mathbf{Rift}_{(-)}(F): \quad \mathbf{Rel}(A, B)^{\text{op}} &\rightarrow \mathbf{Rel}(B, X), \\ \mathbf{Rift}_R: \quad \mathbf{Rel}(A, X) &\rightarrow \mathbf{Rel}(B, X), \\ \mathbf{Rift}_{(-1)}(-2): \mathbf{Rel}(A, X) \times \mathbf{Rel}(A, B)^{\text{op}} &\rightarrow \mathbf{Rel}(B, X). \end{aligned}$$

In other words, given relations

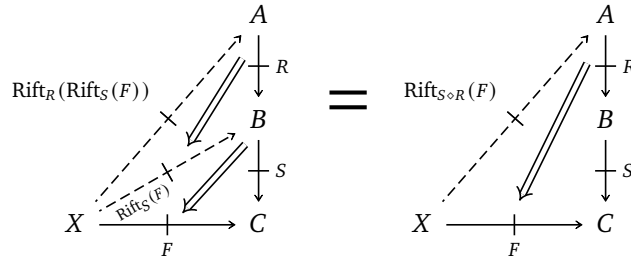
$$A \begin{array}{c} \xrightarrow{R_1} \\ \xleftarrow{R_2} \end{array} B \quad A \begin{array}{c} \xrightarrow{F_1} \\ \xleftarrow{F_2} \end{array} X,$$

if  $R_1 \subset R_2$  and  $F_1 \subset F_2$ , then  $\text{Rift}_{R_2}(F_1) \subset \text{Rift}_{R_1}(F_2)$ .

02G8 2. *Interaction With Composition.* We have

$$\text{Rift}_{S \circ R}(F) = \text{Rift}_R(\text{Ran}_S(F))$$

and an equality



of pasting diagrams in **Rel**.

02G9 3. *Interaction With Converses.* We have

$$\text{Rift}_R(F)^\dagger = \text{Ran}_{R^\dagger}(F^\dagger).$$

02GA 4. *Interaction With Weak Inverse Images.* We have

$$\text{Rift}_R(F)^\dagger = \text{Ran}_{\chi'_B}(F^\dagger) \circ R,$$

where  $\text{Ran}_{\chi_A}(F^\dagger)$  is computed by the formula

$$\begin{aligned} \left[ \text{Ran}_{\chi_A}(F^\dagger) \right](U) &\cong \int_{a \in A} \chi_{\mathcal{P}(B)^{\text{op}}}(U, \chi_a) \pitchfork F^\dagger(a) \\ &\cong \int_{a \in A} \chi_{\mathcal{P}(B)}(\chi_a, U) \pitchfork F^{-1}(a) \end{aligned}$$

$$\begin{aligned}
&\cong \int_{a \in A} \chi_U(a) \pitchfork F(a) \\
&\cong \bigcap_{a \in A} \chi_U(a) \pitchfork F(a) \\
&\cong \bigcap_{a \in U} F(a)
\end{aligned}$$

for each  $U \in \mathcal{P}(A)$ , so we have

$$[\text{Rift}_R(F)]^{-1}(a) = \bigcap_{b \in R(a)} F^{-1}(b)$$

for each  $a \in A$ .

*Proof.* **Item 1, Functoriality:** We have

$$\begin{aligned}
[\text{Rift}_{R_2}(F_1)](x) &= \{a \in A \mid R_2(a) \subset F_1(x)\} \\
&\subset \{a \in A \mid R_1(a) \subset F_1(x)\} \\
&\subset \{a \in A \mid R_1(a) \subset F_2(x)\} \\
&= \text{Rift}_{R_1}(F_2)
\end{aligned}$$

for each  $x \in X$ , so we therefore have  $\text{Rift}_{R_2}(F_1) \subset \text{Rift}_{R_1}(F_2)$ .

**Item 2, Interaction With Composition:** This holds in a general bicategory with the necessary right Kan lifts, being therefore a special case of ??.

**Item 3, Interaction With Converses:** This follows from **Item 3** of **Definition 8.5.17.1.4** by duality.

**Item 4, Interaction With Weak Inverse Images:** We proceed in a few steps.

- We have  $x \in \text{Rift}_R(F)^\dagger(a)$  iff  $a \in \text{Rift}_R(F)(x)$ .
- This holds iff  $R(a) \subset F(x)$ .
- This holds iff, for each  $b \in R(a)$ , we have  $b \in F(x)$ .
- This holds iff, for each  $b \in R(a)$ , we have  $x \in F^{-1}(b)$ .
- This holds iff  $x \in \bigcap_{b \in R(a)} F^{-1}(b)$ .

This finishes the proof. □

### 00MJ 8.5.19 Closedness

00MK **Proposition 8.5.19.1.1.** The 2-category **Rel** is a closed bicategory, there being, for each  $R: A \rightarrowtail B$  and set  $X$ , a pair of adjunctions

$$(R^* \dashv \text{Ran}_R): \text{Rel}(B, X) \begin{array}{c} \xrightarrow{R^*} \\ \perp \\ \xleftarrow{\text{Ran}_R} \end{array} \text{Rel}(A, X),$$

$$(R! \dashv \text{Rift}_R): \text{Rel}(X, A) \begin{array}{c} \xrightarrow{R!} \\ \perp \\ \xleftarrow{\text{Rift}_R} \end{array} \text{Rel}(X, B),$$

witnessed by bijections

$$\begin{aligned} \mathbf{Rel}(S \diamond R, T) &\cong \mathbf{Rel}(S, \text{Ran}_R(T)), \\ \mathbf{Rel}(R \diamond U, V) &\cong \mathbf{Rel}(U, \text{Rift}_R(V)), \end{aligned}$$

natural in  $S \in \text{Rel}(B, X)$ ,  $T \in \text{Rel}(A, X)$ ,  $U \in \text{Rel}(X, A)$ , and  $V \in \text{Rel}(X, B)$ .

*Proof.* This follows from **Constructions With Relations**, ????. □

### 00ML 8.5.20 Rel as a Category of Free Algebras

00MM **Proposition 8.5.20.1.1.** We have an isomorphism of categories

$$\mathbf{Rel} \cong \mathbf{FreeAlg}_{\mathcal{P}_!}(\mathbf{Sets}),$$

where  $\mathcal{P}_!$  is the powerset monad of ??, ??.

*Proof.* Omitted. □

## 02GB 8.6 Properties of the 2-Category of Relations With Apartness Composition

### 02GC 8.6.1 Self-Duality

02GD **Proposition 8.6.1.1.1.** The 2-/category of relations with apartness-composition- is self-dual:

02GE 1. *Self-Duality I.* We have an isomorphism

$$(\mathbf{Rel}^\square)^{\text{op}} \cong \mathbf{Rel}^\square$$

of categories.

02GF 2. *Self-Duality II*. We have a 2-isomorphism

$$(\mathbf{Rel}^\square)^{\mathrm{op}} \cong \mathbf{Rel}^\square$$

of 2-categories.

*Proof.* **Item 1, Self-Duality I**: We claim that the functor

$$(-)^\dagger: (\mathbf{Rel}^\square)^{\mathrm{op}} \rightarrow \mathbf{Rel}^\square$$

given by the identity on objects and by  $R \mapsto R^\dagger$  on morphisms is an isomorphism of categories. Note that this is indeed a functor by **Items 4 and 7** of **Definition 8.1.5.1.3**.

By **Categories, Item 1** of **Definition 11.6.8.1.3**, it suffices to show that  $(-)^{\dagger}$  is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^\dagger: \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(B, A)$$

defined by the assignment  $R \mapsto R^\dagger$  is a bijection by **Item 5** of **Definition 8.1.5.1.3**, showing  $(-)^{\dagger}$  to be fully faithful.

**Item 2, Self-Duality II**: We claim that the 2-functor

$$(-)^\dagger: \mathbf{Rel}^{\mathrm{op}} \rightarrow \mathbf{Rel}$$

given by the identity on objects, by  $R \mapsto R^\dagger$  on morphisms, and by preserving inclusions on 2-morphisms via **Item 1** of **Definition 8.1.5.1.3**, is an isomorphism of categories.

By ??, it suffices to show that  $(-)^{\dagger}$  is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in **Item 1**.
- Bijective on 2-morphisms, which follows from **Item 1** of **Definition 8.1.5.1.3**.

Thus  $(-)^{\dagger}$  is indeed a 2-isomorphism of categories.  $\square$

## 02GG 8.6.2 Isomorphisms and Equivalences

Let  $R: A \rightarrowtail B$  be a relation from  $A$  to  $B$ , and recall that  $R^c \stackrel{\mathrm{def}}{=} B \times A \setminus R$ .

02GH **Lemma 8.6.2.1.1**. The conditions below are row-wise equivalent:

CONDITION	INCLUSION
$R^c$ is functional	$\nabla_B \subset R \sqcap R^\dagger$
$R^c$ is total	$R \sqcap R^\dagger \subset \nabla_A$
$R^c$ is injective	$\nabla_A \subset R^\dagger \sqcap R$
$R^c$ is surjective	$R^\dagger \sqcap R \subset \nabla_B$

*Proof.* This follows from [Definition 8.5.2.1.1](#) and [Item 4](#) of [Definition 8.1.4.1.3](#). For instance:

- Suppose we have  $R \sqcap R^\dagger \subset \nabla_B$ .
- Taking complements, we obtain  $\nabla_B^c \subset (R \sqcap R^\dagger)^c$ .
- Applying [Item 4](#) of [Definition 8.1.4.1.3](#), this becomes  $\Delta_B \subset R^c \diamond (R^\dagger)^c$ .
- Then, by [Definition 8.5.2.1.1](#), this is equivalent to  $R^c$  being total.

The proof of the other equivalences is similar, and thus omitted.  $\square$

**02L2 Remark 8.6.2.1.2.** The statements in [Definition 8.6.2.1.1](#) unwind to the following:

INCLUSION	QUANTIFIER	CONDITION
$\nabla_B \subset R \sqcap R^\dagger$	For each $b_1, b_2 \in B$	If $b_1 \neq b_2$ , then, for each $a \in A$ , we have $a \sim_R b_1$ or $a \sim_R b_2$ .
$R \sqcap R^\dagger \subset \nabla_B$	For each $b_1, b_2 \in B$	If, for each $a \in A$ , $a \sim_R b_1$ or $a \sim_R b_2$ , then $b_1 \neq b_2$ .
$\nabla_A \subset R^\dagger \sqcap R$	For each $a_1, a_2 \in A$	If $a_1 \neq a_2$ , then, for each $b \in B$ , we have $a_1 \sim_R b$ or $a_2 \sim_R b$ .
$R^\dagger \sqcap R \subset \nabla_A$	For each $a_1, a_2 \in A$	If, for each $b \in B$ , $a_1 \sim_R b$ or $a_2 \sim_R b$ , then $a_1 \neq a_2$ .

Equivalently:

INCLUSION	QUANTIFIER	IF	THEN
$\nabla_B \subset R \sqcap R^\dagger$	For each $b_1, b_2 \in B$	$b_1 \neq b_2$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$
$R \sqcap R^\dagger \subset \nabla_B$	For each $b_1, b_2 \in B$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$	$b_1 \neq b_2$
$\nabla_A \subset R^\dagger \sqcap R$	For each $a_1, a_2 \in A$	$a_1 \neq a_2$	$R(a_1) \cup R(a_2) = B$
$R^\dagger \sqcap R \subset \nabla_A$	For each $a_1, a_2 \in A$	$R(a_1) \cup R(a_2) = B$	$a_1 \neq a_2$

**02GJ Proposition 8.6.2.1.3.** The following conditions are equivalent:

**02GK** 1. The relation  $R: A \rightarrowtail B$  is an equivalence in  $\mathbf{Rel}^\square$ , i.e.:

(★) There exists a relation  $R^{-1}: B \rightarrowtail A$  from  $B$  to  $A$  together with isomorphisms

$$R^{-1} \square R \cong \nabla_A,$$

$$R \square R^{-1} \cong \nabla_B.$$

**02GL** 2. The relation  $R: A \rightarrowtail B$  is an isomorphism in  $\mathbf{Rel}$ , i.e.:

(★) There exists a relation  $R^{-1}: B \rightarrowtail A$  from  $B$  to  $A$  such that we have

$$R^{-1} \square R = \nabla_A,$$

$$R \square R^{-1} = \nabla_B.$$

**02GM** 3. There exists a bijection  $f: B \xrightarrow{\sim} A$  with  $R^c = f^{-1}$ .

*Proof.* This follows from [Definition 8.5.2.1.2](#) and [Item 4](#) of [Definition 8.1.4.1.3](#).  $\square$

### **02GN 8.6.3 Internal Adjunctions**

Let  $A$  and  $B$  be sets.

**02L3 Proposition 8.6.3.1.1.** We have a natural bijection

$$\left\{ \begin{array}{c} \text{Adjunctions in } \mathbf{Rel}^\square \\ \text{from } A \text{ to } B \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Functions} \\ \text{from } B \text{ to } A \end{array} \right\},$$

with every adjunction in  $\mathbf{Rel}^\square$  being of the form  $(f^{-1})^c \dashv \text{Gr}(f)^c$  for some function  $f: B \rightarrow A$ .

*Proof.* This follows from [Definition 8.5.3.1.1](#) and [Item 4](#) of [Definition 8.1.4.1.3](#).  $\square$

### **02GP 8.6.4 Internal Monads**

Let  $X$  be a set.

**02L4 Proposition 8.6.4.1.1.** We have a natural identification

$$\left\{ \begin{array}{c} \text{Monads in} \\ \mathbf{Rel}^\square \text{ on } X \end{array} \right\} \cong \{\text{Subsets of } X\}.$$

*Proof.* This follows from [Definition 8.6.4.1.1](#) and [Item 4](#) of [Definition 8.1.4.1.3](#).  $\square$



### 02GQ 8.6.5 Internal Comonads

Let  $X$  be a set.

02L5 **Proposition 8.6.5.1.1.** We have a natural identification

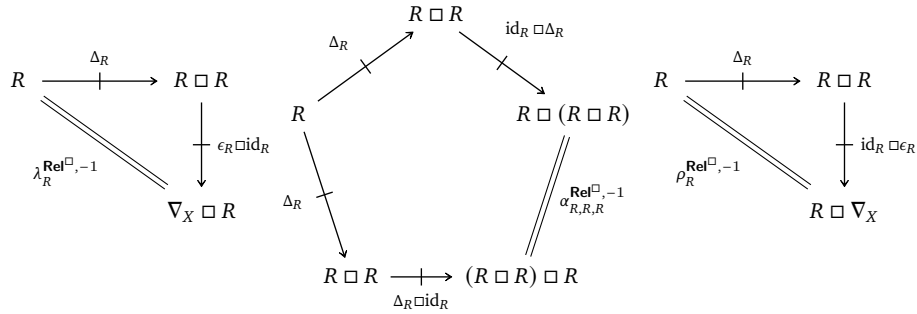
$$\left\{ \begin{array}{c} \text{Comonads in} \\ \mathbf{Rel}^\square \text{ on } X \end{array} \right\} \cong \{ \text{Strict total orders on } X \}.$$

*Proof.* A comonad in  $\mathbf{Rel}^\square$  on  $X$  consists of a relation  $R: X \rightarrowtail X$  together with maps

$$\Delta_R: R \subset R \sqcup R,$$

$$\epsilon_R: R \subset \nabla_X$$

making the diagrams



commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (**Categories**, **Item 4** of **Definition 11.2.7.1.2**), and hence all that is left is the data of the two maps  $\mu_R$  and  $\eta_R$ , which correspond respectively to the following conditions:

02L6 1. For each  $x, z \in X$ , if  $x \sim_R z$ , then, for each  $y \in X$ , we have  $x \sim_R y$  or  $y \sim_R z$ .

02L7 2. For each  $x, y \in X$ , if  $x \sim_R y$ , then  $x \neq y$ .

Replacing  $\sim_R$  with  $<_R$  and taking the contrapositive of each condition, we obtain:

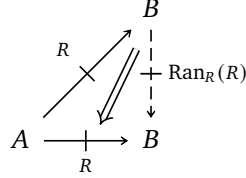
02L8 1. For each  $x, z \in X$ , if there exists some  $y \in X$  such that  $x <_R y$  and  $y <_R z$ , then  $x <_R z$ .

02L9 2. For each  $x \in X$ , we have  $x \not<_R x$ .

These are exactly the requirements for  $R$  to be a strict linear order (??). Conversely, any strict linear order  $<_R$  gives rise to a pair of maps  $\Delta_{<_R}$  and  $\epsilon_{<_R}$ , forming a comonad on  $X$ .  $\square$

02LA **Example 8.6.5.1.2.** Let  $R: A \rightarrowtail B$  be a relation.

02LB 1. The codensity monad  $\text{Ran}_R(R): B \rightarrowtail B$  is given by

$$[\text{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b)$$


for each  $b \in B$ . Thus, it corresponds to the preorder

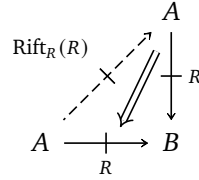
$$\preceq_{\text{Ran}_R(R)}: B \times B \rightarrow \{\text{t}, \text{f}\}$$

on  $B$  obtained by declaring  $b \preceq_{\text{Ran}_R(R)} b'$  iff the following equivalent conditions are satisfied:

02LC (a) For each  $a \in A$ , if  $a \sim_R b$ , then  $a \sim_R b'$ .

02LD (b) We have  $R^{-1}(b) \subset R^{-1}(b')$ .

02LE 2. The dual codensity monad  $\text{Rift}_R(R): A \rightarrowtail A$  is given by

$$[\text{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\}$$


for each  $a \in A$ . Thus, it corresponds to the preorder

$$\preceq_{\text{Rift}_R(R)}: A \times A \rightarrow \{\text{t}, \text{f}\}$$

on  $A$  obtained by declaring  $a \preceq_{\text{Rift}_R(R)} a'$  iff the following equivalent conditions are satisfied:

02LF (a) For each  $a \in A$ , if  $a \sim_R b$ , then  $a' \sim_R b$ .

02LG (b) We have  $R(a') \subset R(a)$ .

- 02LH 8.6.6 Modules Over Internal Monads
- 02LJ 8.6.7 Comodules Over Internal Comonads
- 02LK 8.6.8 Eilenberg–Moore and Kleisli Objects
- 02GS 8.6.9 Monomorphisms
- 02GT 8.6.10 2-Categorical Monomorphisms
- 02GU 8.6.11 Epimorphisms
- 02GV 8.6.12 2-Categorical Epimorphisms
- 02GW 8.6.13 Co/Limits

This will be expanded later on.

- 02LL 8.6.14 Internal Left Kan Extensions
- 02LM 8.6.15 Internal Left Kan Lifts
- 02LN 8.6.16 Internal Right Kan Extensions
- 02LP 8.6.17 Internal Right Kan Lifts
- 02LQ 8.6.18 Coclosedness
- 00R7 8.7 The Adjoint Pairs  $R_! \dashv R_{-1}$  and  $R^{-1} \dashv R_*$

### 00R8 8.7.1 Direct Images

Let  $X$  and  $Y$  be sets and let  $R: X \rightarrowtail Y$  be a relation.

- 00R9 **Definition 8.7.1.1.1.** The **direct image function** associated to  $R$  is the function<sup>22</sup>

$$R_!: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by<sup>23</sup>

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in Y \mid \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

<sup>22</sup>Further Notation: Also written simply  $R: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ .

<sup>23</sup>Further Terminology: The set  $R(U)$  is called the **direct image of  $U$  by  $R$** .

for each  $U \in \mathcal{P}(X)$ .

**02H0 Warning 8.7.1.1.2.** Notation for direct images between powersets is tricky; see [Constructions With Sets](#), [Definition 4.6.1.1.3](#). Here we'll try to align our notation for relations with that for functions.

**00RA Remark 8.7.1.1.3.** Identifying subsets of  $X$  with relations from  $\text{pt}$  to  $X$  via [Constructions With Sets](#), [Item 3](#) of [Definition 4.4.1.1.4](#), we see that the direct image function associated to  $R$  is equivalently the function

$$R_! : \underbrace{\mathcal{P}(X)}_{\cong \text{Rel}(\text{pt}, X)} \rightarrow \underbrace{\mathcal{P}(Y)}_{\cong \text{Rel}(\text{pt}, Y)}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} R \diamond U$$

for each  $U \in \mathcal{P}(X)$ , where  $R \diamond U$  is the composition

$$\text{pt} \xrightarrow{U} X \xrightarrow{R} Y.$$

**00RB Proposition 8.7.1.1.4.** Let  $R: X \rightarrowtail Y$  be a relation.

**00RC** 1. *Functoriality.* The assignment  $U \mapsto R_!(U)$  defines a functor

$$R_! : (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset)$$

where

- *Action on Objects.* For each  $U \in \mathcal{P}(X)$ , we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each  $U, V \in \mathcal{P}(X)$ :

- If  $U \subset V$ , then  $R_!(U) \subset R_!(V)$ .

**00RD** 2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1}) : \mathcal{P}(X) \begin{matrix} \xrightarrow{R_!} \\ \perp \\ \xleftarrow{R_{-1}} \end{matrix} \mathcal{P}(Y),$$

witnessed by:

02H1 (a) Units and counits of the form

$$\begin{aligned}\mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow R_{-1} \circ R_!, \\ R_! \circ R_{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)},\end{aligned}$$

having components of the form

$$\begin{aligned}U &\subset R_{-1}(R_!(U)), \\ R_!(R_{-1}(V)) &\subset V\end{aligned}$$

indexed by  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$

02H2 (b) A bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ . In particular:

(★) The following conditions are equivalent:

- We have  $R_!(U) \subset V$ .
- We have  $U \subset R_{-1}(V)$ .

00RE 3. *Preservation of Colimits.* We have an equality of sets

$$R_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_!(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have equalities

$$\begin{aligned}R_!(U) \cup R_!(V) &= R_!(U \cup V), \\ R_!(\emptyset) &= \emptyset,\end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

00RF 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_!(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have inclusions

$$\begin{aligned}R_!(U \cap V) &\subset R_!(U) \cap R_!(V), \\ R_!(X) &\subset Y,\end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

- 00RG** 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_!, R_!^\otimes, R_{*|\mathbb{1}}^\otimes): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^\otimes: R_!(U) \cup R_!(V) &\xrightarrow{=} R_!(U \cup V), \\ R_{*|\mathbb{1}}^\otimes: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

- 00RH** 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric oplax monoidal structure

$$(R_!, R_!^\otimes, R_{*|\mathbb{1}}^\otimes): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^\otimes: R_!(U \cap V) &\subset R_!(U) \cap R_!(V), \\ R_{*|\mathbb{1}}^\otimes: R_!(X) &\subset Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

- 00RJ** 7. *Relation to Codirect Images.* We have

$$R_!(U) = Y \setminus R_*(X \setminus U)$$

for each  $U \in \mathcal{P}(X)$ .

*Proof.* **Item 1**, *Functoriality*: Clear.

**Item 2**, *Adjointness*: This follows from Kan Extensions, ?? of ??.

**Item 3**, *Preservation of Colimits*: This follows from **Item 2** and ??, ?? of ??.

**Item 4**, *Oplax Preservation of Limits*: Omitted.

**Item 5**, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **Item 3**.

**Item 6**, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **Item 4**.

**Item 7**, *Relation to Codirect Images*: The proof proceeds in the same way as in the case of functions (**Constructions With Sets**, **Item 17** of **Definition 4.6.1.1.5**): applying **Item 7** of **Definition 8.7.4.1.3** to  $A \setminus U$ , we have

$$R_*(X \setminus U) = Y \setminus R_!(X \setminus (X \setminus U))$$

$$= Y \setminus R_!(U).$$

Taking complements, we then obtain

$$\begin{aligned} R_!(U) &= Y \setminus (Y \setminus R_!(U)), \\ &= Y \setminus R_*(X \setminus U), \end{aligned}$$

which finishes the proof.  $\square$

**00RK Proposition 8.7.1.1.5.** Let  $R: X \rightarrowtail Y$  be a relation.

**00RL** 1. *Functionality I.* The assignment  $R \mapsto R_!$  defines a function

$$(-)_!: \text{Rel}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

**00RM** 2. *Functionality II.* The assignment  $R \mapsto R_!$  defines a function

$$(-)_!: \text{Rel}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

**00RN** 3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have<sup>24</sup>

$$(\chi_X)_! = \text{id}_{\mathcal{P}(X)}.$$

**00RP** 4. *Interaction With Composition.* For each pair of composable relations  $R: X \rightarrowtail Y$  and  $S: Y \rightarrowtail C$ , we have<sup>25</sup>

$$(S \diamond R)_! = S_! \circ R_!, \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{R_!} & \mathcal{P}(Y) \\ & \searrow (S \diamond R)_! & \downarrow S_! \\ & & \mathcal{P}(C). \end{array}$$

<sup>24</sup>That is, the postcomposition function

$$(\chi_X)_!: \text{Rel}(\text{pt}, X) \rightarrow \text{Rel}(\text{pt}, X)$$

is equal to  $\text{id}_{\text{Rel}(\text{pt}, X)}$ .

<sup>25</sup>That is, we have

$$(S \diamond R)_! = S_! \circ R_!, \quad \begin{array}{ccc} \text{Rel}(\text{pt}, X) & \xrightarrow{R_!} & \text{Rel}(\text{pt}, Y) \\ & \searrow (S \diamond R)_! & \downarrow S_! \\ & & \text{Rel}(\text{pt}, C). \end{array}$$

*Proof.* **Item 1, Functionality I:** Clear.

**Item 2, Functionality II:** Clear.

**Item 3, Interaction With Identities:** Indeed, we have

$$\begin{aligned}
 (\chi_X)_!(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} \chi_X(a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} \{a\} \\
 &= U \\
 &\stackrel{\text{def}}{=} \text{id}_{\mathcal{P}(X)}(U)
 \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ . Thus  $(\chi_X)_! = \text{id}_{\mathcal{P}(X)}$ .

**Item 4, Interaction With Composition:** Indeed, we have

$$\begin{aligned}
 (S \diamond R)_!(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} [S \diamond R](a) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S(R(a)) \\
 &\stackrel{\text{def}}{=} \bigcup_{a \in U} S_!(R(a)) \\
 &= S_! \left( \bigcup_{a \in U} R(a) \right) \\
 &\stackrel{\text{def}}{=} S_!(R_!(U)) \\
 &\stackrel{\text{def}}{=} [S_! \circ R_!](U)
 \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ , where we used **Item 3** of **Definition 8.7.1.1.4**. Thus  $(S \diamond R)_! = S_! \circ R_!$ .  $\square$

## 00RQ 8.7.2 Strong Inverse Images

Let  $X$  and  $Y$  be sets and let  $R: X \rightarrowtail Y$  be a relation.

**00RR Definition 8.7.2.1.1.** The **strong inverse image function associated to  $R$**  is the function

$$R_{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by<sup>26</sup>

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in X \mid R(a) \subset V\}$$

for each  $V \in \mathcal{P}(Y)$ .

---

<sup>26</sup>*Further Terminology:* The set  $R_{-1}(V)$  is called the **strong inverse image of  $V$  by  $R$** .



**00RS Remark 8.7.2.1.2.** Identifying subsets of  $Y$  with relations from  $\text{pt}$  to  $Y$  via **Constructions With Sets, Item 3** of **Definition 4.4.1.1.4**, we see that the inverse image function associated to  $R$  is equivalently the function

$$R_{-1}: \underbrace{\mathcal{P}(Y)}_{\cong \text{Rel}(\text{pt}, Y)} \rightarrow \underbrace{\mathcal{P}(X)}_{\cong \text{Rel}(\text{pt}, X)}$$

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \text{Rift}_R(V),$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \text{Rift}_R(V) \\ &\cong \int_{b \in Y} \text{Hom}_{\{t, f\}}(R_{-1}^b, V_{-2}^b), \end{aligned}$$

where we have used ??.

*Proof.* We have

$$\begin{aligned} \text{Rift}_R(V) &\cong \int_{b \in Y} \text{Hom}_{\{t, f\}}(R_{-1}^b, V_{-2}^b) \\ &= \left\{ a \in X \mid \int_{b \in Y} \text{Hom}_{\{t, f\}}(R_a^b, V_{-2}^b) = \text{true} \right\} \\ &= \left\{ a \in X \mid \begin{array}{l} \text{for each } b \in Y, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } R_a^b = \text{true} \\ \quad (b) \text{ We have } V_{-2}^b = \text{true} \end{array} \right\} \\ &= \left\{ a \in X \mid \begin{array}{l} \text{for each } b \in Y, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } b \notin R(a) \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } b \in R(a) \\ \quad (b) \text{ We have } b \in V \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
&= \{a \in X \mid \text{for each } b \in R(a), \text{ we have } b \in V\} \\
&= \{a \in X \mid R(a) \subset V\} \\
&\stackrel{\text{def}}{=} R_{-1}(V).
\end{aligned}$$

This finishes the proof.  $\square$

**00RT Proposition 8.7.2.1.3.** Let  $R: X \rightarrowtail Y$  be a relation.

**00RU** 1. *Functoriality.* The assignment  $V \mapsto R_{-1}(V)$  defines a functor

$$R_{-1}: (\mathcal{P}(Y), \subset) \rightarrow (\mathcal{P}(X), \subset)$$

where

- *Action on Objects.* For each  $V \in \mathcal{P}(Y)$ , we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each  $U, V \in \mathcal{P}(Y)$ :
  - If  $U \subset V$ , then  $R_{-1}(U) \subset R_{-1}(V)$ .

**00RV** 2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1}): \mathcal{P}(X) \begin{matrix} \xrightarrow{R_!} \\ \perp \\ \xleftarrow{R_{-1}} \end{matrix} \mathcal{P}(Y),$$

witnessed by a bijections of sets

$$\text{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \text{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ , i.e. such that:

(★) The following conditions are equivalent:

- We have  $R_!(U) \subset V$ .
- We have  $U \subset R_{-1}(V)$ .

**00RW** 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$ . In particular, we have inclusions

$$\begin{aligned}
R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\
\emptyset &\subset R_{-1}(\emptyset),
\end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00RX 4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} R_{-1}(U \cap V) &= R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(Y) &= Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00RY 5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of **Item 1** has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|1}^{\otimes}\right): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{-1|U,V}^{\otimes}: R_{-1}(U) \cup R_{-1}(V) &\subset R_{-1}(U \cup V), \\ R_{-1|1}^{\otimes}: \emptyset &\subset R_{-1}(\emptyset), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00RZ 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|1}^{\otimes}\right): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} R_{-1|U,V}^{\otimes}: R_{-1}(U \cap V) &\xrightarrow{=} R_{-1}(U) \cap R_{-1}(V), \\ R_{-1|1}^{\otimes}: R_{-1}(X) &\xrightarrow{=} Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00S0 7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = X \setminus R^{-1}(Y \setminus V)$$

for each  $V \in \mathcal{P}(Y)$ .

00S1 8. *Interaction With Weak Inverse Images II.* Let  $R: X \rightarrowtail Y$  be a relation from  $X$  to  $Y$ .

00S2 (a) If  $R$  is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in  $V \in \mathcal{P}(Y)$ .

00S3 (b) If  $R$  is total and functional, then the above inclusion is in fact an equality.

00S4 (c) Conversely, if we have  $R_{-1} = R^{-1}$ , then  $R$  is total and functional.

*Proof.* **Item 1, Functoriality:** Clear.

**Item 2, Adjointness:** This follows from Kan Extensions, ?? of ??.

**Item 3, Lax Preservation of Colimits:** Omitted.

**Item 4, Preservation of Limits:** This follows from **Item 2** and ??, ?? of ??.

**Item 5, Symmetric Lax Monoidality With Respect to Unions:** This follows from **Item 3**.

**Item 6, Symmetric Strict Monoidality With Respect to Intersections:** This follows from **Item 4**.

**Item 7, Interaction With Weak Inverse Images I:** We claim we have an equality

$$R_{-1}(Y \setminus V) = X \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(Y \setminus V) &= \{a \in X \mid R(a) \subset Y \setminus V\}, \\ X \setminus R^{-1}(V) &= \{a \in X \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking  $V = Y \setminus V$  then implies the original statement.

**Item 8, Interaction With Weak Inverse Images II:** **Item 8a** is clear, while **Items 8b** and **8c** follow from **Item 6** of **Definition 8.2.2.1.2**.  $\square$

00S5 **Proposition 8.7.2.1.4.** Let  $R: X \rightarrowtail Y$  be a relation.

00S6 1. *Functionality I.* The assignment  $R \mapsto R_{-1}$  defines a function

$$(-)_{-1}: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

00S7 2. *Functionality II.* The assignment  $R \mapsto R_{-1}$  defines a function

$$(-)_{-1}: \text{Sets}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

00S8 3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have

$$(\text{id}_X)_{-1} = \text{id}_{\mathcal{P}(X)}.$$

00S9 4. *Interaction With Composition.* For each pair of composable relations  $R: X \rightarrowtail Y$  and  $S: Y \rightarrowtail C$ , we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S_{-1}} & \mathcal{P}(Y) \\ & \searrow (S \diamond R)_{-1} & \downarrow R_{-1} \\ & & \mathcal{P}(X). \end{array}$$

*Proof.* **Item 1, Functionality I:** Clear.

**Item 2, Functionality II:** Clear.

**Item 3, Interaction With Identities:** Indeed, we have

$$\begin{aligned} (\chi_X)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in X \mid \chi_X(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ . Thus  $(\chi_X)_{-1} = \text{id}_{\mathcal{P}(X)}$ .

**Item 4, Interaction With Composition:** Indeed, we have

$$\begin{aligned} (S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in X \mid [S \diamond R](a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in X \mid S(R(a)) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in X \mid S_!(R(a)) \subset U\} \\ &= \{a \in X \mid R(a) \subset S_{-1}(U)\} \\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U) \end{aligned}$$

for each  $U \in \mathcal{P}(C)$ , where we used **Item 2** of **Definition 8.7.2.1.3**, which implies that the conditions

- We have  $S_!(R(a)) \subset U$ .
- We have  $R(a) \subset S_{-1}(U)$ .

are equivalent. Thus  $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$ . □

### 00SA 8.7.3 Weak Inverse Images

Let  $X$  and  $Y$  be sets and let  $R: X \rightarrowtail Y$  be a relation.

00SB **Definition 8.7.3.1.1.** The **weak inverse image function associated to  $R$** <sup>27</sup> is the function

$$R^{-1}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

defined by<sup>28</sup>

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in X \mid R(a) \cap V \neq \emptyset\}$$

for each  $V \in \mathcal{P}(Y)$ .

00SC **Remark 8.7.3.1.2.** Identifying subsets of  $Y$  with relations from  $Y$  to  $\text{pt}$  via **Constructions With Sets, Item 3** of **Definition 4.4.1.1.4**, we see that the weak inverse image function associated to  $R$  is equivalently the function

$$R^{-1}: \underbrace{\mathcal{P}(Y)}_{\cong \text{Rel}(Y, \text{pt})} \rightarrow \underbrace{\mathcal{P}(X)}_{\cong \text{Rel}(X, \text{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

for each  $V \in \mathcal{P}(X)$ , where  $R \diamond V$  is the composition

$$X \xrightarrow{R} Y \xrightarrow{V} \text{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} V \diamond R \\ &\stackrel{\text{def}}{=} \int^{b \in Y} V_b^{-1} \times R_{-2}^b. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} V \diamond R &\stackrel{\text{def}}{=} \int^{b \in Y} V_b^{-1} \times R_{-2}^b \\ &= \left\{ a \in X \mid \int^{b \in Y} V_b^* \times R_a^b = \text{true} \right\} \end{aligned}$$

<sup>27</sup>Further Terminology: Also called simply the **inverse image function associated to  $R$** .

<sup>28</sup>Further Terminology: The set  $R^{-1}(V)$  is called the **weak inverse image of  $V$  by  $R$**  or simply the **inverse image of  $V$  by  $R$** .

$$\begin{aligned}
&= \left\{ a \in X \mid \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ 1. \text{ We have } V_b^* = \text{true} \\ 2. \text{ We have } R_a^b = \text{true} \end{array} \right\} \\
&= \left\{ a \in X \mid \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ 1. \text{ We have } b \in V \\ 2. \text{ We have } b \in R(a) \end{array} \right\} \\
&= \{a \in X \mid \text{there exists } b \in V \text{ such that } b \in R(a)\} \\
&= \{a \in X \mid R(a) \cap V \neq \emptyset\} \\
&\stackrel{\text{def}}{=} R^{-1}(V)
\end{aligned}$$

This finishes the proof. □

**00SD Proposition 8.7.3.1.3.** Let  $R: X \rightarrowtail Y$  be a relation.

**00SE** 1. *Functoriality.* The assignment  $V \mapsto R^{-1}(V)$  defines a functor

$$R^{-1}: (\mathcal{P}(Y), \subset) \rightarrow (\mathcal{P}(X), \subset)$$

where

- *Action on Objects.* For each  $V \in \mathcal{P}(Y)$ , we have

$$[R^{-1}](V) \stackrel{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each  $U, V \in \mathcal{P}(Y)$ :
  - If  $U \subset V$ , then  $R^{-1}(U) \subset R^{-1}(V)$ .

**00SF** 2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_*) : \mathcal{P}(Y) \begin{array}{c} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_*} \end{array} \mathcal{P}(X),$$

witnessed by a bijections of sets

$$\text{Hom}_{\mathcal{P}(X)}(R^{-1}(U), V) \cong \text{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ , i.e. such that:

(★) The following conditions are equivalent:

- We have  $R^{-1}(U) \subset V$ .
- We have  $U \subset R_*(V)$ .

00SG 3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00SH 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$ . In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(X) &\subset Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00SJ 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1, \otimes}, R_{\mathbb{1}}^{-1, \otimes}\right): (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{U, V}^{-1, \otimes}: R^{-1}(U) \cup R^{-1}(V) &\xrightarrow{=} R^{-1}(U \cup V), \\ R_{\mathbb{1}}^{-1, \otimes}: \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

00SK 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct



image function of **Item 1** has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1, \otimes}, R_{\mathbb{1}}^{-1, \otimes}\right): (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} R_{U,V}^{-1, \otimes} : R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R_{\mathbb{1}}^{-1, \otimes} : R^{-1}(X) &\subset Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(Y)$ .

**00SL** 7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = X \setminus R_{-1}(Y \setminus V)$$

for each  $V \in \mathcal{P}(Y)$ .

**00SM** 8. *Interaction With Strong Inverse Images II.* Let  $R: X \rightarrowtail Y$  be a relation from  $X$  to  $Y$ .

**00SN** (a) If  $R$  is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in  $V \in \mathcal{P}(Y)$ .

**00SP** (b) If  $R$  is total and functional, then the above inclusion is in fact an equality.

**00SQ** (c) Conversely, if we have  $R_{-1} = R^{-1}$ , then  $R$  is total and functional.

*Proof.* **Item 1**, *Functoriality*: Clear.

**Item 2**, *Adjointness*: This follows from Kan Extensions, ?? of ??.

**Item 3**, *Preservation of Colimits*: This follows from **Item 2** and ??, ?? of ??.

**Item 4**, *Oplax Preservation of Limits*: Omitted.

**Item 5**, *Symmetric Strict Monoidality With Respect to Unions*: This follows from **Item 3**.

**Item 6**, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from **Item 4**.

**Item 7**, *Interaction With Strong Inverse Images I*: This follows from **Item 7** of Definition 8.7.2.1.3.

**Item 8**, *Interaction With Strong Inverse Images II*: This was proved in **Item 8** of Definition 8.7.2.1.3.  $\square$

**00SR** **Proposition 8.7.3.1.4.** Let  $R: X \rightarrowtail Y$  be a relation.

- 00SS 1. *Functionality I.* The assignment  $R \mapsto R^{-1}$  defines a function

$$(-)^{-1}: \text{Rel}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

- 00ST 2. *Functionality II.* The assignment  $R \mapsto R^{-1}$  defines a function

$$(-)^{-1}: \text{Rel}(X, Y) \rightarrow \text{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

- 00SU 3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have<sup>29</sup>

$$(\chi_X)^{-1} = \text{id}_{\mathcal{P}(X)}.$$

- 00SV 4. *Interaction With Composition.* For each pair of composable relations  $R: X \rightarrowtail Y$  and  $S: Y \rightarrowtail C$ , we have<sup>30</sup>

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S^{-1}} & \mathcal{P}(Y) \\ & \searrow (S \diamond R)^{-1} & \downarrow R^{-1} \\ & & \mathcal{P}(X). \end{array}$$

*Proof.* **Item 1, Functionality I:** Clear.

**Item 2, Functionality II:** Clear.

**Item 3, Interaction With Identities:** This follows from **Categories, Item 5** of **Definition 11.1.4.1.2**.

**Item 4, Interaction With Composition:** This follows from **Categories, Item 2** of **Definition 11.1.4.1.2**.  $\square$

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<sup>29</sup>That is, the postcomposition

$$(\chi_X)^{-1}: \text{Rel}(\text{pt}, X) \rightarrow \text{Rel}(\text{pt}, X)$$

is equal to  $\text{id}_{\text{Rel}(\text{pt}, X)}$ .

<sup>30</sup>That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1},$$

$$\begin{array}{ccc} \text{Rel}(\text{pt}, C) & \xrightarrow{R^{-1}} & \text{Rel}(\text{pt}, Y) \\ & \searrow (S \diamond R)^{-1} & \downarrow S^{-1} \\ & & \text{Rel}(\text{pt}, X). \end{array}$$

### 00SW 8.7.4 Codirect Images

Let  $X$  and  $Y$  be sets and let  $R: X \rightarrowtail Y$  be a relation.

00SX **Definition 8.7.4.1.1.** The **codirect image function associated to  $R$**  is the function

$$R_*: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

defined by<sup>31,32</sup>

$$\begin{aligned} R_*(U) &\stackrel{\text{def}}{=} \left\{ b \in Y \mid \begin{array}{l} \text{for each } a \in X, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \{ b \in Y \mid R^{-1}(b) \subset U \} \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ .

00SY **Remark 8.7.4.1.2.** Identifying subsets of  $Y$  with relations from  $\text{pt}$  to  $Y$  via **Constructions With Sets, Item 3** of **Definition 4.4.1.1.4**, we see that the codirect image function associated to  $R$  is equivalently the function

$$R_*: \underbrace{\mathcal{P}(X)}_{\cong \text{Rel}(X, \text{pt})} \rightarrow \underbrace{\mathcal{P}(Y)}_{\cong \text{Rel}(Y, \text{pt})}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} \text{Ran}_R(U),$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \text{Ran}_R(U) \\ &\cong \int_{a \in X} \text{Hom}_{\{\text{t}, \text{f}\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used ??.

<sup>31</sup>*Further Terminology:* The set  $R_*(U)$  is called the **codirect image of  $U$  by  $R$** .

<sup>32</sup>We also have

$$R_*(U) = Y \setminus R_!(X \setminus U);$$

see **Item 7** of **Definition 8.7.4.1.3**.

*Proof.* We have

$$\begin{aligned}
 \text{Ran}_R(V) &\cong \int_{a \in X} \text{Hom}_{\{t,f\}}(R_a^{-2}, U_a^{-1}) \\
 &= \left\{ b \in Y \mid \int_{a \in X} \text{Hom}_{\{t,f\}}(R_a^b, U_a^\star) = \text{true} \right\} \\
 &= \left\{ b \in Y \mid \begin{array}{l} \text{for each } a \in X, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } R_a^b = \text{false} \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } R_a^b = \text{true} \\ \quad (b) \text{ We have } U_a^\star = \text{true} \end{array} \right\} \\
 &= \left\{ b \in Y \mid \begin{array}{l} \text{for each } a \in X, \text{ at least one of the} \\ \text{following conditions hold:} \\ \\ 1. \text{ We have } b \notin R(X) \\ 2. \text{ The following conditions hold:} \\ \\ \quad (a) \text{ We have } b \in R(a) \\ \quad (b) \text{ We have } a \in U \end{array} \right\} \\
 &= \left\{ b \in Y \mid \begin{array}{l} \text{for each } a \in X, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\
 &= \{ b \in Y \mid R^{-1}(b) \subset U \} \\
 &\stackrel{\text{def}}{=} R^{-1}(U).
 \end{aligned}$$

This finishes the proof. □

**00SZ Proposition 8.7.4.1.3.** Let  $R: X \rightarrowtail Y$  be a relation.

**00T0** 1. *Functoriality.* The assignment  $U \mapsto R_*(U)$  defines a functor

$$R_*: (\mathcal{P}(X), \subset) \rightarrow (\mathcal{P}(Y), \subset)$$

where

- *Action on Objects.* For each  $U \in \mathcal{P}(X)$ , we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each  $U, V \in \mathcal{P}(X)$ :

- If  $U \subset V$ , then  $R_*(U) \subset R_*(V)$ .

00T1 2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_*\right): \mathcal{P}(Y) \begin{matrix} \xrightarrow{R^{-1}} \\ \perp \\ \xleftarrow{R_*} \end{matrix} \mathcal{P}(X),$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in  $U \in \mathcal{P}(X)$  and  $V \in \mathcal{P}(Y)$ , i.e. such that:

(★) The following conditions are equivalent:

- We have  $R^{-1}(U) \subset V$ .
- We have  $U \subset R_*(V)$ .

00T2 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_*(U_i) \subset R_*\left(\bigcup_{i \in I} U_i\right),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have inclusions

$$\begin{aligned} R_*(U) \cup R_*(V) &\subset R_*(U \cup V), \\ \emptyset &\subset R_*(\emptyset), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

00T3 4. *Preservation of Limits.* We have an equality of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_*(U_i),$$

natural in  $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$ . In particular, we have equalities

$$\begin{aligned} R_*(U \cap V) &= R_*(U) \cap R_*(V), \\ R_*(X) &= Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

- 00T4 5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of **Item 1** has a symmetric lax monoidal structure

$$(R_*, R_*^\otimes, R_{!|\mathbb{1}}^\otimes) : (\mathcal{P}(X), \cup, \emptyset) \rightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{!|U,V}^\otimes : R_*(U) \cup R_*(V) &\subset R_*(U \cup V), \\ R_{!|\mathbb{1}}^\otimes : \emptyset &\subset R_*(\emptyset), \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

- 00T5 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of **Item 1** has a symmetric strict monoidal structure

$$(R_*, R_*^\otimes, R_{!|\mathbb{1}}^\otimes) : (\mathcal{P}(X), \cap, X) \rightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} R_{!|U,V}^\otimes : R_*(U \cap V) &\xrightarrow{=} R_*(U) \cap R_*(V), \\ R_{!|\mathbb{1}}^\otimes : R_*(X) &\xrightarrow{=} Y, \end{aligned}$$

natural in  $U, V \in \mathcal{P}(X)$ .

- 00T6 7. *Relation to Direct Images.* We have

$$R_*(U) = Y \setminus R_!(X \setminus U)$$

for each  $U \in \mathcal{P}(X)$ .

*Proof.* **Item 1**, *Functoriality*: Clear.

**Item 2**, *Adjointness*: This follows from Kan Extensions, ?? of ??.

**Item 3**, *Lax Preservation of Colimits*: Omitted.

**Item 4**, *Preservation of Limits*: This follows from **Item 2** and ??, ?? of ??.

**Item 5**, *Symmetric Lax Monoidality With Respect to Unions*: This follows from **Item 3**.

**Item 6**, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from **Item 4**.

**Item 7**, *Relation to Direct Images*: This follows from **Item 7** of **Definition 8.7.1.1.4**. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (**Constructions With Sets**, **Item 16** of **Definition 4.6.3.1.7**).

We claim that  $R_*(U) = Y \setminus R_!(X \setminus U)$ :

- *The First Implication.* We claim that

$$R_*(U) \subset Y \setminus R_!(X \setminus U).$$

Let  $b \in R_*(U)$ . We need to show that  $b \notin R_!(X \setminus U)$ , i.e. that there is no  $a \in X \setminus U$  such that  $b \in R(a)$ .

This is indeed the case, as otherwise we would have  $a \in R^{-1}(b)$  and  $a \notin U$ , contradicting  $R^{-1}(b) \subset U$  (which holds since  $b \in R_*(U)$ ).

Thus  $b \in Y \setminus R_!(X \setminus U)$ .

- *The Second Implication.* We claim that

$$Y \setminus R_!(X \setminus U) \subset R_*(U).$$

Let  $b \in Y \setminus R_!(X \setminus U)$ . We need to show that  $b \in R_*(U)$ , i.e. that  $R^{-1}(b) \subset U$ .

Since  $b \notin R_!(X \setminus U)$ , there exists no  $a \in X \setminus U$  such that  $b \in R(a)$ , and hence  $R^{-1}(b) \subset U$ .

Thus  $b \in R_*(U)$ .

This finishes the proof. □

**00T7 Proposition 8.7.4.1.4.** Let  $R: X \rightarrowtail Y$  be a relation.

**00T8** 1. *Functionality I.* The assignment  $R \mapsto R_*$  defines a function

$$(-)_*: \text{Sets}(X, Y) \rightarrow \text{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

**00T9** 2. *Functionality II.* The assignment  $R \mapsto R_*$  defines a function

$$(-)_*: \text{Sets}(X, Y) \rightarrow \text{Hom}_{\text{Pos}}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

**00TA** 3. *Interaction With Identities.* For each  $X \in \text{Obj}(\text{Sets})$ , we have

$$(\text{id}_X)_* = \text{id}_{\mathcal{P}(X)}.$$

**00TB** 4. *Interaction With Composition.* For each pair of composable relations  $R: X \rightarrowtail Y$  and  $S: Y \rightarrowtail C$ , we have

$$(S \diamond R)_* = S_* \circ R_*,$$

$$\begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{R_*} & \mathcal{P}(Y) \\ & \searrow (S \diamond R)_* & \downarrow S_* \\ & & \mathcal{P}(C). \end{array}$$

*Proof.* **Item 1**, *Functionality I*: Clear.

**Item 2**, *Functionality II*: Clear.

**Item 3**, *Interaction With Identities*: Indeed, we have

$$\begin{aligned} (\chi_X)_*(U) &\stackrel{\text{def}}{=} \{a \in X \mid \chi_X^{-1}(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each  $U \in \mathcal{P}(X)$ . Thus  $(\chi_X)_* = \text{id}_{\mathcal{P}(X)}$ .

**Item 4**, *Interaction With Composition*: Indeed, we have

$$\begin{aligned} (S \diamond R)_*(U) &\stackrel{\text{def}}{=} \{c \in C \mid [S \diamond R]^{-1}(c) \subset U\} \\ &\stackrel{\text{def}}{=} \{c \in C \mid S^{-1}(R^{-1}(c)) \subset U\} \\ &= \{c \in C \mid R^{-1}(c) \subset S_*(U)\} \\ &\stackrel{\text{def}}{=} R_*(S_*(U)) \\ &\stackrel{\text{def}}{=} [R_* \circ S_*](U) \end{aligned}$$

for each  $U \in \mathcal{P}(C)$ , where we used **Item 2** of **Definition 8.7.4.1.3**, which implies that the conditions

- We have  $S^{-1}(R^{-1}(c)) \subset U$ .
- We have  $R^{-1}(c) \subset S_*(U)$ .

are equivalent. Thus  $(S \diamond R)_* = S_* \circ R_*$ . □

## 00TC 8.7.5 Functoriality of Powersets

**00TD Proposition 8.7.5.1.1.** The assignment  $X \mapsto \mathcal{P}(X)$  defines functors<sup>33</sup>

$$\begin{aligned} \mathcal{P}_! &: \text{Rel} \rightarrow \text{Sets}, \\ \mathcal{P}_{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}^{-1} &: \text{Rel}^{\text{op}} \rightarrow \text{Sets}, \\ \mathcal{P}_* &: \text{Rel} \rightarrow \text{Sets} \end{aligned}$$

where

- *Action on Objects.* For each  $X \in \text{Obj}(\text{Rel})$ , we have

$$\mathcal{P}_!(X) \stackrel{\text{def}}{=} \mathcal{P}(X),$$

<sup>33</sup>The functor  $\mathcal{P}_! : \text{Rel} \rightarrow \text{Sets}$  admits a left adjoint; see **Item 2** of **Definition 8.2.2.1.2**.



$$\begin{aligned}\mathcal{P}_{-1}(X) &\stackrel{\text{def}}{=} \mathcal{P}(X), \\ \mathcal{P}^{-1}(X) &\stackrel{\text{def}}{=} \mathcal{P}(X), \\ \mathcal{P}_*(X) &\stackrel{\text{def}}{=} \mathcal{P}(X).\end{aligned}$$

- *Action on Morphisms.* For each morphism  $R: X \rightarrowtail Y$  of  $\mathbf{Rel}$ , the images

$$\begin{aligned}\mathcal{P}_!(R) &: \mathcal{P}(X) \rightarrow \mathcal{P}(Y), \\ \mathcal{P}_{-1}(R) &: \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \\ \mathcal{P}^{-1}(R) &: \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \\ \mathcal{P}_*(R) &: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)\end{aligned}$$

of  $R$  by  $\mathcal{P}_!$ ,  $\mathcal{P}_{-1}$ ,  $\mathcal{P}^{-1}$ , and  $\mathcal{P}_*$  are defined by

$$\begin{aligned}\mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!, \\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1}, \\ \mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*,\end{aligned}$$

as in [Definitions 8.7.1.1.1](#), [8.7.2.1.1](#), [8.7.3.1.1](#) and [8.7.4.1.1](#).

*Proof.* This follows from [Items 3 and 4 of Definition 8.7.1.1.5](#), [Items 3 and 4 of Definition 8.7.2.1.4](#), [Items 3 and 4 of Definition 8.7.3.1.4](#), and [Items 3 and 4 of Definition 8.7.4.1.4](#).  $\square$

## 00TE 8.7.6 Functoriality of Powersets: Relations on Powersets

Let  $X$  and  $Y$  be sets and let  $R: X \rightarrowtail Y$  be a relation.

00TF **Definition 8.7.6.1.1.** The relation on powersets associated to  $R$  is the relation

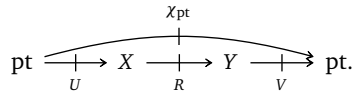
$$\mathcal{P}(R): \mathcal{P}(X) \rightarrowtail \mathcal{P}(Y)$$

defined by<sup>34</sup>

$$\mathcal{P}(R)_U^V \stackrel{\text{def}}{=} \mathbf{Rel}(\chi_{\text{pt}}, V \diamond R \diamond U)$$

for each  $U \in \mathcal{P}(X)$  and each  $V \in \mathcal{P}(Y)$ .

<sup>34</sup>Illustration:



**00TG Remark 8.7.6.1.2.** In detail, we have  $U \sim_{\mathcal{P}(R)} V$  iff the following equivalent conditions hold:

- We have  $\chi_{\text{pt}} \subset V \diamond R \diamond U$ .
- We have  $(V \diamond R \diamond U)_{\star}^{\star} = \text{true}$ , i.e. we have

$$\int^{a \in X} \int^{b \in Y} V_b^{\star} \times R_a^b \times U_{\star}^a = \text{true}.$$

- There exists some  $a \in X$  and some  $b \in Y$  such that:
  - We have  $U_{\star}^a = \text{true}$ .
  - We have  $R_a^b = \text{true}$ .
  - We have  $V_b^{\star} = \text{true}$ .
- There exists some  $a \in X$  and some  $b \in Y$  such that:
  - We have  $a \in U$ .
  - We have  $a \sim_R b$ .
  - We have  $b \in V$ .

**00TH Proposition 8.7.6.1.3.** The assignment  $R \mapsto \mathcal{P}(R)$  defines a functor

$$\mathcal{P}: \text{Rel} \rightarrow \text{Rel}.$$

*Proof.* Omitted. □

## 00MN 8.8 The Left Skew Monoidal Structure on $\text{Rel}(A, B)$

### 00MP 8.8.1 The Left Skew Monoidal Product

Let  $A$  and  $B$  be sets and let  $J: A \rightarrowtail B$  be a relation.

**00MQ Definition 8.8.1.1.1.** The **left  $J$ -skew monoidal product of  $\text{Rel}(A, B)$**  is the functor

$$\triangleleft_J: \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each  $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$ , we have

$$S \triangleleft_J R \stackrel{\text{def}}{=} S \diamond \text{Rift}_J(R),$$

- *Action on Morphisms.* For each  $R, S, R', S' \in \text{Obj}(\mathbf{Rel}(A, B))$ , the action on Hom-sets

$$(\triangleleft_J)_{(G,F),(G',F')} : \text{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \text{Hom}_{\mathbf{Rel}(A,B)}(R, R') \rightarrow \text{Hom}_{\mathbf{Rel}(A,B)}(S \triangleleft_J R, S' \triangleleft_J R')$$

of  $\triangleleft_J$  at  $((R, S), (R', S'))$  is defined by<sup>35</sup>

$$\beta \triangleleft_J \alpha \stackrel{\text{def}}{=} \beta \diamond \text{Rift}_J(\alpha),$$

for each  $\beta \in \text{Hom}_{\mathbf{Rel}(A,B)}(S, S')$  and each  $\alpha \in \text{Hom}_{\mathbf{Rel}(A,B)}(R, R')$ .

## 00MR 8.8.2 The Left Skew Monoidal Unit

Let  $A$  and  $B$  be sets and let  $J: A \rightarrow B$  be a relation.

00MS **Definition 8.8.2.1.1.** The left  $J$ -skew monoidal unit of  $\mathbf{Rel}(A, B)$  is the functor

$$\mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A,B)} : \text{pt} \rightarrow \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}_{\mathbf{Rel}(A,B)}^{\triangleleft_J} \stackrel{\text{def}}{=} J$$

of  $\mathbf{Rel}(A, B)$ .

<sup>35</sup>Since  $\mathbf{Rel}(A, B)$  is posetal, this is to say that if  $S \subset S'$  and  $R \subset R'$ , then  $S \triangleleft_J R \subset S' \triangleleft_J R'$ .

### 00MT 8.8.3 The Left Skew Associators

Let  $A$  and  $B$  be sets and let  $J: A \dashv B$  be a relation.

00MU **Definition 8.8.3.1.1.** The **left  $J$ -skew associator** of  $\mathbf{Rel}(A, B)$  is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B), \triangleleft_J} : \triangleleft_J \circ (\triangleleft_J \times \text{id}) \Longrightarrow \triangleleft_J \circ (\text{id} \times \triangleleft_J) \circ \alpha_{\mathbf{Rel}(A,B), \mathbf{Rel}(A,B), \mathbf{Rel}(A,B)}^{\text{Cats}},$$

as in the diagram

$$\begin{array}{ccc}
 & \mathbf{Rel}(A, B) \times (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) & \\
 \alpha_{\mathbf{Rel}(A,B), \mathbf{Rel}(A,B), \mathbf{Rel}(A,B)}^{\text{Cats}} \nearrow & & \searrow \text{id} \times \triangleleft_J \\
 (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) \times \mathbf{Rel}(A, B) & & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \\
 \triangleleft_J \times \text{id} \searrow & \alpha_{\mathbf{Rel}(A,B), \triangleleft_J} \nearrow & \searrow \triangleleft_J \\
 \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) & \xrightarrow[\triangleleft_J]{} & \mathbf{Rel}(A, B)
 \end{array}$$

whose component

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B), \triangleleft_J} : \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\stackrel{\text{def}}{=} T \diamond \text{Rift}_J(S) \diamond \text{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\stackrel{\text{def}}{=} T \diamond \text{Rift}_J(S \diamond \text{Rift}_J(R))}$$

at  $(T, S, R)$  is given by

$$\alpha_{T,S,R}^{\mathbf{Rel}(A,B), \triangleleft_J} \stackrel{\text{def}}{=} \text{id}_T \diamond \gamma,$$

where

$$\gamma : \text{Rift}_J(S) \diamond \text{Rift}_J(R) \hookrightarrow \text{Rift}_J(S \diamond \text{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \text{id}_{\text{Rift}_J(R)} : \underbrace{J \diamond \text{Rift}_J(S) \diamond \text{Rift}_J(R)}_{\stackrel{\text{def}}{=} J_! (\text{Rift}_J(S) \diamond \text{Rift}_J(R))} \hookrightarrow S \diamond \text{Rift}_J(R)$$

under the adjunction  $J_! \dashv \text{Rift}_J$ , where  $\epsilon : J \diamond \text{Rift}_J \Longrightarrow \text{id}_{\mathbf{Rel}(A,B)}$  is the counit of the adjunction  $J_! \dashv \text{Rift}_J$ .

### 8.8.4 The Left Skew Left Unitors

Let  $A$  and  $B$  be sets and let  $J: A \rightarrowtail B$  be a relation.

**Definition 8.8.4.1.1.** The left  $J$ -skew left unitor of  $\mathbf{Rel}(A, B)$  is the natural transformation

$$\lambda^{\mathbf{Rel}(A, B), \triangleleft_J} : \triangleleft_J \circ \left( \mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A, B)} \times \mathrm{id} \right) \Longrightarrow \lambda_{\mathbf{Rel}(A, B)}^{\mathbf{Cats}_2}$$

as in the diagram

$$\begin{array}{ccc}
 \mathrm{pt} \times \mathbf{Rel}(A, B) & \xrightarrow{\mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A, B)} \times \mathrm{id}} & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \\
 & \searrow \lambda^{\mathbf{Rel}(A, B), \triangleleft_J} & \downarrow \triangleleft_J \\
 & \lambda_{\mathbf{Rel}(A, B)}^{\mathbf{Cats}_2} & \mathbf{Rel}(A, B),
 \end{array}$$

whose component

$$\lambda_R^{\mathbf{Rel}(A, B), \triangleleft_J} : \underbrace{J \triangleleft_J R}_{\stackrel{\mathrm{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at  $R$  is given by

$$\lambda_R^{\mathbf{Rel}(A, B), \triangleleft_J} \stackrel{\mathrm{def}}{=} \epsilon_R,$$

where  $\epsilon : J! \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A, B)}$  is the counit of the adjunction  $J! \dashv \mathrm{Rift}_J$ .

### 8.8.5 The Left Skew Right Unitors

Let  $A$  and  $B$  be sets and let  $J: A \rightarrowtail B$  be a relation.

**Definition 8.8.5.1.1.** The left  $J$ -skew right unitor of  $\mathbf{Rel}(A, B)$  is the natural transformation

$$\rho^{\mathbf{Rel}(A, B), \triangleleft_J} : \rho_{\mathbf{Rel}(A, B)}^{\mathbf{Cats}_2} \Longrightarrow \triangleleft_J \circ \left( \mathrm{id} \times \mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A, B)} \right)$$

as in the diagram

$$\begin{array}{ccc}
 \mathbf{Rel}(A, B) \times \mathbf{pt} & \xrightarrow{\mathrm{id} \times \mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A, B)}} & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B), \\
 & \searrow \rho_{\mathbf{Rel}(A, B)}^{\mathbf{Cats}_2} & \nearrow \rho_{\mathbf{Rel}(A, B), \triangleleft_J}^{\mathbf{Rel}(A, B)} \\
 & & \downarrow \triangleleft_J \\
 & & \mathbf{Rel}(A, B)
 \end{array}$$

whose component

$$\rho_R^{\mathbf{Rel}(A, B), \triangleleft_J} : R \hookrightarrow \underbrace{R \triangleleft_J J}_{\stackrel{\text{def}}{=} R \diamond \mathbf{Rift}_J(J)}$$

at  $R$  is given by the composition

$$\begin{aligned}
 R &\xRightarrow{\sim} R \diamond \chi_A \\
 &\xRightarrow{\mathrm{id}_R \diamond \eta_{\chi_A}} R \diamond \mathbf{Rift}_J(J_!(\chi_A)) \\
 &\stackrel{\text{def}}{=} R \diamond \mathbf{Rift}_J(J \diamond \chi_A) \\
 &\xRightarrow{\sim} R \diamond \mathbf{Rift}_J(J) \\
 &\stackrel{\text{def}}{=} R \triangleleft_J J,
 \end{aligned}$$

where  $\eta: \mathrm{id}_{\mathbf{Rel}(A, A)} \Rightarrow \mathbf{Rift}_J \circ J_!$  is the unit of the adjunction  $J_! \dashv \mathbf{Rift}_J$ .

### 00MZ 8.8.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

00N0 **Proposition 8.8.6.1.1.** The category  $\mathbf{Rel}(A, B)$  admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset  $\mathbf{Rel}(A, B)$  of relations from  $A$  to  $B$  of ?? of ??.
- *The Left Skew Monoidal Product.* The left  $J$ -skew monoidal product

$$\triangleleft_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of **Definition 8.8.1.1.1.**

- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A, B), \triangleleft_J} : \mathbf{pt} \rightarrow \mathbf{Rel}(A, B)$$

of **Definition 8.8.2.1.1.**

- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B), \triangleleft_J} : \triangleleft_J \circ (\triangleleft_J \times \text{id}) \Rightarrow \triangleleft_J \circ (\text{id} \times \triangleleft_J) \circ \alpha_{\mathbf{Rel}(A,B), \mathbf{Rel}(A,B), \mathbf{Rel}(A,B)}^{\text{Cats}}$$

of **Definition 8.8.3.1.1**.

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B), \triangleleft_J} : \triangleleft_J \circ (\mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A,B)} \times \text{id}) \Rightarrow \lambda_{\mathbf{Rel}(A,B)}^{\text{Cats}_2}$$

of **Definition 8.8.4.1.1**.

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B), \triangleleft_J} : \rho_{\mathbf{Rel}(A,B)}^{\text{Cats}_2} \Rightarrow \triangleleft_J \circ (\text{id} \times \mathbb{1}_{\triangleleft_J}^{\mathbf{Rel}(A,B)})$$

of **Definition 8.8.5.1.1**.

*Proof.* Since  $\mathbf{Rel}(A, B)$  is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic (**Categories, Item 4** of **Definition 11.2.7.1.2**), and thus  $\mathbf{Rel}(A, B)$  together with the data in the statement forms a left skew monoidal category.  $\square$

## 00N1 8.9 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let  $A$  and  $B$  be sets and let  $J: A \rightarrow B$  be a relation.

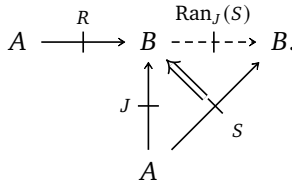
### 00N2 8.9.1 The Right Skew Monoidal Product

00N3 **Definition 8.9.1.1.1.** The right  $J$ -skew monoidal product of  $\mathbf{Rel}(A, B)$  is the functor

$$\triangleright_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each  $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$ , we have

$$S \triangleright_J R \stackrel{\text{def}}{=} \text{Ran}_J(S) \diamond R,$$


- *Action on Morphisms.* For each  $R, S, R', S' \in \text{Obj}(\mathbf{Rel}(A, B))$ , the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} : \text{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \text{Hom}_{\mathbf{Rel}(A,B)}(R, R') \rightarrow \text{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of  $\triangleright_J$  at  $((S, R), (S', R'))$  is defined by<sup>36</sup>

$$\beta \triangleright_J \alpha \stackrel{\text{def}}{=} \text{Ran}_J(\beta) \diamond \alpha,$$

for each  $\beta \in \text{Hom}_{\mathbf{Rel}(A,B)}(S, S')$  and each  $\alpha \in \text{Hom}_{\mathbf{Rel}(A,B)}(R, R')$ .

#### 00N4 8.9.2 The Right Skew Monoidal Unit

00N5 **Definition 8.9.2.1.1.** The right  $J$ -skew monoidal unit of  $\mathbf{Rel}(A, B)$  is the functor

$$\mathbb{1}_{\triangleright_J}^{\mathbf{Rel}(A,B)} : \text{pt} \rightarrow \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}_{\mathbf{Rel}(A,B)}^{\triangleright_J} \stackrel{\text{def}}{=} J$$

of  $\mathbf{Rel}(A, B)$ .

#### 00N6 8.9.3 The Right Skew Associators

00N7 **Definition 8.9.3.1.1.** The right  $J$ -skew associator of  $\mathbf{Rel}(A, B)$  is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B), \triangleright_J} : \triangleright_J \circ (\text{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \text{id}) \circ \alpha_{\mathbf{Rel}(A,B), \mathbf{Rel}(A,B), \mathbf{Rel}(A,B)}^{\text{Cats}, -1}$$

<sup>36</sup>Since  $\mathbf{Rel}(A, B)$  is posetal, this is to say that if  $S \subset S'$  and  $R \subset R'$ , then  $S \triangleright_J R \subset S' \triangleright_J R'$ .



as in the diagram

$$\begin{array}{ccc}
 & (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) \times \mathbf{Rel}(A, B) & \\
 \alpha_{\mathbf{Rel}(A, B), \mathbf{Rel}(A, B), \mathbf{Rel}(A, B)}^{\mathbf{Cats}, -1} \nearrow & & \searrow \triangleright_J \times \text{id} \\
 \mathbf{Rel}(A, B) \times (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) & & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \\
 \downarrow \text{id} \times \triangleright_J & \nearrow \alpha_{\mathbf{Rel}(A, B), \triangleright_J} & \downarrow \triangleright_J \\
 \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) & \xrightarrow[\triangleright_J]{} & \mathbf{Rel}(A, B)
 \end{array}$$

whose component

$$\alpha_{T, S, R}^{\mathbf{Rel}(A, B), \triangleright_J} : \underbrace{T \triangleright_J (S \triangleright_J R)}_{\stackrel{\text{def}}{=} \text{Ran}_J(T) \diamond \text{Ran}_J(S) \diamond R} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\stackrel{\text{def}}{=} \text{Ran}_J(\text{Ran}_J(T) \diamond S) \diamond R}$$

at  $(T, S, R)$  is given by

$$\alpha_{T, S, R}^{\mathbf{Rel}(A, B), \triangleright_J} \stackrel{\text{def}}{=} \gamma \diamond \text{id}_R,$$

where

$$\gamma : \text{Ran}_J(T) \diamond \text{Ran}_J(S) \hookrightarrow \text{Ran}_J(\text{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\text{id}_{\text{Ran}_J(T)} \diamond \epsilon_S : \underbrace{\text{Ran}_J(T) \diamond \text{Ran}_J(S) \diamond J}_{\stackrel{\text{def}}{=} J^*(\text{Ran}_J(T) \diamond \text{Ran}_J(S))} \hookrightarrow \text{Ran}_J(T) \diamond S$$

under the adjunction  $J^* \dashv \text{Ran}_J$ , where  $\epsilon : \text{Ran}_J \diamond J \Rightarrow \text{id}_{\mathbf{Rel}(A, B)}$  is the counit of the adjunction  $J^* \dashv \text{Ran}_J$ .

## 8.9.4 The Right Skew Left Unitors

**Definition 8.9.4.1.1.** The right  $J$ -skew left unitor of  $\mathbf{Rel}(A, B)$  is the natural transformation

$$\lambda^{\mathbf{Rel}(A, B), \triangleright_J} : \lambda_{\mathbf{Rel}(A, B)}^{\mathbf{Cats}_2} \Rightarrow \triangleright_J \circ \left( \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A, B)} \times \text{id} \right),$$

as in the diagram

$$\begin{array}{ccc}
 \text{pt} \times \mathbf{Rel}(A, B) & \xrightarrow{\mathbb{1}_{\triangleright_J}^{\mathbf{Rel}(A, B)} \times \text{id}} & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \\
 & \searrow \lambda_{\mathbf{Rel}(A, B)}^{\text{Cats}_2} & \nearrow \lambda_{\mathbf{Rel}(A, B), \triangleright_J}^{\mathbf{Rel}(A, B)} \\
 & & \downarrow \triangleright_J \\
 & & \mathbf{Rel}(A, B),
 \end{array}$$

whose component

$$\lambda_R^{\mathbf{Rel}(A, B), \triangleright_J} : R \hookrightarrow \underbrace{J \triangleright_J R}_{\stackrel{\text{def}}{=} \text{Ran}_J(J) \diamond R}$$

at  $R$  is given by the composition

$$\begin{aligned}
 R &\xrightarrow{\sim} \chi_B \diamond R \\
 &\xrightarrow{\eta_{\chi_B}} \diamond \text{id}_{\text{Ran}_J(J^*(\chi_A))} \diamond R \\
 &\stackrel{\text{def}}{=} \text{Ran}_J(J^* \diamond \chi_A) \diamond R \\
 &\xrightarrow{\sim} \text{Ran}_J(J) \diamond R \\
 &\stackrel{\text{def}}{=} R \triangleright_J J,
 \end{aligned}$$

where  $\eta : \text{id}_{\mathbf{Rel}(B, B)} \Rightarrow \text{Ran}_J \circ J^*$  is the unit of the adjunction  $J^* \dashv \text{Ran}_J$ .

## 00NA 8.9.5 The Right Skew Right Unitors

00NB **Definition 8.9.5.1.1.** The **right  $J$ -skew right unitor of  $\mathbf{Rel}(A, B)$**  is the natural transformation

$$\rho^{\mathbf{Rel}(A, B), \triangleright_J} : \triangleright_J \circ (\text{id} \times \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A, B)}) \Rightarrow \rho_{\mathbf{Rel}(A, B)}^{\text{Cats}_2},$$

as in the diagram

$$\begin{array}{ccc}
 \mathbf{Rel}(A, B) \times \text{pt} & \xrightarrow{\text{id} \times \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A, B)}} & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B), \\
 & \searrow \rho_{\mathbf{Rel}(A, B), \triangleright_J}^{\mathbf{Rel}(A, B)} & \nearrow \rho_{\mathbf{Rel}(A, B), \triangleright_J}^{\mathbf{Rel}(A, B)} \\
 & & \downarrow \triangleright_J \\
 & & \mathbf{Rel}(A, B)
 \end{array}$$

whose component

$$\rho_S^{\mathbf{Rel}(A,B), \triangleright_J} : \underbrace{S \triangleright_J J}_{\stackrel{\text{def}}{=} \text{Ran}_J(S) \circ J} \hookrightarrow S$$

at  $S$  is given by

$$\rho_S^{\mathbf{Rel}(A,B), \triangleright_J} \stackrel{\text{def}}{=} \epsilon_R,$$

where  $\epsilon : J^* \circ \text{Ran}_J \Rightarrow \text{id}_{\mathbf{Rel}(A,B)}$  is the counit of the adjunction  $J^* \dashv \text{Ran}_J$ .

### 00NC 8.9.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

00ND **Proposition 8.9.6.1.1.** The category  $\mathbf{Rel}(A, B)$  admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset  $\mathbf{Rel}(A, B)$  of relations from  $A$  to  $B$  of ?? of ??.
- *The Right Skew Monoidal Product.* The right  $J$ -skew monoidal product

$$\triangleleft_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \rightarrow \mathbf{Rel}(A, B)$$

of **Definition 8.9.1.1.1.**

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B), \triangleleft_J} : \text{pt} \rightarrow \mathbf{Rel}(A, B)$$

of **Definition 8.9.2.1.1.**

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B), \triangleright_J} : \triangleright_J \circ (\text{id} \times \triangleright_J) \Rightarrow \triangleright_J \circ (\triangleright_J \times \text{id}) \circ \alpha_{\mathbf{Rel}(A,B), \mathbf{Rel}(A,B), \mathbf{Rel}(A,B)}^{\text{Cats}, -1}$$

of **Definition 8.9.3.1.1.**

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B), \triangleright_J} : \lambda_{\mathbf{Rel}(A,B)}^{\text{Cats}_2} \Rightarrow \triangleright_J \circ \left( \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)} \times \text{id} \right)$$

of **Definition 8.9.4.1.1.**

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B), \triangleright_J} : \triangleright_J \circ \left( \text{id} \times \mathbb{1}_{\triangleright}^{\mathbf{Rel}(A,B)} \right) \Rightarrow \rho_{\mathbf{Rel}(A,B)}^{\text{Cats}_2}$$

of **Definition 8.9.5.1.1.**

*Proof.* Since  $\mathbf{Rel}(A, B)$  is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic (**Categories, Item 4** of **Definition 11.2.7.1.2**), and thus  $\mathbf{Rel}(A, B)$  together with the data in the statement forms a right skew monoidal category.  $\square$

## Appendices

### A Other Chapters

#### Preliminaries

1. **Introduction**
2. **A Guide to the Literature**

#### Sets

3. **Sets**
4. **Constructions With Sets**
5. **Monoidal Structures on the Category of Sets**
6. **Pointed Sets**
7. **Tensor Products of Pointed Sets**

#### Relations

8. **Relations**
9. **Constructions With Relations**

#### 10. **Conditions on Relations**

#### Categories

11. **Categories**
12. **Presheaves and the Yoneda Lemma**

#### Monoidal Categories

13. **Constructions With Monoidal Categories**

#### Bicategories

14. **Types of Morphisms in Bicategories**

#### Extra Part

15. **Notes**

### References

- [MO 350788] **Peter LeFanu Lumsdaine**. *Epimorphisms of relations*. MathOverflow. URL: <https://mathoverflow.net/q/455260> (cit. on p. 74).
- [MO 460693] **Tim Champion**. *Answer to “Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I”*. Math-

- Overflow. URL: <https://mathoverflow.net/q/460693> (cit. on p. 80).
- [MO 461592] **Emily**. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. MathOverflow. URL: <https://mathoverflow.net/q/461592> (cit. on pp. 81, 83).
- [MO 467527] **Emily**. *What are the 2-categorical mono/epimorphisms in the 2-category of relations?* MathOverflow. URL: <https://mathoverflow.net/q/467527> (cit. on pp. 72, 79).
- [MSE 350788] **Qiaochu Yuan**. *Mono's and epi's in the category Rel?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/350788> (cit. on pp. 67, 74).
- [Wik25] Wikipedia Contributors. *Multivalued Function* — *Wikipedia, The Free Encyclopedia*. 2025. URL: [https://en.wikipedia.org/wiki/Multivalued\\_function](https://en.wikipedia.org/wiki/Multivalued_function) (cit. on p. 31).