

# Constructions With Monoidal Categories

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July 22, 2025

**01UF** This chapter contains some material on constructions with monoidal categories.

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## 01UG 13.1 Moduli Categories of Monoidal Structures

### 01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let  $C$  be a category.

01UJ **Definition 13.1.1.1.** The **moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow \tilde{\omega} \\ \text{pt} & \xrightarrow{[C]} & \text{Cats}. \end{array}$$

01UK **Remark 13.1.1.2.** In detail, the **moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- *Objects.* The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is  $C$ .
- *Morphisms.* A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  is a strong monoidal functor structure

$$\text{id}_C^\otimes: A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$

$$\text{id}_{\mathbb{1}|C}^\otimes: \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor  $\text{id}_C: C \rightarrow C$  of  $C$ .

- *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{pt} \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $M$  is defined by

$$\text{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \stackrel{\text{def}}{=} (\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes),$$

where  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  is the identity monoidal functor of  $C$  of ??.

- *Composition.* For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{B}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_1}(C)} : \text{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(N, P) \times \text{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M, N) \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M, P)$$

of  $\mathcal{M}_{\mathbb{B}_1}(C)$  at  $(M, N, P)$  is defined by

$$\left( \text{id}_C^{\otimes, '}, \text{id}_{\mathbb{1}|C}^{\otimes, '}, \right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_1}(C)} \left( \text{id}_C^{\otimes}, \text{id}_{\mathbb{1}|C}^{\otimes} \right) \stackrel{\text{def}}{=} \left( \text{id}_C^{\otimes, '}, \text{id}_C^{\otimes}, \text{id}_{\mathbb{1}|C}^{\otimes, '}, \text{id}_{\mathbb{1}|C}^{\otimes} \right).$$

**01UL Remark 13.1.1.3.** In particular, a morphism in  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, '}, \lambda^{C, '}, \rho^{C, '})$  satisfies the following conditions:

- 01UM** 1. *Naturality.* For each pair  $f: A \rightarrow X$  and  $g: B \rightarrow Y$  of morphisms of  $C$ , the diagram

$$\begin{array}{ccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \text{id}_{A,B}^{\otimes} \downarrow & & \downarrow \text{id}_{X,Y}^{\otimes} \\ A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y \end{array}$$

commutes.

- 01UN** 2. *Monoidality.* For each  $A, B, C \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc} & (A \boxtimes_C B) \boxtimes_C C & \\ \text{id}_{A,B}^{\otimes} \boxtimes_C \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^{C, '}, \\ (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \otimes_C B, C}^{\otimes} \downarrow & & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\ \alpha_{A,B,C}^C \swarrow & & \swarrow \text{id}_{A,B \otimes_C C}^{\otimes} \\ & A \otimes_C (B \otimes_C C) & \end{array}$$

commutes.

01UP 3. *Left Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & \mathbb{1}_C \boxtimes_C A & \xrightarrow{\text{id}_{\mathbb{1}_C}^\otimes \otimes \text{id}_A} \mathbb{1}_C \otimes_C A \\
 \text{id}_{\mathbb{1}_C}^\otimes \boxtimes \text{id}_A \nearrow & & \searrow \lambda_A^C \\
 \mathbb{1}_C' \boxtimes_C A & \xrightarrow{\lambda_A^{C,'}} & A
 \end{array}$$

commutes.

01UQ 4. *Right Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & A \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{A, \mathbb{1}_C}^\otimes} A \otimes_C \mathbb{1}_C \\
 \text{id}_A \boxtimes \text{id}_{\mathbb{1}_C}^\otimes \nearrow & & \searrow \rho_A^C \\
 A \boxtimes_C \mathbb{1}_C' & \xrightarrow{\rho_A^{C,'}} & A
 \end{array}$$

commutes.

01UR **Proposition 13.1.1.1.4.** Let  $C$  be a category.

01US 1. *Extra Monoidality Conditions.* Let  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}_C', \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$ .

01UT (a) The diagram

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\text{id}_{A,B}^\otimes \boxtimes \text{id}_C} & (A \otimes_C B) \boxtimes_C C \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A \otimes_C B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\text{id}_{A,B}^\otimes \otimes \text{id}_C} & (A \otimes_C B) \otimes_C C
 \end{array}$$

commutes.

01UU (b) The diagram

$$\begin{array}{ccc}
 A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \boxtimes \text{id}_{B,C}^\otimes} & A \boxtimes_C (B \otimes_C C) \\
 \text{id}_{A, B \boxtimes_C C}^\otimes \downarrow & & \downarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 A \otimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \otimes \text{id}_{B,C}^\otimes} & A \otimes_C (B \otimes_C C)
 \end{array}$$

commutes.

**01WB** 2. *Extra Monoidal Unity Constraints.* Let  $(\text{id}_C^\otimes, \text{id}_{1|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, 1_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, 1'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$ .

**01WC** (a) The diagram

$$\begin{array}{ccc} 1_C \otimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \boxtimes_C 1'_C \\ \lambda_{1'_C}^C \downarrow & & \downarrow \rho_{1_C}^{C'} \\ 1'_C & \xrightarrow{\text{id}_1^\otimes} & 1_C \end{array}$$

commutes.

**01WD** (b) The diagram

$$\begin{array}{ccc} 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1'_C, 1_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C \\ \rho_{1'_C}^C \downarrow & & \downarrow \lambda_{1_C}^{C'} \\ 1'_C & \xrightarrow{\text{id}_1^\otimes} & 1_C \end{array}$$

commutes.

**01WE** (c) The diagram

$$\begin{array}{ccc} 1'_C \boxtimes_C 1_C & \xrightarrow{\text{id}_{1'_C, 1_C}^\otimes} & 1'_C \otimes_C 1_C \\ \lambda_{1_C}^{C'} \downarrow & & \downarrow \rho_{1'_C}^C \\ 1_C & \xrightarrow{\text{id}_1^{\otimes, -1}} & 1'_C \end{array}$$

commutes.

01WF (d) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \rho_{\mathbb{1}_C}^{C, \prime} \downarrow & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

01UV 3. *Mixed Associators.* Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, \prime}, \lambda^{C, \prime}, \rho^{C, \prime})$  be monoidal structures on  $C$  and let

$$\text{id}_{-1, -2}^\otimes: -1 \boxtimes_C -2 \rightarrow -1 \otimes_C -2$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha_{A, B, C}^\otimes: (A \otimes_C B) \boxtimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A, B, C}^\otimes} & A \otimes_C (B \boxtimes_C C) \\
 \text{id}_{A \otimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \otimes_C \text{id}_{B, C}^\otimes \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A, B, C}^C} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A, B, C}^{C, \prime}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A, B}^\otimes \boxtimes_C \text{id}_C \downarrow & & \downarrow \text{id}_{A, B \boxtimes_C C} \\
 (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A, B, C}^\otimes} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidal-ity condition of **Item 2** of **Definition 13.1.1.3**.

01UX

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \\ \text{id}_{A,B}^{\otimes} \otimes \text{id}_C \downarrow & & \downarrow \text{id}_{A,B \otimes_C C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\boxtimes}} & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \boxtimes_C B,C}^{\otimes} \downarrow & & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^{\otimes} \\ (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \end{array}$$

commute, then the natural transformation  $\text{id}^{\otimes}$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

01UY

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes,\otimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \text{id}_{A,B}^{\otimes} \otimes \text{id}_C \downarrow & & \downarrow \text{id}_A \otimes \text{id}_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C'}} & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \boxtimes_C B, C}^{\otimes} & & \downarrow \text{id}_{A, B \boxtimes_C C}^{\otimes} \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^{\otimes}$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

*Proof.* **Item 1, Extra Monoidality Conditions:** We claim that **Items 1a** and **1b** are indeed true:

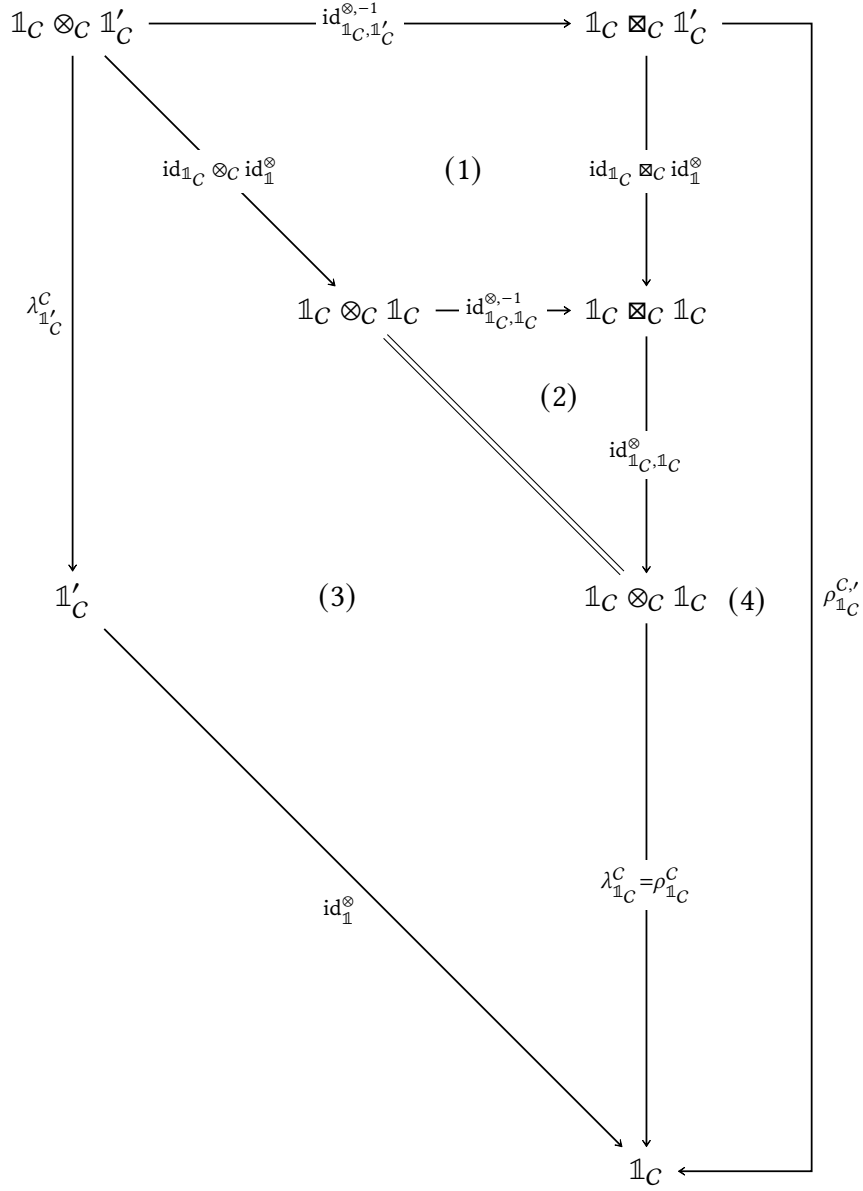
1. *Proof of Item 1a:* This follows from the naturality of  $\text{id}^{\otimes}$  with respect to the morphisms  $\text{id}_{A,B}^{\otimes}$  and  $\text{id}_C$ .
2. *Proof of Item 1b:* This follows from the naturality of  $\text{id}^{\otimes}$  with respect to the morphisms  $\text{id}_A$  and  $\text{id}_{B,C}^{\otimes}$ .

This finishes the proof.

**Item 2, Extra Monoidal Unity Constraints:** We claim that **Items 2a** and **2b** are indeed true:



1. *Proof of Item 1a:* Indeed, consider the diagram



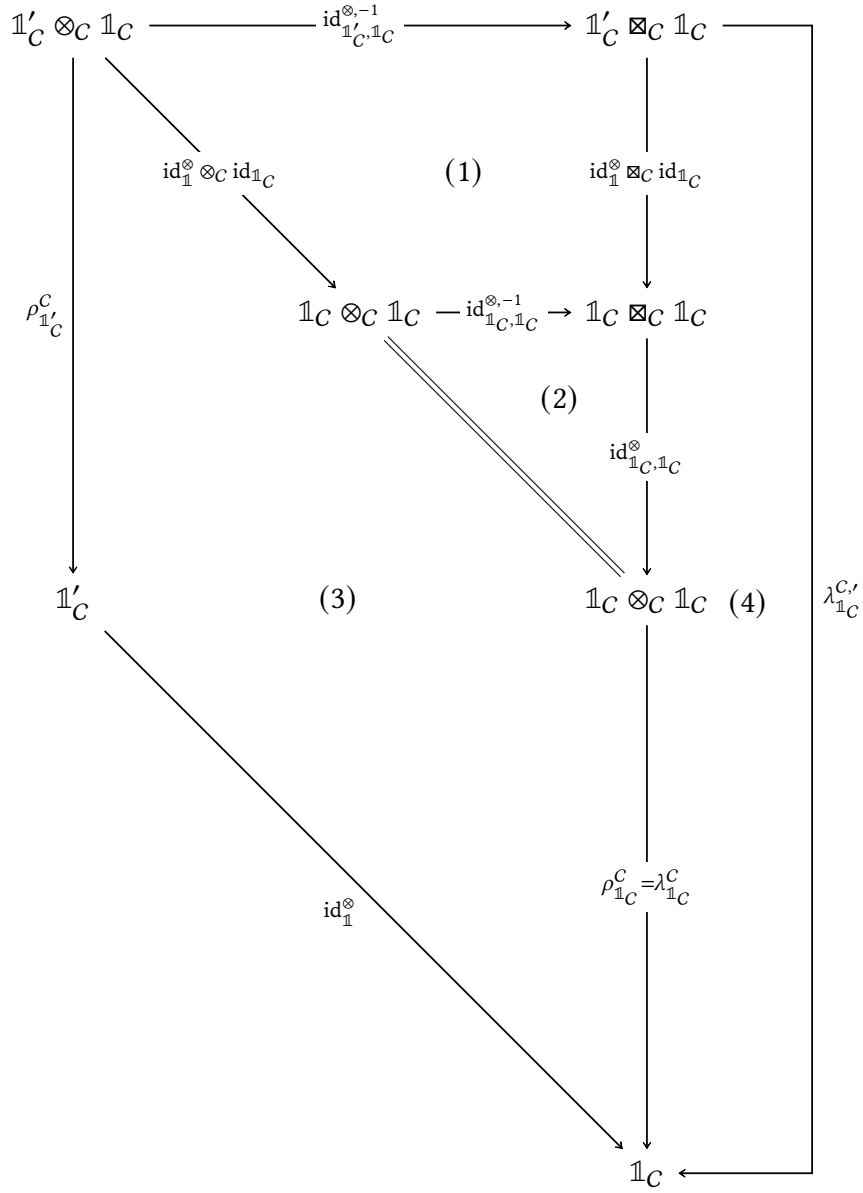
whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of  $(\mathrm{id}_C, \mathrm{id}_C^\otimes, \mathrm{id}_{C|\mathbb{1}}^\otimes)$ ;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of  $(\text{id}_C, \text{id}_C^\otimes, \text{id}_{C|\mathbb{1}}^\otimes)$ ;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \downarrow \rho_{\mathbb{1}'_C}^C & & \downarrow \lambda_{\mathbb{1}_C}^{C, \prime} & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^\otimes} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C.
 \end{array}
 \quad \begin{array}{c} (1) \end{array} \quad \begin{array}{c} (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1b**;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 & & \downarrow \lambda_{\mathbb{1}_C}^{C, \prime} & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}
 \quad \begin{array}{c} (\dagger)$$

commutes. But since  $\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & & \downarrow \rho_{\mathbb{1}_C}^{C, \prime} & & \downarrow \lambda_{\mathbb{1}_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}
 \quad \begin{array}{ccc}
 & (1) & \\
 & & (\dagger)
 \end{array}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1a**;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 & & \downarrow \rho_{\mathbb{1}_C}^{C, \prime} & & \downarrow \lambda_{\mathbb{1}_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}
 \quad (\dagger)$$

commutes. But since  $\text{id}_{\mathbb{1}}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

**Item 3, Mixed Associators:** We claim that **Items 3a to 3c** are indeed true:

**01UZ** 1. *Proof of Item 3a:* We may partition the monoidality diagram for  $\text{id}^{\otimes}$  of

Item 2 of Definition 13.1.1.3 as follows:

$$\begin{array}{ccccc}
 & (A \boxtimes_C B) \boxtimes_C C & & & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & \downarrow & \searrow \alpha_{A,B,C}^C & & \\
 (A \otimes_C B) \boxtimes_C C & \text{id}_{A \boxtimes_C B, C}^\otimes & A \boxtimes_C (B \boxtimes_C C) & & \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) \downarrow & (2) \downarrow & & \\
 & (A \boxtimes_C B) \otimes_C C & & & \\
 \text{id}_{A,B}^\otimes \otimes_C \text{id}_C \swarrow & \downarrow & \searrow \alpha_{A,B,C}^\otimes & & \\
 (A \otimes_C B) \otimes_C C & (3) & A \boxtimes_C (B \otimes_C C) & & \\
 \searrow \alpha_{A,B,C}^{C'} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & & 
 \end{array}$$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.3.

01V0 2. Proof of Item 3b: We may partition the monoidality diagram for  $\text{id}^\otimes$  of

Item 2 of Definition 13.1.1.3 as follows:

$$\begin{array}{ccccc}
 & (A \boxtimes_C B) \boxtimes_C C & & & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^C & & \\
 (A \otimes_C B) \boxtimes_C C & (1) & A \boxtimes_C (B \boxtimes_C C) & & \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \swarrow \alpha_{A,B,C}^\boxtimes & \swarrow \text{id}_{A,B \boxtimes_C C}^\otimes & \searrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes & \\
 & A \otimes_C (B \boxtimes_C C) & & & \\
 & (2) & (3) & & \\
 \downarrow \text{id}_A \otimes_C \text{id}_{B,C}^\otimes & & & & \\
 (A \otimes_C B) \otimes_C C & \downarrow \text{id}_A \otimes_C \text{id}_{B,C}^\otimes & A \boxtimes_C (B \otimes_C C) & & \\
 \searrow \alpha_{A,B,C}^{C'} & \downarrow & \swarrow \text{id}_{A,B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & &
 \end{array}$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.3.

01V1 3. *Proof of Item 3c:* We may partition the monoidality diagram for  $\text{id}^\otimes$  of

Item 2 of Definition 13.1.1.3 as follows:

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^C \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^{C'} & (2) & \swarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.3.

This finishes the proof.  $\square$

01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category

01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category

01V4 13.2 Moduli Categories of Closed Monoidal Structures

01V5 13.3 Moduli Categories of Refinements of Monoidal Structures

01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure



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# Appendices

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2. A Guide to the Literature

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5. Monoidal Structures on the Category of Sets
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8. Relations
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11. Categories
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15. Notes