# Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

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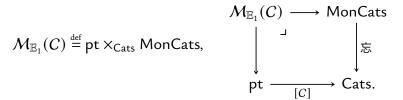
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## 13.1 Moduli Categories of Monoidal Structures

# 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let *C* be a category.

**Definition 13.1.1.1.1.** The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{B}_1}(C)$  defined by



Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- Objects. The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is C.
- *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  is a strong monoidal functor structure

$$\operatorname{id}_C^{\otimes} \colon A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor  $id_C : C \to C$  of C.

• *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\operatorname{def}}{=}(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}),$$

where  $\left(id_{C}^{\otimes}, id_{1|C}^{\otimes}\right)$  is the identity monoidal functor of C of ??.

• Composition. For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\begin{split} \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,N) &\to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,P) \\ & \text{of } \mathcal{M}_{\mathbb{B}_1}(C) \text{ at } (M,N,P) \text{ is defined by} \\ & \left( \operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{\mathbb{B}|C}^{\otimes,\prime} \right) \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \left( \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{B}|C}^{\otimes} \right) \overset{\operatorname{def}}{=} \left( \operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{B}|C}^{\otimes,\prime} \circ \operatorname{id}_{\mathbb{B}|C}^{\otimes} \right). \end{split}$$

**Remark 13.1.1.1.3.** In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

1. *Naturality*. For each pair  $f:A\to X$  and  $g:B\to Y$  of morphisms of C, the diagram

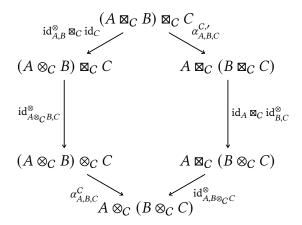
$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow id_{A,B}^{\otimes} \qquad \qquad \downarrow id_{X,Y}^{\otimes}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

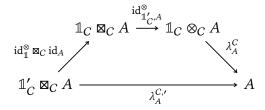
commutes.

2. *Monoidality*. For each  $A, B, C \in Obj(C)$ , the diagram



commutes.

3. *Left Monoidal Unity*. For each  $A \in Obj(C)$ , the diagram



commutes.

4. Right Monoidal Unity. For each  $A \in Obj(C)$ , the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A \otimes_{C} \mathbb{1}_{C}$$

commutes.

#### **Proposition 13.1.1.4.** Let C be a category.

- 1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

commutes.

- 2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

$$\mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow^{\rho_{\mathbb{1}_{C}}^{C,\prime}} \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes.

3. *Mixed Associators*. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{C} -_2 \to -_1 \otimes_{C} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{A.B.C}^{\otimes} \colon (A \otimes_{C} B) \boxtimes_{C} C \to A \otimes_{C} (B \boxtimes_{C} C)$$

making the diagrams

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$id_{A \otimes_{C} B,C}^{\otimes} \downarrow \qquad \qquad \downarrow id_{A} \otimes_{C} id_{B,C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes}: (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_C (B \otimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & \operatorname{id}_{A,B \otimes_C C}^{\otimes} \\ & (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$id_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \downarrow id_{A}\boxtimes_{C} id_{B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes} : (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C} & & & & \operatorname{id}_{A,C}^{\otimes} \\ & (A \otimes_{C} B) \otimes_{C} C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \qquad \qquad \downarrow \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

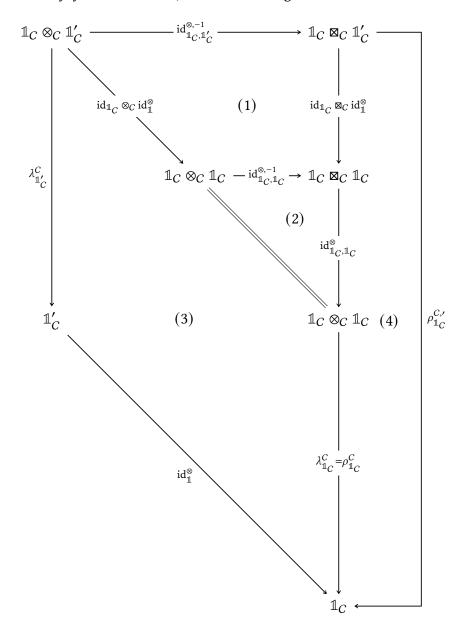
*Proof. Item 1, Extra Monoidality Conditions*: We claim that *Items 1a* and 1b are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of  $\mathrm{id}^\otimes$  with respect to the morphisms  $\mathrm{id}_{A,B}^\otimes$  and  $\mathrm{id}_C$ .
- 2. *Proof of Item 1b*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

*Item 2, Extra Monoidal Unity Constraints*: We claim that *Items 2a* and **2b** are indeed true:





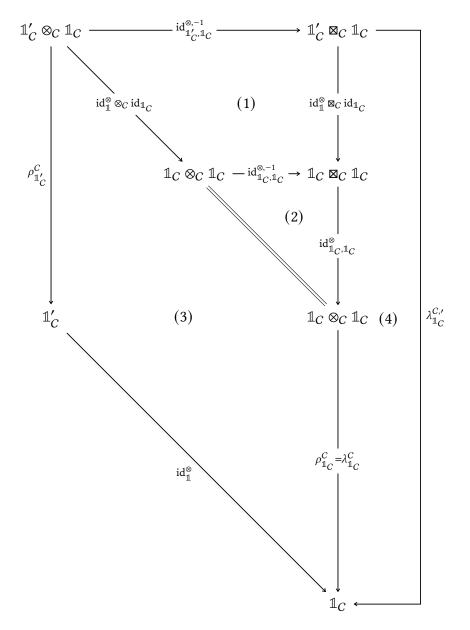
whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$  comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from  $\ref{eq:composition}$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

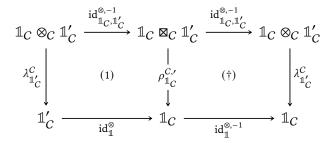
Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d*: Indeed, consider the diagram



Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

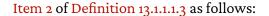
$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

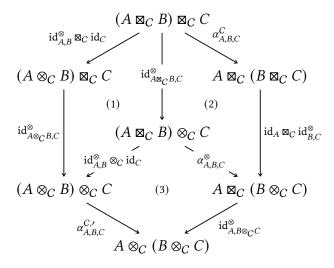
commutes. But since  $id_{1}^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for  $id^{\otimes}$  of



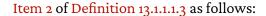


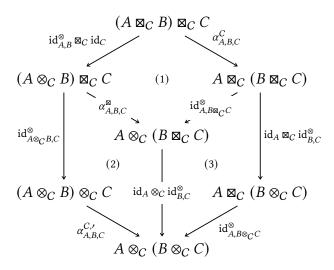
Since:

- Subdiagram (1) commutes by Item 12 of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for  $id^{\otimes}$  of



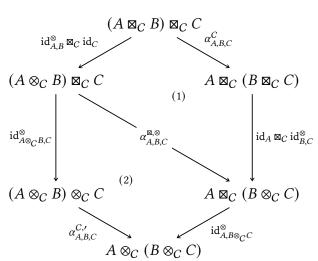


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for  $id^{\otimes}$  of



Item 2 of Definition 13.1.1.1.3 as follows:

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
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