Pointed Sets

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This chapter contains some foundational material on pointed sets.

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6.1   Pointed Sets

6.1.1 Foundations

DEFINITION 6.1.1.1.1 ▶ POINTED SETS

A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

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[1]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$-**modules**.

REMARK 6.1.1.1.2 ▶ UNWINDING DEFINITION 6.1.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).

- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

EXAMPLE 6.1.1.1.3 ▶ THE ZERO SPHERE

The 0-**sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

---

[1]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[2]*Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

EXAMPLE 6.1.1.1.4 ▶ THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$.

- *The Basepoint.* The element $\star$ of pt.

EXAMPLE 6.1.1.1.5 ▶ THE STANDARD POINTED SET WITH $n + 1$ ELEMENTS

The **standard pointed set with** $n + 1$ **elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \overset{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n \rangle$.

6.1.2   Morphisms of Pointed Sets

DEFINITION 6.1.2.1.1 ▶ MORPHISMS OF POINTED SETS

A **morphism of pointed sets**[1,2] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1]*Further Terminology:* Also called a **pointed function**.

[2]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

> **REMARK 6.1.2.1.2 ▶ UNWINDING DEFINITION 6.1.2.1.1**
>
> In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram
>
> $$
> \begin{array}{ccc}
> & \mathrm{pt} & \\
> {\scriptstyle [x_0]} \swarrow & & \searrow {\scriptstyle [y_0]} \\
> X & \xrightarrow{\ f\ } & Y
> \end{array}
> $$
>
> commutes, i.e. such that
> $$f(x_0) = y_0.$$

6.1.3   The Category of Pointed Sets

> **DEFINITION 6.1.3.1.1 ▶ THE CATEGORY OF POINTED SETS**
>
> The **category of pointed sets** is the category Sets_* defined equivalently as:
>
> - The homotopy category of the ∞-category $\mathrm{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**.
>
> - The category Sets_* of Constructions With Categories, **??**.

> **REMARK 6.1.3.1.2 ▶ UNWINDING DEFINITION 6.1.3.1.1**
>
> In detail, the **category of pointed sets** is the category Sets_* where:
>
> - *Objects.* The objects of Sets_* are pointed sets.
>
> - *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.
>
> - *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map
>
> $$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$
>
>   of Sets_* at (X, x_0) is defined by[1]
>
> $$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X.$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} : \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

---

[1]Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.
[2]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

6.1.4   Elementary Properties of Pointed Sets

PROPOSITION 6.1.4.1.1 ▶ ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

   (a) Products, described as in Definition 6.2.3.1.1.

   (b) Pullbacks, described as in Definition 6.2.4.1.1.

   (c) Equalisers, described as in Definition 6.2.5.1.1.

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

   (a) Coproducts, described as in Definition 6.3.3.1.1.

   (b) Pushouts, described as in Definition 6.3.4.1.1;

   (c) Coequalisers, described as in Definition 6.3.5.1.1.

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[1]

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[2]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

   again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[3]

$$\mathsf{Sets}_* \overset{\mathrm{eq.}}{\cong} \mathsf{Sets}^{\mathrm{part.}}$$

   between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

   (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathrm{part.}}$$

   sends:

      i. A pointed set (X, x_0) to X.

     ii. A pointed function

$$f \colon (X, x_0) \longrightarrow (Y, y_0)$$

   to the partial function

$$\xi_f \colon X \to Y$$

   defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\mathrm{def}}{=} f(x)$$

   for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\mathsf{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

  i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

  ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

---

[1]The category Sets_* does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

[2]In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[3] *Warning:* This is not an isomorphism of categories, only an equivalence.

PROOF 6.1.4.1.2 ▶ PROOF OF PROPOSITION 6.1.4.1.1

Item 1: Completeness

This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.

Item 2: Cocompleteness

This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.
The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].

6.1.5   Active and Inert Morphisms of Pointed Sets

DEFINITION 6.1.5.1.1 ▶ ACTIVE AND INERT MORPHISMS OF POINTED SETS

Let $f \colon (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.

2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.
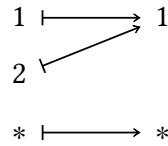

NOTATION 6.1.5.1.2 ▶ THE CATEGORY OF POINTED SETS AND ACTIVE MORPHISMS

We write $\mathsf{Sets}^{\mathrm{actv}}_*$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

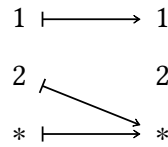**EXAMPLE 6.1.5.1.3 ▶ EXAMPLES OF ACTIVE AND INERT MAPS OF POINTED SETS**

Here are some examples of active and inert maps of pointed sets.

1. The map $\mu\colon \langle 2 \rangle \to \langle 1 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & \nearrow & \\
& & \\
* & \longmapsto & *
\end{array}
$$

   is active but not inert.

2. The map $f\colon \langle 2 \rangle \to \langle 2 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & & 2 \\
* & \longmapsto & *
\end{array}
$$

   is inert but not active.

3. The map $f\colon \langle 3 \rangle \to \langle 1 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & \nearrow & \\
3 & & \\
* & \longmapsto & *
\end{array}
$$
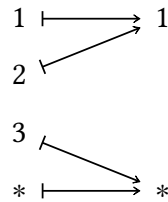
   is neither inert nor active. However, it factors as $f = a \circ i$, where

$$
\begin{aligned}
i &\colon \langle 3 \rangle \to \langle 2 \rangle, \\
a &\colon \langle 2 \rangle \to \langle 1 \rangle
\end{aligned}
$$

are the morphisms of pointed sets given by

$$
\begin{array}{ccc}
1 \longmapsto 1 & \qquad & 1 \longmapsto 1 \\
2 \longmapsto 2 & & 2 \nearrow \\
3 \searrow & & * \longmapsto *, \\
* \longmapsto * &
\end{array}
$$
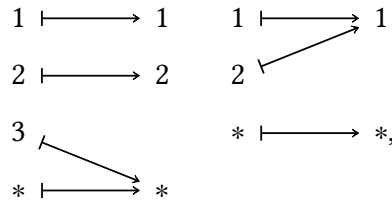
with i being inert and a being active.

PROPOSITION 6.1.5.1.4 ▶ PROPERTIES OF ACTIVE AND INERT MAPS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

   (a) The map $i\colon (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets

   (b) The map $a\colon (K, k_0) \to (Y, y_0)$ is an active morphism of pointed sets.

   Moreover, this determines an orthogonal factorisation system in Sets_*.

PROOF 6.1.5.1.5 ▶ PROOF OF PROPOSITION 6.1.5.1.4

Item 1: Active-Inert Factorisation

Let $f\colon X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$
$$= (X \setminus f^{-1}(y_0)) \cup \{x_0\};$$

- $i \colon X \to K$ is the inert morphism of pointed sets given by

$$i(x) \overset{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

  for each $x \in X$;

- $a \colon K \to Y$ is the active morphism of pointed sets given by

$$a(x) \overset{\text{def}}{=} f(x)$$

  for each $x \in K$.

Next, let

$$
\begin{array}{ccc}
X & \overset{i}{\longrightarrow} & Y \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle g} \\
A & \underset{a}{\longrightarrow} & B
\end{array}
$$

be a commutative diagram in Sets_*. Consider the morphism $\phi \colon Y \to A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$
\begin{array}{ccc}
X & \overset{i}{\longrightarrow} & Y \\
{\scriptstyle f} \downarrow & {\scriptstyle \exists!}\,{\diagup}_{\phi} & \downarrow {\scriptstyle g} \\
A & \underset{a}{\longrightarrow} & B.
\end{array}
$$

Indeed, this diagram commutes, as we have

$$
[\phi \circ i](x) \overset{\text{def}}{=} \phi(i(x))
$$
$$
\overset{\text{def}}{=} f(i^{-1}(i(x)))
$$
$$
= f(x)
$$

for each $x \in X$ and

$$
[a \circ \phi](y) \overset{\text{def}}{=} a(\phi(y))
$$
$$
\overset{\text{def}}{=} a(f(i^{-1}(y)))
$$
$$
\overset{\text{def}}{=} [a \circ f](i^{-1}(y))
$$
$$
= [g \circ i](i^{-1}(y))
$$
$$
\overset{\text{def}}{=} g(i(i^{-1}(y)))
$$
$$
\overset{\text{def}}{=} g(y)
$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & Y \\
{\scriptstyle f}\downarrow & \underset{\psi}{\nearrow} & \downarrow{\scriptstyle g} \\
A & \xrightarrow{\ a\ } & B
\end{array}
$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$
\psi(y) = \psi(i(x))
$$
$$
= f(x)
$$
$$
= f(i^{-1}(y))
$$
$$
\overset{\text{def}}{=} \phi(y).
$$

This finishes the proof.

6.2   Limits of Pointed Sets

6.2.1   The Terminal Pointed Set

DEFINITION 6.2.1.1.1 ▶ THE TERMINAL POINTED SET

The **terminal pointed set** is the terminal object of Sets$_*$ as in Limits and Colimits, **??**.

CONSTRUCTION 6.2.1.1.2 ▶ CONSTRUCTION OF THE TERMINAL POINTED SET

Concretely,   the   **terminal   pointed   set**   is   the   pair
$((\mathrm{pt}, \star), \{!_X\}_{(X,x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \overset{\mathrm{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.2.1.1.3 ▶ PROOF OF CONSTRUCTION 6.2.1.1.2

We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets$_*$. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \; \underset{\exists!}{-\overset{\phi}{-}\to} \; (\mathrm{pt}, \star)$$

commute, namely $!_X$.

6.2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

DEFINITION 6.2.2.1.1 ▶ THE PRODUCT OF A FAMILY OF POINTED SETS

The **product of** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the product of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets$_*$ as in Limits and Colimits, **??**.

CONSTRUCTION 6.2.2.1.2 ▶ CONSTRUCTION OF THE PRODUCT OF A FAMILY OF POINTED SETS

Concretely, the **product of** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right), \left\{\mathrm{pr}_i\right\}_{i \in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$.

- *The Cone.* The collection

$$\left\{\mathrm{pr}_i \colon \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right) \to (X_i, x_0^i)\right\}_{i \in I}$$

  of maps given by

$$\mathrm{pr}_i((x_j)_{j \in I}) \overset{\mathrm{def}}{=} x_i$$

  for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

PROOF 6.2.2.1.3 ▶ PROOF OF CONSTRUCTION 6.2.2.1.2

We claim that $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ is the categorical product of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets$_*$. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{c}
(P, *) \\
 \searrow^{p_i} \\
\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right) \xrightarrow[\mathrm{pr}_i]{} (X_i, x_0^i)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \to (\prod_{i\in I} X_i, (x_0^i)_{i\in I})$$

making the diagram

$$
\begin{array}{c}
(P, *) \\
\phi \downarrow \exists! \qquad \searrow^{p_i} \\
(\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \xrightarrow[\mathrm{pr}_i]{} (X_i, x_0^i)
\end{array}
$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(*) &= (p_i(*))_{i\in I} \\
&= (x_0^i)_{i\in I},
\end{aligned}
$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. ▱

PROPOSITION 6.2.2.1.4 ▶ PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i\in I} \mapsto (\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

PROOF 6.2.2.1.5 ▶ PROOF OF PROPOSITION 6.2.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

6.2.3   Products

Let (X, x_0) and (Y, y_0) be pointed sets.

DEFINITION 6.2.3.1.1 ▶ PRODUCTS OF POINTED SETS

The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets$_*$ as in Limits and Colimits, **??**.

CONSTRUCTION 6.2.3.1.2 ▶ CONSTRUCTION OF PRODUCTS OF POINTED SETS

Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$\mathrm{pr}_1 \colon (X \times Y, (x_0, y_0)) \longrightarrow (X, x_0),$$
$$\mathrm{pr}_2 \colon (X \times Y, (x_0, y_0)) \longrightarrow (Y, y_0)$$

  defined by

$$\mathrm{pr}_1(x, y) \overset{\mathrm{def}}{=} x,$$
$$\mathrm{pr}_2(x, y) \overset{\mathrm{def}}{=} y$$

  for each $(x, y) \in X \times Y$.

PROOF 6.2.3.1.3 ▶ PROOF OF CONSTRUCTION 6.2.3.1.2

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$(P, *)$$

$$p_1 \qquad\qquad p_2$$

$$(X, x_0) \xleftarrow[\mathrm{pr}_1]{} (X \times Y, (x_0, y_0)) \xrightarrow[\mathrm{pr}_2]{} (Y, y_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \longrightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$(P, *)$$

$$p_1 \qquad\qquad p_2$$

$$\phi \;\vert\; \exists!$$

$$(X, x_0) \xleftarrow[\mathrm{pr}_1]{} (X \times Y, (x_0, y_0)) \xrightarrow[\mathrm{pr}_2]{} (Y, y_0)$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

$$= (x_0, y_0),$$

where we have used that p_1 and p_2 are morphisms of pointed sets.   🏳️‍⚧️

PROPOSITION 6.2.3.1.4 ▶ PROPERTIES OF PRODUCTS OF POINTED SETS

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

    $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

    define functors

    $$
    \begin{aligned}
    A \times - &: & \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\
    - \times B &: & \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\
    -_1 \times -_2 &: & \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*,
    \end{aligned}
    $$

    defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

    $$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

    natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Unitality.* We have isomorphisms of pointed sets

    $$
    \begin{aligned}
    (\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\
    (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),
    \end{aligned}
    $$

    natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Commutativity.* We have an isomorphism of pointed sets

    $$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

    natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.

Item 2: Lack of Adjointness

See [MSE 2855868].

Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

Item 4: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.

Item 5: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.

6.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

DEFINITION 6.2.4.1.1 ▶ PULLBACKS OF POINTED SETS

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets$_*$ as in Limits and Colimits, **??**.

CONSTRUCTION 6.2.4.1.2 ▶ CONSTRUCTION OF PULLBACKS OF POINTED SETS

Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$
\mathrm{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),
$$
$$
\mathrm{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)
$$

defined by

$$
\mathrm{pr}_1(x, y) \overset{\mathrm{def}}{=} x,
$$
$$
\mathrm{pr}_2(x, y) \overset{\mathrm{def}}{=} y
$$

for each $(x, y) \in X \times_Z Y$.

PROOF 6.2.4.1.3 ▶ PROOF OF CONSTRUCTION 6.2.4.1.2

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have
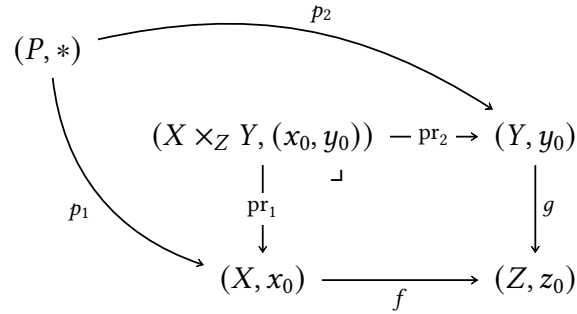
$$
f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,
$$

$$
\begin{array}{ccc}
(X \times_Z Y, (x_0, y_0)) & \xrightarrow{\mathrm{pr}_2} & (Y, y_0) \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
(X, x_0) & \xrightarrow[f]{} & (Z, z_0).
\end{array}
$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$
\begin{aligned}
[f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y)) \\
&= f(x) \\
&= g(y) \\
&= g(\mathrm{pr}_2(x, y)) \\
&= [g \circ \mathrm{pr}_2](x, y),
\end{aligned}
$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a
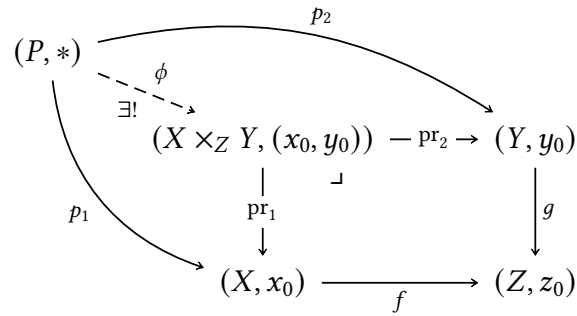
diagram of the form

$$
\begin{array}{c}
(P, *) \xrightarrow{\quad p_2 \quad} \\
\end{array}
$$

$$
(X \times_Z Y, (x_0, y_0)) \; -\mathrm{pr}_2 \to \; (Y, y_0)
$$

$$
\mathrm{pr}_1 \big\downarrow \qquad \qquad \big\downarrow g
$$

$$
(X, x_0) \xrightarrow{\quad f \quad} (Z, z_0)
$$

with p_1 from $(P,*)$ to (X, x_0).

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$
\phi \colon (P, *) \longrightarrow (X \times_Z Y, (x_0, y_0))
$$

making the diagram

$$
\begin{array}{c}
(P, *) \xrightarrow{\quad p_2 \quad} \\
\end{array}
$$

$$
\phi \quad \exists!
$$

$$
(X \times_Z Y, (x_0, y_0)) \; -\mathrm{pr}_2 \to \; (Y, y_0)
$$

$$
\mathrm{pr}_1 \big\downarrow \qquad \qquad \big\downarrow g
$$

$$
(X, x_0) \xrightarrow{\quad f \quad} (Z, z_0)
$$

commute, being uniquely determined by the conditions

$$
\mathrm{pr}_1 \circ \phi = p_1,
$$
$$
\mathrm{pr}_2 \circ \phi = p_2
$$

via

$$
\phi(x) = (p_1(x), p_2(x))
$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$
f \circ p_1 = g \circ p_2,
$$

which gives
$$f(p_1(x)) = g(p_2(x))$$
for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
$$= (x_0, y_0),$$

where we have used that p_1 and p_2 are morphisms of pointed sets.

PROPOSITION 6.2.4.1.4 ▶ PROPERTIES OF PULLBACKS OF POINTED SETS

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor
$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:

$$
\begin{array}{ccc}
 & & \bullet \\
 & & \downarrow \\
\bullet & \longrightarrow & \bullet.
\end{array}
$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism
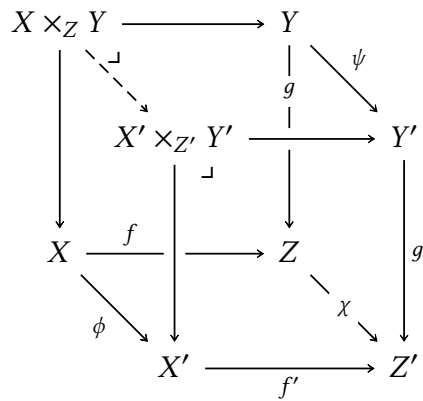
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x_0', y_0'))$$

given by

$$\xi(x, y) \overset{\mathrm{def}}{=} (\phi(x), \psi(y))$$
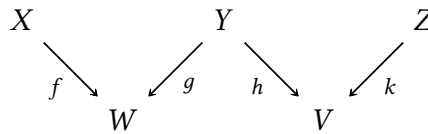
for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

commute.

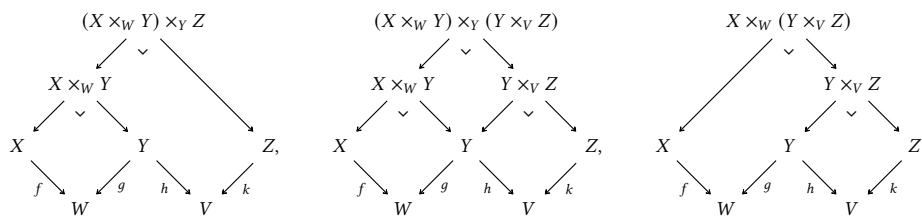2. *Associativity.* Given a diagram

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

3. *Unitality.* We have isomorphisms of pointed sets

$$
\begin{array}{ccc}
A & = & A \\
\scriptstyle f \downarrow & \lrcorner & \downarrow \scriptstyle f \\
X & = & X
\end{array}
\qquad
\begin{array}{c}
X \times_X A \cong A, \\
A \times_X X \cong A,
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
\| & \lrcorner & \| \\
X & \xrightarrow{\ f\ } & X.
\end{array}
$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{array}{ccc}
A \times_X B & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow \scriptstyle g \\
A & \xrightarrow{\ f\ } & X,
\end{array}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{array}{ccc}
B \times_X A & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow \scriptstyle f \\
B & \xrightarrow{\ g\ } & X.
\end{array}
$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow & \lrcorner & \downarrow \scriptstyle !_Y \\
X & \xrightarrow{\ !_X\ } & \mathrm{pt}.
\end{array}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

PROOF 6.2.4.1.5 ► **PROOF OF PROPOSITION 6.2.4.1.4**

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.4.1.7.

Item 3: Unitality

This follows from Constructions With Sets, Item 6 of Proposition 4.1.4.1.7.

Item 4: Commutativity

This follows from Constructions With Sets, Item 7 of Proposition 4.1.4.1.7.

Item 5: Interaction With Products

This follows from Constructions With Sets, Item 10 of Proposition 4.1.4.1.7.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 11 of Proposition 4.1.4.1.7.

6.2.5   Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

DEFINITION 6.2.5.1.1 ▶ EQUALISERS OF POINTED SETS

The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in Limits and Colimits, **??**.

CONSTRUCTION 6.2.5.1.2 ▶ CONSTRUCTION OF EQUALISERS OF POINTED SETS

Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.

- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

PROOF 6.2.5.1.3 ▶ PROOF OF CONSTRUCTION 6.2.5.1.2

We claim that $(\mathrm{Eq}(f,g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\mathrm{Eq}(f,g), x_0) \xrightarrow{\mathrm{eq}(f,g)} (X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$$

$$e$$

$$(E, *)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \longrightarrow (\mathrm{Eq}(f,g), x_0)$$

making the diagram

$$(\mathrm{Eq}(f,g), x_0) \xrightarrow{\mathrm{eq}(f,g)} (X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$$

$$\phi \,\vert\, \exists! \qquad e$$

$$(E, *)$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives
$$f(e(x)) = g(e(x))$$
for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have
$$\phi(*) = e(*)$$
$$= x_0,$$
where we have used that e is a morphism of pointed sets.

PROPOSITION 6.2.5.1.4 ▶ PROPERTIES OF EQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

    where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \; \begin{array}{c} f \\ \xrightarrow{} \\ \xrightarrow{\;-\,g\,\to\;} \\ \xrightarrow{} \\ h \end{array} \; (Y, y_0)$$

    in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

PROOF 6.2.5.1.5 ▶ PROOF OF PROPOSITION 6.2.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.1.5.1.4.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.1.5.1.4.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.1.5.1.4.

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

DEFINITION 6.3.1.1.1 ▶ THE INITIAL POINTED SET

The **initial pointed set** is the initial object of Sets_* as in Limits and Colimits, **??**.

CONSTRUCTION 6.3.1.1.2 ▶ CONSTRUCTION OF THE INITIAL POINTED SET

Concretely, the **initial pointed set** is the pair $((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathrm{pt}, \star) \to (X, x_0)\}_{(X,x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$\iota_X(\star) \overset{\mathrm{def}}{=} x_0.$$

PROOF 6.3.1.1.3 ▶ PROOF OF CONSTRUCTION 6.3.1.1.2

We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathrm{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \ \underset{\exists!}{\overset{\phi}{\dashrightarrow}}\ (X, x_0)$$

commute, namely ι_X.

6.3.2   Coproducts of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

DEFINITION 6.3.2.1.1 ▶ COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family** $\left\{(X_i, x_0^i)\right\}_{i \in I}$[1] is the coproduct of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets_* as in Limits and Colimits, **??**.

---

[1]*Further Terminology:* Also called the **wedge sum of the family** $\left\{(X_i, x_0^i)\right\}_{i \in I}$.

CONSTRUCTION 6.3.2.1.2 ▶ CONSTRUCTION OF COPRODUCTS OF FAMILIES OF POINTED SETS

Concretely, the **coproduct of the family** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \left\{\mathrm{inj}_i\right\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by
  $$\bigvee_{i \in I} X_i \overset{\mathrm{def}}{=} (\coprod_{i \in I} X_i)/{\sim},$$

where $\sim$ is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

– *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \overset{\text{def}}{=} [(i, x_0^i)]$$
$$= [(j, x_0^j)]$$

for any $i, j \in I$.

• *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon (X_i, x_0^i) \to (\bigvee_{i \in I} X_i, p_0) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

PROOF 6.3.2.1.3 ▶ PROOF OF CONSTRUCTION 6.3.2.1.2

We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{ccc}
 & & (C, *) \\
 & \nearrow^{\iota_i} & \\
(X_i, x_0^i) & \xrightarrow[\mathrm{inj}_i]{} & (\bigvee_{i \in I} X_i, p_0)
\end{array}
$$

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$\phi \colon (\bigvee_{i \in I} X_i, p_0) \longrightarrow (C, *)$$

making the diagram

$$
\begin{array}{ccc}
 & & (C, *) \\
 & \nearrow^{\iota_i} & \uparrow \\
 & & \phi \downarrow \exists! \\
(X_i, x_0^i) & \xrightarrow[\mathrm{inj}_i]{} & (\bigvee_{i \in I} X_i, p_0)
\end{array}
$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(p_0) &= \iota_i([(i, x_0^i)]) \\
 &= *,
\end{aligned}
$$

as ι_i is a morphism of pointed sets.

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor
$$\bigvee_{i \in I} \colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \longrightarrow \mathsf{Sets}_*.$$

PROOF 6.3.2.1.5 ▶ PROOF OF PROPOSITION 6.3.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

DEFINITION 6.3.3.1.1 ▶ COPRODUCTS OF POINTED SETS

The **coproduct of** (X, x_0) **and** (Y, y_0)[1] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

---

[1]*Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

CONSTRUCTION 6.3.3.1.2 ▶ CONSTRUCTION OF COPRODUCTS OF POINTED SETS

Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

  - *The Underlying Set.* The set $X \vee Y$ defined by

$$(X \vee Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0)$$
$$\cong (X \coprod_{\text{pt}} Y, p_0)$$
$$\cong (X \coprod Y/{\sim}, p_0),$$

$$\begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow & \ulcorner & \uparrow \\ \Big\uparrow & & \Big\downarrow {\scriptstyle [y_0]} \\ X & \longleftarrow & \text{pt,} \\ & {\scriptstyle [x_0]} & \end{array}$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$
$$= [(1, y_0)].$$

- *The Cocone.* The morphisms of pointed sets

$$\mathrm{inj}_1 \colon (X, x_0) \to (X \vee Y, p_0),$$
$$\mathrm{inj}_2 \colon (Y, y_0) \to (X \vee Y, p_0),$$

given by

$$\mathrm{inj}_1(x) \overset{\mathrm{def}}{=} [(0, x)],$$
$$\mathrm{inj}_2(y) \overset{\mathrm{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.3.1.3 ▶ PROOF OF CONSTRUCTION 6.3.3.1.2

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets$_*$. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccccc}
 & & (C, *) & & \\
 & \iota_1 \nearrow & & \nwarrow \iota_2 & \\
 & & & & \\
(X, x_0) & \underset{\mathrm{inj}_1}{\hookrightarrow} & (X \vee Y, p_0) & \underset{\mathrm{inj}_2}{\hookleftarrow} & (Y, y_0)
\end{array}
$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \to (C, *)$$

making the diagram

$$
\begin{array}{ccccc}
 & & (C, *) & & \\
 & \iota_1 \nearrow & \uparrow \phi \;\; \exists! & \nwarrow \iota_2 & \\
 & & & & \\
(X, x_0) & \underset{\mathrm{inj}_1}{\hookrightarrow} & (X \vee Y, p_0) & \underset{\mathrm{inj}_2}{\hookleftarrow} & (Y, y_0)
\end{array}
$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_X = \iota_X,$$
$$\phi \circ \mathrm{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$
$$= \iota_Y([(1, y_0)])$$
$$= *,$$

as ι_X and ι_Y are morphisms of pointed sets.

PROPOSITION 6.3.3.1.4 ▶ PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

   define functors

$$X \vee - : \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$- \vee Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$-_1 \vee -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, *) \vee (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \vee (\mathrm{pt}, *) \cong (X, x_0),$$

   natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

   natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

   called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

   at X is given by

$$\nabla_X(p) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

   for each $p \in X \vee X$.

PROOF 6.3.3.1.5 ▶ PROOF OF PROPOSITION 6.3.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad
\begin{array}{ccc}
X \vee X & \xrightarrow{\nabla_X} & X \\
{\scriptstyle f \vee f}\downarrow & & \downarrow{\scriptstyle f} \\
Y \vee Y & \xrightarrow[\nabla_Y]{} & Y.
\end{array}$$

Indeed, we have

$$
\begin{aligned}
[\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\
&= f(x) \\
&= f(\nabla_X([(i, x)])) \\
&= [f \circ \nabla_X]([(i, x)])
\end{aligned}
$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation.

6.3.4   Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

DEFINITION 6.3.4.1.1 ▶ PUSHOUTS OF POINTED SETS

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets$_*$ as in Limits and Colimits, **??**.

CONSTRUCTION 6.3.4.1.2 ▶ CONSTRUCTION OF PUSHOUTS OF POINTED SETS

Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:

    - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
    - We have $p_0 = [x_0] = [y_0]$.

- *The Cocone.* The morphisms of pointed sets

$$\mathrm{inj}_1\colon (X, x_0) \to (X \coprod_Z Y, p_0),$$
$$\mathrm{inj}_2\colon (Y, y_0) \to (X \coprod_Z Y, p_0)$$

  given by

$$\mathrm{inj}_1(x) \overset{\mathrm{def}}{=} [(0, x)]$$
$$\mathrm{inj}_2(y) \overset{\mathrm{def}}{=} [(1, y)]$$

  for each $x \in X$ and each $y \in Y$.

PROOF 6.3.4.1.3 ▶ PROOF OF ??

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$
$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g,$$

$$
\begin{array}{ccc}
(X \coprod_Z Y, p_0) & \xleftarrow{\mathrm{inj}_2} & (Y, y_0) \\
{\scriptstyle \mathrm{inj}_1} \uparrow & & \uparrow {\scriptstyle g} \\
(X, x_0) & \xleftarrow{\quad f \quad} & (Z, z_0).
\end{array}
$$

Indeed, given $z \in Z$, we have

$$
\begin{aligned}
[\mathrm{inj}_1 \circ f](z) &= \mathrm{inj}_1(f(z)) \\
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \mathrm{inj}_2(g(z)) \\
&= [\mathrm{inj}_2 \circ g](z),
\end{aligned}
$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram
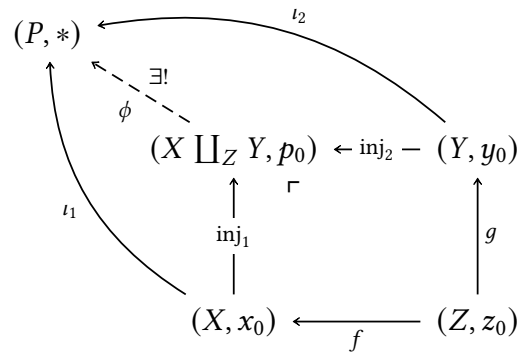
of the form

$$(P, *) \xleftarrow{\quad \iota_2 \quad}$$

$$(X \amalg_Z Y, p_0) \leftarrow \text{inj}_2 - (Y, y_0)$$

$$\uparrow \text{inj}_1 \qquad \ulcorner \qquad \uparrow g$$

$$\iota_1 \qquad (X, x_0) \xleftarrow{\quad f \quad} (Z, z_0)$$
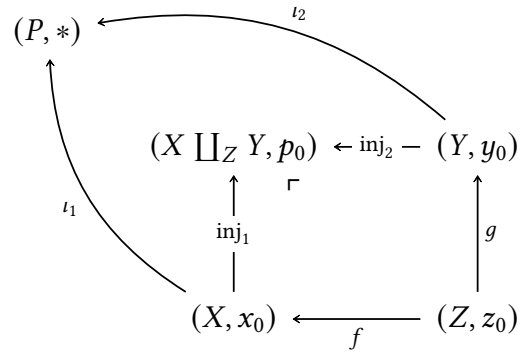
in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \amalg_Z Y, p_0) \to (P, *)$$

making the diagram

$$(P, *) \xleftarrow{\quad \iota_2 \quad}$$

$$\xleftarrow{\quad \exists! \quad}_{\phi}$$

$$(X \amalg_Z Y, p_0) \leftarrow \text{inj}_2 - (Y, y_0)$$

$$\uparrow \text{inj}_1 \qquad \ulcorner \qquad \uparrow g$$

$$\iota_1 \qquad (X, x_0) \xleftarrow{\quad f \quad} (Z, z_0)$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$
$$\phi \circ \text{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)]) \\
&= \iota_1(x_0) \\
&= *,
\end{aligned}$$

or alternatively

$$\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)]) \\
&= \iota_2(y_0) \\
&= *,
\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

PROPOSITION 6.3.4.1.4 ▶ PROPERTIES OF PUSHOUTS OF POINTED SETS

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

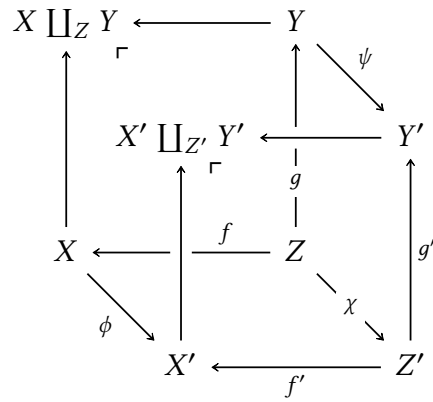1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{array}{c}
\bullet \\
\uparrow \\
\bullet \longleftarrow \bullet.
\end{array}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by
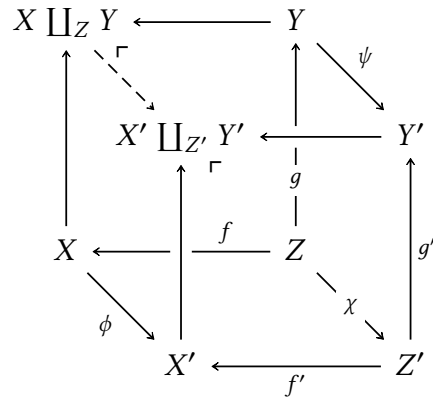
sending a morphism

$$
\begin{array}{ccc}
X \coprod_Z Y & \longleftarrow & Y \\
& & \downarrow \psi \\
X' \coprod_{Z'} Y' & \longleftarrow & Y' \\
& g & \\
X \longleftarrow \overset{f}{} Z & & g' \\
\phi & & \chi \\
X' & \longleftarrow_{f'} & Z'
\end{array}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$
\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p_0')
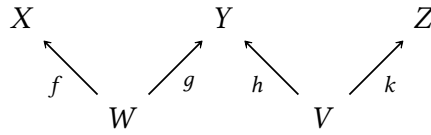$$

given by

$$
\xi(p) \overset{\mathrm{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}
$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
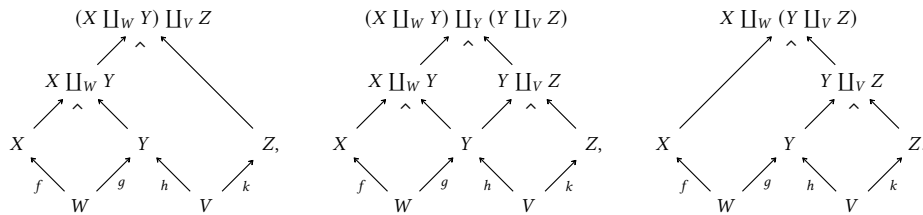\begin{array}{ccc}
X \coprod_Z Y & \longleftarrow & Y \\
& & \downarrow \psi \\
X' \coprod_{Z'} Y' & \longleftarrow & Y' \\
& g & \\
X \longleftarrow \overset{f}{} Z & & g' \\
\phi & & \chi \\
X' & \longleftarrow_{f'} & Z'
\end{array}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{array}{ccccc}
X & & Y & & Z \\
\nwarrow\ f & \nearrow\ g & \nwarrow\ h & \nearrow\ k & \\
& W & & V &
\end{array}
$$

in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams

3. *Unitality.* We have isomorphisms of sets

$$
\begin{array}{ccc}
A & = & A \\
f\uparrow & \ulcorner & \uparrow f \\
X & = & X
\end{array}
\qquad
\begin{array}{l}
X \amalg_X A \cong A, \\
A \amalg_X X \cong A,
\end{array}
\qquad
\begin{array}{ccc}
A & \xleftarrow{\ f\ } & X \\
\| & \ulcorner & \| \\
X & \xleftarrow{\ f\ } & X.
\end{array}
$$

4. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
X \amalg_Z Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow g \\
X & \xleftarrow{\ f\ } & Z,
\end{array}
\qquad
X \amalg_Z Y \cong Y \amalg_Z X
\qquad
\begin{array}{ccc}
Y \amalg_Z X & \longleftarrow & X \\
\uparrow & \ulcorner & \uparrow f \\
Y & \xleftarrow{\ g\ } & Z.
\end{array}
$$

5. *Interaction With Coproducts.* We have

$$X \coprod_{\mathrm{pt}} Y \cong X \vee Y,$$

$$
\begin{array}{ccc}
X \vee Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \uparrow {\scriptstyle [y_0]} \\
X & \xrightarrow{[x_0]} & \mathrm{pt.}
\end{array}
$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

PROOF 6.3.4.1.5 ▶ **PROOF OF PROPOSITION 6.3.4.1.4**

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.

Item 3: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.

Item 4: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.

Item 5: Interaction With Coproducts

Omitted.

Item 6: Symmetric Monoidality

Omitted.

6.3.5   Coequalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

DEFINITION 6.3.5.1.1 ▶ COEQUALISERS OF POINTED SETS

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

CONSTRUCTION 6.3.5.1.2 ▶ CONSTRUCTION OF COEQUALISERS OF POINTED SETS

The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.

- *The Cocone.* The map

$$\mathrm{coeq}(f, g) \colon Y \twoheadrightarrow (\mathrm{CoEq}(f, g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Construction 4.2.5.1.2.

PROOF 6.3.5.1.3 ▶ PROOF OF CONSTRUCTION 6.3.5.1.2

We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$
\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](x) &\stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](f(x)) \\
&\stackrel{\mathrm{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](g(x)) \\
&\stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f, g) \circ g](x)
\end{aligned}
$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \overset{f}{\underset{g}{\rightrightarrows}} (Y, y_0) \xrightarrow{\mathrm{coeq}(f,g)} (\mathrm{CoEq}(f, g), [y_0])$$
$$\overset{c}{\searrow}$$
$$(C, *)$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Proposition 10.6.2.1.3 that there exists a unique map $\phi \colon \mathrm{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$(X, x_0) \overset{f}{\underset{g}{\rightrightarrows}} (Y, y_0) \xrightarrow{\mathrm{coeq}(f,g)} (\mathrm{CoEq}(f, g), [y_0])$$
$$\overset{c}{\searrow} \quad \phi \Big\downarrow \exists!$$
$$(C, *)$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \mathrm{coeq}(f, g)]([y_0])$$
$$= c([y_0])$$
$$= *,$$

where we have used that c is a morphism of pointed sets. ▨

PROPOSITION 6.3.5.1.4 ▶ PROPERTIES OF COEQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f, g, h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \; \substack{f \\ \overset{-g\to}{\longrightarrow} \\ h} \; (Y, y_0)$$

in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

PROOF 6.3.5.1.5 ▶ PROOF OF PROPOSITION 6.3.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.2.5.1.7.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.2.5.1.7.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.5.1.7.

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let X be a set.

DEFINITION 6.4.1.1.1 ▶ FREE POINTED SETS

The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[1]

$$X^+ \overset{\text{def}}{=} X \coprod \text{pt}$$
$$\overset{\text{def}}{=} X \coprod \{\star\}.$$

- *The Basepoint.* The element $\star$ of X^+.

---

[1]*Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

PROPOSITION 6.4.1.1.2 ▶ PROPERTIES OF FREE POINTED SETS

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

   where:

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \overset{\text{def}}{=} X^+,$$

     where X^+ is the pointed set of Definition 6.4.1.1.1.
   - *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image
$$f^+ \colon X^+ \to Y^+$$
     of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘 \right) \colon \quad \mathsf{Sets} \underset{忘}{\overset{(-)^+}{\rightleftarrows}} \bot \; \mathsf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^{+,\coprod}, (-)_{\mathbb{1}}^{+,\coprod}) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} \colon X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \coprod Y)^+,$$
$$(-)_{\mathbb{1}}^{+,\coprod} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)_{\mathbb{1}}^+) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+,$$
$$(-)_{\mathbb{1}}^+ \colon S^0 \overset{\sim}{\dashrightarrow} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.4.1.1.3 ▶ PROOF OF PROPOSITION 6.4.1.1.2

Item 1: Functoriality

We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}_X^+(x) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\mathrm{id}_X^+ = \mathrm{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$f \colon X \to Y,$$
$$g \colon Y \to Z,$$

we have

$$[g^+ \circ f^+](x) \overset{\mathrm{def}}{=} g^+(f^+(x))$$
$$\overset{\mathrm{def}}{=} g^+(f(x))$$
$$\overset{\mathrm{def}}{=} g(f(x))$$
$$\overset{\mathrm{def}}{=} [g \circ f]^+(x)$$

for each $x \in X$ and

$$[g^+ \circ f^+](\star_X) \overset{\mathrm{def}}{=} g^+(f^+(\star_X))$$
$$\overset{\mathrm{def}}{=} g^+(\star_Y)$$
$$\overset{\mathrm{def}}{=} \star_Z$$
$$\overset{\mathrm{def}}{=} [g \circ f]^+(\star_X),$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2: Adjointness

We proceed in a few steps:

- *Map I.* We define a map

$$\Phi_{X,Y}\colon \mathsf{Sets}_*(X^+, Y) \longrightarrow \mathsf{Sets}(X, Y)$$

  by sending a morphism of pointed sets

$$\xi\colon (X^+, \star_X) \longrightarrow (Y, y_0)$$

  to the function

$$\xi^\dagger\colon X \longrightarrow Y$$

  given by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \xi(x)$$

  for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}(X, Y) \longrightarrow \mathsf{Sets}_*(X^+, Y)$$

  given by sending a function $\xi\colon X \to Y$ to the morphism of pointed sets

$$\xi^\dagger\colon (X^+, \star_X) \longrightarrow (Y, y_0)$$

  defined by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

  for each $x \in X^+$.

- *Invertibility I.* Given a morphism of pointed sets

$$\xi\colon (X^+, \star_X) \longrightarrow (Y, y_0),$$

  we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\mathrm{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^\dagger) \end{aligned}$$

$$\stackrel{\text{def}}{=} [\![x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}]\!]$$

$$= [\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}]\!]$$

$$= \xi$$

$$\stackrel{\text{def}}{=} [\text{id}_{\mathsf{Sets}_*(X^+,Y)}](\xi).$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\mathsf{Sets}_*(X^+,Y)} .$$

- *Invertibility II.* Given a map of sets $\xi \colon X \to Y$, we have

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) \stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi))$$

$$= \Phi_{X,Y}(\xi^\dagger)$$

$$= \Phi_{X,Y}([\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}]\!])$$

$$= [\![x \mapsto \xi(x)]\!]$$

$$= \xi$$

$$\stackrel{\text{def}}{=} [\text{id}_{\mathsf{Sets}(X,Y)}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\mathsf{Sets}(X,Y)} .$$

- *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (X', x_0'),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*(X'^{,+}, Y) & \xrightarrow{\ \Phi_{X',Y}\ } & \mathsf{Sets}(X', Y) \\
{\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*(X^+, Y) & \xrightarrow[\ \Phi_{X,Y}\ ]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+} \to Y$, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi))\\
&= \Phi_{X,Y}(\xi \circ f)\\
&= \xi \circ f\\
&= \Phi_{X',Y}(\xi) \circ f\\
&= f^*(\Phi_{X',Y}(\xi))\\
&= f^*(\Phi_{X',Y}(\xi))\\
&= [f^* \circ \Phi_{X',Y}](\xi).
\end{aligned}
$$

Therefore we have

$$
\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}
$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets
$$
g\colon (Y, y_0) \to (Y', y_0'),
$$
the diagram

$$
\begin{array}{ccc}
\mathrm{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \mathrm{Sets}(X, Y)\\
\Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle g_*}\\
\mathrm{Sets}_*(X^+, Y'), & \xrightarrow[\Phi_{X,Y'}]{} & \mathrm{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, given a morphism of pointed sets

$$
\xi^\dagger\colon X^+ \to Y,
$$

we have

$$
[\Phi_{X,Y'} \circ g_*](\xi) = \Phi_{X,Y'}(g_*(\xi))
$$

$$= \Phi_{X,Y'}(g \circ \xi)$$
$$= g \circ \xi$$
$$= g \circ \Phi_{X,Y'}(\xi)$$
$$= g_*(\Phi_{X,Y'}(\xi))$$
$$= [g_* \circ \Phi_{X,Y'}](\xi).$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^+$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{+,\coprod} : X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

  is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

  for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+,\coprod,-1} : (X \coprod Y)^+ \xrightarrow{\sim} X^+ \vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \varnothing^+$$

  is given by sending $\star_X$ to $\star_\varnothing$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to $\times$ and $\wedge$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+$$

  is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

  for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)_{X,Y}^{+,-1} \colon (X \times Y)^+ \overset{\sim}{\dashrightarrow} X^+ \wedge Y^+$$

  given by

$$(-)_{X,Y}^{+,-1}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

  for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\mathbb{1}}_{X,Y} \colon S^0 \dashrightarrow^{\sim} \mathrm{pt}^+$$

  is given by sending 0 to $\star_{\mathrm{pt}}$ and 1 to $\star$, where $\mathrm{pt}^+ = \{\star, \star_{\mathrm{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

6.4.2   Deleting Basepoints

Let (X, x_0) be a pointed set.

DEFINITION 6.4.2.1.1 ▶ SETS WITH DELETED BASEPOINTS

The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\mathrm{def}}{=} X \setminus \{x_0\}.$$

PROPOSITION 6.4.2.1.2 ▶ PROPERTIES OF SETS WITH DELETED BASEPOINTS

Let (X, x_0) be a pointed set.

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- \colon \mathsf{Sets}^{\mathrm{actv}}_* \to \mathsf{Sets},$$

   where:

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets}^{\mathrm{actv}}_*)$, we have

   $$[(-)^-](X) \overset{\mathrm{def}}{=} X^-,$$

     where X^- is the set of Definition 6.4.2.1.1.

   - *Action on Morphisms.* For each morphism $f \colon X \to Y$ of $\mathsf{Sets}^{\mathrm{actv}}_*$, the image

   $$f^- \colon X^- \to Y^-$$

     of f by $(-)^-$ is the map defined by

   $$f^-(x) \overset{\mathrm{def}}{=} f(x)$$

for each $x \in X^-$.

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+): \quad \mathsf{Sets}_*^{\mathrm{actv}} \underset{(-)^+}{\overset{(-)^-}{\underset{\perp_{\mathrm{eq}}}{\rightleftarrows}}} \mathsf{Sets},$$

witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*(X, Y^+),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^-)^+ \cong X,$$
$$(Y^+)^- \cong Y,$$

once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\vee}, (-)_{\mathbb{1}}^{-,\vee}): (\mathsf{Sets}_*^{\mathrm{actv}}, \vee, \mathrm{pt}), \to (\mathsf{Sets}, \coprod, \varnothing),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee}: X^- \coprod Y^- \overset{\sim}{\dashrightarrow} (X \vee Y)^-,$$
$$(-)_{\mathbb{1}}^{-,\vee}: \varnothing \overset{\sim}{\dashrightarrow} \mathrm{pt}^-,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)_{\mathbb{1}}^{-,\times}): (\mathsf{Sets}_*^{\mathrm{actv}}, \wedge, S^0), \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^-: X^- \times Y^- \overset{\sim}{\dashrightarrow} (X \wedge Y)^-,$$
$$(-)_{\mathbb{1}}^-: \mathrm{pt} \overset{\sim}{\dashrightarrow} (S^0)^-,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.4.2.1.3 ▶ PROOF OF PROPOSITION 6.4.2.1.2

Item 1: Functoriality

We claim that $(-)^-$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}_X^-(x) \overset{\mathrm{def}}{=} x$$

  for each $x \in X^-$, so $\mathrm{id}_X^- = \mathrm{id}_{X^-}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f \colon (X, x_0) \longrightarrow (Y, y_0),$$
$$g \colon (Y, y_0) \longrightarrow (Z, z_0),$$

  we have

$$
\begin{aligned}
[g^- \circ f^-](x) &\overset{\mathrm{def}}{=} g^-(f^-(x)) \\
&\overset{\mathrm{def}}{=} g^-(f(x)) \\
&\overset{\mathrm{def}}{=} g(f(x)) \\
&\overset{\mathrm{def}}{=} [g \circ f]^-(x)
\end{aligned}
$$

  for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2: Adjoint Equivalence

We proceed in a few steps:

1. *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-, Y) \longrightarrow \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)$$

   by sending a map $\xi \colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^\dagger \colon X \to Y^+$$

given by

$$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}_*^{\text{actv}}(X, Y^+) \longrightarrow \mathsf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi \colon X \to Y^+$ to the map

$$\xi^\dagger \colon X^- \to Y$$

defined by

$$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi \colon X^- \to Y$, we have

$$
\begin{aligned}
{[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi)} &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} \Psi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]\right) \\
&= [\![x \mapsto \xi(x)]\!] \\
&= \xi \\
&= [\mathrm{id}_{\mathsf{Sets}(X^-,Y)}](\xi).
\end{aligned}
$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}.$$

4. *Invertibility II.* Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) \overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi))$$
$$= \Phi_{X,Y}([\![x \mapsto \xi(x)]\!])$$
$$= [\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}]\!]$$
$$= \xi$$
$$= [\text{id}_{\mathsf{Sets}_*^{\text{actv}}(X,Y^+)}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\mathsf{Sets}_*^{\text{actv}}(X,Y^+)}.$$

5. *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (X', x_0'),$$

the diagram

$$\begin{array}{ccc}
\mathsf{Sets}(X'^{,-}, Y) & \xrightarrow{\Phi_{X',Y}} & \mathsf{Sets}_*^{\text{actv}}(X', Y^+) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*(X^-, Y) & \xrightarrow[\Phi_{X,Y}]{} & \mathsf{Sets}_*^{\text{actv}}(X, Y^+)
\end{array}$$

commutes. Indeed, given a map of sets $\xi \colon X' \to Y$, we have

$$[\Phi_{X,Y} \circ f^*](\xi) = \Phi_{X,Y}(f^*(\xi))$$
$$= \Phi_{X,Y}(\xi \circ f)$$
$$= [\![x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x_0' \end{cases}]\!]$$
$$= f^*([\![x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x_0' \end{cases}]\!])$$

$$= f^*(\Phi_{X',Y}(\xi))$$
$$= [f^* \circ \Phi_{X',Y}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets
$$g \colon (Y, y_0) \longrightarrow (Y', y_0'),$$
the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(X^-, Y) & \xrightarrow{\ \Phi_{X,Y}\ } & \mathsf{Sets}^{\mathrm{actv}}_*(X, Y^+) \\[4pt]
{\scriptstyle g_*}\big\downarrow & & \big\downarrow{\scriptstyle g_*} \\[4pt]
\mathsf{Sets}(X^-, Y') & \xrightarrow[\ \Phi_{X,Y'}\ ]{} & \mathsf{Sets}^{\mathrm{actv}}_*(X, Y'^{,+})
\end{array}
$$

commutes. Indeed, given a map of sets $\xi \colon X^- \to Y$, we have

$$[\Phi_{X,Y'} \circ g_*](\xi) = \Phi_{X,Y'}(g_*(\xi))$$
$$= \Phi_{X,Y'}(g \circ \xi)$$
$$= \left[\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \right]\!\right]$$
$$= g_*\!\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \right]\!\right]\right)$$
$$= g_*(\Phi_{X,Y'}(\xi))$$
$$= [g_* \circ \Phi_{X,Y'}](\xi).$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

7. *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

8. *Fully Faithfulness of $(-)^-$.* We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)^-_{X,Y} \colon \mathsf{Sets}^{\mathrm{actv}}_*(X,Y) \overset{\sim}{\dashrightarrow} \mathsf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)^{-,-1}_{X,Y} \colon \mathsf{Sets}(X^-, Y^-) \overset{\sim}{\dashrightarrow} \mathsf{Sets}^{\mathrm{actv}}_*(X,Y)$$

is given by sending a map of sets $f \colon X^- \to Y^-$ to the active morphism of pointed sets $f^\dagger \colon X \to Y$ defined by

$$f^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

9. *Essential Surjectivity of $(-)^-$.* We need to show that, given an object $X \in \mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}(\mathsf{Sets}^{\mathrm{actv}}_*)$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$\begin{aligned} (X^+)^- &\overset{\mathrm{def}}{=} (X \cup \{\star_X\})^- \\ &\overset{\mathrm{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. *The Functor $(-)^-$ Is an Equivalence.* Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Proposition 11.6.7.1.2.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^-$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-,\vee}_{X,Y} : X^- \coprod Y^- \dashrightarrow^{\sim} (X \vee Y)^-$$

  is given by

$$(-)^{-,\vee}_{X,Y}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

  for each $z \in X^- \coprod Y^-$, with inverse

$$(-)^{-,\vee,-1}_{X,Y} : (X \vee Y)^- \dashrightarrow^{\sim} X^- \coprod Y^-$$

  given by

$$(-)^{-,\vee,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

  for each $z \in (X \vee Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\vee,\mathbb{1}}_{X,Y} : \emptyset \dashrightarrow^{\sim} \text{pt}^-$$

  is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^- \colon X^- \times Y^- \dashrightarrow^{\sim} (X \wedge Y)^-$$

  is given by

$$(-)_{X,Y}^-(x,y) = x \wedge y$$

  for each $(x,y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1} \colon (X \wedge Y)^- \dashrightarrow^{\sim} X^- \times Y^-$$

  given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \overset{\text{def}}{=} (x,y)$$

  for each $x \wedge y \in (X \wedge Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{-,\mathbb{1}} \colon \mathrm{pt} \dashrightarrow^{\sim} (S^0)^-$$

  is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. ⬌

Appendices

A   Other Chapters

References

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[MSE 884460]   Martin Brandenburg. *Why are the category of pointed sets and the category of sets and partial functions "essentially the same"?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/884460 (cit. on p. 8).