Constructions With Sets

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 4.2.4.1.1, 4.2.4.1.3, 4.2.5.1.1 and 4.2.5.1.3).

2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:

   (a) A discussion of the internal Hom of a powerset (Section 4.4.7).

   (b) A 0-categorical version of the Yoneda lemma (Presheaves and the Yoneda Lemma, Definition 12.1.5.1.1), which we term the *Yoneda lemma for sets* (Definition 4.5.5.1.1).

   (c) A characterisation of powersets as free cocompletions (Section 4.4.5), mimicking the corresponding statement for categories of presheaves (**??**).

   (d) A characterisation of powersets as free completions (Section 4.4.6), mimicking the corresponding statement for categories of copresheaves (**??**).

   (e) A (-1)-categorical version of un/straightening (Item 2 of Definition 4.5.1.1.4 and Definition 4.5.1.1.5).

   (f) A 0-categorical form of Isbell duality internal to powersets (Section 4.4.8).

3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* \colon \mathcal{P}(A) \overset{\rightarrow}{\underset{\rightarrow}{}} \mathcal{P}(B)$$

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of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \to B$, including in particular:

(a) How f^{-1} can be described as a precomposition while $f_!$ and f_* can be described as Kan extensions (Definitions 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).

(b) An extensive list of the properties of $f_!$, f^{-1}, and f_* (Definitions 4.6.1.1.5, 4.6.1.1.6, 4.6.2.1.3, 4.6.2.1.4, 4.6.3.1.7 and 4.6.3.1.8).

(c) How the functors $f_!$, f^{-1}, f_*, along with the functors

$$-_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X),$$
$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

may be viewed as a six-functor formalism with the empty set $\emptyset$ as the dualising object (Section 4.6.4).

Contents

4.1 Limits of Sets

4.1.1 The Terminal Set

Definition 4.1.1.1.1. The **terminal set** is the terminal object of Sets as in Limits and Colimits, **??**.

Construction 4.1.1.1.2. Concretely, the terminal set is the pair $(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj(Sets)}})$ consisting of:

1. *The Limit.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.

2. *The Cone.* The collection of maps

$$\{!_A\colon A \to \mathrm{pt}\}_{A\in\mathrm{Obj(Sets)}}$$

   defined by
$$!_A(a) \overset{\mathrm{def}}{=} \star$$
   for each $a \in A$ and each $A \in \mathrm{Obj(Sets)}$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi\colon A \to \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{-\!-\!\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. $\qquad\square$

4.1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 4.1.2.1.1. The **product**[1] **of** $\{A_i\}_{i\in I}$ is the product of $\{A_i\}_{i\in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.1.2.1.2. Concretely, the product of $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of:

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[1]*Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

1. *The Limit.* The set $\prod_{i \in I} A_i$ defined by

$$\prod_{i \in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i \in I} A_i\right) \,\middle|\, \begin{array}{l} \text{for each } i \in I, \text{we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i \right\}_{i \in I}$$

   of maps given by
$$\mathrm{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$
   for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i \in I} A_i$ is the categorical product of $\{A_i\}_{i \in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} P & & \\ & \searrow^{p_i} & \\ \prod_{i \in I} A_i & \xrightarrow[\mathrm{pr}_i]{} & A_i \end{array}$$

in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i \in I} A_i$ making the diagram

$$\begin{array}{ccc} P & & \\ \phi \downarrow {\scriptstyle \exists!} & \searrow^{p_i} & \\ \prod_{i \in I} A_i & \xrightarrow[\mathrm{pr}_i]{} & A_i \end{array}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. $\square$

Remark 4.1.2.1.3. Less formally, we may think of Cartesian products and projection maps as follows:

1. We think of $\prod_{i \in I} A_i$ as the set whose elements are I-indexed collections $(a_i)_{i \in I}$ with $a_i \in A_i$ for each $i \in I$.

2. We view the projection maps

$$\left\{ \mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i \right\}_{i \in I}$$

as being given by

$$\mathrm{pr}_i((a_j)_{j \in I}) \overset{\mathrm{def}}{=} a_i$$

for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

Proposition 4.1.2.1.4. Let $\{A_i\}_{i \in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i \in I}\right]((A_i)_{i \in I}) \overset{\mathrm{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}$, $(B_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I}\right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} \colon \mathsf{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \to \mathsf{Sets}\left(\prod_{i \in I} A_i, \prod_{i \in I} B_i\right)$$

of $\prod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i \in I}$$

in $\mathsf{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\prod_{i \in I} f_i \colon \prod_{i \in I} A_i \to \prod_{i \in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, **??** of **??**.   □

4.1.3   Binary Products of Sets

Let A and B be sets.

Definition 4.1.3.1.1. The **product of A and B**[2] is the product of A and B in Sets as in Limits and Colimits, **??**.

Construction 4.1.3.1.2. Concretely, the product of A and B is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times B$ defined by

$$
\begin{aligned}
A \times B &\overset{\text{def}}{=} \prod_{z\in\{A,B\}} z \\
&\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\
&\cong \{\{\{a\}, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\
&\cong \left\{\begin{array}{l}\text{ordered pairs } (a,b) \text{ with} \\ a \in A \text{ and } b \in B\end{array}\right\}.
\end{aligned}
$$

2. *The Cone.* The maps

$$
\begin{aligned}
\mathrm{pr}_1 &: A \times B \to A, \\
\mathrm{pr}_2 &: A \times B \to B
\end{aligned}
$$

defined by

$$
\begin{aligned}
\mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\
\mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b
\end{aligned}
$$

for each $(a,b) \in A \times B$.

---

[2] *Further Terminology:* Also called the **Cartesian product of A and B**.

Proof. We claim that $A \times B$ is the categorical product of A and B in the category of sets. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccc}
& P & \\
p_1 \swarrow & & \searrow p_2 \\
A \xleftarrow[\mathrm{pr}_1]{} & A \times B & \xrightarrow[\mathrm{pr}_2]{} B
\end{array}
$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times B$ making the diagram

$$
\begin{array}{ccc}
& P & \\
p_1 \swarrow & \phi \downarrow \exists! & \searrow p_2 \\
A \xleftarrow[\mathrm{pr}_1]{} & A \times B & \xrightarrow[\mathrm{pr}_2]{} B
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\mathrm{pr}_1 \circ \phi = p_1,
$$
$$
\mathrm{pr}_2 \circ \phi = p_2
$$

via

$$
\phi(x) = (p_1(x), p_2(x))
$$

for each $x \in P$.   □

Proposition 4.1.3.1.3. Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$
\begin{array}{rll}
A \times - \colon & \mathsf{Sets} & \to \mathsf{Sets}, \\
- \times B \colon & \mathsf{Sets} & \to \mathsf{Sets}, \\
-_1 \times -_2 \colon & \mathsf{Sets} \times \mathsf{Sets} & \to \mathsf{Sets},
\end{array}
$$

   where $-_1 \times -_2$ is the functor where

   · *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$
[-_1 \times -_2](A, B) \overset{\mathrm{def}}{=} A \times B.
$$

· *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g \colon A \times B \to X \times Y$$

defined by

$$[f \times g](a, b) \overset{\mathrm{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness I.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -)) \colon \quad \mathsf{Sets} \underset{\mathsf{Sets}(A,-)}{\overset{A \times -}{\rightleftarrows}} {\perp} \; \mathsf{Sets},$$

$$(- \times B \dashv \mathsf{Sets}(B, -)) \colon \quad \mathsf{Sets} \underset{\mathsf{Sets}(B,-)}{\overset{- \times B}{\rightleftarrows}} {\perp} \; \mathsf{Sets},$$

witnessed by bijections

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$
$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\mathsf{Sets}} \dashv -_1 \times -_2) \colon \quad \mathsf{Sets} \underset{-_1 \times -_2}{\overset{\Delta_{\mathsf{Sets}}}{\rightleftarrows}} {\perp} \; \mathsf{Sets} \times \mathsf{Sets},$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A, A), (B, C)) \cong \mathsf{Sets}(A, B \times C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$.

4. *Associativity.* We have an isomorphism of sets

$$\alpha_{A,B,C}^{\mathsf{Sets}}\colon (A \times B) \times C \overset{\sim}{\dashrightarrow} A \times (B \times C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Unitality.* We have isomorphisms of sets

$$\lambda_A^{\mathsf{Sets}}\colon \mathrm{pt} \times A \overset{\sim}{\dashrightarrow} A,$$
$$\rho_A^{\mathsf{Sets}}\colon A \times \mathrm{pt} \overset{\sim}{\dashrightarrow} A,$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Commutativity.* We have an isomorphism of sets

$$\sigma_{A,B}^{\mathsf{Sets}}\colon A \times B \overset{\sim}{\dashrightarrow} B \times A,$$

   natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\delta_\ell^{\mathsf{Sets}}\colon A \times (B \coprod C) \overset{\sim}{\dashrightarrow} (A \times B) \coprod (A \times C),$$
$$\delta_r^{\mathsf{Sets}}\colon (A \coprod B) \times C \overset{\sim}{\dashrightarrow} (A \times C) \coprod (B \times C),$$

   natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\zeta_\ell^{\mathsf{Sets}}\colon \emptyset \times A \overset{\sim}{\dashrightarrow} \emptyset,$$
$$\zeta_r^{\mathsf{Sets}}\colon A \times \emptyset \overset{\sim}{\dashrightarrow} \emptyset,$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Unions.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cup W) = (U \times V) \cup (U \times W),$$
$$(U \cup V) \times W = (U \times W) \cup (V \times W)$$

   of subsets of $\mathcal{P}(X \times X)$.

10. *Distributivity Over Intersections.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$
U \times (V \cap W) = (U \times V) \cap (U \times W),
$$
$$
(U \cap V) \times W = (U \times W) \cap (V \times W)
$$

of subsets of $\mathcal{P}(X \times X)$.

11. *Distributivity Over Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$
U \times (V \setminus W) = (U \times V) \setminus (U \times W),
$$
$$
(U \setminus V) \times W = (U \times W) \setminus (V \times W)
$$

of subsets of $\mathcal{P}(X \times X)$.

12. *Distributivity Over Symmetric Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$
U \times (V \bigtriangleup W) = (U \times V) \bigtriangleup (U \times W),
$$
$$
(U \bigtriangleup V) \times W = (U \times W) \bigtriangleup (V \times W)
$$

of subsets of $\mathcal{P}(X \times X)$.

13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$
\begin{CD}
(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) @>{\cap \times \cap}>> \mathcal{P}(X) \times \mathcal{P}(X) \\
@V{\mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}}VV @VV{\mathcal{P}^{\times}_{X,X}}V \\
\mathcal{P}(X \times X) \times \mathcal{P}(X \times X) @>{\cap}>> \mathcal{P}(X \times X)
\end{CD}
$$

commutes, i.e. we have

$$
(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).
$$

for each $U, V, W, T \in \mathcal{P}(X)$.

14. *Symmetric Monoidality.* The 8-tuple $\big(\mathsf{Sets}, \times, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\big)$ is a closed symmetric monoidal category.

15. *Symmetric Bimonoidality.* The 18-tuple

$$\left(\mathsf{Sets}, \amalg, \times, \emptyset, \mathsf{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}},\right.$$
$$\left.\alpha^{\mathsf{Sets}, \amalg}, \lambda^{\mathsf{Sets}, \amalg}, \rho^{\mathsf{Sets}, \amalg}, \sigma^{\mathsf{Sets}, \amalg}, \delta_\ell^{\mathsf{Sets}}, \delta_r^{\mathsf{Sets}}, \zeta_\ell^{\mathsf{Sets}}, \zeta_r^{\mathsf{Sets}}\right),$$

is a symmetric closed bimonoidal category, where $\alpha^{\mathsf{Sets}, \amalg}, \lambda^{\mathsf{Sets}, \amalg}, \rho^{\mathsf{Sets}, \amalg}$, and $\sigma^{\mathsf{Sets}, \amalg}$ are the natural transformations from Items 3 to 5 of Definition 4.2.3.1.3.

Proof. Item 1, *Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, *Adjointness*: We prove only that there,s an adjunction $- \times B \dashv \mathsf{Sets}(B, -)$, witnessed by a bijection

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathsf{Sets}(A, -)$ follows almost exactly in the same way.

· *Map I.* We define a map

$$\Phi_{B,C} \colon \mathsf{Sets}(A \times B, C) \longrightarrow \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

by sending a function

$$\xi \colon A \times B \to C$$

to the function
$$\xi^\dagger \colon A \longrightarrow \mathsf{Sets}(B, C),$$
$$a \longmapsto (\xi_a^\dagger \colon B \to C),$$

where we define
$$\xi_a^\dagger(b) \overset{\mathrm{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$\xi^\dagger \overset{\mathrm{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

- *Map II.* We define a map

$$\Psi_{B,C} \colon \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \to \mathsf{Sets}(A \times B, C)$$

given by sending a function

$$\xi \colon A \longrightarrow \mathsf{Sets}(B, C),$$
$$a \longmapsto (\xi_a \colon B \to C),$$

to the function

$$\xi^{\dagger} \colon A \times B \to C$$

defined by

$$\xi^{\dagger}(a, b) \overset{\mathrm{def}}{=} \mathrm{ev}_b(\mathrm{ev}_a(\xi))$$
$$\overset{\mathrm{def}}{=} \mathrm{ev}_b(\xi_a)$$
$$\overset{\mathrm{def}}{=} \xi_a(b)$$

for each $(a, b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A \times B, C)} .$$

Indeed, given a function $\xi \colon A \times B \to C$, we have

$$
\begin{aligned}
[\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\
&= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a, b) \mapsto \xi(a, b) \rrbracket)) \\
&= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a, b) \rrbracket \rrbracket) \\
&= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket) \\
&= \llbracket (a, b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a', b') \rrbracket \rrbracket)) \rrbracket \\
&= \llbracket (a, b) \mapsto \mathrm{ev}_b(\llbracket b' \mapsto \xi(a, b') \rrbracket) \rrbracket \\
&= \llbracket (a, b) \mapsto \xi(a, b) \rrbracket \\
&= \xi.
\end{aligned}
$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A, \mathsf{Sets}(B,C))} .$$

Indeed, given a function

$$\xi \colon A \longrightarrow \mathsf{Sets}(B, C),$$
$$a \longmapsto (\xi_a \colon B \to C),$$

we have

$$
\begin{aligned}
[\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi))\\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a,b) \mapsto \xi_a(b) \rrbracket)\\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket)\\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \mathsf{ev}_{(a,b)}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket\\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket\\
&\overset{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket\\
&\overset{\text{def}}{=} \xi.
\end{aligned}
$$

· *Naturality for Φ, Part I.* We need to show that, given a function $g \colon B \to B'$, the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(A \times B', C) & \xrightarrow{\Phi_{B',C}} & \mathsf{Sets}(A, \mathsf{Sets}(B', C)),\\
{\scriptstyle \mathrm{id}_A \times g^*}\Big\downarrow & & \Big\downarrow{\scriptstyle (g^*)_!}\\
\mathsf{Sets}(A \times B, C) & \xrightarrow[\Phi_{B,C}]{} & \mathsf{Sets}(A, \mathsf{Sets}(B, C))
\end{array}
$$

commutes. Indeed, given a function

$$\xi \colon A \times B' \to C,$$

we have

$$
\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi))\\
&= \Phi_{B,C}(\xi(-_1, g(-_2)))\\
&= [\xi(-_1, g(-_2))]^\dagger\\
&= \xi^\dagger_{-_1}(g(-_2))\\
&= (g^*)_!(\xi^\dagger)\\
&= (g^*)_!(\Phi_{B',C}(\xi))
\end{aligned}
$$

$$= [(g^*)_! \circ \Phi_{B',C}](\xi).$$

Alternatively, using the $[\![\, a \mapsto f(a) \,]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$
\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}([\mathrm{id}_A \times g^*]([\![\, (a,b') \mapsto \xi(a,b') \,]\!])) \\
&= \Phi_{B,C}([\![\, (a,b) \mapsto \xi(a,g(b)) \,]\!]) \\
&= [\![\, a \mapsto [\![\, b \mapsto \xi(a,g(b)) \,]\!] \,]\!] \\
&= [\![\, a \mapsto g^*([\![\, b' \mapsto \xi(a,b') \,]\!]) \,]\!] \\
&= (g^*)_!([\![\, a \mapsto [\![\, b' \mapsto \xi(a,b') \,]\!] \,]\!]) \\
&= (g^*)_!(\Phi_{B',C}([\![\, (a,b') \mapsto \xi(a,b') \,]\!])) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)_! \circ \Phi_{B',C}](\xi).
\end{aligned}
$$

· *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C \to C'$, the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(A \times B, C) & \xrightarrow{\ \Phi_{B,C}\ } & \mathsf{Sets}(A, \mathsf{Sets}(B,C)), \\
\Big\downarrow{\scriptstyle h_!} & & \Big\downarrow{\scriptstyle (h_!)_!} \\
\mathsf{Sets}(A \times B, C') & \xrightarrow[\ \Phi_{B,C'}\ ]{} & \mathsf{Sets}(A, \mathsf{Sets}(B,C'))
\end{array}
$$

commutes. Indeed, given a function

$$\xi\colon A \times B \to C,$$

we have

$$
\begin{aligned}
[\Phi_{B,C} \circ h_!](\xi) &= \Phi_{B,C}(h_!(\xi)) \\
&= \Phi_{B,C}(h_!([\![\, (a,b) \mapsto \xi(a,b) \,]\!])) \\
&= \Phi_{B,C}([\![\, (a,b) \mapsto h(\xi(a,b)) \,]\!]) \\
&= [\![\, a \mapsto [\![\, b \mapsto h(\xi(a,b)) \,]\!] \,]\!] \\
&= [\![\, a \mapsto h_!([\![\, b \mapsto \xi(a,b) \,]\!]) \,]\!] \\
&= (h_!)_!([\![\, a \mapsto [\![\, b \mapsto \xi(a,b) \,]\!] \,]\!])
\end{aligned}
$$

$$= (h_!)_!(\Phi_{B,C}(\llbracket\, (a,b) \mapsto \xi(a,b) \,\rrbracket))$$
$$= (h_!)_!(\Phi_{B,C}(\xi))$$
$$= [(h_!)_! \circ \Phi_{B,C}](\xi).$$

· *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componen-twise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, *Adjointness II*: This follows from the universal property of the product.

Item 4, *Associativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.4.1.1.

Item 5, *Unitality*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.1.5.1.1 and 5.1.6.1.1.

Item 6, *Commutativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.7.1.1.

Item 7, *Distributivity Over Coproducts*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.1.1.1 and 5.3.2.1.1.

Item 8, *Annihilation With the Empty Set*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.3.1.1 and 5.3.4.1.1.

Item 9, *Distributivity Over Unions*: See [Pro25c].

Item 10, *Distributivity Over Intersections*: See [Pro25d, Corollary 1].

Item 11, *Distributivity Over Differences*: See [Pro25a].

Item 12, *Distributivity Over Symmetric Differences*: See [Pro25b].

Item 13, *Middle-Four Exchange With Respect to Intersections*: See [Pro25d, Corollary 1].

Item 14, *Symmetric Monoidality*: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.1.9.1.1, and is proved there.

Item 15, *Symmetric Bimonoidality*: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.3.5.1.1, and is proved there.   □

Remark 4.1.3.1.4. As shown in Item 1 of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

$$-_1 \times -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}.$$
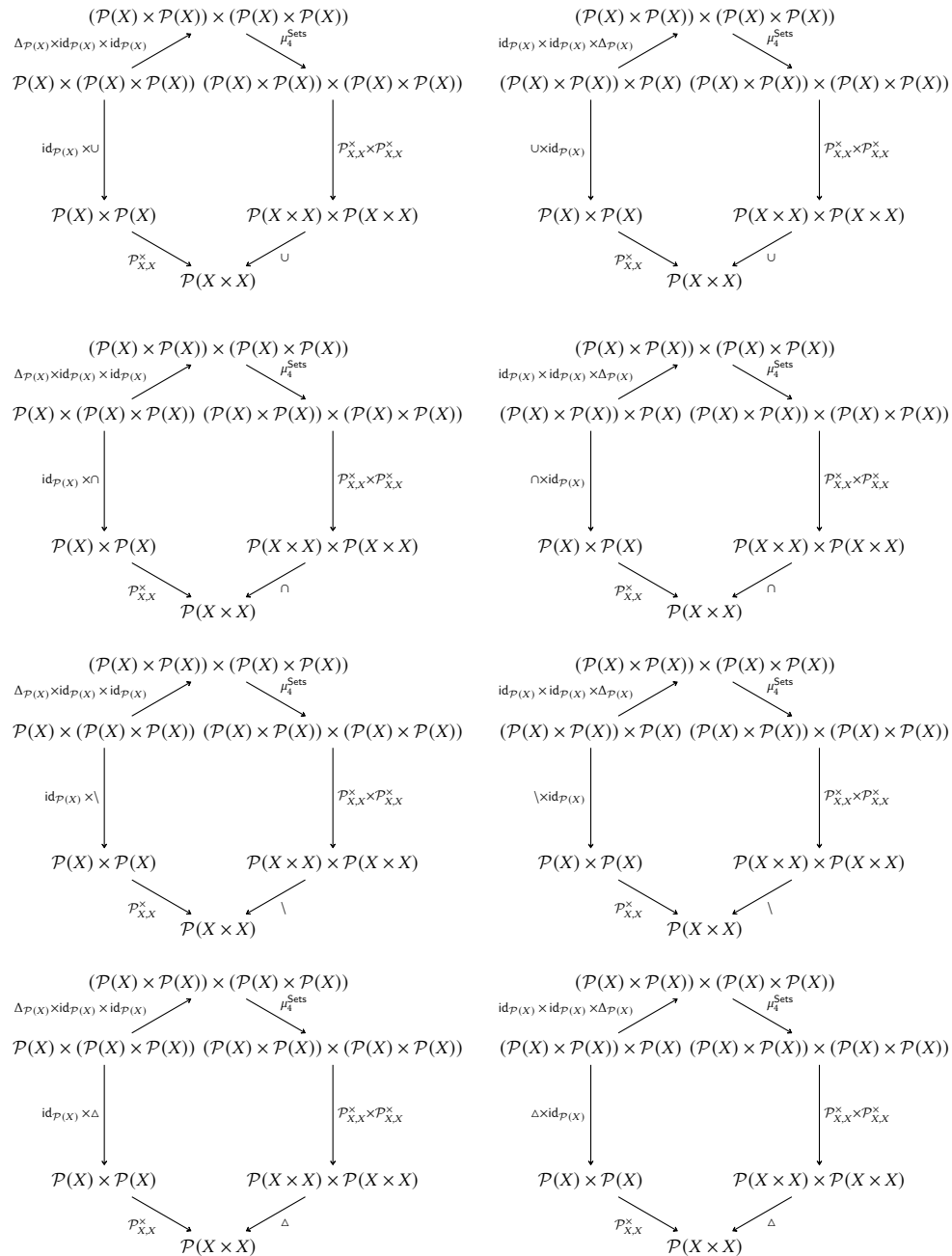
This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k,\ell} \colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets})$$

of tensor products of $\mathbb{E}_k$-monoid objects on Sets with $\mathbb{E}_\ell$-monoid objects on Sets; see **??**.

Remark 4.1.3.1.5. We may state the equalities in Items 9 to 12 of Definition 4.1.3.1.3 as the commutativity of the following diagrams:

$$
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\mathrm{id}_{\mathcal{P}(X)}\times\cup} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\cup} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
\qquad
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\cup\times\mathrm{id}_{\mathcal{P}(X)}} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\cup} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
$$

$$
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\mathrm{id}_{\mathcal{P}(X)}\times\cap} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\cap} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
\qquad
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\cap\times\mathrm{id}_{\mathcal{P}(X)}} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\cap} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
$$

$$
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\mathrm{id}_{\mathcal{P}(X)}\times\backslash} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\backslash} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
\qquad
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\backslash\times\mathrm{id}_{\mathcal{P}(X)}} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\backslash} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
$$

$$
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\mathrm{id}_{\mathcal{P}(X)}\times\triangle} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\triangle} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
\qquad
\begin{array}{c}
(\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[2pt]
{}^{\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}}\nearrow \qquad \searrow^{\mu_4^{\mathsf{Sets}}} \\[2pt]
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \qquad (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \\[4pt]
\Big\downarrow{}^{\triangle\times\mathrm{id}_{\mathcal{P}(X)}} \qquad\qquad\qquad\qquad \Big\downarrow{}^{\mathcal{P}_{X,X}^\times\times\mathcal{P}_{X,X}^\times} \\[4pt]
\mathcal{P}(X)\times\mathcal{P}(X) \qquad\qquad \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \\[2pt]
{}_{\mathcal{P}_{X,X}^\times}\searrow \qquad \swarrow^{\triangle} \\[2pt]
\mathcal{P}(X\times X)
\end{array}
$$

4.1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 4.1.4.1.1. The **pullback of A and B over C along f and g**[3] is the pullback of A and B over C along f and g in Sets as in Limits and Colimits, **??**.

Construction 4.1.4.1.2. Concretely, the pullback of A and B over C along f and g is the pair $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \stackrel{\mathrm{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

2. *The Cone.* The maps[4]

$$\mathrm{pr}_1 \colon A \times_C B \to A,$$
$$\mathrm{pr}_2 \colon A \times_C B \to B$$

   defined by

$$\mathrm{pr}_1(a, b) \stackrel{\mathrm{def}}{=} a,$$
$$\mathrm{pr}_2(a, b) \stackrel{\mathrm{def}}{=} b$$

   for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,$$

$$
\begin{array}{ccc}
A \times_C B & \xrightarrow{\ \mathrm{pr}_2\ } & B \\
{\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
A & \xrightarrow{\ f\ } & C.
\end{array}
$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$[f \circ \mathrm{pr}_1](a, b) = f(\mathrm{pr}_1(a, b))$$

---

[3] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.
[4] *Further Notation:* Also written $\mathrm{pr}_1^{A \times_C B}$ and $\mathrm{pr}_2^{A \times_C B}$.

$$= f(a)$$
$$= g(b)$$
$$= g(\mathrm{pr}_2(a, b))$$
$$= [g \circ \mathrm{pr}_2](a, b),$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form
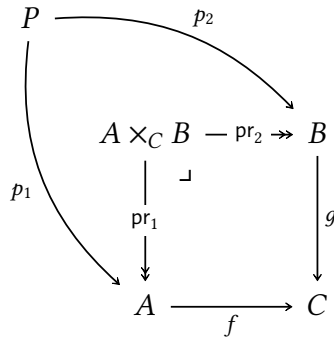
in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$ making the diagram
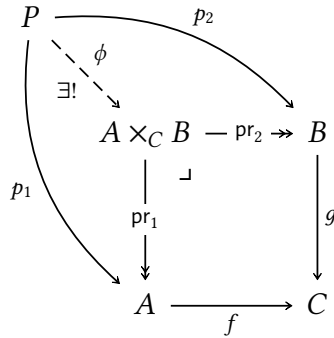
commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

Remark 4.1.4.1.3. It is common practice to write $A \times_C B$ for the pullback of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

   However, the set $A \times_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

Example 4.1.4.1.4. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let X be a set. We have

$$A \cap B \cong A \times_{A \cup B} B,$$

$$\begin{array}{ccc} A \cap B & \longrightarrow & B \\ \downarrow & \lrcorner & \uparrow {\scriptstyle \iota_B} \\ A & \xhookrightarrow{\iota_A} & A \cup B \end{array}$$

   for each $A, B \in \mathcal{P}(X)$.

Proof. Item 1, *Unions via Intersections*: Indeed, we have

$$A \times_{A \cup B} B \cong \{(x, y) \in A \times B \mid x = y\}$$
$$\cong A \cap B.$$

This finishes the proof. □

Proposition 4.1.4.1.5. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

   where $\mathcal{P}$ is the category that looks like this:

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet. \end{array}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$
\begin{array}{ccc}
A \times_C B & \longrightarrow & B \\
& & \downarrow^g \quad \searrow^\psi \\
& A' \times_{C'} B' \longrightarrow & B' \\
\downarrow & & \downarrow \quad \downarrow^{g'} \\
A \xrightarrow{\;f\;} & C & \\
\searrow^\phi & & \searrow^\chi \\
& A' \xrightarrow[f']{} & C'
\end{array}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\mathrm{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$
\begin{array}{ccc}
A \times_C B & \longrightarrow & B \\
 \dashrightarrow & & \downarrow^g \quad \searrow^\psi \\
& A' \times_{C'} B' \longrightarrow & B' \\
\downarrow & & \downarrow \quad \downarrow^{g'} \\
A \xrightarrow{\;f\;} & C & \\
\searrow^\phi & & \searrow^\chi \\
& A' \xrightarrow[f']{} & C'
\end{array}
$$

commute.

2. *Adjointness I.* We have adjunctions

$$
\left(A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)\right) \colon \quad \mathsf{Sets}_{/X} \underset{\mathbf{Sets}_{/X}(A,-)}{\overset{A \times_X -}{\rightleftarrows}} \bot \; \mathsf{Sets}_{/X},
$$

$$
\left(- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)\right) \colon \quad \mathsf{Sets}_{/X} \underset{\mathbf{Sets}_{/X}(B,-)}{\overset{- \times_X B}{\rightleftarrows}} \bot \; \mathsf{Sets}_{/X},
$$

witnessed by bijections

$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}(A, \mathbf{Sets}_{/X}(B, C)),$$
$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}(B, \mathbf{Sets}_{/X}(A, C)),$$

natural in $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$, where $\mathbf{Sets}_{/X}(A, B)$ is the object of $\mathsf{Sets}_{/X}$ consisting of (see Fibred Sets, **??**):

- *The Set.* The set $\mathbf{Sets}_{/X}(A, B)$ defined by

$$\mathbf{Sets}_{/X}(A, B) \overset{\mathrm{def}}{=} \coprod_{x \in X} \mathsf{Sets}(\phi_A^{-1}(x), \phi_Y^{-1}(x))$$

- *The Map to X.* The map

$$\phi_{\mathbf{Sets}_{/X}(A,B)} \colon \mathbf{Sets}_{/X}(A, B) \to X$$

  defined by
$$\phi_{\mathbf{Sets}_{/X}(A,B)}(x, f) \overset{\mathrm{def}}{=} x$$
  for each $(x, f) \in \mathbf{Sets}_{/X}(A, B)$.

3. *Adjointness II.* We have an adjunction

$$\left(\Delta_{\mathsf{Sets}_{/X}} \dashv -_1 \times -_2 \right) \colon \quad \mathsf{Sets}_{/X} \underset{-_1 \times -_2}{\overset{\Delta_{\mathsf{Sets}_{/X}}}{\rightleftarrows}} \bot \; \mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X},$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}}((A, A), (B, C)) \cong \mathsf{Sets}_{/X}(A, B \times_X C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets}_{/X})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X})$.

4. *Associativity.* Given a diagram

$$
\begin{array}{ccccc}
A & & B & & C \\
& \searrow^{f} \; \swarrow^{g} & & \searrow^{h} \; \swarrow^{k} & \\
& X & & Y &
\end{array}
$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$(A \times_X B) \times_Y C$$

$$A \times_X B$$

$$A \qquad B \qquad C,$$

$$f \qquad g \quad h \qquad k$$

$$X \qquad Y$$

$$(A \times_X B) \times_B (B \times_Y C)$$

$$A \times_X B \qquad B \times_Y C$$

$$A \qquad B \qquad C,$$

$$f \qquad g \quad h \qquad k$$

$$X \qquad Y$$

$$A \times_X (B \times_Y C)$$

$$B \times_Y C$$

$$A \qquad B \qquad C.$$

$$f \qquad g \quad h \qquad k$$

$$X \qquad Y$$

5. *Interaction With Composition.* Given a diagram

$$X \qquad\qquad\qquad Y$$
$$\phi \searrow \qquad\qquad \swarrow \psi$$
$$A \qquad\qquad B$$
$$f \searrow \qquad \swarrow g$$
$$K$$
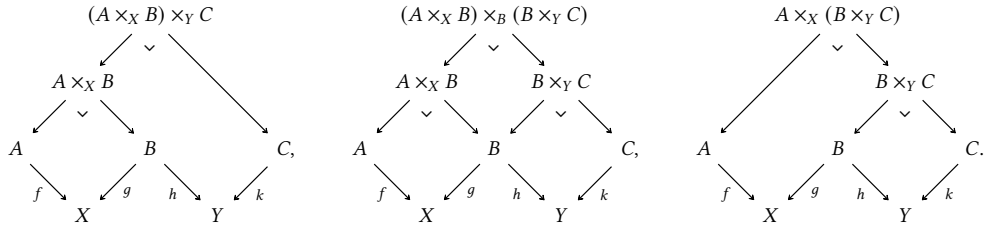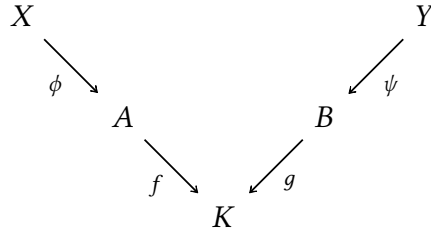
in Sets, we have isomorphisms of sets

$$X \times_K^{f \circ \phi, g \circ \psi} Y \cong (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \times_{A \times_K^{f,g} B}^{p_2, p_1} ((A \times_K^{f,g} B) \times_B^{q_2, \psi} Y)$$

$$\cong X \times_A^{\phi, p} ((A \times_K^{f,g} B) \times_B^{q_2, \psi} Y)$$

$$\cong (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \times_B^{q, \psi} Y$$

where

$$q_1 = \mathrm{pr}_1^{A \times_K^{f,g} B}, \qquad\qquad q_2 = \mathrm{pr}_2^{A \times_K^{f,g} B},$$

$$p_1 = \mathrm{pr}_1^{(A \times_K^{f,g} B) \times_Y^{q_2, \psi}}, \qquad\qquad p_2 = \mathrm{pr}_2^{X \times_{A \times_K^{f,g} B}^{\phi, q_1} (A \times_K^{f,g} B)},$$

$$p = q_1 \circ \mathrm{pr}_1^{(A \times_K^{f,g} B) \times_B^{q_2, \psi} Y}, \qquad q = q_2 \circ \mathrm{pr}_2^{X \times_A^{\phi, q_1} (A \times_K^{f,g} B)},$$

and where these pullbacks are built as in the following diagrams:

6. *Unitality.* We have isomorphisms of sets

$$\lambda_A^{\mathsf{Sets}_{/X}} : X \times_X A \overset{\sim}{\dashrightarrow} A,$$
$$\rho_A^{\mathsf{Sets}_{/X}} : A \times_X X \overset{\sim}{\dashrightarrow} A,$$

natural in $(A, f) \in \mathsf{Obj}(\mathsf{Sets}_{/X})$.

7. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
A \times_X B & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow g \\
A & \xrightarrow{f} & X,
\end{array}
\qquad
\sigma_{A,B}^{\mathsf{Sets}/X} : A \times_X B \xrightarrow{\sim} B \times_X A
\qquad
\begin{array}{ccc}
B \times_X A & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow f \\
B & \xrightarrow{g} & X,
\end{array}
$$

natural in $(A, f), (B, g) \in \mathsf{Obj}(\mathsf{Sets}_{/X})$.

8. *Distributivity Over Coproducts.* Let A, B, and C be sets and let $\phi_A \colon A \to X$, $\phi_B \colon B \to X$, and $\phi_C \colon C \to X$ be morphisms of sets. We have isomorphisms of sets

$$
\delta_\ell^{\mathsf{Sets}/X} : A \times_X (B \coprod C) \xrightarrow{\sim} (A \times_X B) \coprod (A \times_X C),
$$
$$
\delta_r^{\mathsf{Sets}/X} : (A \coprod B) \times_X C \xrightarrow{\sim} (A \times_X C) \coprod (B \times_X C),
$$

as in the diagrams

$$
\begin{array}{ccc}
(A \times_X B) \coprod (A \times_X C) & \longrightarrow & B \coprod C \\
\downarrow & \lrcorner & \downarrow \phi_B \coprod \phi_C \\
A & \xrightarrow{\phi_A} & X
\end{array}
\qquad
\begin{array}{ccc}
(A \times_X C) \coprod (B \times_X C) & \twoheadrightarrow & C \\
\downarrow & \lrcorner & \downarrow \phi_C \\
A \coprod B & \xrightarrow{\phi_A \coprod \phi_B} & X
\end{array}
$$

natural in $A, B, C \in \mathsf{Obj}(\mathsf{Sets}_{/X})$.

9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & \varnothing \\
\downarrow & \lrcorner & \downarrow \\
A & \xrightarrow{f} & X,
\end{array}
\qquad
\begin{array}{c}
\zeta_\ell^{\mathsf{Sets}/X} : A \times_X \varnothing \xrightarrow{\sim} \varnothing, \\[4pt]
\zeta_r^{\mathsf{Sets}/X} : \varnothing \times_X A \xrightarrow{\sim} \varnothing,
\end{array}
\qquad
\begin{array}{ccc}
\varnothing & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow f \\
\varnothing & \longrightarrow & X,
\end{array}
$$

natural in $(A, f) \in \mathsf{Obj}(\mathsf{Sets}_{/X})$.

10. *Interaction With Products.* We have an isomorphism of sets

$$
A \times_{\mathsf{pt}} B \cong A \times B,
\qquad
\begin{array}{ccc}
A \times B & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow {!_B} \\
A & \xrightarrow{!_A} & \mathsf{pt}.
\end{array}
$$

11. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}_{/X}, \times_X, X, \mathbf{Sets}_{/X}, \alpha^{\mathsf{Sets}_{/X}}, \lambda^{\mathsf{Sets}_{/X}}, \rho^{\mathsf{Sets}_{/X}}, \sigma^{\mathsf{Sets}_{/X}}\right)$ is a symmetric closed monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Adjointness I: This is a repetition of Fibred Sets, **??** of **??**, and is proved there.

Item 3, Adjointness II: This follows from the universal property of the product (pullbacks are products in $\mathsf{Sets}_{/X}$).

Item 4, Associativity: We have

$$
\begin{aligned}
(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\
&\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\
&\cong A \times_X (B \times_Y C)
\end{aligned}
$$

and

$$
\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\
&\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{aligned} &f(a) = g(b), b = b', \\ &\text{and } h(b') = k(c) \end{aligned}\right\} \\
&\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{aligned} &f(a) = g(b), b = b', \\ &\text{and } h(b') = k(c) \end{aligned}\right\} \\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{aligned} &f(a) = g(b), b = b', \\ &\text{and } h(b') = k(c) \end{aligned}\right\} \\
&\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{aligned} &f(a) = g(b) \text{ and} \\ &h(b') = k(c) \end{aligned}\right\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}
$$

where we have used Item 6 for the isomorphism $B \times_B B \cong B$.

Item 5, Interaction With Composition: By Item 4, it suffices to construct only the isomorphism

$$
X \times_K^{f \circ \phi, g \circ \psi} Y \cong (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \times_{A \times_K^{f,g} B}^{p_2, p_1} ((A \times_K^{f,g} B) \times_B^{q_2, \psi} Y).
$$

We have

$$
(X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \overset{\text{def}}{=} \left\{(x,(a,b)) \in X \times (A \times_K^{f,g} B) \,\middle|\, \phi(x) = q_1(a,b)\right\}
$$

$$\overset{\text{def}}{=} \left\{ (x, (a,b)) \in X \times (A \times_K^{f,g} B) \,\middle|\, \phi(x) = a \right\}$$

$$\cong \{ (x, (a,b)) \in X \times (A \times B) \mid \phi(x) = a \text{ and } f(a) = g(b) \},$$

$$((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y) \overset{\text{def}}{=} \left\{ ((a,b), y) \in (A \times_K^{f,g} B) \times Y \,\middle|\, q_2(a,b) = \psi(y) \right\}$$

$$\overset{\text{def}}{=} \left\{ ((a,b), y) \in (A \times_K^{f,g} B) \times Y \,\middle|\, b = \psi(y) \right\}$$

$$\cong \{ ((a,b), y) \in (A \times B) \times Y \mid b = \psi(y) \text{ and } f(a) = g(b) \},$$

so writing

$$S = (X \times_A^{\phi,q_1} (A \times_K^{f,g} B))$$
$$S' = ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y),$$

we have

$$S \times_{A \times_K^{f,g} B}^{p_2,p_1} S' \overset{\text{def}}{=} \{ ((x, (a,b)), ((a',b'), y)) \in S \times S' \mid p_1(x, (a,b)) = p_2((a',b'), y) \}$$

$$\overset{\text{def}}{=} \{ ((x, (a,b)), ((a',b'), y)) \in S \times S' \mid (a,b) = (a',b') \}$$

$$\cong \{ ((x, a, b, y)) \in X \times A \times B \times Y \mid \phi(x) = a, \psi(y) = b, \text{ and } f(a) = g(b) \}$$

$$\cong \{ ((x, a, b, y)) \in X \times A \times B \times Y \mid f(\phi(x)) = g(\psi(y)) \}$$

$$\overset{\text{def}}{=} X \times_K Y.$$

This finishes the proof.

Item 6, Unitality: We have

$$X \times_X A \cong \{ (x, a) \in X \times A \mid f(a) = x \},$$
$$A \times_X X \cong \{ (a, x) \in X \times A \mid f(a) = x \},$$

which are isomorphic to A via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$. The proof of the naturality of $\lambda^{\mathsf{Sets}/X}$ and $\rho^{\mathsf{Sets}/X}$ is omitted.

Item 7, Commutativity: We have

$$A \times_C B \overset{\text{def}}{=} \{ (a, b) \in A \times B \mid f(a) = g(b) \}$$
$$= \{ (a, b) \in A \times B \mid g(b) = f(a) \}$$
$$\cong \{ (b, a) \in B \times A \mid g(b) = f(a) \}$$
$$\overset{\text{def}}{=} B \times_C A.$$

The proof of the naturality of $\sigma^{\mathsf{Sets}/X}$ is omitted.

Item 8, Distributivity Over Coproducts: We have

$$A \times_X (B \coprod C) \overset{\text{def}}{=} \left\{ (a, z) \in A \times (B \coprod C) \,\middle|\, \phi_A(a) = \phi_{B \coprod C}(z) \right\}$$

$$= \left\{ (a, z) \in A \times (B \coprod C) \,\Big|\, z = (0, b) \text{ and } \phi_A(a) = \phi_{B \coprod C}(z) \right\}$$
$$\cup \left\{ (a, z) \in A \times (B \coprod C) \,\Big|\, z = (1, c) \text{ and } \phi_A(a) = \phi_{B \coprod C}(z) \right\}$$
$$= \{ (a, z) \in A \times (B \coprod C) \,|\, z = (0, b) \text{ and } \phi_A(a) = \phi_B(b) \}$$
$$\cup \{ (a, z) \in A \times (B \coprod C) \,|\, z = (1, c) \text{ and } \phi_A(a) = \phi_C(c) \}$$
$$\cong \{ (a, b) \in A \times B \,|\, \phi_A(a) = \phi_B(b) \}$$
$$\cup \{ (a, c) \in A \times C \,|\, \phi_A(a) = \phi_C(c) \}$$
$$\overset{\text{def}}{=} (A \times_X B) \cup (A \times_X C)$$
$$\cong (A \times_X B) \coprod (A \times_X C),$$

with the construction of the isomorphism

$$\delta_r^{\mathsf{Sets}/X} \colon (A \coprod B) \times_X C \overset{\sim}{\dashrightarrow} (A \times_X C) \coprod (B \times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\mathsf{Sets}/X}$ and $\delta_r^{\mathsf{Sets}/X}$ is omitted.
Item 9, Annihilation With the Empty Set: We have

$$A \times_X \emptyset \overset{\text{def}}{=} \{ (a, b) \in A \times \emptyset \,|\, f(a) = g(b) \}$$
$$= \{ k \in \emptyset \,|\, f(a) = g(b) \}$$
$$= \emptyset,$$

and similarly for $\emptyset \times_X A$, where we have used Item 8 of Definition 4.1.3.1.3. The proof of the naturality of $\zeta_\ell^{\mathsf{Sets}/X}$ and $\zeta_r^{\mathsf{Sets}/X}$ is omitted.
Item 10, Interaction With Products: We have

$$A \times_{\mathrm{pt}} B \overset{\text{def}}{=} \{ (a, b) \in A \times B \,|\, !_A(a) = !_B(b) \}$$
$$\overset{\text{def}}{=} \{ (a, b) \in A \times B \,|\, \star = \star \}$$
$$= \{ (a, b) \in A \times B \}$$
$$= A \times B.$$

Item 11, Symmetric Monoidality: Omitted. □

4.1.5   Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 4.1.5.1.1. The **equaliser of f and g** is the equaliser of f and g in Sets as in Limits and Colimits, **??**.

Construction 4.1.5.1.2. Concretely, the equaliser of f and g is the pair $(\mathrm{Eq}(f,g),\mathrm{eq}(f,g))$ consisting of:

1. *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \overset{\mathrm{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

2. *The Cone.* The inclusion map

$$\mathrm{eq}(f,g)\colon\ \mathrm{Eq}(f,g) \hookrightarrow A.$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\mathrm{Eq}(f,g) \xhookrightarrow{\ \mathrm{eq}(f,g)\ } A \underset{g}{\overset{f}{\rightrightarrows}} B$$
$$\underset{E}{\overset{\ \ e\ \ }{\nearrow}}$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g)$ making the diagram

$$\mathrm{Eq}(f,g) \xhookrightarrow{\ \mathrm{eq}(f,g)\ } A \underset{g}{\overset{f}{\rightrightarrows}} B$$
$$\phi \Big\uparrow \exists! \qquad \overset{\ \ e\ \ }{\nearrow}$$
$$E$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$.   □

Proposition 4.1.5.1.3. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[5]

$$\underbrace{\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h))}_{=\mathsf{Eq}(f \circ \mathsf{eq}(g,h), h \circ \mathsf{eq}(g,h))} \cong \mathsf{Eq}(f,g,h) \cong \underbrace{\mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g))}_{=\mathsf{Eq}(g \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g))},$$

---

[5]That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$A \xrightarrow[\substack{\longrightarrow \\ -g \to \\ \longrightarrow}]{\substack{f \\ \\ h}} B$$

    in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathsf{Eq}(f,g) \overset{\mathsf{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

    and then take the equaliser of the composition

$$\mathsf{Eq}(f,g) \overset{\mathsf{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

    obtaining a subset

$$\mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)) = \mathsf{Eq}(g \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g))$$

    of $\mathsf{Eq}(f,g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathsf{Eq}(g,h) \overset{\mathsf{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

    and then take the equaliser of the composition

$$\mathsf{Eq}(g,h) \overset{\mathsf{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

    obtaining a subset

$$\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) = \mathsf{Eq}(f \circ \mathsf{eq}(g,h), h \circ \mathsf{eq}(g,h))$$

    of $\mathsf{Eq}(g,h)$.

where $\mathsf{Eq}(f, g, h)$ is the limit of the diagram

$$A \xrightarrow[]{f} \!\! \xrightarrow{-g} \!\! \xrightarrow[h]{} B$$

in Sets, being explicitly given by

$$\mathsf{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathsf{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathsf{Eq}(f, g) \cong \mathsf{Eq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathsf{Eq}(h \circ f \circ \mathsf{eq}(f, g), k \circ g \circ \mathsf{eq}(f, g)) \subset \mathsf{Eq}(h \circ f, k \circ g),$$

where $\mathsf{Eq}(h \circ f \circ \mathsf{eq}(f, g), k \circ g \circ \mathsf{eq}(f, g))$ is the equaliser of the composition

$$\mathsf{Eq}(f, g) \overset{\mathsf{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. Item 1, *Associativity*: We first prove that $\mathsf{Eq}(f, g, h)$ is indeed given by

$$\mathsf{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\mathsf{Eq}(f, g, h) \xleftarrow{\mathsf{eq}(f,g,h)} A \xrightarrow[]{f} \!\! \xrightarrow{-g} \!\! \xrightarrow[h]{} B$$
$$\nwarrow_{e}$$
$$E$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathsf{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathsf{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathsf{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathsf{Eq}(f,g,h)$.

We now check the equalities

$$\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) \cong \mathsf{Eq}(f,g,h) \cong \mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)).$$

Indeed, we have

$$
\begin{aligned}
\mathsf{Eq}(f \circ \mathsf{eq}(g,h), g \circ \mathsf{eq}(g,h)) &\cong \{x \in \mathsf{Eq}(g,h) \mid [f \circ \mathsf{eq}(g,h)](a) = [g \circ \mathsf{eq}(g,h)](a)\} \\
&\cong \{x \in \mathsf{Eq}(g,h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathsf{Eq}(f,g,h).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathsf{Eq}(f \circ \mathsf{eq}(f,g), h \circ \mathsf{eq}(f,g)) &\cong \{x \in \mathsf{Eq}(f,g) \mid [f \circ \mathsf{eq}(f,g)](a) = [h \circ \mathsf{eq}(f,g)](a)\} \\
&\cong \{x \in \mathsf{Eq}(f,g) \mid f(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathsf{Eq}(f,g,h).
\end{aligned}
$$

Item 4, *Unitality*: Indeed, we have

$$
\begin{aligned}
\mathsf{Eq}(f,f) &\overset{\text{def}}{=} \{a \in A \mid f(a) = f(a)\} \\
&= A.
\end{aligned}
$$

Item 5, *Commutativity*: Indeed, we have

$$\mathsf{Eq}(f,g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}$$

$$= \{a \in A \mid g(a) = f(a)\}$$
$$\stackrel{\text{def}}{=} \mathsf{Eq}(g, f).$$

Item 6, Interaction With Composition: Indeed, we have

$$\mathsf{Eq}(h \circ f \circ \mathsf{eq}(f, g), k \circ g \circ \mathsf{eq}(f, g)) \cong \{a \in \mathsf{Eq}(f, g) \mid h(f(a)) = k(g(a))\}$$
$$\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.$$

and

$$\mathsf{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there‚s an inclusion from $\mathsf{Eq}(h \circ f \circ \mathsf{eq}(f, g), k \circ g \circ \mathsf{eq}(f, g))$ to $\mathsf{Eq}(h \circ f, k \circ g)$. $\square$

4.1.6   Inverse Limits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \to \mathsf{Sets}$ be an inverse system of sets.

Definition 4.1.6.1.1. The **inverse limit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.1.6.1.2. Concretely, the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the pair $\left(\varprojlim_{\alpha \in I}(X_\alpha), \{\mathsf{pr}_\alpha\}_{\alpha \in I} \right)$ consisting of:

1. *The Limit.* The set $\varprojlim_{\alpha \in I}(X_\alpha)$ defined by

$$\varprojlim_{\alpha \in I}(X_\alpha) \stackrel{\text{def}}{=} \left\{ (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \;\middle|\; \begin{array}{l} \text{for each } \alpha, \beta \in I, \text{ if } \alpha \preceq \beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta) \end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \mathsf{pr}_\gamma \colon \varprojlim_{\alpha \in I}(X_\alpha) \to X_\gamma \right\}_{\gamma \in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{ \mathsf{pr}_\gamma \colon \prod_{\alpha \in I} X_\alpha \to X_\gamma \right\}_{\gamma \in I}$$

of Item 2 of Definition 4.1.2.1.2 to $\varprojlim_{\alpha \in I}(X_\alpha)$ and hence given by

$$\mathrm{pr}_\gamma((x_\alpha)_{\alpha \in I}) \overset{\mathrm{def}}{=} x_\gamma$$

for each $\gamma \in I$ and each $(x_\alpha)_{\alpha \in I} \in \varprojlim_{\alpha \in I}(X_\alpha)$.

Proof. We claim that $\varprojlim_{\alpha \in I}(X_\alpha)$ is the limit of the inverse system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$. First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta} \circ \mathrm{pr}_\alpha = \mathrm{pr}_\beta,$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma \in I} \in \varprojlim_{\gamma \in I}(X_\gamma)$, we have

$$\begin{aligned}
[f_{\alpha\beta} \circ \mathrm{pr}_\alpha]((x_\gamma)_{\gamma \in I}) &\overset{\mathrm{def}}{=} f_{\alpha\beta}(\mathrm{pr}_\alpha((x_\gamma)_{\gamma \in I})) \\
&\overset{\mathrm{def}}{=} f_{\alpha\beta}(x_\alpha) \\
&= x_\beta \\
&\overset{\mathrm{def}}{=} \mathrm{pr}_\beta((x_\gamma)_{\gamma \in I}),
\end{aligned}$$

where the third equality comes from the definition of $\varprojlim_{\alpha \in I}(X_\alpha)$. Next, we prove that $\varprojlim_{\alpha \in I}(X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

in Sets. Then there indeed exists a unique map $\phi\colon L \xrightarrow{\exists!} \varprojlim_{\alpha\in I}(X_\alpha)$ making the diagram

commute, being uniquely determined by the family of conditions

$$\{p_\alpha = \mathrm{pr}_\alpha \circ \phi\}_{\alpha\in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha\in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha$ indeed lies in $\varprojlim_{\alpha\in I}(X_\alpha)$, as we have

$$f_{\alpha\beta}(p_\alpha(\ell)) \overset{\mathrm{def}}{=} [f_{\alpha\beta} \circ p_\alpha](\ell)$$
$$\overset{\mathrm{def}}{=} p_\beta(\ell)$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $(L, \{p_\alpha\}_{\alpha\in I})$. $\quad\square$

Example 4.1.6.1.3. Here are some examples of inverse limits of sets.

1. *The p-Adic Integers.* The ring of p-adic integers $\mathbb{Z}_p$ of **??** is the inverse limit

$$\mathbb{Z}_p \cong \varprojlim_{n\in\mathbb{N}}(\mathbb{Z}_{/p^n});$$

    see **??**.

2. *Rings of Formal Power Series.* The ring $R[\![t]\!]$ of formal power series in a variable t is the inverse limit

$$R[\![t]\!] \cong \varprojlim_{n\in\mathbb{N}}(R[t]/t^n R[t]);$$

    see **??**.

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see **??**.

4.2 Colimits of Sets

4.2.1 The Initial Set

Definition 4.2.1.1.1. The **initial set** is the initial object of Sets as in Limits and Colimits, **??**.

Construction 4.2.1.1.2. Concretely, the initial set is the pair $(\emptyset, \{\iota_A\}_{A \in \mathrm{Obj}(\mathsf{Sets})})$ consisting of:

1. *The Colimit.* The empty set $\emptyset$ of Definition 4.3.1.1.1.

2. *The Cocone.* The collection of maps

$$\{\iota_A \colon \emptyset \to A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

   given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi \colon \emptyset \to A$ making the diagram

$$\emptyset \overset{\phi}{\underset{\exists!}{\dashrightarrow}} A$$

commute, namely the inclusion map ι_A. □

4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 4.2.2.1.1. The **coproduct of** $\{A_i\}_{i \in I}$[6] is the coproduct of $\{A_i\}_{i \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.2.2.1.2. Concretely, the disjoint union of $\{A_i\}_{i \in I}$ is the pair $(\coprod_{i \in I} A_i, \{\mathrm{inj}_i\}_{i \in I})$ consisting of:

---

[6] *Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i \in I}$.

1. *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \overset{\mathrm{def}}{=} \left\{ (i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \,\middle|\, x \in A_i \right\}.$$

2. *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i : A_i \to \coprod_{i\in I} A_i \right\}_{i\in I}$$

of maps given by

$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i,x)$$

for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{ccc}
 & & C \\
 & \nearrow^{\iota_i} & \\
A_i & \xrightarrow[\mathrm{inj}_i]{} & \coprod_{i\in I} A_i
\end{array}
$$

in Sets. Then there exists a unique map $\phi \colon \coprod_{i\in I} A_i \to C$ making the diagram

$$
\begin{array}{ccc}
 & & C \\
 & \nearrow^{\iota_i} & \uparrow{\scriptstyle\phi}\ {\scriptstyle\exists!} \\
A_i & \xrightarrow[\mathrm{inj}_i]{} & \coprod_{i\in I} A_i
\end{array}
$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i,x)) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$. $\square$

Proposition 4.2.2.1.3. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \coprod_{i \in I} A_i$ defines a functor

$$\coprod_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \mathsf{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i \in I} \right] ((A_i)_{i \in I}) \overset{\mathsf{def}}{=} \coprod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \mathsf{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} \colon \mathsf{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \to \mathsf{Sets}\left(\coprod_{i \in I} A_i, \coprod_{i \in I} B_i \right)$$

  of $\coprod_{i \in I}$ at $((A_i)_{i \in I}, (B_i)_{i \in I})$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i \in I}$$

  in $\mathsf{Nat}((A_i)_{i \in I}, (B_i)_{i \in I})$ to the map of sets

$$\coprod_{i \in I} f_i \colon \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i$$

  defined by

$$\left[\coprod_{i \in I} f_i \right](i, a) \overset{\mathsf{def}}{=} f_i(a)$$

  for each $(i, a) \in \coprod_{i \in I} A_i$.

Proof.   *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.   □

4.2.3   Binary Coproducts

Let A and B be sets.

Definition 4.2.3.1.1. The **coproduct of** A **and** B[7] is the coproduct of A and B in Sets as in Limits and Colimits, **??**.

Construction 4.2.3.1.2. Concretely, the coproduct of A and B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

1. *The Colimit.* The set $A \coprod B$ defined by

$$A \coprod B \overset{\text{def}}{=} \coprod_{z \in \{A,B\}} z$$
$$\overset{\text{def}}{=} \{(0, a) \in S \mid a \in A\} \cup \{(1, b) \in S \mid b \in B\},$$

   where $S = \{0, 1\} \times (A \cup B)$.

2. *The Cocone.* The maps

$$\mathrm{inj}_1 : A \to A \coprod B,$$
$$\mathrm{inj}_2 : B \to A \coprod B,$$

   given by

$$\mathrm{inj}_1(a) \overset{\text{def}}{=} (0, a),$$
$$\mathrm{inj}_2(b) \overset{\text{def}}{=} (1, b),$$

   for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

in Sets. Then there exists a unique map $\phi \colon A \coprod B \to C$ making the diagram

---

[7] *Further Terminology:* Also called the **disjoint union of** A **and** B.

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_A = \iota_A,$$
$$\phi \circ \mathrm{inj}_B = \iota_B$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \coprod B$.   □

Proposition 4.2.3.1.3. Let $A, B, C,$ and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod -: & \quad \mathsf{Sets} & \to \mathsf{Sets}, \\ - \coprod B: & \quad \mathsf{Sets} & \to \mathsf{Sets}, \\ -_1 \coprod -_2: & \ \mathsf{Sets} \times \mathsf{Sets} & \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

   · *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \overset{\text{def}}{=} A \coprod B.$$

   · *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)}: \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

   of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g: A \coprod B \to X \coprod Y$$

   defined by

$$[f \coprod g](x) \overset{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

   for each $x \in A \coprod B$.

   and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have an adjunction

$$(-_1 \coprod -_2 \dashv \Delta_{\mathsf{Sets}}): \quad \mathsf{Sets} \times \mathsf{Sets} \underset{\Delta_{\mathsf{Sets}}}{\overset{-_1 \coprod -_2}{\rightleftarrows}} \perp \mathsf{Sets},$$

witnessed by a bijection

$$\mathsf{Sets}(A \coprod B, C), \cong \mathsf{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A,B),(C,C))$$

natural in $(A, B) \in \mathsf{Obj}(\mathsf{Sets} \times \mathsf{Sets})$ and in $C \in \mathsf{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}: (X \coprod Y) \coprod Z \overset{\sim}{\dashrightarrow} X \coprod (Y \coprod Z),$$

natural in $X, Y, Z \in \mathsf{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\lambda_X^{\mathsf{Sets},\coprod}: \emptyset \coprod X \overset{\sim}{\dashrightarrow} X,$$
$$\rho_X^{\mathsf{Sets},\coprod}: X \coprod \emptyset \overset{\sim}{\dashrightarrow} X,$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}: X \coprod Y \overset{\sim}{\dashrightarrow} Y \coprod X,$$

natural in $X, Y \in \mathsf{Obj}(\mathsf{Sets})$.

6. *Symmetric Monoidality.* The 7-tuple $\left(\mathsf{Sets}, \coprod, \emptyset, \alpha_{\coprod}^{\mathsf{Sets}}, \lambda_{\coprod}^{\mathsf{Sets}}, \rho_{\coprod}^{\mathsf{Sets}}, \sigma_{\coprod}^{\mathsf{Sets}}\right)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, Adjointness: This follows from the universal property of the coproduct.
Item 3, Associativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.3.1.1.
Item 4, Unitality: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.2.4.1.1 and 5.2.5.1.1.
Item 5, Commutativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.6.1.1.
Item 6, Symmetric Monoidality: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.2.7.1.1, and is proved there.   □

4.2.4   Pushouts

Let A, B, and C be sets and let $f\colon C \to A$ and $g\colon C \to B$ be functions.

Definition 4.2.4.1.1. The **pushout of A and B over C along f and g**[8] is the pushout of A and B over C along f and g in Sets as in Limits and Colimits, **??**.

Construction 4.2.4.1.2. Concretely, the pushout of A and B over C along f and g is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

1. *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B/{\sim_C},$$

   where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

2. *The Cocone.* The maps

$$\mathrm{inj}_1\colon A \to A \coprod_C B,$$
$$\mathrm{inj}_2\colon B \to A \coprod_C B$$

   given by

$$\mathrm{inj}_1(a) \stackrel{\text{def}}{=} [(0, a)]$$
$$\mathrm{inj}_2(b) \stackrel{\text{def}}{=} [(1, b)]$$

   for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g,$$

$$\begin{array}{ccc} A \coprod_C B & \xleftarrow{\ \mathrm{inj}_2\ } & B \\ \scriptstyle{\mathrm{inj}_1} \big\uparrow & & \big\uparrow \scriptstyle{g} \\ A & \xleftarrow{\ f\ } & C. \end{array}$$

---

[8] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

Indeed, given $c \in C$, we have

$$
\begin{aligned}
[\mathrm{inj}_1 \circ f](c) &= \mathrm{inj}_1(f(c)) \\
&= [(0, f(c))] \\
&= [(1, g(c))] \\
&= \mathrm{inj}_2(g(c)) \\
&= [\mathrm{inj}_2 \circ g](c),
\end{aligned}
$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form
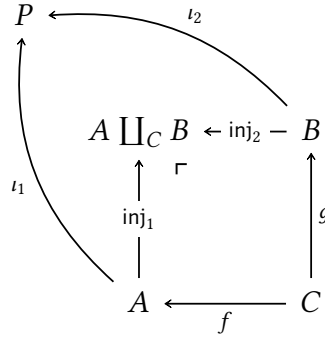
in Sets. Then there exists a unique map $\phi \colon A \coprod_C B \to P$ making the diagram
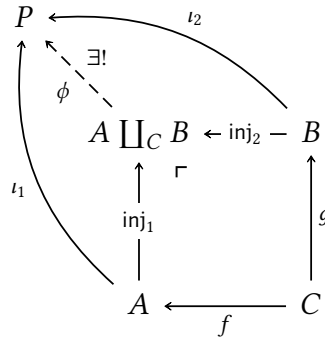
commute, being uniquely determined by the conditions

$$
\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}
$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1*: Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

2. *Case 2*: Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

3. *Case 3*: Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned} \phi([x]) &= \phi([(0, f(c))]) \\ &\overset{\text{def}}{=} \iota_1(f(c)) \\ &= \iota_2(g(c)) \\ &\overset{\text{def}}{=} \phi([(1, g(c))]) \\ &= \phi([y]), \end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned} (0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\ (1, b) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\ (0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b) \end{aligned}$$

gives

$$\phi([(0, a)]) = \phi([(0, a')]),$$
$$\phi([(1, b)]) = \phi([(1, b')]),$$
$$\phi([(0, a)]) = \phi([(1, b)]),$$

showing ϕ to be well-defined.   □

Remark 4.2.4.1.3. In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a, b \in A$ and $a = b$.

2. We have $a, b \in B$ and $a = b$.

3. There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

   (a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.

   (b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

   In other words, there exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

   (c) There exists $c_0 \in C$ satisfying one of the following conditions:

      i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.
      ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.

   (d) For each $1 \leq i \leq n - 1$, there exists $c_i \in C$ satisfying one of the following conditions:

      i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
      ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.

   (e) There exists $c_n \in C$ satisfying one of the following conditions:

      i. We have $x_n = f(c_n)$ and $b = g(c_n)$.
      ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

Remark 4.2.4.1.4. It is common practice to write $A \coprod {}_C B$ for the pushout of A and B over C along f and g, omitting the maps f and g from the notation and instead

leaving them implicit, to be understood from the context.

However, the set $A \coprod_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \coprod_{f,C,g} B$ or $A \coprod_C^{f,g} B$ for $A \coprod_C B$.

Example 4.2.4.1.5. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 6.3.3.1.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let X be a set. We have

$$A \cup B \cong A \coprod_{A \cap B} B,$$

$$\begin{array}{ccc} A \cup B & \longleftarrow & B \\ \uparrow \ulcorner & & \uparrow \\ \big\uparrow & & \big\downarrow \\ A & \longleftarrow & A \cap B \end{array}$$

for each $A, B \in \mathcal{P}(X)$.

Proof. *Item 1, Wedge Sums of Pointed Sets*: This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.
Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. $\square$

Proposition 4.2.4.1.6. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{array}{c} \bullet \\ \uparrow \\ \bullet \longleftarrow \bullet. \end{array}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a

morphism

$$
\begin{array}{ccc}
A \coprod_C B & \longleftarrow & B \\
\end{array}
$$
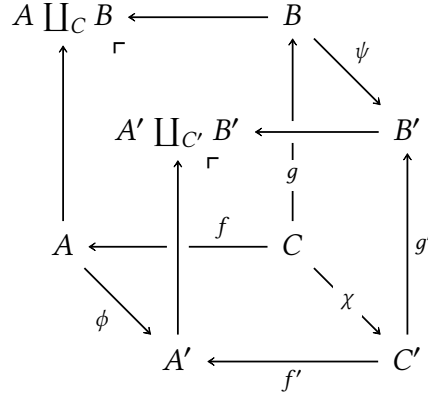
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists !} A' \coprod_{C'} B'$ given by

$$
\xi(x) \overset{\mathrm{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}
$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram
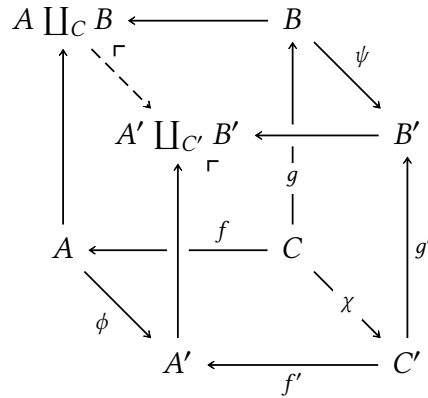
commute.

2. *Adjointness.* We have an adjunction

$$
\left({}_{-1} \coprod x_{-2} \dashv \Delta_{\mathsf{Sets}_{X/}} \right) \colon \quad \mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/} \underset{\Delta_{\mathsf{Sets}_{X/}}}{\overset{{}_{-1} \coprod x_{-2}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Sets}_{X/},
$$

witnessed by a bijection

$$\mathsf{Sets}_{X/}(A \coprod {}_X B, C), \cong \mathsf{Hom}_{\mathsf{Sets}_{X/}\times\mathsf{Sets}_{X/}}((A, B), (C, C))$$

natural in $(A, B) \in \mathsf{Obj}(\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/})$ and in $C \in \mathsf{Obj}(\mathsf{Sets}_{X/})$.

3. *Associativity.* Given a diagram
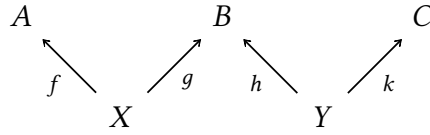
in Sets, we have isomorphisms of sets

$$(A \coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C)$$

where these pullbacks are built as in the diagrams
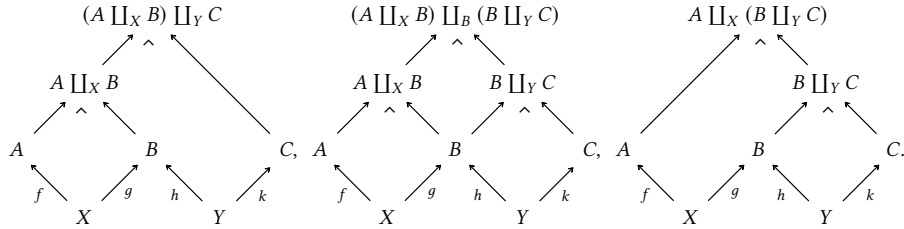
4. *Interaction With Composition.* Given a diagram
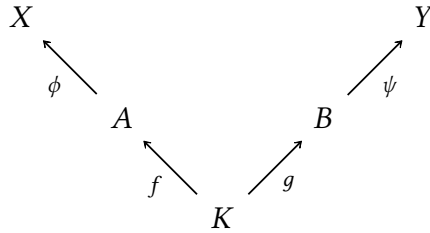
in Sets, we have isomorphisms of sets

$$X \coprod_K^{\phi\circ f,\psi\circ g} Y \cong (X \coprod_A^{\phi,j_1} (A \coprod_K^{f,g} B)) \coprod_{A\coprod_K^{f,g}B}^{i_2,i_1} ((A \coprod_K^{f,g} B) \coprod_B^{j_2,\psi} Y)$$

$$\cong X \coprod_A^{\phi,i} ((A \coprod_K^{f,g} B) \coprod_B^{j_2,\psi} Y)$$

$$\cong (X \coprod_A^{\phi, i_1} (A \coprod_K^{f,g} B)) \coprod_B^{j, \psi} Y$$

where

$$j_1 = \mathrm{inj}_1^{A \times_K^{f,g} B}, \qquad\qquad j_2 = \mathrm{inj}_2^{A \times_K^{f,g} B},$$

$$i_1 = \mathrm{inj}_1^{(A \times_K^{f,g} B) \times_Y^{q_2, \psi}}, \qquad\qquad i_2 = \mathrm{inj}_2^{X \times_{A \times_K^{f,g} B}^{\phi, q_1} (A \times_K^{f,g} B)},$$

$$i = j_1 \circ \mathrm{inj}_1^{(A \times_K^{f,g} B) \times_B^{q_2, \psi} Y}, \qquad\qquad j = j_2 \circ \mathrm{inj}_2^{X \times_A^{\phi, q_1} (A \times_K^{f,g} B)},$$

and where these pullbacks are built as in the diagrams
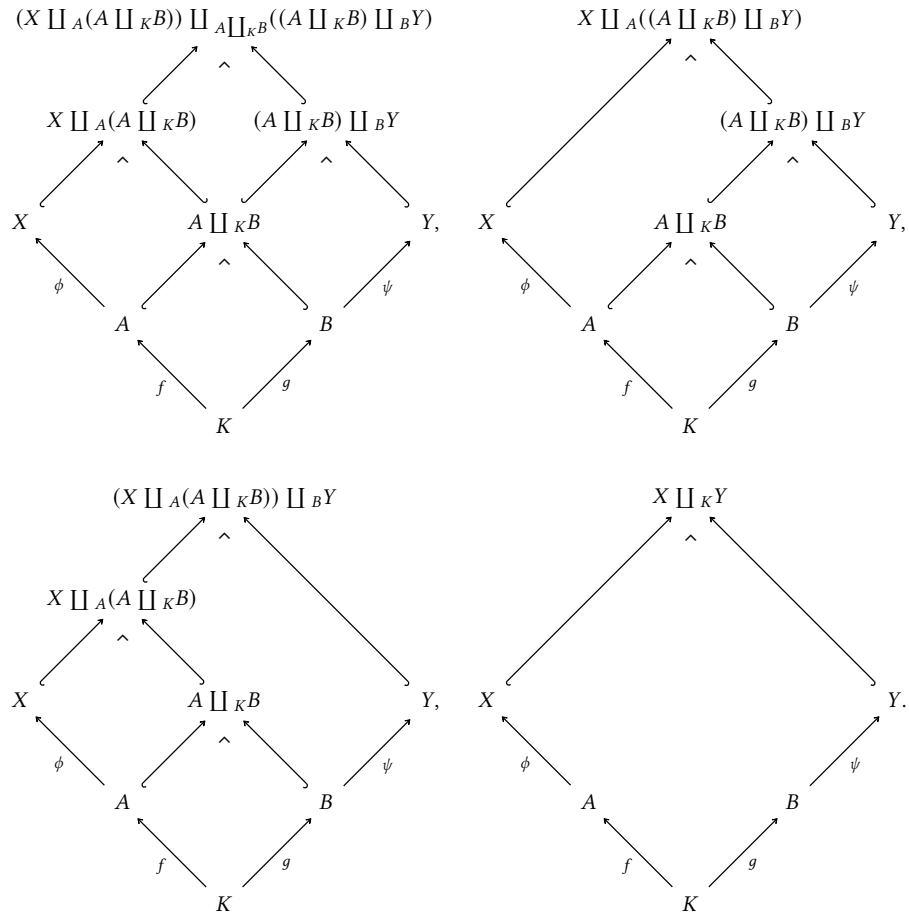
5. *Unitality.* We have isomorphisms of sets

$$
\begin{array}{ccc}
A & =\!\!=\!\!= & A \\
\uparrow\scriptstyle{\ulcorner} & & \uparrow \\
f\big| & & \big|f \\
X & =\!\!=\!\!= & X
\end{array}
\qquad
\begin{array}{l}
\lambda_A^{\mathsf{Sets}_{X/}} : X \coprod_X A \overset{\sim}{\dashrightarrow} A, \\[2mm]
\rho_A^{\mathsf{Sets}_{X/}} : A \coprod_X X \overset{\sim}{\dashrightarrow} A,
\end{array}
\qquad
\begin{array}{ccc}
A & \overset{f}{\longleftarrow} & X \\
\big\| \scriptstyle{\ulcorner} & & \big\| \\
X & \underset{f}{\longleftarrow} & X,
\end{array}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

6. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
A \coprod_X B & \longleftarrow & B \\
\uparrow\scriptstyle{\ulcorner} & & \uparrow\scriptstyle{g} \\
A & \underset{f}{\longleftarrow} & X,
\end{array}
\qquad
\sigma_A^{\mathsf{Sets}_{X/}} : A \coprod_X B \overset{\sim}{\dashrightarrow} B \coprod_X A
\qquad
\begin{array}{ccc}
B \coprod_X A & \longleftarrow & A \\
\uparrow\scriptstyle{\ulcorner} & & \uparrow\scriptstyle{f} \\
B & \underset{g}{\longleftarrow} & X.
\end{array}
$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

7. *Interaction With Coproducts.* We have

$$
A \coprod_\emptyset B \cong A \coprod B,
\qquad
\begin{array}{ccc}
A \coprod B & \longleftarrow & B \\
\uparrow\scriptstyle{\ulcorner} & & \big\downarrow\scriptstyle{\iota_B} \\
A & \underset{\iota_A}{\longleftarrow} & \emptyset.
\end{array}
$$

8. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_{X/}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, : Adjointness: This follows from the universal property of the coproduct (pushouts are coproducts in $\mathsf{Sets}_{X/}$).
Item 3, Associativity: Omitted.
Item 4, Interaction With Composition: Omitted.
Item 5, Unitality: Omitted.
Item 6, Commutativity: Omitted.
Item 7, Interaction With Coproducts: Omitted.
Item 8, Symmetric Monoidality: Omitted.   □

4.2.5   Coequalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 4.2.5.1.1. The **coequaliser of f and g** is the coequaliser of f and g in Sets as in Limits and Colimits, **??**.

Construction 4.2.5.1.2. Concretely, the coequaliser of f and g is the pair $(\mathrm{CoEq}(f, g), \mathrm{coeq}(f, g))$ consisting of:

1. *The Colimit.* The set $\mathrm{CoEq}(f, g)$ defined by

$$\mathrm{CoEq}(f, g) \overset{\mathrm{def}}{=} B/{\sim},$$

   where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

2. *The Cocone.* The map

$$\mathrm{coeq}(f, g)\colon B \twoheadrightarrow \mathrm{CoEq}(f, g)$$

   given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f, g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$
\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](a) &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](f(a)) \\
&\overset{\mathrm{def}}{=} [f(a)] \\
&= [g(a)] \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](g(a)) \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g) \circ g](a)
\end{aligned}
$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\hookrightarrow} \mathrm{CoEq}(f, g)$$
$$\searrow{\scriptstyle c}$$
$$C$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\mathsf{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\ \mathsf{coeq}(f,g)\ } \mathsf{CoEq}(f,g)$$

$$\overset{c}{\searrow} \quad \Big\downarrow \exists!$$

$$C$$

commute.   □

Remark 4.2.5.1.3. In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a = b$;

2. There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

    (a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.

    (b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

    In other words, there exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

    (a) There exists $z_0 \in A$ satisfying one of the following conditions:

        i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.
        ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.

    (b) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:

        i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
        ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.

    (c) There exists $z_n \in A$ satisfying one of the following conditions:

        i. We have $x_n = f(z_n)$ and $b = g(z_n)$.
        ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 4.2.5.1.4. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathsf{CoEq}(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X).$$

Proof. *Item 1, Quotients by Equivalence Relations*: See [Pro25z].   □

Proposition 4.2.5.1.5. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[9]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{-g\,\to}{\rightrightarrows}}} B$$

---

[9]That is, the following three ways of forming "the' coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

    $$A \overset{f}{\underset{h}{\overset{-g\,\to}{\rightrightarrows}}} B$$

    in Sets.

2. First take the coequaliser of f and g, forming a diagram

    $$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

    and then take the coequaliser of the composition

    $$A \overset{f}{\underset{h}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

    obtaining a quotient

    $$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

    of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

    $$A \overset{g}{\underset{h}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

    and then take the coequaliser of the composition

    $$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

    obtaining a quotient

    $$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

    of $\mathrm{CoEq}(g,h)$.

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\mathsf{CoEq}(f, f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathsf{CoEq}(f, g) \cong \mathsf{CoEq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathsf{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathsf{CoEq}(\mathsf{coeq}(h, k) \circ h \circ f, \mathsf{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathsf{CoEq}(\mathsf{coeq}(h, k) \circ h \circ f, \mathsf{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathsf{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof.   Item 1, *Associativity*: Omitted.
Item 4, *Unitality*: Omitted.
Item 5, *Commutativity*: Omitted.
Item 6, *Interaction With Composition*: Omitted. $\square$

4.2.6   Direct Colimits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \to \mathsf{T}$ be a direct system of sets.

Definition 4.2.6.1.1. The **direct colimit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.2.6.1.2. Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the pair $\left(\underset{\alpha \in I}{\mathrm{colim}}(X_\alpha), \{\mathsf{inj}_\alpha\}_{\alpha \in I} \right)$ consisting of:

1. *The Colimit.* The set $\operatorname*{colim}_{\alpha \in I}(X_\alpha)$ defined by

$$\operatorname*{colim}_{\alpha \in I}(X_\alpha) \overset{\text{def}}{=} \left(\coprod_{\alpha \in I} X_\alpha\right)\Big/ {\sim},$$

where $\sim$ is the equivalence relation on $\coprod_{\alpha \in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

(a) We have $\alpha \preceq \gamma$.

(b) We have $\beta \preceq \gamma$.

(c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

2. *The Cocone.* The collection

$$\left\{ \operatorname{inj}_\gamma \colon X_\gamma \longrightarrow \operatorname*{colim}_{\alpha \in I}(X_\alpha) \right\}_{\gamma \in I}$$

of maps of sets defined by

$$\operatorname{inj}_\gamma(x) \overset{\text{def}}{=} [(\gamma, x)]$$

for each $\gamma \in I$ and each $x \in X_\gamma$.

Proof. We will prove Definition 4.2.6.1.2 below in a bit, but first we need a lemma (which is interesting in its own right). $\qquad\square$

Lemma 4.2.6.1.3. For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have

$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in $\operatorname*{colim}_{\alpha \in I}(X_\alpha)$.

Proof. Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \overset{\text{def}}{=} \operatorname{id}_{X_\beta}$, and we have

$$\begin{aligned}
f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\
&\overset{\text{def}}{=} \operatorname{id}_{X_\beta}(f_{\alpha\beta}(x)), \\
&= f_{\alpha\beta}(x).
\end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, Items 1a to 1c of Definition 4.2.6.1.2 are met. Thus we have $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$. $\qquad\square$

We can now prove Definition 4.2.6.1.2:

Proof. We claim that $\operatorname*{colim}_{\alpha \in I}(X_\alpha)$ is the colimit of the direct system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$.
Commutativity of the Colimit Diagram: First, we need to check that the colimit diagram defined by $\operatorname*{colim}_{\alpha \in I}(X_\alpha)$ commutes, i.e. that we have

$$\operatorname{inj}_\alpha = \operatorname{inj}_\beta \circ f_{\alpha\beta},$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$
\begin{aligned}
[\operatorname{inj}_\beta \circ f_{\alpha\beta}](x) &\overset{\text{def}}{=} \operatorname{inj}_\beta(f_{\alpha\beta}(x)) \\
&\overset{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\
&= [(\alpha, x)] \\
&\overset{\text{def}}{=} \operatorname{inj}_\alpha(x),
\end{aligned}
$$

where we have used Definition 4.2.6.1.3 for the third equality.
Proof of the Universal Property of the Colimit: Next, we prove that $\operatorname*{colim}_{\alpha \in I}(X_\alpha)$ as constructed in Definition 4.2.6.1.2 satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

in Sets. We claim that there exists a unique map $\phi \colon \operatorname*{colim}_{\alpha \in I}(X_\alpha) \overset{\exists!}{\longrightarrow} C$ making the

diagram
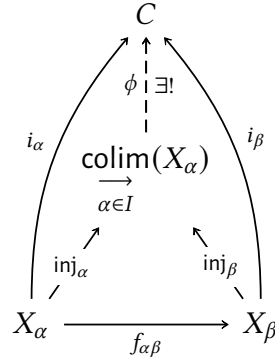
commute. To this end, first consider the diagram

Lemma. If $(\alpha, x) \sim (\beta, y)$, then we have

$$\left[\coprod_{\alpha \in I} i_\alpha\right](x) = \left[\coprod_{\alpha \in I} i_\alpha\right](y).$$

Proof. Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

1. We have $\alpha \preceq \gamma$.

2. We have $\beta \preceq \gamma$.

3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$\left[\coprod_{\alpha \in I} i_\alpha\right](x) \overset{\text{def}}{=} i_\alpha(x)$$

$$\overset{\text{def}}{=} [i_\gamma \circ f_{\alpha\gamma}](x)$$

$$\overset{\text{def}}{=} i_\gamma(f_{\alpha\gamma}(x))$$
$$= i_\gamma(f_{\beta\gamma}(x))$$
$$\overset{\text{def}}{=} [i_\gamma \circ f_{\beta\gamma}](x)$$
$$= i_\beta(y)$$
$$\overset{\text{def}}{=} \left[\coprod_{\alpha\in I} i_\alpha\right](y).$$

This finishes the proof of the lemma. Continuing, by Conditions on Relations, **??** of Definition 10.6.2.1.3, there then exists a map $\phi\colon \underset{\alpha\in I}{\mathrm{colim}}(X_\alpha) \overset{\exists!}{\longrightarrow} C$ making the diagram

$$\coprod_{\alpha\in I} X_\alpha \overset{\mathrm{pr}}{\twoheadrightarrow} \underset{\alpha\in I}{\mathrm{colim}}(X_\alpha)$$

with $\coprod_{\alpha\in I} i_\alpha$ and ϕ to C

commute. In particular, this implies that the diagram

$$X_\alpha \overset{\mathrm{inj}_\alpha}{\twoheadrightarrow} \underset{\alpha\in I}{\mathrm{colim}}(X_\alpha)$$

with i_α and ϕ to C

also commutes, and thus so does the diagram

$$C$$
$$\phi \;\vdots\; \exists!$$
$$i_\alpha \qquad \underset{\alpha\in I}{\mathrm{colim}}(X_\alpha) \qquad i_\beta$$
$$\mathrm{inj}_\alpha \qquad \mathrm{inj}_\beta$$
$$X_\alpha \overset{f_{\alpha\beta}}{\longrightarrow} X_\beta.$$

This finishes the proof.[10]   □

Example 4.2.6.1.4. Here are some examples of direct colimits of sets.

1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \overset{\text{def}}{=} \underset{n\in\mathbb{N}}{\mathrm{colim}}(\mathbb{Z}_{/p^n});$$

see **??**.

4.3   Operations With Sets

4.3.1   The Empty Set

Definition 4.3.1.1.1. The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where X is the set in the set existence axiom, **??** of **??**.

4.3.2   Singleton Sets

Let X be a set.

Definition 4.3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself of Definition 4.3.3.1.1.

---

[10]Incidentally, the conditions

$$\{i_\alpha = \phi \circ \mathrm{inj}_\alpha\}_{\alpha \in I}$$

show that ϕ must be given by

$$\phi([(\alpha, x)]) = (i_\alpha(x))_{\alpha \in I}$$

for each $[(\alpha, x)] \in \underset{\alpha \in I}{\mathrm{colim}}(X_\alpha)$, although we would need to show that this assignment is well-defined were we to prove Definition 4.2.6.1.2 in this way. Instead, invoking Conditions on Relations, **??** of Definition 10.6.2.1.3 gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

4.3.3   Pairings of Sets

Let X and Y be sets.

Definition 4.3.3.1.1. The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

4.3.4   Ordered Pairs

Let A and B be sets.

Definition 4.3.4.1.1. The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 4.3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let $A, B, C,$ and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.
   (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3].   □

4.3.5   Sets of Maps

Let A and B be sets.

Definition 4.3.5.1.1. The **set of maps from A to B**[11] is the set $\mathsf{Sets}(A, B)$[12] whose elements are the functions from A to B.

Proposition 4.3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define func-

---

[11] *Further Terminology:* Also called the **Hom set from A to B**.
[12] *Further Notation:* Also written $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$.

tors

$$\begin{aligned}
\mathsf{Sets}(X, -) &: & \mathsf{Sets} & \to \mathsf{Sets}, \\
\mathsf{Sets}(-, Y) &: & \mathsf{Sets}^{\mathsf{op}} & \to \mathsf{Sets}, \\
\mathsf{Sets}(-_1, -_2) &: & \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} & \to \mathsf{Sets}.
\end{aligned}$$

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -)) : \quad \mathsf{Sets} \xrightarrow[\mathsf{Sets}(A,-)]{\overset{A \times -}{\perp}} \mathsf{Sets},$$

$$(- \times B \dashv \mathsf{Sets}(B, -)) : \quad \mathsf{Sets} \xrightarrow[\mathsf{Sets}(B,-)]{\overset{- \times B}{\perp}} \mathsf{Sets},$$

witnessed by bijections

$$\begin{aligned}
\mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\
\mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),
\end{aligned}$$

natural in $A, B, C \in \mathsf{Obj}(\mathsf{Sets})$.

3. *Maps From the Punctual Set.* We have a bijection

$$\mathsf{Sets}(\mathsf{pt}, A) \cong A,$$

natural in $A \in \mathsf{Obj}(\mathsf{Sets})$.

4. *Maps to the Punctual Set.* We have a bijection

$$\mathsf{Sets}(A, \mathsf{pt}) \cong \mathsf{pt},$$

natural in $A \in \mathsf{Obj}(\mathsf{Sets})$.

Proof.   *Item 1, Functoriality:* This follows from Categories, Items 2 and 5 of Definition 11.1.4.1.2.
Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.1.3.1.3 and is proved there.
Item 3, Maps From the Punctual Set: The bijection

$$\Phi_A : \mathsf{Sets}(\mathsf{pt}, A) \xdashrightarrow{\sim} A$$

is given by

$$\Phi_A(f) \overset{\text{def}}{=} f(\star)$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$, admitting an inverse

$$\Phi_A^{-1} \colon A \overset{\sim}{\dashrightarrow} \mathsf{Sets}(\mathrm{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \overset{\text{def}}{=} [\![\star \mapsto a]\!]$$

for each $a \in A$. Indeed, we have

$$\begin{aligned}
[\Phi_A^{-1} \circ \Phi_A](f) &\overset{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\
&\overset{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\
&\overset{\text{def}}{=} [\![\star \mapsto f(\star)]\!] \\
&\overset{\text{def}}{=} f \\
&\overset{\text{def}}{=} [\mathrm{id}_{\mathsf{Sets}(\mathrm{pt},A)}](f)
\end{aligned}$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$ and

$$\begin{aligned}
[\Phi_A \circ \Phi_A^{-1}](a) &\overset{\text{def}}{=} \Phi_A(\Phi_A^{-1}(a)) \\
&\overset{\text{def}}{=} \Phi_A([\![\star \mapsto a]\!]) \\
&\overset{\text{def}}{=} \mathrm{ev}_\star([\![\star \mapsto a]\!]) \\
&\overset{\text{def}}{=} a \\
&\overset{\text{def}}{=} [\mathrm{id}_A](a)
\end{aligned}$$

for each $a \in A$, and thus we have

$$\begin{aligned}
\Phi_A^{-1} \circ \Phi_A &= \mathrm{id}_{\mathsf{Sets}(\mathrm{pt},A)} \\
\Phi_A \circ \Phi_A^{-1} &= \mathrm{id}_A \,.
\end{aligned}$$

To prove naturality, we need to show that the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(\mathrm{pt}, A) & \overset{f_!}{\longrightarrow} & \mathsf{Sets}(\mathrm{pt}, B) \\
\Phi_A \downarrow {\scriptstyle \wr} & & {\scriptstyle \wr} \downarrow \Phi_B \\
A & \underset{f}{\longrightarrow} & B
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[f \circ \Phi_A](\phi) &\overset{\text{def}}{=} f(\Phi_A(\phi)) \\
&\overset{\text{def}}{=} f(\phi(\star)) \\
&\overset{\text{def}}{=} [f \circ \phi](\star) \\
&\overset{\text{def}}{=} \Phi_B(f \circ \phi) \\
&\overset{\text{def}}{=} \Phi_B(f_!(\phi)) \\
&\overset{\text{def}}{=} [\Phi_B \circ f_!](\phi)
\end{aligned}
$$

for each $\phi \in \mathsf{Sets}(\mathrm{pt}, A)$. This finishes the proof.
Item 4, Maps to the Punctual Set: This follows from the universal property of pt as the terminal set, Definition 4.1.1.1.1.   □

4.3.6   Unions of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.6.1.1. The **union of** $\mathcal{U}$ is the set $\bigcup_{U \in \mathcal{U}} U$ defined by

$$
\bigcup_{U \in \mathcal{U}} U \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}.
$$

Proposition 4.3.6.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$ defines a functor

$$
\bigcup \colon (\mathcal{P}(\mathcal{P}(X)), \subset) \to (\mathcal{P}(X), \subset).
$$

   In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

   (★) If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V$.

2. *Associativity.* The diagram

$$
\begin{CD}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) @>{\mathrm{id}_{\mathcal{P}(X)} \star \bigcup}>> \mathcal{P}(\mathcal{P}(X)) \\
@V{\bigcup \star \mathrm{id}_{\mathcal{P}(X)}}VV @VV{\bigcup}V \\
\mathcal{P}(\mathcal{P}(X)) @>>{\bigcup}> \mathcal{P}(X)
\end{CD}
$$

commutes, i.e. we have

$$\bigcup_{U\in\bigcup_{A\in\mathcal{A}}A} U = \bigcup_{A\in\mathcal{A}}\left(\bigcup_{U\in A}U\right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) & & \\
\chi_{\mathcal{P}(X)}\Big\downarrow & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcup]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcup_{V\in\{U\}} V = U$$

for each $U \in \mathcal{P}(X)$.

4. *Right Unitality.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) & & \\
\mathcal{P}(\chi_X)\Big\downarrow & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcup]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcup_{\{u\}\in\chi_X(U)} \{u\} = U$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Unions I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \cup\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup \times \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \cup\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right)
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} & \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) & \xrightarrow[\ U \cup -\ ]{} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} & \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) & \xrightarrow[\ - \cup V\ ]{} & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$
U \cup \left(\bigcup_{V \in \mathcal{V}} V \right) = \bigcup_{V \in \mathcal{V}} (U \cup V),
$$

$$
\left(\bigcup_{U \in \mathcal{U}} U \right) \cup V = \bigcup_{U \in \mathcal{U}} (U \cup V)
$$

for each nonempty $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \cap\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup \times \cup} \downarrow & \quad\supset & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \cap\ ]{} & \mathcal{P}(X),
\end{array}
$$

with components

$$\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \subset \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$U \cap \left(\bigcup_{V \in \mathcal{V}} V \right) = \bigcup_{V \in \mathcal{V}} (U \cap V),$$

$$\left(\bigcup_{U \in \mathcal{U}} U \right) \cap V = \bigcup_{U \in \mathcal{U}} (U \cap V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \setminus\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup \times \cup} \downarrow & \textcolor{orange}{\times} & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\ \setminus\ } & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \xrightarrow{\;(-)^{\mathsf{c}}\;} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cup^{\mathsf{op}}}\Big\downarrow & {\color{orange}\times} & \Big\downarrow{\scriptstyle \cup} \\
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow[\;(-)^{\mathsf{c}}\;]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$
\bigcup_{U \in \mathcal{U}^{\mathsf{c}}} U \neq \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}
$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\
{\scriptstyle \mathsf{id}_{\mathcal{P}(X)}\,\star(-)^{\mathsf{c}}}\nearrow & & \searrow{\scriptstyle \sim} \\
\mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\
{\scriptstyle \cup}\searrow & & \swarrow{\scriptstyle \cap^{\mathsf{op}}} \\
& \mathcal{P}(X) \xrightarrow[\;(-)^{\mathsf{c}}\;]{} \mathcal{P}(X)^{\mathsf{op}}, &
\end{array}
$$

commutes, i.e. we have

$$
\left(\bigcup_{U \in \mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}
$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)}\,\star\,(-)^{\mathsf{c}}}\nearrow & & \searrow{\scriptstyle \sim} \\
\mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\
{\scriptstyle \cap}\searrow & & \swarrow{\scriptstyle \bigcup^{\mathsf{op}}} \\
& \mathcal{P}(X) \xrightarrow[(-)^{\mathsf{c}}]{} \mathcal{P}(X)^{\mathsf{op}}, &
\end{array}
$$

commutes, i.e. we have

$$
\left(\bigcap_{U\in\mathcal{U}} U \right)^{\mathsf{c}} = \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}
$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\,\triangle\,} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcup\times\bigcup}\Big\downarrow & \textcolor{orange}{\times} & \Big\downarrow{\scriptstyle \bigcup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\triangle]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$
\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcup_{U\in\mathcal{U}} U \right) \triangle \left(\bigcup_{V\in\mathcal{V}} V \right)
$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-_1,-_2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcup^{\mathsf{op}}\times\bigcup^{\mathsf{op}}}\Big\downarrow & \textcolor{orange}{\times} & \Big\downarrow{\scriptstyle \bigcup} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[{[-_1,-_2]_X}]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}}$$

$$\nearrow \qquad \searrow \cup^{\mathsf{op}}$$

$$\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \qquad \mathcal{P}(X)^{\mathsf{op}}$$

$$\mathsf{id}_{\mathcal{P}(X)} \star [-,V]_X \downarrow \qquad \qquad \downarrow [-,V]_X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\quad \cap \quad} \mathcal{P}(X)$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\quad \cup \quad} \mathcal{P}(X)$$

$$\mathsf{id}_{\mathcal{P}(X)} \star [U,-]_X \downarrow \qquad \qquad \downarrow [U,-]_X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\quad \cup \quad} \mathcal{P}(X)$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\
{\scriptstyle\cup}\big\downarrow & & \big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V
$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cup}\big\downarrow & & \big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U
$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\
{\scriptstyle\cup}\big\downarrow & & \big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V
$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Intersections of Families I.* The diagram

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\mathrm{id}_{\mathcal{P}(X)} \star \bigcap} \mathcal{P}(\mathcal{P}(x))$$

$$\downarrow{\scriptstyle\bigcup \star \mathrm{id}_{\mathcal{P}(X)}} \qquad\qquad\qquad \downarrow{\scriptstyle\bigcap}$$

$$\mathcal{P}(X) \xrightarrow{\qquad\bigcap\qquad} X$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Intersections of Families II.* Let X be a set and consider the compositions

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$$

with maps labelled $\bigcap \star \mathrm{id}_{\mathcal{P}(X)}$, $\bigcup \star \mathrm{id}_{\mathcal{P}(X)}$, $\mathrm{id}_{\mathcal{P}(X)} \star \bigcap$, $\mathrm{id}_{\mathcal{P}(X)} \star \bigcup$ to four copies of $\mathcal{P}(\mathcal{P}(X))$, then down via $\bigcup$, $\bigcap$, $\bigcup$, $\bigcap$ to $\mathcal{P}(X)$.

given by

$$\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \qquad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

$$\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \qquad\qquad \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U$$

All other possible inclusions fail to hold in general.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition (⋆). So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

Item 2, Associativity: We have

$$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U \overset{\text{def}}{=} \left\{ x \in X \ \middle|\ \begin{array}{l} \text{there exists some } U \in \bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\}$$

$$= \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\}$$

$$= \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right\}$$

$$\overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right).$$

This finishes the proof.

Item 3, Left Unitality: We have

$$\bigcup_{V \in \{U\}} V \overset{\text{def}}{=} \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in U \end{array} \right\}$$

$$= \{ x \in X \mid x \in U \}$$

$$= U.$$

This finishes the proof.

Item 4, Right Unitality: We have

$$\bigcup_{\{u\} \in \chi_X(U)} \{u\} \overset{\text{def}}{=} \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right\}$$

$$= \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right\}$$

$$= \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right\}$$

$$= \{ x \in X \mid x \in U \}$$

$$= U.$$

This finishes the proof.

Item 5, Interaction With Unions I: We have

$$\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W \overset{\text{def}}{=} \left\{ x \in X \middle| \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\}$$

$$= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right\}$$

$$\overset{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } W \in \mathcal{U} \\ \text{such that we have } x \in W \end{array} \right\}$$

$$\cup \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } W \in \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\}$$

$$\overset{\text{def}}{=} \left(\bigcup_{W \in \mathcal{U}} W \right) \cup \left(\bigcup_{W \in \mathcal{V}} W \right)$$

$$= \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 6, Interaction With Unions II: Assume $\mathcal{V}$ is nonempty. We have

$$U \cup \bigcup_{V \in \mathcal{V}} V \overset{\text{def}}{=} \left\{ x \in X \,\middle|\, x \in U \text{ or } x \in \bigcup_{V \in \mathcal{V}} V \right\}$$

$$= \left\{ x \in X \,\middle|\, \begin{array}{l} x \in U \text{ or there exists some} \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\}$$

$$= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ or } x \in V \end{array} \right\}$$

$$= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cup V \end{array} \right\}$$

$$\overset{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cup V.$$

This concludes the proof of the first statement. For the second statement, use Item 4 of Definition 4.3.8.1.2 to rewrite

$$\left(\bigcup_{U \in \mathcal{U}} U \right) \cup V = V \cup \left(\bigcup_{U \in \mathcal{U}} U \right),$$

$$\bigcup_{U \in \mathcal{U}} (U \cup V) = \bigcup_{U \in \mathcal{U}} (V \cup U).$$

But these two sets are equal by the first statement.

Item 7, Interaction With Intersections I: We have

$$
\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \overset{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cap \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\}
$$

$$
\subset \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } U \in \mathcal{U} \text{ and some } V \in \mathcal{V} \\ \text{such that we have } x \in U \text{ and } x \in V \end{array} \right\}
$$

$$
= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}
$$

$$
\cup \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that we have } x \in V \end{array} \right\}
$$

$$
\overset{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right).
$$

This finishes the proof.

Item 8, Interaction With Intersections II: We have

$$
U \cap \bigcup_{V \in \mathcal{V}} V \overset{\text{def}}{=} \left\{ x \in X \,\middle|\, x \in U \text{ and } x \in \bigcup_{V \in \mathcal{V}} V \right\}
$$

$$
= \left\{ x \in X \,\middle|\, \begin{array}{l} x \in U \text{ and there exists some} \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\}
$$

$$
= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ and } x \in V \end{array} \right\}
$$

$$
= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cap V \end{array} \right\}
$$

$$
\overset{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cap V.
$$

This concludes the proof of the first statement. For the second statement, use Item 5 of Definition 4.3.9.1.2 to rewrite

$$
\left(\bigcup_{U \in \mathcal{U}} U \right) \cap V = V \cap \left(\bigcup_{U \in \mathcal{U}} U \right),
$$

$$\bigcup_{U \in \mathcal{U}} (U \cap V) = \bigcup_{U \in \mathcal{U}} (V \cap U).$$

But these two sets are equal by the first statement.

Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} U = \bigcup_{W \in \{\{0,1\}\}} W$$

$$= \{0, 1\},$$

whereas

$$\left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right) = \{0, 1\} \setminus \{0\}$$

$$= \{1\}.$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \{1\} = \left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0, 1\}$ and let $\mathcal{U} = \{0\}$. We have

$$\bigcup_{U \in \mathcal{U}^c} U = \bigcup_{U \in \{\emptyset, \{1\}, \{0,1\}\}} U$$

$$= \{0, 1\},$$

whereas

$$\bigcup_{U \in \mathcal{U}} U^c = \{0\}^c$$

$$= \{1\}.$$

Thus we have

$$\bigcup_{U \in \mathcal{U}^c} U = \{0, 1\} \neq \{1\} = \bigcup_{U \in \mathcal{U}} U^c.$$

This finishes the proof.

Item 11, *Interaction With Complements II*: We have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathrm{c}} \overset{\mathrm{def}}{=} \left\{x \in X \,\middle|\, \begin{array}{l} \text{there exists no } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array}\right\}$$

$$= \left\{x \in X \,\middle|\, \begin{array}{l} \text{for all } U \in \mathcal{U} \\ \text{we have } x \notin U \end{array}\right\}$$

$$\overset{\mathrm{def}}{=} \left\{x \in X \,\middle|\, \begin{array}{l} \text{for all } U \in \mathcal{U} \\ \text{we have } x \in U^{\mathrm{c}} \end{array}\right\}$$

$$\overset{\mathrm{def}}{=} \bigcap_{U\in\mathcal{U}} U^{\mathrm{c}}.$$

Item 12, *Interaction With Complements III*: By Item 11 Item 3 of Definition 4.3.11.1.2, we have

$$\left(\bigcap_{U\in\mathcal{U}} U\right)^{\mathrm{c}} = \left(\bigcap_{U\in\mathcal{U}} (U^{\mathrm{c}})^{\mathrm{c}}\right)^{\mathrm{c}}$$

$$= \left(\left(\bigcup_{U\in\mathcal{U}} U^{\mathrm{c}}\right)^{\mathrm{c}}\right)^{\mathrm{c}}$$

$$= \bigcup_{U\in\mathcal{U}} U^{\mathrm{c}}.$$

Item 13, *Interaction With Symmetric Differences*: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W = \bigcup_{W\in\{\{0\}\}} W$$

$$= \{0\},$$

whereas

$$\left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right) = \{0, 1\} \triangle \{0, 1\}$$

$$= \emptyset,$$

Thus we have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of Item 7 of Definition 4.4.7.1.3 and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of Item 8 of Definition 4.4.7.1.3 and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of Item 9 of Definition 4.4.7.1.3 and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of Item 3 of Definition 4.6.1.1.5 and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of Item 3 of Definition 4.6.2.1.3 and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of Item 3 of Definition 4.6.3.1.7 and is proved there.

Item 20, Interaction With Intersections of Families I: We have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}$$

$$= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\}$$

$$\stackrel{\text{def}}{=} \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right).$$

This finishes the proof.

Item 21, Interaction With Intersections of Families II: Omitted.   □

4.3.7   Intersections of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.7.1.1. The **intersection of** $\mathcal{U}$ is the set $\bigcap_{U \in \mathcal{U}} U$ defined by

$$\bigcap_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\}.$$

Proposition 4.3.7.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcap_{U \in \mathcal{U}} U$ defines a functor

$$\bigcap \colon (\mathcal{P}(\mathcal{P}(X)), \supset) \to (\mathcal{P}(X), \subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

($\star$) If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U$.

2. *Oplax Associativity.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\mathrm{id}_{\mathcal{P}(X)} \star \cap} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap \star \mathrm{id}_{\mathcal{P}(X)}} \downarrow & \supset & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\cap]{} & \mathcal{P}(X)
\end{array}
$$

with components

$$\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \subset \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(X) & \\
{\scriptstyle \chi_{\mathcal{P}(X)}} \downarrow & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\cap]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{V \in \{U\}} V = U.$$

for each $U \in \mathcal{P}(X)$.

4. *Oplax Right Unitality.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) & & \\
\Big\downarrow{\scriptstyle\mathcal{P}(\chi_X)} & \searrow{\scriptstyle\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \underset{\cap}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

×

does not commute in general, i.e. we may have

$$
\bigcap_{\{x\}\in\chi_X(U)} \{x\} \neq U
$$

in general, where $U \in \mathcal{P}(X)$. However, when U is nonempty, we have

$$
\bigcap_{\{x\}\in\chi_X(U)} \{x\} \subset U.
$$

5. *Interaction With Unions I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \cup\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap\times\cap}\Big\downarrow & & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \cap\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcap_{W\in\mathcal{U}\cup\mathcal{V}} W = \left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right)
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap}\Big\downarrow & & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) & \xrightarrow[\ U\cup-\ ]{} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap}\Big\downarrow & & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) & \xrightarrow[\ -\cup V\ ]{} & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) = \bigcap_{V \in \mathcal{V}} (U \cup V),$$

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cup V = \bigcap_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\cap} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap \times \cap} \downarrow & \supset & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X),
\end{array}
$$

with components

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) \subset \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap} \downarrow & & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X) & \xrightarrow{U \cap -} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap} \downarrow & & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X) & \xrightarrow{- \cap V} & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) = \bigcap_{V \in \mathcal{V}} (U \cup V),$$

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cup V = \bigcap_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \backslash\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap \times \cap}\Big\downarrow & \textcolor{orange}{\times} & \Big\downarrow{\scriptstyle \cap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \backslash\ ]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$
\bigcap_{W \in \mathcal{U}\backslash\mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \backslash \left(\bigcap_{V \in \mathcal{V}} V \right)
$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \xrightarrow{\ (-)^{\mathsf{c}}\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap^{\mathsf{op}}}\Big\downarrow & \textcolor{orange}{\times} & \Big\downarrow{\scriptstyle \cap} \\
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow[\ (-)^{\mathsf{c}}\ ]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$
\bigcap_{W \in \mathcal{U}^{\mathsf{c}}} W \neq \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}
$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}}\nearrow & & \searrow{\scriptstyle \sim} \\
\mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\
{\scriptstyle \cap}\searrow & & \swarrow{\scriptstyle \cup^{\mathsf{op}}} \\
\mathcal{P}(X) & \xrightarrow[(-)^{\mathsf{c}}]{} & \mathcal{P}(X)^{\mathsf{op}},
\end{array}
$$

commutes, i.e. we have

$$\left(\bigcap_{U\in\mathcal{U}} U\right)^{c} = \bigcup_{U\in\mathcal{U}} U^{c}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\
\mathrm{id}_{\mathcal{P}(X)}\star(-)^{c}\nearrow & & \overset{\sim}{\dashrightarrow} \\
\mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\
\cup\downarrow & & \downarrow\cap^{\mathsf{op}} \\
\mathcal{P}(X) & \xrightarrow[(-)^{c}]{} & \mathcal{P}(X)^{\mathsf{op}},
\end{array}
$$

commutes, i.e. we have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{c} = \bigcap_{U\in\mathcal{U}} U^{c}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\triangle} & \mathcal{P}(\mathcal{P}(X)) \\
\cap\times\cap\downarrow & \times & \downarrow\cap \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\triangle]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcap_{U\in\mathcal{U}} U\right) \triangle \left(\bigcap_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U},\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \xrightarrow{[-_1,-_2]_{\mathcal{P}(X)}} \mathcal{P}(\mathcal{P}(X))$$

$$\bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}} \downarrow \qquad \textcolor{orange}{\times} \qquad \downarrow \bigcap$$

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \xrightarrow[{[-_1,-_2]_X}]{} \mathcal{P}(X),$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}}$$

$$\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \qquad\qquad \mathcal{P}(X)^{\mathsf{op}}$$

with arrows $\simeq$ (dashed), $\bigcap^{\mathsf{op}}$, $\mathsf{id}_{\mathcal{P}(X)} \star [-,V]_X$, $[-,V]_X$,

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\cup} \mathcal{P}(X)$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V\right]_X = \bigcup_{U \in \mathcal{U}} [U,V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\bigcap} \mathcal{P}(X)$$

$$\mathsf{id}_{\mathcal{P}(X)} \star [U,-]_X \downarrow \qquad\qquad \downarrow [U,-]_X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow[{\bigcap}]{} \mathcal{P}(X)$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\mathrm{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\mathrm{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U\in\mathcal{U}} f_*(U) = \bigcap_{V\in f_*(\mathcal{U})} V$$

for each $\mathcal{U}\in\mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Unions of Families I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \mathrm{id}_{\mathcal{P}(X)}\,\star\,\cap\ } & \mathcal{P}(\mathcal{P}(x)) \\
{\scriptstyle \bigcup\,\star\,\mathrm{id}_{\mathcal{P}(X)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \cap} \\
\mathcal{P}(X) & \xrightarrow[\ \cap\ ]{} & X
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U\in\bigcup_{A\in\mathcal{A}} A} U = \bigcap_{A\in\mathcal{A}}\left(\bigcap_{U\in A} U\right)$$

for each $\mathcal{A}\in\mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Unions of Families II.* Let X be a set and consider the compositions

given by

$$\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

$$\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \qquad\qquad \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U$$

All other possible inclusions fail to hold in general.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition $(\star)$. So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion. *Item 2, Oplax Associativity*: We have

$$\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \bigcap_{U \in A} U \end{array} \right\}$$

$$
\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \displaystyle\bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}
$$

$$
\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \displaystyle\bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}
$$

$$
\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.
$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 3, Left Unitality: We have

$$
\bigcap_{V \in \{U\}} V \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in U \end{array} \right\}
$$
$$
= \{ x \in X \mid x \in U \}
$$
$$
= U.
$$

This finishes the proof.

Item 4, Oplax Right Unitality: If $U = \emptyset$, then we have

$$
\bigcap_{\{u\} \in \chi_X(U)} \{u\} = \bigcap_{\{u\} \in \emptyset} \{u\}
$$
$$
= X,
$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$. When U is nonempty, we have two cases:

1. If U is a singleton, say $U = \{u\}$, we have

$$
\bigcap_{\{u\} \in \chi_X(U)} \{u\} = \{u\}
$$
$$
\stackrel{\text{def}}{=} U.
$$

2. If U contains at least two elements, we have

$$\bigcap_{\{u\}\in\chi_X(U)} \{u\} = \emptyset$$

$$\subset U.$$

This finishes the proof.

Item 5, Interaction With Unions I: We have

$$\bigcap_{W\in\mathcal{U}\cup\mathcal{V}} W \overset{\text{def}}{=} \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array}\right\}$$

$$= \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } W \in \mathcal{U} \text{ and each} \\ W \in \mathcal{V}, \text{we have } x \in W \end{array}\right\}$$

$$\overset{\text{def}}{=} \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } W \in \mathcal{U}, \\ \text{we have } x \in W \end{array}\right\}$$

$$\cap \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } W \in \mathcal{V}, \\ \text{we have } x \in W \end{array}\right\}$$

$$\overset{\text{def}}{=} \left(\bigcap_{W\in\mathcal{U}} W\right) \cap \left(\bigcap_{W\in\mathcal{V}} W\right)$$

$$= \left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right) \overset{\text{def}}{=} \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array}\right\}$$

$$\cup \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } V \in \mathcal{V}, \\ \text{we have } x \in V \end{array}\right\}$$

$$= \left\{x \in X \,\middle|\, \begin{array}{l} \text{for each } W \in \mathcal{U} \cap \mathcal{V}, \\ \text{we have } x \in W \end{array}\right\}$$

$$\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\}$$

$$\overset{\text{def}}{=} \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W.$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0\}, \{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} U = \bigcap_{W \in \{\{0,1\}\}} W$$

$$= \{0, 1\},$$

whereas

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right) = \{0\} \setminus \{0\}$$

$$= \emptyset.$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \emptyset = \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0, 1\}$ and let $\mathcal{U} = \{\{0\}\}$. We have

$$\bigcap_{W \in \mathcal{U}^{\mathsf{c}}} U = \bigcap_{W \in \{\emptyset, \{1\}, \{0,1\}\}} W$$

$$= \emptyset,$$

whereas

$$\bigcap_{U \in \mathcal{U}} U^{\mathsf{c}} = \{0\}^{\mathsf{c}}$$

$$= \{1\}.$$

Thus we have

$$\bigcap_{W \in \mathcal{U}^{\mathsf{c}}} U = \emptyset \neq \{1\} = \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.

Item 11, Interaction With Complements II: This is a repetition of Item 12 of Definition 4.3.6.1.2 and is proved there.

Item 12, Interaction With Complements III: This is a repetition of Item 11 of Definition 4.3.6.1.2 and is proved there.

Item 13, Interaction With Symmetric Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\bigcap_{W \in \mathcal{U} \triangle \mathcal{V}} W = \bigcap_{W \in \{\{0\}\}} W$$
$$= \{0\},$$

whereas

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \triangle \left(\bigcap_{V \in \mathcal{V}} V \right) = \{0, 1\} \triangle \{0\}$$
$$= \emptyset,$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \triangle \mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcap_{U \in \mathcal{U}} U \right) \triangle \left(\bigcap_{V \in \mathcal{V}} V \right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of Item 10 of Definition 4.4.7.1.3 and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of Item 11 of Definition 4.4.7.1.3 and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of Item 12 of Definition 4.4.7.1.3 and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of Item 4 of Definition 4.6.1.1.5 and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of Item 4 of Definition 4.6.2.1.3 and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of Item 4 of Definition 4.6.3.1.7 and is proved there.

Item 20, Interaction With Unions of Families I: This is a repetition of Item 20 of Definition 4.3.6.1.2 and is proved there.
Item 21, Interaction With Unions of Families II: This is a repetition of Item 21 of Definition 4.3.6.1.2 and is proved there.   □

4.3.8   Binary Unions

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.8.1.1. The **union of U and V** is the set $U \cup V$ defined by

$$U \cup V \overset{\text{def}}{=} \bigcup_{z \in \{U,V\}} z$$
$$\overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}.$$

Proposition 4.3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$
\begin{aligned}
U \cup - : && (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\
- \cup V : && (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\
-_1 \cup -_2 : & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \cup V \subset A \cup V$.

   (b) If $V \subset B$, then $U \cup V \subset U \cup B$.

   (c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

2. *Associativity.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
\overset{\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}}{\nearrow} & & \searrow {\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \times \cup} \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X) \\
\downarrow {\scriptstyle \cup \times \mathrm{id}_{\mathcal{P}(X)}} & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \underset{\cup}{\to} & \mathcal{P}(X),
\end{array}
$$

commutes, i.e. we have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

3. *Unitality*. The diagrams

commute, i.e. we have equalities of sets

$$\emptyset \cup U = U,$$
$$U \cup \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

4. *Commutativity*. The diagram

commutes, i.e. we have an equality of sets

$$U \cup V = V \cup U$$

for each $U, V \in \mathcal{P}(X)$.

5. *Annihilation With X*. The diagrams

commute, i.e. we have equalities of sets

$$U \cup X = X,$$
$$X \cup V = X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Distributivity of Unions Over Intersections.* The diagrams
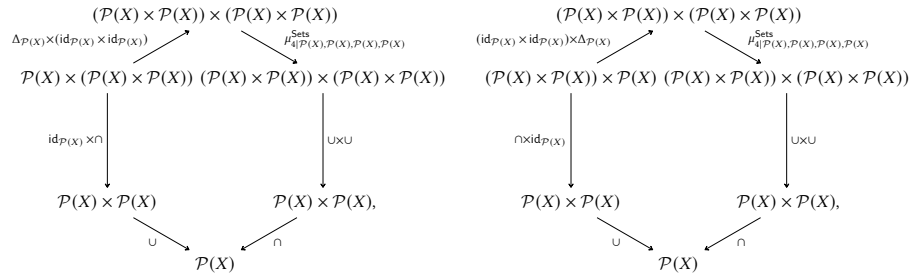
commute, i.e. we have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Distributivity of Intersections Over Unions.* The diagrams
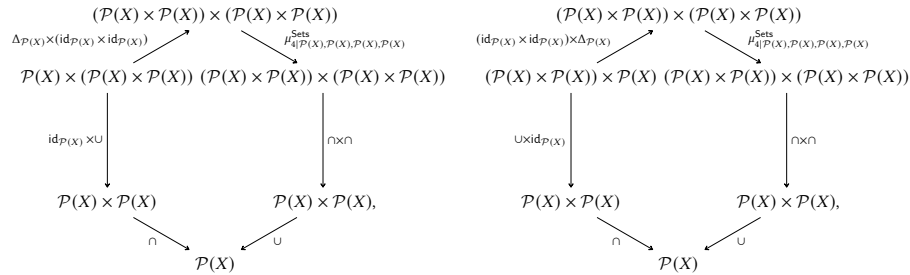
commute, i.e. we have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Idempotency.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\Delta_{\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
 & {\scriptstyle \mathrm{id}_{\mathcal{P}(X)}} \searrow & \downarrow {\scriptstyle \cup} \\
 & & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$
U \cup U = U
$$

for each $U \in \mathcal{P}(X)$.

9. *Via Intersections and Symmetric Differences.* The diagram

$$
\begin{array}{ccc}
(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \xrightarrow{\Delta \times \cap} & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle \Delta_{\mathcal{P}(X) \times \mathcal{P}(X)}} \nearrow & & \searrow {\scriptstyle \Delta} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\hspace{3cm} \cup \hspace{3cm}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$
U \cup V = (U \mathbin{\triangle} V) \mathbin{\triangle} (U \cap V)
$$

for each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$
\chi_{U \cup V} = \max(\chi_U, \chi_V)
$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$
\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}
$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\ f_! \times f_!\ } & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle \cup}\big\downarrow & & \big\downarrow{\scriptstyle \cup} \\
\mathcal{P}(X) & \xrightarrow[\ f_!\ ]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
f_!(U \cup V) = f_!(U) \cup f_!(V)
$$

for each $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\ f^{-1} \times f^{-1}\ } & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle \cup}\big\downarrow & & \big\downarrow{\scriptstyle \cup} \\
\mathcal{P}(Y) & \xrightarrow[\ f^{-1}\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)
$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\ f_* \times f_*\ } & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle \cup}\big\downarrow & \subset & \big\downarrow{\scriptstyle \cup} \\
\mathcal{P}(X) & \xrightarrow[\ f_*\ ]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_*(U) \cup f_*(V) \subset f_*(U \cup V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro25an].
Item 2, Associativity: See [Pro25ba].
Item 3, Unitality: This follows from [Pro25bd] and Item 4.
Item 4, Commutativity: See [Pro25bb].
Item 5, Annihilation With X: We have

$$U \cup X \overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\}$$
$$= \{x \in X \mid x \in X\},$$
$$= X$$

and

$$X \cup V \overset{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\}$$
$$= \{x \in X \mid x \in X\}$$
$$= X.$$

This finishes the proof.
Item 6, Distributivity of Unions Over Intersections: See [Pro25az].
Item 7, Distributivity of Intersections Over Unions: See [Pro25aj].
Item 8, Idempotency: See [Pro25am].
Item 9, Via Intersections and Symmetric Differences: See [Pro25ay].
Item 10, Interaction With Characteristic Functions I: See [Pro25h].
Item 11, Interaction With Characteristic Functions II: See [Pro25h].
Item 12, Interaction With Direct Images: See [Pro25p].
Item 13, Interaction With Inverse Images: See [Pro25y].
Item 14, Interaction With Codirect Images: This is a repetition of Item 5 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 of this propostition and Items 3 to 6 and 8 of Definition 4.3.9.1.2.   □

4.3.9 Binary Intersections

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.9.1.1. The **intersection of** U **and** V is the set $U \cap V$ defined by

$$U \cap V \overset{\text{def}}{=} \bigcap_{z \in \{U,V\}} z$$

$$\overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}.$$

Proposition 4.3.9.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$
\begin{aligned}
U \cap -: && (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\
- \cap V: && (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\
-_1 \cap -_2: && (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \cap V \subset A \cap V$.

   (b) If $V \subset B$, then $U \cap V \subset U \cap B$.

   (c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X): \quad \mathcal{P}(X) \underset{[U,-]_X}{\overset{U \cap -}{\rightleftarrows}} \bot \; \mathcal{P}(X),$$

$$(- \cap V \dashv [V, -]_X): \quad \mathcal{P}(X) \underset{[V,-]_X}{\overset{- \cap V}{\rightleftarrows}} \bot \; \mathcal{P}(X),$$

witnessed by bijections

$$\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X),$$

natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X: \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor of Section 4.4.7. In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

  i. We have $U \cap V \subset W$.

  ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

  i. We have $U \cap V \subset W$.

  ii. We have $V \subset [U, W]_X$.

3. *Associativity.* The diagram

$$\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X))$$

$$\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)} \qquad\qquad \mathrm{id}_{\mathcal{P}(X)} \times \cap$$

$$(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \qquad \mathcal{P}(X) \times \mathcal{P}(X)$$

$$\cap \times \mathrm{id}_{\mathcal{P}(X)} \qquad\qquad\qquad \cap$$

$$\mathcal{P}(X) \times \mathcal{P}(X) \underset{\cap}{\to} \mathcal{P}(X),$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* The diagrams

$$\mathrm{pt} \times \mathcal{P}(X) \xrightarrow{[X] \times \mathrm{id}_{\mathcal{P}(X)}} \mathcal{P}(X) \times \mathcal{P}(X) \qquad \mathcal{P}(X) \times \mathrm{pt} \xrightarrow{\mathrm{id}_{\mathcal{P}(X)} \times [X]} \mathcal{P}(X) \times \mathcal{P}(X)$$

$$\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)} \qquad\qquad \cap \qquad\qquad\qquad \rho^{\mathsf{Sets}}_{\mathcal{P}(X)} \qquad\qquad \cap$$

$$\mathcal{P}(X) \qquad\qquad\qquad\qquad \mathcal{P}(X)$$

commute, i.e. we have equalities of sets

$$X \cap U = U,$$
$$U \cap X = U$$

for each $U \in \mathcal{P}(X)$.

5. *Commutativity.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
& {\scriptstyle \cap} \searrow & \downarrow {\scriptstyle \cap} \\
& & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$
U \cap V = V \cap U
$$

for each $U, V \in \mathcal{P}(X)$.

6. *Annihilation With the Empty Set.* The diagrams

$$
\begin{array}{ccccc}
& & \mathsf{pt} \times \mathsf{pt} & & \\
{\scriptstyle \mathrm{id}_{\mathsf{pt}} \times \epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}} \nearrow & & & \searrow {\scriptstyle \mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}} & \\
\mathsf{pt} \times \mathcal{P}(X) & & & & \mathsf{pt} \\
{\scriptstyle [\varnothing] \times \mathrm{id}_{\mathcal{P}(X)}} \downarrow & & & & \downarrow {\scriptstyle [\varnothing]} \\
& \mathcal{P}(X) \times \mathcal{P}(X) & \underset{\cap}{\rightarrow} & \mathcal{P}(X) &
\end{array}
\qquad
\begin{array}{ccccc}
& & \mathsf{pt} \times \mathsf{pt} & & \\
{\scriptstyle \epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathsf{pt}}} \nearrow & & & \searrow {\scriptstyle \mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}} & \\
\mathcal{P}(X) \times \mathsf{pt} & & & & \mathsf{pt} \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \times [\varnothing]} \downarrow & & & & \downarrow {\scriptstyle [\varnothing]} \\
& \mathcal{P}(X) \times \mathcal{P}(X) & \underset{\cap}{\rightarrow} & \mathcal{P}(X) &
\end{array}
$$

commute, i.e. we have equalities of sets

$$
\varnothing \cap X = \varnothing,
$$
$$
X \cap \varnothing = \varnothing
$$

for each $U \in \mathcal{P}(X)$.

7. *Distributivity of Unions Over Intersections.* The diagrams

$$
\begin{array}{ccc}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
{\scriptstyle \Delta_{\mathcal{P}(X)} \times (\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)})} \nearrow & & \searrow {\scriptstyle \mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}} \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \quad & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \times \cap} \downarrow & & \downarrow {\scriptstyle \cup \times \cup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X), \\
{\scriptstyle \cup} \searrow & & \swarrow {\scriptstyle \cap} \\
& \mathcal{P}(X) &
\end{array}
$$

$$
\begin{array}{ccc}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) & \\
{\scriptstyle (\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)}) \times \Delta_{\mathcal{P}(X)}} \nearrow & & \searrow {\scriptstyle \mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}} \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) & \quad & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\
{\scriptstyle \cap \times \mathrm{id}_{\mathcal{P}(X)}} \downarrow & & \downarrow {\scriptstyle \cup \times \cup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & & \mathcal{P}(X) \times \mathcal{P}(X), \\
{\scriptstyle \cup} \searrow & & \swarrow {\scriptstyle \cap} \\
& \mathcal{P}(X) &
\end{array}
$$

commute, i.e. we have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Distributivity of Intersections Over Unions.* The diagrams
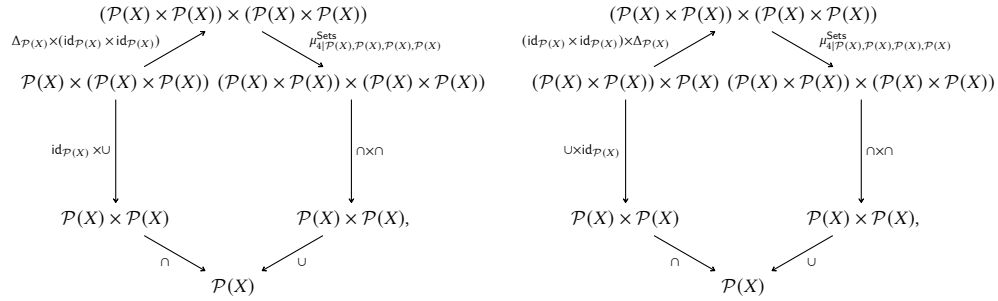
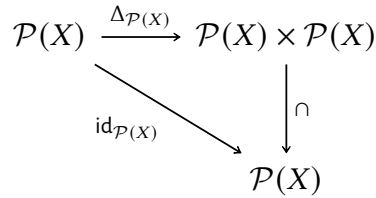commute, i.e. we have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Idempotency.* The diagram

commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f: X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle \cap} \downarrow & \subseteq & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f: X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle \cap} \downarrow & & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f: X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle \cap} \downarrow & & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof.   *Item 1, Functoriality*: See [Pro25al].
Item 2, Adjointness: See [MSE 267469].
Item 3, Associativity: See [Pro25r].
Item 4, Unitality: This follows from [Pro25v] and Item 5.
Item 5, Commutativity: See [Pro25s].
Item 6, Annihilation With the Empty Set: This follows from [Pro25t] and Item 5.
Item 7, Distributivity of Unions Over Intersections: See [Pro25az].
Item 8, Distributivity of Intersections Over Unions: See [Pro25aj].
Item 9, Idempotency: See [Pro25ak].
Item 10, Interaction With Characteristic Functions I: See [Pro25e].
Item 11, Interaction With Characteristic Functions II: See [Pro25e].
Item 12, Interaction With Direct Images: See [Pro25n].
Item 13, Interaction With Inverse Images: See [Pro25w].
Item 14, Interaction With Codirect Images: This is a repetition of Item 6 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 6.
Item 16, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 and Items 3 to 6 and 8 of Definition 4.3.9.1.2.   □

4.3.10   Differences

Let X and Y be sets.

Definition 4.3.10.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 4.3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$
\begin{aligned}
U \setminus -: & \quad (\mathcal{P}(X), \supset) & \to (\mathcal{P}(X), \subset), \\
- \setminus V: & \quad (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\
-_1 \setminus -_2: & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.

   (b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.

   (c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

2. *De Morgan's Laws.* We have equalities of sets

$$
\begin{aligned}
X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\
X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V)
\end{aligned}
$$

   for each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$
U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)
$$

   for each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$
(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)
$$

   for each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$
\begin{aligned}
U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\
&= (U \setminus V) \setminus W \\
&= (U \setminus W) \setminus V
\end{aligned}
$$

   for each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$
(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)
$$

   for each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$(U \setminus V) \cap W = (U \cap W) \setminus V$$
$$= U \cap (W \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \vartriangle (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

13. *Right Annihilation.* We have

$$U \setminus X = \emptyset$$

for each $U \in \mathcal{P}(X)$.

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

15. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Direct Images.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathrm{op}} \times f_!} & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \\
\setminus \downarrow & \quad\supset & \downarrow \setminus \\
\mathcal{P}(X) & \xrightarrow{\quad f_! \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

18. *Interaction With Inverse Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\mathrm{op},-1} \times f^{-1}} & \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \\
\setminus \downarrow & & \downarrow \setminus \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

19. *Interaction With Codirect Images.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathrm{op}} \times f_!} & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \\
\Big\downarrow{\scriptstyle\backslash} & \supset & \Big\downarrow{\scriptstyle\backslash} \\
\mathcal{P}(X) & \xrightarrow{\quad f_! \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof.   *Item 1, Functoriality*: See [Pro25ad] and [Pro25ah].
Item 2, De Morgan's Laws: See [Pro25k].
Item 3, Interaction With Unions I: See [Pro25l].
Item 4, Interaction With Unions II: We have

$$
\begin{aligned}
(U \setminus V) \cup W &\overset{\mathrm{def}}{=} \{x \in X \mid (x \in U \text{ and } x \notin V) \text{ or } x \in W\} \\
&= \{x \in X \mid (x \in U \text{ or } x \in W) \text{ and } (x \notin V \text{ or } x \in W)\} \\
&= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \text{ and } x \notin W)\} \\
&= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \setminus W)\} \\
&= \{x \in X \mid (x \in (U \cup W) \setminus (V \setminus W))\} \\
&= (U \cup W) \setminus (V \setminus W).
\end{aligned}
$$

Item 5, Interaction With Unions III: See [Pro25ai].
Item 6, Interaction With Unions IV: See [Pro25ac].
Item 7, Interaction With Intersections: See [Pro25u].
Item 8, Interaction With Complements: See [Pro25aa].
Item 9, Interaction With Symmetric Differences: See [Pro25ab].
Item 10, Triple Differences: See [Pro25ag].
Item 11, Left Annihilation: The direction $\emptyset \subset \emptyset \setminus U$ always holds. Now assume $x \in \emptyset \setminus U$. Then, $x \in \emptyset$ and $x \notin U$. Hence $\emptyset \setminus U \subset \emptyset$ must hold and the sets are equal.
Item 12, Right Unitality: See [Pro25ae].
Item 13, Right Annihilation: It suffices to show that no $x \in X$ can be an element of $U \setminus X$. Assume $x \in U \setminus X$. Then $x \notin X$, contradicting $x \in X$. This completes the proof.

Item 14, Invertibility: See [Pro25af].

Item 15, Interaction With Containment: The conditions are symmetric in U, W, hence it suffices to show that $V \setminus U \subset W$ implies $V \setminus W \subset U$. So assume $V \setminus U \subset W, x \in V \setminus W$. Then $x \in V, x \notin W$. So by contraposition, $x \notin V \setminus U$. But $x \in V$, so we must have $x \in U$, completing the proof.

Item 16, Interaction With Characteristic Functions: See [Pro25f].

Item 17, Interaction With Direct Images: See [Pro25o].

Item 18, Interaction With Inverse Images: See [Pro25x].   □

4.3.11   Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.3.11.1.1. The **complement of** U is the set U^{c} defined by

$$U^{\mathsf{c}} \overset{\text{def}}{=} X \setminus U$$
$$\overset{\text{def}}{=} \{a \in X \mid a \notin U\}.$$

Proposition 4.3.11.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X).$$

   In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

   (★) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\cup^{\mathsf{op}}} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}} \times (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cap} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\cap^{\mathsf{op}}} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}} \times (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\cup} & \mathcal{P}(X)
\end{array}
$$

   commute, i.e. we have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

   for each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} & \xrightarrow{(-)^{\mathrm{c}}} & \mathcal{P}(X) \\
& \searrow{\scriptstyle \mathrm{id}_{\mathcal{P}(X)^{\mathrm{op}}}} & \ \ \downarrow{\scriptstyle (-)^{\mathrm{c,op}}} \\
& & \mathcal{P}(X)^{\mathrm{op}}
\end{array}
$$

commutes, i.e. we have

$$(U^{\mathrm{c}})^{\mathrm{c}} = U$$

for each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathrm{c}}} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Direct Images.* Let $f : X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} & \xrightarrow{f_*^{\mathrm{op}}} & \mathcal{P}(Y)^{\mathrm{op}} \\
{\scriptstyle (-)^{\mathrm{c}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{c}}} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(U^{\mathrm{c}}) = f_*(U)^{\mathrm{c}}$$

for each $U \in \mathcal{P}(X)$.

6. *Interaction With Inverse Images.* Let $f : X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathrm{op}} & \xrightarrow{f^{-1,\mathrm{op}}} & \mathcal{P}(X)^{\mathrm{op}} \\
{\scriptstyle (-)^{\mathrm{c}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathrm{c}}} \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\ f_!^{\mathsf{op}}\ } & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) & \xrightarrow[\ f_*\ ]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: This follows from Item 1 of Definition 4.3.10.1.2.
Item 2, De Morgan's Laws: See [Pro25k].
Item 3, Involutority: See [Pro25i].
Item 4, Interaction With Characteristic Functions: We consider the two cases $x \in U, x \notin U$.

1. If $x \in U$, then $x \notin U^{\mathsf{c}}$. So $\chi_U(x) = 1$ and

$$
\begin{aligned}
\chi_{U^{\mathsf{c}}}(x) &= 0 \\
&= 1 - \chi_U(x).
\end{aligned}
$$

2. If $x \notin U$, then $x \in U^{\mathsf{c}}$. So $\chi_U(x) = 0$ and

$$
\begin{aligned}
\chi_{U^{\mathsf{c}}}(x) &= 1 \\
&= 1 - \chi_U(x).
\end{aligned}
$$

Hence, the equation holds for all $x \in X$.
Item 5, Interaction With Direct Images: This is a repetition of Item 8 of Definition 4.6.1.1.5 and is proved there.
Item 6, Interaction With Inverse Images: This is a repetition of Item 8 of Definition 4.6.2.1.3 and is proved there.
Item 7, Interaction With Codirect Images: This is a repetition of Item 7 of Definition 4.6.3.1.7 and is proved there. $\qquad\square$

4.3.12   Symmetric Differences

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.12.1.1. The **symmetric difference of** U **and** V is the set $U \triangle V$ defined by[13]

$$U \triangle V \overset{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

Proposition 4.3.12.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ **does not** in general define functors

$$\begin{array}{rll} U \triangle - : & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\ - \triangle V : & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\ -_1 \triangle -_2 : & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) & \to (\mathcal{P}(X), \subset). \end{array}$$

2. *Via Unions and Intersections.* We have

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

   for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram

3. *Symmetric Differences of Disjoint Sets.* If U and V are disjoint, then we have

$$U \triangle V = U \cup V.$$

4. *Associativity.* The diagram

---

[13] *Illustration:*
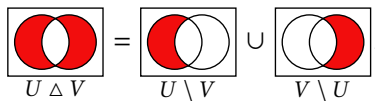
commutes, i.e. we have

$$(U \bigtriangleup V) \bigtriangleup W = U \bigtriangleup (V \bigtriangleup W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram

5. *Unitality.* The diagrams

commute, i.e. we have

$$U \bigtriangleup \emptyset = U,$$
$$\emptyset \bigtriangleup U = U$$

for each $U \in \mathcal{P}(X)$.

6. *Commutativity.* The diagram

commutes, i.e. we have

$$U \bigtriangleup V = V \bigtriangleup U$$

for each $U, V \in \mathcal{P}(X)$.

7. *Invertibility.* We have
$$U \mathbin{\triangle} U = \emptyset$$
for each $U \in \mathcal{P}(X)$.

8. *Interaction With Unions.* We have
$$(U \mathbin{\triangle} V) \cup (V \mathbin{\triangle} T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$
for each $U, V, W \in \mathcal{P}(X)$.

9. *Interaction With Complements I.* We have
$$U \mathbin{\triangle} U^{\mathsf{c}} = X$$
for each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements II.* We have
$$U \mathbin{\triangle} X = U^{\mathsf{c}},$$
$$X \mathbin{\triangle} U = U^{\mathsf{c}}$$
for each $U \in \mathcal{P}(X)$.

11. *Interaction With Complements III.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\ \triangle\ } & \mathcal{P}(X) \\
{\scriptstyle (-)^{\mathsf{c}} \times (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \triangle\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have
$$U^{\mathsf{c}} \mathbin{\triangle} V^{\mathsf{c}} = U \mathbin{\triangle} V$$
for each $U, V \in \mathcal{P}(X)$.

12. *"Transitivity".* We have
$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$
for each $U, V, W \in \mathcal{P}(X)$.

13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Distributivity Over Intersections.* We have

$$U \cap (V \triangle W) = (U \cap V) \triangle (U \cap W),$$
$$(U \triangle V) \cap W = (U \cap W) \triangle (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

16. *Bijectivity.* Given $U, V \in \mathcal{P}(X)$, the maps

$$U \triangle - : \mathcal{P}(X) \to \mathcal{P}(X),$$
$$- \triangle V : \mathcal{P}(X) \to \mathcal{P}(X)$$

are self-inverse bijections. Moreover, the map

$$\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$
$$C \longmapsto C \triangle (U \triangle V)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending U to V and V to U.

17. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[14]

---

[14] Here are some examples:

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

· The group $\mathcal{P}(X)$ of Item 17;
· The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$0 \cdot U \overset{\mathrm{def}}{=} \emptyset,$$
$$1 \cdot U \overset{\mathrm{def}}{=} U;$$

is an $\mathbb{F}_2$-vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

(a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of Item 4.

(b) We have
$$\dim(\mathcal{P}(X)) = \#X.$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[15]

---

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}) \cong \mathrm{pt}.$$

2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}) \cong \mathbb{Z}_{/2}.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[15] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25aw] for a proof.

7. *Interaction With Direct Images.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{\ f_!^{\mathsf{op}} \times f_!\ } & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
\Big\downarrow{\scriptstyle \triangle} & \supset & \Big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow[\ f_!\ ]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_!(U) \mathbin{\triangle} f_!(V) \subset f_!(U \mathbin{\triangle} V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Inverse Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{\ f^{\mathsf{op},-1} \times f^{-1}\ } & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
\Big\downarrow{\scriptstyle \triangle} & & \Big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(Y) & \xrightarrow[\ f^{-1}\ ]{} & \mathcal{P}(X)
\end{array}
$$

i.e. we have
$$
f^{-1}(U) \mathbin{\triangle} f^{-1}(V) = f^{-1}(U \mathbin{\triangle} V)
$$
for each $U, V \in \mathcal{P}(Y)$.

9. *Interaction With Codirect Images.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{\ f_*^{\mathsf{op}} \times f_*\ } & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
\Big\downarrow{\scriptstyle \triangle} & \supset & \Big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow[\ f_*\ ]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_*(U \mathbin{\triangle} V) \subset f_*(U) \mathbin{\triangle} f_*(V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. Item 1, *Lack of Functoriality*: Let $X = \{0, 1\}, U = \{0\}$. Then $\emptyset \subset U$, but $U \triangle \emptyset = U \not\subset \emptyset = U \triangle U$ from Item 5 and Item 7. This gives a counterexample to the first statement. By using Item 6, we can adapt it to the second and third statement.

Item 2, *Via Unions and Intersections*: See [Pro25m].

Item 3, *Symmetric Differences of Disjoint Sets*: Since U and V are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$
\begin{aligned}
U \triangle V &= (U \cup V) \setminus (U \cap V) \\
&= (U \cup V) \setminus \emptyset \\
&= U \cup V,
\end{aligned}
$$

where we,ve used Item 2 and Item 12 of Definition 4.3.10.1.2.

Item 4, *Associativity*: See [Pro25ao].

Item 5, *Unitality*: This follows from Item 6 and [Pro25at].

Item 6, *Commutativity*: See [Pro25ap].

Item 7, *Invertibility*: See [Pro25av].

Item 8, *Interaction With Unions*: See [Pro25bc].

Item 9, *Interaction With Complements I*: See [Pro25as].

Item 10, *Interaction With Complements II*: This follows from Item 6 and [Pro25ax].

Item 11, *Interaction With Complements III*: See [Pro25aq].

Item 12, *"Transitivity"*: We have

$$
\begin{aligned}
(U \triangle V) \triangle (V \triangle W) \quad &= \quad U \triangle (V \triangle (V \triangle W)) \quad && \text{(by Item 4)} \\
&= \quad U \triangle ((V \triangle V) \triangle W) \quad && \text{(by Item 4)} \\
&= \quad U \triangle (\emptyset \triangle W) \quad && \text{(by Item 7)} \\
&= \quad U \triangle W. \quad && \text{(by Item 5)}
\end{aligned}
$$

This finishes the proof.

Item 13, *The Triangle Inequality for Symmetric Differences*: This follows from Items 2 and 12.

Item 14, *Distributivity Over Intersections*: See [Pro25q].

Item 15, *Interaction With Characteristic Functions*: See [Pro25g].

Item 16, *Bijectivity*:

- We show that
$$
(U \triangle -) \colon \mathcal{P}(X) \to \mathcal{P}(X)
$$
  is self-inverse.

  Let $W \in \mathcal{P}(X)$. Then,

$$\begin{aligned}
U \triangle (U \triangle W) &= (U \triangle U) \triangle W && \text{(by Item 4)} \\
&= \emptyset \triangle W && \text{(by Item 7)} \\
&= W. && \text{(by Item 5)}
\end{aligned}$$

· By Item 6, $(- \triangle V) = (V \triangle -)$, hence the former is also self-inverse by the first point.

· The map $- \triangle (U \triangle V)$ is a bijection as a special case of the second point. From the first two points and Item 6, we get

$$U \triangle (U \triangle V) = V, \; V \triangle (U \triangle V) = V \triangle (V \triangle U) = U.$$

Hence the function maps U to V and V to U.

Item 17, Interaction With Powersets and Groups: Item 17a follows from Items 4 to 7, while Item 3b follows from Item 7.[16]
Item 4, Interaction With Powersets and Vector Spaces I: See [MSE 2719059].
Item 5, Interaction With Powersets and Vector Spaces II: See [MSE 2719059].
Item 6, Interaction With Powersets and Rings: This follows from Items 6 and 15 of Definition 4.3.9.1.2 and Items 14 and 17.[17]
Item 7, Interaction With Direct Images: This is a repetition of Item 9 of Definition 4.6.1.1.5 and is proved there.
Item 8, Interaction With Inverse Images: This is a repetition of Item 9 of Definition 4.6.2.1.3 and is proved there.
Item 9, Interaction With Codirect Images: This is a repetition of Item 8 of Definition 4.6.3.1.7 and is proved there. □

4.4 Powersets

4.4.1 Foundations

Let X be a set.

Definition 4.4.1.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

---

[16] *Reference*: [Pro25ar].
[17] *Reference*: [Pro25au].

Remark 4.4.1.1.2. Under the analogy that $\{t, f\}$ should be the (-1)-categorical analogue of Sets, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set X is equivalently (Item 2 of Definition 4.5.1.1.4) the set

$$\mathsf{Sets}(X, \{t, f\})$$

  of functions from X to the set $\{t, f\}$ of classical truth values.

- The category of presheaves on a category C is the category

$$\mathsf{Fun}(C^{\mathrm{op}}, \mathsf{Sets})$$

  of functors from C^{op} to the category Sets of sets.

Notation 4.4.1.1.3. Let X be a set.

1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of X.

2. We write $\mathcal{P}_{\mathsf{fin}}(X)$ for the set of finite subsets of X.

Proposition 4.4.1.1.4. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

   (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

   (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.

3. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathsf{Rel}(\mathsf{pt}, X),$$
$$\mathcal{P}(X) \cong \mathsf{Rel}(X, \mathsf{pt}),$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets})$.

4. *Interaction With Products I.* The map

$$\mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \coprod Y)$$
$$(U, V) \longmapsto U \cup V$$

is an isomorphism of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!, \mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!, \mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

5. *Interaction With Products II.* The map

$$\mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \coprod Y)$$
$$(U, V) \longmapsto U \boxtimes_{X \times Y} V,$$

where[18]

$$U \boxtimes_{X \times Y} V \overset{\mathrm{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

is an inclusion of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!, \mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!, \mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

6. *Interaction With Products III.* We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!, \mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1, where $\otimes$ denotes the tensor product of suplattices of **??**. Moreover, this makes each of $\mathcal{P}_!, \mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

Proof. Item 1, *Co/Completeness*: Omitted.
Item 2, *Cartesian Closedness*: See Section 4.4.7.
Item 3, *Powersets as Sets of Relations*: Indeed, we have

$$\mathsf{Rel}(\mathrm{pt}, X) \overset{\mathrm{def}}{=} \mathcal{P}(\mathrm{pt} \times X)$$

---

[18]The set $U \boxtimes_{X \times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see also Section 4.6.4).

$$\cong \mathcal{P}(X)$$

and

$$\mathrm{Rel}(X, \mathrm{pt}) \overset{\mathrm{def}}{=} \mathcal{P}(X \times \mathrm{pt})$$
$$\cong \mathcal{P}(X),$$

where we have used Item 5 of Definition 4.1.3.1.3.
Item 4, Interaction With Products I: The inverse of the map in the statement is the map

$$\Phi \colon \mathcal{P}(X \coprod Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \overset{\mathrm{def}}{=} (S_X, S_Y)$$

for each $S \in \mathcal{P}(X \coprod Y)$, where

$$S_X \overset{\mathrm{def}}{=} \{x \in X \mid (0, x) \in S\}$$
$$S_Y \overset{\mathrm{def}}{=} \{y \in Y \mid (1, y) \in S\}.$$

The rest of the proof is omitted.
Item 5, Interaction With Products II: Omitted.
Item 6, Interaction With Products III: Omitted. $\square$

4.4.2   Functoriality of Powersets

Proposition 4.4.2.1.1. Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

· *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \overset{\mathrm{def}}{=} \mathcal{P}(A).$$

· *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}_!(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \overset{\text{def}}{=} f_!,$$

as in Definition 4.6.1.1.1.

2. *Functoriality II*. The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1}\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}^{-1}(f)\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \overset{\text{def}}{=} f^{-1},$$

as in Definition 4.6.2.1.1.

3. *Functoriality III*. The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_*\colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

· *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \overset{\mathrm{def}}{=} f_*,$$

as in Definition 4.6.3.1.1.

Proof. *Item 1, Functoriality I*: This follows from Items 3 and 4 of Definition 4.6.1.1.6.
Item 2, Functoriality II: This follows from Items 3 and 4 of Definition 4.6.2.1.4.
Item 3, Functoriality III: This follows from Items 3 and 4 of Definition 4.6.3.1.8.   □

4.4.3   Adjointness of Powersets I

Proposition 4.4.3.1.1. We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathrm{op}}\right) \colon \quad \mathsf{Sets}^{\mathrm{op}} \underset{\mathcal{P}^{-1,\mathrm{op}}}{\overset{\mathcal{P}^{-1}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Sets},$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathrm{op}}(\mathcal{P}(X), Y)}_{\overset{\mathrm{def}}{=}\,\mathsf{Sets}(Y,\mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathrm{op}})$.

Proof. We have

$$
\begin{aligned}
\mathsf{Sets}^{\mathrm{op}}(\mathcal{P}(A), B) \quad &\overset{\mathrm{def}}{=} \quad \mathsf{Sets}(B, \mathcal{P}(A)) && \\
&\cong \quad \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\
&\cong \quad \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \quad \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \quad \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{aligned}
$$

where all bijections are natural in A and B.[19]   □

---

[19] Here we are using Item 3 of Definition 4.5.1.1.4.

4.4.4 Adjointness of Powersets II

Proposition 4.4.4.1.1. We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_!): \quad \mathrm{Sets} \underset{\mathcal{P}_!}{\overset{\mathrm{Gr}}{\rightleftarrows}} \perp \mathrm{Rel},$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(X), Y) \cong \mathrm{Sets}(X, \mathcal{P}(Y))$$

natural in $X \in \mathrm{Obj}(\mathrm{Sets})$ and $Y \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, Item 1 of Definition 8.2.2.1.2 and $\mathcal{P}_!$ is the functor of Relations, Definition 8.7.5.1.1.

Proof. We have

$$
\begin{aligned}
\mathrm{Rel}(\mathrm{Gr}(A), B) \quad &\cong \quad \mathcal{P}(A \times B) \\
&\cong \quad \mathrm{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\
&\cong \quad \mathrm{Sets}(A, \mathrm{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \quad \mathrm{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{aligned}
$$

where all bijections are natural in A, (where we are using Item 3 of Definition 4.5.1.1.4). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Relations, Definition 8.1.1.1.1.

Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$
\begin{array}{ccc}
\mathrm{Rel}(\mathrm{Gr}(A), B) & \xrightarrow{R\diamond-} & \mathrm{Rel}(\mathrm{Gr}(A), B') \\
\Big\downarrow{\wr} & & \Big\downarrow{\wr} \\
\mathrm{Sets}(A, \mathcal{P}(B)) & \xrightarrow[R_!]{} & \mathrm{Sets}(A, \mathcal{P}(B'))
\end{array}
$$

commutes, which follows from Relations, Definition 8.7.1.1.3. □

4.4.5 Powersets as Free Cocompletions

Let X be a set.

Proposition 4.4.5.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of X of Definition 4.4.1.1.1;

- The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

($\star$) Given another pair (Y, f) consisting of

- A suplattice $(Y, \preceq)$;
- A function $f \colon X \to Y$;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$
\begin{array}{ccc}
 & & \mathcal{P}(X) \\
 & \nearrow^{\chi_X} & \Big| {\scriptstyle \exists!} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

commute.

Proof. This is a rephrasing of Definition 4.4.5.1.2, which we prove below.[20] $\square$

Proposition 4.4.5.1.2. We have an adjunction

$$\left(\mathcal{P} \dashv \text{忘}\right) \colon \quad \mathsf{Sets} \underset{\text{忘}}{\overset{\mathcal{P}}{\rightleftarrows}} {\scriptstyle \perp} \mathsf{SupLat},$$

witnessed by a bijection

$$\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{SupLat})$, where:

---

[20] Here we only remark that the unique morphism of suplattices in the statement is given by the

· The category SupLat is the category of suplattices of **??**.

· The map

$$\chi_X^* \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \overset{\mathrm{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of suplattices $f \colon \mathcal{P}(X) \to Y$ to the composition

$$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \overset{f}{\longrightarrow} Y.$$

· The map

$$\mathsf{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function $f \colon X \to Y$ to its left Kan extension along χ_X,

$$\mathsf{Lan}_{\chi_X}(f) \colon \mathcal{P}(X) \to Y,$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathsf{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$[\mathsf{Lan}_{\chi_X}(f)](U) = \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)$$

$$= \int^{x \in X} \chi_U(x) \odot f(x)$$

$$= \bigvee_{x \in X} (\chi_U(x) \odot f(x))$$

$$= \left(\bigvee_{x \in U} (\chi_U(x) \odot f(x)) \right) \vee \left(\bigvee_{x \in U^{\mathsf{c}}} (\chi_U(x) \odot f(x)) \right)$$

$$= \left(\bigvee_{x \in U} f(x) \right) \vee \left(\bigvee_{x \in U^c} \emptyset_Y \right)$$

$$= \bigvee_{x \in U} f(x)$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigvee$ denotes the join in $(Y, \preceq)$.
- The symbol $\odot$ denotes the tensor of an element of Y by a truth value as in **??**. In particular, we have

$$\text{true} \odot f(x) \overset{\text{def}}{=} f(x),$$
$$\text{false} \odot f(x) \overset{\text{def}}{=} \emptyset_Y,$$

  where $\emptyset_Y$ is the bottom element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\text{Lan}_{\chi_X}(f)$ is given by

$$[\text{Lan}_{\chi_X}(f)](U) = \bigvee_{x \in U} f(x)$$
$$= \bigcup_{x \in U} f(x)$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y} \colon \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \text{Sets}(X, Y)$$

as in the statement, i.e. by
$$\Phi_{X,Y}(f) \overset{\text{def}}{=} f \circ \chi_X$$
for each $f \in \text{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

---
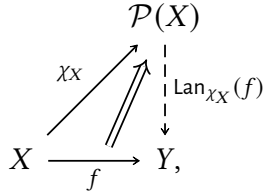
left Kan extension $\text{Lan}_{\chi_X}(f)$ of f along χ_X.

Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \overset{\mathrm{def}}{=} \mathsf{Lan}_{\chi_X}(f),$$

for each $f \in \mathsf{Sets}(X, Y)$.

Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathsf{id}_{\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))}.$$

We have

$$
\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\overset{\mathrm{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\
&\overset{\mathrm{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\
&\overset{\mathrm{def}}{=} \mathsf{Lan}_{\chi_X}(f \circ \chi_X)
\end{aligned}
$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. We now claim that

$$\mathsf{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. Indeed, we have

$$
\begin{aligned}
\left[\mathsf{Lan}_{\chi_X}(f \circ \chi_X)\right](U) &= \bigvee_{x \in U} f(\chi_X(x)) \\
&= f\left(\bigvee_{x \in U} \chi_X(x)\right) \\
&= f\left(\bigcup_{x \in U} \{x\}\right) \\
&= f(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))}$ of $\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)} .$$

We have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\overset{\mathrm{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\
&\overset{\mathrm{def}}{=} \Phi_{X,Y}(\mathsf{Lan}_{\chi_X}(f)) \\
&\overset{\mathrm{def}}{=} \mathsf{Lan}_{\chi_X}(f) \circ \chi_X
\end{aligned}
$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathsf{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$
\begin{aligned}
[\mathsf{Lan}_{\chi_X}(f) \circ \chi_X](x) &= \bigvee_{y \in \{x\}} f(y) \\
&= f(x)
\end{aligned}
$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, Part I: We need to show that, given a function $f \colon X \to X'$, the diagram

$$
\begin{array}{ccc}
\mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) & \xrightarrow{\;\Phi_{X',Y}\;} & \mathsf{Sets}(X', Y) \\
\Big\downarrow{\scriptstyle \mathcal{P}_!(f)^*} & & \Big\downarrow{\scriptstyle f^*} \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow[\;\Phi_{X,Y}\;]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
&\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
&= \xi \circ (f_! \circ \chi_X) \\
&\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\overset{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}
$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, Part II: We need to show that, given a morphism of suplattices

$$
g \colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),
$$

the diagram

$$
\begin{array}{ccc}
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow{\ \Phi_{X,Y}\ } & \mathsf{Sets}(X, Y) \\
\Big\downarrow{\scriptstyle g_!} & & \Big\downarrow{\scriptstyle g_!} \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y', \preceq)) & \xrightarrow[\ \Phi_{X,Y'}\ ]{} & \mathsf{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[\Phi_{X,Y'} \circ g_!](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} g_!(\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}
$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. $\square$

Warning 4.4.5.1.3. Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion of* X, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.

4.4.6   Powersets as Free Completions

Let X be a set.

Proposition 4.4.6.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- · The powerset of X together with reverse inclusion $\mathcal{P}(X)^{\mathsf{op}} = (\mathcal{P}(X), \supset)$ of Definition 4.4.1.1.1;

- · The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

($\star$) Given another pair (Y, f) consisting of

- An inflattice $(Y, \preceq)$;
- A function $f \colon X \to Y$;

there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(X)^{\mathsf{op}} & \\
\chi_X \nearrow & & \Big\downarrow \exists! \\
X \xrightarrow[f]{} & Y &
\end{array}
$$

commute.

Proof. This is a rephrasing of Definition 4.4.6.1.2, which we prove below.[21]   □

Proposition 4.4.6.1.2. We have an adjunction

$$\left(\mathcal{P} \dashv \text{忘}\right): \quad \text{Sets} \underset{\text{忘}}{\overset{\mathcal{P}}{\underset{\perp}{\rightleftarrows}}} \text{InfLat},$$

witnessed by a bijection

$$\text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, \preceq) \in \text{Obj}(\text{InfLat})$, where:

- The category InfLat is the category of inflattices of **??**.

- The map

  $$\chi_X^* \colon \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \text{Sets}(X, Y)$$

  witnessing the above bijection is defined by

  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of inflattices $f \colon \mathcal{P}(X)^{\text{op}} \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X)^{\text{op}} \overset{f}{\longrightarrow} Y.$$

- The map

  $$\text{Ran}_{\chi_X} \colon \text{Sets}(X, Y) \to \text{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

  witnessing the above bijection is given by sending a function $f \colon X \to Y$ to its right Kan extension along χ_X,

  $$\text{Ran}_{\chi_X}(f) \colon \mathcal{P}(X)^{\text{op}} \to Y,$$

---

[21] Here we only remark that the unique morphism of inflattices in the statement is given by the

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$
\begin{aligned}
[\mathrm{Ran}_{\chi_X}(f)](U) &= \int_{x \in X} \chi_{\mathcal{P}(X)^{\mathrm{op}}}(\chi_x, U) \pitchfork f(x) \\
&= \int_{x \in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \pitchfork f(x) \\
&= \int_{x \in X} \chi_U(x) \pitchfork f(x) \\
&= \bigwedge_{x \in X} \chi_U(x) \pitchfork f(x) \\
&= \left(\bigwedge_{x \in U} \chi_U(x) \pitchfork f(x) \right) \wedge \left(\bigwedge_{x \in U^c} \chi_U(x) \pitchfork f(x) \right) \\
&= \left(\bigwedge_{x \in U} f(x) \right) \wedge \left(\bigwedge_{x \in U^c} \infty_Y \right) \\
&= \left(\bigwedge_{x \in U} f(x) \right) \wedge \infty_Y \\
&= \bigwedge_{x \in U} f(x)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigwedge$ denotes the meet in $(Y, \preceq)$.
- The symbol $\pitchfork$ denotes the cotensor of an element of Y by a truth value as in **??**. In particular, we have

$$
\begin{aligned}
\mathsf{true} \pitchfork f(x) &\overset{\mathrm{def}}{=} f(x), \\
\mathsf{false} \pitchfork f(x) &\overset{\mathrm{def}}{=} \infty_Y,
\end{aligned}
$$

where ∞_Y is the top element of $(Y, \preceq)$.

---

right Kan extension $\mathrm{Ran}_{\chi_X}(f)$ of f along χ_X.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathsf{Ran}_{\chi_X}(f)$ is given by

$$[\mathsf{Ran}_{\chi_X}(f)](U) = \bigwedge_{x \in U} f(x)$$
$$= \bigcap_{x \in U} f(x)$$

for each $U \in \mathcal{P}(X)$.

Proof.   *Map I*: We define a map

$$\Phi_{X,Y} \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \overset{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \overset{\text{def}}{=} \mathsf{Ran}_{\chi_X}(f),$$

for each $f \in \mathsf{Sets}(X, Y)$.
Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathsf{id}_{\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))} .$$

We have

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) \overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f))$$
$$\overset{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X)$$
$$\overset{\text{def}}{=} \mathsf{Ran}_{\chi_X}(f \circ \chi_X)$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\mathsf{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$
\begin{aligned}
\left[\mathsf{Ran}_{\chi_X}(f \circ \chi_X)\right](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\
&= f\left(\bigwedge_{x \in U} \chi_X(x)\right) \\
&= f\left(\bigcup_{x \in U} \{x\}\right) \\
&= f(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathsf{id}_{\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}$ of $\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathsf{id}_{\mathsf{Sets}(X,Y)} .$$

We have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\
&\overset{\text{def}}{=} \Phi_{X,Y}(\mathsf{Ran}_{\chi_X}(f)) \\
&\overset{\text{def}}{=} \mathsf{Ran}_{\chi_X}(f) \circ \chi_X
\end{aligned}
$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathsf{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$[\mathsf{Ran}_{\chi_X}(f) \circ \chi_X](x) = \bigwedge_{y \in \{x\}} f(y)$$

$$= f(x)$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, *Part I*: We need to show that, given a function $f \colon X \to X'$, the diagram

$$
\begin{array}{ccc}
\mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) & \xrightarrow{\ \Phi_{X',Y}\ } & \mathsf{Sets}(X', Y) \\[2pt]
{\scriptstyle \mathcal{P}_!(f)^*} \Big\downarrow & & \Big\downarrow {\scriptstyle f^*} \\[2pt]
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow[\ \Phi_{X,Y}\ ]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\overset{\mathrm{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\overset{\mathrm{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
&\overset{\mathrm{def}}{=} (\xi \circ f_!) \circ \chi_X \\
&= \xi \circ (f_! \circ \chi_X) \\
&\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\overset{\mathrm{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\overset{\mathrm{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\overset{\mathrm{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}
$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, *Part II*: We need to show that, given a cocontinuous morphism of posets

$$g \colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\ \Phi_{X,Y}\ } & \mathsf{Sets}(X, Y) \\[2pt]
{\scriptstyle g_!} \Big\downarrow & & \Big\downarrow {\scriptstyle g_!} \\[2pt]
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) & \xrightarrow[\ \Phi_{X,Y'}\ ]{} & \mathsf{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[\Phi_{X,Y'} \circ g_!](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} g_!(\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}
$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.   □

Warning 4.4.6.1.3. Although the assignment $X \mapsto \mathcal{P}(X)^{\mathsf{op}}$ is called the *free completion of* X, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}})^{\mathsf{op}} \neq \mathcal{P}(X)^{\mathsf{op}}$.

4.4.7   The Internal Hom of a Powerset

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Proposition 4.4.7.1.1. The **internal Hom of** $\mathcal{P}(X)$ **from** U **to** V is the subset $[U, V]_X$[22] of X given by

$$
\begin{aligned}
[U, V]_X &= U^{\mathsf{c}} \cup V \\
&= (U \setminus V)^{\mathsf{c}}
\end{aligned}
$$

where U^{c} is the complement of U of Definition 4.3.11.1.1.

Proof. Proof of the Equality $U^{\mathsf{c}} \cup V = (U \setminus V)^{\mathsf{c}}$: We have

$$
\begin{aligned}
(U \setminus V)^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus (U \setminus V) \\
&= (X \cap V) \cup (X \setminus U) \\
&= V \cup (X \setminus U) \\
&\overset{\text{def}}{=} V \cup U^{\mathsf{c}} \\
&= U^{\mathsf{c}} \cup V,
\end{aligned}
$$

where we have used:

---

[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$.

1. Item 10 of Definition 4.3.10.1.2 for the second equality.

2. Item 4 of Definition 4.3.9.1.2 for the third equality.

3. Item 4 of Definition 4.3.8.1.2 for the last equality.

This finishes the proof.

Proof that $U^c \cup V$ Is Indeed the Internal Hom: This follows from Item 2 of Definition 4.3.9.1.2. □

Remark 4.4.7.1.2. Henning Makholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from U to V ([MSE 267365]):

1. Since products in $\mathcal{P}(X)$ are given by binary intersections (Item 1 of Definition 4.4.1.1.4), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.

2. Under the Curry–Howard correspondence (**??**), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.

3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.

4. The expression $\neg U \vee V$ then corresponds to the set $U^c \cup V$ in $\mathcal{P}(X)$.

5. The set $U^c \vee V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

Proposition 4.4.7.1.3. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$
\begin{aligned}
[U, -]_X : & \quad (\mathcal{P}(X), \supset) & \to (\mathcal{P}(X), \subset), \\
[-, V]_X : & \quad (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\
[-_1, -_2]_X : & \ (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

(a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.

(b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.

(c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U,-]_X): \quad \mathcal{P}(X) \underset{[U,-]_X}{\overset{U \cap -}{\rightleftarrows}} \mathcal{P}(X),$$

$$(- \cap V \dashv [V,-]_X): \quad \mathcal{P}(X) \underset{[V,-]_X}{\overset{- \cap V}{\rightleftarrows}} \mathcal{P}(X),$$

witnessed by bijections

$$\mathsf{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathsf{Hom}_{\mathcal{P}(X)}(U, [V,W]_X),$$
$$\mathsf{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathsf{Hom}_{\mathcal{P}(X)}(V, [U,W]_X).$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

  (a) The following conditions are equivalent:

   i. We have $U \cap V \subset W$.
   ii. We have $U \subset [V,W]_X$.

  (b) The following conditions are equivalent:

   i. We have $U \cap V \subset W$.
   ii. We have $V \subset [U,W]_X$.

3. *Interaction With the Empty Set I.* We have

$$[U,\emptyset]_X = U^{\mathsf{c}},$$
$$[\emptyset,V]_X = X,$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Interaction With X.* We have

$$[U,X]_X = X,$$
$$[X,V]_X = V,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Interaction With the Empty Set II.* The functor

$$D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

defined by

$$
\begin{aligned}
D_X &\overset{\text{def}}{=} [-, \emptyset]_X \\
&= (-)^{\mathsf{c}}
\end{aligned}
$$

is an involutory isomorphism of categories, making $\emptyset$ into a dualising object for $(\mathcal{P}(X), \cap, X, [-, -]_X)$ in the sense of **??**. In particular:

(a) The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\ D_X\ } & \mathcal{P}(X) \\
& \searrow{\scriptstyle \mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}} & \downarrow{\scriptstyle D_X} \\
& & \mathcal{P}(X)^{\mathsf{op}}
\end{array}
$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\overset{\text{def}}{=}[[U,\emptyset]_X,\emptyset]_X} = U$$

for each $U \in \mathcal{P}(X)$.

(b) The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \xrightarrow{\cap^{\mathsf{op}}} \mathcal{P}(X)^{\mathsf{op}} & \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X} \nearrow & & \searrow {\scriptstyle D_X} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[{[-_1,-_2]_X}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\text{def}}{=}[U\cap[V,\emptyset]_X,\emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With the Empty Set III.* Let $f \colon X \to Y$ be a function.

   (a) *Interaction With Direct Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\; f_*^{\mathsf{op}} \;} & \mathcal{P}(Y)^{\mathsf{op}} \\[2pt]
\Big\downarrow{\scriptstyle D_X} & & \Big\downarrow{\scriptstyle D_Y} \\[6pt]
\mathcal{P}(X) & \xrightarrow[\; f_! \;]{} & \mathcal{P}(Y)
\end{array}
$$

   commutes, i.e. we have

$$
f_!(D_X(U)) = D_Y(f_*(U))
$$

   for each $U \in \mathcal{P}(X)$.

   (b) *Interaction With Inverse Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} & \xrightarrow{\; f^{-1,\mathsf{op}} \;} & \mathcal{P}(X)^{\mathsf{op}} \\[2pt]
\Big\downarrow{\scriptstyle D_Y} & & \Big\downarrow{\scriptstyle D_X} \\[6pt]
\mathcal{P}(Y) & \xrightarrow[\; f^{-1} \;]{} & \mathcal{P}(X)
\end{array}
$$

   commutes, i.e. we have

$$
f^{-1}(D_Y(U)) = D_X(f^{-1}(U))
$$

   for each $U \in \mathcal{P}(X)$.

   (c) *Interaction With Codirect Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\; f_!^{\mathsf{op}} \;} & \mathcal{P}(Y)^{\mathsf{op}} \\[2pt]
\Big\downarrow{\scriptstyle D_X} & & \Big\downarrow{\scriptstyle D_Y} \\[6pt]
\mathcal{P}(X) & \xrightarrow[\; f_* \;]{} & \mathcal{P}(Y)
\end{array}
$$

   commutes, i.e. we have

$$
f_*(D_X(U)) = D_Y(f_!(U))
$$

   for each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathrm{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ [-_1,-_2]_{\mathcal{P}(X)}\ } & \mathcal{P}(\mathcal{P}(X)) \\
\big\downarrow{\scriptstyle\cup^{\mathrm{op}}\times\cup^{\mathrm{op}}} & \textcolor{orange}{\times} & \big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow[\ [-_1,-_2]_X\ ]{} & \mathcal{P}(X),
\end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W\in[\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U\in\mathcal{U}} U,\ \bigcup_{V\in\mathcal{V}} V\right]_X$$

in general, where $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\mathrm{op}} & \\
 & & \searrow{\scriptstyle\cup^{\mathrm{op}}} \\
\mathcal{P}(\mathcal{P}(X)^{\mathrm{op}}) & & \mathcal{P}(X)^{\mathrm{op}} \\
\searrow{\scriptstyle\mathrm{id}_{\mathcal{P}(X)}\star[-,V]_X} & & \swarrow{\scriptstyle[-,V]_X} \\
 & \mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \cap\ } \mathcal{P}(X) &
\end{array}$$

commutes, i.e. we have

$$\left[\bigcup_{U\in\mathcal{U}} U,\ V\right]_X = \bigcap_{U\in\mathcal{U}}[U,V]_X$$

for each $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$ and each $V\in\mathcal{P}(X)$.

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \cup\ } & \mathcal{P}(X) \\
\big\downarrow{\scriptstyle\mathrm{id}_{\mathcal{P}(X)}\star[U,-]_X} & & \big\downarrow{\scriptstyle[U,-]_X} \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\ \cup\ ]{} & \mathcal{P}(X)
\end{array}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V \right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-_1,-_2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap^{\mathsf{op}} \times \cap^{\mathsf{op}}} \downarrow & \textcolor{orange}{\times} & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[{[-_1,-_2]_X}]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \\
 & {\scriptstyle \cong} \nearrow \qquad \searrow {\scriptstyle \cap^{\mathsf{op}}} & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X} \searrow & & \swarrow {\scriptstyle [-,V]_X} \\
 & \mathcal{P}(\mathcal{P}(X)) \xrightarrow[\cup]{} \mathcal{P}(X) &
\end{array}
$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\;\cap\;} & \mathcal{P}(X) \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X}\Big\downarrow & & \Big\downarrow {\scriptstyle [U,-]_X} \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\;\cap\;]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\left[U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X
$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Binary Unions.* We have equalities of sets

$$
[U \cap V, W]_X = [U, W]_X \cup [V, W]_X,
$$
$$
[U, V \cap W]_X = [U, V]_X \cap [U, W]_X
$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Binary Intersections.* We have equalities of sets

$$
[U \cup V, W]_X = [U, W]_X \cap [V, W]_X,
$$
$$
[U, V \cup W]_X = [U, V]_X \cup [U, W]_X
$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Differences.* We have equalities of sets

$$
\begin{aligned}
[U \setminus V, W]_X &= [U, W]_X \cup [V^{\mathsf{c}}, W]_X \\
&= [U, W]_X \cup [U, V]_X, \\
[U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)
\end{aligned}
$$

for each $U, V, W \in \mathcal{P}(X)$.

16. *Interaction With Complements.* We have equalities of sets

$$
\begin{aligned}
[U^{\mathsf{c}}, V]_X &= U \cup V, \\
[U, V^{\mathsf{c}}]_X &= U \cap V, \\
[U, V]_X^{\mathsf{c}} &= U \setminus V
\end{aligned}
$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \ (\mathrm{mod}\ 2), \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

18. *Interaction With Direct Images.* Let $f: X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathrm{op}} \times f_!} & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \\
\downarrow{\scriptstyle [-_1,-_2]_X} & & \downarrow{\scriptstyle [-_1,-_2]_Y} \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

19. *Interaction With Inverse Images.* Let $f: X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1,\mathrm{op}} \times f^{-1}} & \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \\
\downarrow{\scriptstyle [-_1,-_2]_Y} & & \downarrow{\scriptstyle [-_1,-_2]_X} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

20. *Interaction With Codirect Images.* Let $f: X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathrm{op}} \times f_*} & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \\
\downarrow{\scriptstyle [-_1,-_2]_X} & \quad \subset & \downarrow{\scriptstyle [-_1,-_2]_Y} \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove Items 1a to 1c.

1. *Proof of Item 1a*: We have

$$
\begin{aligned}
[A, V]_X &\overset{\text{def}}{=} A^{\mathsf{c}} \cup V \\
&\subset U^{\mathsf{c}} \cup V \\
&\overset{\text{def}}{=} [U, V]_X,
\end{aligned}
$$

where we have used:

(a) Item 1 of Definition 4.3.11.1.2, which states that if $U \subset A$, then $A^{\mathsf{c}} \subset U^{\mathsf{c}}$.

(b) Item 1a of Item 1 of Definition 4.3.11.1.2, which states that if $A^{\mathsf{c}} \subset U^{\mathsf{c}}$, then $A^{\mathsf{c}} \cup K \subset U^{\mathsf{c}} \cup K$ for any $K \in \mathcal{P}(X)$.

2. *Proof of Item 1b*: We have

$$
\begin{aligned}
[U, V]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup V \\
&\subset U^{\mathsf{c}} \cup B \\
&\overset{\text{def}}{=} [U, B]_X,
\end{aligned}
$$

where we have used Item 1b of Item 1 of Definition 4.3.11.1.2, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

3. *Proof of Item 1c*: We have

$$
\begin{aligned}
[A, V]_X &\subset [U, V]_X \\
&\subset [U, B]_X,
\end{aligned}
$$

where we have used Items 1a and 1b.

This finishes the proof.
Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.3.9.1.2 and is proved there.

Item 3, Interaction With the Empty Set I: We have

$$[U, \emptyset]_X \overset{\text{def}}{=} U^c \cup \emptyset$$
$$= U^c,$$

where we have used Item 3 of Definition 4.3.8.1.2, and we have

$$[\emptyset, V]_X \overset{\text{def}}{=} \emptyset^c \cup V$$
$$\overset{\text{def}}{=} (X \setminus \emptyset) \cup V$$
$$= X \cup V$$
$$= X,$$

where we have used:

1. Item 12 of Definition 4.3.10.1.2 for the first equality.

2. Item 5 of Definition 4.3.8.1.2 for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
Item 4, Interaction With X: We have

$$[U, X]_X \overset{\text{def}}{=} U^c \cup X$$
$$= X,$$

where we have used Item 5 of Definition 4.3.8.1.2, and we have

$$[X, V]_X \overset{\text{def}}{=} X^c \cup V$$
$$\overset{\text{def}}{=} (X \setminus X) \cup V$$
$$= \emptyset \cup V$$
$$= V,$$

where we have used Item 3 of Definition 4.3.8.1.2 for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
Item 5, Interaction With the Empty Set II: We have

$$D_X(D_X(U)) \overset{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X$$
$$= [U^c, \emptyset]_X$$
$$= (U^c)^c$$
$$= U,$$

where we have used:

1. Item 3 for the second and third equalities.

2. Item 3 of Definition 4.3.11.1.2 for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and thus we have

$$[[-,\emptyset]_X,\emptyset]_X \cong \mathrm{id}_{\mathcal{P}(X)}$$

This finishes the proof.

Item 6, Interaction With the Empty Set III: Since $D_X = (-)^{\mathrm{c}}$, this is essentially a repetition of the corresponding results for $(-)^{\mathrm{c}}$, namely Items 5 to 7 of Definition 4.3.11.1.2.

Item 7, Interaction With Unions of Families of Subsets I: By Item 3 of Definition 4.4.7.1.3, we have

$$[\mathcal{U},\emptyset]_{\mathcal{P}(X)} = \mathcal{U}^{\mathrm{c}},$$
$$[U,\emptyset]_X = U^{\mathrm{c}}.$$

With this, the counterexample given in the proof of Item 10 of Definition 4.3.6.1.2 then applies.

Item 8, Interaction With Unions of Families of Subsets II: We have

$$\left[\bigcup_{U\in\mathcal{U}} U, V\right]_X \overset{\mathrm{def}}{=} \left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathrm{c}} \cup V$$
$$= \left(\bigcap_{U\in\mathcal{U}} U^{\mathrm{c}}\right) \cup V$$
$$= \bigcap_{U\in\mathcal{U}} (U^{\mathrm{c}} \cup V)$$
$$\overset{\mathrm{def}}{=} \bigcap_{U\in\mathcal{U}} [U,V]_X,$$

where we have used:

1. Item 11 of Definition 4.3.6.1.2 for the second equality.

2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.

Item 9, Interaction With Unions of Families of Subsets III: We have

$$\bigcup_{V\in\mathcal{V}} [U,V]_X \overset{\mathrm{def}}{=} \bigcup_{V\in\mathcal{V}} (U^{\mathrm{c}} \cup V)$$

$$= U^{\mathsf{c}} \cup \left(\bigcup_{V \in \mathcal{V}} V \right)$$

$$\overset{\text{def}}{=} \left[U, \bigcup_{V \in \mathcal{V}} V \right]_X .$$

where we have used Item 6. This finishes the proof.

Item 10, Interaction With Intersections of Families of Subsets I: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \bigcap_{W \in \mathcal{P}(X)} W$$

$$= \{0, 1\},$$

whereas

$$\left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X = [\{0, 1\}, \{0\}]$$

$$= \{0\},$$

Thus we have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X .$$

This finishes the proof.

Item 11, Interaction With Intersections of Families of Subsets II: We have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X \overset{\text{def}}{=} \left(\bigcap_{U \in \mathcal{U}} U \right)^{\mathsf{c}} \cup V$$

$$= \left(\bigcup_{U \in \mathcal{U}} U^{\mathsf{c}} \right) \cup V$$

$$= \bigcup_{U \in \mathcal{U}} (U^{\mathsf{c}} \cup V)$$

$$\overset{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X,$$

where we have used:

1. Item 12 of Definition 4.3.6.1.2 for the second equality.

2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.

Item 12, Interaction With Intersections of Families of Subsets III: We have

$$\bigcap_{V \in \mathcal{V}} [U, V]_X \overset{\text{def}}{=} \bigcap_{V \in \mathcal{V}} (U^c \cup V)$$

$$= U^c \cup \left(\bigcap_{V \in \mathcal{V}} V \right)$$

$$\overset{\text{def}}{=} \left[U, \bigcap_{V \in \mathcal{V}} V \right]_X.$$

where we have used Item 6. This finishes the proof.

Item 13, Interaction With Binary Unions: We have

$$[U \cap V, W]_X \overset{\text{def}}{=} (U \cap V)^c \cup W$$

$$= (U^c \cup V^c) \cup W$$

$$= (U^c \cup V^c) \cup (W \cup W)$$

$$= (U^c \cup W) \cup (V^c \cup W)$$

$$\overset{\text{def}}{=} [U, W]_X \cup [V, W]_X,$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.

2. Item 8 of Definition 4.3.8.1.2 for the third equality.

3. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the fourth equality.

For the second equality in the statement, we have

$$[U, V \cap W]_X \overset{\text{def}}{=} U^c \cup (V \cap W)$$

$$= (U^c \cup V) \cap (U^c \cap W)$$

$$\overset{\text{def}}{=} [U, V]_X \cap [U, W]_X,$$

where we have used Item 6 of Definition 4.3.8.1.2 for the second equality.

Item 14, Interaction With Binary Intersections: We have

$$
\begin{aligned}
[U \cup V, W]_X &\stackrel{\text{def}}{=} (U \cup V)^{\mathsf{c}} \cup W \\
&= (U^{\mathsf{c}} \cap V^{\mathsf{c}}) \cup W \\
&= (U^{\mathsf{c}} \cup W) \cap (V^{\mathsf{c}} \cup W) \\
&\stackrel{\text{def}}{=} [U, W]_X \cap [V, W]_X,
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.

2. Item 6 of Definition 4.3.8.1.2 for the third equality.

Now, for the second equality in the statement, we have

$$
\begin{aligned}
[U, V \cup W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup (V \cup W) \\
&= (U^{\mathsf{c}} \cup U^{\mathsf{c}}) \cup (V \cup W) \\
&= (U^{\mathsf{c}} \cup V) \cup (U^{\mathsf{c}} \cup W) \\
&\stackrel{\text{def}}{=} [U, V]_X \cup [U, W]_X,
\end{aligned}
$$

where we have used:

1. Item 8 of Definition 4.3.8.1.2 for the second equality.

2. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the third equality.

This finishes the proof.

Item 15, Interaction With Differences: We have

$$
\begin{aligned}
[U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W \\
&\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\
&= (V \cup (U^{\mathsf{c}} \cup U^{\mathsf{c}})) \cup W \\
&= (U^{\mathsf{c}} \cup W) \cup (U^{\mathsf{c}} \cup V) \\
&\stackrel{\text{def}}{=} [U, W]_X \cup [U, V]_X,
\end{aligned}
$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.

2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.

3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.

4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

We also have

$$
\begin{aligned}
[U \setminus V, W]_X &\overset{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\overset{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W \\
&\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\
&= (V \cup U^{\mathsf{c}}) \cup (W \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup (V \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup ((V^{\mathsf{c}})^{\mathsf{c}} \cup W) \\
&\overset{\text{def}}{=} [U, W]_X \cup [V^{\mathsf{c}}, W]_X,
\end{aligned}
$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.

2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.

3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.

4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

5. Item 3 of Definition 4.3.11.1.2 for the eighth equality.

Now, for the second equality in the statement, we have

$$
\begin{aligned}
[U, V \setminus W]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup (V \setminus W) \\
&= (V \setminus W) \cup U^{\mathsf{c}} \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \setminus U^{\mathsf{c}}) \\
&\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \setminus (W \setminus (X \setminus U))
\end{aligned}
$$

$$
\begin{aligned}
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup (W \setminus X)) \\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup \emptyset) \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \cap U) \\
&= (V \cup U^{\mathsf{c}}) \setminus (U \cap W) \\
&\overset{\text{def}}{=} [U, V]_X \setminus (U \cap W)
\end{aligned}
$$

where we have used:

1. Item 4 of Definition 4.3.8.1.2 for the second equality.

2. Item 4 of Definition 4.3.10.1.2 for the third equality.

3. Item 10 of Definition 4.3.10.1.2 for the fifth equality.

4. Item 13 of Definition 4.3.10.1.2 for the sixth equality.

5. Item 3 of Definition 4.3.8.1.2 for the seventh equality.

6. Item 5 of Definition 4.3.9.1.2 for the eighth equality.

This finishes the proof.
Item 16, Interaction With Complements: We have

$$
\begin{aligned}
[U^{\mathsf{c}}, V]_X &\overset{\text{def}}{=} (U^{\mathsf{c}})^{\mathsf{c}} \cup V, \\
&= U \cup V,
\end{aligned}
$$

where we have used Item 3 of Definition 4.3.11.1.2. We also have

$$
\begin{aligned}
[U, V^{\mathsf{c}}]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup V^{\mathsf{c}} \\
&= U \cap V
\end{aligned}
$$

where we have used Item 2 of Definition 4.3.11.1.2. Finally, we have

$$
\begin{aligned}
[U, V]_X^{\mathsf{c}} &= ((U \setminus V)^{\mathsf{c}})^{\mathsf{c}} \\
&= U \setminus V,
\end{aligned}
$$

where we have used Item 2 of Definition 4.3.11.1.2.
Item 17, Interaction With Characteristic Functions: We have

$$
\chi_{[U,V]_{\mathcal{P}(X)}}(x) \overset{\text{def}}{=} \chi_{U^{\mathsf{c}} \cup V}(x)
$$

$$= \max(\chi_{U^c}, \chi_V)$$
$$= \max(1 - \chi_U \ (\text{mod } 2), \chi_V),$$

where we have used:

1. Item 10 of Definition 4.3.8.1.2 for the second equality.

2. Item 4 of Definition 4.3.11.1.2 for the third equality.

This finishes the proof.
Item 18, Interaction With Direct Images: This is a repetition of Item 10 of Definition 4.6.1.1.5 and is proved there.
Item 19, Interaction With Inverse Images: This is a repetition of Item 10 of Definition 4.6.2.1.3 and is proved there.
Item 20, Interaction With Codirect Images: This is a repetition of Item 9 of Definition 4.6.3.1.7 and is proved there. $\square$

4.4.8   Isbell Duality for Sets

Let X be a set.

Definition 4.4.8.1.1. The **Isbell function** of X is the map

$$\mathsf{I} \colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

defined by

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [\![x \mapsto [U, \{x\}]_X]\!]$$

for each $U \in \mathcal{P}(X)$.

Remark 4.4.8.1.2. Recall from Definition 4.4.1.1.2 that we may view the powerset $\mathcal{P}(X)$ of a set X as the decategorification of the category of presheaves $\mathsf{PSh}(\mathcal{C})$ of a category $\mathcal{C}$. Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\mathsf{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \mathsf{Nat}(\mathcal{F}, h_{(-)})$$

for each $\mathcal{F} \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$. To this end, we could define

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [U, \chi_{(-)}]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of C into $\mathsf{PSh}(C)$ with the characteristic embedding $x \mapsto \chi_x$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1.

- The internal Hom Nat of $\mathsf{PSh}(C)$ with the internal Hom $[-,-]_X$ of $\mathcal{P}(X)$ of Definition 4.4.7.1.1.

However, since $[U, \chi_x]_X$ is a subset of U instead of a truth value, we get a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathcal{P}(X).$$

This makes some of the properties involving I a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $\mathsf{I}_! \circ \mathsf{I}$ evaluates to $\mathrm{id}_{\mathcal{P}(X)}$ in the sense of Definition 4.4.8.1.3.

Proposition 4.4.8.1.3. The diagram

$$\mathcal{P}(X) \xrightarrow{\;\mathsf{I}\;} \mathsf{Sets}(X, \mathcal{P}(X))$$

$$\Delta_{\Delta_{\mathrm{id}_{\mathcal{P}(X)}}} \searrow \qquad \downarrow \mathsf{I}_!$$

$$\mathsf{Sets}(X, \mathsf{Sets}(X, \mathcal{P}(X)))$$

commutes, i.e. we have

$$\mathsf{I}_!(\mathsf{I}(U)) = [\![x \mapsto [\![y \mapsto U]\!]]\!]$$

for each $U \in \mathcal{P}(X)$.

Proof. We have

$$\begin{aligned}
\mathsf{I}_!(\mathsf{I}(U)) &\overset{\mathrm{def}}{=} \mathsf{I}_!([\![x \mapsto U^{\mathsf{c}} \cup \{x\}]\!]) \\
&\overset{\mathrm{def}}{=} [\![x \mapsto \mathsf{I}(U^{\mathsf{c}} \cup \{x\})]\!] \\
&\overset{\mathrm{def}}{=} [\![x \mapsto [\![y \mapsto (U^{\mathsf{c}} \cup \{x\})^{\mathsf{c}} \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \setminus \{x\}) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto U]\!]]\!],
\end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2 for the fourth equality above.   □

4.5 Characteristic Functions

4.5.1 The Characteristic Function of a Subset

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.5.1.1.1. The **characteristic function of** U[23] is the function $\chi_U \colon X \to \{t, f\}$[24] defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in U, \\ \text{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

Remark 4.5.1.1.2. Under the analogy that $\{t, f\}$ should be the (-1)-categorical analogue of Sets, we may view a function

$$f \colon X \to \{t, f\}$$

as a decategorification of presheaves and copresheaves

$$\mathcal{F} \colon C^{\text{op}} \to \text{Sets},$$
$$F \colon C \to \text{Sets}.$$

The characteristic functions χ_U of the subsets of X are then the primordial examples of such functions (and, in fact, all of them).

Notation 4.5.1.1.3. We will often employ the bijection $\{t, f\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when disucssing characteristic functions.

Examples of this include Items 4 to 11 of Definition 4.5.1.1.4 below.

Proposition 4.5.1.1.4. Let X be a set.

1. *Functionality.* The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)} \colon \mathcal{P}(X) \to \text{Sets}(X, \{t, f\}).$$

2. *Bijectivity.* The function $\chi_{(-)}$ from Item 1 is bijective.

---

[23] *Further Terminology:* Also called the **indicator function of** U.
[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

3. *Naturality.* The collection

$$\left\{\chi_{(-)}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})\right\}_{X \in \mathrm{Obj}(\mathsf{Sets})}$$

defines a natural isomorphism between $\mathcal{P}^{-1}$ and $\mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\})$. In particular, given a function $f\colon X \to Y$, the diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) & \xrightarrow{\;\;f^{-1}\;\;} & \mathcal{P}(X) \\
{\scriptstyle \chi_{(-)}}\Big\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\Big\downarrow{\scriptstyle \chi_{(-)}} \\
\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) & \xrightarrow[\;\;f^{*}\;\;]{} & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})
\end{array}
$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Complements.* We have

$$\chi_{U^c} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Internal Homs.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functionality*: There is nothing to prove.
Item 2, Bijectivity: We proceed in three steps:

1. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi \colon \mathsf{Sets}(X, \{t, f\}) \overset{\sim}{\dashrightarrow} \mathcal{P}(X),$$

defined by

$$\begin{aligned}
\Phi(f) &\overset{\text{def}}{=} U_f \\
&\overset{\text{def}}{=} f^{-1}(\text{true}) \\
&\overset{\text{def}}{=} \{x \in X \mid f(x) = \text{true}\}
\end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{t, f\})$.

2. *Invertibility I.* We have

$$
\begin{aligned}
[\Phi \circ \chi_{(-)}](U) &\overset{\text{def}}{=} \Phi(\chi_U) \\
&\overset{\text{def}}{=} \chi_U^{-1}(\text{true}) \\
&\overset{\text{def}}{=} \{x \in X \mid \chi_U(x) = \text{true}\} \\
&\overset{\text{def}}{=} \{x \in X \mid x \in U\} \\
&= U \\
&\overset{\text{def}}{=} [\text{id}_{\mathcal{P}(X)}](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$. Thus, we have

$$
\Phi \circ \chi_{(-)} = \text{id}_{\mathcal{P}(X)} \, .
$$

3. *Invertibility II.* We have

$$
\begin{aligned}
[\chi_{(-)} \circ \Phi](U) &\overset{\text{def}}{=} \chi_{\Phi(f)} \\
&\overset{\text{def}}{=} \chi_{f^{-1}(\text{true})} \\
&\overset{\text{def}}{=} \left[\!\!\left[x \mapsto \begin{cases} \text{true} & \text{if } x \in f^{-1}(\text{true}) \\ \text{false} & \text{otherwise} \end{cases} \right]\!\!\right] \\
&= \left[\!\!\left[x \mapsto f(x) \right]\!\!\right] \\
&= f \\
&\overset{\text{def}}{=} [\text{id}_{\text{Sets}(X,\{t,f\})}](f)
\end{aligned}
$$

for each $f \in \text{Sets}(X, \{t, f\})$. Thus, we have

$$
\chi_{(-)} \circ \Phi = \text{id}_{\text{Sets}(X,\{t,f\})} \, .
$$

This finishes the proof.
Item 3, *Naturality*: We proceed in two steps:

1. *Naturality of $\chi_{(-)}$.* We have

$$
\begin{aligned}
[\chi_V \circ f](v) &\overset{\text{def}}{=} \chi_V(f(v)) \\
&= \begin{cases} \text{true} & \text{if } f(v) \in V, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } v \in f^{-1}(V), \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

$$\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v)$$

for each $v \in V$.

2. *Naturality of* Φ. Since $\chi_{(-)}$ is natural and a componentwise inverse to Φ, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Φ is also natural in each argument.

This finishes the proof.

Item 4, Interaction With Unions I: This is a repetition of Item 10 of Definition 4.3.8.1.2 and is proved there.

Item 5, Interaction With Unions II: This is a repetition of Item 11 of Definition 4.3.8.1.2 and is proved there.

Item 6, Interaction With Intersections I: This is a repetition of Item 10 of Definition 4.3.9.1.2 and is proved there.

Item 7, Interaction With Intersections II: This is a repetition of Item 11 of Definition 4.3.9.1.2 and is proved there.

Item 8, Interaction With Differences: This is a repetition of Item 16 of Definition 4.3.10.1.2 and is proved there.

Item 9, Interaction With Complements: This is a repetition of Item 4 of Definition 4.3.11.1.2 and is proved there.

Item 10, Interaction With Symmetric Differences: This is a repetition of Item 15 of Definition 4.3.12.1.2 and is proved there.

Item 11, Interaction With Internal Homs: This is a repetition of Item 17 of Definition 4.4.7.1.3 and is proved there.   □

Remark 4.5.1.1.5. The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 2 of Definition 4.5.1.1.4, which

· Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U \colon X \to \{\mathsf{true}, \mathsf{false}\}$;

· Takes a function $f \colon X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\mathsf{FibSets}_X}_{\stackrel{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}_X}_{\stackrel{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}}, \mathsf{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, **??**. Here we view:

- Subsets $U \hookrightarrow X$ as being analogous to X-fibred sets $\phi_X \colon A \to X$.

- Functions $f \colon X \to \{t, f\}$ as being analogous to X-indexed sets $A \colon X_{\text{disc}} \to$ Sets.

4.5.2   The Characteristic Function of a Point

Let X be a set and let $x \in X$.

Definition 4.5.2.1.1. The **characteristic function of** x is the function[25]

$$\chi_x \colon X \to \{t, f\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

Remark 4.5.2.1.2. Expanding upon Definition 4.5.1.1.2, we may think of the characteristic function

$$\chi_x \colon X \to \{t, f\}$$

of an *element* x of X as a decategorification of the representable presheaf and of the representable copresheaf

$$h_X \colon C^{\text{op}} \to \text{Sets},$$
$$h^X \colon C \to \text{Sets}$$

associated of an *object* X of a category C.

4.5.3   The Characteristic Relation of a Set

Let X be a set.

---

[25] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

Definition 4.5.3.1.1. The **characteristic relation on** X[26] is the relation[27]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

Remark 4.5.3.1.2. Expanding upon Definitions 4.5.1.1.2 and 4.5.2.1.2, we may view the characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

of X as a decategorification of the Hom profunctor

$$\mathrm{Hom}_C(-_1, -_2) \colon C^{\mathrm{op}} \times C \to \mathsf{Sets}$$

of a category C.

Proposition 4.5.3.1.3. Let $f \colon X \to Y$ be a function.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f \colon A \to B$ be a function. We have an inclusion[29]

$$\chi_B \circ (f \times f) \subset \chi_A,$$

$$\{\mathsf{t}, \mathsf{f}\}.$$

Proof. Item 1, *The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$', which is true.   □

---

[26] *Further Terminology:* Also called the **identity relation on** X.

[27] *Further Notation:* Also written $\chi^{-1}_{-_2}$, or $\sim_{\mathrm{id}}$ in the context of relations.

[28] Under the bijection $\mathsf{Sets}(X \times X, \{\mathsf{t}, \mathsf{f}\}) \cong \mathcal{P}(X \times X)$ of Item 2 of Definition 4.5.1.1.4, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

[29] *Note:* This is the 0-categorical version of Categories, Definition 11.5.4.1.1.

4.5.4 The Characteristic Embedding of a Set

Let X be a set.

Definition 4.5.4.1.1. The **characteristic embedding**[30] **of X into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by[31]

$$\chi_{(-)}(x) \overset{\mathrm{def}}{=} \chi_x$$
$$= \{x\}$$

for each $x \in X$.

Remark 4.5.4.1.2. Expanding upon Definitions 4.5.1.1.2, 4.5.2.1.2 and 4.5.3.1.2, we may view the characteristic embedding

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$よ \colon C^{\mathrm{op}} \hookrightarrow \mathsf{PSh}(C)$$

of a category C into $\mathsf{PSh}(C)$.

Proposition 4.5.4.1.3. Let $f \colon X \to Y$ be a map of sets.

1. *Interaction With Functions.* We have

$$f_! \circ \chi_X = \chi_Y \circ f, \qquad
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\chi_X \downarrow & & \downarrow \chi_Y \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y).
\end{array}$$

Proof. Item 1, *Interaction With Functions*: Indeed, we have

$$[f_! \circ \chi_X](x) \overset{\mathrm{def}}{=} f_!(\chi_X(x))$$

---

[30]The name "characteristic *embedding*" is justified by Definition 4.5.5.1.2, which gives an analogue of fully faithfulness for $\chi_{(-)}$.

[31]Here we are identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ as per Item 2 of Definition 4.5.1.1.4.

$$\overset{\text{def}}{=} f_!(\{x\})$$
$$= \{f(x)\}$$
$$\overset{\text{def}}{=} \chi_{X'}(f(x))$$
$$\overset{\text{def}}{=} [\chi_{X'} \circ f](x),$$

for each $x \in X$, showing the desired equality. $\square$

4.5.5   The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 4.5.5.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } U \subset V, \\ \text{false} & \text{otherwise.} \end{cases}$$

Proof. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } \{x\} \subset U, \\ \text{false} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \text{true} & \text{if } x \in U \\ \text{false} & \text{otherwise} \end{cases}$$
$$\overset{\text{def}}{=} \chi_U(x).$$

This finishes the proof. $\square$

Corollary 4.5.5.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) \cong \chi_X(x, y)$$

for each $x, y \in X$.

Proof. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_y(x)$$

$$\overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in \{y\} \\ \text{false} & \text{otherwise} \end{cases}$$

$$= \begin{cases} \text{true} & \text{if } x = y \\ \text{false} & \text{otherwise} \end{cases}$$

$$\overset{\text{def}}{=} \chi_X(x, y).$$

where we have used Definition 4.5.5.1.1 for the first equality. $\square$

4.6   The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

4.6.1   Direct Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.1.1.1. The **direct image function associated to** f is the function[32]

$$f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[33]

$$f_!(U) \overset{\text{def}}{=} \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\}$$

$$= \{f(x) \in Y \mid x \in U\}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.1.1.2. Sometimes one finds the notation

$$\exists_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

---

[32] *Further Notation:* Also written simply $f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$.

[33] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

- We have $y \in \exists_f(U)$.

- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.1.1.3. Notation for direct images between powersets is tricky:

1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:

   (a) *Powersets.* Given a function $f\colon X \to Y$, we have an inverse image functor
   $$f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X).$$
   The *left* adjoint of this functor is the usual direct image, defined above in Definition 4.6.1.1.1.

   (b) *Presheaves.* Given a morphism of topological spaces $f\colon X \to Y$, we have an inverse image functor
   $$f^{-1}\colon \mathsf{PSh}(Y) \to \mathsf{PSh}(X).$$
   The *right* adjoint of this functor is the direct image functor of presheaves, defined in **??**.

2. The presheaf direct image functor is denoted f_*, but the direct image functor for powersets is denoted $f_!$ (as it,s a left adjoint).

3. Adding to the confusion, it,s somewhat common for $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ to be denoted f_*.

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

SITUATION	ADJOINT STRING
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*)\colon \mathcal{P}(X) \overset{\rightarrow}{\underset{\rightarrow}{}} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*)\colon \mathsf{PSh}(X) \overset{\rightarrow}{\underset{\rightarrow}{}} \mathsf{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*)\colon C_{/X} \overset{\rightarrow}{\underset{\rightarrow}{}} C_{/Y}$
Kan Extensions	$(F_! \dashv F^* \dashv F_*)\colon \mathsf{Fun}(C, \mathcal{E}) \overset{\rightarrow}{\underset{\rightarrow}{}} \mathsf{Fun}(\mathcal{D}, \mathcal{E})$

Remark 4.6.1.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the direct image function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$
\begin{aligned}
f_!(\chi_U) &\overset{\text{def}}{=} \mathsf{Lan}_f(\chi_U) \\
&= \mathrm{colim}((f \overset{\to}{\times} \underline{(-_1)}) \overset{\mathrm{pr}}{\twoheadrightarrow} A \overset{\chi_U}{\longrightarrow} \{\mathsf{t}, \mathsf{f}\}) \\
&= \operatorname*{colim}_{\substack{x \in X \\ f(x) =-_1}} (\chi_U(x)) \\
&= \bigvee_{\substack{x \in X \\ f(x) =-_1}} (\chi_U(x)),
\end{aligned}
$$

where we have used **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\
&= \begin{cases} \text{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $y \in Y$.

Proposition 4.6.1.1.5. Let $f \colon X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

   ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_*\right): \quad \mathcal{P}(X) \xleftarrow{\;\; f^{-1} \;\;} \mathcal{P}(Y),$$

witnessed by:

   (a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \hookrightarrow f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \hookrightarrow f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \hookrightarrow \mathrm{id}_{\mathcal{P}(X)},$$

   having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*(f^{-1}(V)),$$
$$f_!(f^{-1}(V)) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

   (b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

   natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

   i. The following conditions are equivalent:
      A. We have $f_!(U) \subset V$.
      B. We have $U \subset f^{-1}(V)$.
   ii. The following conditions are equivalent:
      A. We have $f^{-1}(U) \subset V$.
      B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\;(f_!)_!\;} & \mathcal{P}(\mathcal{P}(Y)) \\
\cup \downarrow & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{\;\;f_!\;\;} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cup \downarrow & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cap \downarrow & \subseteq & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

7. *Interaction With Differences.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{\ f_!^{\mathsf{op}} \times f_! \ } & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \backslash}\big\downarrow & \supset & \big\downarrow{\scriptstyle \backslash} \\
\mathcal{P}(X) & \xrightarrow[\ f_! \ ]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\ f_*^{\mathsf{op}} \ } & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}}}\big\downarrow & & \big\downarrow{\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) & \xrightarrow[\ f_! \ ]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{\ f_!^{\mathsf{op}} \times f_! \ } & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \triangle}\big\downarrow & \supset & \big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow[\ f_! \ ]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U) \vartriangle f_!(V) \subset f_!(U \vartriangle V)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathrm{op}} \times f_!} & \mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \\
\downarrow{\scriptstyle [-_1,-_2]_X} & & \downarrow{\scriptstyle [-_1,-_2]_Y} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$f_!(U) \cup f_!(V) = f_!(U \cup V),$$
$$f_!(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V),$$
$$f_!(X) \subset Y,$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_!, f_!^\otimes, f_{!|\mathbb{1}}^\otimes) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$f_{!|U,V}^\otimes \colon f_!(U) \cup f_!(V) \xrightarrow{=} f_!(U \cup V),$$

$$f_{!|\mathbb{1}}^\otimes \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(f_!, f_!^\otimes, f_{!|\mathbb{1}}^\otimes) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$f_{!|U,V}^\otimes \colon f_!(U \cap V) \hookrightarrow f_!(U) \cap f_!(V),$$

$$f_{!|\mathbb{1}}^\otimes \colon f_!(X) \hookrightarrow Y,$$

natural in $U, V \in \mathcal{P}(X)$.

15. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Interaction With Products.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

17. *Relation to Codirect Images.* We have

$$f_!(U) = f_*(U^c)^c$$
$$\overset{\text{def}}{=} Y \setminus f_*(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Omitted.

Item 2, *Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.

Item 3, *Interaction With Unions of Families of Subsets*: We have

$$\bigcup_{V \in f_!(\mathcal{U})} V = \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$
$$= \bigcup_{U \in \mathcal{U}} f_!(U).$$

This finishes the proof.

Item 4, *Interaction With Intersections of Families of Subsets*: We have

$$\bigcap_{V \in f_!(\mathcal{U})} V = \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$
$$= \bigcap_{U \in \mathcal{U}} f_!(U).$$

This finishes the proof.

Item 5, *Interaction With Binary Unions*: See [Pro25p].

Item 6, *Interaction With Binary Intersections*: See [Pro25n].

Item 7, *Interaction With Differences*: See [Pro25o].

Item 8, *Interaction With Complements*: Applying Item 17 to $X \setminus U$, we have

$$\begin{aligned}
f_!(U^{\mathsf{c}}) &= f_!(X \setminus U) \\
&= Y \setminus f_*(X \setminus (X \setminus U)) \\
&= Y \setminus f_*(U) \\
&= f_*(U)^{\mathsf{c}}.
\end{aligned}$$

This finishes the proof.

Item 9, *Interaction With Symmetric Differences*: We have

$$\begin{aligned}
f_!(U) \bigtriangleup f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\
&\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\
&= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\
&\subset f_!((U \cup V) \setminus (U \cap V)) \\
&= f_!(U \bigtriangleup V),
\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.

2. Item 6 of this proposition together with Item 1 of Definition 4.3.10.1.2 for the first inclusion.

3. Item 5 for the second equality.

4. Item 7 for the second inclusion.

5. Item 2 of Definition 4.3.12.1.2 for the tchird equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
Item 10, Interaction With Internal Homs of Powersets: We have

$$
\begin{aligned}
f_!([U, V]_X) &\overset{\text{def}}{=} f_!(U^{\mathsf{c}} \cup V) \\
&= f_!(U^{\mathsf{c}}) \cup f_!(V) \\
&= f_*(U)^{\mathsf{c}} \cup f_!(V) \\
&\overset{\text{def}}{=} [f_*(U), f_!(V)]_Y,
\end{aligned}
$$

where we have used:

1. Item 5 for the second equality.

2. Item 17 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
Item 11, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.[34]
Item 12, Oplax Preservation of Limits: The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25n] for the other inclusions.
Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.
Item 14, Symmetric Oplax Monoidality With Respect to Intersections: The inclusions in the statement follow from Item 12. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
Item 15, Interaction With Coproducts: Omitted.
Item 16, Interaction With Products: Omitted.
Item 17, Relation to Codirect Images: Applying Item 16 of Definition 4.6.3.1.7 to $X \setminus U$,

---

[34]*Reference*: [Pro25p].

we have

$$f_*(X \setminus U) = B \setminus f_!(X \setminus (X \setminus U))$$
$$= B \setminus f_!(U).$$

Taking complements, we then obtain

$$f_!(U) = B \setminus (B \setminus f_!(U)),$$
$$= B \setminus f_*(X \setminus U),$$

which finishes the proof.   □

Proposition 4.6.1.1.6. Let $f \colon X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subseteq), (\mathcal{P}(Y), \subseteq)).$$

3. *Interaction With Identities.* For each $X \in \mathsf{Obj}(\mathsf{Sets})$, we have

$$(\mathsf{id}_X)_! = \mathsf{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon X \to Y$ and $g \colon Y \to Z$, we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$\mathcal{P}(X) \xrightarrow{f_!} \mathcal{P}(Y)$$
$$\underset{(g \circ f)_!}{\searrow} \quad \downarrow g_!$$
$$\mathcal{P}(Z).$$

Proof. Item 1, *Functionality I*: There is nothing to prove.
Item 2, *Functionality II*: This follows from Item 1 of Definition 4.6.1.1.5.
Item 3, *Interaction With Identities*: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**.   □

4.6.2   Inverse Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.2.1.1. The **inverse image function associated to** f is the function[35]

$$f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[36]

$$f^{-1}(V) \overset{\mathrm{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 4.6.2.1.2. Identifying $\mathcal{P}(Y)$ with $\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by

$$f^*(\chi_V) \overset{\mathrm{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ \chi_V\ } \{\text{true}, \text{false}\}$$

in Sets.

Proposition 4.6.2.1.3. Let $f\colon X \to Y$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

    $$f^{-1}\colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset).$$

    In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

    ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

---

[35] *Further Notation:* Also written $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$.
[36] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_*\right): \quad \mathcal{P}(X) \overset{f_!}{\underset{\scriptstyle\perp}{\longleftarrow}} f^{-1} \overset{\scriptstyle\perp}{\underset{f_*}{\longrightarrow}} \mathcal{P}(Y),$$

witnessed by:

(a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \hookrightarrow f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \hookrightarrow f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \hookrightarrow \mathrm{id}_{\mathcal{P}(X)},$$

having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*(f^{-1}(V)),$$
$$f_!(f^{-1}(V)) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

  i. The following conditions are equivalent:
    A. We have $f_!(U) \subset V$.
    B. We have $U \subset f^{-1}(V)$.
  ii. The following conditions are equivalent:
    A. We have $f^{-1}(U) \subset V$.
    B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
\Big\downarrow{\scriptstyle\cup} & & \Big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{(f^{-1})^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap}\downarrow & & \downarrow{\scriptstyle\cap} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

5. *Interaction With Binary Unions.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle\cup}\downarrow & & \downarrow{\scriptstyle\cup} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

6. *Interaction With Binary Intersections.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle\cap}\downarrow & & \downarrow{\scriptstyle\cap} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\mathsf{op},-1} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle \backslash} \downarrow & & \downarrow {\scriptstyle \backslash} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} & \xrightarrow{f^{-1,\mathsf{op}}} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\mathsf{op},-1} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle \triangle} \downarrow & & \downarrow {\scriptstyle \triangle} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

i.e. we have

$$f^{-1}(U) \vartriangle f^{-1}(V) = f^{-1}(U \vartriangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1,\mathsf{op}} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle [-_1,-_2]_Y} \Big\downarrow & & \Big\downarrow {\scriptstyle [-_1,-_2]_X} \\
\mathcal{P}(Y) & \xrightarrow[\quad f^{-1} \quad]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i \right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V),$$
$$f^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(Y)$.

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i \right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V),$$
$$f^{-1}(Y) = X,$$

natural in $U, V \in \mathcal{P}(Y)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}) \colon (\mathcal{P}(Y), \cup, \emptyset) \to (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$

$$f_{\mathbb{1}}^{-1,\otimes} \colon \emptyset \xrightarrow{=} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(Y)$.

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}) \colon (\mathcal{P}(Y), \cap, Y) \to (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$

$$f_{\mathbb{1}}^{-1,\otimes} \colon X \xrightarrow{=} f^{-1}(Y),$$

natural in $U, V \in \mathcal{P}(Y)$.

15. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \amalg g)^{-1}(U' \amalg V') = f^{-1}(U') \amalg g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

16. *Interaction With Products.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

Proof. Item 1, *Functoriality*: Omitted.
Item 2, *Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.

Item 3, *Interaction With Unions of Families of Subsets*: We have

$$\bigcup_{U \in f^{-1}(\mathcal{V})} U = \bigcup_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U$$

$$= \bigcup_{V \in \mathcal{V}} f^{-1}(V).$$

This finishes the proof.

Item 4, *Interaction With Intersections of Families of Subsets*: We have

$$\bigcap_{U \in f^{-1}(\mathcal{V})} U = \bigcap_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U$$

$$= \bigcap_{V \in \mathcal{V}} f^{-1}(V).$$

This finishes the proof.

Item 5, *Interaction With Binary Unions*: See [Pro25y].

Item 6, *Interaction With Binary Intersections*: See [Pro25w].

Item 7, *Interaction With Differences*: See [Pro25x].

Item 8, *Interaction With Complements*: See [Pro25j].

Item 9, *Interaction With Symmetric Differences*: We have

$$
\begin{aligned}
f^{-1}(U \bigtriangleup V) &= f^{-1}((U \cup V) \setminus (U \cap V)) \\
&= f^{-1}(U \cup V) \setminus f^{-1}(U \cap V) \\
&= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U \cap V) \\
&= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U) \cap f^{-1}(V) \\
&= f^{-1}(U) \bigtriangleup f^{-1}(V),
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.

2. Item 7 for the second equality.

3. Item 5 for the third equality.

4. Item 6 for the fourth equality.

5. Item 2 of Definition 4.3.12.1.2 for the fifth equality.

This finishes the proof.

Item 10, Interaction With Internal Homs of Powersets: We have

$$f^{-1}([U, V]_Y) \overset{\mathrm{def}}{=} f^{-1}(U^{\mathsf{c}} \cup V)$$
$$= f^{-1}(U^{\mathsf{c}}) \cup f^{-1}(V)$$
$$= f^{-1}(U)^{\mathsf{c}} \cup f^{-1}(V)$$
$$\overset{\mathrm{def}}{=} [f^{-1}(U), f^{-1}(V)]_X,$$

where we have used:

1. Item 8 for the second equality.

2. Item 5 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 11, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.[37]

Item 12, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.[38]

Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.

Item 14, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 12.

Item 15, Interaction With Coproducts: Omitted.

Item 16, Interaction With Products: Omitted. $\square$

Proposition 4.6.2.1.4. Let $f: X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{X,Y}^{-1}: \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)_{X,Y}^{-1}: \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathsf{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_X^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

---

[37] *Reference:* [Pro25y].

[38] *Reference:* [Pro25w].

4. *Interaction With Composition.* For each pair of composable functions $f\colon X \to Y$ and $g\colon Y \to Z$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1},$$

$$
\begin{array}{ccc}
\mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y) \\
& \searrow_{(g\circ f)^{-1}} & \downarrow_{f^{-1}} \\
& & \mathcal{P}(X).
\end{array}
$$

Proof. *Item 1, Functionality I*: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.2.1.3.
Item 3, Interaction With Identities: This follows from Definition 4.6.2.1.2 and Categories, Item 5 of Definition 11.1.4.1.2.
Item 4, Interaction With Composition: This follows from Definition 4.6.2.1.2 and Categories, Item 2 of Definition 11.1.4.1.2. □

4.6.3   Codirect Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.3.1.1. The **codirect image function associated to** f is the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[39],[40]

$$
f_*(U) \overset{\text{def}}{=} \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right\}
$$
$$
= \left\{ y \in Y \;\middle|\; \text{we have } f^{-1}(y) \subset U \right\}
$$

for each $U \in \mathcal{P}(X)$.

---

[39] *Further Terminology:* The set $f_*(U)$ is called the **codirect image of** U **by** f.
[40] We also have

$$
f_*(U) = f_!(U^{\mathsf{c}})^{\mathsf{c}}
$$
$$
\overset{\text{def}}{=} Y \setminus f_!(X \setminus U);
$$

see Item 16 of Definition 4.6.3.1.7.

Notation 4.6.3.1.2. Sometimes one finds the notation

$$\forall_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.

- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.3.1.3. See Definition 4.6.1.1.3.

Remark 4.6.3.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the codirect image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$
\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\
&= \lim\left(((-_1) \overset{\rightarrow}{\times} f) \overset{\text{pr}}{\twoheadrightarrow} X \overset{\chi_U}{\longrightarrow} \{\text{true}, \text{false}\}\right) \\
&= \lim_{\substack{x \in X \\ f(x)=-_1}} (\chi_U(x)) \\
&= \bigwedge_{\substack{x \in X \\ f(x)=-_1}} (\chi_U(x)).
\end{aligned}
$$

where we have used **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](y) &= \bigwedge_{\substack{x \in X \\ f(x)=y}} (\chi_U(x)) \\
&= \begin{cases} \text{true} & \text{if, for each } x \in X \text{ such that} \\ & \quad f(x) = y, \text{ we have } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } f^{-1}(y) \subset U \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $y \in Y$.

Definition 4.6.3.1.5. Let U be a subset of X.[41,42]

1. The **image part of the codirect image** $f_*(U)$ **of** U is the set $f_{*,\mathrm{im}}(U)$ defined by

    $$f_{*,\mathrm{im}}(U) \overset{\mathrm{def}}{=} f_*(U) \cap \mathrm{Im}(f)$$
    $$= \left\{ y \in Y \;\middle|\; \begin{matrix} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{matrix} \right\}.$$

2. The **complement part of the codirect image** $f_*(U)$ **of** U is the set $f_{*,\mathrm{cp}}(U)$ defined by

    $$f_{*,\mathrm{cp}}(U) \overset{\mathrm{def}}{=} f_*(U) \cap (Y \setminus \mathrm{Im}(f))$$
    $$= Y \setminus \mathrm{Im}(f)$$
    $$= \left\{ y \in Y \;\middle|\; \begin{matrix} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{matrix} \right\}$$
    $$= \left\{ y \in Y \;\middle|\; f^{-1}(y) = \emptyset \right\}.$$

Example 4.6.3.1.6. Here are some examples of codirect images.

1. *Multiplication by Two.* Consider the function $f \colon \mathbb{N} \to \mathbb{N}$ given by

    $$f(n) \overset{\mathrm{def}}{=} 2n$$

---

[41] Note that we have

$$f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U),$$

as

$$\begin{aligned} f_*(U) &= f_*(U) \cap Y \\ &= f_*(U) \cap (\mathrm{Im}(f) \cup (Y \setminus \mathrm{Im}(f))) \\ &= (f_*(U) \cap \mathrm{Im}(f)) \cup (f_*(U) \cap (Y \setminus \mathrm{Im}(f))) \\ &\overset{\mathrm{def}}{=} f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U). \end{aligned}$$

[42] In terms of the meet computation of $f_*(U)$ of Definition 4.6.3.1.4, namely

$$f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$f_{*,\mathrm{im}}(U) = f_!(U)$$
$$f_{*,\mathrm{cp}}(U) = \{\text{odd natural numbers}\}$$

for any $U \subset \mathbb{N}$. In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

2. *Parabolas.* Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$f_{*,\mathrm{im}}([0,1]) = \{0\},$$
$$f_{*,\mathrm{im}}([-1,1]) = [0,1],$$
$$f_{*,\mathrm{im}}([1,2]) = \emptyset,$$
$$f_{*,\mathrm{im}}([-2,-1] \cup [1,2]) = [1,4].$$

3. *Circles.* Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$f_{*,\mathrm{im}}([-1,1] \times [-1,1]) = [0,1],$$
$$f_{*,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) = \emptyset.$$

Proposition 4.6.3.1.7. Let $f\colon X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

   ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_*\right)\colon \quad \mathcal{P}(X) \overset{f_!}{\underset{f_*}{\overset{\perp}{\underset{\perp}{\longleftarrow f^{-1} \longrightarrow}}}} \mathcal{P}(Y),$$

   witnessed by:

   (a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \hookrightarrow f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \hookrightarrow f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \hookrightarrow \mathrm{id}_{\mathcal{P}(X)},$$

   having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*(f^{-1}(V)),$$
$$f_!(f^{-1}(V)) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

   (b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

   natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

   i. The following conditions are equivalent:

       A. We have $f_!(U) \subset V$.

       B. We have $U \subset f^{-1}(V)$.

  ii. The following conditions are equivalent:

       A. We have $f^{-1}(U) \subset V$.

       B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$
\begin{CD}
\mathcal{P}(\mathcal{P}(X)) @>{(f_*)_*}>> \mathcal{P}(\mathcal{P}(Y)) \\
@V{\cup}VV @VV{\cup}V \\
\mathcal{P}(X) @>>{f_*}> \mathcal{P}(Y)
\end{CD}
$$

commutes, i.e. we have

$$
\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V
$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$
\begin{CD}
\mathcal{P}(\mathcal{P}(X)) @>{(f_*)_*}>> \mathcal{P}(\mathcal{P}(Y)) \\
@V{\cap}VV @VV{\cap}V \\
\mathcal{P}(X) @>>{f_*}> \mathcal{P}(Y)
\end{CD}
$$

commutes, i.e. we have

$$
\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V
$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

---

we see that $f_{*,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

5. *Interaction With Binary Unions.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cup \downarrow & \subset & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{ f_* } & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_*(U) \cup f_*(V) \subset f_*(U \cup V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{ f_* } & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
f_*(U) \cap f_*(V) = f_*(U \cap V)
$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_!^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
(-)^{\mathsf{c}} \downarrow & & \downarrow (-)^{\mathsf{c}} \\
\mathcal{P}(X) & \xrightarrow{ f_* } & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}
$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathsf{op}} \times f_*} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \triangle} \downarrow & \quad\supset\quad & \downarrow {\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow{\quad f_* \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_*(U \bigtriangleup V) \subset f_*(U) \bigtriangleup f_*(V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathsf{op}} \times f_*} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle [-_1,-_2]_X} \downarrow & \quad\supset\quad & \downarrow {\scriptstyle [-_1,-_2]_Y} \\
\mathcal{P}(X) & \xrightarrow{\quad f_* \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)
$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$
\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i \right),
$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$
f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V),
$$
$$
\emptyset \hookrightarrow f_*(\emptyset),
$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Limits.* We have an equality of sets

$$
f_*\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_*(U_i),
$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$
f^{-1}(U \cap V) = f_*(U) \cap f^{-1}(V),
$$
$$
f_*(X) = Y,
$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$
(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) : (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),
$$

being equipped with inclusions

$$
f_{*|U,V}^{\otimes} : f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V),
$$
$$
f_{*|\mathbb{1}}^{\otimes} : \emptyset \hookrightarrow f_*(\emptyset),
$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$
(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) : (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),
$$

being equipped with equalities

$$
f_{*|U,V}^{\otimes} : f_*(U \cap V) \xrightarrow{=} f_*(U) \cap f_*(V),
$$
$$
f_{*|\mathbb{1}}^{\otimes} : f_*(X) \xrightarrow{=} Y,
$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Interaction With Coproducts.* Let $f : X \to X'$ and $g : Y \to Y'$ be maps of sets. We have

$$
(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)
$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

15. *Interaction With Products.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_*(U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Relation to Direct Images.* We have

$$f_*(U) = f_!(U^c)^c$$
$$= Y \setminus f_!(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

17. *Interaction With Injections.* If f is injective, then we have

$$f_{*,\mathrm{im}}(U) = f_!(U),$$
$$f_{*,\mathrm{cp}}(U) = Y \setminus \mathrm{Im}(f),$$

and so

$$f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U)$$
$$= f_!(U) \cup (Y \setminus \mathrm{Im}(f))$$

for each $U \in \mathcal{P}(X)$.

18. *Interaction With Surjections.* If f is surjective, then we have

$$f_{*,\mathrm{im}}(U) \subset f_!(U),$$
$$f_{*,\mathrm{cp}}(U) = \emptyset,$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality:* Omitted.
Item 2, Triple Adjointness: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.

Item 3, Interaction With Unions of Families of Subsets: We have

$$\bigcup_{V \in f_*(\mathcal{U})} V = \bigcup_{V \in \{f_*(U) \in \mathcal{P}(X) \,|\, U \in \mathcal{U}\}} V$$
$$= \bigcup_{U \in \mathcal{U}} f_*(U).$$

This finishes the proof.

Item 4, Interaction With Intersections of Families of Subsets: We have

$$\bigcap_{V \in f_*(\mathcal{U})} V = \bigcap_{V \in \{f_*(U) \in \mathcal{P}(X) \,|\, U \in \mathcal{U}\}} V$$
$$= \bigcap_{U \in \mathcal{U}} f_*(U).$$

This finishes the proof.

Item 5, Interaction With Binary Unions: We have

$$\begin{aligned}
f_*(U) \cup f_*(V) &= f_!(U^\mathsf{c})^\mathsf{c} \cup f_!(V^\mathsf{c})^\mathsf{c} \\
&= (f_!(U^\mathsf{c}) \cap f_!(V^\mathsf{c}))^\mathsf{c} \\
&\subset (f_!(U^\mathsf{c} \cap V^\mathsf{c}))^\mathsf{c} \\
&= f_!((U \cup V)^\mathsf{c})^\mathsf{c} \\
&= f_*(U \cup V),
\end{aligned}$$

where:

1. We have used Item 16 for the first equality.

2. We have used Item 2 of Definition 4.3.11.1.2 for the second equality.

3. We have used Item 6 of Definition 4.6.1.1.5 for the third equality.

4. We have used Item 2 of Definition 4.3.11.1.2 for the fourth equality.

5. We have used Item 16 for the last equality.

This finishes the proof.
Item 6, Interaction With Binary Intersections: This follows from Item 11.
Item 7, Interaction With Complements: Omitted.
Item 8, Interaction With Symmetric Differences: Omitted.

Item 9, *Interaction With Internal Homs of Powersets*: We have

$$\left[f_!(U), f^!(V)\right]_X \overset{\text{def}}{=} f_!(U)^{\complement} \cup f_*(V)$$
$$= f_*(U^{\complement}) \cup f_*(V)$$
$$\subset f_*(U^{\complement} \cup V)$$
$$\overset{\text{def}}{=} f_*([U, V]_X),$$

where we have used:

1. Item 7 of Definition 4.6.3.1.7 for the second equality.

2. Item 5 of Definition 4.6.3.1.7 for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
This finishes the proof.

Item 10, *Lax Preservation of Colimits*: Omitted.

Item 11, *Preservation of Limits*: This follows from Item 2 and **??**, **??** of **??**.

Item 12, *Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 10.

Item 13, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 11.

Item 14, *Interaction With Coproducts*: Omitted.

Item 15, *Interaction With Products*: Omitted.

Item 16, *Relation to Direct Images*: We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication*. We claim that

$$f_*(U) \subset Y \setminus f_!(X \setminus U).$$

   Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.

   This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).

   Thus $y \in Y \setminus f_!(X \setminus U)$.

- *The Second Implication*. We claim that

$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$

   Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.

   Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.

   Thus $y \in f_*(U)$.

This finishes the proof of Item 16.
Item 17, Interaction With Injections: Omitted.
Item 18, Interaction With Surjections: Omitted.   □

Proposition 4.6.3.1.8. Let $f\colon X \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y}\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y}\colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon X \to Y$ and $g\colon Y \to Z$, we have

$$(g \circ f)_* = g_* \circ f_*,$$

$$
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\ f_*\ } & \mathcal{P}(Y) \\
& \searrow{\scriptstyle (g\circ f)_*} & \downarrow{\scriptstyle g_*} \\
& & \mathcal{P}(Z).
\end{array}
$$

Proof. *Item 1, Functionality I*: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.3.1.7.
Item 3, Interaction With Identities: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**.
Item 4, Interaction With Composition: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**.   □

4.6.4   A Six-Functor Formalism for Sets

Remark 4.6.4.1.1. The assignment $X \mapsto \mathcal{P}(X)$ together with the functors f_*, f^{-1}, and $f_!$ of Item 1 of Definition 4.6.1.1.5, Item 1 of Definition 4.6.2.1.3, and Item 1 of Definition 4.6.3.1.7, and the functors

$$-_1 \cap -_2\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X),$$

$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

of Item 1 of Definition 4.3.9.1.2 and Item 1 of Definition 4.4.7.1.3 satisfy several properties reminiscent of a six functor formalism in the sense of **??**.

We collect these properties in Definition 4.6.4.1.2 below.[43]

Proposition 4.6.4.1.2. Let X be a set.

1. *The Beck–Chevalley Condition.* Let

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\mathrm{pr}_2} & Y \\
\scriptstyle\mathrm{pr}_1 \downarrow & \lrcorner & \downarrow \scriptstyle g \\
X & \xrightarrow{f} & Z
\end{array}
$$

be a pullback diagram in Sets. We have

$$
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\mathrm{pr}_1^{-1}} & \mathcal{P}(X \times_Z Y) \\
\scriptstyle f_! \downarrow & & \downarrow \scriptstyle (\mathrm{pr}_2)_! \\
\mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y),
\end{array}
\qquad
\begin{aligned}
g^{-1} \circ f_! &= (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}, \\
f^{-1} \circ g_! &= (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1},
\end{aligned}
\qquad
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\mathrm{pr}_2^{-1}} & \mathcal{P}(X \times_Z Y) \\
\scriptstyle g_! \downarrow & & \downarrow \scriptstyle (\mathrm{pr}_1)_! \\
\mathcal{P}(Z) & \xrightarrow{f^{-1}} & \mathcal{P}(Y).
\end{array}
$$

2. *The Projection Formula I.* The diagram

$$
\begin{array}{ccccc}
& & \mathcal{P}(X) \times \mathcal{P}(X) & & \\
& \scriptstyle \mathrm{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \scriptstyle \cap & \\
\mathcal{P}(X) \times \mathcal{P}(Y) & & & & \mathcal{P}(X) \\
\scriptstyle f_! \times \mathrm{id}_{\mathcal{P}(Y)} \searrow & & & & \nearrow \scriptstyle f_! \\
& & \mathcal{P}(Y) \times \mathcal{P}(Y) \xrightarrow[\cap]{} \mathcal{P}(Y), & &
\end{array}
$$

commutes, i.e. we have

$$f_!(U \cap f^{-1}(V)) = f_!(U) \cap V$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

---

[43] See also [nLa25].

3. *The Projection Formula II.* We have a natural transformation

$$
\begin{array}{ccc}
 & \mathcal{P}(X) \times \mathcal{P}(X) & \\
\mathrm{id}_{\mathcal{P}(X)} \times f^{-1} \nearrow & & \searrow \cap \\
\mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
f_* \times \mathrm{id}_{\mathcal{P}(Y)} \searrow & \curvearrowleft & \downarrow f_* \\
 & \mathcal{P}(Y) \times \mathcal{P}(Y) \underset{\cap}{\to} \mathcal{P}(Y),
\end{array}
$$

   with components

$$
f_*(U) \cap V \subset f_*(U \cap f^{-1}(V))
$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

4. *Strong Closed Monoidality.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1,\mathsf{op}} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle [-_1,-_2]_Y} \downarrow & & \downarrow {\scriptstyle [-_1,-_2]_X} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

   commutes, i.e. we have an equality of sets

$$
f^{-1}([U,V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,
$$

   natural in $U, V \in \mathcal{P}(X)$.

5. *The External Tensor Product.* We have an external tensor product

$$
-_1 \boxtimes_{X \times Y} -_2 \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)
$$

   given by

$$
\begin{aligned}
U \boxtimes_{X \times Y} V &\overset{\mathrm{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\
&= \{(u,v) \in X \times Y \mid u \in U \text{ and } v \in V\}.
\end{aligned}
$$

   This is the same map as the one in Item 5 of Definition 4.4.1.1.4. Moreover, the following conditions are satisfied:

(a) *Interaction With Direct Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be
   functions. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_! \times g_!} & \mathcal{P}(X') \times \mathcal{P}(Y') \\
\Big\downarrow{\scriptstyle \boxtimes_{X \times Y}} & & \Big\downarrow{\scriptstyle \boxtimes_{X' \times Y'}} \\
\mathcal{P}(X \times Y) & \xrightarrow[f_! \times g_!]{} & \mathcal{P}(X' \times Y')
\end{array}
$$

   commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

   for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(b) *Interaction With Inverse Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be
   functions. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X') \times \mathcal{P}(Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X) \times \mathcal{P}(Y) \\
\Big\downarrow{\scriptstyle \boxtimes_{X' \times Y'}} & & \Big\downarrow{\scriptstyle \boxtimes_{X \times Y}} \\
\mathcal{P}(X' \times Y') & \xrightarrow[f^{-1} \times g^{-1}]{} & \mathcal{P}(X \times Y)
\end{array}
$$

   commutes, i.e. we have

$$[f^{-1} \times g^{-1}](U \boxtimes_{X' \times Y'} V) = f^{-1}(U) \boxtimes_{X \times Y} g^{-1}(V)$$

   for each $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$.

(c) *Interaction With Codirect Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be
   functions. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\
\Big\downarrow{\scriptstyle \boxtimes_{X \times Y}} & & \Big\downarrow{\scriptstyle \boxtimes_{X' \times Y'}} \\
\mathcal{P}(X \times Y) & \xrightarrow[f_* \times g_*]{} & \mathcal{P}(X' \times Y')
\end{array}
$$

   commutes, i.e. we have

$$[f_* \times g_*](U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

   for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(d) *Interaction With Diagonals.* The diagram

$$\mathcal{P}(X) \times \mathcal{P}(X) \xrightarrow{\boxtimes_{X \times X}} \mathcal{P}(X \times X)$$

$$\cap \searrow \quad \downarrow \Delta_X^{-1}$$

$$\mathcal{P}(X),$$

i.e. we have

$$U \cap V = \Delta_X^{-1}(U \boxtimes_{X \times X} V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *The Dualisation Functor.* We have a functor

$$D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

given by

$$D_X(U) \overset{\text{def}}{=} [U, \emptyset]_X$$
$$\overset{\text{def}}{=} U^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$, as in Item 5 of Definition 4.4.7.1.3, satisfying the following conditions:

(a) *Duality.* We have

$$D_X(D_X(U)) = U, \qquad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{D_X} & \mathcal{P}(X) \\ & \mathsf{id}_{\mathcal{P}(X)} \searrow & \downarrow D_X \\ & & \mathcal{P}(X). \end{array}$$

(b) *Duality.* The diagram

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \xrightarrow{\cap^{\mathsf{op}}} \mathcal{P}(X)^{\mathsf{op}}$$

$$\mathsf{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X \nearrow \qquad\qquad \searrow D_X$$

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \xrightarrow[{[-_1, -_2]_X}]{} \mathcal{P}(X)$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\text{def}}{=}[U\cap[V,\emptyset]_X,\emptyset]_X} = [U,V]_X$$

for each $U, V \in \mathcal{P}(X)$.

(c) *Interaction With Direct Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_*^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle D_X}\downarrow & & \downarrow{\scriptstyle D_Y} \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(d) *Interaction With Inverse Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} & \xrightarrow{f^{-1,\mathsf{op}}} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle D_Y}\downarrow & & \downarrow{\scriptstyle D_X} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

(e) *Interaction With Codirect Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_!^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle D_X}\downarrow & & \downarrow{\scriptstyle D_Y} \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

Proof. Item 1, *The Beck–Chevalley Condition*: We have

$$
\begin{aligned}
[g^{-1} \circ f_!](U) &\overset{\text{def}}{=} g^{-1}(f_!(U)) \\
&\overset{\text{def}}{=} \{y \in Y \mid g(y) \in f_!(U)\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } y = y \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \{(x, y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } \mathrm{pr}_2(x, y) = y \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\{(x, y) \in X \times_Z Y \mid x \in U\}) \\
&= (\mathrm{pr}_2)_!(\{(x, y) \in X \times_Z Y \mid \mathrm{pr}_1(x, y) \in U\}) \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\mathrm{pr}_1^{-1}(U)) \\
&\overset{\text{def}}{=} [(\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$g^{-1} \circ f_! = (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}.$$

For the second equality, we have

$$
\begin{aligned}
[f^{-1} \circ g_!](U) &\overset{\text{def}}{=} f^{-1}(g_!(U)) \\
&\overset{\text{def}}{=} \{x \in X \mid f(x) \in g_!(V)\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right\}
\end{aligned}
$$

$$= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid y \in V\} \end{array} \right\}$$

$$= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } x = x \end{array} \right\}$$

$$= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } \mathrm{pr}_1(x,y) = x \end{array} \right\}$$

$$\overset{\mathrm{def}}{=} (\mathrm{pr}_1)_!(\{(x,y) \in X \times_Z Y \mid y \in V\})$$

$$= (\mathrm{pr}_1)_!(\{(x,y) \in X \times_Z Y \mid \mathrm{pr}_2(x,y) \in V\})$$

$$\overset{\mathrm{def}}{=} (\mathrm{pr}_1)_!(\mathrm{pr}_2^{-1}(V))$$

$$\overset{\mathrm{def}}{=} [(\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}](V)$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$f^{-1} \circ g_! = (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}.$$

This finishes the proof.

Item 2, *The Projection Formula I*: We claim that

$$f_!(U) \cap V \subset f_!(U \cap f^{-1}(V)).$$

Indeed, we have

$$f_!(U) \cap V \subset f_!(U) \cap f_!(f^{-1}(V))$$
$$= f_!(U \cap f^{-1}(V)),$$

where we have used:

1. Item 2 of Definition 4.6.1.1.5 for the inclusion.

2. Item 6 of Definition 4.6.1.1.5 for the equality.

Conversely, we claim that

$$f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V.$$

Indeed:

1. Let $y \in f_!(U \cap f^{-1}(V))$.

2. Since $y \in f_!(U \cap f^{-1}(V))$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.

3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.

4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.

5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.

6. But $y = f(x)$, so $y \in f_!(U) \cap V$.

7. Thus $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$.

This finishes the proof.
Item 3, The Projection Formula II: We have

$$f_*(U) \cap V \subset f_*(U) \cap f_*(f^{-1}(V))$$
$$= f_*(U \cap f^{-1}(V)),$$

where we have used:

1. Item 2 of Definition 4.6.3.1.7 for the inclusion.

2. Item 6 of Definition 4.6.3.1.7 for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
Item 4, Strong Closed Monoidality: This is a repetition of Item 19 of Definition 4.4.7.1.3 and is proved there.
Item 5, The External Tensor Product: We have

$$
\begin{aligned}
U \boxtimes_{X \times Y} V &\overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\
&\overset{\text{def}}{=} \{(x, y) \in X \times Y \mid \mathrm{pr}_1(x, y) \in U\} \\
&\quad \cup \{(x, y) \in X \times Y \mid \mathrm{pr}_2(x, y) \in V\} \\
&= \{(x, y) \in X \times Y \mid x \in U\} \\
&\quad \cup \{(x, y) \in X \times Y \mid y \in V\} \\
&= \{(x, y) \in X \times Y \mid x \in U \text{ and } y \in V\} \\
&\overset{\text{def}}{=} U \times V.
\end{aligned}
$$

Next, we claim that Items 5a to 5d are indeed true:

1. *Proof of Item 5a:* This is a repetition of Item 16 of Definition 4.6.1.1.5 and is proved there.

2. *Proof of Item 5b:* This is a repetition of Item 16 of Definition 4.6.2.1.3 and is proved there.

3. *Proof of Item 5c:* This is a repetition of Item 15 of Definition 4.6.3.1.7 and is proved there.

4. *Proof of Item 5d:* We have

$$
\begin{aligned}
\Delta_X^{-1}(U \boxtimes_{X \times X} V) &\overset{\text{def}}{=} \{x \in X \mid (x, x) \in U \boxtimes_{X \times X} V\} \\
&= \{x \in X \mid (x, x) \in \{(u, v) \in X \times X \mid u \in U \text{ and } v \in V\}\} \\
&= U \cap V.
\end{aligned}
$$

This finishes the proof.

Item 6, The Dualisation Functor: This is a repetition of Items 5 and 6 of Definition 4.4.7.1.3 and is proved there. $\qquad\qquad\square$

4.7 Miscellany

4.7.1 Injective Functions

Let A and B be sets.

Definition 4.7.1.1.1. A function $f\colon A \to B$ is **injective** if it satisfies the following condition:

($\star$) For each $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

Proposition 4.7.1.1.2. Let $f\colon A \to B$ be a function.

1. *Characterisations.* The following conditions are equivalent:[44]

---

[44] Items 1c to 1f unwind respectively to the following statements:

· For each $U, V \in \mathcal{P}(A)$, if $f_!(U) = f_!(V)$, then $U = V$.

· For each $U, V \in \mathcal{P}(A)$, if $f_*(U) = f_*(V)$, then $U = V$.

· For each $U, V \in \mathcal{P}(A)$, if $f_!(U) \subset f_!(V)$, then $U \subset V$.

· For each $U, V \in \mathcal{P}(A)$, if $f_*(U) \subset f_*(V)$, then $U \subset V$.

(a) The function f is injective.

(b) The function f is a monomorphism in Sets.

(c) The direct image function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to f is injective.

(d) The codirect image function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to f is injective.

(e) The direct image functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

associated to f is full.

(f) The codirect image function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to f is full.

(g) The diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
& \chi_A \searrow & \Big\downarrow f^{-1} \\
& & \mathcal{P}(A)
\end{array}
$$

commutes. That is, we have

$$f^{-1}(f(a)) = \{a\}$$

for each $a \in A$.

(h) We have

$$f^{-1} \circ f_! = \mathrm{id}_{\mathcal{P}(A)}$$

$$
\begin{array}{ccc}
\mathcal{P}(A) & \xrightarrow{\;f_!\;} & \mathcal{P}(B) \\
& \diagdown & \Big\downarrow f^{-1} \\
& & \mathcal{P}(A).
\end{array}
$$

In other words, we have

$$\{a \in A \mid f(a) \in f(U)\} = U$$

for each $U \in \mathcal{P}(A)$.

(i) We have

$$f^{-1} \circ f_* = \mathrm{id}_{\mathcal{P}(A)}$$

$$\begin{array}{ccc}
\mathcal{P}(A) & \xrightarrow{f_*} & \mathcal{P}(B) \\
 & & \Big\downarrow{f^{-1}} \\
 & & \mathcal{P}(A).
\end{array}$$

In other words, we have

$$\left\{a \in A \mid f^{-1}(f(a)) \subset U\right\} = U$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Characterisations*: We will proceed by showing:

- Step 1: Item 1a $\iff$ Item 1b.

- Step 2: Item 1a $\iff$ Item 1c.

- Step 3: Item 1a $\iff$ Item 1d.

- Step 4: Item 1c $\iff$ Item 1e.

- Step 5: Item 1e $\iff$ Item 1f.

- Step 6: Item 1a $\iff$ Item 1g.

- Step 7: Item 1g $\iff$ Item 1h.

- Step 8: Item 1a $\iff$ Item 1i.

Step 1: Item 1a $\iff$ Item 1b. We claim that Items 1a and 1b are equivalent:

- *Item 1a $\implies$ Item 1b*: We proceed in a few steps:

    − Proceeding by contrapositive, we claim that given a pair of maps $g, h \colon C \rightrightarrows A$ such that $g \neq h$, we have $f \circ g \neq f \circ h$.

- Indeed, as g and h are different maps, there must exist at least one element $x \in C$ such that $g(x) \neq h(x)$.
- But then we have $f(g(x)) \neq f(h(x))$, as f is injective.
- Thus $f \circ g \neq f \circ h$, and we are done.

· *Item 1b $\implies$ Item 1a*: We proceed in a few steps:

  - Consider the diagram

  $$\mathrm{pt} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{\ f\ } B,$$

  where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A.
  - Note that we have $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$.
  - Since f is assumed to be a monomorphism, if $f(x) = f(y)$, then $f \circ [x] = f \circ [y]$ and therefore $[x] = [y]$.
  - This shows that if $f(x) = f(y)$, then $x = y$, so f is injective.

Step 2: Item 1a $\iff$ Item 1c. We claim that Items 1a and 1c are indeed equivalent:

· *Item 1a $\implies$ Item 1c*: We proceed in a few steps:

  - Assume that f is injective and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$. We wish to show that $U = V$.
  - To show that $U \subset V$, let $u \in U$.
  - By the definition of the direct image, we have $f(u) \in f_!(U)$.
  - Since $f_!(U) = f_!(V)$, it follows that $f(u) \in f_!(V)$.
  - Thus, there exists some $v \in V$ such that $f(v) = f(u)$.
  - Since f is injective, the equality $f(v) = f(u)$ implies that $v = u$.
  - Thus $u \in V$ and $U \subset V$.
  - A symmetric argument shows that $V \subset U$.
  - Therefore $U = V$, showing $f_!$ to be injective.

· *Item 1c $\implies$ Item 1a*: We proceed in a few steps:

— Assume that the direct image function $f_!$ is injective and let $a, a' \in A$ such that $f(a) = f(a')$. We wish to show that $a = a'$.

— Since

$$f_!(\{a\}) = \{f(a)\}$$
$$= \{f(a')\}$$
$$= f_!(\{a'\}),$$

we must have $\{a\} = \{a'\}$, as $f_!$ is injective, so $a = a'$, showing f to be injective.

Step 3: Item 1c $\Longleftrightarrow$ Item 1d. This follows from Item 17 of Definition 4.6.1.1.5.

Step 4: Item 1c $\Longleftrightarrow$ Item 1e. We claim that Items 1c and 1e are equivalent:

· *Item 1c $\Longrightarrow$ Item 1e*: We proceed in a few steps:

— Let $U, V \in \mathcal{P}(A)$ such that $f_!(U) \subset f_!(V)$, assume $f_!$ to be injective, and consider the set $U \cup V$.

— Since $f_!(U) \subset f_!(V)$, we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$
$$= f_!(V),$$

where we have used Item 5 of Definition 4.6.1.1.5 for the first equality.

— Since $f_!$ is injective, this implies $U \cup V = V$.

— Thus $U \subset V$, as we wished to show.

· *Item 1c $\Longrightarrow$ Item 1e*: We proceed in a few steps:

— Suppose Item 1e holds, and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$.

— Since $f_!(U) = f_!(V)$, we have $f_!(U) \subset f_!(V)$ and $f_!(V) \subset f_!(U)$.

— By assumption, this implies $U \subset V$ and $V \subset U$.

— Thus $U = V$, showing $f_!$ to be injective.

Step 5: Item 1e $\Longleftrightarrow$ Item 1f. This follows from Item 17 of Definition 4.6.1.1.5.

Step 6: Item 1a $\Longleftrightarrow$ Item 1g. We have

$$f^{-1}(f(a)) = \{a' \in A \mid f(a') = f(a)\}$$

so the condition $f^{-1}(f(a)) = \{a\}$ states precisely that if $f(a') = f(a)$, then $a' = a$.

Step 7: Item 1g $\Longleftrightarrow$ Item 1h. We claim that Items 1g and 1h are indeed equivalent:

· *Item 1g* $\Longrightarrow$ *Item 1h:* We have

$$[f^{-1} \circ f_!](U) \overset{\text{def}}{=} f^{-1}(f_!(U))$$

$$= f^{-1}\left(f_!\left(\bigcup_{u \in U}\{u\}\right)\right)$$

$$= f^{-1}\left(\bigcup_{u \in U} f_!(\{u\})\right)$$

$$= \bigcup_{u \in U} f^{-1}(f_!(\{u\}))$$

$$= \bigcup_{u \in U} f^{-1}(f_!(u))$$

$$= \bigcup_{u \in U}\{u\}$$

$$= U$$

for each $U \in \mathcal{P}(A)$, where we have used Item 5 of Definition 4.6.1.1.5 for the third equality and Item 5 of Definition 4.6.2.1.3 for the fourth equality.

· *Item 1h* $\Longrightarrow$ *Item 1g:* Applying the condition $f^{-1} \circ f_! = \text{id}_{\mathcal{P}(A)}$ to $U = \{a\}$ gives

$$f^{-1}(f_!(\{a\})) = \{a\}.$$

Step 8: Item 1a $\Longleftrightarrow$ Item 1i. We claim that Items 1a and 1i are equivalent:

· *Item 1a* $\Longrightarrow$ *Item 1i:* If f is injective, then $f^{-1}(f(a)) = \{a\}$, so we have

$$f^{-1}(f_*(a)) = \{a \in A \mid \{a\} \subset U\}$$
$$= U.$$

· *Item 1i* $\Longrightarrow$ *Item 1a:* For $U = \{a\}$, the condition $f^{-1}(f_*(U)) = U$ becomes

$$\left\{a' \in A \mid f^{-1}(f(a')) \subset \{a\}\right\} = \{a\}.$$

Since the set $f^{-1}(f(a'))$ is given by

$$\{a \in A \mid f(a) = f(a')\},$$

it follows that f is injective.

This finishes the proof. $\square$

4.7.2   Surjective Functions

Let A and B be sets.

Definition 4.7.2.1.1. A function $f: A \to B$ is **surjective** if it satisfies the following condition:

($\star$)   For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Proposition 4.7.2.1.2. Let $f: A \to B$ be a function.

1. *Characterisations.* The following conditions are equivalent:

   (a) The function f is surjective.

   (b) The function f is an epimorphism in Sets.

   (c) The inverse image function

   $$f^{-1}: \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to f is injective.

   (d) The inverse image functor

   $$f^{-1}: (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   associated to f is full.

   (e) The diagram

   $$B \xrightarrow{\ f^{-1}\ } \mathcal{P}(A)$$

   with χ_B and $f_!$ to $\mathcal{P}(B)$

   commutes. That is, we have

   $$f_!(f^{-1}(b)) = \{b\}$$

   for each $b \in B$.

(f) We have

$$P(B) \xrightarrow{f^{-1}} P(A)$$

$$f_! \circ f^{-1} = \mathrm{id}_{P(B)}$$

$$\downarrow f_!$$

$$P(B).$$

In other words, we have

$$\left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in f^{-1}(U) \\ \text{such that } f(a) = b \end{array} \right\} = U$$

for each $U \in P(A)$.

(g) We have

$$P(B) \xrightarrow{f^{-1}} P(A)$$

$$f_* \circ f^{-1} = \mathrm{id}_{P(B)}$$

$$\downarrow f_*$$

$$P(B).$$

In other words, we have

$$\left\{ b \in B \,\middle|\, f^{-1}(b) \subset f^{-1}(U) \right\} = U$$

for each $U \in P(B)$.

Proof. *Item 1, Characterisations*: We will proceed by showing:

· Step 1: Item 1a $\Longleftrightarrow$ Item 1b.

· Step 2: Item 1a $\Longleftrightarrow$ Item 1c.

· Step 3: Item 1c $\Longleftrightarrow$ Item 1d.

· Step 4: Item 1a $\Longleftrightarrow$ Item 1e.

· Step 5: Item 1e $\Longleftrightarrow$ Item 1f.

· Step 6: Item 1a $\Longleftrightarrow$ Item 1g.

Step 1: Item 1a ⟺ Item 1b. We claim Items 1a and 1b are indeed equivalent:

- *Item 1a ⟹ Item 1b:* We proceed in a few steps:

  - Let $g, h\colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$.
  - For each $a \in A$, we have

  $$g(f(a)) = h(f(a)).$$

  - However, this implies that

  $$g(b) = h(b)$$

  for each $b \in B$, as f is surjective.

  - Thus $g = h$ and f is an epimorphism.

- *Item 1b ⟹ Item 1a:* We proceed by contrapositive. Consider the diagram

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$

  where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by

  $$g(b) = \begin{cases} 1 & \text{if } b \in \mathsf{Im}(f), \\ 0 & \text{otherwise.} \end{cases}$$

  Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathsf{Im}(f)$, we have

  $$g(b) = 0 \neq 1 = h(b).$$

  Therefore $g \neq h$ and f is not an epimorphism.

Step 2: Item 1a ⟺ Item 1c. We claim Items 1a and 1c are indeed equivalent:

- *Item 1a ⟹ Item 1c:* We proceed in a few steps:

  - Assume that f is surjective. Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$. We wish to show that $U = V$.
  - To show that $U \subset V$, let $b \in U$.
  - Since f is surjective, there must exist some $a \in A$ such that $f(a) = b$.

- By the definition of the inverse image, since $f(a) = b$ and $b \in U$, we have $a \in f^{-1}(U)$.

- By our initial assumption, $f^{-1}(U) = f^{-1}(V)$, so it follows that $a \in f^{-1}(V)$.

- Again, by the definition of the inverse image, $a \in f^{-1}(V)$ means that $f(a) \in V$.

- Since $f(a) = b$, we have shown that $b \in V$.

- This establishes that $U \subset V$. A symmetric argument shows that $V \subset U$.

- Thus $U = V$, proving that f^{-1} is injective.

· *Item 1c* $\implies$ *Item 1a*: We proceed in a few steps:

- Assume that the inverse image function f^{-1} is injective. Suppose, for the sake of contradiction, that f is not surjective.

- The assumption that f is not surjective means there exists some $b_0 \in B$ such that for all $a \in A$, we have $f(a) \neq b_0$.

- By the definition of the inverse image, this is equivalent to stating that $f^{-1}(\{b_0\}) = \emptyset$.

- Since $f^{-1}(\emptyset) = \emptyset$, we have $f^{-1}(\{b_0\}) = f^{-1}(\emptyset)$.

- Since f^{-1} is injective, this implies that $\{b_0\} = \emptyset$.

- This is a contradiction, as the singleton set $\{b_0\}$ is non-empty.

- Therefore, f is surjective.

Step 3: Item 1c $\iff$ Item 1d. We claim that Items 1c and 1d are equivalent:

· *Item 1c* $\implies$ *Item 1d*: We proceed in a few steps:

- Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) \subset f^{-1}(V)$, assume f^{-1} to be injective, and consider the set $U \cup V$.

- Since $f^{-1}(U) \subset f^{-1}(V)$, we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$
$$= f^{-1}(V),$$

where we have used Item 5 of Definition 4.6.2.1.3 for the first equality.

- Since f^{-1} is injective, this implies $U \cup V = V$.

- Thus $U \subset V$, as we wished to show.

· *Item 1d $\Longrightarrow$ Item 1c*: We proceed in a few steps:

- Suppose Item 1d holds, and let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$.

- Since $f^{-1}(U) = f^{-1}(V)$, we have $f^{-1}(U) \subset f^{-1}(V)$ and $f^{-1}(V) \subset f^{-1}(U)$.

- By assumption, this implies $U \subset V$ and $V \subset U$.

- Thus $U = V$, showing f^{-1} to be injective.

Step 4: Item 1a $\Longleftrightarrow$ Item 1e. We have

$$f_!(f^{-1}(b)) = \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in f^{-1}(b) \\ \text{such that } f(a) = b \end{array} \right\},$$

so the condition $f_!(f^{-1}(b)) = \{b\}$ holds iff f is surjective.

Step 5: Item 1e $\Longleftrightarrow$ Item 1f. We claim that Items 1e and 1f are indeed equivalent:

· *Item 1e $\Longrightarrow$ Item 1f*: We have

$$[f_! \circ f^{-1}](U) \overset{\text{def}}{=} f_!(f^{-1}(U))$$

$$= f_!\left(f^{-1}\left(\bigcup_{u \in U} \{u\}\right)\right)$$

$$= f_!\left(\bigcup_{u \in U} f^{-1}(\{u\})\right)$$

$$= \bigcup_{u \in U} f_!(f^{-1}(\{u\}))$$

$$= \bigcup_{u \in U} f_!(f^{-1}(u))$$

$$= \bigcup_{u \in U} \{u\}$$

$$= U$$

for each $U \in \mathcal{P}(B)$, where we have used Item 5 of Definition 4.6.1.1.5 for the third equality and Item 5 of Definition 4.6.2.1.3 for the fourth equality.

· *Item 1f* $\implies$ *Item 1e:* Applying the condition $f_! \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)}$ to $U = \{b\}$ gives

$$f_!(f^{-1}(\{b\})) = \{b\}.$$

Step 6: Item 1a $\iff$ Item 1g. First, note that for the condition $f^{-1}(b) \subset f^{-1}(U)$ to hold, we must have $b \in U$ or $f^{-1}(b) = \emptyset$. Thus

$$f_*(f^{-1}(U)) = (U \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)).$$

We now claim that Items 1a and 1g are indeed equivalent:

· *Item 1a* $\implies$ *Item 1g:* If f is surjective, we have

$$\begin{aligned}
(U \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)) &= U \cup \emptyset \\
&= U,
\end{aligned}$$

so $f_* \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)}$.

· *Item 1g* $\implies$ *Item 1a:* Taking $U = \emptyset$ gives

$$\begin{aligned}
f_*(f^{-1}(\emptyset)) &= (\emptyset \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)) \\
&= B \setminus \mathrm{Im}(f),
\end{aligned}$$

so the condition $f_*(f^{-1}(\emptyset)) = \emptyset$ implies $B \setminus \mathrm{Im}(f) = \emptyset$. Thus $\mathrm{Im}(f) = B$ and f is surjective.

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Appendices

A   Other Chapters

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