

Pointed Sets

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This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

DEFINITION 6.1.1.1.1 ► POINTED SETS

A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A pointed object in $(\mathbf{Sets}, \text{pt})$.

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -**modules**.

REMARK 6.1.1.1.2 ► UNWINDING DEFINITION 6.1.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X , called the **underlying set of** (X, x_0) .
- *The Basepoint.* A morphism

$$[x_0]: \text{pt} \rightarrow X$$

in \mathbf{Sets} , determining an element $x_0 \in X$, called the **basepoint of** X .

EXAMPLE 6.1.1.1.3 ► THE ZERO SPHERE

The **0-sphere**¹ is the pointed set $(S^0, 0)$ ² consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0 .

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

²*Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

EXAMPLE 6.1.1.1.4 ► THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- *The Underlying Set.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element \star of pt .

EXAMPLE 6.1.1.1.5 ► THE STANDARD POINTED SET WITH $n + 1$ ELEMENTS

The **standard pointed set with $n + 1$ elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{\star\} \cup \{1, \dots, n\}.$$

- *The Basepoint.* The element \star of $\langle n \rangle$.

6.1.2 Morphisms of Pointed Sets

DEFINITION 6.1.2.1.1 ► MORPHISMS OF POINTED SETS

A **morphism of pointed sets**^{1,2} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A morphism of pointed objects in $(\mathbf{Sets}, \text{pt})$.

¹*Further Terminology:* Also called a **pointed function**.

²*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of \mathbb{F}_1 -modules**.

REMARK 6.1.2.1.2 ► UNWINDING DEFINITION 6.1.2.1.1

In detail, a **morphism of pointed sets** $f: (X, x_0) \rightarrow (Y, y_0)$ is a

morphism of sets $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

DEFINITION 6.1.3.1.1 ► THE CATEGORY OF POINTED SETS

The **category of pointed sets** is the category \mathbf{Sets}_* defined equivalently as:

- The homotopy category of the ∞ -category $\mathbf{Mon}_{\mathbb{E}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$ of ??, ??.
- The category \mathbf{Sets}_* of Constructions With Categories, ??.

REMARK 6.1.3.1.2 ► UNWINDING DEFINITION 6.1.3.1.1

In detail, the **category of pointed sets** is the category \mathbf{Sets}_* where:

- *Objects.* The objects of \mathbf{Sets}_* are pointed sets.
- *Morphisms.* The morphisms of \mathbf{Sets}_* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*}: \text{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

of \mathbf{Sets}_* at (X, x_0) is defined by¹

$$\text{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\text{def}}{=} \text{id}_X.$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$, the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*}: \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of \mathbf{Sets}_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by²

$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\text{def}}{=} g \circ f.$$

¹Note that id_X is indeed a morphism of pointed sets, as we have $\text{id}_X(x_0) = x_0$.

²Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & \downarrow [y_0] & \searrow [z_0] \\ X & \xrightarrow{f} Y & \xrightarrow{g} Z \end{array}$$

$g(f(x_0)) = g(y_0) = z_0,$

6.1.4 Elementary Properties of Pointed Sets

PROPOSITION 6.1.4.1.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

1. *Completeness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is complete, having in particular:
 - (a) Products, described as in [Definition 6.2.3.1.1](#).
 - (b) Pullbacks, described as in [Definition 6.2.4.1.1](#).
 - (c) Equalisers, described as in [Definition 6.2.5.1.1](#).
2. *Cocompleteness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is cocomplete, having in particular:
 - (a) Coproducts, described as in [Definition 6.3.3.1.1](#).
 - (b) Pushouts, described as in [Definition 6.3.4.1.1](#);
 - (c) Coequalisers, described as in [Definition 6.3.5.1.1](#).
3. *Failure To Be Cartesian Closed.* The category \mathbf{Sets}_* is not Cartesian closed.¹
4. *Morphisms From the Monoidal Unit.* We have a bijection of sets²

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories³

$$\mathbf{Sets}_* \xrightarrow{\text{eq.}} \mathbf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

- (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \mathbf{Sets}_* \xrightarrow{\cong} \mathbf{Sets}^{\text{part.}}$$

sends:

- i. A pointed set (X, x_0) to X .
- ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

- (b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \mathbf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathbf{Sets}_*$$

sends:

- i. A set X is to the pointed set (X, \star) with \star an element that is not in X .
- ii. A partial function

$$f: X \rightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$


for each $x \in X$.

¹The category \mathbf{Sets}_* does admit a natural monoidal closed structure, however; see [Tensor Products of Pointed Sets](#).

²In other words, the forgetful functor

$$\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

³ *Warning:* This is not an isomorphism of categories, only an equivalence.

PROOF 6.1.4.1.2 ► PROOF OF PROPOSITION 6.1.4.1.1

Item 1: Completeness

This follows from (the proofs) of [Definitions 6.2.3.1.1](#), [6.2.4.1.1](#) and [6.2.5.1.1](#) and ??.

Item 2: Cocompleteness

This follows from (the proofs) of [Definitions 6.3.3.1.1](#), [6.3.4.1.1](#) and [6.3.5.1.1](#) and ??.

Item 3: Failure To Be Cartesian Closed

See [\[MSE 2855868\]](#).

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X , we obtain a bijection between pointed maps $S^0 \rightarrow X$ and the elements of X .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}: S^0 \rightarrow X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \rightarrow X$ picking the element x_0 of X , and thus we see that the bijection between pointed maps $S^0 \rightarrow X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [\[MSE 884460\]](#).



6.1.5 Active and Inert Morphisms of Pointed Sets

DEFINITION 6.1.5.1.1 ► ACTIVE AND INERT MORPHISMS OF POINTED SETS

Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a morphism of pointed sets.

1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

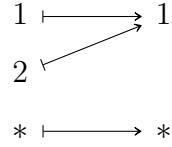
NOTATION 6.1.5.1.2 ► THE CATEGORY OF POINTED SETS AND ACTIVE MORPHISMS

We write $\mathbf{Sets}_*^{\text{actv}}$ for the wide subcategory of \mathbf{Sets}_* spanned by pointed sets and the active maps between them.

EXAMPLE 6.1.5.1.3 ► EXAMPLES OF ACTIVE AND INERT MAPS OF POINTED SETS

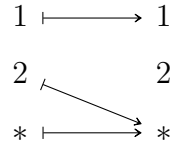
Here are some examples of active and inert maps of pointed sets.

1. The map $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$ given by



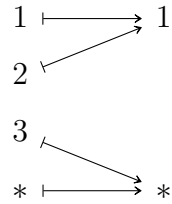
is active but not inert.

2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by



is inert but not active.

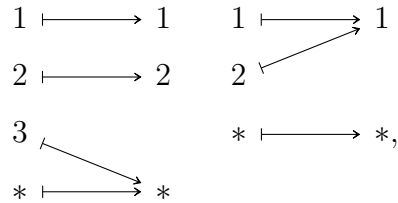
3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by



is neither inert nor active. However, it factors as $f = a \circ i$, where

$$\begin{aligned} i: \langle 3 \rangle &\rightarrow \langle 2 \rangle, \\ a: \langle 2 \rangle &\rightarrow \langle 1 \rangle \end{aligned}$$

are the morphisms of pointed sets given by



with i being inert and a being active.

PROPOSITION 6.1.5.1.4 ► PROPERTIES OF ACTIVE AND INERT MAPS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

- (a) The map $i: (X, x_0) \rightarrow (K, k_0)$ is an inert morphism of pointed sets
- (b) The map $a: (K, k_0) \rightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in \mathbf{Sets}_* .

PROOF 6.1.5.1.5 ► PROOF OF PROPOSITION 6.1.5.1.4**Item 1: Active-Inert Factorisation**

Let $f: X \rightarrow Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= (X \setminus f^{-1}(y_0)) \cup \{x_0\}; \end{aligned}$$

- $i: X \rightarrow K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

- $a: K \rightarrow Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{a} & B \end{array}$$

be a commutative diagram in \mathbf{Sets}_* . Consider the morphism $\phi: Y \rightarrow A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & \exists! \nearrow \phi & \downarrow g \\ A & \xrightarrow{a} & B. \end{array}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\stackrel{\text{def}}{=} \phi(i(x)) \\ &\stackrel{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [a \circ \phi](y) &\stackrel{\text{def}}{=} a(\phi(y)) \\ &\stackrel{\text{def}}{=} a(f(i^{-1}(y))) \\ &\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y)) \end{aligned}$$

$$\begin{aligned}
&= [g \circ i](i^{-1}(y)) \\
&\stackrel{\text{def}}{=} g(i(i^{-1}(y))) \\
&\stackrel{\text{def}}{=} g(y)
\end{aligned}$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
f \downarrow & \swarrow \psi & \downarrow g \\
A & \xrightarrow{a} & B
\end{array}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned}
\psi(y) &= \psi(i(x)) \\
&= f(x) \\
&= f(i^{-1}(y)) \\
&\stackrel{\text{def}}{=} \phi(y).
\end{aligned}$$

This finishes the proof. 

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

DEFINITION 6.2.1.1.1 ► THE TERMINAL POINTED SET

The **terminal pointed set** is the terminal object of \mathbf{Sets}_* as in Limits and Colimits, ??.

CONSTRUCTION 6.2.1.1.2 ► CONSTRUCTION OF THE TERMINAL POINTED SET

Concretely, the **terminal pointed set** is the pair $((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\text{Sets})$.

PROOF 6.2.1.1.3 ► PROOF OF CONSTRUCTION 6.2.1.1.2

We claim that (pt, \star) is the terminal object of Sets_* . Indeed, suppose we have a diagram of the form


$$(X, x_0) \quad (\text{pt}, \star)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely $!_X$. 

6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

DEFINITION 6.2.2.1.1 ► THE PRODUCT OF A FAMILY OF POINTED SETS

The **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* as in Limits and Colimits, ??.

CONSTRUCTION 6.2.2.1.2 ► CONSTRUCTION OF THE PRODUCT OF A FAMILY OF POINTED SETS

Concretely, the **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right), \{\text{pr}_i\}_{i \in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$.
- *The Cone.* The collection

$$\left\{ \text{pr}_i : \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i((x_j)_{j \in I}) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

PROOF 6.2.2.1.3 ► PROOF OF CONSTRUCTION 6.2.2.1.2

We claim that $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ & \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) & \xrightarrow{\text{pr}_i} (X_i, x_0^i) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi : (P, *) \rightarrow \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$$

making the diagram

$$\begin{array}{ccc}
 (P, *) & & \\
 \downarrow \phi \exists! & \searrow p_i & \\
 \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i)
 \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_i(*))_{i \in I} \\
 &= (x_0^i)_{i \in I},
 \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$.



PROPOSITION 6.2.2.1.4 ► PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$ defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

PROOF 6.2.2.1.5 ► PROOF OF PROPOSITION 6.2.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.



6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* as in Limits and Colimits, ??.

CONSTRUCTION 6.2.3.1.2 ► CONSTRUCTION OF PRODUCTS OF POINTED SETS

Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1 &: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0), \\ \text{pr}_2 &: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times Y$.

PROOF 6.2.3.1.3 ► PROOF OF CONSTRUCTION 6.2.3.1.2

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & (P, *) & \\ p_1 \swarrow & & \searrow p_2 \\ (X, x_0) & \xleftarrow{\text{pr}_1} (X \times Y, (x_0, y_0)) \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccc}
 & (P, *) & \\
 p_1 \swarrow & \downarrow \phi \mid \exists! & \searrow p_2 \\
 (X, x_0) & \xleftarrow{\text{pr}_1} (X \times Y, (x_0, y_0)) \xrightarrow{\text{pr}_2} & (Y, y_0)
 \end{array}$$

commute, being uniquely determined by the conditions


$$\begin{aligned}
 \text{pr}_1 \circ \phi &= p_1, \\
 \text{pr}_2 \circ \phi &= p_2
 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_1(*), p_2(*)) \\
 &= (x_0, y_0),
 \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. 

PROPOSITION 6.2.3.1.4 ► PROPERTIES OF PRODUCTS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned}
 A \times -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\
 - \times B: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\
 -_1 \times -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*,
 \end{aligned}$$

defined in the same way as the functors of **Constructions With Sets**, Item 1 of Proposition 4.1.3.1.4.

2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

4. *Unitality.* We have isomorphisms of pointed sets

$$(\text{pt}, \star) \times (X, x_0) \cong (X, x_0),$$

$$(X, x_0) \times (\text{pt}, \star) \cong (X, x_0),$$

natural in $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$.

5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\mathbf{Sets}_*)$.

6. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \times, (\text{pt}, \star))$ is a symmetric monoidal category.

PROOF 6.2.3.1.5 ► PROOF OF PROPOSITION 6.2.3.1.4

Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2: Lack of Adjointness

See [MSE 2855868].

Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

Item 4: Unitality

This follows from **Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.**

Item 5: Commutativity

This follows from **Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.**

Item 6: Symmetric Monoidality

This follows from **Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.** 

6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \rightarrow (Z, z_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ be morphisms of pointed sets.

DEFINITION 6.2.4.1.1 ► PULLBACKS OF POINTED SETS

The **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in **Sets_{*}** as in Limits and Colimits, ??.

CONSTRUCTION 6.2.4.1.2 ► CONSTRUCTION OF PULLBACKS OF POINTED SETS

Concretely, the **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\text{pr}_1: (X \times_Z Y, (x_0, y_0)) \rightarrow (X, x_0),$$

$$\text{pr}_2: (X \times_Z Y, (x_0, y_0)) \rightarrow (Y, y_0)$$

defined by

$$\text{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

$$\text{pr}_2(x, y) \stackrel{\text{def}}{=} y$$

for each $(x, y) \in X \times_Z Y$.

PROOF 6.2.4.1.3 ► PROOF OF CONSTRUCTION 6.2.4.1.2

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$\begin{array}{ccc}
 (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\
 \text{pr}_1 \downarrow & & \downarrow g \\
 (X, x_0) & \xrightarrow{f} & (Z, z_0).
 \end{array}$$

$f \circ \text{pr}_1 = g \circ \text{pr}_2,$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned}
 [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\
 &= f(x) \\
 &= g(y) \\
 &= g(\text{pr}_2(x, y)) \\
 &= [g \circ \text{pr}_2](x, y),
 \end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{array}{ccccc}
 (P, *) & & \xrightarrow{p_2} & & (Y, y_0) \\
 & \searrow p_1 & & \swarrow \text{pr}_2 & \\
 & & (X \times_Z Y, (x_0, y_0)) & & \\
 & & \downarrow \text{pr}_1 & \lrcorner & \downarrow g \\
 & & (X, x_0) & \xrightarrow{f} & (Z, z_0)
 \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc}
 (P, *) & & & & \\
 \swarrow \phi & \xrightarrow{p_2} & & & \\
 \exists! \searrow & & (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\
 & & \downarrow \text{pr}_1 & \lrcorner & \downarrow g \\
 & & (X, x_0) & \xrightarrow{f} & (Z, z_0) \\
 \searrow p_1 & & & &
 \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
 \text{pr}_1 \circ \phi &= p_1, \\
 \text{pr}_2 \circ \phi &= p_2
 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition


$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_1(*), p_2(*)) \\
 &= (x_0, y_0),
 \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. 

PROPOSITION 6.2.4.1.4 ► PROPERTIES OF PULLBACKS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f, Z, g} Y$ defines

a functor

$$-_1 \times_{-3} -_1 : \mathbf{Fun}(\mathcal{P}, \mathbf{Sets}_*) \rightarrow \mathbf{Sets}_*,$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & \bullet & \\ & \downarrow & \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \\ & X' & \xrightarrow{f'} & Z' & \end{array}$$

in $\mathbf{Fun}(\mathcal{P}, \mathbf{Sets}_*)$ to the morphism of pointed sets

$$\xi : (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed

sets making the diagram

$$\begin{array}{ccccc}
 X \times_Z Y & \longrightarrow & Y & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & X' \times_{Z'} Y' & \longrightarrow & Y' & \\
 & \downarrow & \downarrow & \downarrow g' & \\
 X & \xrightarrow{f} & Z & & \\
 \downarrow \phi & & \downarrow \chi & & \\
 & X' & \xrightarrow{f'} & Z' &
 \end{array}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{array}{ccccc}
 X & & Y & & Z \\
 & \searrow f & \swarrow g & \searrow h & \swarrow k \\
 & W & & V &
 \end{array}$$

in \mathbf{Sets}_* , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} (X \times_W Y) \times_Y Z \\ \swarrow \quad \searrow \\ X \times_W Y \quad Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} &
 \begin{array}{c} (X \times_W Y) \times_Y (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} &
 \begin{array}{c} X \times_W (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array}
 \end{array}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow f & \lrcorner & \downarrow f \\ X & \xlongequal{\quad} & X \end{array} &
 \begin{array}{l} X \times_X A \cong A, \\ A \times_X X \cong A, \end{array} &
 \begin{array}{ccc} A & \xrightarrow{f} & X \\ \parallel & \lrcorner & \parallel \\ X & \xrightarrow{f} & X. \end{array}
 \end{array}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow \lrcorner & & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad X \times_{\text{pt}} Y \cong X \times Y,$$

6. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

PROOF 6.2.4.1.5 ► PROOF OF PROPOSITION 6.2.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from [Constructions With Sets, Item 4 of Proposition 4.1.4.1.7](#).

Item 3: Unitality

This follows from [Constructions With Sets, Item 6 of Proposition 4.1.4.1.7](#).

Item 4: Commutativity

This follows from [Constructions With Sets, Item 7 of Proposition 4.1.4.1.7](#).

Item 5: Interaction With Products

This follows from **Constructions With Sets**, **Item 10** of **Proposition 4.1.4.1.7**.

Item 6: Symmetric Monoidality

This follows from **Constructions With Sets**, **Item 11** of **Proposition 4.1.4.1.7**. 

6.2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

DEFINITION 6.2.5.1.1 ► EQUALISERS OF POINTED SETS

The **equaliser of** (f, g) is the equaliser of f and g in \mathbf{Sets}_* as in Limits and Colimits, ??.

CONSTRUCTION 6.2.5.1.2 ► CONSTRUCTION OF EQUALISERS OF POINTED SETS

Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g): (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

PROOF 6.2.5.1.3 ► PROOF OF CONSTRUCTION 6.2.5.1.2

We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in \mathbf{Sets}_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose

we have a diagram of the form

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \\ & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \\ \uparrow \phi \exists! & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition


$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. 

PROPOSITION 6.2.5.1.4 ► PROPERTIES OF EQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{} \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in \mathbf{Sets}_* , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

PROOF 6.2.5.1.5 ► PROOF OF PROPOSITION 6.2.5.1.4

Item 1: Associativity

This follows from [Constructions With Sets, Item 1 of Proposition 4.1.5.1.4](#).

Item 2: Unitality

This follows from [Constructions With Sets, Item 2 of Proposition 4.1.5.1.4](#).

Item 3: Commutativity

This follows from [Constructions With Sets, Item 3 of Proposition 4.1.5.1.4.](#) 

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

DEFINITION 6.3.1.1.1 ► THE INITIAL POINTED SET

The **initial pointed set** is the initial object of \mathbf{Sets}_* as in Limits and Colimits, ??.

CONSTRUCTION 6.3.1.1.2 ► CONSTRUCTION OF THE INITIAL POINTED SET

Concretely, the **initial pointed set** is the pair $((\text{pt}, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (\text{pt}, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

PROOF 6.3.1.1.3 ► PROOF OF CONSTRUCTION 6.3.1.1.2

We claim that (pt, \star) is the initial object of \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$(\text{pt}, \star) \quad (X, x_0)$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi : (\text{pt}, \star) \rightarrow (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X .



6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

DEFINITION 6.3.2.1.1 ► COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ ¹ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **wedge sum of the family** $\{(X_i, x_0^i)\}_{i \in I}$.

CONSTRUCTION 6.3.2.1.2 ► CONSTRUCTION OF COPRODUCTS OF FAMILIES OF POINTED SETS

Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \{\text{inj}_i\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
 - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\coprod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{ \text{inj}_i : (X_i, x_0^i) \rightarrow \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

PROOF 6.3.2.1.3 ► PROOF OF CONSTRUCTION 6.3.2.1.2

We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi : \left(\bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram


$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \uparrow \exists! \phi \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([i, x]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \iota_i\left(\left[(i, x_0^i)\right]\right) \\ &= *,\end{aligned}$$

as ι_i is a morphism of pointed sets. 

PROPOSITION 6.3.2.1.4 ► PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS


Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

PROOF 6.3.2.1.5 ► PROOF OF PROPOSITION 6.3.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??. 

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

DEFINITION 6.3.3.1.1 ► COPRODUCTS OF POINTED SETS

The **coproduct of (X, x_0) and (Y, y_0)** ¹ is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, ??.

¹*Further Terminology:* Also called the **wedge sum of (X, x_0) and (Y, y_0)** .

CONSTRUCTION 6.3.3.1.2 ► CONSTRUCTION OF COPRODUCTS OF POINTED SETS

Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
 - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}
 (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) & X \vee Y &\longleftarrow Y \\
 &\cong (X \amalg_{\text{pt}} Y, p_0) & \uparrow \ulcorner & \uparrow [y_0] \\
 &\cong (X \amalg Y / \sim, p_0), & X &\longleftarrow_{[x_0]} \text{pt},
 \end{aligned}$$

where \sim is the equivalence relation on $X \amalg Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}
 p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\
 &= [(1, y_0)].
 \end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}
 \text{inj}_1 &: (X, x_0) \rightarrow (X \vee Y, p_0), \\
 \text{inj}_2 &: (Y, y_0) \rightarrow (X \vee Y, p_0),
 \end{aligned}$$

given by

$$\begin{aligned}
 \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\
 \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],
 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.3.1.3 ► PROOF OF CONSTRUCTION 6.3.3.1.2

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & (C, *) & \\ \iota_1 \nearrow & & \nwarrow \iota_2 \\ (X, x_0) & \xrightarrow{\text{inj}_1} (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} (Y, y_0) \end{array}$$

in \mathbf{Sets} . Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccc} & (C, *) & \\ \iota_1 \nearrow & \uparrow \phi \exists! & \nwarrow \iota_2 \\ (X, x_0) & \xrightarrow{\text{inj}_1} (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \text{inj}_X &= \iota_X, \\ \phi \circ \text{inj}_Y &= \iota_Y \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

$$= \iota_Y([(1, y_0)])$$

$$= *,$$

as ι_X and ι_Y are morphisms of pointed sets.



PROPOSITION 6.3.3.1.4 ► PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\text{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\text{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \vee, \text{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}} \Rightarrow \mathrm{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

PROOF 6.3.3.1.5 ► PROOF OF PROPOSITION 6.3.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a

morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \quad \begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation.



6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \rightarrow (X, x_0)$ and $g: (Z, z_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

DEFINITION 6.3.4.1.1 ► PUSHOUTS OF POINTED SETS

The **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in \mathbf{Sets}_* as in Limits and Colimits, ??.

CONSTRUCTION 6.3.4.1.2 ► CONSTRUCTION OF PUSHOUTS OF POINTED SETS

Concretely, the **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Colimit.* The pointed set $(X \amalg_{f, Z, g} Y, p_0)$, where:
 - The set $X \amalg_{f, Z, g} Y$ is the pushout (of unpointed sets) of X

- and Y over Z with respect to f and g ;
- We have $p_0 = [x_0] = [y_0]$.
 - *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1 &: (X, x_0) \rightarrow (X \amalg_Z Y, p_0), \\ \text{inj}_2 &: (Y, y_0) \rightarrow (X \amalg_Z Y, p_0) \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)] \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.4.1.3 ► PROOF OF ??

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \amalg_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \amalg_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} & (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} (Y, y_0) \\ \text{inj}_1 \circ f = \text{inj}_2 \circ g, & \uparrow \text{inj}_1 & \uparrow g \\ & (X, x_0) & \xleftarrow{f} (Z, z_0). \end{array}$$

Indeed, given $z \in Z$, we have

$$[\text{inj}_1 \circ f](z) = \text{inj}_1(f(z))$$

$$\begin{aligned}
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \text{inj}_2(g(z)) \\
&= [\text{inj}_2 \circ g](z),
\end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation \sim on $X \amalg Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \amalg_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{array}{ccc}
(P, *) & \xleftarrow{\quad \iota_2 \quad} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\
& \nwarrow \iota_1 & \uparrow \text{inj}_1 \quad \sqcap \quad \uparrow g \\
& & (X, x_0) \xleftarrow{\quad f \quad} (Z, z_0)
\end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram

$$\begin{array}{ccc}
(P, *) & \xleftarrow{\quad \iota_2 \quad} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\
& \nwarrow \iota_1 \quad \swarrow \phi \quad \exists! & \uparrow \text{inj}_1 \quad \sqcap \quad \uparrow g \\
& & (X, x_0) \xleftarrow{\quad f \quad} (Z, z_0)
\end{array}$$

commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via


$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of **Constructions With Sets, Definition 4.2.4.1.1**. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. 

PROPOSITION 6.3.4.1.4 ► PROPERTIES OF PUSHOUTS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \amalg_{f, Z, g} Y$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & & \bullet \\ & \uparrow & \\ \bullet & \longleftarrow & \bullet \end{array}$$

In particular, the action on morphisms of ${}_1 \amalg_{-3} -_1$ is given by

sending a morphism

$$\begin{array}{ccccc}
 X \amalg_Z Y & \longleftarrow & Y & & \\
 \uparrow & & \uparrow & \searrow \psi & \\
 & & X' \amalg_{Z'} Y' & \longleftarrow & Y' \\
 & & \uparrow g & & \uparrow g' \\
 X & \xleftarrow{f} & Z & \searrow \chi & \\
 \searrow \phi & & & & \\
 & & X' & \xleftarrow{f'} & Z'
 \end{array}$$

in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi: (X \amalg_Z Y, p_0) \xrightarrow{\exists!} (X' \amalg_{Z'} Y', p'_0)$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{array}{ccccc}
 X \amalg_Z Y & \longleftarrow & Y & & \\
 \uparrow & \searrow \xi & \uparrow & \searrow \psi & \\
 & & X' \amalg_{Z'} Y' & \longleftarrow & Y' \\
 & & \uparrow g & & \uparrow g' \\
 X & \xleftarrow{f} & Z & \searrow \chi & \\
 \searrow \phi & & & & \\
 & & X' & \xleftarrow{f'} & Z'
 \end{array}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{array}{ccccc} X & & Y & & Z \\ & \nwarrow f & \nearrow g & \nwarrow h & \nearrow k \\ & W & & V & \end{array}$$

in **Sets**, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc} \begin{array}{ccccc} & (X \amalg_W Y) \amalg_V Z & & & \\ & \nearrow \wedge & \nwarrow & & \\ X \amalg_W Y & & Y & & Z \\ \nearrow \wedge & & \nwarrow & & \\ X & & Y & & Z \\ \nwarrow f & & \nearrow g & & \nwarrow h & & \nearrow k \\ & W & & V & \end{array} & \begin{array}{ccccc} & (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) & & & \\ & \nearrow \wedge & \nwarrow & & \\ X \amalg_W Y & & Y \amalg_V Z & & Z \\ \nearrow \wedge & & \nwarrow & & \\ X & & Y & & Z \\ \nwarrow f & & \nearrow g & & \nwarrow h & & \nearrow k \\ & W & & V & \end{array} & \begin{array}{ccccc} & X \amalg_W (Y \amalg_V Z) & & & \\ & \nearrow \wedge & \nwarrow & & \\ X & & Y \amalg_V Z & & Z \\ \nearrow \wedge & & \nwarrow & & \\ X & & Y & & Z \\ \nwarrow f & & \nearrow g & & \nwarrow h & & \nearrow k \\ & W & & V & \end{array} \end{array}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc} \begin{array}{ccc} A & \xlongequal{\quad} & A \\ \uparrow f & \lrcorner & \uparrow f \\ X & \xlongequal{\quad} & X \end{array} & \begin{array}{c} X \amalg_X A \cong A, \\ A \amalg_X X \cong A, \end{array} & \begin{array}{ccc} A & \xleftarrow{f} & X \\ \parallel \lrcorner & & \parallel \\ X & \xleftarrow{f} & X. \end{array} \end{array}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc} \begin{array}{ccc} X \amalg_Z Y & \xleftarrow{\quad} & Y \\ \uparrow \lrcorner & & \uparrow g \\ X & \xleftarrow{f} & Z, \end{array} & X \amalg_Z Y \cong Y \amalg_Z X & \begin{array}{ccc} Y \amalg_Z X & \xleftarrow{\quad} & X \\ \uparrow \lrcorner & & \uparrow f \\ Y & \xleftarrow{g} & Z. \end{array} \end{array}$$

5. *Interaction With Coproducts.* We have

$$\begin{array}{ccc} X \amalg_{\text{pt}} Y \cong X \vee Y, & \begin{array}{ccc} X \vee Y & \xleftarrow{\quad} & Y \\ \uparrow \lrcorner & & \uparrow [y_0] \\ X & \xleftarrow{[x_0]} & \text{pt.} \end{array} \end{array}$$

6. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \amalg_X, (X, x_0))$ is a symmetric monoidal category.

PROOF 6.3.4.1.5 ► PROOF OF PROPOSITION 6.3.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from **Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.**

Item 3: Unitality

This follows from **Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.**


Item 4: Commutativity

This follows from **Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.**

Item 5: Interaction With Coproducts

Omitted.

Item 6: Symmetric Monoidality

Omitted. 

6.3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

DEFINITION 6.3.5.1.1 ► COEQUALISERS OF POINTED SETS

The **coequaliser of (f, g)** is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

CONSTRUCTION 6.3.5.1.2 ► CONSTRUCTION OF COEQUALISERS OF POINTED SETS

The **coequaliser of** (f, g) is the pair $((\text{CoEq}(f, g), [y_0]), \text{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\text{CoEq}(f, g), [y_0])$, where $\text{CoEq}(f, g)$ is the coequaliser of f and g as in **Constructions With Sets, Definition 4.2.5.1.1**.
- *The Cocone.* The map

$$\text{coeq}(f, g): Y \twoheadrightarrow (\text{CoEq}(f, g), [y_0])$$

given by the quotient map, as in **Constructions With Sets, Item 2 of Construction 4.2.5.1.2**.

PROOF 6.3.5.1.3 ► PROOF OF CONSTRUCTION 6.3.5.1.2

We claim that $(\text{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in \mathbf{Sets}_* . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\ &\stackrel{\text{def}}{=} [f(x)] \\ &= [g(x)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x) \end{aligned}$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form


$$\begin{array}{ccc} (X, x_0) & \xrightarrow[f]{g} & (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} & (\text{CoEq}(f, g), [y_0]) \\ & & & \searrow c & \\ & & & & (C, *) \end{array}$$

in **Sets**. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **Conditions on Relations**, **Items 4** and **5** of **Proposition 10.6.2.1.3** that there exists a unique map $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\ & \searrow c & \downarrow \phi \exists! \\ & & (C, *) \end{array}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that c is a morphism of pointed sets. 

PROPOSITION 6.3.5.1.4 ► PROPERTIES OF COEQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \xrightarrow[g]{f} (Y, y_0) \xrightarrow[h]{g}$$

in **Sets**_{*}.

2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

|

|

Item 1: Associativity

This follows from **Constructions With Sets**, Item 1 of Proposition 4.2.5.1.7.

Item 2: Unitality

This follows from **Constructions With Sets**, Item 2 of Proposition 4.2.5.1.7.

Item 3: Commutativity

This follows from **Constructions With Sets**, Item 3 of Proposition 4.2.5.1.7. 

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let X be a set.

DEFINITION 6.4.1.1.1 ► FREE POINTED SETS

The **free pointed set on X** is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by¹

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \amalg \text{pt} \\ &\stackrel{\text{def}}{=} X \amalg \{\star\}. \end{aligned}$$

- *The Basepoint.* The element \star of X^+ .

¹*Further Notation:* We sometimes write \star_X for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

PROPOSITION 6.4.1.1.2 ► PROPERTIES OF FREE POINTED SETS

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \text{Obj}(\mathbf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of [Definition 6.4.1.1.1](#).

- *Action on Morphisms.* For each morphism $f: X \rightarrow Y$ of \mathbf{Sets} , the image

$$f^+: X^+ \rightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv \overset{\sim}{\omega} \right): \quad \mathbf{Sets} \begin{array}{c} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\overset{\sim}{\omega}} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathbf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathbf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathbf{Sets}_*)$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of [Item 1](#) has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, \amalg, (-)^+_1, \amalg \right): (\mathbf{Sets}, \amalg, \emptyset) \rightarrow (\mathbf{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} \amalg: X^+ \vee Y^+ &\xrightarrow{\sim} (X \amalg Y)^+, \\ (-)^+_1 \amalg: \text{pt} &\xrightarrow{\sim} \emptyset^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, (-)_{\mathbb{1}}^+\right): (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^+ &: X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+, \\ (-)_{\mathbb{1}}^+ &: S^0 \xrightarrow{\sim} \text{pt}^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

PROOF 6.4.1.1.3 ► PROOF OF PROPOSITION 6.4.1.1.2

Item 1: Functoriality

We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \text{Obj}(\mathbf{Sets})$. We have

$$\text{id}_X^+(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\text{id}_X^+ = \text{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$\begin{aligned} f &: X \rightarrow Y, \\ g &: Y \rightarrow Z, \end{aligned}$$

we have

$$\begin{aligned} [g^+ \circ f^+](x) &\stackrel{\text{def}}{=} g^+(f^+(x)) \\ &\stackrel{\text{def}}{=} g^+(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^+(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [g^+ \circ f^+](\star_X) &\stackrel{\text{def}}{=} g^+(f^+(\star_X)) \\ &\stackrel{\text{def}}{=} g^+(\star_Y) \\ &\stackrel{\text{def}}{=} \star_Z \\ &\stackrel{\text{def}}{=} [g \circ f]^+(\star_X), \end{aligned}$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2: Adjointness

We proceed in a few steps:

- *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}_*(X^+, Y) \rightarrow \mathbf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger: X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y}: \mathbf{Sets}(X, Y) \rightarrow \mathbf{Sets}_*(X^+, Y)$$

given by sending a function $\xi: X \rightarrow Y$ to the morphism of pointed sets

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* Given a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0),$$

we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^\dagger) \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\ &= \xi \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathbf{Sets}_*(X^+, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\mathbf{Sets}_*(X^+, Y)}.$$

- *Invertibility II.* Given a map of sets $\xi: X \rightarrow Y$, we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\xi^\dagger) \\ &= \Phi_{X,Y}\left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket\right) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &\stackrel{\text{def}}{=} [\text{id}_{\mathbf{Sets}(X, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\mathbf{Sets}(X, Y)}.$$

- *Naturality for Φ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc} \mathbf{Sets}_*(X', +, Y) & \xrightarrow{\Phi_{X', Y}} & \mathbf{Sets}(X', Y) \\ f^* \downarrow & & \downarrow f^* \\ \mathbf{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X, Y}} & \mathbf{Sets}(X, Y) \end{array}$$

commutes. Indeed, given a morphism of pointed sets $\xi: X'^+ \rightarrow Y$, we have

$$\begin{aligned} [\Phi_{X, Y} \circ f^*](\xi) &= \Phi_{X, Y}(f^*(\xi)) \\ &= \Phi_{X, Y}(\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X', Y}(\xi) \circ f \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= [f^* \circ \Phi_{X', Y}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X, Y} \circ f^* = f^* \circ \Phi_{X', Y}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Φ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
 \mathbf{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \mathbf{Sets}(X, Y) \\
 g_* \downarrow & & \downarrow g_* \\
 \mathbf{Sets}_*(X^+, Y') & \xrightarrow{\Phi_{X,Y'}} & \mathbf{Sets}(X, Y')
 \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger: X^+ \rightarrow Y,$$

we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
 &= \Phi_{X,Y'}(g \circ \xi) \\
 &= g \circ \xi \\
 &= g \circ \Phi_{X,Y'}(\xi) \\
 &= g_*(\Phi_{X,Y'}(\xi)) \\
 &= [g_* \circ \Phi_{X,Y'}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that Ψ is also natural in each argument.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^+$ with respect to \amalg and \vee as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{+,\amalg}: X^+ \vee Y^+ \xrightarrow{\sim} (X \amalg Y)^+$$

is given by

$$(-)_{X,Y}^{+,\amalg}(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+,\amalg,-1}: (X \amalg Y)^+ \xrightarrow{\sim} X^+ \vee Y^+$$

given by

$$(-)_{X,Y}^{+,\amalg,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(1, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg_Y \end{cases}$$

for each $z \in (X \amalg Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\amalg,1}: \text{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_\emptyset .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+ : X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)_{X,Y}^{+,-1} : (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by


$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\mathbb{1}} : S^0 \xrightarrow{\sim} \text{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $\text{pt}^+ = \{\star, \star_{\text{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. 

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

DEFINITION 6.4.2.1.1 ► SETS WITH DELETED BASEPOINTS

The **set with deleted basepoint associated to X** is the set X^- defined by

$$X^- \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

PROPOSITION 6.4.2.1.2 ► PROPERTIES OF SETS WITH DELETED BASEPOINTS

Let (X, x_0) be a pointed set.

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- : \mathbf{Sets}_*^{\text{actv}} \rightarrow \mathbf{Sets},$$

where:

- *Action on Objects.* For each $X \in \text{Obj}(\mathbf{Sets}_*^{\text{actv}})$, we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where X^- is the set of [Definition 6.4.2.1.1](#).

- *Action on Morphisms.* For each morphism $f : X \rightarrow Y$ of $\mathbf{Sets}_*^{\text{actv}}$, the image

$$f^- : X^- \rightarrow Y^-$$

of f by $(-)^-$ is the map defined by

$$f^-(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$(((-)^- \dashv (-)^+) : \mathbf{Sets}_*^{\text{actv}} \begin{array}{c} \xrightarrow{(-)^-} \\ \perp_{\text{eq}} \\ \xleftarrow{(-)^+} \end{array} \mathbf{Sets},$$

witnessed by a bijection of sets

$$\mathbf{Sets}(X^-, Y) \cong \mathbf{Sets}_*(X, Y^+),$$

natural in $X \in \text{Obj}(\mathbf{Sets}_*)$ and $Y \in \text{Obj}(\mathbf{Sets})$, and by isomorphisms

$$\begin{aligned} (X^-)^+ &\cong X, \\ (Y^+)^- &\cong Y, \end{aligned}$$

once again natural in $X \in \text{Obj}(\mathbf{Sets}_*)$ and $Y \in \text{Obj}(\mathbf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^-, (-)^{-,\vee}, (-)_{\mathbb{1}}^{-,\vee}\right): (\mathbf{Sets}_*^{\text{actv}}, \vee, \text{pt}) \rightarrow (\mathbf{Sets}, \amalg, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^{-,\vee}: X^- \amalg Y^- &\xrightarrow{\sim} (X \vee Y)^-, \\ (-)_{\mathbb{1}}^{-,\vee}: \emptyset &\xrightarrow{\sim} \text{pt}^-, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$\left((-)^-, (-)^{-,\times}, (-)_{\mathbb{1}}^{-,\times}\right): (\mathbf{Sets}_*^{\text{actv}}, \wedge, S^0) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)_{X,Y}^-: X^- \times Y^- &\xrightarrow{\sim} (X \wedge Y)^-, \\ (-)_{\mathbb{1}}^-: \text{pt} &\xrightarrow{\sim} (S^0)^-, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

PROOF 6.4.2.1.3 ► PROOF OF PROPOSITION 6.4.2.1.2

Item 1: Functoriality

We claim that $(-)^-$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \text{Obj}(\mathbf{Sets})$. We have

$$\text{id}_X^-(x) \stackrel{\text{def}}{=} x$$

for each $x \in X^-$, so $\text{id}_X^- = \text{id}_X$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

$$g: (Y, y_0) \rightarrow (Z, z_0),$$

we have

$$\begin{aligned} [g^- \circ f^-](x) &\stackrel{\text{def}}{=} g^-(f^-(x)) \\ &\stackrel{\text{def}}{=} g^-(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^-(x) \end{aligned}$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2: Adjoint Equivalence

We proceed in a few steps:

1. *Map I.* We define a map

$$\Phi_{X,Y}: \mathbf{Sets}(X^-, Y) \rightarrow \mathbf{Sets}_*^{\text{actv}}(X, Y^+)$$

by sending a map $\xi: X^- \rightarrow Y$ to the active morphism of pointed sets

$$\xi^\dagger: X \rightarrow Y^+$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. *Map II.* We define a map

$$\Psi_{X,Y}: \mathbf{Sets}_*^{\text{actv}}(X, Y^+) \rightarrow \mathbf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi: X \rightarrow Y^+$ to the map

$$\xi^\dagger: X^- \rightarrow Y$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi: X^- \rightarrow Y$, we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \right) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}(X^-, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}(X^-, Y)}.$$

4. *Invertibility II.* Given a morphism of pointed sets

$$\xi: (X, x_0) \rightarrow (Y^+, \star_Y),$$

we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}.$$

5. *Naturality for Φ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc} \mathbf{Sets}(X', -, Y) & \xrightarrow{\Phi_{X', Y}} & \mathbf{Sets}_*^{\text{actv}}(X', Y^+) \\ f^* \downarrow & & \downarrow f^* \\ \mathbf{Sets}_*(X^-, Y) & \xrightarrow{\Phi_{X, Y}} & \mathbf{Sets}_*^{\text{actv}}(X, Y^+) \end{array}$$

commutes. Indeed, given a map of sets $\xi: X' \rightarrow Y$, we have

$$\begin{aligned} [\Phi_{X, Y} \circ f^*](\xi) &= \Phi_{X, Y}(f^*(\xi)) \\ &= \Phi_{X, Y}(\xi \circ f) \\ &= \llbracket x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases} \rrbracket \\ &= f^* \left(\llbracket x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{-} \\ \star_Y & \text{if } x' = x'_0 \end{cases} \rrbracket \right) \\ &= f^*(\Phi_{X', Y}(\xi)) \\ &= [f^* \circ \Phi_{X', Y}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X, Y} \circ f^* = f^* \circ \Phi_{X', Y},$$

and the naturality diagram for Φ above indeed commutes.

6. *Naturality for Φ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
 \mathbf{Sets}(X^-, Y) & \xrightarrow{\Phi_{X,Y}} & \mathbf{Sets}_*^{\text{actv}}(X, Y^+) \\
 g_* \downarrow & & \downarrow g_* \\
 \mathbf{Sets}(X^-, Y') & \xrightarrow{\Phi_{X,Y'}} & \mathbf{Sets}_*^{\text{actv}}(X, Y'^+)
 \end{array}$$

commutes. Indeed, given a map of sets $\xi: X^- \rightarrow Y$, we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
 &= \Phi_{X,Y'}(g \circ \xi) \\
 &= \llbracket x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \rrbracket \\
 &= g_* \left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \right) \\
 &= g_*(\Phi_{X,Y}(\xi)) \\
 &= [g_* \circ \Phi_{X,Y}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y},$$

and the naturality diagram for Φ above indeed commutes.

7. *Naturality for Ψ* . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that Ψ is also natural in each argument.
8. *Fully Faithfulness of $(-)^-$* . We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{X,Y}^-: \mathbf{Sets}_*^{\text{actv}}(X, Y) \xrightarrow{\sim} \mathbf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)_{X,Y}^{-,-1}: \mathbf{Sets}(X^-, Y^-) \xrightarrow{\sim} \mathbf{Sets}_*^{\text{actv}}(X, Y)$$

is given by sending a map of sets $f: X^- \rightarrow Y^-$ to the active morphism of pointed sets $f^\dagger: X \rightarrow Y$ defined by

$$f^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

9. *Essential Surjectivity of $(-)^-$.* We need to show that, given an object $X \in \text{Obj}(\mathbf{Sets})$, there exists some $X' \in \text{Obj}(\mathbf{Sets}_*^{\text{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$\begin{aligned} (X^+)^- &\stackrel{\text{def}}{=} (X \cup \{\star_X\})^- \\ &\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. *The Functor $(-)^-$ Is an Equivalence.* Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by [Categories, Item 1](#) of [Proposition 11.6.7.1.2](#).

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^-$ with respect to \vee and \amalg as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{-,\vee}: X^- \amalg Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0, x)] & \text{if } z = (0, x) \text{ with } x \in X, \\ [(1, y)] & \text{if } z = (1, y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \amalg Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1}: (X \vee Y)^- \xrightarrow{\sim} X^- \amalg Y^-$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = [(0, x)], \\ (1, y) & \text{if } z = [(1, y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\vee,1}: \emptyset \xrightarrow{\sim} \text{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^-: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)_{X,Y}^-(x, y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1}: (X \wedge Y)^- \xrightarrow{\sim} X^- \times Y^-$$

given by


$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x, y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{-,1} : \text{pt} \xrightarrow{\sim} (S^0)^{-}$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. 

Appendices

A Other Chapters

Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

Relations

8. [Relations](#)
9. [Constructions With Relations](#)

10. [Conditions on Relations](#)

Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

Monoidal Categories

13. [Constructions With Monoidal Categories](#)

Bicategories

14. [Types of Morphisms in Bicategories](#)

Extra Part

15. [Notes](#)

References

- [MSE 2855868] **Qiaochu Yuan**. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on pp. 7, 19).
- [MSE 884460] **Martin Brandenburg**. *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 8).