# Presheaves and the Yoneda Lemma

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This chapter contains some material about presheaves and the Yoneda lemma. This chapter is under revision. TODO:

1. Subsection properties of categories of copresheaves
2. Adjointness of tensor product of functors
3. Limit of category of elements (instead of colimit)
4. Category of elements where objects are natural transformations $\mathcal{F} \Rightarrow h_X$ instead of the other way around. Is this related to Isbell duality?
5. Motivate the proof of the Yoneda lemma as in Martin's comment here: https://mathoverflow.net/questions/130883/are-there-proofs-that-you-feel-you-did-not-understand-for-a-long-time#comment360113_131050
6. Add discussion of universal properties
7. Add $h_{g \circ f} = h_g \circ h_f$ to properties of representable natural transformations

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## 12.1 Presheaves

### 12.1.1 Foundations

Let $C$ be a category.

**Definition 12.1.1.1.1.** A **presheaf on** $C$ is a functor $\mathcal{F}\colon C^{\mathsf{op}} \to \mathsf{Sets}$.

**Example 12.1.1.1.2.** Presheaves on the delooping $\mathsf{B}A$ of a monoid $A$ are precisely the left $A$-sets; see Monoid Actions, **??**.

**Definition 12.1.1.1.3.** A **morphism of presheaves** on $C$ from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation $\alpha\colon \mathcal{F} \Rightarrow \mathcal{G}$.

**Definition 12.1.1.1.4.** The **category of presheaves on** $C$ is the category $\mathsf{PSh}(C)$[1] defined by

$$\mathsf{PSh}(C) \overset{\text{def}}{=} \mathsf{Fun}\left(C^{\mathsf{op}}, \mathsf{Sets}\right).$$

**Remark 12.1.1.1.5.** In detail, the **category of presheaves on** $C$ is the category $\mathsf{PSh}(C)$ where

- *Objects.* The objects of $\mathsf{PSh}(C)$ are presheaves on $C$ as in Definition 12.1.1.1.1.

[1] *Further Notation:* Also written $\widehat{C}$ in some parts of the literature.

- *Morphisms.* The morphisms of $\mathsf{PSh}(\mathcal{C})$ are morphisms of presheaves as in Definition 12.1.1.1.3, i.e. we have

$$\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\mathcal{F},\mathcal{G}) \stackrel{\text{def}}{=} \mathrm{Nat}(\mathcal{F},\mathcal{G})$$

  for each $\mathcal{F},\mathcal{G} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

- *Identities.* For each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}} : \mathrm{pt} \to \mathrm{Nat}(\mathcal{F},\mathcal{F})$$

  of $\mathsf{PSh}(\mathcal{C})$ at $\mathcal{F}$ is defined by

$$\mathrm{id}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{F}},$$

  where $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \Rightarrow \mathcal{F}$ is the identity natural transformation of Categories, Definition 11.9.3.1.1.

- *Composition.* For each $\mathcal{F},\mathcal{G},\mathcal{H} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} : \mathrm{Nat}(\mathcal{G},\mathcal{H}) \times \mathrm{Nat}(\mathcal{F},\mathcal{G}) \to \mathrm{Nat}(\mathcal{F},\mathcal{H})$$

  of $\mathsf{PSh}(\mathcal{C})$ at $(\mathcal{F},\mathcal{G},\mathcal{H})$ is defined by

$$\beta \circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha : \mathcal{F} \Rightarrow \mathcal{H}$ is the vertical composition of $\alpha$ and $\beta$ of Categories, Definition 11.9.4.1.1.

### 12.1.2 Representable Presheaves

Let $\mathcal{C}$ be a category.

**Definition 12.1.2.1.1.** Let $A \in \mathrm{Obj}(\mathcal{C})$.

1. The **representable presheaf associated to** $A$ is the presheaf

$$h_A : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$

   where

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathcal{C})$, we have

$$h_A(X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X, A).$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(C)$, the action on morphisms

  $$h_{A|X,Y} \colon \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_{\mathsf{Sets}}(h_A(Y), h_A(X))$$

  of $h_A$ at $(X,Y)$ is given by sending a morphism

  $$f \colon X \to Y$$

  of $C$ to the map of sets

  $$h_A(f) \colon \underbrace{h_A(Y)}_{\overset{\text{def}}{=}\mathrm{Hom}_C(Y,A)} \to \underbrace{h_A(X)}_{\overset{\text{def}}{=}\mathrm{Hom}_C(X,A)}$$

  defined by

  $$h_A(f) \overset{\text{def}}{=} f^*,$$

  where $f^*$ is the precomposition by $f$ morphism of Categories, Item 1 of Definition 11.1.4.1.1.

2. A **representing object** for a presheaf $\mathcal{F} \colon C^{\mathsf{op}} \to \mathsf{Sets}$ on $C$ is an object $A$ of $C$ such that we have $\mathcal{F} \cong h_A$.

3. A presheaf $\mathcal{F} \colon C^{\mathsf{op}} \to \mathsf{Sets}$ on $C$ is **representable** if $\mathcal{F}$ admits a representing object.

**Example 12.1.2.1.2.** The representable presheaf on the delooping $\mathrm{B}A$ of a monoid $A$ associated to the unique object $\bullet$ of $\mathrm{B}A$ is the left regular representation of $A$ of Monoid Actions, **??**.

**Proposition 12.1.2.1.3.** Let $\mathcal{F} \colon C^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf. If there exist $A, B \in \mathrm{Obj}(C)$ such that we have natural isomorphisms

$$h_A \cong \mathcal{F},$$
$$h_B \cong \mathcal{F},$$

then $A \cong B$.

*Proof.* By composing the isomorphisms $h_A \cong \mathcal{F} \cong h_B$, we get a natural isomorphism $h_A \cong h_B$. By Item 2 of Definition 12.1.4.1.3, we have $A \cong B$. □

### 12.1.3 Representable Natural Transformations

Let $C$ be a category, let $A, B \in \mathrm{Obj}(C)$, and let $f \colon A \to B$ be a morphism of $C$.

**Definition 12.1.3.1.1.** The **representable natural transformation associated to** $f$ is the natural transformation

$$h_f \colon h_A \Rightarrow h_B$$

consisting of the collection

$$\left\{ h_{f|X} \colon \underbrace{h_A(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,A)} \longrightarrow \underbrace{h_B(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,B)} \right\}_{X \in \mathrm{Obj}(\mathcal{C})}$$

with

$$h_{f|X} \stackrel{\text{def}}{=} f_*,$$

where $f_*$ is the postcomposition by $f$ morphism of Categories, Item 2 of Definition 11.1.4.1.1.

### 12.1.4 The Yoneda Embedding

**Definition 12.1.4.1.1.** The **Yoneda embedding of** $\mathcal{C}$[2] is the functor[3]

$$\text{よ}_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

  $$\text{よ}_{\mathcal{C}}(A) \stackrel{\text{def}}{=} h_A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

  $$\text{よ}_{\mathcal{C}|A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Nat}(h_A, h_B)$$

  of $\text{よ}_{\mathcal{C}}$ at $(A,B)$ is given by

  $$\text{よ}_{\mathcal{C}|A,B}(f) \stackrel{\text{def}}{=} h_f$$

  for each $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$, where $h_f$ is the representable natural transformation associated to $f$ of Definition 12.1.3.1.1.

[2] *Further Terminology:* Also called the **covariant Yoneda embedding** to distinguish it from the contravariant Yoneda embedding of Definition 12.2.5.1.1.

[3] *Further Notation:* Also written $h_{(-)}$, or simply よ.

**Remark 12.1.4.1.2.** The notation よ for the Yoneda embedding was first introduced in [JS17]. The symbol よ is the hiragana for *yo*, and comes from "Yoneda" in Nobuo Yoneda (米田信夫).

It is pronounced *yo* but without letting the "o" in *yo* sound like an o-u diphthong:

- See here.
- IPA transcription: [jo̞].

**Proposition 12.1.4.1.3.** Let $C$ be a category.

1. *Fully Faithfulness.* The Yoneda embedding

$$\text{よ}_C\colon C \hookrightarrow \mathsf{PSh}(C)$$

is fully faithful.

2. *Preservation and Reflection of Isomorphisms.* The Yoneda embedding

$$\text{よ}_C\colon C \hookrightarrow \mathsf{PSh}(C)$$

preserves and reflects isomorphisms, i.e. given $A, B \in \mathrm{Obj}(C)$, the following conditions are equivalent:

(a) We have $A \cong B$.

(b) We have $h_A \cong h_B$.

3. *Density.* The Yoneda embedding

$$\text{よ}_C\colon C \hookrightarrow \mathsf{PSh}(C)$$

is dense.

4. *Interaction With Density Comonads.* We have

$$\mathrm{Lan}_{\text{よ}}(\text{よ}) \cong \mathrm{id}_{\mathsf{PSh}(C)}, \qquad \begin{tikzcd} & \mathsf{PSh}(C) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}}(\text{よ})"] \\ C \arrow[ru, "\text{よ}_C"] \arrow[r, "\text{よ}_C"'] \arrow[r, phantom, shift left=3, "\Uparrow"] & \mathsf{PSh}(C). \end{tikzcd}$$

5. *Interaction With Codensity Monads.* We have

$$\mathrm{Ran}_{\text{よ}}(\text{よ}) \cong \mathsf{Spec} \circ \mathsf{O},$$

where $\mathsf{Spec}$ and $\mathsf{O}$ are the functors of **??**.

*Proof.* *Item 1, Fully Faithfulness*: Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Applying the Yoneda lemma (Definition 12.1.5.1.1) to the functor $h_B$ (i.e. in the case $\mathcal{F} = h_B$), we have

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Nat}(h_A, h_B),$$

and the natural isomorphism

$$\xi_{A,B} : h_B(A) \Rightarrow \mathrm{Nat}(h_A, h_B)$$

witnessing this bijection is given by

$$\begin{aligned} \xi_{A,B}(g)_X &\stackrel{\text{def}}{=} h_g^X \\ &\stackrel{\text{def}}{=} g_* \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathcal{C})$ and each $g \in h_B^X$, i.e. we have $\xi_{A,B} = よ_{\mathcal{C}|A,B}$. Thus $よ_{\mathcal{C}}$ is fully faithful.

*Item 2, Preservation and Reflection of Isomorphisms*: This follows from Categories, Item 1 of Definition 11.5.1.1.6 and Item 3 of Definition 11.6.3.1.2.

*Item 3, Density*: Omitted.

*Item 4, Interaction With Density Comonads*: Omitted.

*Item 5, Interaction With Codensity Monads*: Omitted. □

### 12.1.5 The Yoneda Lemma

Let $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$.

**Theorem 12.1.5.1.1.** We have a bijection

$$\mathrm{Nat}(h_A, \mathcal{F}) \cong \mathcal{F}(A),$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}(h_{(-)}, \mathcal{F}) \cong \mathcal{F}.$$

*Proof.* *The Transformation ev: $\mathrm{Nat}(h_{(-)}, \mathcal{F}) \Rightarrow \mathcal{F}$*: Let

$$\mathrm{ev} : \mathrm{Nat}(h_{(-)}, \mathcal{F}) \Rightarrow \mathcal{F}$$

be the transformation consisting of the collection

$$\{\mathrm{ev}_A : \mathrm{Nat}(h_A, \mathcal{F}) \to \mathcal{F}(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$\mathrm{ev}_A(\alpha) = \alpha_A(\mathrm{id}_A)$$

for each $\alpha \in \mathrm{Nat}(h_A, \mathcal{F})$, where $\alpha_A$ is the component

$$\alpha_A \colon \mathrm{Hom}_C(A, A) \to \mathcal{F}(A)$$

of $\alpha$ at $A$.
*The Transformation* $\xi \colon \mathcal{F} \Rightarrow \mathit{Nat}(h_{(-)}, \mathcal{F})$: Let

$$\xi \colon \mathcal{F} \Rightarrow \mathrm{Nat}(h_{(-)}, \mathcal{F})$$

be the transformation consisting of the collection

$$\{\xi_A \colon \mathcal{F}(A) \to \mathrm{Nat}(h_A, \mathcal{F})\}_{A \in \mathrm{Obj}(C)},$$

where $\xi_A$ is the map sending an element $\phi \in \mathcal{F}(A)$ to the transformation

$$\xi_A(\phi) \colon h_A \Rightarrow \mathcal{F}$$

(which we will show is natural in a bit) consisting of the collection

$$\{\xi_A(\phi)_X \colon h_A(X) \to \mathcal{F}(X)\}_{X \in \mathrm{Obj}(C)},$$

with

$$\xi_A(\phi)_X(f) \stackrel{\text{def}}{=} [\mathcal{F}(f)](\phi)$$

for each $f \in h_A(X)$, where

$$\mathcal{F}(f) \colon \mathcal{F}(A) \to \mathcal{F}(X)$$

is the image of $f$ by $\mathcal{F}$.
*Naturality of* $\xi_A(\phi) \colon h_A \Rightarrow \mathcal{F}$: The transformation

$$\xi_A(\phi) \colon h_A \Rightarrow \mathcal{F}$$

is indeed natural, as the diagram

$$\begin{tikzcd}
h_A^Y \arrow[r, "f^*"] \arrow[d, "\xi_A(\phi)_Y"'] & h_A^X \arrow[d, "\xi_A(\phi)_X"] \\
\mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X)
\end{tikzcd}$$

commutes for each morphism $f \colon X \to Y$ of $C$, acting on elements as

$$\begin{tikzcd}
h \arrow[d, mapsto] & & h \arrow[r, mapsto] & h \circ f \arrow[d, mapsto] \\
{[\mathcal{F}(h)](\phi)} \arrow[r, mapsto] & {[\mathcal{F}(f)]([\mathcal{F}(h)](\phi))} & & {[\mathcal{F}(h \circ f)(\phi)],}
\end{tikzcd}$$

where we have

$$[\mathcal{F}(f)]([\mathcal{F}(h)](\phi)) = [\mathcal{F}(h \circ f)(\phi)]$$

by the functoriality of $\mathcal{F}$.

*Naturality of ev*: $Nat(h_{(-)}, \mathcal{F}) \Rightarrow \mathcal{F}$: Let $f \colon X \to Y$ be a morphism of $\mathcal{C}$. We claim the naturality diagram

$$\begin{tikzcd} \mathrm{Nat}(h_Y, \mathcal{F}) \arrow[r, "(h_f)^*"] \arrow[d, "\mathrm{ev}_Y"'] & \mathrm{Nat}(h_X, \mathcal{F}) \arrow[d, "\mathrm{ev}_X"] \\ \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

for ev at $f$, acting on elements as

$$\begin{tikzcd} \alpha \arrow[d, mapsto] & & \alpha \arrow[r, mapsto] & \alpha \circ h_f \arrow[d, mapsto] \\ \alpha_Y(\mathrm{id}_Y) \arrow[r, mapsto] & {[\mathcal{F}(f)](\alpha_Y(\mathrm{id}_Y))} & & {[\alpha \circ h_f]_X(\mathrm{id}_X),} \end{tikzcd}$$

commutes. Indeed:

- We have

$$\begin{aligned} [\alpha \circ h_f]_X(\mathrm{id}_X) &\overset{\text{def}}{=} [\alpha_X \circ h_{f|X}](\mathrm{id}_X) \\ &\overset{\text{def}}{=} [\alpha_X \circ f_*](\mathrm{id}_X) \\ &\overset{\text{def}}{=} \alpha_X(f_*(\mathrm{id}_X)) \\ &\overset{\text{def}}{=} \alpha_X(f). \end{aligned}$$

- Applying the naturality diagram

$$\begin{tikzcd} h_Y^Y \arrow[r, "f^*"] \arrow[d, "\alpha_Y"'] & h_Y^X \arrow[d, "\alpha_X"] \\ \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

  of $\alpha \colon h_Y \Rightarrow \mathcal{F}$ at $f \colon X \to Y$ to the element $\mathrm{id}_Y$ of $h_Y^Y$, we have

$$\begin{tikzcd} \mathrm{id}_Y \arrow[d, mapsto] & & \mathrm{id}_Y \arrow[r, mapsto] & f \arrow[d, mapsto] \\ \alpha_Y(\mathrm{id}_Y) \arrow[r, mapsto] & {[\mathcal{F}(f)](\alpha_Y(\mathrm{id}_Y))} & & \alpha_X(f), \end{tikzcd}$$

showing that we have

$$[\mathcal{F}(f)](\alpha_Y(\mathrm{id}_Y)) = \alpha_X(f).$$

Thus the naturality diagram for ev at $f$ commutes, and ev is natural.
*Naturality of* $\xi\colon \mathcal{F} \Rightarrow \mathit{Nat}(h_{(-)}, \mathcal{F})$: Let $f\colon X \to Y$ be a morphism of $C$. We claim the naturality diagram

$$\begin{tikzcd}
\mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\xi_Y"'] & \mathcal{F}(X) \arrow[d, "\xi_X"] \\
\mathrm{Nat}(h_Y, \mathcal{F}) \arrow[r, "(h_f)^*"'] & \mathrm{Nat}(h_X, \mathcal{F})
\end{tikzcd}$$

for $\xi$ at $f$, acting on elements as

$$\begin{tikzcd}
\phi \arrow[d, mapsto] & & \\
\xi_Y(\phi) \arrow[r, mapsto] & \xi_Y(\phi) \circ h_f
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi \arrow[r, mapsto] & {[\mathcal{F}(f)](\phi)} \arrow[d, mapsto] \\
& \xi_X([\mathcal{F}(f)](\phi))
\end{tikzcd}$$

commutes. Indeed, for each $X \in \mathrm{Obj}(C)$ and each $g \in h_X^A$, we have

$$\begin{aligned}
[\xi_Y(\phi) \circ h_f]_X(g) &\overset{\text{def}}{=} [\xi_Y(\phi)_X \circ h_{f|X}](g) \\
&\overset{\text{def}}{=} [\xi_Y(\phi)_X \circ f_*](g) \\
&\overset{\text{def}}{=} \xi_Y(\phi)_X(f_*(g)) \\
&\overset{\text{def}}{=} \xi_Y(\phi)_X(f \circ g) \\
&\overset{\text{def}}{=} [\mathcal{F}(f \circ g)](\phi)
\end{aligned}$$

and

$$\begin{aligned}
[\xi_X([\mathcal{F}(f)](\phi))]_X(g) &\overset{\text{def}}{=} \mathcal{F}(g)([\mathcal{F}(f)](\phi)) \\
&= [\mathcal{F}(f \circ g)](\phi),
\end{aligned}$$

where we have used the functoriality of $\mathcal{F}$. Thus $\xi_Y(\phi) \circ h_f$ and $\xi_X([\mathcal{F}(f)](\phi))$ are equal, and the naturality diagram for $\xi$ at $f$ above commutes, showing $\xi$ to be natural.
*Invertibility I: ev* $\circ\, \xi = \mathit{id}_{\mathcal{F}}$: We claim that $\mathrm{ev} \circ \xi = \mathrm{id}_{\mathcal{F}}$, i.e. that we have

$$(\mathrm{ev} \circ \xi)_A = \mathrm{id}_{\mathcal{F}(A)}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$. Indeed, we have

$$\begin{aligned}[\mathrm{ev} \circ \xi]_A(\phi) &\stackrel{\text{def}}{=} [\mathrm{ev}_A \circ \xi_A](\phi)\\ &\stackrel{\text{def}}{=} \mathrm{ev}_A(\xi_A(\phi))\\ &\stackrel{\text{def}}{=} \xi_A(\phi)_A(\mathrm{id}_A)\\ &\stackrel{\text{def}}{=} [\mathcal{F}(\mathrm{id}_A)](\phi)\\ &= [\mathrm{id}_{\mathcal{F}(A)}](\phi)\end{aligned}$$

for each $\phi \in \mathcal{F}(A)$.
*Invertibility II: $\xi \circ ev = id_{Nat(h_{(-)},\mathcal{F})}$:* We claim that $\xi \circ \mathrm{ev} = \mathrm{id}_{\mathrm{Nat}(h_{(-)},\mathcal{F})}$, i.e. that we have

$$(\xi \circ \mathrm{ev})_A = \mathrm{id}_{\mathrm{Nat}(h_A,\mathcal{F})}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$. Indeed:

- We have

$$\begin{aligned}[\xi \circ \mathrm{ev}]_A(\alpha) &\stackrel{\text{def}}{=} [\xi_A \circ \mathrm{ev}_A](\alpha)\\ &\stackrel{\text{def}}{=} \xi_A(\mathrm{ev}_A(\alpha))\\ &\stackrel{\text{def}}{=} \xi_A(\alpha_A(\mathrm{id}_A))\end{aligned}$$

  for each $\alpha \in \mathrm{Nat}(h_A, \mathcal{F})$.

- For each $X \in \mathrm{Obj}(\mathcal{C})$, we have

$$\xi_A(\alpha_A(\mathrm{id}_A))_X = \alpha_X,$$

  since we have

$$\begin{aligned}\xi_A(\alpha_A(\mathrm{id}_A))_X(f) &\stackrel{\text{def}}{=} [\mathcal{F}(f)](\alpha_A(\mathrm{id}_A))\\ &\stackrel{(\dagger)}{=} \alpha_X(f)\end{aligned}$$

  for each $f \in h_A(X)$, where the equality marked with (†) follows from the commutativity of the naturality diagram

$$\begin{tikzcd} h_A^A \arrow[r, "f_*"] \arrow[d, "\alpha_A"'] & h_X^A \arrow[d, "\alpha_X"] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

of $\alpha$ at $f\colon A \to X$, which acts on $\mathrm{id}_A$ as

$$\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\
\alpha_A(\mathrm{id}_A) \arrow[r, mapsto] & {[\mathcal{F}(f)](\alpha_A(\mathrm{id}_A)) = \alpha_X(f).}
\end{tikzcd}$$

This finishes the proof. □

### 12.1.6 Properties of Categories of Presheaves

**Proposition 12.1.6.1.1.** Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{PSh}(\mathcal{C})$ defines a functor

$$\mathsf{PSh}\colon \mathsf{Cats} \to \mathsf{Cats}$$

   up to some set-theoretic considerations.[4]

2. *Interaction With Slice Categories.* Let $X \in \mathrm{Obj}(\mathcal{C})$. We have an equivalence of categories

$$\mathsf{PSh}(\mathcal{C}_{/X}) \overset{\text{eq.}}{\cong} \mathsf{PSh}(\mathcal{C})_{/h_X}.$$

3. *Interaction With Categories of Elements.* Let $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$. We have an equivalence of categories

$$\mathsf{PSh}\left(\textstyle\int_{\mathcal{C}} \mathcal{F}\right) \overset{\text{eq.}}{\cong} \mathsf{PSh}(\mathcal{C})_{/\mathcal{F}}.$$

*Proof.* *Item 1, Functoriality*: Omitted.
*Item 2, Interaction With Slice Categories*: Omitted.
*Item 3, Interaction With Categories of Elements*: Omitted. □

## 12.2 Copresheaves

### 12.2.1 Foundations

Let $\mathcal{C}$ be a category.

[4]For instance:

- The $\mathsf{Cats}$ in the source of $\mathsf{PSh}$ could be small categories, and then the $\mathsf{Cats}$ in the right would be locally small categories.

**Definition 12.2.1.1.1.** A **copresheaf on** $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathsf{Sets}$.

**Example 12.2.1.1.2.** Copresheaves on the delooping $\mathsf{B}A$ of a monoid $A$ are precisely the right $A$-sets; see Monoid Actions, **??**.

**Definition 12.2.1.1.3.** A **morphism of copresheaves** on $\mathcal{C}$ from $F$ to $G$ is a natural transformation $\alpha\colon F \Rightarrow G$.

**Definition 12.2.1.1.4.** The **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ defined by

$$\mathsf{CoPSh}(\mathcal{C}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}).$$

**Remark 12.2.1.1.5.** In detail, the **category of copresheaves on** $\mathcal{C}$ is the category $\mathsf{CoPSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{CoPSh}(\mathcal{C})$ are copresheaves on $\mathcal{C}$ as in Definition 12.2.1.1.1.
- *Morphisms.* The morphisms of $\mathsf{CoPSh}(\mathcal{C})$ are morphisms of copresheaves as in Definition 12.2.1.1.3, i.e. we have

  $$\mathrm{Hom}_{\mathsf{CoPSh}(\mathcal{C})}(F, G) \stackrel{\text{def}}{=} \mathrm{Nat}(F, G)$$

  for each $F, G \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$.
- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{CoPSh}(\mathcal{C})}_{F}\colon \mathrm{pt} \to \mathrm{Nat}(F, F)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at $F$ is defined by

  $$\mathrm{id}^{\mathsf{CoPSh}(\mathcal{C})}_{F} \stackrel{\text{def}}{=} \mathrm{id}_{F},$$

  where $\mathrm{id}_F\colon F \Rightarrow F$ is the identity natural transformation of Categories, Definition 11.9.3.1.1.
- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the composition map

  $$\circ^{\mathsf{CoPSh}(\mathcal{C})}_{F,G,H}\colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H)$$

  of $\mathsf{CoPSh}(\mathcal{C})$ at $(F, G, H)$ is defined by

  $$\beta \circ^{\mathsf{CoPSh}(\mathcal{C})}_{F,G,H} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha\colon F \Rightarrow H$ is the vertical composition of $\alpha$ and $\beta$ of Categories, Definition 11.9.4.1.1.

### 12.2.2 Corepresentable Copresheaves

Let $\mathcal{C}$ be a category.

**Definition 12.2.2.1.1.** Let $A \in \text{Obj}(\mathcal{C})$.

1. The **corepresentable copresheaf associated to** $A$ is the copresheaf

$$h^A : \mathcal{C} \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $X \in \text{Obj}(\mathcal{C})$, we have

$$h^A(X) \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{C}}(A, X).$$

- *Action on Morphisms.* For each $X, Y \in \text{Obj}(\mathcal{C})$, the action on morphisms

$$h^A_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathsf{Sets}}(h^A(X), h^A(Y))$$

of $h^A$ at $(X, Y)$ is given by sending a morphism

$$f : X \to Y$$

of $\mathcal{C}$ to the map of sets

$$h^A(f) : \underbrace{h^A(X)}_{\stackrel{\text{def}}{=} \text{Hom}_{\mathcal{C}}(A,X)} \to \underbrace{h^A(Y)}_{\stackrel{\text{def}}{=} \text{Hom}_{\mathcal{C}}(A,Y)}$$

defined by

$$h^A(f) \stackrel{\text{def}}{=} f_*,$$

where $f_*$ is the postcomposition by $f$ morphism of Categories, Item 2 of Definition 11.1.4.1.1.

2. A **corepresenting object** for a copresheaf $F : \mathcal{C} \to \mathsf{Sets}$ on $\mathcal{C}$ is an object $A$ of $\mathcal{C}$ such that we have $F \cong h^A$.

3. A copresheaf $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ on $\mathcal{C}$ is **corepresentable** if $F$ admits a corepresenting object.

---

- The $\mathsf{Cats}$ in the source of $\mathsf{PSh}$ could be locally small categories, and then the $\mathsf{Cats}$ on the right would be large categories.

In general, one can systematise and formalise this using Grothendieck universes.

**Example 12.2.2.1.2.** The corepresentable copresheaf on the delooping $\mathsf{B}A$ of a monoid $A$ associated to the unique object $\bullet$ of $\mathsf{B}A$ is the right regular representation of $A$ of Monoid Actions, **??**.

**Proposition 12.2.2.1.3.** Let $F\colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf. If there exist $A, B \in \mathrm{Obj}(\mathcal{C})$ such that we have natural isomorphisms

$$h^A \cong F,$$
$$h^B \cong F,$$

then $A \cong B$.

*Proof.* By composing the isomorphisms $h^A \cong F \cong h^B$, we get a natural isomorphism $h^A \cong h^B$. By Item 2 of Definition 12.2.4.1.2, we have $A \cong B$. □

### 12.2.3 Corepresentable Natural Transformations

Let $\mathcal{C}$ be a category, let $A, B \in \mathrm{Obj}(\mathcal{C})$, and let $f\colon A \to B$ be a morphism of $\mathcal{C}$.

**Definition 12.2.3.1.1.** The **corepresentable natural transformation associated to** $f$ is the natural transformation

$$h^f\colon h^B \Rightarrow h^A$$

consisting of the collection

$$\left\{ h^f_X\colon \underbrace{h^B(X)}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(B,X)} \to \underbrace{h^A(X)}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,X)} \right\}_{X \in \mathrm{Obj}(\mathcal{C})}$$

with

$$h^f_X \overset{\mathrm{def}}{=} f^*,$$

where $f_*$ is the precomposition by $f$ morphism of Categories, Item 1 of Definition 11.1.4.1.1.

### 12.2.4 The Contravariant Yoneda Embedding

**Definition 12.2.4.1.1.** The **contravariant Yoneda embedding of** $\mathcal{C}$ is the functor[5]

$$Ⲩ_{\mathcal{C}}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})$$

where

[5] *Further Notation:* Also written $h^{(-)}$, or simply Ⲩ.

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$よ_{\mathcal{C}}(A) \overset{\mathrm{def}}{=} h^A.$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$よ_{\mathcal{C}|A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h^B, h^A)$$
of $よ_{\mathcal{C}}$ at $(A, B)$ is given by
$$よ_{\mathcal{C}|A,B}(f) \overset{\mathrm{def}}{=} h^f$$
for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, where $h^f$ is the corepresentable natural transformation associated to $f$ of Definition 12.2.3.1.1.

**Proposition 12.2.4.1.2.** Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The contravariant Yoneda embedding
$$よ_{\mathcal{C}} \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})$$
is fully faithful.

2. *Preservation and Reflection of Isomorphisms.* The contravariant Yoneda embedding
$$よ_{\mathcal{C}} \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})$$
preserves and reflects isomorphisms, i.e. given $A, B \in \mathrm{Obj}(\mathcal{C})$, the following conditions are equivalent:
    (a) We have $A \cong B$.
    (b) We have $h^A \cong h^B$.

*Proof.* *Item 1, Fully Faithfulness*: The proof is dual to that of Item 1 of Definition 12.1.4.1.3, and is therefore omitted.
*Item 2, Preservation and Reflection of Isomorphisms*: This follows from Categories, Item 1 of Definition 11.5.1.1.6 and Item 3 of Definition 11.6.3.1.2. □

## 12.2.5 The Contravariant Yoneda Lemma

Let $F \colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf on $\mathcal{C}$.

**Theorem 12.2.5.1.1.** We have a bijection
$$\mathrm{Nat}(h^A, F) \cong F(A),$$
natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors
$$\mathrm{Nat}(h^{(-)}, F) \cong F.$$

*Proof.* The proof is dual to that of Definition 12.1.5.1.1, and is therefore omitted. □

## 12.3 Restricted Yoneda Embeddings and Yoneda Extensions

### 12.3.1 Foundations

let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

**Definition 12.3.1.1.1.** The **restricted Yoneda embedding associated to** $F$ is the functor

$$よ_F\colon \mathcal{D} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

defined as the composition

$$\mathcal{D} \xhookrightarrow{よ_{\mathcal{D}}} \mathsf{PSh}(\mathcal{D}) \xrightarrow{F^{\mathsf{op},*}} \mathsf{PSh}(\mathcal{C}).$$

**Remark 12.3.1.1.2.** In detail, the **restricted Yoneda embedding associated to** $F$ is the functor

$$よ_F\colon \mathcal{D} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{D})$, we have

$$\begin{aligned} よ_F(A) &\overset{\text{def}}{=} h_A \circ F^{\mathsf{op}} \\ &\overset{\text{def}}{=} h_A^{F(-)}. \end{aligned}$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{D})$, the action on morphisms

$$よ_{F|A,B}\colon \mathrm{Hom}_{\mathcal{D}}(A,B) \to \mathrm{Nat}(h_A^{F(-)}, h_B^{F(-)})$$

  of $よ_F$ at $(A,B)$ is given by

$$\begin{aligned} よ_{F|A,B}(f) &\overset{\text{def}}{=} h_f^{F(-)} \\ &\overset{\text{def}}{=} h_f \star \mathrm{id}_{F^{\mathsf{op}}} \end{aligned}$$

  for each $f \in \mathrm{Hom}_{\mathcal{D}}(A,B)$, where $h_f$ is the representable natural transformation associated to $f$ of Definition 12.1.3.1.1.

**Example 12.3.1.1.3.** Here are some examples of restricted Yoneda embeddings.

1. *The Nerve Functor.* Let

$$\iota\colon \mathbb{\Delta} \hookrightarrow \mathsf{Cats}$$

   be the functor given by $[n] \to \mathbb{n}$. Then the restricted Yoneda embedding

$$よ_\iota\colon \mathsf{Cats} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\text{def}}{=}\mathsf{sSets}}$$

   of $\iota$ is given by the nerve functor $\mathrm{N}_\bullet$ of **??**, **??**.

2. *The Singular Simplicial Set Associated to a Topological Space.* Let

$$\iota\colon \mathbb{\Delta} \hookrightarrow \mathbb{T}$$

be the functor given by $[n] \to |\Delta^n|$. Then the restricted Yoneda embedding

$$\text{よ}_\iota\colon \mathbb{T} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\text{def}}{=}\mathsf{sSets}}$$

of $\iota$ is given by the singular simplicial set functor $\mathrm{Sing}_\bullet$ of **??**, **??**.

3. *The Coherent Nerve Functor.* Let

$$\iota\colon \mathbb{\Delta} \hookrightarrow \mathsf{sCats}$$

be the functor given by $[n] \to \mathsf{Path}(\Delta^n)$, where $\mathsf{Path}(\Delta^n)$ is the simplicial category of **??**, **??**. Then the restricted Yoneda embedding

$$\text{よ}_\iota\colon \mathsf{sCats} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\text{def}}{=}\mathsf{sSets}}$$

of $\iota$ is given by the coherent nerve functor $\mathrm{N}^{\mathrm{hc}}_\bullet$ of **??**, **??**.

4. *Kan's Ex Functor.* Let

$$\mathrm{sd}\colon \mathbb{\Delta} \hookrightarrow \mathsf{sSets}$$

be the functor given by $[n] \to \mathrm{Sd}(\Delta^n)$, where $\mathrm{Sd}(\Delta^n)$ is the barycentric subdivision of $\Delta^n$ of **??**. Then the restricted Yoneda embedding

$$\text{よ}_{\mathrm{sd}}\colon \mathsf{sSets} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\text{def}}{=}\mathsf{sSets}}$$

of sd is given by Kan's Ex functor of **??**.

**Proposition 12.3.1.1.4.** let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Fully Faithfulness.* The following conditions are equivalent:
   (a) The restricted Yoneda embedding $\text{よ}_F$ is fully faithful.
   (b) The functor $F$ is dense (Limits and Colimits, **??**).

2. *As a Left Kan Extension.* We have a natural isomorphism of functors

$$\text{よ}_F \cong \mathrm{Lan}_F(\text{よ}), \qquad \begin{tikzcd} & \mathcal{D} \arrow[d, dashed, "\text{よ}_F"] \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "\text{よ}_\mathcal{C}"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathsf{PSh}(\mathcal{C}). \end{tikzcd}$$

*Proof.* Item 1, *Interaction With Fully Faithfulness*: Omitted.
Item 2, *As a Left Kan Extension*: Omitted. □

### 12.3.2 The Yoneda Extension Functor

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with $\mathcal{C}$ small and $\mathcal{D}$ cocomplete.

**Definition 12.3.2.1.1.** The **Yoneda extension functor associated to** $F$ is the left Kan extension

$$\mathrm{Lan}_{よ}(F)\colon \mathsf{PSh}(\mathcal{C}) \to \mathcal{D}, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "\mathrm{Lan}_{よ}(F)"] \\ \mathcal{C} \arrow[ur, "よ_{\mathcal{C}}"] \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{D}. \end{tikzcd}$$

**Example 12.3.2.1.2.** Here are some examples of Yoneda extensions.

1. *The Homotopy Category Functor.* Let

   $$\iota\colon \mathbb{\Delta} \hookrightarrow \mathsf{Cats}$$

   be the functor given by $[n] \to \mathbb{n}$. Then the Yoneda extension

   $$\mathrm{Lan}_{よ}(\iota)\colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\mathrm{def}}{=}\mathsf{sSets}} \to \mathsf{Cats}$$

   of $\iota$ is given by the homotopy category functor $\mathsf{Ho}$ of **??**, **??**.

2. *The Geometric Realisation Functor.* Let

   $$\iota\colon \mathbb{\Delta} \hookrightarrow \mathbb{T}$$

   be the functor given by $[n] \to |\Delta^n|$. Then the Yoneda extension

   $$\mathrm{Lan}_{よ}(\iota)\colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\mathrm{def}}{=}\mathsf{sSets}} \to \mathbb{T}$$

   of $\iota$ is given by the geometric realisation functor $|-|$ of **??**, **??**.

3. *The Path Simplicial Category Functor.* Let

   $$\iota\colon \mathbb{\Delta} \hookrightarrow \mathsf{sCats}$$

   be the functor given by $[n] \to \mathsf{Path}(\Delta^n)$, where $\mathsf{Path}(\Delta^n)$ is the simplicial category of **??**, **??**. Then the Yoneda extension

   $$\mathrm{Lan}_{よ}(\iota)\colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\mathrm{def}}{=}\mathsf{sSets}} \to \mathsf{sCats}$$

   of $\iota$ is given by the path simplicial category functor $\mathsf{Path}$ of **??**, **??**.

4. *The Barycentric Subdivision Functor.* Let

$$\mathsf{sd}\colon \mathbb{\Delta} \hookrightarrow \mathsf{sSets}$$

be the functor given by $[n] \to \mathrm{Sd}(\Delta^n)$, where $\mathrm{Sd}(\Delta^n)$ is the barycentric subdivision of $\Delta^n$ of **??**. Then the Yoneda extension

$$\mathrm{Lan}_{よ}(\mathrm{sd})\colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\mathrm{def}}{=}\mathsf{sSets}} \to \mathsf{sSets}$$

of sd is given by the barycentric subdivision functor Sd of **??**.

**Proposition 12.3.2.1.3.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with $\mathcal{C}$ small and $\mathcal{D}$ cocomplete.

1. *Functoriality.* The assignment $F \mapsto \mathrm{Lan}_{よ}(F)$ defines a functor

$$\mathrm{Lan}_{よ}\colon \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}).$$

2. *Adjointness.* We have an adjunction[6]

$$\left(\mathrm{Lan}_{よ}(F) \dashv よ_F\right)\colon \quad \begin{tikzcd} \mathsf{PSh}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Lan}_{よ}(F)"] \arrow[r, phantom, "\perp"] & \mathcal{D}, \arrow[l, bend left, "よ_F"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathcal{D}}([\mathrm{Lan}_{よ}(F)](\mathcal{F}), D) \cong \mathrm{Nat}(\mathcal{F}, よ_F(D)),$$

natural in $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$ and $D \in \mathrm{Obj}(\mathcal{D})$.

3. *Interaction With the Yoneda Embedding.* We have a natural isomorphism of functors

$$\mathrm{Lan}_{よ}(F) \circ よ_{\mathcal{C}} \cong F, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "\mathrm{Lan}_{よ}(F)"] \\ \mathcal{C} \arrow[ur, "よ_{\mathcal{C}}"] \arrow[r, "F"'] & \mathcal{D}. \end{tikzcd}$$

[6] Applying Item 2 of Definition 12.3.1.1.4, we see that this adjunction has the form $\mathrm{Lan}_{よ}(F) \dashv \mathrm{Lan}_F(よ)$.

4. *As a Coend*. We have

$$\begin{aligned}[\mathrm{Lan}_{よ}(F)](\mathcal{F}) &\cong \int^{A\in C} \mathrm{Nat}(h_A, \mathcal{F}) \odot F(A)\\ &\cong \int^{A\in C} \mathcal{F}(A) \odot F(A)\end{aligned}$$

for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(C))$.

5. *Interaction With Tensors of Presheaves With Functors*. We have a natural isomorphism

$$\mathrm{Lan}_{よ}(F) \cong (-) \odot_C F,$$

natural in $F \in \mathrm{Obj}(\mathsf{Fun}(C, \mathcal{D}))$.

6. *Interaction With Finite Limits*. Let $F : C \to \mathsf{Sets}$ be a functor. The following conditions are equivalent:

   (a) The functor $F$ preserves finite limits.

   (b) The functor $\mathrm{Lan}_{よ}(F)$ preserves finite limits.

   (c) The category of elements $\int_C F$ of $F$ is cofiltered.

*Proof*. *Item 1, Functoriality*: This follows from Kan Extensions, **??** of **??**.
*Item 2, Adjointness*: Omitted.
*Item 3, Interaction With the Yoneda Embedding*: This follows from Kan Extensions, **??** of **??**.
*Item 4, As a Coend*: This follows from Kan Extensions, **??** of **??** and Definition 12.1.5.1.1.
*Item 5, Interaction With Tensors of Presheaves With Functors*: This follows from Item 4.
*Item 6, Interaction With Finite Limits*: See [**coend-calculus**]. □

## 12.4 Functor Tensor Products

### 12.4.1 The Tensor Product of Presheaves With Copresheaves

Let $C$ be a category, let $\mathcal{F} : C^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $C$, and let $G : C \to \mathsf{Sets}$ be a copresheaf on $C$.

**Definition 12.4.1.1.1.** The **tensor product** of $\mathcal{F}$ with $G$ is the set $\mathcal{F} \boxtimes_C G$[7] defined by

$$\mathcal{F} \boxtimes_C G \stackrel{\mathrm{def}}{=} \int^{A\in C} \mathcal{F}(A) \times G(A).$$

[7] *Further Notation:* Also written simply $\mathcal{F} \boxtimes G$.

**Remark 12.4.1.1.2.** In other words, the tensor product of $\mathcal{F}$ with $G$ is the set $\mathcal{F} \boxtimes_C G$ defined as the coend of the functor

$$C^{\mathsf{op}} \times C \xrightarrow{\mathcal{F} \times G} \mathsf{Sets} \times \mathsf{Sets} \xrightarrow{\times} \mathsf{Sets},$$

which is equivalently the composition

$$\times \circ (\mathcal{F} \times G) \cong \mathcal{F} \diamond F, \qquad \begin{tikzcd} C \arrow[r, "F", "\shortmid"{marking}] \arrow[dr, "\times\circ(\mathcal{F}\times G)"', "\shortmid"{marking}] & \mathrm{pt} \arrow[d, "\mathcal{F}", "\shortmid"{marking}] \\ & C \end{tikzcd}$$

in $\mathsf{Prof}$.

**Example 12.4.1.1.3.**

**Proposition 12.4.1.1.4.** Let $C$ be a category.

1. *Functoriality.* The assignments $\mathcal{F}, G, (\mathcal{F}, G) \mapsto \mathcal{F} \boxtimes_C G$ define functors

$$\begin{aligned} \mathcal{F} \boxtimes_C -\colon && \mathsf{PSh}(C) &\to \mathsf{Sets}, \\ - \boxtimes_C G\colon && \mathsf{CoPSh}(C) &\to \mathsf{Sets}, \\ -_1 \boxtimes_C -_2\colon && \mathsf{PSh}(C) \times \mathsf{CoPSh}(C) &\to \mathsf{Sets}. \end{aligned}$$

2. *As a Composition of Profunctors.* Let $C$ be a category and let:

   - $\mathcal{F}\colon \mathrm{pt} \nrightarrow C$ be a presheaf on $C$, viewed as a profunctor.
   - $F\colon C \nrightarrow \mathrm{pt}$ be a copresheaf on $C$, viewed as a profunctor.

   We have a natural isomorphism of profunctors

$$\mathcal{F} \boxtimes_C F \cong F \diamond \mathcal{F}, \qquad \begin{tikzcd} & C \arrow[d, "F", "\shortmid"{marking}] \\ \mathrm{pt} \arrow[ur, "\mathcal{F}", "\shortmid"{marking}] \arrow[r, "\mathcal{F}\boxtimes_C F"', "\shortmid"{marking}] & \mathrm{pt}, \end{tikzcd}$$

   natural in $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(C))$ and $F \in \mathrm{Obj}(\mathsf{CoPSh}(C))$.

3. *Interaction With Representable Presheaves.* Let $\mathcal{F}$ be a presheaf on $C$. We have a bijection of sets

$$\mathcal{F} \boxtimes_C h^X \cong \mathcal{F}(X),$$

natural in $X \in \mathrm{Obj}(\mathcal{C})$, giving a natural isomorphism of functors

$$\mathcal{F} \boxtimes_{\mathcal{C}} h^{(-)} \cong \mathcal{F}, \qquad \begin{tikzcd} & \mathsf{CoPSh}(\mathcal{C}) \arrow[d, "\mathcal{F}\boxtimes_{\mathcal{C}}-"] \\ \mathcal{C}^{\mathsf{op}} \arrow[ur, "\rotatebox[origin=c]{180}{よ}_{\mathcal{C}}"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, shift left=3, "\Leftrightarrow"] & \mathsf{Sets}. \end{tikzcd}$$

4. *Interaction With Corepresentable Copresheaves.* Let $G$ be a copresheaf on $\mathcal{C}$. We have a bijection of sets

$$h_X \boxtimes_{\mathcal{C}} G \cong G(X),$$

natural in $X \in \mathrm{Obj}(\mathcal{C})$, giving a natural isomorphism of functors

$$h_{(-)} \boxtimes_{\mathcal{C}} G \cong G, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, "-\boxtimes_{\mathcal{C}}G"] \\ \mathcal{C} \arrow[ur, "\text{よ}_{\mathcal{C}}"] \arrow[r, "G"'] \arrow[ur, phantom, shift left=3, "\Leftrightarrow"] & \mathsf{Sets}. \end{tikzcd}$$

5. *Interaction With Yoneda Extensions.* Let $G \colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf on $\mathcal{C}$. We have a natural isomorphism

$$\mathrm{Lan}_{\text{よ}}(G) \cong (-) \boxtimes_{\mathcal{C}} G, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "(-)\boxtimes_{\mathcal{C}}G"] \\ \mathcal{C} \arrow[ur, "\text{よ}_{\mathcal{C}}"] \arrow[r, "G"'] \arrow[ur, phantom, shift left=3, "\Uparrow"] & \mathsf{Sets}, \end{tikzcd}$$

natural in $G \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$.

6. *Interaction With Contravariant Yoneda Extensions.* Let $\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$. We have a natural isomorphism

$$\mathrm{Lan}_{\rotatebox[origin=c]{180}{よ}}(\mathcal{F}) \cong \mathcal{F} \boxtimes_{\mathcal{C}} (-), \qquad \begin{tikzcd} & \mathsf{CoPSh}(\mathcal{C}) \arrow[d, dashed, "\mathcal{F}\boxtimes_{\mathcal{C}}(-)"] \\ \mathcal{C}^{\mathsf{op}} \arrow[ur, "\rotatebox[origin=c]{180}{よ}_{\mathcal{C}}"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, shift left=3, "\Uparrow"] & \mathsf{Sets}, \end{tikzcd}$$

natural in $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

*Proof.* *Item 1, Functoriality*: Omitted.
*Item 2, As a Composition of Profunctors*: Clear.
*Item 3, Interaction With Representable Presheaves*: This follows from **??**.
*Item 4, Interaction With Corepresentable Copresheaves*: This follows from **??**.
*Item 5, Interaction With Yoneda Extensions*: This is a special case of Item 5 of Definition 12.3.2.1.3.
*Item 6, Interaction With Contravariant Yoneda Extensions*: This is a special case of **??** of **??**. □

### 12.4.2 The Tensor of a Presheaf With a Functor

Let $\mathcal{C}$ be a category, let $\mathcal{D}$ be a category with coproducts, let $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$, and let $G\colon \mathcal{C} \to \mathcal{D}$ be a functor.

**Definition 12.4.2.1.1.** The **tensor** of $\mathcal{F}$ with $G$ is the object $\mathcal{F} \odot_{\mathcal{C}} G$[8] of $\mathcal{D}$ defined by

$$\mathcal{F} \odot_{\mathcal{C}} G \stackrel{\text{def}}{=} \int^{A\in\mathcal{C}} \mathcal{F}(A) \odot G(A).$$

**Remark 12.4.2.1.2.** In other words, the tensor of $\mathcal{F}$ with $G$ is the object $\mathcal{F} \odot_{\mathcal{C}} G$ of $\mathcal{D}$ defined as the coend of the functor

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{F}\times G} \mathsf{Sets} \times \mathcal{D} \xrightarrow{\odot} \mathcal{D}.$$

**Proposition 12.4.2.1.3.** Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignments $\mathcal{F}, G, (\mathcal{F}, G) \mapsto \mathcal{F} \odot_{\mathcal{C}} G$ define functors

$$\begin{aligned} \mathcal{F} \odot_{\mathcal{C}} -\colon &\quad \mathsf{PSh}(\mathcal{C}) \to \mathcal{D},\\ - \odot_{\mathcal{C}} G\colon &\quad \mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathcal{D},\\ -_1 \odot_{\mathcal{C}} -_2\colon &\quad \mathsf{PSh}(\mathcal{C}) \times \mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathcal{D}. \end{aligned}$$

2. *Interaction With Corepresentable Copresheaves.* We have an isomorphism

$$h_X \odot_{\mathcal{C}} G \cong G(X),$$

natural in $X \in \mathrm{Obj}(\mathcal{C})$, giving a natural isomorphism of functors

$$h_{(-)} \odot_{\mathcal{C}} G \cong G.$$

3. *Interaction With Yoneda Extensions.* We have a natural isomorphism

$$\mathrm{Lan}_{\text{よ}}(G) \cong (-) \odot_{\mathcal{C}} G,$$

natural in $G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$.

[8] *Further Notation:* Also written simply $\mathcal{F} \odot G$.

*Proof.* *Item 1, Functoriality*: Omitted.
**??**, *Interaction With Corepresentable Copresheaves*: This follows from **??**.
*Item 3, Interaction With Yoneda Extensions*: This is a repetition of Item 5 of Definition 12.3.2.1.3, and is proved there. □

### 12.4.3 The Tensor of a Copresheaf With a Functor

Let $C$ be a category, let $\mathcal{D}$ be a category with coproducts, let $F\colon C \to \mathsf{Sets}$ be a copresheaf on $C$, and let $G\colon C^{\mathsf{op}} \to \mathcal{D}$ be a functor.

**Definition 12.4.3.1.1.** The **tensor** of $F$ with $G$ is the set $F \odot_C G$[9] defined by

$$F \odot_C G \overset{\text{def}}{=} \int^{A \in C} F(A) \odot G(A).$$

**Remark 12.4.3.1.2.** In other words, the tensor of $F$ with $G$ is the object $F \odot_C G$ of $\mathcal{D}$ defined as the coend of the functor

$$C^{\mathsf{op}} \times C \overset{\sim}{\dashrightarrow} C \times C^{\mathsf{op}} \xrightarrow{F \times G} \mathsf{Sets} \times \mathcal{D} \xrightarrow{\odot} \mathcal{D}.$$

**Proposition 12.4.3.1.3.** Let $C$ be a category.

1. *Functoriality.* The assignments $F, G, (F, G) \mapsto F \odot_C G$ define functors

$$\begin{aligned} F \odot_C -\colon & \qquad \mathsf{CoPSh}(C) \qquad \to \mathcal{D}, \\ - \odot_C \mathcal{G}\colon & \qquad \mathsf{Fun}(C^{\mathsf{op}}, \mathcal{D}) \qquad \to \mathcal{D}, \\ -_1 \odot_C -_2\colon & \ \mathsf{Fun}(C^{\mathsf{op}}, \mathcal{D}) \times \mathsf{CoPSh}(C) \to \mathcal{D}. \end{aligned}$$

2. *Interaction With Corepresentable Copresheaves.* We have an isomorphism

$$h^X \odot_C G \cong G(X),$$

natural in $X \in \mathrm{Obj}(C)$, giving a natural isomorphism of functors

$$h^{(-)} \odot_C G \cong G.$$

3. *Interaction With Contravariant Yoneda Extensions.* We have a natural isomorphism

$$\mathrm{Lan}_{\mathcal{Y}}(G) \cong G \odot_C (-),$$

natural in $G \in \mathrm{Obj}(\mathsf{Fun}(C^{\mathsf{op}}, \mathcal{D}))$.

[9] *Further Notation:* Also written simply $F \odot G$.

*Proof.* *Item 1, Functoriality*: Omitted.
??, *Interaction With Representable Presheaves*: This follows from ??.
??, *Interaction With Corepresentable Copresheaves*: This follows from ??.
??, *Interaction With Yoneda Extensions*: Omitted.
*Item 3, Interaction With Contravariant Yoneda Extensions*: Omitted. □

# Appendices

## A Other Chapters

**Preliminaries**

1. Introduction
2. A Guide to the Literature

**Sets**

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

**Relations**

8. Relations
9. Constructions With Relations
10. Conditions on Relations

**Categories**

11. Categories
12. Presheaves and the Yoneda Lemma

**Monoidal Categories**

13. Constructions With Monoidal Categories

**Bicategories**

14. Types of Morphisms in Bicategories

**Extra Part**

15. Notes

## References

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