

# Monoidal Structures on the Category of Sets

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This chapter contains some material on monoidal structures on Sets.

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## 5.1 The Monoidal Category of Sets and Products

### 5.1.1 Products of Sets

See [Constructions With Sets, Section 4.1.3](#).

### 5.1.2 The Internal Hom of Sets

See [Constructions With Sets, Section 4.3.5](#).

### 5.1.3 The Monoidal Unit

**Definition 5.1.3.1.1.** The **monoidal unit of the product of sets** is the functor

$$\mathbb{1}^{\text{Sets}} : \text{pt} \rightarrow \text{Sets}$$

defined by

$$\mathbb{1}_{\text{Sets}} \stackrel{\text{def}}{=} \text{pt},$$

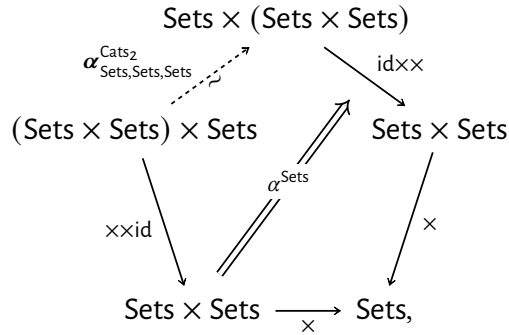
where pt is the terminal set of [Constructions With Sets, Definition 4.1.1.1.1](#).

### 5.1.4 The Associator

**Definition 5.1.4.1.1.** The **associator of the product of sets** is the natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2},$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\text{Sets}}: (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at  $(X, Y, Z)$  is given by

$$\alpha_{X,Y,Z}^{\text{Sets}}((x, y), z) \stackrel{\text{def}}{=} (x, (y, z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

*Proof. Invertibility:* The inverse of  $\alpha_{X,Y,Z}^{\text{Sets}}$  is the morphism

$$\alpha_{X,Y,Z}^{\text{Sets},-1}: X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \stackrel{\text{def}}{=} ((x, y), z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[ \alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} \right]((x, y), z) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets},-1} \left( \alpha_{X,Y,Z}^{\text{Sets}}((x, y), z) \right) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \\ &\stackrel{\text{def}}{=} ((x, y), z) \\ &\stackrel{\text{def}}{=} [\text{id}_{(X \times Y) \times Z}]((x, y), z) \end{aligned}$$

for each  $((x, y), z) \in (X \times Y) \times Z$ , and therefore we have

$$\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} = \text{id}_{(X \times Y) \times Z}.$$

- *Invertibility II.* We have

$$\begin{aligned}
 \left[ \alpha_{X,Y,Z}^{\text{Sets}} \circ \alpha_{X,Y,Z}^{\text{Sets},-1} \right] (x, (y, z)) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets}} \left( \alpha_{X,Y,Z}^{\text{Sets},-1} (x, (y, z)) \right) \\
 &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets}} ((x, y), z) \\
 &\stackrel{\text{def}}{=} (x, (y, z)) \\
 &\stackrel{\text{def}}{=} [\text{id}_{(X \times Y) \times Z}] (x, (y, z))
 \end{aligned}$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ , and therefore we have

$$\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} = \text{id}_{X \times (Y \times Z)} .$$

Therefore  $\alpha_{X,Y,Z}^{\text{Sets}}$  is indeed an isomorphism.

*Naturality:* We need to show that, given functions

$$\begin{aligned}
 f &: X \rightarrow X', \\
 g &: Y \rightarrow Y', \\
 h &: Z \rightarrow Z'
 \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 (X \times Y) \times Z & \xrightarrow{(f \times g) \times h} & (X' \times Y') \times Z' \\
 \alpha_{X,Y,Z}^{\text{Sets}} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}} \\
 X \times (Y \times Z) & \xrightarrow{f \times (g \times h)} & X' \times (Y' \times Z')
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 ((x, y), z) & & ((x, y), z) \longmapsto ((f(x), g(y)), h(z)) \\
 \downarrow & & \downarrow \\
 (x, (y, z)) \longmapsto (f(x), (g(y), h(z))) & & (f(x), (g(y), h(z)))
 \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\alpha^{\text{Sets}}$  is natural and  $\alpha^{\text{Sets},-1}$  is a component-wise inverse to  $\alpha^{\text{Sets}}$ , it follows from **Categories, Item 2 of Definition 11.9.7.1.2** that  $\alpha^{\text{Sets},-1}$  is also natural. Thus  $\alpha^{\text{Sets}}$  is a natural isomorphism.  $\square$

### 5.1.5 The Left Unitor

**Definition 5.1.5.1.1.** The **left unitor of the product of sets** is the natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}} : \text{pt} \times X \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\lambda_X^{\text{Sets}}(\star, x) \stackrel{\text{def}}{=} x$$

for each  $(\star, x) \in \text{pt} \times X$ .

*Proof. Invertibility:* The inverse of  $\lambda_X^{\text{Sets}}$  is the morphism

$$\lambda_X^{\text{Sets}, -1} : X \xrightarrow{\sim} \text{pt} \times X$$

defined by

$$\lambda_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each  $x \in X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}, -1} \circ \lambda_X^{\text{Sets}}](\text{pt}, x) &= \lambda_X^{\text{Sets}, -1}(\lambda_X^{\text{Sets}}(\text{pt}, x)) \\ &= \lambda_X^{\text{Sets}, -1}(x) \\ &= (\star, x) \\ &= [\text{id}_{\text{pt} \times X}](\text{pt}, x) \end{aligned}$$

for each  $(\text{pt}, x) \in \text{pt} \times X$ , and therefore we have

$$\lambda_X^{\text{Sets}, -1} \circ \lambda_X^{\text{Sets}} = \text{id}_{\text{pt} \times X}.$$

- *Invertibility II.* We have

$$\begin{aligned}
 [\lambda_X^{\text{Sets}} \circ \lambda_X^{\text{Sets}, -1}](x) &= \lambda_X^{\text{Sets}}(\lambda_X^{\text{Sets}, -1}(x)) \\
 &= \lambda_X^{\text{Sets}, -1}(\text{pt}, x) \\
 &= x \\
 &= [\text{id}_X](x)
 \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\text{Sets}} \circ \lambda_X^{\text{Sets}, -1} = \text{id}_X.$$

Therefore  $\lambda_X^{\text{Sets}}$  is indeed an isomorphism.

*Naturality:* We need to show that, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc}
 \text{pt} \times X & \xrightarrow{\text{id}_{\text{pt}} \times f} & \text{pt} \times Y \\
 \lambda_X^{\text{Sets}} \downarrow & & \downarrow \lambda_Y^{\text{Sets}} \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (\star, x) & & (\star, x) \mapsto (\star, f(x)) \\
 \downarrow & & \downarrow \\
 x & \mapsto & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\text{Sets}}$  is a natural transformation.

*Being a Natural Isomorphism:* Since  $\lambda^{\text{Sets}}$  is natural and  $\lambda^{\text{Sets}, -1}$  is a component-wise inverse to  $\lambda^{\text{Sets}}$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\lambda^{\text{Sets}}$  is a natural isomorphism.  $\square$

### 5.1.6 The Right Unitor

**Definition 5.1.6.1.1.** The **right unitor of the product of sets** is the natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}} : X \times \text{pt} \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\rho_X^{\text{Sets}}(x, \star) \stackrel{\text{def}}{=} x$$

for each  $(x, \star) \in X \times \text{pt}$ .

*Proof. Invertibility:* The inverse of  $\rho_X^{\text{Sets}}$  is the morphism

$$\rho_X^{\text{Sets}, -1} : X \xrightarrow{\sim} X \times \text{pt}$$

defined by

$$\rho_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (x, \star)$$

for each  $x \in X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} [\rho_X^{\text{Sets}, -1} \circ \rho_X^{\text{Sets}}](x, \star) &= \rho_X^{\text{Sets}, -1}(\rho_X^{\text{Sets}}(x, \star)) \\ &= \rho_X^{\text{Sets}, -1}(x) \\ &= (x, \star) \\ &= [\text{id}_{X \times \text{pt}}](x, \star) \end{aligned}$$

for each  $(x, \star) \in X \times \text{pt}$ , and therefore we have

$$\rho_X^{\text{Sets}, -1} \circ \rho_X^{\text{Sets}} = \text{id}_{X \times \text{pt}}.$$

- *Invertibility II.* We have

$$\begin{aligned}
 [\rho_X^{\text{Sets}} \circ \rho_X^{\text{Sets}, -1}](x) &= \rho_X^{\text{Sets}}(\rho_X^{\text{Sets}, -1}(x)) \\
 &= \rho_X^{\text{Sets}, -1}(x, \star) \\
 &= x \\
 &= [\text{id}_X](x)
 \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\text{Sets}} \circ \rho_X^{\text{Sets}, -1} = \text{id}_X.$$

Therefore  $\rho_X^{\text{Sets}}$  is indeed an isomorphism.

*Naturality:* We need to show that, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc}
 X \times \text{pt} & \xrightarrow{f \times \text{id}_{\text{pt}}} & Y \times \text{pt} \\
 \rho_X^{\text{Sets}} \downarrow & & \downarrow \rho_Y^{\text{Sets}} \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x, \star) & & (x, \star) \mapsto (f(x), \star) \\
 \downarrow & & \downarrow \\
 x & \mapsto & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\rho^{\text{Sets}}$  is a natural transformation.

*Being a Natural Isomorphism:* Since  $\rho^{\text{Sets}}$  is natural and  $\rho^{\text{Sets}, -1}$  is a component-wise inverse to  $\rho^{\text{Sets}}$ , it follows from **Categories, Item 2 of Definition 11.9.7.1.2** that  $\rho^{\text{Sets}, -1}$  is also natural. Thus  $\rho^{\text{Sets}}$  is a natural isomorphism.  $\square$



### 5.1.7 The Symmetry

**Definition 5.1.7.1.1.** The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2},$$

$$\begin{array}{ccc} \text{Sets} \times \text{Sets} & \xrightarrow{\times} & \text{Sets} \\ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2} \searrow & \Downarrow \sigma^{\text{Sets}} & \nearrow \times \\ & \text{Sets} \times \text{Sets} & \end{array}$$

whose component

$$\sigma_{X,Y}^{\text{Sets}} : X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in \text{Obj}(\text{Sets})$  is defined by

$$\sigma_{X,Y}^{\text{Sets}}(x, y) \stackrel{\text{def}}{=} (y, x)$$

for each  $(x, y) \in X \times Y$ .

*Proof. Invertibility:* The inverse of  $\sigma_{X,Y}^{\text{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\text{Sets}, -1} : Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets}, -1}(y, x) \stackrel{\text{def}}{=} (x, y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[ \sigma_{X,Y}^{\text{Sets}, -1} \circ \sigma_{X,Y}^{\text{Sets}} \right](x, y) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets}, -1} \left( \sigma_{X,Y}^{\text{Sets}}(x, y) \right) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets}, -1}(y, x) \\ &\stackrel{\text{def}}{=} (x, y) \\ &\stackrel{\text{def}}{=} [\text{id}_{X \times Y}](x, y) \end{aligned}$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets}, -1} \circ \sigma_{X,Y}^{\text{Sets}} = \text{id}_{X \times Y}.$$

- *Invertibility II.* We have

$$\begin{aligned}
 \left[ \sigma_{X,Y}^{\text{Sets}} \circ \sigma_{X,Y}^{\text{Sets},-1} \right] (y, x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets},-1} \left( \sigma_{X,Y}^{\text{Sets}} (y, x) \right) \\
 &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets},-1} (x, y) \\
 &\stackrel{\text{def}}{=} (y, x) \\
 &\stackrel{\text{def}}{=} [\text{id}_{Y \times X}] (y, x)
 \end{aligned}$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets}} \circ \sigma_{X,Y}^{\text{Sets},-1} = \text{id}_{Y \times X}.$$

Therefore  $\sigma_{X,Y}^{\text{Sets}}$  is indeed an isomorphism.

*Naturality:* We need to show that, given functions

$$\begin{aligned}
 f &: X \rightarrow A, \\
 g &: Y \rightarrow B
 \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{f \times g} & A \times B \\
 \sigma_{X,Y}^{\text{Sets}} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}} \\
 Y \times X & \xrightarrow{g \times f} & B \times A
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x, y) & & (x, y) \mapsto (f(x), g(y)) \\
 \downarrow & & \downarrow \\
 (y, x) \mapsto (g(y), f(x)) & & (g(y), f(x))
 \end{array}$$

and hence indeed commutes, showing  $\sigma^{\text{Sets}}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\sigma^{\text{Sets}}$  is natural and  $\sigma^{\text{Sets},-1}$  is a component-wise inverse to  $\sigma^{\text{Sets}}$ , it follows from [Categories, Item 2](#) of [Definition 11.9.7.1.2](#) that  $\sigma^{\text{Sets},-1}$  is also natural. Thus  $\sigma^{\text{Sets}}$  is a natural isomorphism.  $\square$

### 5.1.8 The Diagonal

**Definition 5.1.8.1.1.** The **diagonal of the product of sets** is the natural transformation

$$\Delta: \text{id}_{\text{Sets}} \Rightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2},$$

whose component

$$\Delta_X: X \rightarrow X \times X$$

at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .

*Proof.* We need to show that, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X \downarrow & & \downarrow \Delta_Y \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & & x \longmapsto f(x) \\ \downarrow & & \downarrow \\ (x, x) \longmapsto (f(x), f(x)) & & (f(x), f(x)) \end{array}$$

and hence indeed commutes, showing  $\Delta$  to be natural.  $\square$

**Proposition 5.1.8.1.2.** Let  $X$  be a set.

1. *Monoidality.* The diagonal map

$$\Delta: \text{id}_{\text{Sets}} \Longrightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2},$$

is a monoidal natural transformation:

(a) *Compatibility With Strong Monoidality Constraints.* For each  $X, Y \in \text{Obj}(\text{Sets})$ , the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\Delta_X \times \Delta_Y} & (X \times X) \times (Y \times Y) \\ & \searrow \Delta_{X \times Y} & \downarrow \wr \\ & & (X \times Y) \times (X \times Y) \end{array}$$

commutes.

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{array}{ccc} \text{pt} & \xrightarrow{\Delta_{\text{pt}}} & \text{pt} \times \text{pt} \\ & \searrow & \downarrow \lambda_{\text{pt}}^{\text{Sets}} \\ & & \text{pt} \end{array} \quad \begin{array}{ccc} \text{pt} & \xrightarrow{\Delta_{\text{pt}}} & \text{pt} \times \text{pt} \\ & \searrow & \downarrow \rho_{\text{pt}}^{\text{Sets}} \\ & & \text{pt} \end{array}$$

commute, i.e. we have

$$\begin{aligned} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets}, -1} \\ &= \rho_{\text{pt}}^{\text{Sets}, -1}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned} \lambda_{\text{pt}}^{\text{Sets}} &= \rho_{\text{pt}}^{\text{Sets}}, \\ \lambda_{\text{pt}}^{\text{Sets}, -1} &= \rho_{\text{pt}}^{\text{Sets}, -1} \end{aligned}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

2. *The Diagonal of the Unit.* The component

$$\Delta_{\text{pt}}: \text{pt} \xrightarrow{\sim} \text{pt} \times \text{pt}$$

of  $\Delta$  at  $\text{pt}$  is an isomorphism.

*Proof.* **Item 1, Monoidality:** We claim that  $\Delta$  is indeed monoidal:

1. **Item 1a: Compatibility With Strong Monoidality Constraints:** We need to show that the diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\Delta_X \times \Delta_Y} & (X \times X) \times (Y \times Y) \\
 & \searrow \Delta_{X \times Y} & \downarrow \wr \\
 & & (X \times Y) \times (X \times Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 (x, y) & \longmapsto & ((x, x), (y, y)) & & (x, y) \\
 & & \downarrow & & \searrow \\
 & & ((x, y), (x, y)) & & ((x, y), (x, y))
 \end{array}$$

and hence indeed commutes.

2. **Item 1b: Compatibility With Strong Unitality Constraints:** As shown in the proof of **Definition 5.1.5.1.1**, the inverse of the left unitor of **Sets** with respect to the product at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\lambda_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each  $x \in X$ , so when  $X = \text{pt}$ , we have

$$\lambda_{\text{pt}}^{\text{Sets}, -1}(\star) \stackrel{\text{def}}{=} (\star, \star),$$

and also

$$\Delta_{\text{pt}}^{\text{Sets}}(\star) \stackrel{\text{def}}{=} (\star, \star),$$

so we have  $\Delta_{\text{pt}} = \lambda_{\text{pt}}^{\text{Sets}, -1}$ .

This finishes the proof.

**Item 2, The Diagonal of the Unit:** This follows from **Item 1** and the invertibility of the left/right unitor of **Sets** with respect to  $\times$ , proved in the proof of **Definition 5.1.5.1.1** for the left unitor or the proof of **Definition 5.1.6.1.1** for the right unitor.  $\square$

### 5.1.9 The Monoidal Category of Sets and Products

**Proposition 5.1.9.1.1.** The category  $\mathbf{Sets}$  admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category  $\mathbf{Sets}$  of pointed sets.
- *The Monoidal Product.* The product functor

$$\times: \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of [Constructions With Sets, Item 1 of Definition 4.1.3.1.3.](#)

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}: \mathbf{Sets}^{\text{op}} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of [Constructions With Sets, Item 1 of Definition 4.3.5.1.2.](#)

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}}: \text{pt} \rightarrow \mathbf{Sets}$$

of [Definition 5.1.3.1.1.](#)

- *The Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}}: \times \circ (\times \times \text{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\mathbf{Sets}} \times \times) \circ \alpha_{\mathbf{Sets}, \mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}}$$

of [Definition 5.1.4.1.1.](#)

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Sets}}: \times \circ (\mathbb{1}^{\mathbf{Sets}} \times \text{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}}^{\mathbf{Cats}_2}$$

of [Definition 5.1.5.1.1.](#)

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathbf{Sets}}: \times \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}}) \xrightarrow{\sim} \rho_{\mathbf{Sets}}^{\mathbf{Cats}_2}$$

of [Definition 5.1.6.1.1.](#)

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}} : \times \xrightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.7.1.1.**

- *The Diagonals.* The monoidal natural transformation

$$\Delta : \text{id}_{\text{Sets}} \Rightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.8.1.1.**

*Proof. The Pentagon Identity:* Let  $W, X, Y$  and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \times (X \times Y)) \times Z & \\
 \alpha_{W,X,Y}^{\text{Sets}} \times \text{id}_Z \nearrow & & \searrow \alpha_{W,X \times Y,Z}^{\text{Sets}} \\
 ((W \times X) \times Y) \times Z & & W \times ((X \times Y) \times Z) \\
 \downarrow \alpha_{W \times X,Y,Z}^{\text{Sets}} & & \downarrow \text{id}_W \times \alpha_{X,Y,Z}^{\text{Sets}} \\
 (W \times X) \times (Y \times Z) & \xrightarrow{\alpha_{W,X,Y \times Z}^{\text{Sets}}} & W \times (X \times (Y \times Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 & & ((w, (x, y)), z) & & \\
 & \swarrow & & \searrow & \\
 (((w, x), y), z) & & & & (w, ((x, y), z)) \\
 \searrow & & \swarrow & & \searrow \\
 ((w, x), (y, z)) \mapsto (w, (x, (y, z))) & & & & (w, (x, (y, z))),
 \end{array}$$

and thus the pentagon identity is satisfied.

*The Triangle Identity:* Let  $X$  and  $Y$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \times \text{pt}) \times Y & \xrightarrow{\alpha_{X,\text{pt},Y}^{\text{Sets}}} & X \times (\text{pt} \times Y) \\
 \searrow \rho_X^{\text{Sets}} \times \text{id}_Y & & \swarrow \text{id}_X \times \lambda_Y^{\text{Sets}} \\
 & X \times Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 ((x, \star), y) & & ((x, \star), y) & \xrightarrow{\quad} & (x, (\star, y)) \\
 & \searrow & & & \swarrow \\
 & (x, y) & & & (x, y)
 \end{array}$$

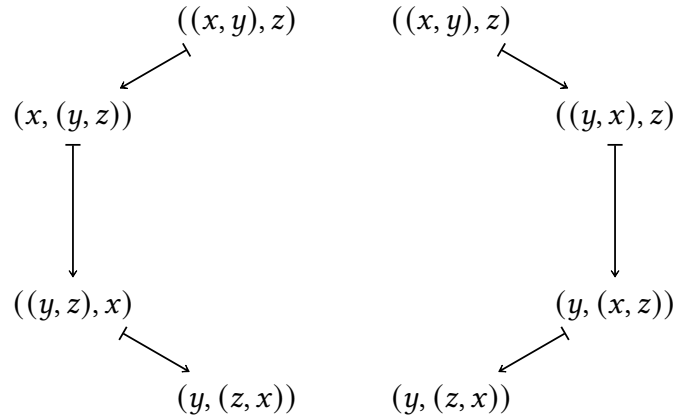
and thus the triangle identity is satisfied.

*The Left Hexagon Identity:* Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \times Y) \times Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}} \swarrow & & \searrow \sigma_{X,Y}^{\text{Sets}} \times \text{id}_Z \\
 X \times (Y \times Z) & & (Y \times X) \times Z \\
 \downarrow \sigma_{X,Y \times Z}^{\text{Sets}} & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}} \\
 (Y \times Z) \times X & & Y \times (X \times Z) \\
 \alpha_{Y,Z,X}^{\text{Sets}} \swarrow & & \swarrow \text{id}_Y \times \sigma_{X,Z}^{\text{Sets}} \\
 & Y \times (Z \times X) &
 \end{array}$$

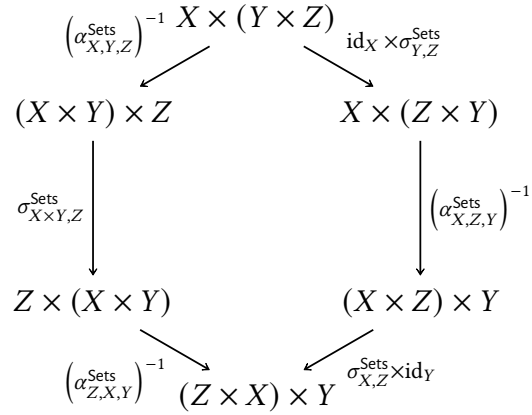


commutes. Indeed, this diagram acts on elements as

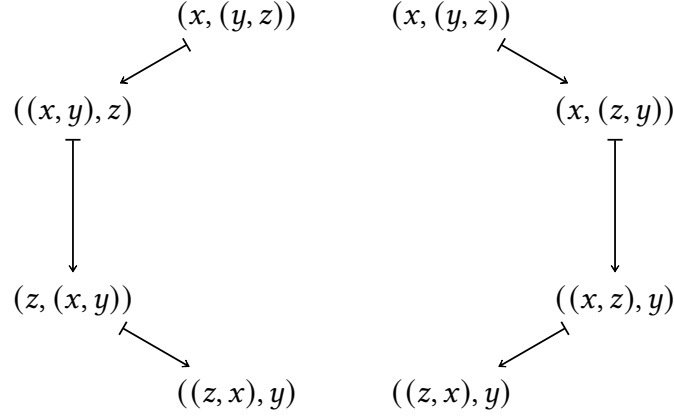


and thus the left hexagon identity is satisfied.

*The Right Hexagon Identity:* Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the right hexagon identity is satisfied.

*Monoidal Closedness:* This follows from **Constructions With Sets, Item 2 of Definition 4.3.5.1.2**

*Existence of Monoidal Diagonals:* This follows from **Items 1 and 2 of Definition 5.1.8.1.2**.  $\square$

### 5.1.10 The Universal Property of $(\mathbf{Sets}, \times, \text{pt})$

**Theorem 5.1.10.1.1.** The symmetric monoidal structure on the category  $\mathbf{Sets}$  of **Definition 5.1.9.1.1** is uniquely determined by the following requirements:

1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathbf{Sets}} : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of  $\mathbf{Sets}$  admits an internal Hom  $[-1, -2]_{\mathbf{Sets}}$ .

2. *The Unit Object Is pt.* We have  $\mathbb{1}_{\mathbf{Sets}} \cong \text{pt}$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}^{\text{cld}}(\mathbf{Sets})$  of ?? spanned by the closed symmetric monoidal categories  $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$  satisfying **Items 1 and 2** is contractible (i.e. equivalent to the punctual category).

*Proof. Unwinding the Statement:* Let  $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda', \rho', \sigma')$  be a closed symmetric monoidal category satisfying **Items 1 and 2**. We need to show that the identity functor

$$\text{id}_{\mathbf{Sets}} : \mathbf{Sets} \rightarrow \mathbf{Sets}$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \text{id}_{\mathbf{Sets}}^{\otimes} &: A \otimes_{\mathbf{Sets}} B \xrightarrow{\sim} A \times B, \\ \text{id}_{\mathbf{Sets}}^{\text{Hom}} &: [A, B]_{\mathbf{Sets}} \xrightarrow{\sim} \mathbf{Sets}(A, B), \\ \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes} &: \mathbb{1}_{\mathbf{Sets}} \xrightarrow{\sim} \text{pt}, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from  $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda', \rho', \sigma')$  to the closed symmetric monoidal category  $(\mathbf{Sets}, \times, \mathbf{Sets}(-1, -2), \mathbb{1}_{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$  of **Definition 5.1.9.1.1**.

*Constructing an Isomorphism  $[-1, -2]_{\mathbf{Sets}} \cong \mathbf{Sets}(-1, -2)$ :* By ??, we have a natural isomorphism

$$\mathbf{Sets}(\text{pt}, [-1, -2]_{\mathbf{Sets}}) \cong \mathbf{Sets}(-1, -2).$$

By **Constructions With Sets, Item 3** of **Definition 4.3.5.1.2**, we also have a natural isomorphism

$$\mathbf{Sets}(\text{pt}, [-1, -2]_{\mathbf{Sets}}) \cong [-1, -2]_{\mathbf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathbf{Sets}(-1, -2) \cong [-1, -2]_{\mathbf{Sets}}.$$

Given  $A, B \in \text{Obj}(\mathbf{Sets})$ , we will write

$$\text{id}_{A,B}^{\text{Hom}} : \mathbf{Sets}(A, B) \xrightarrow{\sim} [A, B]_{\mathbf{Sets}}$$

for the component of this isomorphism at  $(A, B)$ .

*Constructing an Isomorphism  $\otimes_{\mathbf{Sets}} \cong \times$ :* Since  $\otimes_{\mathbf{Sets}}$  is adjoint in each variable to  $[-1, -2]_{\mathbf{Sets}}$  by assumption and  $\times$  is adjoint in each variable to  $\mathbf{Sets}(-1, -2)$  by **Constructions With Sets, Item 2** of **Definition 4.3.5.1.2**, uniqueness of adjoints (??) gives us natural isomorphisms

$$\begin{aligned} A \otimes_{\mathbf{Sets}} - &\cong A \times -, \\ - \otimes_{\mathbf{Sets}} B &\cong B \times -. \end{aligned}$$

By ??, we then have  $\otimes_{\mathbf{Sets}} \cong \times$ . We will write

$$\text{id}_{\mathbf{Sets}|A,B}^{\otimes} : A \otimes_{\mathbf{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at  $(A, B)$ .

*Alternative Construction of an Isomorphism  $\otimes_{\mathbf{Sets}} \cong \times$ :* Alternatively, we may construct a natural isomorphism  $\otimes_{\mathbf{Sets}} \cong \times$  as follows:

1. Let  $A \in \text{Obj}(\text{Sets})$ .
2. Since  $\otimes_{\text{Sets}}$  is part of a closed monoidal structure, it preserves colimits in each variable by ??.
3. Since  $A \cong \coprod_{a \in A} \text{pt}$  and  $\otimes_{\text{Sets}}$  preserves colimits in each variable, we have

$$\begin{aligned}
 A \otimes_{\text{Sets}} B &\cong \left( \coprod_{a \in A} \text{pt} \right) \otimes_{\text{Sets}} B \\
 &\cong \coprod_{a \in A} (\text{pt} \otimes_{\text{Sets}} B) \\
 &\cong \coprod_{a \in A} B \\
 &\cong A \times B,
 \end{aligned}$$

naturally in  $B \in \text{Obj}(\text{Sets})$ , where we have used that  $\text{pt}$  is the monoidal unit for  $\otimes_{\text{Sets}}$ . Thus  $A \otimes_{\text{Sets}} - \cong A \times -$  for each  $A \in \text{Obj}(\text{Sets})$ .

4. Similarly,  $- \otimes_{\text{Sets}} B \cong - \times B$  for each  $B \in \text{Obj}(\text{Sets})$ .
5. By ??, we then have  $\otimes_{\text{Sets}} \cong \times$ .

Below, we'll show that if a natural isomorphism  $\otimes_{\text{Sets}} \cong \times$  exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\text{id}_{\text{Sets}|A,B}^{\otimes} : A \otimes_{\text{Sets}} B \rightarrow A \times B$  from before.

*Constructing an Isomorphism  $\text{id}_{\mathbb{1}}^{\otimes} : \mathbb{1}_{\text{Sets}} \rightarrow \text{pt}$ :* We define an isomorphism  $\text{id}_{\mathbb{1}}^{\otimes} : \mathbb{1}_{\text{Sets}} \rightarrow \text{pt}$  as the composition

$$\mathbb{1}_{\text{Sets}} \xrightarrow[\sim]{\rho_{\mathbb{1}_{\text{Sets}}}^{\text{Sets}, -1}} \mathbb{1}_{\text{Sets}} \times \text{pt} \xrightarrow[\sim]{\text{id}_{\text{Sets}|\mathbb{1}_{\text{Sets}}}^{\otimes}} \mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} \text{pt} \xrightarrow[\sim]{\lambda'_{\text{pt}}} \text{pt}$$

in  $\text{Sets}$ .

*Monoidal Left Unity of the Isomorphism  $\otimes_{\text{Sets}} \cong \times$ :* We have to show that the diagram

$$\begin{array}{ccc}
 & \text{pt} \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},A}^{\otimes}} \text{pt} \times A \\
 \text{id}_{\mathbb{1}_{\text{Sets}}}^{\otimes} \otimes_{\text{Sets}} \text{id}_A \nearrow & & \searrow \lambda_A^{\text{Sets}} \\
 \mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} A & \xrightarrow{\lambda_A^{\otimes}} & A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & \mathbf{pt} \otimes_{\mathbf{Sets}} \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|\mathbf{pt},\mathbf{pt}}^{\otimes}} \mathbf{pt} \times \mathbf{pt} \\
 \mathrm{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes} \otimes_{\mathbf{Sets}} \mathrm{id}_{\mathbf{pt}} \nearrow & & \searrow \lambda_{\mathbf{pt}}^{\mathbf{Sets}} \\
 \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \mathbf{pt} & \xrightarrow{\lambda'_{\mathbf{pt}}} & \mathbf{pt},
 \end{array}$$

corresponding to the case  $A = \mathbf{pt}$ , commutes by the terminality of  $\mathbf{pt}$  (**Constructions With Sets, Definition 4.1.1.2**). Since this diagram commutes, so does the diagram

$$\begin{array}{ccc}
 & \mathbf{pt} \times \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|\mathbf{pt},\mathbf{pt}}^{\otimes,-1}} \mathbf{pt} \otimes_{\mathbf{Sets}} \mathbf{pt} \\
 \lambda_{\mathbf{pt}}^{\mathbf{Sets},-1} \nearrow & & \searrow \mathrm{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes,-1} \otimes_{\mathbf{Sets}} \mathrm{id}_{\mathbf{pt}} \\
 \mathbf{pt} & \xrightarrow{\lambda'^{-1}_{\mathbf{pt}}} & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \mathbf{pt}.
 \end{array}
 \quad (\dagger)$$

Now, let  $A \in \mathrm{Obj}(\mathbf{Sets})$ , let  $a \in A$ , and consider the diagram

$$\begin{array}{ccccc}
 & \mathbf{pt} \times \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|\mathbf{pt},\mathbf{pt}}^{\otimes,-1}} & \mathbf{pt} \otimes_{\mathbf{Sets}} \mathbf{pt} & \\
 \lambda_{\mathbf{pt}}^{\mathbf{Sets},-1} \nearrow & & & & \searrow \mathrm{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes,-1} \times \mathrm{id}_{\mathbf{pt}} \\
 \mathbf{pt} & \xrightarrow{\lambda'^{-1}_{\mathbf{pt}}} & & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \mathbf{pt} & \\
 \downarrow [a] & \downarrow \mathrm{id}_{\mathbf{pt}} \times [a] & (1) & \downarrow \mathrm{id}_{\mathbf{pt}} \otimes_{\mathbf{Sets}} [a] & \downarrow \mathrm{id}_{\mathbb{1}_{\mathbf{Sets}}} \times [a] \\
 & \mathbf{pt} \times A & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|\mathbf{pt},A}^{\otimes,-1}} & \mathbf{pt} \otimes_{\mathbf{Sets}} A & \\
 \lambda_A^{\mathbf{Sets},-1} \nearrow & & & & \searrow \mathrm{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes,-1} \times \mathrm{id}_A \\
 A & \xrightarrow{\lambda'^{-1}_A} & & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A &
 \end{array}$$

$(3)$   $(5)$   $(2)$   $(4)$

Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'^{-1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.

- Subdiagram (4) commutes by the naturality of  $\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}_{\mathbf{Sets}}^{\otimes, -1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\mathbf{Sets}, -1}$ .

it follows that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times A & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} A \\
 & \nearrow \lambda_A^{\mathbf{Sets}, -1} & & & \searrow \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_A \\
 \text{pt} & \xrightarrow{[a]} & A & \xrightarrow{\lambda_A'^{-1}} & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}
 \lambda_A'^{-1}(a) &= [\lambda_A'^{-1} \circ [a]](\star) \\
 &= \left[ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \times \text{id}_A \right) \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\mathbf{Sets}, -1} \circ [a] \right](\star) \\
 &= \left[ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \times \text{id}_A \right) \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\mathbf{Sets}, -1} \right](a)
 \end{aligned}$$

for each  $a \in A$ , and thus we have

$$\lambda_A'^{-1} = \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \times \text{id}_A \right) \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\mathbf{Sets}, -1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathbf{Sets}} \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes} \circ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \times \text{id}_A \right),$$

showing that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \otimes_{\mathbf{Sets}} A & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes}} & \text{pt} \times A \\
 & \nearrow \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_A & & & \searrow \lambda_A^{\mathbf{Sets}} \\
 \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A & \xrightarrow{\lambda_A'} & & & A
 \end{array}$$

indeed commutes.

*Monoidal Right Unity of the Isomorphism  $\otimes_{\text{Sets}} \cong \times$ :* We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\text{Sets}} \cong \times$  above. For completeness, we repeat it below.

We have to show that the diagram

$$\begin{array}{ccccc}
 & & A \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|A,\text{pt}}^{\otimes}} & A \times \text{pt} \\
 \text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}}}^{\otimes} \nearrow & & & & \searrow \rho_A^{\text{Sets}} \\
 A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho'_A} & & & A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}}}^{\otimes} \nearrow & & & & \searrow \rho_{\text{pt}}^{\text{Sets}} \\
 \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho'_{\text{pt}}} & & & \text{pt},
 \end{array}$$

corresponding to the case  $A = \text{pt}$ , commutes by the terminality of  $\text{pt}$  (**Constructions With Sets, Definition 4.1.1.1.2**). Since this diagram commutes, so does the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes,-1}} & \text{pt} \otimes_{\text{Sets}} \text{pt} \\
 \rho_{\text{pt}}^{\text{Sets},-1} \nearrow & & & & \searrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}}}^{\otimes,-1} \\
 \text{pt} & \xrightarrow{\rho_{\text{pt}}'^{-1}} & & & \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}.
 \end{array}$$

(†)

Now, let  $A \in \mathbf{Obj}(\mathbf{Sets})$ , let  $a \in A$ , and consider the diagram

$$\begin{array}{ccccc}
 & & \mathbf{pt} \times \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|\mathbf{pt},\mathbf{pt}}^{\otimes,-1}} & \mathbf{pt} \otimes_{\mathbf{Sets}} \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{pt}} \times \mathrm{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes,-1}} & \mathbf{pt} \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}} \\
 & \nearrow \rho_{\mathbf{pt}}^{\mathbf{Sets},-1} & \downarrow & \scriptstyle (\dagger) & \downarrow & \searrow & \\
 \mathbf{pt} & \xrightarrow{\quad \quad \quad} & \mathbf{pt} & \xrightarrow{\rho_{\mathbf{pt}}'^{-1}} & \mathbf{pt} & \xrightarrow{\quad \quad \quad} & \mathbf{pt} \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}} \\
 & \downarrow \mathrm{id}_{\mathbf{pt}} \times [a] & & (1) & \downarrow \mathrm{id}_{\mathbf{pt}} \otimes_{\mathbf{Sets}} [a] & & \downarrow \mathrm{id}_{\mathbb{1}_{\mathbf{Sets}}} \times [a] \\
 & (3) & & & (4) & & \\
 & & A \times \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|A,\mathbf{pt}}^{\otimes,-1}} & A \otimes_{\mathbf{Sets}} \mathbf{pt} & & \\
 & \nearrow \rho_A^{\mathbf{Sets},-1} & \downarrow & \scriptstyle (2) & \searrow \mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes,-1} & & \\
 A & \xrightarrow{\quad \quad \quad} & A & \xrightarrow{\rho_A'^{-1}} & A & \xrightarrow{\quad \quad \quad} & A \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}} \\
 & \downarrow [a] & & & & & \\
 & & A & & & & 
 \end{array}$$

Since:

- Subdiagram (5) commutes by the naturality of  $\rho'^{-1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\mathrm{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{\mathbf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\mathbf{Sets},-1}$ .

it follows that the diagram

$$\begin{array}{ccccc}
 & & A \times \mathbf{pt} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}|A,\mathbf{pt}}^{\otimes,-1}} & A \otimes_{\mathbf{Sets}} \mathbf{pt} \\
 & \nearrow \rho_A^{\mathbf{Sets},-1} & & & \searrow \mathrm{id}_A \otimes_{\mathbf{Sets}} \mathrm{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes,-1} \\
 \mathbf{pt} & \xrightarrow{[a]} & A & \xrightarrow{\rho_A'^{-1}} & A \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}}
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\rho_A'^{-1}(a) = [\rho_A'^{-1} \circ [a]](\star)$$



$$\begin{aligned}
&= \left[ \left( \text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1} \circ [a] \right] (\star) \\
&= \left[ \left( \text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1} \right] (a)
\end{aligned}$$

for each  $a \in A$ , and thus we have

$$\rho'_A{}^{-1} = \left( \text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1}.$$

Taking inverses then gives

$$\rho'_A = \rho_A^{\text{Sets}} \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes} \circ \left( \text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \right),$$

showing that the diagram

$$\begin{array}{ccc}
& A \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|A, \text{pt}}^{\otimes}} A \times \text{pt} \\
\text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \nearrow & & \searrow \rho_A^{\text{Sets}} \\
A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho'_A} & A
\end{array}$$

indeed commutes.

*Monoidality of the Isomorphism  $\otimes_{\text{Sets}} \cong \times$ :* We have to show that the diagram

$$\begin{array}{ccc}
& (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & \\
\text{id}_{\text{Sets}|A, B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C^{\otimes} \swarrow & & \searrow \alpha'_{A, B, C} \\
(A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
\downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes} & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B, C}^{\otimes} \\
(A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
\searrow \alpha_{A, B, C}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \\
& A \times (B \times C) &
\end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & \\
 \text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\text{pt}} \swarrow & & \searrow \alpha'_{\text{pt},\text{pt},\text{pt}} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes} & & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \searrow \alpha_{\text{pt},\text{pt},\text{pt}}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes} \\
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 & \downarrow !_{\text{pt} \times (\text{pt} \times \text{pt})} & \\
 & \text{pt} &
 \end{array}$$

commutes by the terminality of  $\text{pt}$  ([Constructions With Sets, Definition 4.1.1.1.2](#)). Since the map  $!_{\text{pt} \times (\text{pt} \times \text{pt})} : \text{pt} \times (\text{pt} \times \text{pt}) \rightarrow \text{pt}$  is an isomorphism (e.g. having inverse  $\lambda_{\text{pt}}^{\text{Sets}, -1} \circ \lambda_{\text{pt}}^{\text{Sets}, -1}$ ), it follows that the diagram

$$\begin{array}{ccc}
 & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & \\
 \text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\text{pt}} \swarrow & & \searrow \alpha'_{\text{pt},\text{pt},\text{pt}} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes} & & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \searrow \alpha_{\text{pt},\text{pt},\text{pt}}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes} \\
 & \text{pt} \times (\text{pt} \times \text{pt}) &
 \end{array}$$

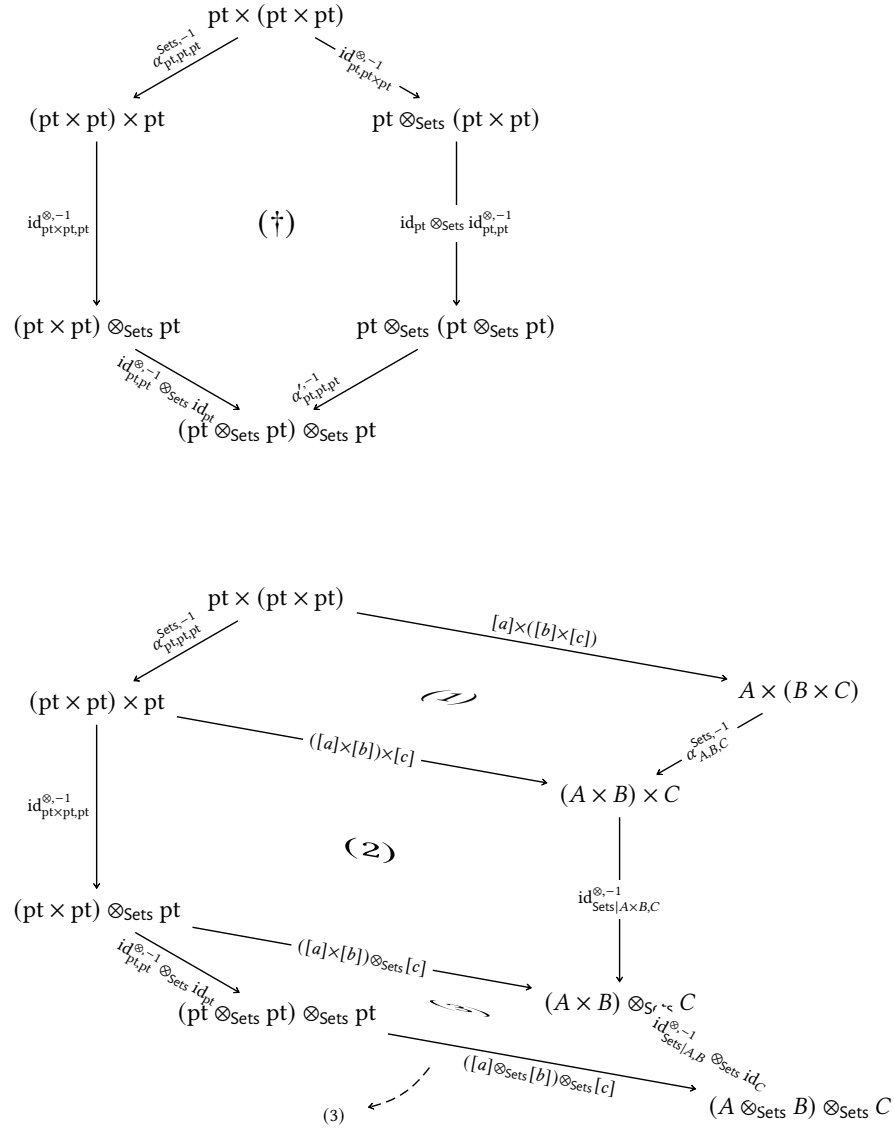
also commutes. Taking inverses, we see that the diagram

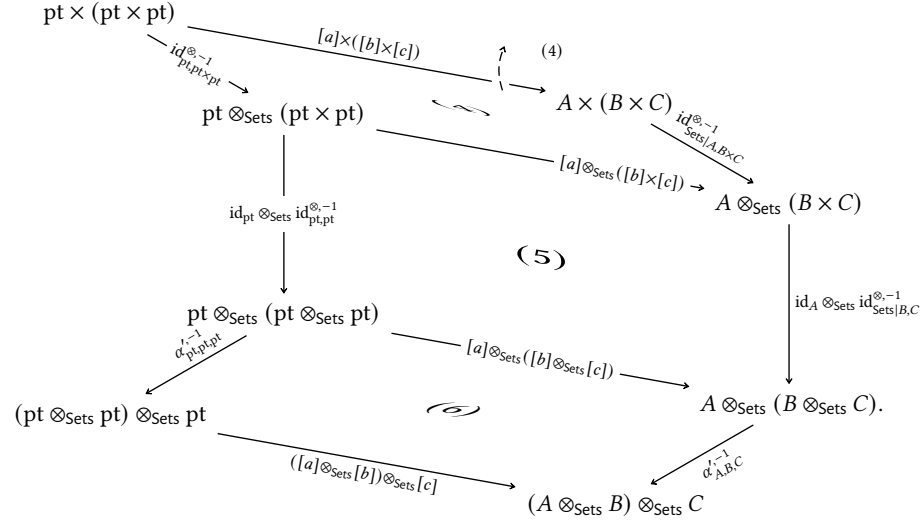
$$\begin{array}{ccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\text{Sets}, -1} \swarrow & & \searrow \text{id}_{\text{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes, -1} & (\dagger) & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_{\text{pt}} & & \swarrow \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\prime, -1} \\
 & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} &
 \end{array}$$

commutes as well. Now, let  $A, B, C \in \text{Obj}(\text{Sets})$ , let  $a \in A$ , let  $b \in B$ , let  $c \in C$ , and consider the diagram

$$\begin{array}{ccccccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & & & & & \\
 \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\text{Sets}, -1} \swarrow & & \text{id}_{\text{pt}, \text{pt} \times \text{pt}}^{\otimes, -1} \searrow & & [a] \times ([b] \times [c]) \searrow & & \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) & & A \times (B \times C) & & \\
 \downarrow \text{id}_{\text{pt} \times \text{pt}, \text{pt}}^{\otimes, -1} & & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{pt}, \text{pt}}^{\otimes, -1} & & \downarrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} & & \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) & & (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \downarrow \text{id}_{\text{pt}, \text{pt}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_{\text{pt}} & & \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} & & \downarrow [a] \otimes_{\text{Sets}} ([b] \otimes_{\text{Sets}} [c]) & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \\
 (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & & (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) & & \\
 \downarrow \text{id}_{\text{pt}, \text{pt}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_{\text{pt}} & & \downarrow \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C & & \downarrow \alpha_{A, B, C}^{\prime, -1} & & \\
 & & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & & & &
 \end{array}$$

which we partition into subdiagrams as follows:





Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\text{Sets}, -1}$ .
- Subdiagram (2) commutes by the naturality of  $\text{id}_{\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (3) commutes by the naturality of  $\text{id}_{\text{Sets}}^{\otimes, -1}$ .
- Subdiagram ( $\dagger$ ) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\text{id}_{\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (5) commutes by the naturality of  $\text{id}_{\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .

it follows that the diagram

$$\begin{array}{ccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 & \downarrow [a] \times ([b] \times [c]) & \\
 & A \times (B \times C) & \\
 \swarrow \alpha_{A,B,C}^{\text{Sets}, -1} & & \searrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} & & \downarrow \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \swarrow \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C & & \swarrow \alpha'_{A,B,C}{}'^{-1} \\
 & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C &
 \end{array}$$

also commutes. We then have

$$\begin{aligned}
 & \left[ \left( \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C \right) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{A,B,C}^{\text{Sets}, -1} \right] (a, (b, c)) = \left[ \left( \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C \right) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{A,B,C}^{\text{Sets}, -1} \circ ([a] \times ([b] \times [c])) \right] (\star, (\star, \star)) \\
 & = \left[ \alpha'_{A,B,C}{}'^{-1} \circ \left( \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \right) \right. \\
 & \quad \left. \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \circ ([a] \times ([b] \times [c])) \right] (\star, (\star, \star)) \\
 & = \left[ \alpha'_{A,B,C}{}'^{-1} \circ \left( \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \right] (a, (b, c))
 \end{aligned}$$

for each  $(a, (b, c)) \in A \times (B \times C)$ , and thus we have

$$\left( \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C \right) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \circ \alpha_{A,B,C}^{\text{Sets}, -1} = \alpha'_{A,B,C}{}'^{-1} \circ \left( \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\text{Sets}} \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes} \circ \left( \text{id}_{\text{Sets}|A, B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \right) = \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \circ \left( \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes} \right) \circ \alpha'_{A,B,C},$$

showing that the diagram

$$\begin{array}{ccc}
 (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & & \\
 \text{id}_{\text{Sets}|A,B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \swarrow & & \searrow \alpha'_{A,B,C} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B,C}^{\otimes} & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B,C}^{\otimes} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \searrow \alpha_{A,B,C}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|A,B \times C}^{\otimes} \\
 & A \times (B \times C) &
 \end{array}$$

indeed commutes.

*Braidedness of the Isomorphism  $\otimes_{\text{Sets}} \cong \times$ :* We have to show that the diagram

$$\begin{array}{ccc}
 A \otimes_{\text{Sets}} B & \xrightarrow{\text{id}_{\text{Sets}|A,B}^{\otimes}} & A \times B \\
 \sigma'_{A,B} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}} \\
 B \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|B,A}^{\otimes}} & B \times A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|pt,pt}^{\otimes}} & \text{pt} \times \text{pt} \\
 \sigma'_{pt,pt} \downarrow & & \downarrow \sigma_{pt,pt}^{\text{Sets}} \\
 \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|pt,pt}^{\otimes}} & \text{pt} \times \text{pt} \\
 & & \searrow !_{pt \times pt} \\
 & & \text{pt}
 \end{array}$$

commutes by the terminality of  $\text{pt}$  (**Constructions With Sets, Definition 4.1.1.1.2**).

Since the map  $!_{pt \times pt}: \text{pt} \times \text{pt} \rightarrow \text{pt}$  is invertible (e.g. with inverse  $\lambda_{pt}^{\text{Sets}, -1}$ ), the

diagram

$$\begin{array}{ccc}
 \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 \sigma'_{\text{pt},\text{pt}} \downarrow & & \downarrow \sigma_{\text{pt},\text{pt}}^{\mathbf{Sets}} \\
 \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt}
 \end{array}$$

also commutes. Taking inverses, we see that the diagram

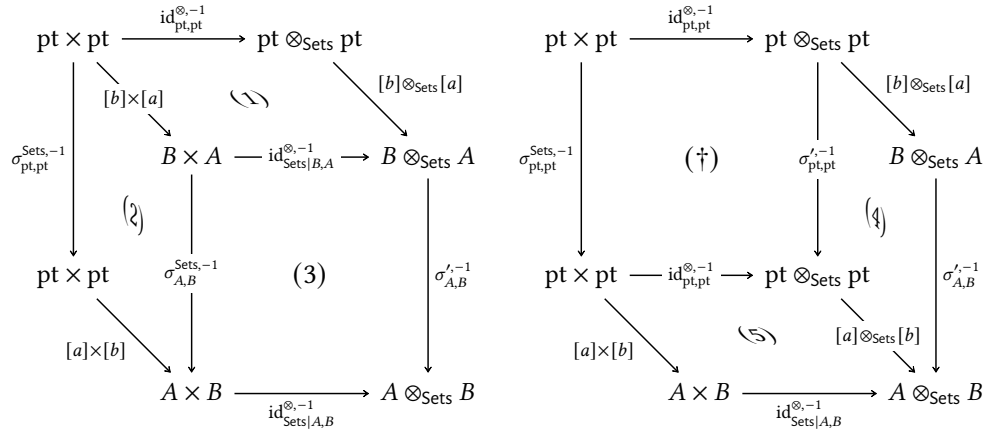
$$\begin{array}{ccc}
 \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} \\
 \sigma_{\text{pt},\text{pt}}^{\mathbf{Sets}, -1} \downarrow & (\dagger) & \downarrow \sigma'_{\text{pt},\text{pt}}{}^{-1} \\
 \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt}
 \end{array}$$

commutes as well. Now, let  $A, B \in \text{Obj}(\mathbf{Sets})$ , let  $a \in A$ , let  $b \in B$ , and consider the diagram

$$\begin{array}{ccccc}
 \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{pt},\text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & & \\
 \downarrow \sigma_{\text{pt},\text{pt}}^{\mathbf{Sets}, -1} & \searrow [b] \times [a] & \downarrow \sigma'_{\text{pt},\text{pt}}{}^{-1} & \searrow [b] \otimes_{\mathbf{Sets}} [a] & \\
 & B \times A & \xrightarrow{\text{id}_{\mathbf{Sets}|B,A}^{\otimes, -1}} & B \otimes_{\mathbf{Sets}} A & \\
 & \downarrow \sigma_{A,B}^{\mathbf{Sets}, -1} & \downarrow & \downarrow \sigma'_{A,B}{}^{-1} & \\
 \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{pt},\text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & & \\
 \searrow [a] \times [b] & \downarrow & \searrow [a] \otimes_{\mathbf{Sets}} [b] & & \\
 & A \times B & \xrightarrow{\text{id}_{\mathbf{Sets}|A,B}^{\otimes, -1}} & A \otimes_{\mathbf{Sets}} B &
 \end{array}$$



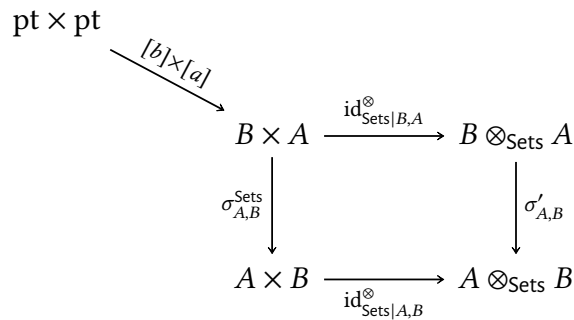
which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of  $\sigma^{\text{Sets}, -1}$ .
- Subdiagram (5) commutes by the naturality of  $\text{id}^{\otimes, -1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{-1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}^{\otimes, -1}$ .

it follows that the diagram



commutes. We then have

$$\left[ \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \circ \sigma_{A, B}^{\text{Sets}, -1} \right] (b, a) = \left[ \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \circ \sigma_{A, B}^{\text{Sets}, -1} \circ ([b] \times [a]) \right] (\star, \star)$$

$$\begin{aligned}
&= \left[ \sigma'_{A,B}{}^{-1} \circ \text{id}_{\mathbf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a]) \right] (\star, \star) \\
&= \left[ \sigma'_{A,B}{}^{-1} \circ \text{id}_{\mathbf{Sets}|B,A}^{\otimes,-1} \right] (b, a)
\end{aligned}$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\text{id}_{\mathbf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathbf{Sets},-1} = \sigma'_{A,B}{}^{-1} \circ \text{id}_{\mathbf{Sets}|B,A}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathbf{Sets}} \circ \text{id}_{\mathbf{Sets}|A,B}^{\otimes} = \text{id}_{\mathbf{Sets}|B,A}^{\otimes} \circ \sigma'_{A,B},$$

showing that the diagram

$$\begin{array}{ccc}
A \otimes_{\mathbf{Sets}} B & \xrightarrow{\text{id}_{\mathbf{Sets}|A,B}^{\otimes}} & A \times B \\
\sigma'_{A,B} \downarrow & & \downarrow \sigma_{A,B}^{\mathbf{Sets}} \\
B \otimes_{\mathbf{Sets}} A & \xrightarrow{\text{id}_{\mathbf{Sets}|B,A}^{\otimes}} & B \times A
\end{array}$$

indeed commutes.

*Uniqueness of the Isomorphism  $\otimes_{\mathbf{Sets}} \cong \times$ :* Let  $\phi, \psi: -_1 \otimes_{\mathbf{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of  $\mathbf{Sets}$  with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\begin{aligned}
\lambda'_B &= \lambda_B^{\mathbf{Sets}} \circ \phi_{\text{pt},B} \circ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_Y \right), \\
\lambda'_B &= \lambda_B^{\mathbf{Sets}} \circ \psi_{\text{pt},B} \circ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_Y \right).
\end{aligned}$$

Postcomposing both sides with  $\lambda_B^{\mathbf{Sets},-1}$  gives

$$\begin{aligned}
\lambda_B^{\mathbf{Sets},-1} \circ \lambda'_B \circ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes,-1} \otimes_{\mathbf{Sets}} \text{id}_Y \right) &= \phi_{\text{pt},B}, \\
\lambda_B^{\mathbf{Sets},-1} \circ \lambda'_B \circ \left( \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes,-1} \otimes_{\mathbf{Sets}} \text{id}_Y \right) &= \psi_{\text{pt},B},
\end{aligned}$$

and thus we have

$$\phi_{\text{pt},B} = \psi_{\text{pt},B}$$

for each  $B \in \text{Obj}(\text{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams

$$\begin{array}{ccc} \text{pt} \times B & \xrightarrow{[a] \times \text{id}_B} & A \times B \\ \downarrow \phi_{\text{pt}, B} & & \downarrow \phi_{A, B} \\ \text{pt} \otimes_{\text{Sets}} B & \xrightarrow{[a] \otimes_{\text{Sets}} \text{id}_B} & A \otimes_{\text{Sets}} B \end{array} \quad \begin{array}{ccc} \text{pt} \times B & \xrightarrow{[a] \times \text{id}_B} & A \times B \\ \downarrow \psi_{\text{pt}, B} & & \downarrow \psi_{A, B} \\ \text{pt} \otimes_{\text{Sets}} B & \xrightarrow{[a] \otimes_{\text{Sets}} \text{id}_B} & A \otimes_{\text{Sets}} B \end{array}$$

for  $\phi$  and  $\psi$  with respect to the morphisms  $[a]$  and  $\text{id}_B$ . Having shown that  $\phi_{\text{pt}, B} = \psi_{\text{pt}, B}$ , we have

$$\begin{aligned} \phi_{A, B}(a, b) &= [\phi_{A, B} \circ ([a] \times \text{id}_B)](\star, b) \\ &= [([a] \otimes_{\text{Sets}} \text{id}_B) \circ \phi_{\text{pt}, B}](\star, b) \\ &= [([a] \otimes_{\text{Sets}} \text{id}_B) \circ \psi_{\text{pt}, B}](\star, b) \\ &= [\psi_{A, B} \circ ([a] \times \text{id}_B)](\star, b) \\ &= \psi_{A, B}(a, b) \end{aligned}$$

for each  $(a, b) \in A \times B$ . Therefore we have

$$\phi_{A, B} = \psi_{A, B}$$

for each  $A, B \in \text{Obj}(\text{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\text{Sets}} \cong \times$  to be unique.  $\square$

**Corollary 5.1.10.1.2.** The symmetric monoidal structure on the category  $\text{Sets}$  of [Definition 5.1.9.1.1](#) is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\text{Sets}} : \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of  $\text{Sets}$  preserves colimits separately in each variable.

2. *The Unit Object Is pt.* We have  $\mathbb{1}_{\text{Sets}} \cong \text{pt}$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\text{Sets})$  of ?? spanned by the symmetric monoidal categories  $(\text{Sets}, \otimes_{\text{Sets}}, \mathbb{1}_{\text{Sets}}, \lambda^{\text{Sets}}, \rho^{\text{Sets}}, \sigma^{\text{Sets}})$  satisfying [Items 1](#) and [2](#) is contractible.

*Proof.* Since  $\text{Sets}$  is locally presentable (??), it follows from ?? that [Item 1](#) is equivalent to the existence of an internal Hom as in [Item 1](#) of [Definition 5.1.10.1.1](#). The result then follows from [Definition 5.1.10.1.1](#).  $\square$

## 5.2 The Monoidal Category of Sets and Coproducts

### 5.2.1 Coproducts of Sets

See [Constructions With Sets, Section 4.2.3](#).

### 5.2.2 The Monoidal Unit

**Definition 5.2.2.1.1.** The **monoidal unit of the coproduct of sets** is the functor

$$\mathbb{0}^{\text{Sets}} : \text{pt} \rightarrow \text{Sets}$$

defined by

$$\mathbb{0}_{\text{Sets}} \stackrel{\text{def}}{=} \emptyset,$$

where  $\emptyset$  is the empty set of [Constructions With Sets, Definition 4.3.1.1.1](#).

### 5.2.3 The Associator

**Definition 5.2.3.1.1.** The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\text{Sets}, \amalg} : \amalg \circ (\amalg \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \amalg \circ (\text{id}_{\text{Sets}} \times \amalg) \circ \alpha^{\text{Cats}_{\text{Sets}, \text{Sets}, \text{Sets}}}$$

as in the diagram

$$\begin{array}{ccccc}
 & & \text{Sets} \times (\text{Sets} \times \text{Sets}) & & \\
 & \nearrow \alpha^{\text{Cats}_{\text{Sets}, \text{Sets}, \text{Sets}}} & \searrow \text{id} \times \amalg & & \\
 (\text{Sets} \times \text{Sets}) \times \text{Sets} & & \text{Sets} \times \text{Sets} & & \\
 \downarrow \amalg \times \text{id} & \nearrow \alpha^{\text{Sets}, \amalg} & \downarrow \amalg & & \\
 \text{Sets} \times \text{Sets} & \xrightarrow{\amalg} & \text{Sets} & & 
 \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg} : (X \amalg Y) \amalg Z \xrightarrow{\sim} X \amalg (Y \amalg Z)$$

at  $(X, Y, Z)$  is given by

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg}(a) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } a = (0, (0, x)), \\ (1, (0, y)) & \text{if } a = (0, (1, y)), \\ (1, (1, a)) & \text{if } a = (1, z) \end{cases}$$

for each  $a \in (X \amalg Y) \amalg Z$ .

*Proof. Unwinding the Definitions of  $(X \amalg Y) \amalg Z$  and  $X \amalg (Y \amalg Z)$ :* Firstly, we unwind the expressions for  $(X \amalg Y) \amalg Z$  and  $X \amalg (Y \amalg Z)$ . We have

$$\begin{aligned} (X \amalg Y) \amalg Z &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \amalg Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ &= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ &\quad \cup \{(1, z) \in S \mid z \in Z\}, \end{aligned}$$

where  $S = \{0, 1\} \times ((X \amalg Y) \cup Z)$  and

$$\begin{aligned} X \amalg (Y \amalg Z) &\stackrel{\text{def}}{=} \{(0, x) \in S' \mid x \in X\} \cup \{(1, a) \in S' \mid a \in Y \amalg Z\} \\ &= \{(0, x) \in S' \mid x \in X\} \cup \{(1, (0, y)) \in S' \mid y \in Y\} \\ &\quad \cup \{(1, (1, z)) \in S' \mid z \in Z\}, \end{aligned}$$

where  $S' = \{0, 1\} \times (X \cup (Y \amalg Z))$ .

*Invertibility:* The inverse of  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg}$  is the map

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1} : X \amalg (Y \amalg Z) \rightarrow (X \amalg Y) \amalg Z$$

given by

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1}(a) \stackrel{\text{def}}{=} \begin{cases} (0, (0, x)) & \text{if } a = (0, x), \\ (0, (1, y)) & \text{if } a = (1, (0, y)), \\ (1, z) & \text{if } a = (1, (1, z)) \end{cases}$$

for each  $a \in X \amalg (Y \amalg Z)$ . Indeed:

- *Invertibility I.* The map  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1} \circ \alpha_{X,Y,Z}^{\text{Sets}, \amalg}$  acts on elements as

$$\begin{aligned} (0, (0, x)) &\mapsto (0, x) \mapsto (0, (0, x)), \\ (0, (0, y)) &\mapsto (1, (0, y)) \mapsto (0, (0, y)), \\ (1, z) &\mapsto (1, (1, z)) \mapsto (1, z) \end{aligned}$$

and hence is equal to the identity map of  $(X \amalg Y) \amalg Z$ .

- *Invertibility II.* The map  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg} \circ \alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1}$  acts on elements as

$$\begin{aligned} (0, x) &\mapsto (0, (0, x)) \mapsto (0, x), \\ (1, (0, y)) &\mapsto (0, (0, y)) \mapsto (1, (0, y)), \\ (1, (1, z)) &\mapsto (1, z) \mapsto (1, (1, z)) \end{aligned}$$

and hence is equal to the identity map of  $X \amalg (Y \amalg Z)$ .

Therefore  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg}$  is indeed an isomorphism.

*Naturality:* We need to show that, given functions

$$\begin{aligned} f &: X \rightarrow X', \\ g &: Y \rightarrow Y', \\ h &: Z \rightarrow Z' \end{aligned}$$

the diagram

$$\begin{array}{ccc} (X \amalg Y) \amalg Z & \xrightarrow{(f \amalg g) \amalg h} & (X' \amalg Y') \amalg Z' \\ \downarrow \alpha_{X,Y,Z}^{\text{Sets}, \amalg} & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}, \amalg} \\ X \amalg (Y \amalg Z) & \xrightarrow{f \amalg (g \amalg h)} & X' \amalg (Y' \amalg Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0, (0, x)) & & (0, (0, x)) \mapsto (0, (0, f(x))) \\
 \downarrow & & \downarrow \\
 (0, x) \mapsto (0, f(x)) & & (0, f(x)) \\
 \\ 
 (0, (1, y)) & & (0, (1, y)) \mapsto (0, (1, g(y))) \\
 \downarrow & & \downarrow \\
 (1, (0, y)) \mapsto (1, (0, g(y))) & & (1, (0, g(y))) \\
 \\ 
 (1, z) & & (1, z) \mapsto (1, h(z)) \\
 \downarrow & & \downarrow \\
 (1, (1, z)) \mapsto (1, (1, h(z))) & & (1, (1, h(z)))
 \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}, \amalg}$  to be a natural transformation. *Being a Natural Isomorphism:* Since  $\alpha^{\text{Sets}, \amalg}$  is natural and  $\alpha^{\text{Sets}, \amalg, -1}$  is a componentwise inverse to  $\alpha^{\text{Sets}, \amalg}$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\alpha^{\text{Sets}, \amalg}$  is a natural isomorphism.  $\square$

## 5.2.4 The Left Unitor

**Definition 5.2.4.1.1.** The **left unitor of the coproduct of sets** is the natural isomorphism

$$\lambda^{\text{Sets}, \amalg} : \amalg \circ (\amalg^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}, \amalg} : \emptyset \amalg X \xrightarrow{\sim} X$$

at  $X$  is given by

$$\lambda_X^{\text{Sets}, \amalg}((1, x)) \stackrel{\text{def}}{=} x$$

for each  $(1, x) \in \emptyset \amalg X$ .

*Proof. Unwinding the Definition of  $\emptyset \amalg X$ :* Firstly, we unwind the expressions for  $\emptyset \amalg X$ . We have

$$\begin{aligned} \emptyset \amalg X &\stackrel{\text{def}}{=} \{(0, z) \in S \mid z \in \emptyset\} \cup \{(1, x) \in S \mid x \in X\} \\ &= \emptyset \cup \{(1, x) \in S \mid x \in X\} \\ &= \{(1, x) \in S \mid x \in X\}, \end{aligned}$$

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

*Invertibility:* The inverse of  $\lambda_X^{\text{Sets}, \amalg}$  is the map

$$\lambda_X^{\text{Sets}, \amalg, -1} : X \rightarrow \emptyset \amalg X$$

given by

$$\lambda_X^{\text{Sets}, \amalg, -1}(x) \stackrel{\text{def}}{=} (1, x)$$

for each  $x \in X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[ \lambda_X^{\text{Sets}, \amalg, -1} \circ \lambda_X^{\text{Sets}, \amalg} \right](1, x) &= \lambda_X^{\text{Sets}, \amalg, -1} \left( \lambda_X^{\text{Sets}, \amalg}(1, x) \right) \\ &= \lambda_X^{\text{Sets}, \amalg, -1}(x) \\ &= (1, x) \\ &= \left[ \text{id}_{\emptyset \amalg X} \right](1, x) \end{aligned}$$

for each  $(1, x) \in \emptyset \amalg X$ , and therefore we have

$$\lambda_X^{\text{Sets}, \amalg, -1} \circ \lambda_X^{\text{Sets}, \amalg} = \text{id}_{\emptyset \amalg X}.$$



- *Invertibility II.* We have

$$\begin{aligned}
 \left[ \lambda_X^{\text{Sets}, \coprod} \circ \lambda_X^{\text{Sets}, \coprod, -1} \right] (x) &= \lambda_X^{\text{Sets}, \coprod} \left( \lambda_X^{\text{Sets}, \coprod, -1} (x) \right) \\
 &= \lambda_X^{\text{Sets}, \coprod, -1} (1, x) \\
 &= x \\
 &= [\text{id}_X] (x)
 \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\text{Sets}, \coprod} \circ \lambda_X^{\text{Sets}, \coprod, -1} = \text{id}_X.$$

Therefore  $\lambda_X^{\text{Sets}, \coprod}$  is indeed an isomorphism.

*Naturality:* We need to show that, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc}
 \emptyset \coprod X & \xrightarrow{\text{id}_{\emptyset} \coprod f} & \emptyset \coprod Y \\
 \lambda_X^{\text{Sets}, \coprod} \downarrow & & \downarrow \lambda_Y^{\text{Sets}, \coprod} \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (1, x) & \longmapsto & (1, f(x)) \\
 \downarrow & & \downarrow \\
 x & \longmapsto & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\text{Sets}, \coprod}$  is a natural transformation.  
*Being a Natural Isomorphism:* Since  $\lambda^{\text{Sets}, \coprod}$  is natural and  $\lambda^{\text{Sets}, -1}$  is a component-wise inverse to  $\lambda^{\text{Sets}, \coprod}$ , it follows from [Categories, Item 2 of Definition 11.9.7.1.2](#) that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\lambda^{\text{Sets}, \coprod}$  is a natural isomorphism.  $\square$

### 5.2.5 The Right Unitor

**Definition 5.2.5.1.1.** The **right unitor of the coproduct of sets** is the natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \mathbb{0}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}, \amalg} : X \amalg \emptyset \xrightarrow{\sim} X$$

at  $X$  is given by

$$\rho_X^{\text{Sets}, \amalg}((0, x)) \stackrel{\text{def}}{=} x$$

for each  $(0, x) \in X \amalg \emptyset$ .

*Proof. Unwinding the Definition of  $X \amalg \emptyset$ :* Firstly, we unwind the expression for  $X \amalg \emptyset$ . We have

$$\begin{aligned} X \amalg \emptyset &\stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, z) \in S \mid z \in \emptyset\} \\ &= \{(0, x) \in S \mid x \in X\} \cup \emptyset \\ &= \{(0, x) \in S \mid x \in X\}, \end{aligned}$$

where  $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$ .

*Invertibility:* The inverse of  $\rho_X^{\text{Sets}, \amalg}$  is the map

$$\rho_X^{\text{Sets}, \amalg, -1} : X \rightarrow X \amalg \emptyset$$

given by

$$\rho_X^{\text{Sets}, \amalg, -1}(x) \stackrel{\text{def}}{=} (0, x)$$

for each  $x \in X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned}
 \left[ \rho_X^{\text{Sets}, \coprod, -1} \circ \rho_X^{\text{Sets}, \coprod} \right] (0, x) &= \rho_X^{\text{Sets}, \coprod, -1} \left( \rho_X^{\text{Sets}, \coprod} (0, x) \right) \\
 &= \rho_X^{\text{Sets}, \coprod, -1} (x) \\
 &= (0, x) \\
 &= \left[ \text{id}_X \coprod \emptyset \right] (0, x)
 \end{aligned}$$

for each  $(0, x) \in \emptyset \coprod X$ , and therefore we have

$$\rho_X^{\text{Sets}, \coprod, -1} \circ \rho_X^{\text{Sets}, \coprod} = \text{id}_{\emptyset \coprod X}.$$

- *Invertibility II.* We have

$$\begin{aligned}
 \left[ \rho_X^{\text{Sets}, \coprod} \circ \rho_X^{\text{Sets}, \coprod, -1} \right] (x) &= \rho_X^{\text{Sets}, \coprod} \left( \rho_X^{\text{Sets}, \coprod, -1} (x) \right) \\
 &= \rho_X^{\text{Sets}, \coprod, -1} (0, x) \\
 &= x \\
 &= [\text{id}_X] (x)
 \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\text{Sets}, \coprod} \circ \rho_X^{\text{Sets}, \coprod, -1} = \text{id}_X.$$

Therefore  $\rho_X^{\text{Sets}, \coprod}$  is indeed an isomorphism.

*Naturality:* We need to show that, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc}
 X \coprod \emptyset & \xrightarrow{f \coprod \text{id}_{\emptyset}} & Y \coprod \emptyset \\
 \rho_X^{\text{Sets}, \coprod} \downarrow & & \downarrow \rho_Y^{\text{Sets}, \coprod} \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0, x) & & (0, x) \mapsto (1, f(x)) \\
 \downarrow & & \downarrow \\
 x & \mapsto & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\rho^{\text{Sets}, \coprod}$  is a natural transformation. *Being a Natural Isomorphism:* Since  $\rho^{\text{Sets}, \coprod}$  is natural and  $\rho^{\text{Sets}, -1}$  is a componentwise inverse to  $\rho^{\text{Sets}, \coprod}$ , it follows from **Categories, Item 2 of Definition 11.9.7.1.2** that  $\rho^{\text{Sets}, -1}$  is also natural. Thus  $\rho^{\text{Sets}, \coprod}$  is a natural isomorphism.  $\square$

### 5.2.6 The Symmetry

**Definition 5.2.6.1.1.** The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\text{Sets}, \coprod} : \coprod \xrightarrow{\sim} \coprod \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}, \quad \begin{array}{ccc} \text{Sets} \times \text{Sets} & \xrightarrow{\coprod} & \text{Sets}, \\ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2} \searrow & \Downarrow \sigma_{\text{Sets}, \coprod} & \nearrow \coprod \\ & \text{Sets} \times \text{Sets} & \end{array}$$

whose component

$$\sigma_{X,Y}^{\text{Sets}, \coprod} : X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at  $X, Y \in \text{Obj}(\text{Sets})$  is defined by

$$\sigma_{X,Y}^{\text{Sets}, \coprod}(x, y) \stackrel{\text{def}}{=} (y, x)$$

for each  $(x, y) \in X \times Y$ .

*Proof. Unwinding the Definitions of  $X \coprod Y$  and  $Y \coprod X$ :* Firstly, we unwind the expressions for  $X \coprod Y$  and  $Y \coprod X$ . We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},$$

where  $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$ .

*Invertibility:* The inverse of  $\sigma_{X,Y}^{\text{Sets}, \coprod}$  is the map

$$\sigma_{X,Y}^{\text{Sets}, \coprod, -1} : Y \coprod X \rightarrow X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets}, \coprod, -1} \stackrel{\text{def}}{=} \sigma_{Y,X}^{\text{Sets}, \coprod}$$

and hence given by

$$\sigma_{X,Y}^{\text{Sets}, \coprod, -1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = (1, x), \\ (1, y) & \text{if } z = (0, y) \end{cases}$$

for each  $z \in Y \coprod X$ . Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[ \sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} \right](0, x) &= \sigma_X^{\text{Sets}, \coprod, -1} \left( \sigma_X^{\text{Sets}, \coprod} (0, x) \right) \\ &= \sigma_X^{\text{Sets}, \coprod, -1} (1, x) \\ &= (0, x) \\ &= \left[ \text{id}_X \coprod \text{id}_Y \right](0, x) \end{aligned}$$

for each  $(0, x) \in X \coprod Y$  and

$$\begin{aligned} \left[ \sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} \right](1, y) &= \sigma_X^{\text{Sets}, \coprod, -1} \left( \sigma_X^{\text{Sets}, \coprod} (1, y) \right) \\ &= \sigma_X^{\text{Sets}, \coprod, -1} (0, y) \\ &= (1, y) \\ &= \left[ \text{id}_X \coprod \text{id}_Y \right](1, y) \end{aligned}$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} = \text{id}_{X \coprod Y}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[ \sigma_{X,Y}^{\text{Sets}, \coprod} \circ \sigma_{X,Y}^{\text{Sets}, \coprod, -1} \right](0, y) &= \sigma_X^{\text{Sets}, \coprod} \left( \sigma_X^{\text{Sets}, \coprod, -1} (0, y) \right) \\ &= \sigma_X^{\text{Sets}, \coprod, -1} (1, y) \\ &= (0, y) \\ &= \left[ \text{id}_Y \coprod \text{id}_X \right](0, y) \end{aligned}$$

for each  $(0, y) \in Y \amalg X$  and

$$\begin{aligned} \left[ \sigma_{X,Y}^{\text{Sets}, \amalg} \circ \sigma_{X,Y}^{\text{Sets}, \amalg, -1} \right] (1, x) &= \sigma_X^{\text{Sets}, \amalg} \left( \sigma_X^{\text{Sets}, \amalg, -1} (1, x) \right) \\ &= \sigma_X^{\text{Sets}, \amalg, -1} (0, x) \\ &= (1, x) \\ &= \left[ \text{id}_Y \amalg \text{id}_X \right] (1, x) \end{aligned}$$

for each  $(1, x) \in Y \amalg X$ , and therefore we have

$$\sigma_X^{\text{Sets}, \amalg} \circ \sigma_X^{\text{Sets}, \amalg, -1} = \text{id}_{Y \amalg X}.$$

Therefore  $\sigma_{X,Y}^{\text{Sets}, \amalg}$  is indeed an isomorphism.

*Naturality:* We need to show that, given functions  $f: A \rightarrow X$  and  $g: B \rightarrow Y$ , the diagram

$$\begin{array}{ccc} A \amalg B & \xrightarrow{f \amalg g} & X \amalg Y \\ \sigma_{A,B}^{\text{Sets}, \amalg} \downarrow & & \downarrow \sigma_{X,Y}^{\text{Sets}, \amalg} \\ B \amalg A & \xrightarrow{g \amalg f} & Y \amalg X \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (0, a) & \xrightarrow{\quad} & (0, f(a)) \\ \downarrow & & \downarrow \\ (1, a) & \xrightarrow{\quad} & (1, f(a)) \\ \\ (1, b) & \xrightarrow{\quad} & (1, g(b)) \\ \downarrow & & \downarrow \\ (0, b) & \xrightarrow{\quad} & (0, g(b)) \end{array}$$

and hence indeed commutes. Therefore  $\sigma^{\text{Sets}, \amalg}$  is a natural transformation.

*Being a Natural Isomorphism:* Since  $\sigma^{\text{Sets}, \amalg}$  is natural and  $\sigma^{\text{Sets}, -1}$  is a componentwise inverse to  $\sigma^{\text{Sets}, \amalg}$ , it follows from [Categories, Item 2 of Definition 11.9.7.1.2](#) that  $\sigma^{\text{Sets}, -1}$  is also natural. Thus  $\sigma^{\text{Sets}, \amalg}$  is a natural isomorphism.  $\square$

## 5.2.7 The Monoidal Category of Sets and Coproducts

**Proposition 5.2.7.1.1.** The category Sets admits a closed symmetric monoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Monoidal Product.* The coproduct functor

$$\amalg : \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of [Constructions With Sets, Item 1 of Definition 4.2.3.1.3](#).

- *The Monoidal Unit.* The functor

$$\mathbb{0}^{\text{Sets}} : \text{pt} \rightarrow \text{Sets}$$

of [Definition 5.2.2.1.1](#).

- *The Associators.* The natural isomorphism

$$\alpha^{\text{Sets}, \amalg} : \amalg \circ (\amalg \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \amalg \circ (\text{id}_{\text{Sets}} \times \amalg) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of [Definition 5.2.3.1.1](#).

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}, \amalg} : \amalg \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.4.1.1](#).

- *The Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \mathbb{0}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.5.1.1](#).

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}, \coprod} : \times \xrightarrow{\sim} \times \circ \sigma^{\text{Cats}_2}_{\text{Sets}, \text{Sets}}$$

of **Definition 5.2.6.1.1**.

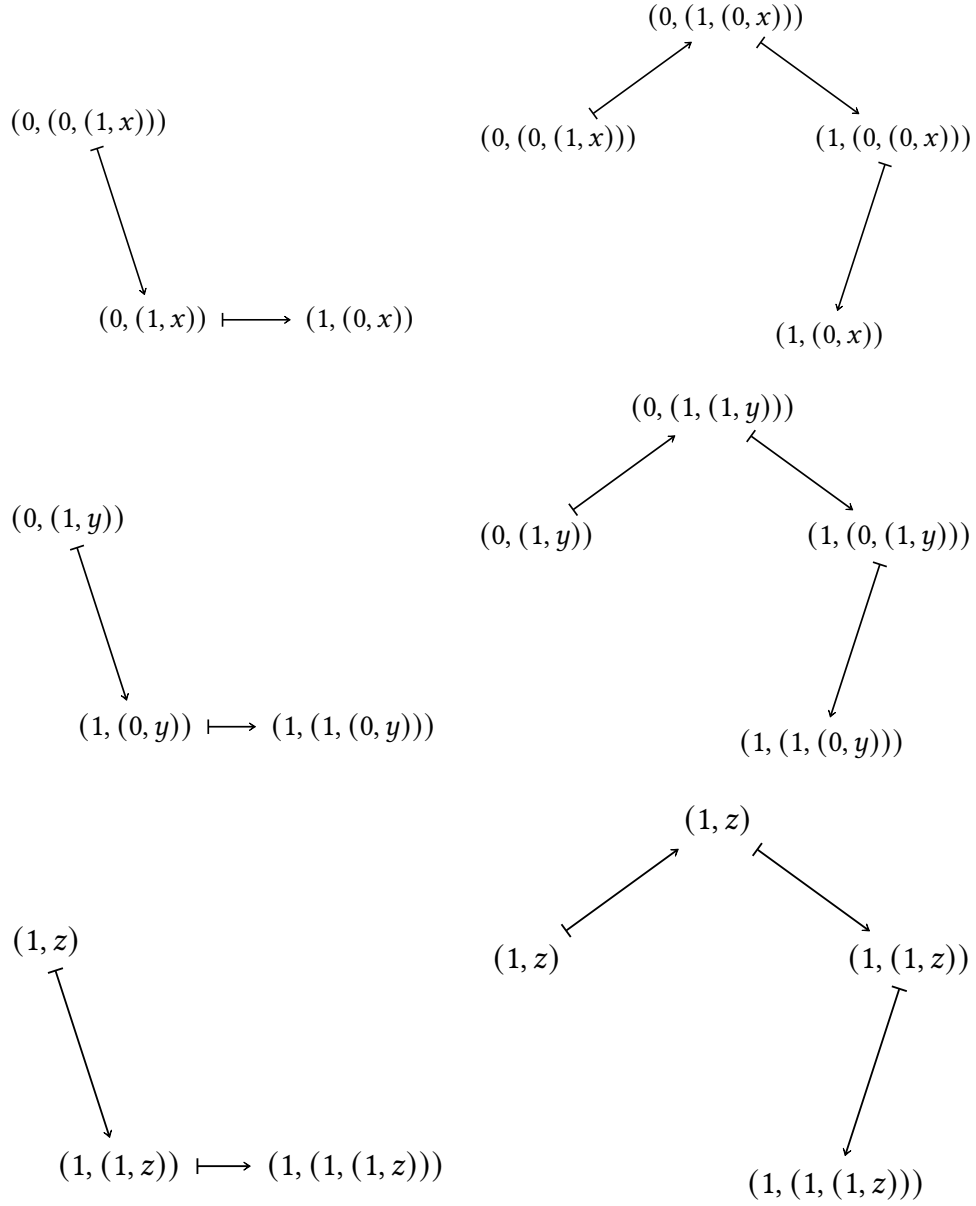
*Proof. The Pentagon Identity:* Let  $W, X, Y$  and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & (W \coprod (X \coprod Y)) \coprod Z & & \\
 & \nearrow^{\alpha^{\text{Sets}, \coprod}_{W, X, Y} \coprod \text{id}_Z} & & \searrow^{\alpha^{\text{Sets}, \coprod}_{W, X} \coprod \text{id}_{Y, Z}} & \\
 ((W \coprod X) \coprod Y) \coprod Z & & & & W \coprod ((X \coprod Y) \coprod Z) \\
 \searrow^{\alpha^{\text{Sets}, \coprod}_W \coprod \text{id}_{X, Y, Z}} & & & & \swarrow^{\text{id}_W \coprod \alpha^{\text{Sets}, \coprod}_{X, Y, Z}} \\
 (W \coprod X) \coprod (Y \coprod Z) & \xrightarrow{\alpha^{\text{Sets}, \coprod}_{W, X, Y} \coprod Z} & W \coprod (X \coprod (Y \coprod Z)) & & 
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 & & (0, (0, w)) & & \\
 & \swarrow & & \searrow & \\
 (0, (0, (0, w))) & & & & (0, w) \\
 \searrow & & & & \swarrow \\
 (0, (0, w)) & \longmapsto & (0, w) & & 
 \end{array}$$





and therefore the pentagon identity is satisfied.

*The Triangle Identity:* Let  $X$  and  $Y$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \amalg \emptyset) \amalg Y & \xrightarrow{\alpha_{X,\emptyset,Y}^{\text{Sets}, \amalg}} & X \amalg (\emptyset \amalg Y) \\
 \searrow \rho_X^{\text{Sets}, \amalg} \amalg \text{id}_Y & & \swarrow \text{id}_X \amalg \lambda_Y^{\text{Sets}, \amalg} \\
 & X \amalg Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

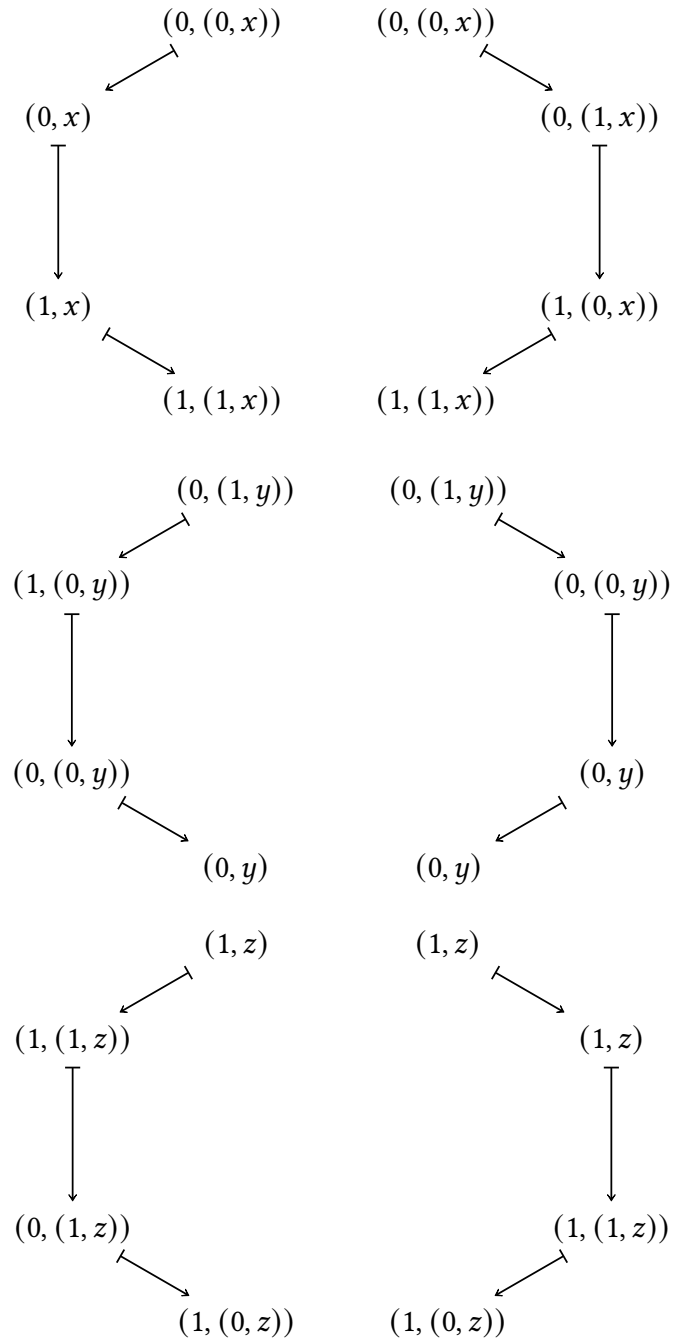
$$\begin{array}{ccc}
 (0, (1, x)) & & (1, (0, x)) \xrightarrow{\quad} (0, x) \\
 \searrow & & \swarrow \\
 & (0, x) & \\
 \\ 
 (1, y) & & (1, y) \xrightarrow{\quad} (1, (1, y)) \\
 \searrow & & \swarrow \\
 & (1, y) &
 \end{array}$$

and therefore the triangle identity is satisfied.

*The Left Hexagon Identity:* Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \amalg Y) \amalg Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}} \swarrow & & \searrow \sigma_{X,Y}^{\text{Sets}} \amalg \text{id}_Z \\
 X \amalg (Y \amalg Z) & & (Y \amalg X) \amalg Z \\
 \downarrow \sigma_{X,Y}^{\text{Sets}} \amalg Z & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}} \\
 (Y \amalg Z) \amalg X & & Y \amalg (X \amalg Z) \\
 \alpha_{Y,Z,X}^{\text{Sets}} \swarrow & & \swarrow \text{id}_Y \amalg \sigma_{X,Z}^{\text{Sets}} \\
 & Y \amalg (Z \amalg X) &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

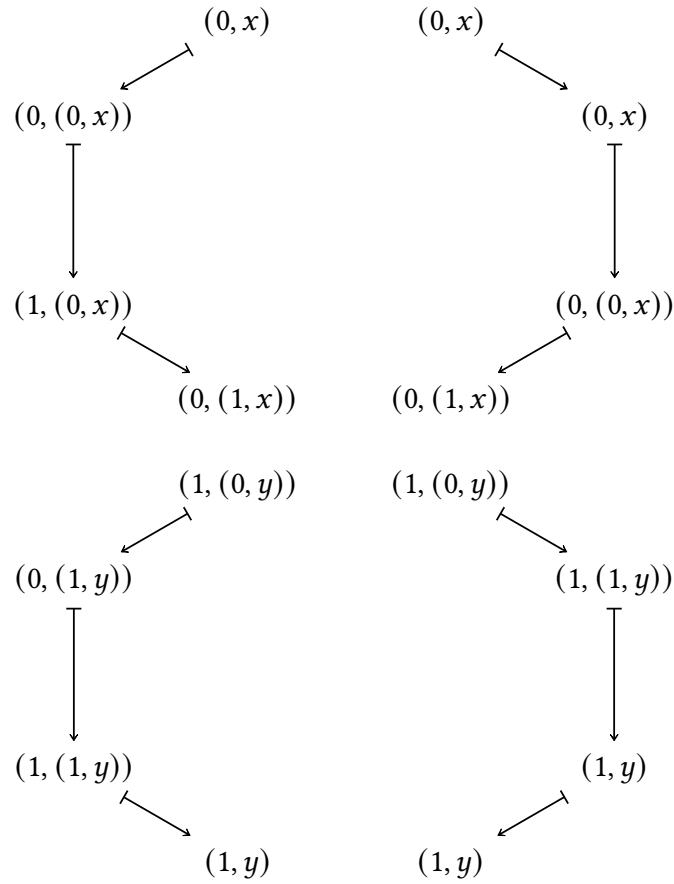


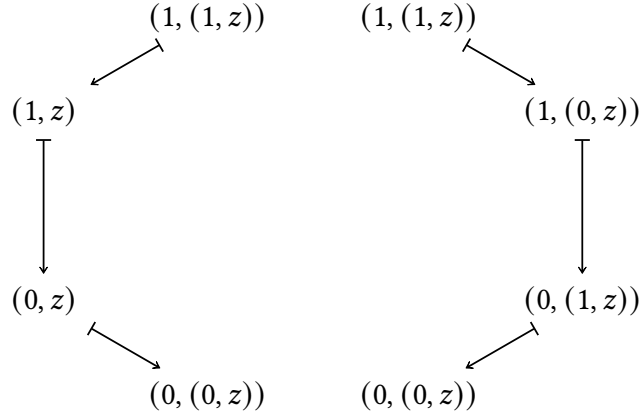
and thus the left hexagon identity is satisfied.

*The Right Hexagon Identity:* Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (\alpha_{X,Y,Z}^{\text{Sets}})^{-1} X \amalg (Y \amalg Z) & \\
 \swarrow & & \searrow \text{id}_X \amalg \sigma_{Y,Z}^{\text{Sets}} \\
 (X \amalg Y) \amalg Z & & X \amalg (Z \amalg Y) \\
 \downarrow \sigma_{X \amalg Y, Z}^{\text{Sets}} & & \downarrow (\alpha_{X,Z,Y}^{\text{Sets}})^{-1} \\
 Z \amalg (X \amalg Y) & & (X \amalg Z) \amalg Y \\
 \searrow (\alpha_{Z,X,Y}^{\text{Sets}})^{-1} & & \swarrow \sigma_{X,Z}^{\text{Sets}} \amalg \text{id}_Y \\
 & (Z \amalg X) \amalg Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as





and thus the right hexagon identity is satisfied.  $\square$

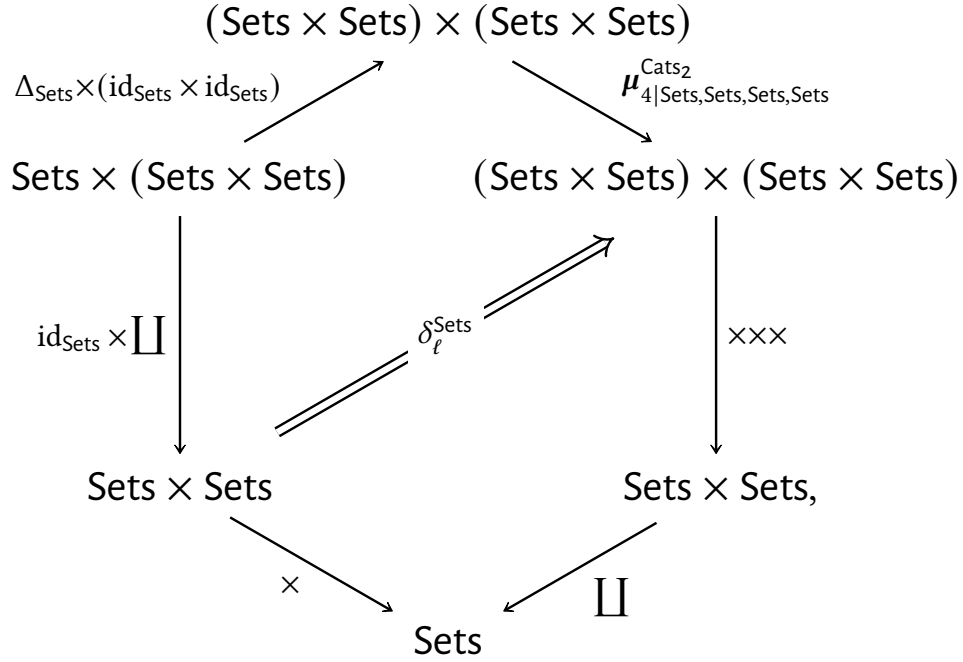
## 5.3 The Bimonoidal Category of Sets, Products, and Coproducts

### 5.3.1 The Left Distributor

**Definition 5.3.1.1.1.** The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_\ell^{\text{Sets}} : \times \circ (\text{id}_{\text{Sets}} \times \coprod) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\Delta_{\text{Sets}} \times (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}))$$

as in the diagram



whose component

$$\delta_{\ell|X,Y,Z}^{\text{Sets}}: X \times (Y \amalg Z) \xrightarrow{\sim} (X \times Y) \amalg (X \times Z)$$

at  $(X, Y, Z)$  is defined by

$$\delta_{\ell|X,Y,Z}^{\text{Sets}}(x, a) \stackrel{\text{def}}{=} \begin{cases} (0, (x, y)) & \text{if } a = (0, y), \\ (1, (x, z)) & \text{if } a = (1, z) \end{cases}$$

for each  $(x, a) \in X \times (Y \amalg Z)$ .

*Proof.* Omitted. □

### 5.3.2 The Right Distributor

**Definition 5.3.2.1.1.** The **right distributor of the product of sets over the co-product of sets** is the natural isomorphism

$$\delta_r^{\text{Sets}}: \times \circ (\amalg \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \amalg \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ ((\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}})$$

as in the diagram

$$\begin{array}{ccc}
 & (\text{Sets} \times \text{Sets}) \times (\text{Sets} \times \text{Sets}) & \\
 (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}} \nearrow & & \searrow \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \\
 (\text{Sets} \times \text{Sets}) \times \text{Sets} & & (\text{Sets} \times \text{Sets}) \times (\text{Sets} \times \text{Sets}) \\
 \downarrow \coprod \times \text{id}_{\text{Sets}} & \delta_r^{\text{Sets}} \nearrow & \downarrow \times \times \times \\
 \text{Sets} \times \text{Sets} & & \text{Sets} \times \text{Sets}, \\
 \searrow \times & & \swarrow \coprod \\
 & \text{Sets} &
 \end{array}$$

whose component

$$\delta_{r|X,Y,Z}^{\text{Sets}} : (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at  $(X, Y, Z)$  is defined by

$$\delta_{r|X,Y,Z}^{\text{Sets}}(a, z) \stackrel{\text{def}}{=} \begin{cases} (0, (x, z)) & \text{if } a = (0, x), \\ (1, (y, z)) & \text{if } a = (1, y) \end{cases}$$

for each  $(a, z) \in (X \coprod Y) \times Z$ .

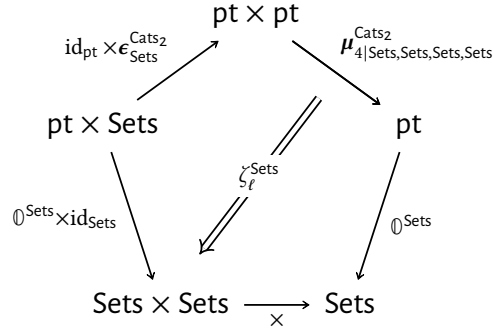
*Proof.* Omitted. □

### 5.3.3 The Left Annihilator

**Definition 5.3.3.1.1.** The **left annihilator of the product of sets** is the natural isomorphism

$$\gamma_{\ell}^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\text{id}_{\text{pt}} \times \epsilon_{\text{Sets}}^{\text{Cats}_2}) \xrightarrow{\sim} \times \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}})$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\text{Sets}} : \emptyset \times A \dashrightarrow \emptyset.$$

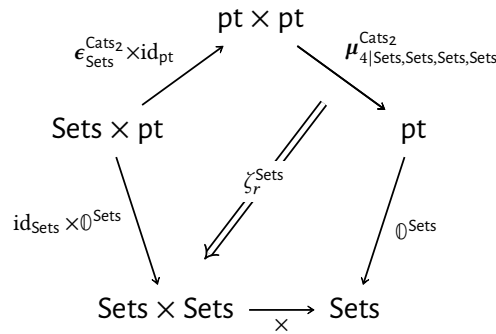
*Proof.* Omitted. For a partial proof, see [Pro25]. □

### 5.3.4 The Right Annihilator

**Definition 5.3.4.1.1.** The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\text{Sets}} : \emptyset^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ \left( \epsilon_{\text{Sets}}^{\text{Cats}_2} \times \text{id}_{\text{pt}} \right) \dashrightarrow \times \circ \left( \text{id}_{\text{Sets}} \times \emptyset^{\text{Sets}} \right)$$

as in the diagram



with components

$$\zeta_{r|A}^{\text{Sets}} : A \times \emptyset \dashrightarrow \emptyset.$$

*Proof.* Omitted. For a partial proof, see [Pro25]. □



### 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

**Proposition 5.3.5.1.1.** The category **Sets** admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category.* The category **Sets** of pointed sets.
- *The Additive Monoidal Product.* The coproduct functor

$$\amalg : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Definition 4.2.3.1.3**.

- *The Multiplicative Monoidal Product.* The product functor

$$\times : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Definition 4.1.3.1.3**.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}} : \mathbf{pt} \rightarrow \mathbf{Sets}$$

of **Definition 5.1.3.1.1**.

- *The Monoidal Zero.* The functor

$$\mathbb{0}^{\mathbf{Sets}} : \mathbf{pt} \rightarrow \mathbf{Sets}$$

of **Definition 5.1.3.1.1**.

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets} : \mathbf{Sets}^{\mathbf{op}} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, ?? of ??.

- *The Additive Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}, \amalg} : \amalg \circ (\amalg \times \mathrm{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \amalg \circ (\mathrm{id}_{\mathbf{Sets}} \times \amalg) \circ \alpha_{\mathbf{Sets}, \mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}}$$

of **Definition 5.2.3.1.1**.

- *The Additive Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}, \amalg} : \amalg \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.4.1.1](#).

- *The Additive Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \mathbb{0}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.5.1.1](#).

- *The Additive Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}, \amalg} : \amalg \xRightarrow{\sim} \amalg \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.6.1.1](#).

- *The Multiplicative Associators.* The natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of [Definition 5.1.4.1.1](#).

- *The Multiplicative Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.1.5.1.1](#).

- *The Multiplicative Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.1.6.1.1](#).

- *The Multiplicative Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.1.7.1.1](#).

- *The Left Distributor.* The natural isomorphism

$$\delta_\ell^{\text{Sets}} : \times \circ (\text{id}_{\text{Sets}} \times \coprod) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\Delta_{\text{Sets}} \times (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}))$$

of [Definition 5.3.1.1.1](#).

- *The Right Distributor.* The natural isomorphism

$$\delta_r^{\text{Sets}} : \times \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ ((\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}})$$

of [Definition 5.3.2.1.1](#).

- *The Left Annihilator.* The natural isomorphism

$$\zeta_\ell^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\text{id}_{\text{pt}} \times \epsilon_{\text{Sets}}^{\text{Cats}_2}) \xrightarrow{\sim} \times \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}})$$

of [Definition 5.3.3.1.1](#).

- *The Right Annihilator.* The natural isomorphism

$$\zeta_r^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\epsilon_{\text{Sets}}^{\text{Cats}_2} \times \text{id}_{\text{pt}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \mathbb{0}^{\text{Sets}})$$

of [Definition 5.3.4.1.1](#).

*Proof.* Omitted. □

# Appendices

## A Other Chapters

### Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

### Sets

3. [Sets](#)

### 4. [Constructions With Sets](#)

### 5. [Monoidal Structures on the Category of Sets](#)

### 6. [Pointed Sets](#)

### 7. [Tensor Products of Pointed Sets](#)

**Relations**

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

**Categories**

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

**Monoidal Categories**

- 13. Constructions With Monoidal Categories

**Bicategories**

- 14. Types of Morphisms in Bicat-egories

**Extra Part**

- 15. Notes

**References**

- [Pro25] ProofWiki Contributors. *Cartesian Product Is Empty Iff Factor Is Empty*—*ProofWiki*. 2025. URL: [https://proofwiki.org/wiki/Cartesian\\_Product\\_is\\_Empty\\_iff\\_Factor\\_is\\_Empty](https://proofwiki.org/wiki/Cartesian_Product_is_Empty_iff_Factor_is_Empty) (cit. on p. 56).