

# Types of Morphisms in Bicategories

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**019H** In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories* (*Sections 14.1 and 14.2*). There is a large number of different notions capturing the idea of a “monomorphism” or of an “epimorphism” in a bicategory.

Arguably, the notion that best captures these concepts is that of a *pseudomononic morphism* (*Definition 14.1.10.1.1*) and of a *pseudoepic morphism* (*Definition 14.2.10.1.1*), although the other notions introduced in *Sections 14.1* and *14.2* are also interesting on their own.

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## 019J 14.1 Monomorphisms in Bicategories

### 019K 14.1.1 Representably Faithful Morphisms

Let  $\mathcal{C}$  be a bicategory.

**019L Definition 14.1.1.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably faithful**<sup>1</sup> if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is faithful.

**019M Remark 14.1.1.1.2.** In detail,  $f$  is representably faithful if, for all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

**019N Example 14.1.1.1.3.** Here are some examples of representably faithful morphisms.

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<sup>1</sup>*Further Terminology:* Also called simply a **faithful morphism**, based on [Item 1](#) of [Definition 14.1.1.1.3](#).

- 019P 1. *Representably Faithful Morphisms in  $\mathbf{Cats}_2$* . The representably faithful morphisms in  $\mathbf{Cats}_2$  are precisely the faithful functors; see [Categories](#), [Item 2](#) of [Definition 11.6.1.1.2](#).
- 019Q 2. *Representably Faithful Morphisms in  $\mathbf{Rel}$* . Every morphism of  $\mathbf{Rel}$  is representably faithful; see [Relations](#), [Item 1](#) of [Definition 8.5.11.1.1](#).

### 019R 14.1.2 Representably Full Morphisms

Let  $\mathcal{C}$  be a bicategory.

- 019S **Definition 14.1.2.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably full**<sup>2</sup> if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is full.

- 019T **Remark 14.1.2.1.2.** In detail,  $f$  is representably full if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

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<sup>2</sup>*Further Terminology:* Also called simply a **full morphism**, based on [Item 1](#) of

**019U Example 14.1.2.1.3.** Here are some examples of representably full morphisms.

**019V** 1. *Representably Full Morphisms in  $\mathbf{Cats}_2$ .* The representably full morphisms in  $\mathbf{Cats}_2$  are precisely the full functors; see [Categories](#), ?? of [Definition 11.6.2.1.2](#).

**019W** 2. *Representably Full Morphisms in  $\mathbf{Rel}$ .* The representably full morphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), [Item 2](#) of [Definition 8.5.11.1.1](#).

### **019X 14.1.3 Representably Fully Faithful Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**019Y Definition 14.1.3.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably fully faithful**<sup>3</sup> if the following equivalent conditions are satisfied:

**019Z** 1. The 1-morphism  $f$  is representably faithful ([Definition 14.1.1.1.1](#)) and representably full ([Definition 14.1.2.1.1](#)).

**01A0** 2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is fully faithful.

**01A1 Remark 14.1.3.1.2.** In detail,  $f$  is representably fully faithful if the conditions in [Definition 14.1.1.1.2](#) and [Definition 14.1.2.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

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**Definition 14.1.2.1.3.**

<sup>3</sup>*Further Terminology:* Also called simply a **fully faithful morphism**, based on [Item 1](#) of [Definition 14.1.3.1.3](#).

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

**01A2 Example 14.1.3.1.3.** Here are some examples of representably fully faithful morphisms.

**01A3** 1. *Representably Fully Faithful Morphisms in  $\mathbf{Cats}_2$ .* The representably fully faithful morphisms in  $\mathbf{Cats}_2$  are precisely the fully faithful functors; see [Categories, Item 6](#) of [Definition 11.6.3.1.2](#).

**01A4** 2. *Representably Fully Faithful Morphisms in  $\mathbf{Rel}$ .* The representably fully faithful morphisms of  $\mathbf{Rel}$  coincide ([Relations, Item 3](#) of [Definition 8.5.11.1.1](#)) with the representably full morphisms in  $\mathbf{Rel}$ , which are characterised in [Relations, Item 2](#) of [Definition 8.5.11.1.1](#).

## **01A5 14.1.4 Morphisms Representably Faithful on Cores**

Let  $\mathcal{C}$  be a bicategory.

**01A6 Definition 14.1.4.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably faithful on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is faithful.

**01A7 Remark 14.1.4.1.2.** In detail,  $f$  is representably faithful on cores if, for all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then  $\alpha = \beta$ .

### 01A8 14.1.5 Morphisms Representably Full on Cores

Let  $\mathcal{C}$  be a bicategory.

**01A9 Definition 14.1.5.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably full on cores** if, for each  $X \in \mathrm{Obj}(\mathcal{C})$ , the functor

$$f_*: \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \rightarrow \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is full.

**01AA Remark 14.1.5.1.2.** In detail,  $f$  is representably full on cores if, for each  $X \in \mathrm{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

**01AB 14.1.6 Morphisms Representably Fully Faithful on Cores**

Let  $\mathcal{C}$  be a bicategory.

**01AC Definition 14.1.6.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

- 01AD** 1. The 1-morphism  $f$  is representably faithful on cores ([Definition 14.1.5.1.1](#)) and representably full on cores ([Definition 14.1.4.1.1](#)).
- 01AE** 2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by  $f$  is fully faithful.

**01AF Remark 14.1.6.1.2.** In detail,  $f$  is representably fully faithful on cores if the conditions in [Definition 14.1.4.1.2](#) and [Definition 14.1.5.1.2](#) hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

### 01AG 14.1.7 Representably Essentially Injective Morphisms

Let  $\mathcal{C}$  be a bicategory.

**01AH Definition 14.1.7.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably essentially injective** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is essentially injective.

**01AJ Remark 14.1.7.1.2.** In detail,  $f$  is representably essentially injective if, for each pair of morphisms  $\phi, \psi: X \rightrightarrows A$  of  $\mathcal{C}$ , the following condition is satisfied:

$$(\star) \text{ If } f \circ \phi \cong f \circ \psi, \text{ then } \phi \cong \psi.$$

### 01AK 14.1.8 Representably Conservative Morphisms

Let  $\mathcal{C}$  be a bicategory.

**01AL Definition 14.1.8.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **representably conservative** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is conservative.

**01AM Remark 14.1.8.1.2.** In detail,  $f$  is representably conservative if, for each pair of morphisms  $\phi, \psi: X \rightrightarrows A$  and each 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$



of  $\mathcal{C}$ , if the 2-morphism

$$\mathrm{id}_f \star \alpha: f \circ \phi \Longrightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \parallel \\ \mathrm{id}_f \star \alpha \\ \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

is a 2-isomorphism, then so is  $\alpha$ .

### 01AN 14.1.9 Strict Monomorphisms

Let  $\mathcal{C}$  be a bicategory.

**01AP Definition 14.1.9.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is a **strict monomorphism** if, for each  $X \in \mathrm{Obj}(\mathcal{C})$ , the functor

$$f_*: \mathrm{Hom}_{\mathcal{C}}(X, A) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is injective on objects, i.e. its action on objects

$$f_*: \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \rightarrow \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

is injective.

**01AQ Remark 14.1.9.1.2.** In detail,  $f$  is a strict monomorphism in  $\mathcal{C}$  if, for each diagram in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \psi \end{array} A \xrightarrow{f} B,$$

if  $f \circ \phi = f \circ \psi$ , then  $\phi = \psi$ .

**01AR Example 14.1.9.1.3.** Here are some examples of strict monomorphisms.

- 01AS** 1. *Strict Monomorphisms in  $\mathbf{Cats}_2$ .* The strict monomorphisms in  $\mathbf{Cats}_2$  are precisely the functors which are injective on objects and injective on morphisms; see [Categories](#), [Item 1](#) of [Definition 11.7.2.1.2](#).
- 01AT** 2. *Strict Monomorphisms in  $\mathbf{Rel}$ .* The strict monomorphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), [Definition 8.5.10.1.1](#).

### 01AU 14.1.10 Pseudomononic Morphisms

Let  $\mathcal{C}$  be a bicategory.

**01AV Definition 14.1.10.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **pseudomononic** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by  $f$  is pseudomononic.

**01AW Remark 14.1.10.1.2.** In detail, a 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is pseudomononic if it satisfies the following conditions:

**01AX** 1. For all diagrams in  $\mathcal{C}$  of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then  $\alpha = \beta$ .

**01AY** 2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of  $\mathcal{C}$  such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

**01AZ Proposition 14.1.10.1.3.** Let  $f: A \rightarrow B$  be a 1-morphism of  $\mathcal{C}$ .

**01B0** 1. *Characterisations.* The following conditions are equivalent:

**01B1** (a) The morphism  $f$  is pseudomonic.

**01B2** (b) The morphism  $f$  is representably full on cores and representably faithful.

**01B3** (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \times_B A, \quad \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & \nearrow \text{dashed} & \downarrow F \\ A & \xrightarrow{F} & B \end{array}$$

in  $\mathcal{C}$  up to equivalence.

**01B4** 2. *Interaction With Cotensors.* If  $\mathcal{C}$  has cotensors with  $\mathbb{1}$ , then the following conditions are equivalent:

(a) The morphism  $f$  is pseudomonic.

(b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \times_{\mathbb{1} \pitchfork F} B, \quad \begin{array}{ccc} A & \hookrightarrow & \mathbb{1} \pitchfork A \\ F \downarrow & \nearrow \text{dashed} & \downarrow \mathbb{1} \pitchfork F \\ B & \hookrightarrow & \mathbb{1} \pitchfork B \end{array}$$

in  $\mathcal{C}$  up to equivalence.

*Proof.* **Item 1, Characterisations:** Omitted.

**Item 2, Interaction With Cotensors:** Omitted. □

## **01B5 14.2 Epimorphisms in Bicategories**

### **01B6 14.2.1 Corepresentably Faithful Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**01B7 Definition 14.2.1.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably faithful** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is faithful.

**01B8 Remark 14.2.1.1.2.** In detail,  $f$  is corepresentably faithful if, for all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

**01B9 Example 14.2.1.1.3.** Here are some examples of corepresentably faithful morphisms.

**01BA** 1. *Corepresentably Faithful Morphisms in  $\mathbf{Cats}_2$ .* The corepresentably faithful morphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 5 of Definition 11.6.1.1.2](#).

**01BB** 2. *Corepresentably Faithful Morphisms in  $\mathbf{Rel}$ .* Every morphism of  $\mathbf{Rel}$  is corepresentably faithful; see [Relations, Item 1 of Definition 8.5.13.1.1](#).

## **01BC 14.2.2 Corepresentably Full Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**01BD Definition 14.2.2.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably full** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is full.

**01BE Remark 14.2.2.1.2.** In detail,  $f$  is corepresentably full if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: \phi \circ f \Longrightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

**01BF Example 14.2.2.1.3.** Here are some examples of corepresentably full morphisms.

- 01BG** 1. *Corepresentably Full Morphisms in  $\mathbf{Cats}_2$ .* The corepresentably full morphisms in  $\mathbf{Cats}_2$  are characterised in [Categories](#), [Item 7](#) of [Definition 11.6.2.1.2](#).
- 01BH** 2. *Corepresentably Full Morphisms in  $\mathbf{Rel}$ .* The corepresentably full morphisms in  $\mathbf{Rel}$  are characterised in [Relations](#), [Item 2](#) of [Definition 8.5.13.1.1](#).

### **01BJ 14.2.3 Corepresentably Fully Faithful Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**01BK Definition 14.2.3.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably fully faithful**<sup>4</sup> if the following equivalent conditions are satisfied:

- 01BL** 1. The 1-morphism  $f$  is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).
- 01BM** 2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is fully faithful.

**01BN Remark 14.2.3.1.2.** In detail,  $f$  is corepresentably fully faithful if the conditions in Definition 14.2.1.1.2 and Definition 14.2.2.1.2 hold:

- 1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

- 2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

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<sup>4</sup>*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name “corepresentably fully faithful morphism” instead in this work.

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

**01BP Example 14.2.3.1.3.** Here are some examples of corepresentably fully faithful morphisms.

- 01BQ** 1. *Corepresentably Fully Faithful Morphisms in  $\mathbf{Cats}_2$ .* The fully faithful epimorphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 10](#) of [Definition 11.6.3.1.2](#).
- 01BR** 2. *Corepresentably Fully Faithful Morphisms in  $\mathbf{Rel}$ .* The corepresentably fully faithful morphisms of  $\mathbf{Rel}$  coincide ([Relations, Item 3](#) of [Definition 8.5.13.1.1](#)) with the corepresentably full morphisms in  $\mathbf{Rel}$ , which are characterised in [Relations, Item 2](#) of [Definition 8.5.13.1.1](#).

## **01BS 14.2.4 Morphisms Corepresentably Faithful on Cores**

Let  $\mathcal{C}$  be a bicategory.

**01BT Definition 14.2.4.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably faithful on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is faithful.

**01BU Remark 14.2.4.1.2.** In detail,  $f$  is corepresentably faithful on cores if, for all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

**01BV 14.2.5 Morphisms Corepresentably Full on Cores**

Let  $\mathcal{C}$  be a bicategory.

**01BW Definition 14.2.5.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably full on cores** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is full.

**01BX Remark 14.2.5.1.2.** In detail,  $f$  is corepresentably full on cores if, for each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

**01BY 14.2.6 Morphisms Corepresentably Fully Faithful on Cores**

Let  $\mathcal{C}$  be a bicategory.

**01BZ Definition 14.2.6.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:



- 01C0 1. The 1-morphism  $f$  is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

- 01C1 2. For each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by  $f$  is fully faithful.

- 01C2 **Remark 14.2.6.1.2.** In detail,  $f$  is corepresentably fully faithful on cores if the conditions in Definition 14.2.4.1.2 and Definition 14.2.5.1.2 hold:

1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if  $\alpha$  and  $\beta$  are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

**01C3 14.2.7 Corepresentably Essentially Injective Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**01C4 Definition 14.2.7.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably essentially injective** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is essentially injective.

**01C5 Remark 14.2.7.1.2.** In detail,  $f$  is corepresentably essentially injective if, for each pair of morphisms  $\phi, \psi: B \rightrightarrows X$  of  $\mathcal{C}$ , the following condition is satisfied:

$$(\star) \text{ If } \phi \circ f \cong \psi \circ f, \text{ then } \phi \cong \psi.$$

**01C6 14.2.8 Corepresentably Conservative Morphisms**

Let  $\mathcal{C}$  be a bicategory.

**01C7 Definition 14.2.8.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **corepresentably conservative** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is conservative.

**01C8 Remark 14.2.8.1.2.** In detail,  $f$  is corepresentably conservative if, for each pair of morphisms  $\phi, \psi: B \rightrightarrows X$  and each 2-morphism

$$\alpha: \phi \rightrightarrows \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$ , if the 2-morphism

$$\alpha \star \text{id}_f: \phi \circ f \rightrightarrows \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \parallel \\ \alpha \star \text{id}_f \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

is a 2-isomorphism, then so is  $\alpha$ .

### 01C9 14.2.9 Strict Epimorphisms

Let  $\mathcal{C}$  be a bicategory.

**01CA Definition 14.2.9.1.1.** A 1-morphism  $f: A \rightarrow B$  is a **strict epimorphism** in  $\mathcal{C}$  if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is injective on objects, i.e. its action on objects

$$f_*: \text{Obj}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Obj}(\text{Hom}_{\mathcal{C}}(A, X))$$

is injective.

**01CB Remark 14.2.9.1.2.** In detail,  $f$  is a strict epimorphism if, for each diagram in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \xrightleftharpoons[\psi]{\phi} X,$$

if  $\phi \circ f = \psi \circ f$ , then  $\phi = \psi$ .

**01CC Example 14.2.9.1.3.** Here are some examples of strict epimorphisms.

- 01CD** 1. *Strict Epimorphisms in  $\mathbf{Cats}_2$* . The strict epimorphisms in  $\mathbf{Cats}_2$  are characterised in [Categories, Item 1](#) of [Definition 11.7.3.1.2](#).
- 01CE** 2. *Strict Epimorphisms in  $\mathbf{Rel}$* . The strict epimorphisms in  $\mathbf{Rel}$  are characterised in [Relations, Definition 8.5.12.1.1](#).

### 01CF 14.2.10 Pseudoepic Morphisms

Let  $\mathcal{C}$  be a bicategory.

**01CG Definition 14.2.10.1.1.** A 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is **pseudoepic** if, for each  $X \in \text{Obj}(\mathcal{C})$ , the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by  $f$  is pseudomonc.

**01CH Remark 14.2.10.1.2.** In detail, a 1-morphism  $f: A \rightarrow B$  of  $\mathcal{C}$  is pseudoepic if it satisfies the following conditions:

- 01CJ 1. For all diagrams in  $\mathcal{C}$  of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then  $\alpha = \beta$ .

- 01CK 2. For each  $X \in \text{Obj}(\mathcal{C})$  and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of  $\mathcal{C}$ , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of  $\mathcal{C}$  such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in  $\mathcal{C}$ , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

01CL **Proposition 14.2.10.1.3.** Let  $f: A \rightarrow B$  be a 1-morphism of  $\mathcal{C}$ .

- 01CM 1. *Characterisations.* The following conditions are equivalent:

01CN (a) The morphism  $f$  is pseudoepic.

01CP (b) The morphism  $f$  is corepresentably full on cores and corepresentably faithful.

01CQ

(c) We have an isococcomma square of the form

$$B \stackrel{\text{eq.}}{\cong} B \coprod_A B, \quad \begin{array}{ccc} B & \xleftarrow{\text{id}_B} & B \\ \text{id}_B \uparrow & \nearrow \text{dashed} & \uparrow F \\ B & \xleftarrow{F} & A \end{array}$$

in  $\mathcal{C}$  up to equivalence.*Proof.* *Item 1, Characterisations:* Omitted. □

# Appendices

## A Other Chapters

### Preliminaries

1. Introduction
2. A Guide to the Literature

### Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
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### Relations

8. Relations
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### 10. Conditions on Relations

### Categories

11. Categories
12. Presheaves and the Yoneda Lemma

### Monoidal Categories

13. Constructions With Monoidal Categories

### Bicategories

14. Types of Morphisms in Bicategories

### Extra Part

15. Notes

## References

- [Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. “On Functors Which Are Lax Epimorphisms”. In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 14).