Types of Morphisms in Bicategories

The Clowder Project Authors

July 21, 2025

In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

Contents

14.1   Monomorphisms in Bicategories

14.1.1   Representably Faithful Morphisms

Let C be a bicategory.

DEFINITION 14.1.1.1.1 ▶ REPRESENTABLY FAITHFUL MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

---

[1]*Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 14.1.1.1.3.

REMARK 14.1.1.1.2 ▶ UNWINDING DEFINITION 14.1.1.1.1

In detail, f is representably faithful if, for all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}}{}_{\alpha \Downarrow\ \Downarrow \beta} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

EXAMPLE 14.1.1.1.3 ▶ EXAMPLES OF REPRESENTABLY FAITHFUL MORPHISMS

Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Categories, Item 2 of Proposition 11.6.1.1.2.

2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Proposition 8.5.11.1.1.

14.1.2   Representably Full Morphisms

Let C be a bicategory.

DEFINITION 14.1.2.1.1 ▶ REPRESENTABLY FULL MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **representably full**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is full.

---

[1]*Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Example 14.1.2.1.3.

REMARK 14.1.2.1.2 ▶ UNWINDING DEFINITION 14.1.2.1.1

In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B \quad \beta\big\Downarrow$$

of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad X \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha} A$$

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha} A \overset{f}{\longrightarrow} B \quad = \quad X \overset{f \circ \phi}{\underset{f \circ \psi}{\Longrightarrow}}_{\beta} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

EXAMPLE 14.1.2.1.3 ▶ EXAMPLES OF REPRESENTABLY FULL MORPHISMS

Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Categories, **??** of Proposition 11.6.2.1.2.

2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 8.5.11.1.1.

14.1.3   Representably Fully Faithful Morphisms

Let C be a bicategory.

DEFINITION 14.1.3.1.1 ▶ REPRESENTABLY FULLY FAITHFUL MORPHISMS

A 1-morphism $f \colon A \to B$ of C is **representably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1)

and representably full (Definition 14.1.2.1.1).

2. For each $X \in \mathsf{Obj}(C)$, the functor

$$f_* \colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

---

[1]*Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Example 14.1.3.1.3.

REMARK 14.1.3.1.2 ▸ UNWINDING REPRESENTABLY FULLY FAITHFUL MORPHISMS

In detail, f is representably fully faithful if the conditions in Remark 14.1.1.1.2 and Remark 14.1.2.1.2 hold:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow \Downarrow \beta}{\rightrightarrows}}} A \xrightarrow{f} B,$$

    if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

    then $\alpha = \beta$.

2. For each $X \in \mathsf{Obj}(C)$ and each 2-morphism

$$\beta \colon f \circ \phi \Longrightarrow f \circ \psi, \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\underset{\beta \Downarrow}{\rightrightarrows}}} B$$

    of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \quad X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow}{\rightrightarrows}}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Rightarrow}} \underset{\alpha\Downarrow}{} A \xrightarrow{f} B \quad = \quad X \underset{f\circ\psi}{\overset{f\circ\phi}{\Rightarrow}} \underset{\beta\Downarrow}{} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

EXAMPLE 14.1.3.1.3 ▶ EXAMPLES OF REPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 6 of Proposition 11.6.3.1.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 8.5.11.1.1) with the representably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 8.5.11.1.1.

14.1.4   Morphisms Representably Faithful on Cores

Let C be a bicategory.

DEFINITION 14.1.4.1.1 ▶ MORPHISMS REPRESENTABLY FAITHFUL ON CORES

A 1-morphism $f \colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

> **REMARK 14.1.4.1.2 ▶ UNWINDING DEFINITION 14.1.4.1.1**
>
> In detail, f is representably faithful on cores if, for all diagrams in C of the form
>
> $$X \; \overset{\phi}{\underset{\psi}{\alpha\Downarrow\Downarrow\beta}} \; A \xrightarrow{\;f\;} B,$$
>
> if α and β are 2-isomorphisms and we have
>
> $$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
>
> then $\alpha = \beta$.

14.1.5   Morphisms Representably Full on Cores

Let C be a bicategory.

> **DEFINITION 14.1.5.1.1 ▶ MORPHISMS REPRESENTABLY FULL ON CORES**
>
> A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f_*\colon \mathrm{Core}(\mathrm{Hom}_C(X,A)) \to \mathrm{Core}(\mathrm{Hom}_C(X,B))$$
>
> given by postcomposition by f is full.

> **REMARK 14.1.5.1.2 ▶ UNWINDING DEFINITION 14.1.5.1.1**
>
> In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism
>
> $$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \; \overset{f\circ\phi}{\underset{f\circ\psi}{\beta\Downarrow}} \; B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \overset{\phi}{\underset{\psi}{\Longrightarrow}} A$$

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow}} A \overset{f}{\longrightarrow} B \quad = \quad X \overset{f\circ\phi}{\underset{f\circ\psi}{\Longrightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.6   Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

DEFINITION 14.1.6.1.1 ▶ MORPHISMS REPRESENTABLY FULLY FAITHFUL ON CORES

A 1-morphism $f \colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is fully faithful.

REMARK 14.1.6.1.2 ▶ UNWINDING DEFINITION 14.1.6.1.1

In detail, f is representably fully faithful on cores if the conditions in Remark 14.1.4.1.2 and Remark 14.1.5.1.2 hold:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}} \alpha\Downarrow\Downarrow\beta \; A \xrightarrow{f} B,$$

    if α and β are 2-isomorphisms and we have

    $$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

    then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\beta\Downarrow}} B$$

    of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\alpha\Downarrow}} A$$

    of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\alpha\Downarrow}} A \xrightarrow{f} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\beta\Downarrow}} B$$

    of pasting diagrams in C, i.e. such that we have

    $$\beta = \mathrm{id}_f \star \alpha.$$

14.1.7   Representably Essentially Injective Morphisms

Let C be a bicategory.

DEFINITION 14.1.7.1.1 ▶ REPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

REMARK 14.1.7.1.2 ▶ UNWINDING DEFINITION 14.1.7.1.1

In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

  ($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

14.1.8   Representably Conservative Morphisms

Let C be a bicategory.

DEFINITION 14.1.8.1.1 ▶ REPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

> **REMARK 14.1.8.1.2 ▶ UNWINDING DEFINITION 14.1.8.1.1**
>
> In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism
>
> $$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A \quad \alpha\Downarrow$$
>
> of C, if the 2-morphism
>
> $$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \quad \mathrm{id}_f \star \alpha \quad B \quad \overset{f \circ \phi}{\underset{f \circ \psi}{\Downarrow}}$$
>
> is a 2-isomorphism, then so is α.

14.1.9   Strict Monomorphisms

Let C be a bicategory.

> **DEFINITION 14.1.9.1.1 ▶ STRICT MONOMORPHISMS**
>
> A 1-morphism $f\colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$
>
> given by postcomposition by f is injective on objects, i.e. its action on objects
>
> $$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$
>
> is injective.

> **REMARK 14.1.9.1.2 ▶ UNWINDING DEFINITION 14.1.9.1.1**
>
> In detail, f is a strict monomorphism in C if, for each diagram in C of the form
>
> $$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$
>
> if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

> **EXAMPLE 14.1.9.1.3 ▶ EXAMPLES OF STRICT MONOMORPHISMS**
>
> Here are some examples of strict monomorphisms.
>
> 1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Proposition 11.7.2.1.2.
>
> 2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, Proposition 8.5.10.1.1.

14.1.10   Pseudomonic Morphisms

Let C be a bicategory.

> **DEFINITION 14.1.10.1.1 ▶ PSEUDOMONIC MORPHISMS**
>
> A 1-morphism $f : A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f_* : \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$
>
> given by postcomposition by f is pseudomonic.

> **REMARK 14.1.10.1.2 ▶ UNWINDING DEFINITION 14.1.10.1.1**
>
> In detail, a 1-morphism $f : A \to B$ of C is pseudomonic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\Rrightarrow}} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rightarrow}} B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\Rightarrow}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Rightarrow}} A \xrightarrow{f} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Rightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

PROPOSITION 14.1.10.1.3 ▶ PROPERTIES OF PSEUDOMONIC MORPHISMS

Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) The morphism f is representably full on cores and representably faithful.

(c) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \stackrel{\leftrightarrow}{\times}_B A,$$

$$\begin{array}{ccc} A & \xrightarrow{\ \text{id}_A\ } & A \\ {\scriptstyle \text{id}_A}\downarrow & \nearrow & \downarrow{\scriptstyle F} \\ A & \xrightarrow[F]{} & B \end{array}$$

in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \stackrel{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B,$$

$$\begin{array}{ccc} A & \longrightarrow & \mathbb{1} \pitchfork A \\ {\scriptstyle F}\downarrow & \nearrow & \downarrow{\scriptstyle \mathbb{1}\pitchfork F} \\ B & \longrightarrow & \mathbb{1} \pitchfork B \end{array}$$

in C up to equivalence.

PROOF 14.1.10.1.4 ▶ PROOF OF PROPOSITION 14.1.10.1.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted.

14.2   Epimorphisms in Bicategories

14.2.1   Corepresentably Faithful Morphisms

Let C be a bicategory.

DEFINITION 14.2.1.1.1 ▶ COREPRESENTABLY FAITHFUL MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is faithful.

REMARK 14.2.1.1.2 ▶ UNWINDING DEFINITION 14.2.1.1.1

In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \Downarrow \ \Downarrow \beta}}} X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

EXAMPLE 14.2.1.1.3 ▶ EXAMPLES OF COREPRESENTABLY FAITHFUL MORPHISMS

Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 5 of Proposition 11.6.1.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Relations, Item 1 of Proposition 8.5.13.1.1.

14.2.2   Corepresentably Full Morphisms

Let C be a bicategory.

DEFINITION 14.2.2.1.1 ► COREPRESENTABLY FULL MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **corepresentably full** if, for each $X \in$ Obj(C), the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is full.

REMARK 14.2.2.1.2 ► UNWINDING DEFINITION 14.2.2.1.1

In detail, f is corepresentably full if, for each $X \in$ Obj(C) and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \quad \beta\Downarrow$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \quad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad \alpha\Downarrow$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \; \alpha\Downarrow \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X \; \beta\Downarrow$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

> **EXAMPLE 14.2.2.1.3 ▶ EXAMPLES OF COREPRESENTABLY FULL MORPHISMS**
>
> Here are some examples of corepresentably full morphisms.
>
> 1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 7 of Proposition 11.6.2.1.2.
>
> 2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in Relations, Item 2 of Proposition 8.5.13.1.1.

14.2.3   Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

> **DEFINITION 14.2.3.1.1 ▶ COREPRESENTABLY FULLY FAITHFUL MORPHISMS**
>
> A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[1] if the following equivalent conditions are satisfied:
>
> 1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).
>
> 2. For each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$
>
> given by precomposition by f is fully faithful.
>
> _____
>
> [1]*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

> **REMARK 14.2.3.1.2 ▶ UNWINDING DEFINITION 14.2.3.1.1**
>
> In detail, f is corepresentably fully faithful if the conditions in Remark 14.2.1.1.2 and Remark 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X,$$

   if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

   of C, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

   of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

EXAMPLE 14.2.3.1.3 ▶ EXAMPLES OF COREPRESENTABLY FULLY FAITHFUL MORPHISMS

Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful

epimorphisms in Cats_2 are characterised in Categories, Item 10 of Proposition 11.6.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Proposition 8.5.13.1.1) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Proposition 8.5.13.1.1.

14.2.4   Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

DEFINITION 14.2.4.1.1 ▶ MORPHISMS COREPRESENTABLY FAITHFUL ON CORES

A 1-morphism $f \colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Core}(\mathsf{Hom}_C(B, X)) \to \mathsf{Core}(\mathsf{Hom}_C(A, X))$$

given by precomposition by f is faithful.

REMARK 14.2.4.1.2 ▶ UNWINDING DEFINITION 14.2.4.1.1

In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \overset{\phi}{\underset{\psi}{\underset{}{\overrightarrow{\alpha \Downarrow \Downarrow \beta}}}} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

14.2.5   Morphisms Corepresentably Full on Cores

Let C be a bicategory.

DEFINITION 14.2.5.1.1 ▶ MORPHISMS COREPRESENTABLY FULL ON CORES

A 1-morphism $f\colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is full.

REMARK 14.2.5.1.2 ▶ UNWINDING DEFINITION 14.2.5.1.1

In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

of C such that we have an equality

$$A \overset{f}{\longrightarrow} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f\,.$$

14.2.6   Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

DEFINITION 14.2.6.1.1 ▶ MORPHISMS COREPRESENTABLY FULLY FAITHFUL ON CORES

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is fully faithful.

REMARK 14.2.6.1.2 ▶ UNWINDING DEFINITION 14.2.6.1.1

In detail, f is corepresentably fully faithful on cores if the conditions in Remark 14.2.4.1.2 and Remark 14.2.5.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\;f\;} B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha\!\Downarrow\;\Downarrow\beta}}} X,$$

   if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\underset{\Longrightarrow}{\beta\Downarrow}}} X$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha}}} X$$

of C such that we have an equality

$$A \overset{f}{\longrightarrow} B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha}}} X \quad = \quad A \underset{\psi\circ f}{\overset{\phi\circ f}{\underset{\Longrightarrow}{\beta}}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

14.2.7   Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

DEFINITION 14.2.7.1.1 ▶ COREPRESENTABLY ESSENTIALLY INJECTIVE MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

REMARK 14.2.7.1.2 ▶ UNWINDING DEFINITION 14.2.7.1.1

In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of C, the following condition is satisfied:

(★) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

14.2.8   Corepresentably Conservative Morphisms

Let C be a bicategory.

DEFINITION 14.2.8.1.1 ▶ COREPRESENTABLY CONSERVATIVE MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is conservative.

REMARK 14.2.8.1.2 ▶ UNWINDING DEFINITION 14.2.8.1.1

In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \;\underset{\psi}{\overset{\phi}{\Longrightarrow}}\; X$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \;\underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}}\; X$$

is a 2-isomorphism, then so is α.

14.2.9   Strict Epimorphisms

Let C be a bicategory.

> **DEFINITION 14.2.9.1.1 ▶ STRICT EPIMORPHISMS**
>
> A 1-morphism $f\colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor
>
> $$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$
>
> given by precomposition by f is injective on objects, i.e. its action on objects
> $$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_C(A, X))$$
> is injective.

> **REMARK 14.2.9.1.2 ▶ UNWINDING DEFINITION 14.2.9.1.1**
>
> In detail, f is a strict epimorphism if, for each diagram in C of the form
>
> $$A \xrightarrow{\ f\ } B \mathrel{\substack{\phi \\ \longrightarrow \\ \psi}} X,$$
>
> if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

> **EXAMPLE 14.2.9.1.3 ▶ EXAMPLES OF STRICT EPIMORPHISMS**
>
> Here are some examples of strict epimorphisms.
>
> 1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Proposition 11.7.3.1.2.
>
> 2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, Proposition 8.5.12.1.1.

14.2.10   Pseudoepic Morphisms

Let C be a bicategory.

DEFINITION 14.2.10.1.1 ▶ PSEUDOEPIC MORPHISMS

A 1-morphism $f\colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is pseudomonic.

REMARK 14.2.10.1.2 ▶ UNWINDING DEFINITION 14.2.10.1.1

In detail, a 1-morphism $f\colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$A \xrightarrow{\;f\;} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

 $\alpha \| \, \| \beta$

 if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

 then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Rightarrow}} X$$

 of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Rightarrow}} X$$

 of C such that we have an equality

$$A \xrightarrow{\;f\;} B \underset{\psi}{\overset{\phi}{\Rightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Rightarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f \,.$$

PROPOSITION 14.2.10.1.3 ▶ PROPERTIES OF PSEUDOEPIC MORPHISMS

Let $f \colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.

   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

   (c) We have an isococomma square of the form

   $$B \overset{\mathrm{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod_A} B,$$

   in C up to equivalence.

PROOF 14.2.10.1.4 ▶ PROOF OF PROPOSITION 14.2.10.1.3

Item 1: Characterisations

Omitted.

Appendices

A   Other Chapters

References

[Adá+01]   Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. issn: 1201-561X (cit. on p. 17).