## Constructions With Monoidal Categories

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**O1UF** This chapter contains some material on constructions with monoidal categories.

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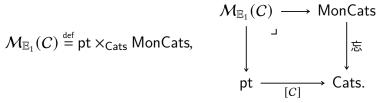
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	Let $C$ be a category.		

01UJ

DEFINITION 13.1.1.1.1 ► THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATE-

The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C)\stackrel{\mathsf{def}}{=}\mathsf{pt} imes_{\mathsf{Cats}}\mathsf{MonCats},$$



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#### REMARK 13.1.1.1.2 ► UNWINDING DEFINITION 13.1.1.1.1, I

In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- · Objects. The objects of  $\mathcal{M}_{\mathbb{B}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C,$  $\alpha^C$ ,  $\lambda^C$ ,  $\rho^C$ ) whose underlying category is C.
- · Morphisms. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^C, \lambda^C, \rho^C)$  $\alpha^{C,\prime}$  ,  $\lambda^{C,\prime}$  ,  $\,\rho^{C,\prime}\big)$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C : C \to C$  of C.

· Identities. For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{B}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\operatorname{def}}{=}\left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right),$$

where  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  is the identity monoidal functor of C of ??.

· Composition. For each  $M, N, P \in \mathsf{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,P)$$
 of  $\mathcal{M}_{\mathbb{B}_{1}}(C)$  at  $(M,N,P)$  is defined by

$$\left(\operatorname{id}_{C}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)}\left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right)\stackrel{\text{\tiny def}}{=}\left(\operatorname{id}_{C}^{\otimes,\prime}\circ\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{B}|C}^{\otimes}\right).$$

#### 01UL REMARK 13.1.1.1.3 ➤ UNWINDING DEFINITION 13.1.1.1.1, II

In particular, a morphism in  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

1. Naturality. For each pair  $f: A \to X$  and  $g: B \to Y$  of morphisms of C, the diagram

$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow \operatorname{id}_{A,B}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{X,Y}^{\otimes}$$

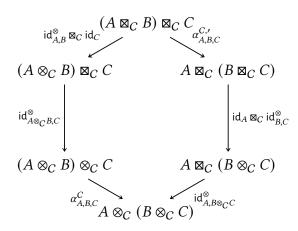
$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

commutes.

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2. Monoidality. For each  $A, B, C \in Obj(C)$ , the diagram



commutes.

**01UP** 

3. Left Monoidal Unity. For each  $A \in Obj(C)$ , the diagram

$$\mathbb{1}_{C} \boxtimes_{C} A \xrightarrow{\operatorname{id}_{\mathbb{1}'_{C}, A}^{\otimes}} \mathbb{1}_{C} \otimes_{C} A$$

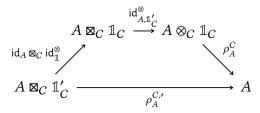
$$\operatorname{id}_{\mathbb{1}}^{\otimes} \boxtimes_{C} \operatorname{id}_{A} \xrightarrow{\lambda_{A}^{C}} A$$

$$\mathbb{1}'_{C} \boxtimes_{C} A \xrightarrow{\lambda_{A}^{C,'}} A$$

commutes.

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4. Right Monoidal Unity. For each  $A \in Obj(C)$ , the diagram



commutes.

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#### PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUC-TURES ON A CATEGORY

Let C be a category.

**01US** 

- 1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
- 01UT

commutes.

(a) The diagram

**01UU** 

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B \boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B \otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

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- 2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
- (a) The diagram

commutes.

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(b) The diagram

commutes.

01WE

(c) The diagram

commutes.

01WF

(d) The diagram

commutes.

**01UV** 

3. Mixed Associators. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\mathsf{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

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(a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} \colon (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \\ \operatorname{id}_{A \otimes_C B,C}^{\otimes} & & & \operatorname{id}_{A,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A\boxtimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_C (B\boxtimes_C C) \\ \operatorname{id}_{A,B}^\otimes\boxtimes_C \operatorname{id}_C & & & & & & & & & \\ (A\otimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^\otimes} A\otimes_C (B\boxtimes_C C) & & & & & \end{array}$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{A.B.C}^{\boxtimes} \colon (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A\boxtimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A\boxtimes_C (B\otimes_C C) \\ \operatorname{id}_{A,B}^{\otimes}\otimes_C \operatorname{id}_C & & & & \operatorname{id}_{A,B\otimes_C C}^{\otimes} \\ (A\otimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A\otimes_C (B\otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A\boxtimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A\boxtimes_C (B\boxtimes_C C) \\ \operatorname{id}_{A\boxtimes_C B,C}^\otimes & & & & & & & & & & \\ (A\boxtimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} & A\boxtimes_C (B\otimes_C C) & & & & & \end{array}$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{AB,C}^{\boxtimes,\otimes} \colon (A\boxtimes_C B) \otimes_C C \to A \otimes_C (B\boxtimes_C C)$$

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making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_C (B \boxtimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & \operatorname{id}_{A,C}^{\otimes} \\ & (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

#### PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

#### Item 1: Extra Monoidality Conditions

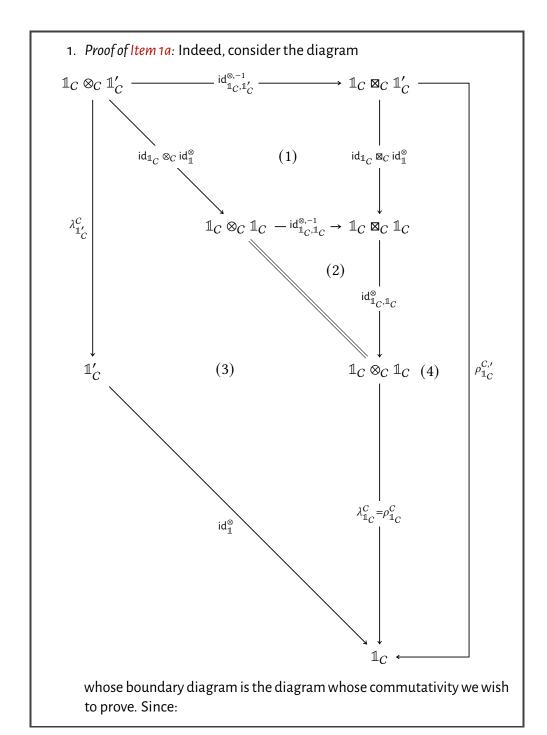
We claim that Items 1a and 1b are indeed true:

- 1. Proof of Item 1a: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{AB}^{\otimes}$  and  $id_{C}$ .
- 2. Proof of Item 1b: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{BC}^{\otimes}$ .

This finishes the proof.

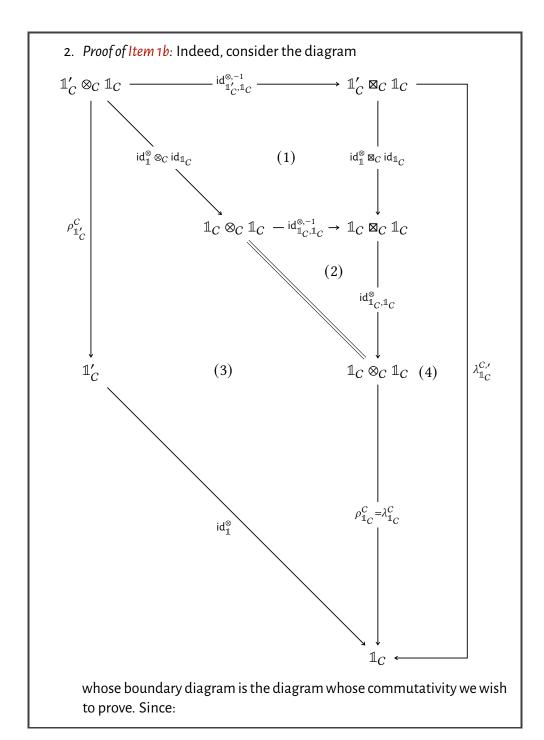
#### Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:



- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- · Subdiagram (2) commutes trivially;
- · Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from  $\ref{eq:composition}$ ;
- · Subdiagram (4) commutes by the right monoidal unity of  $\left(\mathrm{id}_C,\mathrm{id}_C^\otimes,\mathrm{id}_{C|\mathbb{1}}^\otimes\right);$

so does the boundary diagram, and we are done.



- · Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{C}^{\otimes,-1}$ ;
- · Subdiagram (2) commutes trivially;
- · Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from  $\ref{eq:condition}$ ;
- · Subdiagram (4) commutes by the left monoidal unity of  $\left(\mathrm{id}_{C},\mathrm{id}_{C}^{\otimes},\mathrm{id}_{C|\mathbb{I}}^{\otimes}\right);$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- · The boundary diagram commutes trivially;
- · Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes_{C},\mathbb{1}'_{C}}}
\mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes_{C}}}
\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{C}_{\mathbb{1}'_{C}} \qquad \qquad (\dagger) \qquad \qquad \downarrow^{\rho^{C}_{\mathbb{1}'_{C}}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}}
\mathbb{1}'_{C}$$

commutes. But since  $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- · The boundary diagram commutes trivially;
- · Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

commutes. But since  $id_{1}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

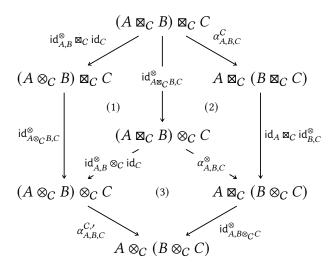
#### Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for  $id^{\otimes}$  of

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#### Item 2 of Remark 13.1.1.1.3 as follows:



#### Since:

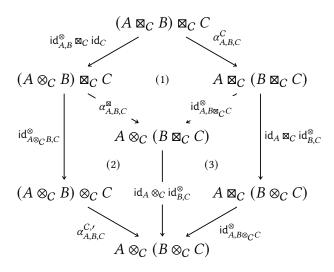
- · Subdiagram (1) commutes by Item 1a of Item 1.
- · Subdiagram (2) commutes by assumption.
- · Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for  $id^{\otimes}$  of

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#### Item 2 of Remark 13.1.1.1.3 as follows:



#### Since:

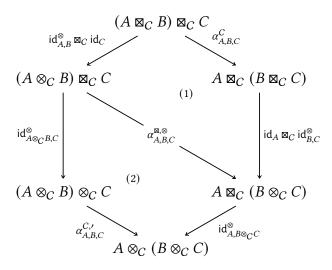
- · Subdiagram (1) commutes by assumption.
- · Subdiagram (2) commutes by assumption.
- · Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for  $id^{\otimes}$  of

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Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- **13.2** Moduli Categories of Closed Monoidal Structures
- 91V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# **Appendices**

## **A** Other Chapters

#### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
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10. Conditions on Relations

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- 11. Categories
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14. Types of Morphisms in Bicategories

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15. Notes