Constructions With Monoidal Categories

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O1UF This chapter contains some material on constructions with monoidal categories.

Contents

	13.1	Moduli Categories of Monoidal Structures	1
			1
	Cate	13.1.2 The Moduli Category of Braided Monoidal Structures on a gory	15
	Gute	13.1.3 The Moduli Category of Symmetric Monoidal Structures on	10
	a Cai	tegory	15
	13.2	Moduli Categories of Closed Monoidal Structures	15
	13.3 Moduli Categories of Refinements of Monoidal Structures 13.3.1 The Moduli Category of Braided Refinements of a Monoidal		15
	Stru	cture	15
	A	Other Chapters	15
01UG	13.1	Moduli Categories of Monoidal Structures	
01UH	13.1	1 The Moduli Category of Monoidal Structures on a Ca	te-
		gory	
	Let <i>C</i> be a category.		

Definition 13.1.1.1.1. The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{B}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathrm{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, \qquad \bigvee_{\mathbb{E}_1} \mathcal{M}_{\mathbb{E}_1}(C) \longrightarrow \mathsf{MonCats}$$

$$\mathsf{pt} \xrightarrow{[C]} \mathsf{Cats}.$$

- **Q1UK** Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:
 - Objects. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
 - *Morphisms*. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor $id_C: C \to C$ of C.

• *Identities*. For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \stackrel{\operatorname{def}}{=} (\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes}),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of ??.

• *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,P)$$
of $\mathcal{M}_{\mathbb{E}_1}(C)$ at (M,N,P) is defined by

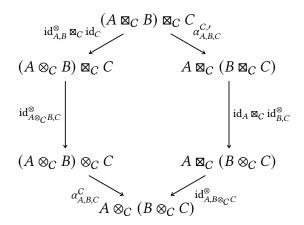
$$\left(\operatorname{id}_{\mathcal{C}}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{H}_{1}}(\mathcal{C})}\left(\operatorname{id}_{\mathcal{C}}^{\otimes},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)\stackrel{\text{def}}{=}\left(\operatorname{id}_{\mathcal{C}}^{\otimes,\prime}\circ\operatorname{id}_{\mathcal{C}}^{\otimes},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right).$$

- **Remark 13.1.1.13.** In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:
- 01UM 1. *Naturality*. For each pair $f: A \to X$ and $g: B \to Y$ of morphisms of C, the diagram

$$\begin{array}{ccccc} A \boxtimes_C B & \xrightarrow{f\boxtimes_C g} & X \boxtimes_C Y \\ \operatorname{id}_{A,B}^{\otimes} & & & & & & & & & & & \\ A \otimes_C B & \xrightarrow{f\otimes_C g} & X \otimes_C Y & & & & & & \end{array}$$

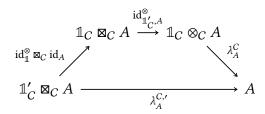
commutes.

Olum 2. *Monoidality.* For each $A, B, C \in Obj(C)$, the diagram



commutes.

01UP 3. Left Monoidal Unity. For each $A \in Obj(C)$, the diagram



commutes.

01UQ 4. *Right Monoidal Unity.* For each $A \in Obj(C)$, the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \xrightarrow{\rho_{A}^{C}} A \otimes_{C} \mathbb{1}_{C}$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A \otimes_{C} \mathbb{1}_{C}$$

commutes.

- **10 Proposition 13.1.1.1.4.** Let *C* be a category.
- 01US 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
- 01UT (a) The diagram

commutes.

01UU (b) The diagram

$$A\boxtimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\boxtimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\boxtimes_{C}(B\otimes_{C}C)$$

$$\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}\downarrow \qquad \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C}C}^{\otimes}$$

$$A\otimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\otimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\otimes_{C}(B\otimes_{C}C)$$

commutes.

01WB 2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

01WC (a) The diagram

commutes.

01WD (b) The diagram

commutes.

01WE (c) The diagram

commutes.

01WF (d) The diagram

commutes.

01UV 3. Mixed Associators. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon \mathrel{-_1} \boxtimes_{C} \mathrel{-_2} \to \mathrel{-_1} \otimes_{C} \mathrel{-_2}$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes}: (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ id_{A \otimes_C B,C}^{\otimes} & & & \downarrow id_A \otimes_C id_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_C (B \boxtimes_C C) \\ \operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C & & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UX (b) If there exists a natural transformation

$$\alpha_{A.B.C}^{\boxtimes} \colon (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$id_{A,B}^{\otimes} \otimes_{C} id_{C} \downarrow \qquad \qquad \downarrow id_{A,B \otimes_{C} C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C'}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$id_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \downarrow id_{A}\boxtimes_{C} id_{B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_C (B \boxtimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & & \operatorname{id}_{A,C}^{\otimes} \\ & (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

Proof. Item **1**, *Extra Monoidality Conditions*: We claim that *Items* **1a** and **1b** are indeed true:

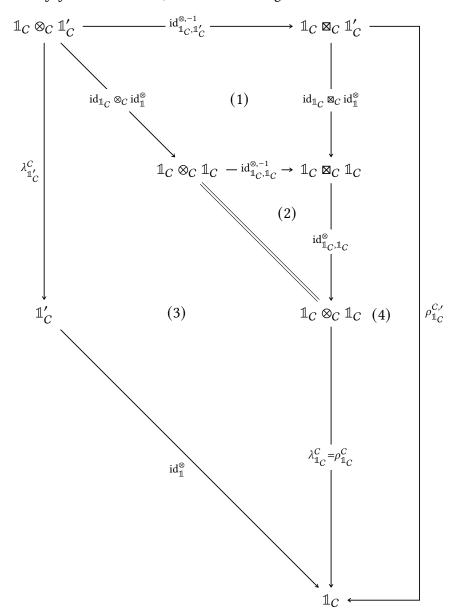
1. Proof of Item 1a: This follows from the naturality of id^{\otimes} with respect to the morphisms $id_{A,B}^{\otimes}$ and id_{C} .

2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and id_{RC}^{\otimes} .

This finishes the proof.

Item **2**, *Extra Monoidal Unity Constraints*: We claim that *Items* **2a** and **2b** are indeed true:

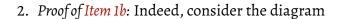
1. Proof of Item 1a: Indeed, consider the diagram

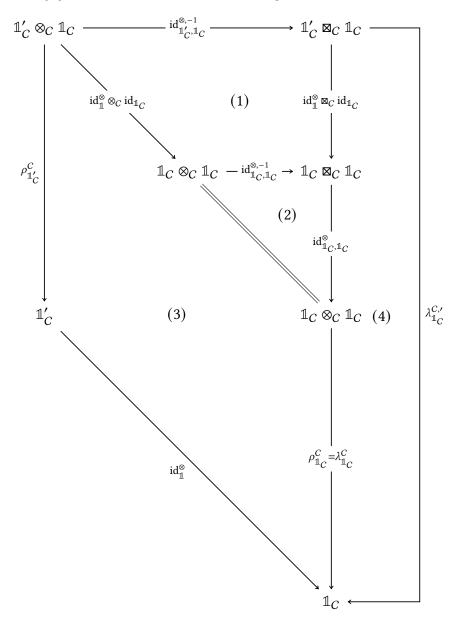


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:composition}$;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$;

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from $\ref{eq:composition}$;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$; so does the boundary diagram, and we are done.
- 3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

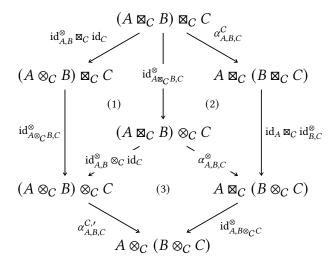
commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

Item 3, *Mixed Associators*: We claim that *Items 3a* to *3c* are indeed true:

01UZ 1. Proof of Item 3a: We may partition the monoidality diagram for id^{\otimes} of



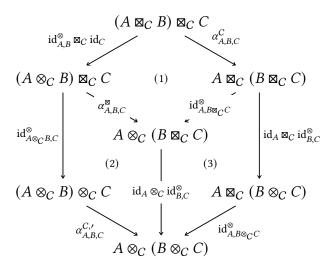


Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for id[®] of Item 2 of Definition 13.1.1.1.3 as follows:

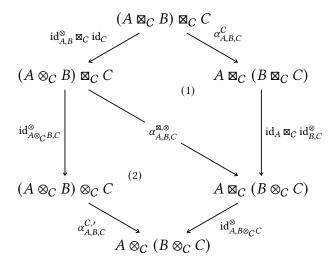


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item 3c*: We may partition the monoidality diagram for id[⊗] of Item 2 of Definition 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- o1V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets

7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal gories
Categories

Bicategories

Extra Part

14. Types of Morphisms in Bicate- 15. Notes