# Monoidal Structures on the Category of Sets

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**O1NK** This chapter contains some material on monoidal structures on Sets.

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# 01NM 5.1.1 Products of Sets

See Constructions With Sets, Section 4.1.3.

# **01NN 5.1.2** The Internal Hom of Sets

See Constructions With Sets, Section 4.3.5.

# **01NP 5.1.3** The Monoidal Unit

# 01NQ DEFINITION 5.1.3.1.1 ► THE MONOIDAL UNIT OF ×

The monoidal unit of the product of sets is the functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{1}_{\mathsf{Sets}} \stackrel{\mathsf{def}}{=} \mathsf{pt},$$

where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.1.

5.1.4 The Associator 3

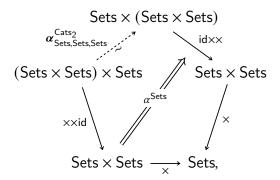
# **01NR 5.1.4** The Associator

# 01NS DEFINITION 5.1.4.1.1 ► THE ASSOCIATOR OF ×

The associator of the product of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}'}$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \stackrel{\mathsf{def}}{=} (x,(y,z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

# PROOF 5.1.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.4.1.1

# Invertibility

The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is the morphism

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

5.1.4 The Associator 4

defined by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \stackrel{\mathsf{def}}{=} ((x,y),z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}}\right] &((x,y),z) \stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1} \Big(\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z)\Big) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} ((x,y),z) \\ &\stackrel{\mathsf{def}}{=} \left[\mathrm{id}_{(X\times Y)\times Z}\right] &((x,y),z) \end{split}$$

for each  $((x, y), z) \in (X \times Y) \times Z$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathsf{id}_{(X \times Y) \times Z}$$
.

· Invertibility II. We have

$$\begin{split} \left[\alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1}\right] (x,(y,z)) &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} \left(\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z))\right) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} ((x,y),z) \\ &\stackrel{\mathsf{def}}{=} (x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} \left[ \mathrm{id}_{(X \times Y) \times Z} \right] (x,(y,z)) \end{split}$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathsf{id}_{X \times (Y \times Z)} \ .$$

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is indeed an isomorphism.

### **Naturality**

We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$ 

5.1.5 The Left Unitor

$$h: Z \to Z'$$

5

the diagram

$$\begin{array}{c|c} (X \times Y) \times Z & \xrightarrow{(f \times g) \times h} & (X' \times Y') \times Z' \\ \\ \alpha^{\mathsf{Sets}}_{X,Y,Z} & & & & & \\ \alpha^{\mathsf{Sets}}_{X',Y',Z'} \\ X \times (Y \times Z) & \xrightarrow{f \times (g \times h)} & X' \times (Y' \times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes, showing  $\alpha^{\text{Sets}}$  to be a natural transformation.

# Being a Natural Isomorphism

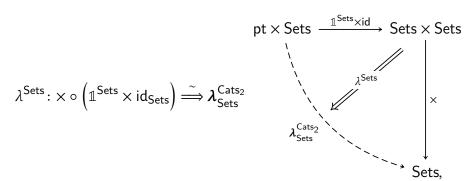
Since  $\alpha^{\text{Sets}}$  is natural and  $\alpha^{\text{Sets},-1}$  is a componentwise inverse to  $\alpha^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\alpha^{\text{Sets},-1}$  is also natural. Thus  $\alpha^{\text{Sets}}$  is a natural isomorphism.

# **01NT** 5.1.5 The Left Unitor

**01NU DEFINITION 5.1.5.1.1** ► THE LEFT UNITOR OF ×

5.1.5 The Left Unitor 6

The **left unitor of the product of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets}} \colon \mathsf{pt} \times X \xrightarrow{\sim} X$$

at  $X \in \mathsf{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\mathsf{def}}{=} x$$

for each  $(\star, x) \in pt \times X$ .

### PROOF 5.1.5.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.5.1.1

# Invertibility

The inverse of  $\lambda_X^{\mathrm{Sets}}$  is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathsf{pt} \times X$$

defined by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathsf{def}}{=} (\star, x)$$

for each  $x \in X$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}\right] (\mathsf{pt},x) &= \lambda_X^{\mathsf{Sets},-1} \Big(\lambda_X^{\mathsf{Sets}} (\mathsf{pt},x)\Big) \\ &= \lambda_X^{\mathsf{Sets},-1} (x) \end{split}$$

5.1.5 The Left Unitor

$$= (pt, x)$$
$$= [id_{pt \times X}](pt, x)$$

for each  $(pt, x) \in pt \times X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}} = \mathsf{id}_{\mathsf{pt} \times X} \,.$$

· Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}\right](x) &= \lambda_X^{\mathsf{Sets}} \Big(\lambda_X^{\mathsf{Sets},-1}(x)\Big) \\ &= \lambda_X^{\mathsf{Sets},-1}(\mathsf{pt},x) \\ &= x \\ &= [\mathsf{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathsf{id}_X \,.$$

Therefore  $\lambda_X^{\mathrm{Sets}}$  is indeed an isomorphism.

# Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc}
\operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\
\lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{pmatrix}
\star, x \\
\downarrow \\
x \\
\longleftarrow
 \end{pmatrix}
 f(x)$$

$$(\star, x) \\
\downarrow \\
f(x)$$

and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets}}$  is a natural transformation.

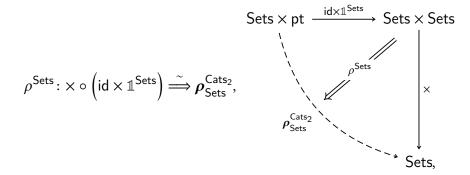
# Being a Natural Isomorphism

Since  $\lambda^{\text{Sets}}$  is natural and  $\lambda^{\text{Sets},-1}$  is a componentwise inverse to  $\lambda^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\lambda^{\text{Sets},-1}$  is also natural. Thus  $\lambda^{\text{Sets}}$  is a natural isomorphism.

# 01NV 5.1.6 The Right Unitor

# **01NW DEFINITION 5.1.6.1.1** ► THE RIGHT UNITOR OF ×

The right unitor of the product of sets is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \mathsf{pt} \xrightarrow{\sim} X$$

at  $X \in \mathsf{Obj}(\mathsf{Sets})$  is given by

$$\rho_X^{\mathsf{Sets}}(x, \star) \stackrel{\mathsf{def}}{=} x$$

for each  $(x, \star) \in X \times pt$ .

# PROOF 5.1.6.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.6.1.1

# Invertibility

The inverse of  $\rho_X^{\rm Sets}$  is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\mathsf{def}}{=} (x, \star)$$

for each  $x \in X$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} \right] (x, \star) &= \rho_X^{\mathsf{Sets},-1} \Big( \rho_X^{\mathsf{Sets}} (x, \star) \Big) \\ &= \rho_X^{\mathsf{Sets},-1} (x) \\ &= (x, \star) \\ &= \left[ \mathsf{id}_{X \times \mathsf{pt}} \right] (x, \star) \end{split}$$

for each  $(x, \star) \in X \times pt$ , and therefore we have

$$ho_X^{\mathsf{Sets},-1} \circ 
ho_X^{\mathsf{Sets}} = \mathsf{id}_{X \times \mathsf{pt}} \,.$$

· Invertibility II. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} \right] (x) &= \rho_X^{\mathsf{Sets}} \Big( \rho_X^{\mathsf{Sets},-1} (x) \Big) \\ &= \rho_X^{\mathsf{Sets},-1} (x, \bigstar) \\ &= x \\ &= \lceil \mathsf{id}_X \rceil (x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathsf{id}_X \,.$$

Therefore  $\rho_X^{\rm Sets}$  is indeed an isomorphism.

**Naturality** 

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc} X \times \operatorname{pt} & \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} & Y \times \operatorname{pt} \\ & & & & & & & \\ \rho_X^{\operatorname{Sets}} & & & & & & \\ \rho_Y^{\operatorname{Sets}} & & & & & \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(x, \star) & (x, \star) & \longmapsto (f(x), \star) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\rho^{\rm Sets}$  is a natural transformation.

# Being a Natural Isomorphism

Since  $\rho^{\mathsf{Sets}}$  is natural and  $\rho^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\rho^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\rho^{\mathsf{Sets},-1}$  is also natural. Thus  $\rho^{\mathsf{Sets}}$  is a natural isomorphism.

# **01NX 5.1.7** The Symmetry

# **O1NY DEFINITION 5.1.7.1.1** ► THE SYMMETRY OF ×

The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{array}{c} \mathsf{Sets} \times \mathsf{Sets} & \stackrel{\times}{\longrightarrow} \mathsf{Sets}, \\ & \parallel & \\ & \sigma^{\mathsf{Sets}} & \downarrow & \\ & \mathsf{Sets} \times \mathsf{Sets} & \end{array}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets}} \colon X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in \mathsf{Obj}(\mathsf{Sets})$  is defined by

$$\sigma_{X,Y}^{\mathsf{Sets}}(x,y) \stackrel{\mathsf{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

# PROOF 5.1.7.1.2 ▶ Proof of the Claims Made in Definition 5.1.7.1.1

# Invertibility

The inverse of  $\sigma_{X,Y}^{\mathsf{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \colon Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},-1}(y,x) \stackrel{\mathsf{def}}{=} (x,y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}}\right] &(x,y) \stackrel{\mathsf{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \Big(\sigma_{X,Y}^{\mathsf{Sets}}(x,y)\Big) \\ \stackrel{\mathsf{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} (y,x) \\ \stackrel{\mathsf{def}}{=} (x,y) \\ \stackrel{\mathsf{def}}{=} [\mathsf{id}_{X\times Y}] (x,y) \end{split}$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}} = \mathsf{id}_{X \times Y} \,.$$

· Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1}\right] (y,x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \Big(\sigma_{X,Y}^{\mathsf{Sets}} (y,x)\Big) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} (x,y) \\ &\stackrel{\text{def}}{=} (y,x) \end{split}$$

$$\stackrel{\text{def}}{=} [\mathsf{id}_{Y \times X}](y, x)$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \mathsf{id}_{Y \times X} \,.$$

Therefore  $\sigma_{X,Y}^{\mathsf{Sets}}$  is indeed an isomorphism.

# **Naturality**

We need to show that, given functions

$$f: X \to A$$
,  $g: Y \to B$ 

the diagram

$$X \times Y \xrightarrow{f \times g} A \times B$$

$$\sigma_{X,Y}^{\mathsf{Sets}} \qquad \qquad \int_{\sigma_{A,B}^{\mathsf{Sets}}} \sigma_{X,Y}^{\mathsf{Sets}} dx$$

$$Y \times X \xrightarrow{g \times f} B \times A$$

commutes. Indeed, this diagram acts on elements as

$$(x,y) \qquad (x,y) \longmapsto (f(x),g(y))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(y,x) \longmapsto (g(y),f(x)) \qquad (g(y),f(x))$$

and hence indeed commutes, showing  $\sigma^{\text{Sets}}$  to be a natural transformation.

# Being a Natural Isomorphism

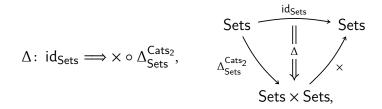
Since  $\sigma^{\text{Sets}}$  is natural and  $\sigma^{\text{Sets},-1}$  is a componentwise inverse to  $\sigma^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\sigma^{\text{Sets},-1}$  is also natural. Thus  $\sigma^{\text{Sets}}$  is a natural isomorphism.

# 01NZ 5.1.8 The Diagonal

### 01P0

# **DEFINITION 5.1.8.1.1** ► THE DIAGONAL OF ×

The diagonal of the product of sets is the natural transformation



whose component

$$\Delta_X \colon X \to X \times X$$

at  $X \in \mathsf{Obj}(\mathsf{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .

### PROOF 5.1.8.1.2 ▶ Proof of the Claims Made in Definition 5.1.8.1.1

We need to show that, given a function  $f: X \to Y$ , the diagram

$$X \xrightarrow{f} Y$$

$$\Delta_X \downarrow \qquad \qquad \downarrow \Delta_Y$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{cccc}
x & & x & \longrightarrow f(x) \\
\downarrow & & \downarrow \\
(x,x) & \longmapsto (f(x),f(x)) & & (f(x),f(x))
\end{array}$$

and hence indeed commutes, showing  $\Delta$  to be natural.

### 01P1 PROPOSITION 5.1.8.1.3 ▶ PROPERTIES OF THE DIAGONAL MAP

Let *X* be a set.

01P2

01P3

01P4

1. Monoidality. The diagonal map

$$\Delta \colon \operatorname{id}_{\mathsf{Sets}} \Longrightarrow \mathsf{X} \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

is a monoidal natural transformation:

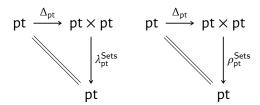
(a) Compatibility With Strong Monoidality Constraints. For each  $X, Y \in \mathsf{Obj}(\mathsf{Sets})$ , the diagram

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} (X \times X) \times (Y \times Y)$$

$$\downarrow (X \times Y) \times (X \times Y)$$

commutes.

(b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets},-1} \\ &= \rho_{\text{pt}}^{\text{Sets},-1}, \end{split}$$

where we recall that the equalities

$$\begin{split} \lambda_{\text{pt}}^{\text{Sets}} &= \rho_{\text{pt}}^{\text{Sets}}, \\ \lambda_{\text{pt}}^{\text{Sets},-1} &= \rho_{\text{pt}}^{\text{Sets},-1} \end{split}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

01P5

2. The Diagonal of the Unit. The component

$$\Delta_{pt} : pt \xrightarrow{\sim} pt \times pt$$

of  $\Delta$  at pt is an isomorphism.

### PROOF 5.1.8.1.4 ▶ PROOF OF PROPOSITION 5.1.8.1.3

# Item 1: Monoidality

We claim that  $\Delta$  is indeed monoidal:

024S

1. *Item 1a*: Compatibility With Strong Monoidality Constraints: We need to show that the diagram

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} (X \times X) \times (Y \times Y)$$

$$\downarrow^{\lambda}$$

$$(X \times Y) \times (X \times Y)$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes.

024T

2. Item 1b: Compatibility With Strong Unitality Constraints: As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathsf{def}}{=} (\star,x)$$

for each  $x \in X$ , so when X = pt, we have

$$\lambda_{pt}^{Sets,-1}(\star) \stackrel{\text{def}}{=} (\star,\star),$$

and also

$$\Delta_{\rm pt}^{\rm Sets}(\star)\stackrel{\rm def}{=}(\star,\star),$$

so we have  $\Delta_{pt} = \lambda_{pt}^{Sets,-1}$ .

This finishes the proof.

# Item 2: The Diagonal of the Unit

This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to ×, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor.

# 01P6 5.1.9 The Monoidal Category of Sets and Products

01P7

PROPOSITION 5.1.9.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO THE PROD-UCT

The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- · The Underlying Category. The category Sets of pointed sets.
- · The Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

· The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.3.5.1.2.

· The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

· The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \mathsf{\times} \circ (\mathsf{x} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \mathsf{\times} \circ (\mathsf{id}_{\mathsf{Sets}} \mathsf{x} \mathsf{x}) \circ \pmb{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

· The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

· The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \times \circ \left( \mathsf{id} \times \mathbb{1}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

· The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

· The Diagonals. The monoidal natural transformation

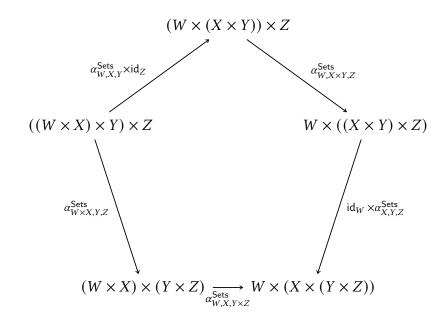
$$\Delta : id_{Sets} \Longrightarrow \times \circ \Delta^{Cats_2}_{Sets}$$

of Definition 5.1.8.1.1.

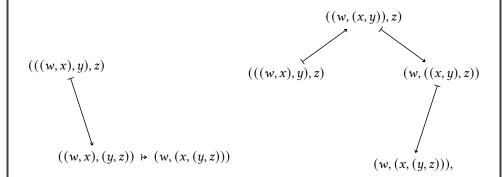
# PROOF 5.1.9.1.2 ► PROOF OF PROPOSITION 5.1.9.1.1

The Pentagon Identity

Let W, X, Y and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the pentagon identity is satisfied.

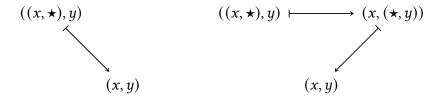
The Triangle Identity

Let X and Y be sets. We have to show that the diagram

$$(X \times \mathsf{pt}) \times Y \xrightarrow{\alpha_{X,\mathsf{pt},Y}^{\mathsf{Sets}}} X \times (\mathsf{pt} \times Y)$$

$$\rho_X^{\mathsf{Sets}} \times \mathsf{id}_Y \xrightarrow{\mathsf{id}_X \times \lambda_Y^{\mathsf{Sets}}} X \times Y$$

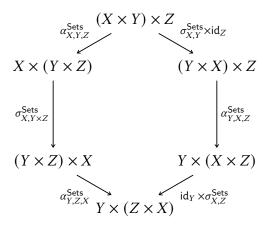
commutes. Indeed, this diagram acts on elements as



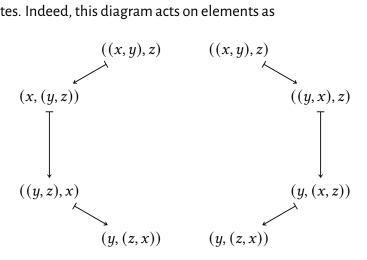
and thus the triangle identity is satisfied.

# The Left Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram



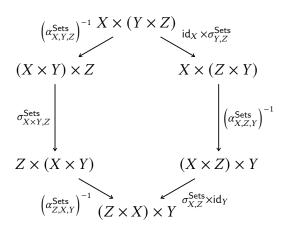
commutes. Indeed, this diagram acts on elements as

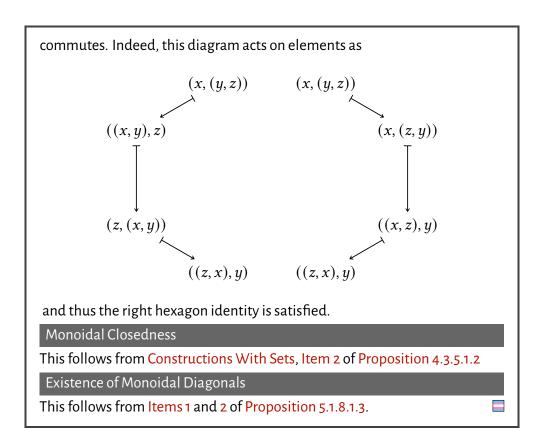


and thus the left hexagon identity is satisfied.

# The Right Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram





# **10** 5.1.10 The Universal Property of (Sets, $\times$ , pt)

# 01P9 THEOREM 5.1.10.1.1 ► THE UNIVERSAL PROPERTY OF (Sets, ×, PT)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Existence of an Internal Hom. The tensor product

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$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom  $[-1, -2]_{Sets}$ .

2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

More precisely, the full subcategory of the category  $\mathcal{M}^{cld}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:spanned}$  symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,  $[-_1, -_2]_{\mathsf{Sets}}$ ,  $\mathbb{1}_{\mathsf{Sets}}$ ,

 $\lambda^{\rm Sets}$ ,  $\rho^{\rm Sets}$ ,  $\sigma^{\rm Sets}$ ) satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

### PROOF 5.1.10.1.2 ▶ PROOF OF THEOREM 5.1.10.1.1

# Unwinding the Statement

Let (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$id_{Sets} : Sets \rightarrow Sets$$

admits a unique closed symmetric monoidal functor structure

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) to the closed symmetric monoidal category (Sets,  $\times$ , Sets $(-_1, -_2)$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda^{Sets}$ ,  $\rho^{Sets}$ ,  $\sigma^{Sets}$ ) of Proposition 5.1.9.1.1.

# Constructing an Isomorphism $[-1, -2]_{Sets} \cong Sets(-1, -2)$

By ??, we have a natural isomorphism

$$Sets(pt, [-1, -2]_{Sets}) \cong Sets(-1, -2).$$

By Constructions With Sets, Item 3 of Proposition 4.3.5.1.2, we also have a natural isomorphism

Sets
$$(pt, [-1, -2]_{Sets}) \cong [-1, -2]_{Sets}$$
.

Composing both natural isomorphisms, we obtain a natural isomorphism

$$Sets(-_1, -_2) \cong [-_1, -_2]_{Sets}.$$

Given  $A, B \in \mathsf{Obj}(\mathsf{Sets})$ , we will write

$$id_{A,B}^{Hom} : Sets(A,B) \xrightarrow{\sim} [A,B]_{Sets}$$

for the component of this isomorphism at (A, B).

# Constructing an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Since  $\otimes_{Sets}$  is adjoint in each variable to  $[-1, -2]_{Sets}$  by assumption and  $\times$  is adjoint in each variable to Sets(-1, -2) by Constructions With Sets, Item 2 of Proposition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -,$$
  
 $- \otimes_{\mathsf{Sets}} B \cong B \times -.$ 

By  $\ref{eq:special}$ , we then have  $\otimes_{\mathsf{Sets}} \cong \times$ . We will write

$$\operatorname{id}_{\operatorname{Sets}(A)B}^{\otimes} \colon A \otimes_{\operatorname{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

# Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Alternatively, we may construct a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  as follows:

1. Let  $A \in Obj(Sets)$ .

2. Since  $\otimes_{Sets}$  is part of a closed monoidal structure, it preserves colimits in each variable by  $\ref{eq:sets}$ .

3. Since  $A \cong \coprod_{a \in A} \operatorname{pt}$  and  $\otimes_{\operatorname{Sets}}$  preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong \left( \bigsqcup_{a \in A} \mathsf{pt} \right) \otimes_{\mathsf{Sets}} B$$

$$\cong \bigsqcup_{a \in A} (\mathsf{pt} \otimes_{\mathsf{Sets}} B)$$

$$\cong \bigsqcup_{a \in A} B$$

$$\cong A \times B,$$

naturally in  $B \in \mathsf{Obj}(\mathsf{Sets})$ , where we have used that pt is the monoidal unit for  $\otimes_{\mathsf{Sets}}$ . Thus  $A \otimes_{\mathsf{Sets}} - \cong A \times -$  for each  $A \in \mathsf{Obj}(\mathsf{Sets})$ .

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- 4. Similarly,  $\otimes_{\mathsf{Sets}} B \cong \times B$  for each  $B \in \mathsf{Obj}(\mathsf{Sets})$ .
- 01PG
- 5. By ??, we then have  $\otimes_{Sets} \cong \times$ .

Below, we'll show that if a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times \mathsf{exists}$ , then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\mathsf{id}_{\mathsf{Sets}|A,B}^\otimes \colon A \otimes_{\mathsf{Sets}} B \to A \times B$  from before.

# Constructing an Isomorphism $id_1^{\otimes} : \mathbb{1}_{Sets} \to pt$

We define an isomorphism  $id_1^{\otimes} \colon \mathbb{1}_{\mathsf{Sets}} \to \mathsf{pt}$  as the composition

in Sets.

# Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$\operatorname{pt} \otimes_{\mathsf{Sets}} A \xrightarrow{\operatorname{id}_{\mathsf{Sets}}^{\otimes} | \operatorname{pt}, A} \operatorname{pt} \times A$$

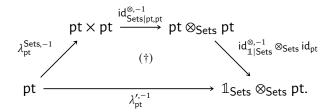
$$\operatorname{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \operatorname{id}_{A} \xrightarrow{\lambda_{A}'} A$$

$$\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \xrightarrow{\lambda_{A}'} A$$

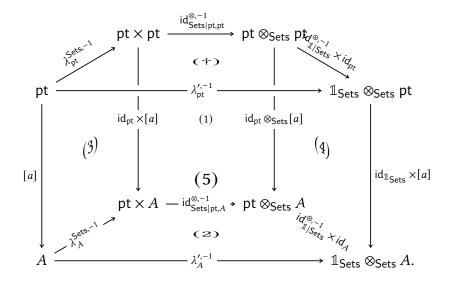
commutes. First, note that the diagram

corresponding to the case A = pt, commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so

# does the diagram



Now, let  $A \in Obj(Sets)$ , let  $a \in A$ , and consider the diagram



# Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'$ . $^{-1}$ .
- · Subdiagram (†) commutes, as proved above.
- · Subdiagram (4) commutes by the naturality of  $\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- · Subdiagram (3) commutes by the naturality of  $\lambda^{\mathsf{Sets},-1}$ .

it follows that the diagram

$$\operatorname{pt} \times A \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1}} \operatorname{pt} \otimes_{\mathsf{Sets}} A$$

$$\operatorname{pt} \xrightarrow{[a]} A \xrightarrow{\lambda_A'^{,-1}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\lambda_{A}^{\prime,-1}(a) = \left[\lambda_{A}^{\prime,-1} \circ [a]\right](\star)$$

$$= \left[\left(\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_{A}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_{A}^{\mathsf{Sets},-1} \circ [a]\right](\star)$$

$$= \left[\left(\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_{A}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_{A}^{\mathsf{Sets},-1}\right](a)$$

for each  $a \in A$ , and thus we have

$$\lambda_A^{\prime,-1} = \left( \mathsf{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathsf{id}_A \right) \circ \mathsf{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathsf{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ \left(\mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \times \mathsf{id}_A\right)$$

showing that the diagram

$$\operatorname{pt} \otimes_{\mathsf{Sets}} A \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes}} \operatorname{pt} \times A$$

$$\operatorname{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \operatorname{id}_{A} \xrightarrow{\lambda_{A}'} A$$

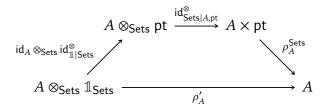
$$\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \xrightarrow{\lambda_{A}'} A$$

indeed commutes.

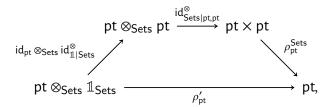
# Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  above. For completeness, we repeat it below.

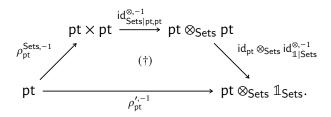
We have to show that the diagram

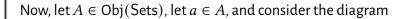


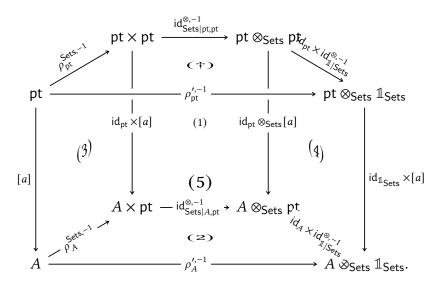
commutes. First, note that the diagram



corresponding to the case  $A=\operatorname{pt}$ , commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so does the diagram



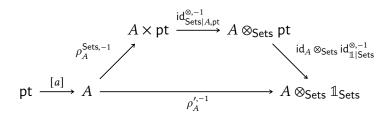




### Since:

- · Subdiagram (5) commutes by the naturality of  $\rho'^{-1}$ .
- · Subdiagram (†) commutes, as proved above.
- · Subdiagram (4) commutes by the naturality of  $\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\rm Sets,-1}.$

# it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{split} \rho_A^{\prime,-1}(a) &= \left[\rho_A^{\prime,-1} \circ [a]\right](\bigstar) \\ &= \left[\left(\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1} \circ [a]\right](\bigstar) \\ &= \left[\left(\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}\right](a) \end{split}$$

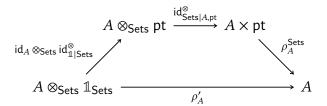
for each  $a \in A$ , and thus we have

$$\rho_A^{\prime,-1} = \left( \mathsf{id}_A \times \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \right) \circ \mathsf{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathsf{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathsf{id}_A \times \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes}),$$

showing that the diagram



indeed commutes.

# Monoidality of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$(A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C$$

$$(A \times B) \otimes_{\mathsf{Sets}} C$$

$$(A \times B) \otimes_{\mathsf{Sets}} C$$

$$\mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \qquad \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes}$$

$$(A \times B) \times C$$

$$A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C)$$

$$\mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \qquad \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes}$$

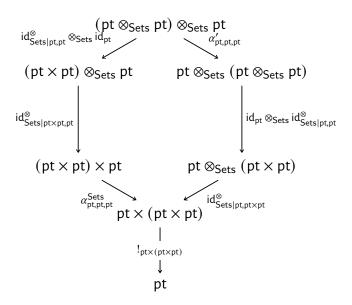
$$(A \times B) \times C$$

$$A \otimes_{\mathsf{Sets}} (B \times C)$$

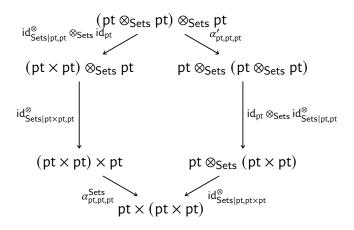
$$\mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes}$$

$$\mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes}$$

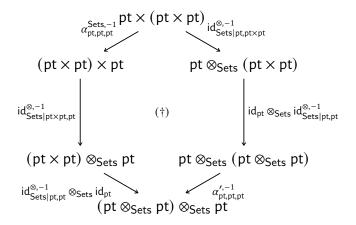
# commutes. First, note that the diagram



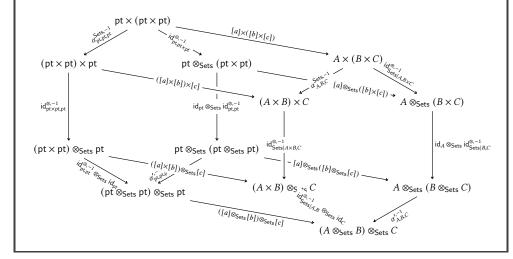
commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since the map  $!_{pt \times (pt \times pt)} : pt \times (pt \times pt) \rightarrow pt$  is an isomorphism (e.g. having inverse  $\lambda_{pt}^{\mathsf{Sets},-1} \circ \lambda_{pt}^{\mathsf{Sets},-1}$ ), it follows that the diagram

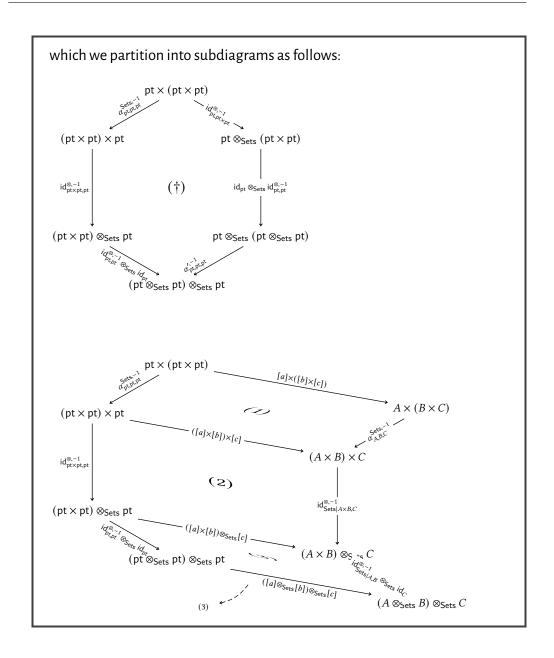


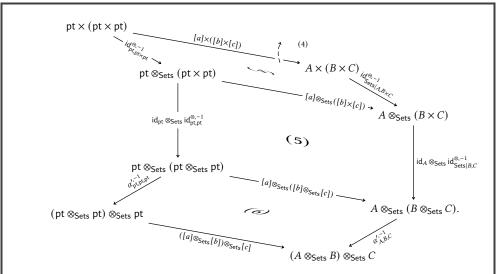
also commutes. Taking inverses, we see that the diagram



commutes as well. Now, let  $A, B, C \in \mathsf{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , let  $c \in C$ , and consider the diagram



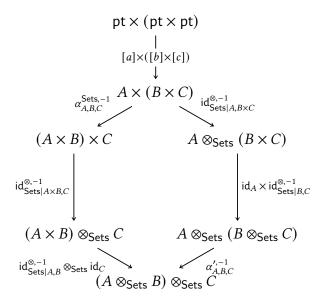




### Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\text{Sets,-1}}$ .
- Subdiagram (2) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- · Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (5) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$  .
- · Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .

# it follows that the diagram



also commutes. We then have

$$\begin{split} \left[ \left( \mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathsf{id}_C \right) \circ \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ &\circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) = \left[ \left( \mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathsf{id}_C \right) \circ \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ &\circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ \left( [a] \times ([b] \times [c]) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \mathsf{id}_A \times \mathsf{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \\ &\circ \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ \left( [a] \times ([b] \times [c]) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \mathsf{id}_A \times \mathsf{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \circ \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

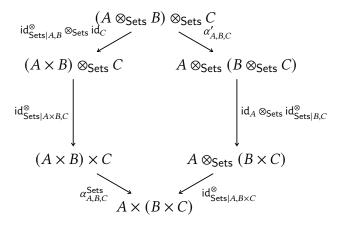
for each  $(a, (b, c)) \in A \times (B \times C)$ , and thus we have

$$\left(\mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathsf{id}_{C}\right) \circ \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} = \alpha_{A,B,C}'^{,-1} \circ \left(\mathsf{id}_{A} \times \mathsf{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}\right) \circ \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \,.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ \left( \mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathsf{id}_{C} \right) = \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ \left( \mathsf{id}_{A} \times \mathsf{id}_{\mathsf{Sets}|B,C}^{\otimes} \right) \circ \alpha_{A,B,C}',$$

# showing that the diagram



indeed commutes.

# Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{c|c} A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B \\ \\ \sigma'_{A,B} \downarrow & & \downarrow \sigma^{\mathsf{Sets}}_{A,B} \\ B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathsf{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A \end{array}$$

commutes. First, note that the diagram

commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since the map  $!_{pt \times pt} : pt \times pt \rightarrow pt$  is invertible (e.g. with inverse  $\lambda_{pt}^{Sets,-1}$ ), the diagram

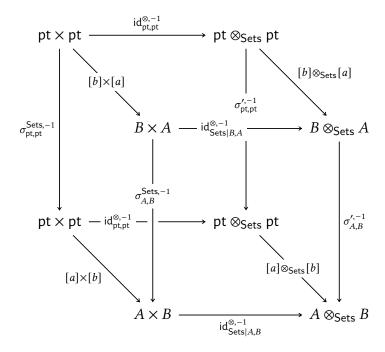
$$\begin{array}{c|c} \text{pt} \otimes_{\mathsf{Sets}} \text{pt} & \xrightarrow{\mathsf{id}^{\otimes}_{\mathsf{Sets}|\mathsf{pt},\mathsf{pt}}} & \mathsf{pt} \times \mathsf{pt} \\ \\ \sigma'_{\mathsf{pt},\mathsf{pt}} & & & & & \\ \sigma'_{\mathsf{pt},\mathsf{pt}} & & & & \\ \mathsf{pt} \otimes_{\mathsf{Sets}} \text{pt} & \xrightarrow{\mathsf{id}^{\otimes}_{\mathsf{Sets}|\mathsf{pt},\mathsf{pt}}} & \mathsf{pt} \times \mathsf{pt} \end{array}$$

also commutes. Taking inverses, we see that the diagram

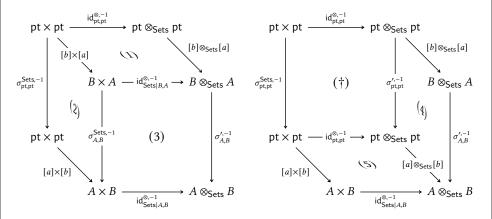
$$\begin{array}{c|c} \mathsf{pt} \times \mathsf{pt} & \xrightarrow{\mathsf{id}_{\mathsf{Sets}|\mathsf{pt},\mathsf{pt}}^{\otimes,-1}} \mathsf{pt} \otimes_{\mathsf{Sets}} \mathsf{pt} \\ \sigma_{\mathsf{pt},\mathsf{pt}}^{\mathsf{Sets},-1} & (\dagger) & & & \sigma_{\mathsf{pt},\mathsf{pt}}^{\prime,-1} \\ \mathsf{pt} \times \mathsf{pt} & \xrightarrow{\mathsf{id}_{\mathsf{Sets}|\mathsf{pt}}^{\otimes,-1}} \mathsf{pt} \otimes_{\mathsf{Sets}} \mathsf{pt} \end{array}$$

commutes as well. Now, let  $A, B \in \mathsf{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , and

#### consider the diagram



which we partition into subdiagrams as follows:

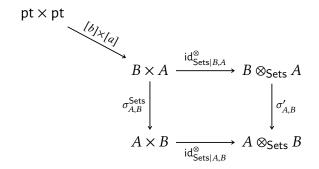


Since:

· Subdiagram (2) commutes by the naturality of  $\sigma^{\text{Sets},-1}$ .

- · Subdiagram (5) commutes by the naturality of  $id^{\otimes,-1}$ .
- · Subdiagram (†) commutes, as proved above.
- · Subdiagram (4) commutes by the naturality of  $\sigma'^{,-1}$ .
- · Subdiagram (1) commutes by the naturality of  $id^{\otimes,-1}$ .

it follows that the diagram



commutes. We then have

$$\begin{split} \left[ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \right] (b,a) &= \left[ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a]) \right] (\star, \star) \\ &= \left[ \sigma_{A,B}^{\prime,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a]) \right] (\star, \star) \\ &= \left[ \sigma_{A,B}^{\prime,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \right] (b,a) \end{split}$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}'^{,-1} \circ \operatorname{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathsf{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}',$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\sigma'_{A,B} \downarrow \qquad \qquad \qquad \downarrow \sigma^{\mathsf{Sets}}_{A,B}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathsf{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

indeed commutes.

## Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Let  $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ \left( \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathsf{id}_Y \right),$$
$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ \left( \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathsf{id}_Y \right).$$

Postcomposing both sides with  $\lambda_B^{\mathsf{Sets},-1}$  gives

$$\lambda_{B}^{\mathsf{Sets},-1} \circ \lambda_{B}' \circ \left( \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathsf{id}_{Y} \right) = \phi_{\mathsf{pt},B},$$
$$\lambda_{B}^{\mathsf{Sets},-1} \circ \lambda_{B}' \circ \left( \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathsf{id}_{Y} \right) = \psi_{\mathsf{pt},B},$$

and thus we have

$$\phi_{\mathsf{pt},B} = \psi_{\mathsf{pt},B}$$

for each  $B \in \mathsf{Obj}(\mathsf{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams

for  $\phi$  and  $\psi$  with respect to the morphisms [a] and  $\mathrm{id}_B$ . Having shown that  $\phi_{\mathsf{pt},B} = \psi_{\mathsf{pt},B}$ , we have

$$\begin{aligned} \phi_{A,B}(a,b) &= \left[\phi_{A,B} \circ ([a] \times \mathrm{id}_B)\right] (\star,b) \\ &= \left[([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \phi_{\mathsf{pt},B}\right] (\star,b) \\ &= \left[([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \psi_{\mathsf{pt},B}\right] (\star,b) \\ &= \left[\psi_{A,B} \circ ([a] \times \mathrm{id}_B)\right] (\star,b) \\ &= \psi_{A,B}(a,b) \end{aligned}$$

for each  $(a, b) \in A \times B$ . Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each  $A, B \in \mathsf{Obj}(\mathsf{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\mathsf{Sets}} \cong \mathsf{x}$  to be unique.

#### **O1PH** COROLLARY 5.1.10.1.3 ► A SECOND UNIVERSAL PROPERTY FOR (Sets, ×, PT)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sets}} : \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

01PK 2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

01PJ

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:Sets}$  spanned by the symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,  $\mathbb{1}_{\mathsf{Sets}}$ ,  $\lambda^{\mathsf{Sets}}$ ,  $\rho^{\mathsf{Sets}}$ ,  $\sigma^{\mathsf{Sets}}$ ) satisfying Items 1 and 2 is contractible.

#### PROOF 5.1.10.1.4 ► PROOF OF COROLLARY 5.1.10.1.3

Since Sets is locally presentable (??), it follows from ?? that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Theorem 5.1.10.1.1. The result then follows from Theorem 5.1.10.1.1.

# **O1PL** 5.2 The Monoidal Category of Sets and Coproducts

## **01PM 5.2.1** Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

#### 01PN 5.2.2 The Monoidal Unit

### 01PP DEFINITION 5.2.2.1.1 ► THE MONOIDAL UNIT OF [

The monoidal unit of the coproduct of sets is the functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{O}_{\mathsf{Sets}} \stackrel{\mathsf{def}}{=} \emptyset$$

where Ø is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

## 01PQ 5.2.3 The Associator

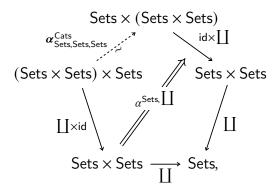
#### 01PR DEFINITION 5.2.3.1.1 ► THE ASSOCIATOR OF [

The associator of the coproduct of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} \colon \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \overset{\widetilde{}}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \pmb{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$

5.2.3 The Associator 42

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} : (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each  $a \in (X \mid \mid Y) \mid \mid Z$ .

#### PROOF 5.2.3.1.2 ▶ Proof of the Claims Made in Definition 5.2.3.1.1

#### Unwinding the Definitions of $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$

Firstly, we unwind the expressions for  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ . We have

$$(X \coprod Y) \coprod Z \stackrel{\text{def}}{=} \{ (0, a) \in S \mid a \in X \coprod Y \} \cup \{ (1, z) \in S \mid z \in Z \}$$

$$= \{ (0, (0, x)) \in S \mid x \in X \} \cup \{ (0, (1, y)) \in S \mid y \in Y \}$$

$$\cup \{ (1, z) \in S \mid z \in Z \},$$

5.2.3 The Associator 43

where  $S = \{0, 1\} \times ((X \coprod Y) \cup Z)$  and

$$\begin{split} X \coprod (Y \coprod Z) &\stackrel{\text{def}}{=} \{(0, x) \in S' \mid x \in X\} \cup \{(1, a) \in S' \mid a \in Y \coprod Z\} \\ &= \{(0, x) \in S' \mid x \in X\} \cup \{(1, (0, y)) \in S' \mid y \in Y\} \\ & \cup \{(1, (1, z)) \in S' \mid z \in Z\}, \end{split}$$

where  $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$ .

#### Invertibility

The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\mathsf{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each  $a \in X \coprod Y(\coprod Z)$ . Indeed:

· Invertibility I. The map  $\alpha_{X,Y,Z}^{\text{Sets},\coprod,-1}\circ\alpha_{X,Y,Z}^{\text{Sets},\coprod}$  acts on elements as

and hence is equal to the identity map of  $(X \coprod Y) \coprod Z$ .

· Invertibility II. The map  $lpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ lpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$  acts on elements as

$$(0,x) \mapsto (0,(0,x)) \mapsto (0,x),$$

$$(1,(0,y)) \mapsto (0,(0,y)) \mapsto (1,(0,y)),$$

$$(1,(1,z)) \mapsto (1,z) \mapsto (1,(1,z))$$

and hence is equal to the identity map of  $X \coprod (Y \coprod Z)$ .

5.2.3 The Associator 44

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

## Naturality

We need to show that, given functions

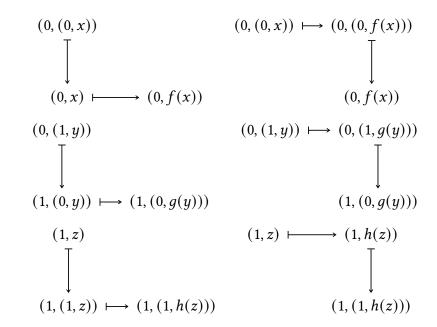
$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$\begin{array}{c|c} (X \coprod Y) \coprod Z & \xrightarrow{\Big(f \coprod g\Big) \coprod h} & (X' \coprod Y') \coprod Z' \\ & \xrightarrow{\text{Sets,} \coprod} & & & & & \\ X \coprod (Y \coprod Z) & \xrightarrow{f \coprod \Big(g \coprod h\Big)} & X' \coprod (Y' \coprod Z') \end{array}$$

5.2.4 The Left Unitor 45

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing  $\alpha^{\text{Sets},\coprod}$  to be a natural transformation.

## Being a Natural Isomorphism

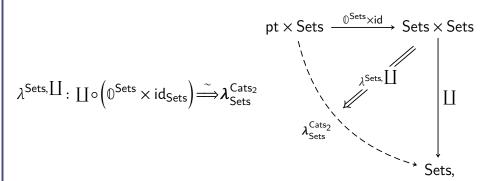
Since  $\alpha^{\text{Sets},\coprod}$  is natural and  $\alpha^{\text{Sets},\coprod,-1}$  is a componentwise inverse to  $\alpha^{\text{Sets},\coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\lambda^{\text{Sets},-1}$  is also natural. Thus  $\alpha^{\text{Sets},\coprod}$  is a natural isomorphism.

#### **01PS 5.2.4** The Left Unitor

01PT DEFINITION 5.2.4.1.1 ► THE LEFT UNITOR OF \[ \]

5.2.4 The Left Unitor 46

The left unitor of the coproduct of sets is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} : \emptyset \coprod X \xrightarrow{\sim} X$$

at X is given by

$$\lambda_X^{\mathsf{Sets},\coprod}((1,x))\stackrel{\mathsf{def}}{=} x$$

for each  $(1, x) \in \emptyset \coprod X$ .

#### PROOF 5.2.4.1.2 ▶ Proof of the Claims Made in Definition 5.2.4.1.1

#### Unwinding the Definition of $\emptyset \coprod X$

Firstly, we unwind the expressions for  $\emptyset \coprod X$ . We have

$$\emptyset \coprod X \stackrel{\text{def}}{=} \{(0, z) \in S \mid z \in \emptyset\} \cup \{(1, x) \in S \mid x \in X\}$$
$$= \emptyset \cup \{(1, x) \in S \mid x \in X\}$$
$$= \{(1, x) \in S \mid x \in X\},$$

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

## Invertibility

The inverse of  $\lambda_X^{\mathsf{Sets},\coprod}$  is the map

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \colon X \to \emptyset \coprod X$$

5.2.4 The Left Unitor 47

given by

$$\lambda_X^{\mathsf{Sets},\coprod,-1}(x)\stackrel{\mathsf{def}}{=} (1,x)$$

for each  $x \in X$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod}\right] (1, x) &= \lambda_X^{\mathsf{Sets}, \coprod, -1} \left(\lambda_X^{\mathsf{Sets}, \coprod} (1, x)\right) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (1, x) \\ &= \left[\mathsf{id}_{\emptyset \coprod X}\right] (1, x) \end{split}$$

for each  $(1, x) \in \emptyset \coprod X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod} = \mathsf{id}_{\emptyset \coprod X}.$$

· Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}, \coprod} \circ \lambda_X^{\mathsf{Sets}, \coprod, -1}\right](x) &= \lambda_X^{\mathsf{Sets}, \coprod} \left(\lambda_X^{\mathsf{Sets}, \coprod, -1}(x)\right) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1}(1, x) \\ &= x \\ &= [\mathsf{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathsf{id}_X$$
 .

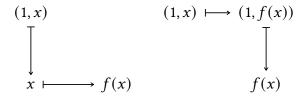
Therefore  $\lambda_X^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

**Naturality** 

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{c|c}
\emptyset \coprod X & \xrightarrow{\operatorname{id}_{\emptyset} \coprod f} \emptyset \coprod Y \\
\downarrow_{\lambda_{X}^{\mathsf{Sets}, \coprod}} & & \downarrow_{\lambda_{Y}^{\mathsf{Sets}, \coprod}} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets}, \coprod}$  is a natural transformation.

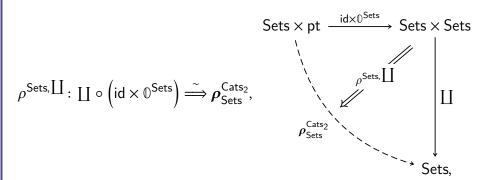
## Being a Natural Isomorphism

Since  $\lambda^{\text{Sets}, \coprod}$  is natural and  $\lambda^{\text{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\text{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\lambda^{\text{Sets}, \coprod}$  is a natural isomorphism.

# 01PU 5.2.5 The Right Unitor

01PV DEFINITION 5.2.5.1.1 ► THE RIGHT UNITOR OF

The **right unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets},\coprod}\colon X\coprod \not \!\!\! \bigcirc \stackrel{\sim}{\longrightarrow} X$$

at X is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x))\stackrel{\mathrm{def}}{=} x$$

for each  $(0, x) \in X \coprod \emptyset$ .

#### PROOF 5.2.5.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.5.1.1

#### Unwinding the Definition of $X \coprod \emptyset$

Firstly, we unwind the expression for  $X \coprod \emptyset$ . We have

$$X \coprod \emptyset \stackrel{\text{def}}{=} \{ (0, x) \in S \mid x \in X \} \cup \{ (1, z) \in S \mid z \in \emptyset \}$$

$$= \{ (0, x) \in S \mid x \in X \} \cup \emptyset$$

$$= \{ (0, x) \in S \mid x \in X \},$$

where  $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$ .

#### Invertibility

The inverse of  $\rho_X^{\mathrm{Sets},\coprod}$  is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \emptyset$$

given by

$$\rho_X^{\mathsf{Sets},\coprod,-1}(x)\stackrel{\mathrm{def}}{=} (0,x)$$

for each  $x \in X$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} \right] (0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} \Big( \rho_X^{\mathsf{Sets}, \coprod} (0, x) \Big) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (0, x) \\ &= \Big[ \mathsf{id}_{X \coprod \emptyset} \Big] (0, x) \end{split}$$

for each  $(0, x) \in \emptyset \coprod X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} = \mathsf{id}_{\emptyset \coprod X} \,.$$

· Invertibility II. We have

$$\begin{split} \left[\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1}\right](x) &= \rho_X^{\mathsf{Sets}, \coprod} \left(\rho_X^{\mathsf{Sets}, \coprod, -1}(x)\right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= x \\ &= [\mathsf{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1} = \mathsf{id}_X \,.$$

Therefore  $\rho_X^{\mathrm{Sets},\coprod}$  is indeed an isomorphism.

**Naturality** 

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{c|c} X \coprod \emptyset & \xrightarrow{f \coprod \mathrm{id}_{\emptyset}} & Y \coprod \emptyset \\ & & \downarrow \rho_X^{\mathsf{Sets}, \coprod} & & \downarrow \rho_Y^{\mathsf{Sets}, \coprod} \\ & X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(0,x) & (0,x) & \longmapsto (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $ho^{\mathsf{Sets},\coprod}$  is a natural transformation.

# Being a Natural Isomorphism

Since  $\rho^{\text{Sets},\coprod}$  is natural and  $\rho^{\text{Sets},-1}$  is a componentwise inverse to  $\rho^{\text{Sets},\coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\rho^{\text{Sets},-1}$  is also natural. Thus  $\rho^{\text{Sets},\coprod}$  is a natural isomorphism.

## **01PW 5.2.6** The Symmetry

#### 01PX DEFINITION 5.2.6.1.1 ► THE SYMMETRY OF \( \)

The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}, \coprod} : \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}, \qquad \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}} \qquad \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}} \qquad \sigma^{\mathsf{Sets}, \coprod}_{\mathsf{Sets}}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod} : X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at  $X, Y \in \mathsf{Obj}(\mathsf{Sets})$  is defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}(x,y)\stackrel{\mathsf{def}}{=}(y,x)$$

for each  $(x, y) \in X \times Y$ .

#### PROOF 5.2.6.1.2 ▶ Proof of the Claims Made in Definition 5.2.6.1.1

# Unwinding the Definitions of $X \coprod Y$ and $Y \coprod X$

Firstly, we unwind the expressions for  $X \coprod Y$  and  $Y \coprod X$ . We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},\$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},\$$

where 
$$S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$$
.

#### Invertibility

The inverse of  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is the map

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\mathsf{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}(z) \stackrel{\mathsf{def}}{=} \begin{cases} (0,x) & \mathsf{if}\, z = (1,x), \\ (1,y) & \mathsf{if}\, z = (0,y) \end{cases}$$

for each  $z \in Y \coprod X$ . Indeed:

· Invertibility I. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}\right] (0, x) &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} \left(\sigma_{X}^{\mathsf{Sets}, \coprod} (0, x)\right) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (1, x) \\ &= (0, x) \\ &= \left[\mathsf{id}_{X \coprod Y}\right] (0, x) \end{split}$$

for each  $(0, x) \in X \coprod Y$  and

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}\right] (1,y) &= \sigma_X^{\mathsf{Sets}, \coprod, -1} \Big(\sigma_X^{\mathsf{Sets}, \coprod} (1,y) \Big) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (0,y) \\ &= (1,y) \\ &= \Big[ \mathsf{id}_{X \coprod Y} \Big] (1,y) \end{split}$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod} = \mathsf{id}_{X\coprod Y} \,.$$

· Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}\right] (0, y) &= \sigma_{X}^{\mathsf{Sets}, \coprod} \left(\sigma_{X}^{\mathsf{Sets}, \coprod, -1} (0, y)\right) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (1, y) \\ &= (0, y) \\ &= \left[\mathsf{id}_{Y \coprod X}\right] (0, y) \end{split}$$

for each  $(0, y) \in Y \coprod X$  and

$$\begin{split} \Big[\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \Big] (1, x) &= \sigma_{X}^{\mathsf{Sets}, \coprod} \Big(\sigma_{X}^{\mathsf{Sets}, \coprod, -1} (1, x) \Big) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (0, x) \end{split}$$

$$= (1, x)$$

$$= \left[ id_{Y \coprod X} \right] (1, x)$$

for each  $(1, x) \in Y \coprod X$ , and therefore we have

$$\sigma_X^{\mathsf{Sets},\coprod} \circ \sigma_X^{\mathsf{Sets},\coprod,-1} = \mathsf{id}_{Y\coprod X}$$
 .

Therefore  $\sigma_{X,Y}^{\text{Sets},\coprod}$  is indeed an isomorphism.

#### **Naturality**

We need to show that, given functions  $f:A\to X$  and  $g:B\to Y$ , the diagram

$$A \coprod B \xrightarrow{f \coprod g} X \coprod Y$$

$$\downarrow_{\sigma_{A,B}^{\mathsf{Sets}, \coprod}} \qquad \qquad \downarrow_{\sigma_{X,Y}^{\mathsf{Sets}, \coprod}}$$

$$B \coprod A \xrightarrow{g \coprod f} Y \coprod X$$

commutes. Indeed, this diagram acts on elements as

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes. Therefore  $\sigma^{\mathsf{Sets},\coprod}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\sigma^{\text{Sets},\coprod}$  is natural and  $\sigma^{\text{Sets},-1}$  is a componentwise inverse to  $\sigma^{\text{Sets},\coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\sigma^{\text{Sets},-1}$  is also natural. Thus  $\sigma^{\text{Sets},\coprod}$  is a natural isomorphism.

## **01PY** 5.2.7 The Monoidal Category of Sets and Coproducts

#### 01PZ PROPOSITION 5.2.7.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO [

The category Sets admits a closed symmetric monoidal category structure consisting of:

- · The Underlying Category. The category Sets of pointed sets.
- · The Monoidal Product. The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

· The Monoidal Unit. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

· The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
 of Definition 5.2.3.1.1.

· The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

· The Right Unitors. The natural isomorphism

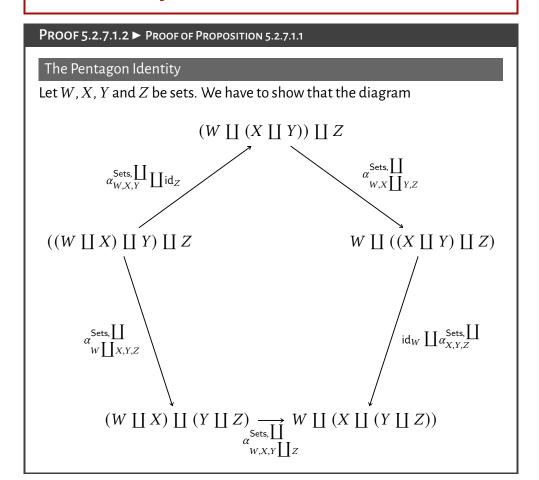
$$\rho^{\mathsf{Sets}, \coprod} \colon \coprod \circ \left(\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}\right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

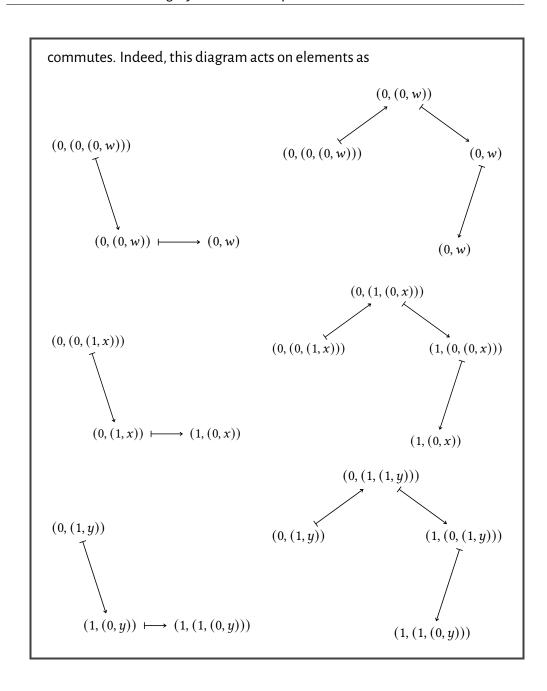
of Definition 5.2.5.1.1.

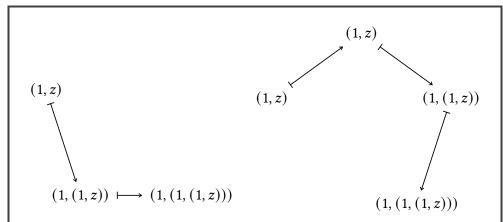
· The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}, \coprod} : \times \stackrel{\widetilde{\longrightarrow}}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}$$

of Definition 5.2.6.1.1.



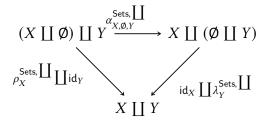




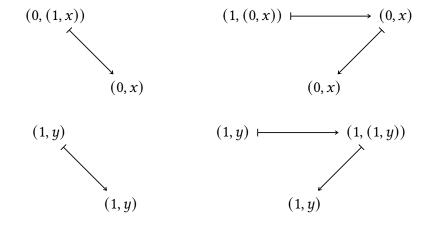
and therefore the pentagon identity is satisfied.

## The Triangle Identity

Let X and Y be sets. We have to show that the diagram



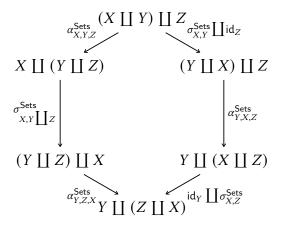
commutes. Indeed, this diagram acts on elements as



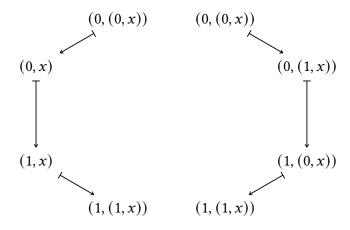
and therefore the triangle identity is satisfied.

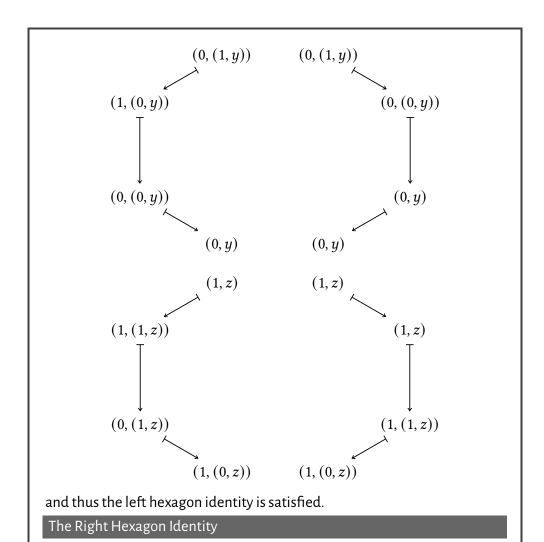
## The Left Hexagon Identity

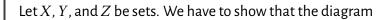
Let X, Y, and Z be sets. We have to show that the diagram

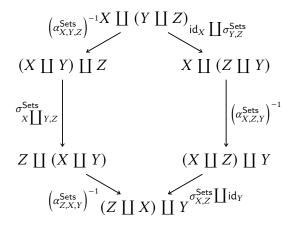


commutes. Indeed, this diagram acts on elements as

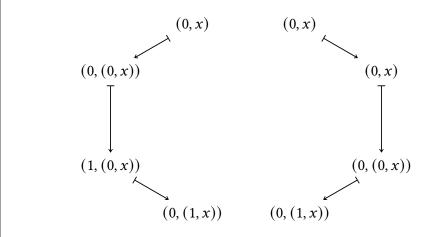


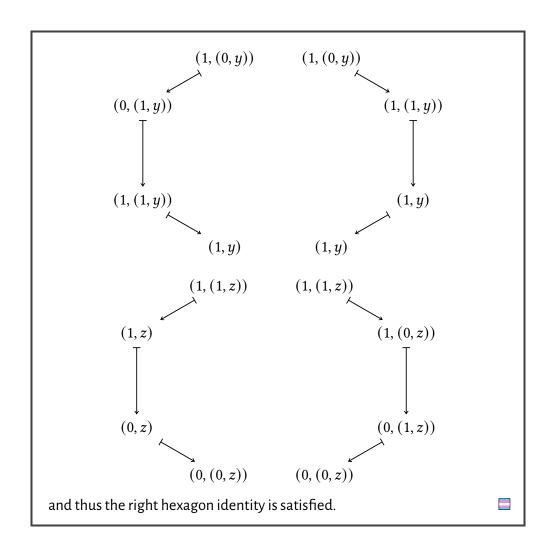






commutes. Indeed, this diagram acts on elements as





- 5.3 The Bimonoidal Category of Sets, Products, and Coproducts
- 01Q1 5.3.1 The Left Distributor

#### 01Q2 DEFINITION 5.3.1.1.1 ► THE LEFT DISTRIBUTOR OF × OVER []

The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

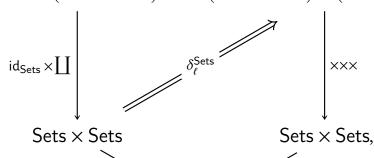
 $\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$  as in the diagram

$$(\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets})$$

$$\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}})$$

$$\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}})$$

$$\Delta_{\mathsf{Sets}} \times (\mathsf{Sets} \times \mathsf{Sets})$$



× Sets

whose component

$$\delta^{\mathsf{Sets}}_{\ell \mid X,Y,Z} \colon X \times (Y \coprod Z) \xrightarrow{\sim} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}(x,a) \stackrel{\mathsf{def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each  $(x, a) \in X \times (Y \coprod Z)$ .

#### PROOF 5.3.1.1.2 ▶ Proof of the Claims Made in Definition 5.3.1.1.1

Omitted.

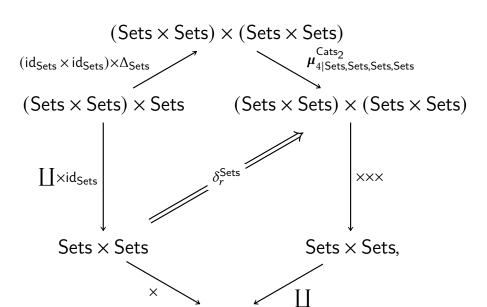
## 01Q3 5.3.2 The Right Distributor

## 0104 DEFINITION 5.3.2.1.1 ► THE RIGHT DISTRIBUTOR OF × OVER ∐

The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ ( \coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{-}}{\Longrightarrow} \coprod \circ ( \times \times \times) \circ \boldsymbol{\mu}_{4 \mid \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}_2} \circ ( (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$

as in the diagram



whose component

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each  $(a, z) \in (X \coprod Y) \times Z$ .

#### PROOF 5.3.2.1.2 ▶ Proof of the Claims Made in Definition 5.3.2.1.1

Omitted.

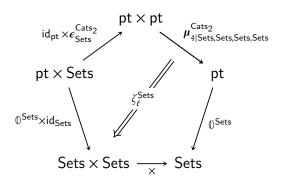
## 0105 5.3.3 The Left Annihilator

## **01Q6 DEFINITION 5.3.3.1.1** ► THE LEFT ANNIHILATOR OF ×

The left annihilator of the product of sets is the natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{4 \mid \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathsf{id}_{\mathsf{pt}} \times \pmb{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}\right)$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \emptyset \times A \xrightarrow{\sim} \emptyset.$$

#### PROOF 5.3.3.1.2 ▶ Proof of the Claims Made in Definition 5.3.3.1.1

Omitted. For a partial proof, see [Pro25].

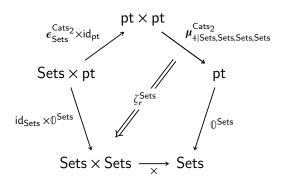
## 01Q7 5.3.4 The Right Annihilator

#### 0108 DEFINITION 5.3.4.1.1 ▶ THE RIGHT ANNIHILATOR OF $\times$

The right annihilator of the product of sets is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\pmb{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathsf{id}_{\mathsf{pt}}\right) \overset{\sim}{\dashrightarrow} \times \circ \left(\mathsf{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$

as in the diagram



with components

$$\zeta_{r|A}^{\mathsf{Sets}} \colon A \times \emptyset \xrightarrow{\sim} \emptyset.$$

#### PROOF 5.3.4.1.2 ▶ Proof of the Claims Made in Definition 5.3.4.1.1

Omitted. For a partial proof, see [Pro25].

# 0109 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

#### **PROPOSITION 5.3.5.1.1** ► THE BIMONOIDAL STRUCTURE ON SETS ASSOCIATED TO × AND \[ \]

The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- · The Underlying Category. The category Sets of pointed sets.
- · The Additive Monoidal Product. The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

· The Multiplicative Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

· The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

· The Monoidal Zero. The functor

$$\mathbb{0}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

· The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, ?? of ??.

· The Additive Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
 of Definition 5.2.3.1.1.

· The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{^{\sim}}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

· The Additive Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}, \coprod} \colon \coprod \circ \left(\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}\right) \stackrel{\widetilde{\longrightarrow}}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.2.5.1.1.

· The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} : \coprod \stackrel{\widetilde{}}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

· The Multiplicative Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

· The Multiplicative Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

· The Multiplicative Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} : \times \circ \left( \mathsf{id} \times \mathbb{1}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

· The Multiplicative Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}\,\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

· The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$$
 of Definition 5.3.1.1.1.

· The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
 of Definition 5.3.2.1.1.

· The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathsf{id}_{\mathsf{pt}} \times \boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}\right)$$

of Definition 5.3.3.1.1.

· The Right Annihilator. The natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\pmb{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathsf{id}_{\mathsf{pt}}\right) \overset{\sim}{\dashrightarrow} \times \circ \left(\mathsf{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$

of Definition 5.3.4.1.1.

PROOF 5.3.5.1.2 ► PROOF OF PROPOSITION 5.3.5.1.1

Omitted.



# A Other Chapters

**Appendices** 

#### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

#### Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

#### **Categories**

- 11. Categories
- Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

13. Constructions With Monoidal Categories

References 71

## **Bicategories**

#### **Extra Part**

14. Types of Morphisms in Bicategories

15. Notes

# References

[Pro25] Proof Wiki Contributors. Cartesian Product Is Empty Iff Factor Is Empty—
Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Cartesian\_
Product\_is\_Empty\_iff\_Factor\_is\_Empty (cit. on pp. 66, 67).