Pointed Sets

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This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

Definition 6.1.1.1.1. A **pointed set**¹ is equivalently:

- · An \mathbb{E}_0 -monoid in (N_•(Sets), pt).
- · A pointed object in (Sets, pt).

Remark 6.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- · The Underlying Set. A set X, called the **underlying set of** (X, x_0) .
- · The Basepoint. A morphism

$$[x_0]: pt \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 6.1.1.1.3. The 0-sphere² is the pointed set $(S^0, 0)^3$ consisting of:

· The Underlying Set. The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

• The Basepoint. The element 0 of S^0 .

Example 6.1.1.1.4. The **trivial pointed set** is the pointed set (pt, \star) consisting of:

¹Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -modules.

² Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

³Further Notation: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

- The Underlying Set. The punctual set pt $\stackrel{\text{def}}{=} \{ \star \}$.
- · *The Basepoint*. The element ★ of pt.

Example 6.1.1.1.5. The **standard pointed set with** n+1 **elements** is the pointed set $\langle n \rangle$ consisting of

· The Underlying Set. The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{ * \} \cup \{ 1, \dots, n \}.$$

• The Basepoint. The element * of $\langle n \rangle$.

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1. A morphism of pointed sets^{4,5} is equivalently:

- · A morphism of \mathbb{E}_0 -monoids in (N_•(Sets), pt).
- · A morphism of pointed objects in (Sets, pt).

Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f:(X,x_0)\to (Y,y_0)$ is a morphism of sets $f:X\to Y$ such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & & [y_0] \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1. The **category of pointed sets** is the category Sets_{*} defined equivalently as:

⁴ Further Terminology: Also called a **pointed function**.

⁵ Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of** \mathbb{F}_1 -**modules**.

- · The homotopy category of the ∞ -category $\mathsf{Mon}_{\mathbb{B}_0}(\mathsf{N}_{\bullet}(\mathsf{Sets}),\mathsf{pt})$ of ??, ??.
- The category Sets* of Constructions With Categories, ??.

Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category Sets* where:

- · Objects. The objects of Sets* are pointed sets.
- · Morphisms. The morphisms of Sets, are morphisms of pointed sets.
- · *Identities.* For each $(X, x_0) \in Obj(Sets_*)$, the unit map

$$\mathbb{1}_{(X,x_0)}^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets_{*} at (X, x_0) is defined by⁶

$$id_{(X,x_0)}^{Sets_*} \stackrel{\text{def}}{=} id_X$$
.

· Composition. For each $(X,x_0),(Y,y_0),(Z,z_0)\in \mathsf{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ_{(X,x_0),(Y,y_0),(Z,z_0)}^{\mathsf{Sets}_*} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

of Sets_{*} at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by⁷

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

6.1.4 Elementary Properties of Pointed Sets

Proposition 6.1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness*. The category Sets* of pointed sets and morphisms between them is complete, having in particular:

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

⁶Note that id_X is indeed a morphism of pointed sets, as we have id_X(x_0) = x_0 .

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

- (a) Products, described as in Definition 6.2.3.1.1.
- (b) Pullbacks, described as in Definition 6.2.4.1.1.
- (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets* of pointed sets and morphisms between them is cocomplete, having in particular:
 - (a) Coproducts, described as in Definition 6.3.3.1.1.
 - (b) Pushouts, described as in Definition 6.3.4.1.1;
 - (c) Coequalisers, described as in Definition 6.3.5.1.1.
- 3. Failure To Be Cartesian Closed. The category Sets, is not Cartesian closed.8
- 4. Morphisms From the Monoidal Unit. We have a bijection of sets⁹

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0,X)\cong (X,x_0),$$

again natural in $(X, x_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$.

5. Relation to Partial Functions. We have an equivalence of categories 10

$$\mathsf{Sets}_* \stackrel{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

⁸The category Sets_{*} does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

⁹In other words, the forgetful functor

¹⁰ Warning: This is not an isomorphism of categories, only an equivalence.

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \stackrel{\cong}{\to} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

- i. A pointed set (X, x_0) to X.
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1}$$
: Sets^{part.} $\stackrel{\cong}{\to}$ Sets_{*}

sends:

- i. A set X is to the pointed set (X, \star) with \star an element that is not in X.
- ii. A partial function

$$f: X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, *Completeness*: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathsf{Sets}_*\big(S^0,X\big)\cong (X,x_0)$$

follows by noting that $\Delta_{x_0}\colon S^0\to X$, the basepoint of $\mathbf{Sets}_*(S^0,X)$, corresponds to the pointed map $S^0\to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0\to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [MSE 884460].

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1. Let $f: (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

- 1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
- 2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2. We write $Sets^{actv}_*$ for the wide subcategory of $Sets_*$ spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

1. The map $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$ given by

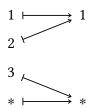


is active but not inert.

2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by

is inert but not active.

3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by

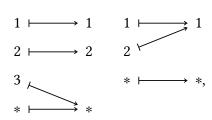


is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i: \langle 3 \rangle \to \langle 2 \rangle,$$

 $a: \langle 2 \rangle \to \langle 1 \rangle$

are the morphisms of pointed sets given by



with *i* being inert and *a* being active.

Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. Active-Inert Factorisation. Every morphism of pointed sets $f\colon (X,x_0)\to (Y,y_0)$ factors uniquely as

$$f = a \circ i$$
,

where:

- (a) The map $i: (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets
- (b) The map $a: (K, k_0) \to (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_{*}.

Proof. Item 1, Active-Inert Factorisation: Let $f: X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\longrightarrow} K \stackrel{a}{\longrightarrow} Y$$
,

where:

 $\cdot K$ is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$

= $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$

 $i: X \to K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

 $\cdot a: K \to Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$X \xrightarrow{i} Y$$

$$f \downarrow \qquad \qquad \downarrow g$$

$$A \xrightarrow{g} B$$

be a commutative diagram in Sets_* . Consider the morphism $\phi\colon Y\to A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{array}{c|c}
X & \xrightarrow{i} & Y \\
f & \exists ! & \downarrow g \\
A & \xrightarrow{g} & B.
\end{array}$$

Indeed, this diagram commutes, as we have

$$[\phi \circ i](x) \stackrel{\text{def}}{=} \phi(i(x))$$
$$\stackrel{\text{def}}{=} f(i^{-1}(i(x)))$$
$$= f(x)$$

for each $x \in X$ and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & & \downarrow g \\
A & \xrightarrow{a} & B
\end{array}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1. The **terminal pointed set** is the terminal object of Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.1.1.2. Concretely, the **terminal pointed set** is the pair (pt, \star) , $\{!_X\}_{(X,x_0) \in Obj(Sets_*)}$ consisting of:

- · The Limit. The pointed set (pt, \star) .
- · The Cone. The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathsf{pt}, \star)\}_{(X, x_0) \in \mathsf{Obi}(\mathsf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in Obj(Sets)$.

Proof. We claim that (pt, \star) is the terminal object of Sets*. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt, \star)

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathsf{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists !}} (\mathsf{pt}, \star)$$

commute, namely $!_X$.

6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 6.2.2.1.1. The **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*} as in Limits and Colimits, ??.

Construction 6.2.2.1.2. Concretely, the **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$ consisting of:

- · The Limit. The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- · The Cone. The collection

$$\left\{ \operatorname{pr}_{i} : \left(\prod_{i \in I} X_{i}, \left(x_{0}^{i} \right)_{i \in I} \right) \to \left(X_{i}, x_{0}^{i} \right) \right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_i\Big(\big(x_j\big)_{j\in I}\Big)\stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j\in I}\in\prod_{i\in I}X_i$ and each $i\in I$.

Proof. We claim that $\left(\prod_{i\in I}X_i,\left(x_0^i\right)_{i\in I}\right)$ is the categorical product of $\left\{\left(X_i,x_0^i\right)\right\}_{i\in I}$ in Sets $_*$. Indeed, suppose we have, for each $i\in I$, a diagram of the form

$$(P, *) \xrightarrow{p_i} (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \xrightarrow{\mathsf{pr}_i} (X_i, x_0^i)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left(\prod_{i \in I} X_i, \left(x_0^i \right)_{i \in I} \right)$$

making the diagram

$$(P, *)$$

$$\downarrow \qquad \qquad p_i$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \xrightarrow{\mathsf{pr}_i} (X_i, x_0^i)$$

commute, being uniquely determined by the condition $\operatorname{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

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for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= (x_0^i)_{i \in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$.

Proposition 6.2.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??.

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets* as in Limits and Colimits, ??.

Construction 6.2.3.1.2. Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- · The Limit. The pointed set $(X \times Y, (x_0, y_0))$.
- · The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 \colon (X \times Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

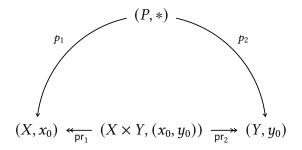
 $\operatorname{pr}_2(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and

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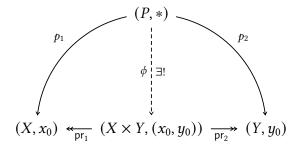
 (Y, y_0) in Sets_* . Indeed, suppose we have a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$

 $\operatorname{pr}_2 \circ \phi = p_2$

via

$$\phi(x)=(p_1(x),p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 6.2.3.1.3. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

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1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$A \times -:$$
 Sets_{*} \rightarrow Sets_{*},
 $- \times B:$ Sets_{*} \rightarrow Sets_{*},
 $-_1 \times -_2:$ Sets_{*} \times Sets_{*} \rightarrow Sets_{*},

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

- 2. Lack of Adjointness. The functors $X \times -$ and $\times Y$ do not admit right adjoints.
- 3. Associativity. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in Obj(Sets_*).$

4. Unitality. We have isomorphisms of pointed sets

$$(\mathsf{pt}, \star) \times (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \times (\mathsf{pt}, \star) \cong (X, x_0),$

natural in $(X, x_0) \in Obj(Sets_*)$.

5. *Commutativity*. We have an isomorphism of pointed sets

$$(X\times Y,(x_0,y_0))\cong (Y\times X,(y_0,x_0)),$$
 natural in $(X,x_0),(Y,y_0)\in \mathsf{Obj}(\mathsf{Sets}_*).$

6. Symmetric Monoidality. The triple (Sets_{*}, \times , (pt, \star)) is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2, Lack of Adjointness: See [MSE 2855868].

Item 3, *Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.

Item 4, *Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.

Item 5, *Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.

Item 6, *Symmetric Monoidality*: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3.

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6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets $_*$ as in Limits and Colimits, ??.

Construction 6.2.4.1.2. Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- · The Limit. The pointed set $(X \times_Z Y, (x_0, y_0))$.
- · The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_2(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets $_*$. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad (X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$

$$\downarrow g \qquad \qquad \downarrow g \qquad \qquad \downarrow g \qquad \qquad (X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

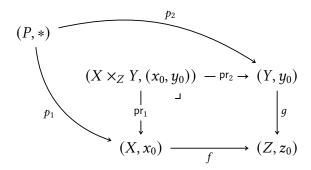
Indeed, given $(x, y) \in X \times_Z Y$, we have

$$[f \circ \operatorname{pr}_1](x, y) = f(\operatorname{pr}_1(x, y))$$
$$= f(x)$$
$$= g(y)$$

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$$= g(\operatorname{pr}_2(x, y))$$
$$= [g \circ \operatorname{pr}_2](x, y),$$

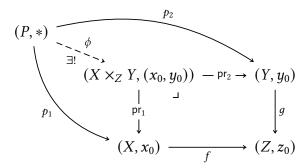
where f(x) = g(y) since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$

 $\operatorname{pr}_2 \circ \phi = p_2$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f\circ p_1=g\circ p_2,$$

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which gives

$$f(p_1(x)) = q(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 6.2.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

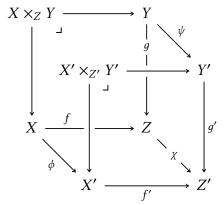
1. Functoriality. The assignment $(X,Y,Z,f,g)\mapsto X\times_{f,Z,g}Y$ defines a functor

$$-_1 \times_{-_3} -_1 : \operatorname{\mathsf{Fun}}(\mathcal{P}, \operatorname{\mathsf{Sets}}_*) \to \operatorname{\mathsf{Sets}}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \times_{-3} -1$ is given by sending a morphism



in $Fun(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

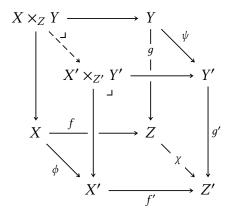
$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x,y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

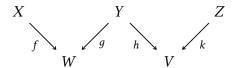
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for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

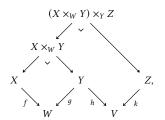
2. Associativity. Given a diagram

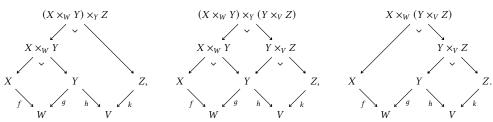


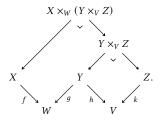
in Sets*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams





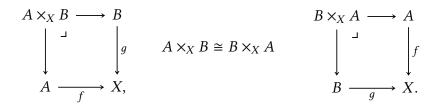


3. *Unitality*. We have isomorphisms of pointed sets



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4. Commutativity. We have an isomorphism of pointed sets



5. Interaction With Products. We have an isomorphism of pointed sets

$$X \times_{\mathsf{pt}} Y \cong X \times Y, \qquad \qquad \begin{matrix} X \times Y & \longrightarrow & Y \\ & & & \downarrow !_{Y} \\ & X & \xrightarrow{!_{Y}} & \mathsf{pt}. \end{matrix}$$

6. Symmetric Monoidality. The triple (Sets_{*}, \times_X , X) is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, *Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.

Item 3, *Unitality*: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5. *Item* 4, *Commutativity*: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.

Item 5, Interaction With Products: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.

Item 6, *Symmetric Monoidality*: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5.

6.2.5 Equalisers

Let $f, g: (X, x_0) \Rightarrow (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1. The **equaliser of** (f, g) is the equaliser of f and g in Sets $_*$ as in Limits and Colimits, ??.

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Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, q) is the pair consisting of:

- · The Limit. The pointed set $(Eq(f, g), x_0)$.
- · The Cone. The morphism of pointed sets

$$eq(f,q): (Eq(f,q),x_0) \hookrightarrow (X,x_0)$$

given by the canonical inclusion $eq(f,g) \hookrightarrow Eq(f,g) \hookrightarrow X$.

Proof. We claim that $(Eq(f,g),x_0)$ is the categorical equaliser of f and g in $Sets_*$. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ eq(f,g) = g \circ eq(f,g),$$

which indeed holds by the definition of the set Eq(f,g). Next, we prove that Eq(f,g) satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\mathsf{Eq}(f,g),x_0) \xrightarrow{\mathsf{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$(E,*)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathsf{Eq}(f, q), x_0)$$

making the diagram

$$(\mathsf{Eq}(f,g),x_0) \xrightarrow{\mathsf{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow^{\phi} \exists ! \qquad e$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f, g) \circ \phi = e$$

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via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in Eq(f, g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that *e* is a morphism of pointed sets.

Proposition 6.2.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\operatorname{Eq}(f \circ \operatorname{eq}(g,h), g \circ \operatorname{eq}(g,h))}_{=\operatorname{Eq}(f \circ \operatorname{eq}(g,h), h \circ \operatorname{eq}(g,h))} \cong \underbrace{\operatorname{Eq}(f,g,h)}_{=\operatorname{Eq}(g \circ \operatorname{eq}(f,g), h \circ \operatorname{eq}(f,g))} \cong \underbrace{\operatorname{Eq}(f \circ \operatorname{eq}(f,g), h \circ \operatorname{eq}(f,g))}_{=\operatorname{Eq}(g \circ \operatorname{eq}(f,g), h \circ \operatorname{eq}(f,g))}$$

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets_{*}, being explicitly given by

$$Eq(f, a, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, f) \cong X.$$

3. Commutativity. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, q) \cong \operatorname{Eq}(q, f).$$

Proof. Item 1, *Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3. Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3.

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1. The **initial pointed set** is the initial object of Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.1.1.2. Concretely, the **initial pointed set** is the pair $(pt, \star), \{\iota_X\}_{(X,x_0) \in \text{Obj}(\mathsf{Sets}_*)}$ consisting of:

- · The Limit. The pointed set (pt, \star) .
- · The Cone. The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathsf{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathsf{Obi}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_{*}. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
 (X, x_0)

in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathsf{pt}, \star) \xrightarrow{-\frac{\phi}{\exists !}} (X, x_0)$$

commute, namely ι_X .

6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

Definition 6.3.2.1.1. The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}^{11}$ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*} as in Limits and Colimits, ??.

¹¹ Further Terminology: Also called the **wedge sum of the family** $\left\{\left(X_{i},x_{0}^{i}\right)\right\}_{i\in I}$

Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \{\inf_i\}_{i \in I})$ consisting of:

- · The Colimit. The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
 - The Underlying Set. The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

– The Basepoint. The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \stackrel{\text{def}}{=} \left[\left(i, x_0^i \right) \right] \\ = \left[\left(j, x_0^j \right) \right]$$

for any $i, j \in I$.

· The Cocone. The collection

$$\left\{\mathsf{inj}_i\colon \left(X_i,x_0^i\right)\to \left(\bigvee_{i\in I}X_i,p_0\right)\right\}_{i\in I}$$

of morphism of pointed sets given by

$$\mathsf{inj}_i(x) \stackrel{\text{\tiny def}}{=} (i,x)$$

for each $x \in X_i$ and each $i \in I$.

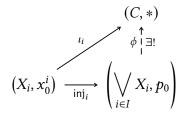
Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_{*}. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$(X_i, x_0^i) \xrightarrow[\text{inj}_i]{l_i} \left(\bigvee_{i \in I} X_i, p_0\right)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i\in I} X_i, p_0\right) \to (C, *)$$

making the diagram



commute, being uniquely determined by the condition $\phi \circ \operatorname{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$
= *,

as ι_i is a morphism of pointed sets.

Proposition 6.3.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. Functoriality. The assignment $\left\{\left(X_i,x_0^i\right)\right\}_{i\in I}\mapsto (\bigvee_{i\in I}X_i,p_0)$ defines a functor

$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??.

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** $(Y, y_0)^{12}$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- · The Colimit. The pointed set $(X \vee Y, p_0)$ consisting of:
 - The Underlying Set. The set $X \vee Y$ defined by

where \sim is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- The Basepoint. The element p_0 of $X \vee Y$ defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$

= $[(1, y_0)].$

· The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \vee Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \vee Y, p_0),$

given by

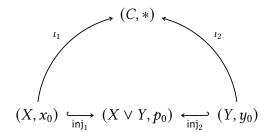
$$\begin{aligned} &\inf_{1}(x) \stackrel{\text{def}}{=} [(0,x)], \\ &\inf_{2}(y) \stackrel{\text{def}}{=} [(1,y)], \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0)

¹² Further Terminology: Also called the **wedge sum of** (X, x_0) **and** (Y, y_0) .

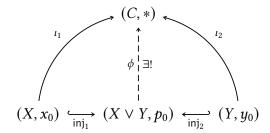
in Sets_{*}. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$

 $\phi \circ \operatorname{inj}_Y = \iota_Y$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

= $\iota_Y([(1, y_0)])$
= *,

as ι_X and ι_Y are morphisms of pointed sets.

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Proposition 6.3.3.1.3. Let (X, x_0) and (Y, y_0) be pointed sets.

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$

2. Associativity. We have an isomorphism of pointed sets

$$(X \lor Y) \lor Z \cong X \lor (Y \lor Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in Sets_*$.

3. Unitality. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \lor (pt, *) \cong (X, x_0),$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$
,

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

- 5. Symmetric Monoidality. The triple (Sets $_*$, \vee , pt) is a symmetric monoidal category.
- 6. The Fold Map. We have a natural transformation

$$\nabla\colon \vee\circ\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}\Longrightarrow \mathsf{id}_{\mathsf{Sets}_*}, \qquad \begin{array}{c} \mathsf{Sets}_*\times\mathsf{Sets}_*\\ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}& & \\ & & \vee\\ \mathsf{Sets}_*& & \\ & & & \vee\\ \mathsf{Sets}_*& & \\ & & & & \\ \mathsf{Sets}_*, \end{array}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

Item 5, Symmetric Monoidality: Omitted.

Item 6, *The Fold Map*: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f:(X,x_0)\to (Y,y_0)$, we have

$$\nabla_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad \begin{array}{c} X \vee X \xrightarrow{\nabla_{X}} X \\ \downarrow^{f} \\ Y \vee Y \xrightarrow{\nabla_{Y}} Y. \end{array}$$

Indeed, we have

$$\begin{split} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{split}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation.

6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

Construction 6.3.4.1.2. Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- · The Colimit. The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
 - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
 - We have $p_0 = [x_0] = [y_0]$.
- · The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$

given by

$$\operatorname{inj}_{1}(x) \stackrel{\text{def}}{=} [(0, x)]$$

 $\operatorname{inj}_{2}(y) \stackrel{\text{def}}{=} [(1, y)]$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \xleftarrow{\inf_{2}} (Y, y_{0})$$

$$\inf_{1} \circ f = \inf_{2} \circ g, \qquad \inf_{1} \qquad \int_{g} g$$

$$(X, x_{0}) \xleftarrow{f} (Z, z_{0}).$$

Indeed, given $z \in Z$, we have

$$[\inf_{1} \circ f](z) = \inf_{1}(f(z))$$

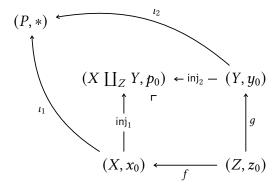
$$= [(0, f(z))]$$

$$= [(1, g(z))]$$

$$= \inf_{2}(g(z))$$

$$= [\inf_{2} \circ g](z),$$

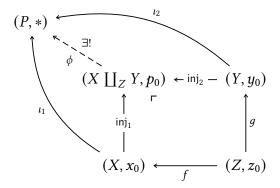
where [(0, f(z))] = [(1, g(z))] by the definition of the relation \sim on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets_{*}. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1$$
,

$$\phi \circ \operatorname{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$

= $\iota_1(x_0)$
= *,

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$

= $\iota_2(y_0)$
= *,

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

Proposition 6.3.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

1. Functoriality. The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f, Z, g} Y$ defines a functor

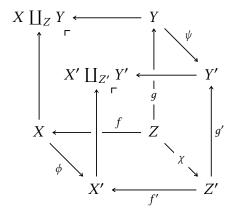
$$\mathsf{-}_1 \coprod_{\mathsf{-}_3} \mathsf{-}_1 \colon \mathsf{Fun}(\mathcal{P},\mathsf{Sets}) \to \mathsf{Sets}_*,$$

where ${\cal P}$ is the category that looks like this:



In particular, the action on morphisms of $-1 \coprod_{-3} -1$ is given by sending a

morphism



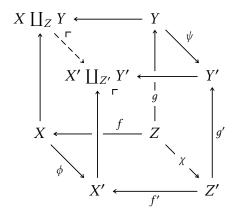
in $Fun(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

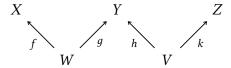
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

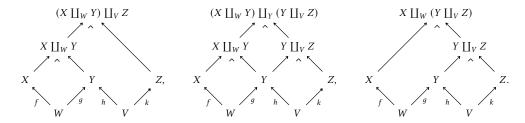
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_V (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

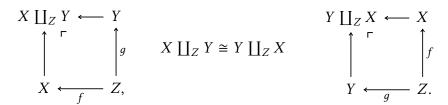
where these pullbacks are built as in the diagrams



3. *Unitality*. We have isomorphisms of sets



4. Commutativity. We have an isomorphism of sets



5. Interaction With Coproducts. We have

$$X \coprod_{\mathsf{pt}} Y \cong X \vee Y, \qquad \uparrow \qquad \downarrow [y_0]$$

$$X \longleftarrow_{[x_0]} \mathsf{pt}.$$

6. Symmetric Monoidality. The triple (Sets_{*}, \coprod_X , (X, x_0)) is a symmetric monoidal category.

Proof. Item 1, Functoriality: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, *Associativity*: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.

Item 3, *Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.

Item 4, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.

Item 5, *Interaction With Coproducts*: Omitted.

Item 6, Symmetric Monoidality: Omitted.

6.3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(CoEq(f, g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f,g) is the pair $((CoEq(f,g),[y_0]), coeq(f,g))$ consisting of:

- The Colimit. The pointed set $(CoEq(f, g), [y_0])$, where CoEq(f, g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
- · The Cocone. The map

$$coeq(f,g): Y \rightarrow (CoEq(f,g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(CoEq(f,g),[y_0])$ is the categorical coequaliser of f and g in $Sets_*$. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f, g) \circ f = coeq(f, g) \circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g) \circ g](x)$$

for each $x \in X$. Next, we prove that CoEq(f,g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\phi \colon \mathsf{CoEq}(f,g) \overset{\exists !}{\longrightarrow} C$ making the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$\downarrow c \qquad \qquad \downarrow \downarrow \exists ! \qquad \qquad \downarrow \downarrow (C, *)$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ coeq(f, g)]([y_0])$$

= $c([y_0])$
= *,

where we have used that c is a morphism of pointed sets.

Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathsf{CoEq}(\mathsf{coeq}(f,g) \circ f, \mathsf{coeq}(f,g) \circ h)}_{=\mathsf{CoEq}(\mathsf{coeq}(f,g) \circ g, \mathsf{coeq}(f,g) \circ h)} \cong \mathsf{CoEq}(f,g,h) \cong \underbrace{\mathsf{CoEq}(\mathsf{coeq}(g,h) \circ f, \mathsf{coeq}(g,h) \circ g, \mathsf{coeq}(g,h) \circ h)}_{=\mathsf{CoEq}(\mathsf{coeq}(g,h) \circ f, \mathsf{coeq}(g,h) \circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets_{*}.

2. Unitality. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f, g) \cong CoEq(g, f)$$
.

Proof. Item 1, *Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5. Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5.

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let *X* be a set.

Definition 6.4.1.1.1. The free pointed set on X is the pointed set X^+ consisting of:

• The Underlying Set. The set X^+ defined by ¹³

$$X^+ \stackrel{\text{def}}{=} X \coprod \text{pt}$$

 $\stackrel{\text{def}}{=} X \coprod \{ \star \}.$

· The Basepoint. The element \star of X^+ .

Proposition 6.4.1.1.2. Let *X* be a set.

1. Functoriality. The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where:

¹³ Further Notation: We sometimes write \star_X for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

· Action on Objects. For each $X \in Obj(Sets)$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

· Action on Morphisms. For each morphism $f: X \to Y$ of Sets, the image

$$f^+: X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. Adjointness. We have an adjunction

$$((-)^+ \dashv \overline{\Xi})$$
: Sets $\stackrel{(-)^+}{\sqsubseteq}$ Sets_{*},

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$.

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+,(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon(\mathsf{Sets}, \coprod, \emptyset)\to(\mathsf{Sets}_*, \vee, \mathsf{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} : X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod} : \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in $X, Y \in Obj(Sets)$.

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)^+_{\mathbb{1}}) : (\mathsf{Sets}, \times, \mathsf{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_{1}^+ \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+,$$

natural in $X, Y \in Obj(Sets)$.

Proof. Item 1, Functoriality: We claim that $(-)^+$ is indeed a functor:

· Preservation of Identities. Let $X \in \mathsf{Obj}(\mathsf{Sets})$. We have

$$\operatorname{id}_X^+(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $id_X^+ = id_{X^+}$.

· Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$
, $g: Y \to Z$,

we have

$$[g^+ \circ f^+](x) \stackrel{\text{def}}{=} g^+(f^+(x))$$

$$\stackrel{\text{def}}{=} g^+(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^+(x)$$

for each $x \in X$ and

$$[g^{+} \circ f^{+}](\star_{X}) \stackrel{\text{def}}{=} g^{+}(f^{+}(\star_{X}))$$

$$\stackrel{\text{def}}{=} g^{+}(\star_{Y})$$

$$\stackrel{\text{def}}{=} \star_{Z}$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(\star_{X}),$$

$$\mathrm{so}\left(g\circ f\right)^{+}=g^{+}\circ f^{+}.$$

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

· Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger} \colon X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

· Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function $\xi\colon X\to Y$ to the morphism of pointed sets

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, y_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

· Invertibility I. Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\Phi_{X,Y}(\xi)\right) \\ &= \Psi_{X,Y} \left(\xi^{\dagger}\right) \end{split}$$

$$\stackrel{\text{def}}{=} [x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}]$$

$$= [x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}]$$

$$= \xi$$

$$\stackrel{\text{def}}{=} [\text{id}_{\mathsf{Sets}_*(X^+,Y)}](\xi).$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}$$
.

· Invertibility II. Given a map of sets $\xi \colon X \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big(\Psi_{X,Y} (\xi) \big) \\ &= \Phi_{X,Y} \bigg(\xi^{\dagger} \bigg) \\ &= \Phi_{X,Y} \bigg(\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right] \bigg) \\ &= \left[x \mapsto \xi(x) \right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[\text{id}_{\mathsf{Sets}(X,Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathsf{id}_{\mathsf{Sets}(X,Y)}$$
.

· Naturality for Φ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\mathsf{Sets}_*(X'^+, Y) \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}(X', Y)$$

$$f^* \downarrow \qquad \qquad \downarrow f^*$$

$$\mathsf{Sets}_*(X^+, Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X, Y)$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+}\to Y$, we have

$$[\Phi_{X,Y} \circ f^*](\xi) = \Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \xi \circ f$$

$$= \Phi_{X',Y}(\xi) \circ f$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= [f^* \circ \Phi_{X',Y}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

· Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\begin{array}{ccc} \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X,Y) \\ & g_* & & \downarrow g_* \\ & & \downarrow g_* \\ & \mathsf{Sets}_*(X^+,Y'), & \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}(X,Y') \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger} \colon X^+ \to Y$$
,

we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \end{split}$$

$$= g_* (\Phi_{X,Y'}(\xi))$$

= $[g_* \circ \Phi_{X,Y'}](\xi)$.

Therefore we have

$$\Phi_{X,Y'} \circ q_* = q_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

• Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to [] and \lor as follows:

· The Strong Monoidality Constraints. The isomorphism

$$(-)_{X,Y}^{+,\coprod}: X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{XY}^{+,\coprod,-1} \colon (X \coprod Y)^+ \xrightarrow{\sim} X^+ \lor Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

· The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} \colon \mathsf{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_{\emptyset} .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

· The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \land y \in X^+ \land Y^+$, with inverse

$$(-)_{YY}^{+,-1} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \bigstar_X \wedge \bigstar_Y & \text{if } z = \bigstar_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

· The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $pt^+ = \{\star, \star_{pt}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1. The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^{-} \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2. Let (X, x_0) be a pointed set.

1. Functoriality. The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^-: \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets}_*$$

where:

· Action on Objects. For each $X \in Obj(Sets_*^{actv})$, we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where X^- is the set of Definition 6.4.2.1.1.

· Action on Morphisms. For each morphism $f\colon X\to Y$ of $\mathsf{Sets}^\mathsf{actv}_*$, the image

$$f^-\colon X^-\to Y^-$$

of f by $(-)^-$ is the map defined by

$$f^{-}(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

2. Adjoint Equivalence. We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+)$$
: Sets* $(-)^-$ Sets,

witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*(X, Y^+),$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathsf{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^{-})^{+} \cong X,$$
$$(Y^{+})^{-} \cong Y,$$

once again natural in $X \in \mathsf{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathsf{Obj}(\mathsf{Sets})$.

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The functor of Item1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\vee}, (-)^{-,\vee}_{\mathfrak{1}}) : (\mathsf{Sets}^{\mathsf{actv}}_*, \vee, \mathsf{pt}), \to (\mathsf{Sets}, \coprod, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-,$$
$$(-)_{1}^{-,\vee} \colon \emptyset \xrightarrow{\sim} \mathsf{pt}^-,$$

natural in $X, Y \in Obj(Sets)$.

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\times},(-)_{\mathbb{1}}^{-,\times}\right)\colon \left(\mathsf{Sets}^{\mathsf{actv}}_*,\wedge,S^0\right), \to \left(\mathsf{Sets},\times,\mathsf{pt}\right)$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-},$$
$$(-)_{1}^{-} \colon \mathsf{pt} \xrightarrow{\sim} (S^{0})^{-},$$

natural in $X, Y \in Obj(Sets)$.

Proof. Item 1, Functoriality: We claim that $(-)^-$ is indeed a functor:

· Preservation of Identities. Let $X \in \mathsf{Obj}(\mathsf{Sets})$. We have

$$\operatorname{id}_X^-(x)\stackrel{\scriptscriptstyle\rm def}{=} x$$

for each $x \in X^-$, so $id_X^- = id_{X^-}$.

· Preservation of Composition. Given morphisms of pointed sets

$$f: (X, x_0) \to (Y, y_0),$$

 $g: (Y, y_0) \to (Z, z_0),$

we have

$$[g^{-} \circ f^{-}](x) \stackrel{\text{def}}{=} g^{-}(f^{-}(x))$$

$$\stackrel{\text{def}}{=} g^{-}(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^{-}(x)$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2, *Adjoint Equivalence*: We proceed in a few steps:

1. Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-,Y) \to \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

by sending a map $\xi \colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^{\dagger} \colon X \to Y^{+}$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$

given by sending an active morphism of pointed sets $\xi \colon X \to Y^+$ to the map

$$\xi^{\dagger} \colon X^{-} \to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi \colon X^- \to Y$, we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y} \right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\Phi_{X,Y} (\xi) \right) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^{-} \\ \star_{Y} & \text{if } x = x_{0} \end{cases} \right] \right) \\ &= \left[\left[x \mapsto \xi(x) \right] \right] \\ &= \xi \\ &= \left[\text{id}_{\mathsf{Sets}(X^{-},Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathsf{id}_{\mathsf{Sets}(X^-,Y)}$$
.

4. Invertibility II. Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \big(\Psi_{X,Y}(\xi)\big) \\ &= \Phi_{X,Y} (\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= \left[\text{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} \right] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_{*}(X,Y^{+})}$$
.

5. Naturality for Φ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\mathsf{Sets}\big(X^{',-},Y\big) \xrightarrow{\Phi_{X',Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X',Y^+)$$

$$f^* \qquad \qquad \qquad \downarrow f^*$$

$$\mathsf{Sets}_*(X^-,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

commutes. Indeed, given a map of sets $\xi \colon X' \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \left[\!\left[x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases}\right]\!\right] \\ &= f^* \left(\!\left[\!\left[x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x'_0 \end{cases}\right]\!\right) \end{split}$$

$$= f^* (\Phi_{X',Y}(\xi))$$

= $[f^* \circ \Phi_{X',Y}](\xi).$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\mathsf{Sets}(X^-,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

$$\downarrow^{g_*} \qquad \qquad \downarrow^{g_*}$$

$$\mathsf{Sets}(X^-,Y') \xrightarrow{\Phi_{X,Y'}} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y'^{,+})$$

commutes. Indeed, given a map of sets $\xi \colon X^- \to Y$, we have

$$\begin{split} \big[\Phi_{X,Y'} \circ g_*\big](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= \big[\!\big[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \big]\!\big] \\ &= g_* \bigg(\big[\!\big[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases} \big]\!\big) \\ &= g_* \big(\Phi_{X,Y'}(\xi)\big) \\ &= \big[g_* \circ \Phi_{X,Y'}\big](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

- 7. Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.
- 8. Fully Faithfulness of $(-)^-$. We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{XY}^- : \mathsf{Sets}^{\mathsf{actv}}_*(X,Y) \xrightarrow{\sim} \mathsf{Sets}(X^-,Y^-).$$

Indeed, the inverse map

$$(-)^{-,-1}_{X,Y} \colon \mathsf{Sets}(X^-,Y^-) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X,Y)$$

is given by sending a map of sets $f\colon X^-\to Y^-$ to the active morphism of pointed sets $f^\dagger\colon X\to Y$ defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

9. Essential Surjectivity of $(-)^-$. We need to show that, given an object $X \in \mathsf{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathsf{Obj}(\mathsf{Sets}^{\mathsf{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$(X^{+})^{-} \stackrel{\text{def}}{=} (X \cup \{\star_{X}\})^{-}$$
$$\stackrel{\text{def}}{=} (X \cup \{\star_{X}\}) \setminus \{\star_{X}\}$$
$$= X,$$

and thus we have in fact an equality $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. The Functor $(-)^-$ is an Equivalence. Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^-$ with respect to \vee and \coprod as follows:

· The Strong Monoidality Constraints. The isomorphism

$$(-)^{-,\vee}_{X,Y} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{\neg,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \coprod Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1} \colon (X \vee Y)^- \xrightarrow{\sim} X^- \coprod Y^-$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

· The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\vee,1} \colon \emptyset \xrightarrow{\sim} \mathsf{pt}^{-}$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

*Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

· The Strong Monoidality Constraints. The isomorphism

$$(-)^-_{XY} \colon X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{X,Y}(x,y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1} \colon (X \land Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)^{-,-1}_{XY}(x \wedge y) \stackrel{\text{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

· The Strong Monoidal Unity Constraint. The isomorphism

$$(-)^{-,\mathbb{1}}_{X,Y} \colon \mathsf{pt} \overset{\sim}{\longrightarrow} \left(S^0\right)^-$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

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Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

Constructions With Monoidal Categories

Bicategories

 Types of Morphisms in Bicategories

Extra Part

15. Notes

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