Monoidal Structures on the Category of Sets

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This chapter contains some material on monoidal structures on Sets.

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Products of Sets 5.I.I

See Constructions With Sets, Section 4.1.3.

The Internal Hom of Sets

See Constructions With Sets, Section 4.3.5.

The Monoidal Unit 5.1.3

Definition 5.1.3.1.1. The monoidal unit of the product of sets is the functor

$$1^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$1_{Sets} \stackrel{\text{def}}{=} pt$$
,

where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.1.

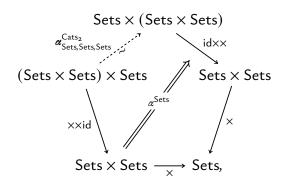
The Associator **5.I.4**

Definition 5.1.4.1.1. The associator of the product of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{\longrightarrow}}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}'}$$

5.1.4 The Associator

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \stackrel{\mathrm{def}}{=} (x,(y,z))$$

for each $((x, y), z) \in (X \times Y) \times Z$.

Proof. Invertibility: The inverse of $\alpha_{X,Y,Z}^{\mathsf{Sets}}$ is the morphism

$$\alpha_{X,Y,Z}^{\text{Sets,-1}} : X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \stackrel{\text{def}}{=} ((x,y),z)$$

for each $(x, (y, z)) \in X \times (Y \times Z)$. Indeed:

• *Invertibility I.* We have

$$\begin{split} [\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}}]((x,y),z) &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z)) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} ((x,y),z) \\ &\stackrel{\mathsf{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}]((x,y),z) \end{split}$$

for each $((x, y), z) \in (X \times Y) \times Z$, and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{(X \times Y) \times Z}.$$

• *Invertibility II.* We have

$$\begin{split} [\alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1}](x,(y,z)) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}}(\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z))) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \\ &\stackrel{\text{def}}{=} (x,(y,z)) \\ &\stackrel{\text{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}](x,(y,z)) \end{split}$$

for each $(x, (y, z)) \in X \times (Y \times Z)$, and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{X \times (Y \times Z)}$$
.

Therefore $\alpha_{X,Y,Z}^{Sets}$ is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to X',$$

 $g: Y \to Y',$
 $h: Z \to Z'$

the diagram

commutes. Indeed, this diagram acts on elements as

$$((x, y), z) \qquad ((x, y), z) \longmapsto ((f(x), g(y)), h(z))$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

and hence indeed commutes, showing α^{Sets} to be a natural transformation. Being a Natural Isomorphism: Since α^{Sets} is natural and $\alpha^{Sets,-1}$ is a component-wise inverse to α^{Sets} , it follows from Categories, Item 2 of Definition II.9.7.1.2 that $\alpha^{Sets,-1}$ is also natural. Thus α^{Sets} is a natural isomorphism.

5.1.5 The Left Unitor

Definition 5.1.5.1.1. The **left unitor of the product of sets** is the natural isomorphism

$$pt \times Sets \xrightarrow{1^{Sets} \times id} Sets \times Sets$$

$$\lambda^{Sets} : \times \circ (1^{Sets} \times id_{Sets}) \xrightarrow{\tilde{\lambda}} \lambda^{Cats_2} \times \lambda^{Cats_2}_{Sets}$$

$$\lambda^{Cats_2} \times \delta^{Cats_2}$$

$$\lambda^{Cats_2} \times \delta^{Cats_2} \times \delta^{Cats_2}$$

whose component

$$\lambda_X^{\mathsf{Sets}} \colon \mathsf{pt} \times X \xrightarrow{\sim} X$$

at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\text{def}}{=} x$$

for each $(\star, x) \in pt \times X$.

Proof. Invertibility: The inverse of $\lambda_X^{\mathsf{Sets}}$ is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathsf{pt} \times X$$

defined by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star, x)$$

for each $x \in X$. Indeed:

• Invertibility I. We have

$$\begin{split} [\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}](\mathsf{pt},x) &= \lambda_X^{\mathsf{Sets},-1}(\lambda_X^{\mathsf{Sets}}(\mathsf{pt},x)) \\ &= \lambda_X^{\mathsf{Sets},-1}(x) \\ &= (\mathsf{pt},x) \\ &= [\mathrm{id}_{\mathsf{pt} \times X}](\mathsf{pt},x) \end{split}$$

for each $(pt, x) \in pt \times X$, and therefore we have

$$\lambda_X^{\text{Sets},-1} \circ \lambda_X^{\text{Sets}} = \mathrm{id}_{\mathrm{pt} \times X}.$$

• Invertibility II. We have

$$\begin{split} [\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}](x) &= \lambda_X^{\mathsf{Sets}}(\lambda_X^{\mathsf{Sets},-1}(x)) \\ &= \lambda_X^{\mathsf{Sets},-1}(\mathsf{pt},x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each $x \in X$, and therefore we have

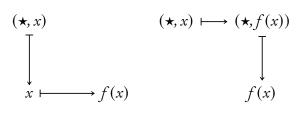
$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X$$
.

Therefore $\lambda_X^{\mathsf{Sets}}$ is indeed an isomorphism.

Naturality: We need to show that, given a function $f: X \to Y$, the diagram

$$\begin{array}{c|c} \operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\ \lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

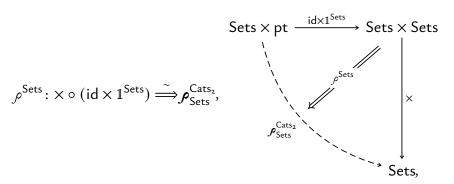
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore λ^{Sets} is a natural transformation. Being a Natural Isomorphism: Since λ^{Sets} is natural and $\lambda^{\mathsf{Sets},-1}$ is a componentwise inverse to λ^{Sets} , it follows from Categories, Item 2 of Definition II.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus λ^{Sets} is a natural isomorphism.

5.1.6 The Right Unitor

Definition 5.1.6.1.1. The **right unitor of the product of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \mathsf{pt} \stackrel{\sim}{\dashrightarrow} X$$

at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\rho_X^{\mathsf{Sets}}(x, \star) \stackrel{\mathrm{def}}{=} x$$

for each $(x, \star) \in X \times pt$.

Proof. Invertibility: The inverse of $\rho_X^{\sf Sets}$ is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\mathsf{def}}{=} (x, \star)$$

for each $x \in X$. Indeed:

• Invertibility I. We have

$$[\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}}](x, \star) = \rho_X^{\mathsf{Sets},-1}(\rho_X^{\mathsf{Sets}}(x, \star))$$

$$= \rho_X^{\mathsf{Sets},-1}(x)$$

$$= (x, \star)$$

$$= [\mathrm{id}_{X \times \mathrm{pt}}](x, \star)$$

for each $(x, \star) \in X \times pt$, and therefore we have

$$\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} = \mathrm{id}_{X \times \mathrm{pt}}.$$

• Invertibility II. We have

$$\begin{aligned} \left[\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} \right] (x) &= \rho_X^{\mathsf{Sets}} (\rho_X^{\mathsf{Sets},-1} (x)) \\ &= \rho_X^{\mathsf{Sets},-1} (x, \bigstar) \\ &= x \\ &= [\mathrm{id}_X] (x) \end{aligned}$$

for each $x \in X$, and therefore we have

$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X.$$

Therefore ρ_X^{Sets} is indeed an isomorphism.

Naturality: We need to show that, given a function $f: X \to Y$, the diagram

$$\begin{array}{ccc} X \times \operatorname{pt} & \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} & Y \times \operatorname{pt} \\ & & & \downarrow^{\rho_{X}^{\operatorname{Sets}}} & & & \downarrow^{\rho_{Y}^{\operatorname{Sets}}} \\ & X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x, \star) \qquad (x, \star) \longmapsto (f(x), \star)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$x \longmapsto f(x) \qquad \qquad f(x)$$

and hence indeed commutes. Therefore ρ^{Sets} is a natural transformation. Being a Natural Isomorphism: Since ρ^{Sets} is natural and $\rho^{\text{Sets},-1}$ is a componentwise inverse to ρ^{Sets} , it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\rho^{\text{Sets},-1}$ is also natural. Thus ρ^{Sets} is a natural isomorphism.

5.1.7 The Symmetry

Definition 5.1.7.1.1. The **symmetry of the product of sets** is the natural isomorphism

whose component

$$\sigma_{X,Y}^{\mathsf{Sets}} \colon X \times Y \xrightarrow{\sim} Y \times X$$

at $X, Y \in \text{Obj}(\mathsf{Sets})$ is defined by

$$\sigma_{X,Y}^{\text{Sets}}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each $(x, y) \in X \times Y$.

Proof. Invertibility: The inverse of $\sigma_{X,Y}^{\mathsf{Sets}}$ is the morphism

$$\sigma_{YY}^{\mathsf{Sets},-1} \colon Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets},-1}(y,x) \stackrel{\text{def}}{=} (x,y)$$

for each $(y, x) \in Y \times X$. Indeed:

• Invertibility I. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}}](x,y) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(\sigma_{X,Y}^{\mathsf{Sets}}(x,y)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(y,x) \\ &\stackrel{\text{def}}{=} (x,y) \\ &\stackrel{\text{def}}{=} [\mathrm{id}_{X\times Y}](x,y) \end{split}$$

for each $(x, y) \in X \times Y$, and therefore we have

$$\sigma_{X,Y}^{\text{Sets},-1} \circ \sigma_{X,Y}^{\text{Sets}} = \mathrm{id}_{X \times Y}$$
.

• Invertibility II. We have

$$\begin{split} \big[\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1}\big](y,x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(\sigma_{X,Y}^{\mathsf{Sets}}(y,x)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(x,y) \\ &\stackrel{\text{def}}{=} (y,x) \\ &\stackrel{\text{def}}{=} [\mathrm{id}_{Y \times X}](y,x) \end{split}$$

for each $(y, x) \in Y \times X$, and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \mathrm{id}_{Y \times X}$$
.

Therefore $\sigma_{X,Y}^{\text{Sets}}$ is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to A$$
, $g: Y \to B$

the diagram

$$\begin{array}{c} X \times Y \xrightarrow{f \times g} A \times B \\ \\ \sigma_{X,Y}^{\mathsf{Sets}} \Big| & & \Big| \sigma_{A,B}^{\mathsf{Sets}} \\ Y \times X \xrightarrow{\sigma \times f} B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes, showing σ^{Sets} to be a natural transformation. Being a Natural Isomorphism: Since σ^{Sets} is natural and $\sigma^{Sets,-1}$ is a component-wise inverse to σ^{Sets} , it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\sigma^{Sets,-1}$ is also natural. Thus σ^{Sets} is a natural isomorphism.

5.1.8 The Diagonal

Definition 5.1.8.1.1. The **diagonal of the product of sets** is the natural transformation



whose component

$$\Delta_X : X \to X \times X$$

at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each $x \in X$.

Proof. We need to show that, given a function $f: X \to Y$, the diagram

$$X \xrightarrow{f} Y$$

$$\Delta_X \downarrow \qquad \qquad \downarrow \Delta_Y$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
x & & x & & \\
\downarrow & & & \downarrow \\
(x,x) & \longmapsto (f(x),f(x)) & & (f(x),f(x))
\end{array}$$

and hence indeed commutes, showing Δ to be natural.

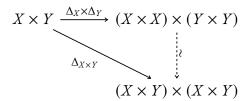
Proposition 5.1.8.1.2. Let *X* be a set.

1. Monoidality. The diagonal map

$$\Delta : id_{Sets} \Longrightarrow \times \circ \Delta^{Cats_2}_{Sets}$$

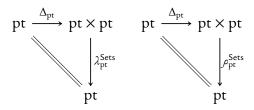
is a monoidal natural transformation:

(a) Compatibility With Strong Monoidality Constraints. For each $X, Y \in Obj(Sets)$, the diagram



commutes.

(b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets,-1}} \\ &= \rho_{\text{pt}}^{\text{Sets,-1}}, \end{split}$$

where we recall that the equalities

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets}} = \rho_{\mathrm{pt}}^{\mathrm{Sets}},$$

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets,-1}} = \rho_{\mathrm{pt}}^{\mathrm{Sets,-1}}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

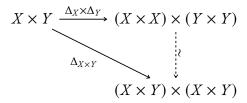
2. The Diagonal of the Unit. The component

$$\Delta_{pt} \colon pt \xrightarrow{\sim} pt \times pt$$

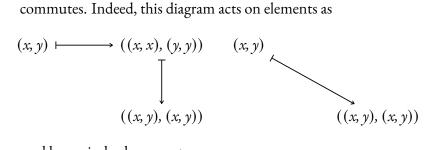
of Δ at pt is an isomorphism.

Proof. Item 1, *Monoidality*: We claim that Δ is indeed monoidal:

I. Item Ia: Compatibility With Strong Monoidality Constraints: We need to show that the diagram



commutes. Indeed, this diagram acts on elements as



and hence indeed commutes.

2. Item 1b: Compatibility With Strong Unitality Constraints: As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathsf{def}}{=} (\star, x)$$

for each $x \in X$, so when X = pt, we have

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets},-1}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

and also

$$\Delta_{\mathrm{pt}}^{\mathrm{Sets}}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

so we have $\Delta_{pt} = \lambda_{pt}^{Sets,-1}$.

This finishes the proof.

Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to X, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor.

5.1.9 The Monoidal Category of Sets and Products

Proposition 5.1.9.1.1. The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category*. The category Sets of pointed sets.
- The Monoidal Product. The product functor

$$\times$$
: Sets \times Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Definition 4.3.5.1.2.

• The Monoidal Unit. The functor

$$1^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• *The Associators.* The natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}}_{\text{Sets},\text{Sets},\text{Sets}}$$
of Definition 5.I.4.I.I.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of Definition 5.1.6.1.1.

• *The Symmetry*. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

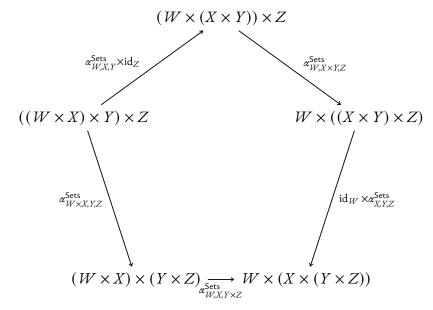
of Definition 5.1.7.1.1.

• The Diagonals. The monoidal natural transformation

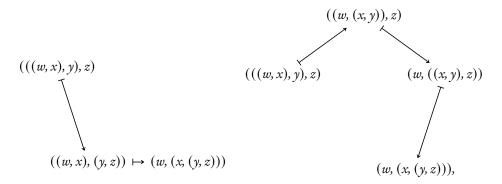
$$\Delta \colon \operatorname{id}_{\mathsf{Sets}} \Longrightarrow \mathsf{X} \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

Proof. The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

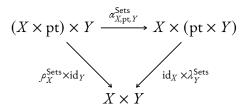


commutes. Indeed, this diagram acts on elements as

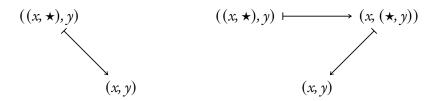


and thus the pentagon identity is satisfied.

The Triangle Identity: Let *X* and *Y* be sets. We have to show that the diagram

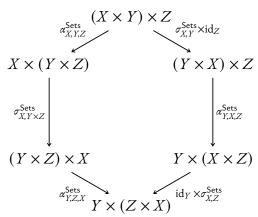


commutes. Indeed, this diagram acts on elements as

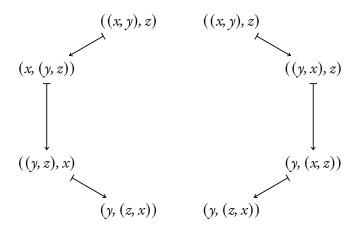


and thus the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

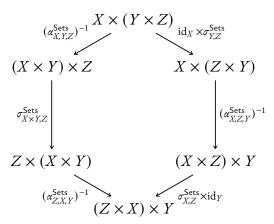


commutes. Indeed, this diagram acts on elements as

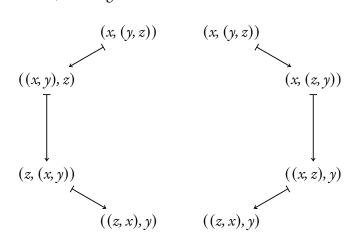


and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the right hexagon identity is satisfied.

Monoidal Closedness: This follows from Constructions With Sets, Item 2 of Definition 4.3,5,1.2

Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 5.1.8.1.2.

5.1.10 The Universal Property of (Sets, \times , pt)

Theorem 5.1.10.1.1. The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

I. Existence of an Internal Hom. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom $[-1, -2]_{Sets}$.

2. The Unit Object Is pt. We have $\mathbb{1}_{Sets} \cong pt$.

More precisely, the full subcategory of the category $\mathcal{M}^{cld}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$ of $\ref{eq:spanned}$ by the closed symmetric monoidal categories (Sets , \otimes_{Sets} , $[-1, -2]_{\mathsf{Sets}}$, $\mathbb{I}_{\mathsf{Sets}}$, λ^{Sets} , ρ^{Sets}) satisfying $\ref{eq:spanned}$ and $\mathbf{2}$ is contractible (i.e. equivalent to the punctual category).

Proof. Unwinding the Statement: Let (Sets, \otimes_{Sets} , $[-1, -2]_{\mathsf{Sets}}$, $\mathbb{1}_{\mathsf{Sets}}$, λ' , ρ' , σ')

be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$id_{Sets} : Sets \rightarrow Sets$$

admits a unique closed symmetric monoidal functor structure

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets, \otimes_{Sets} , $[-_1, -_2]_{Sets}$, $\mathbb{1}_{Sets}$, λ', ρ', σ') to the closed symmetric monoidal category (Sets, \times , Sets $(-_1, -_2)$, $\mathbb{1}_{Sets}$, λ^{Sets} , ρ^{Sets} , σ^{Sets}) of Definition 5.1.9.1.1. Constructing an Isomorphism $[-_1, -_2]_{Sets} \cong Sets(-_1, -_2)$: By ??, we have a natural isomorphism

Sets(pt,
$$[-1, -2]_{Sets}$$
) \cong Sets($-1, -2$).

By Constructions With Sets, Item 3 of Definition 4.3.5.1.2, we also have a natural isomorphism

Sets
$$(pt, [-1, -2]_{Sets}) \cong [-1, -2]_{Sets}$$
.

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}(-_1, -_2) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Given $A, B \in \text{Obj}(\mathsf{Sets})$, we will write

$$id_{A,B}^{Hom} : Sets(A, B) \xrightarrow{\sim} [A, B]_{Sets}$$

for the component of this isomorphism at (A, B).

Constructing an Isomorphism $\otimes_{Sets} \cong \times$: Since \otimes_{Sets} is adjoint in each variable to $[-1, -2]_{Sets}$ by assumption and \times is adjoint in each variable to Sets(-1, -2) by Constructions With Sets, Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -,$$

 $- \otimes_{\mathsf{Sets}} B \cong B \times -.$

By ??, we then have $\otimes_{\mathsf{Sets}} \cong \times$. We will write

$$\operatorname{id}_{\operatorname{\mathsf{Sets}}|A,B}^{\otimes} \colon A \otimes_{\operatorname{\mathsf{Sets}}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Alternatively, we may construct a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ as follows:

- I. Let $A \in \text{Obj}(\mathsf{Sets})$.
- 2. Since \otimes_{Sets} is part of a closed monoidal structure, it preserves colimits in each variable by $\ref{eq:sets}$.
- 3. Since $A \cong \coprod_{a \in A} \text{pt}$ and \bigotimes_{Sets} preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong (\coprod_{a \in A} \mathsf{pt}) \otimes_{\mathsf{Sets}} B$$

$$\cong \coprod_{a \in A} (\mathsf{pt} \otimes_{\mathsf{Sets}} B)$$

$$\cong \coprod_{a \in A} B$$

$$\cong A \times B.$$

naturally in $B \in \text{Obj}(\mathsf{Sets})$, where we have used that pt is the monoidal unit for \otimes_{Sets} . Thus $A \otimes_{\mathsf{Sets}} - \cong A \times -$ for each $A \in \mathsf{Obj}(\mathsf{Sets})$.

- 4. Similarly, $-\otimes_{\mathsf{Sets}} B \cong -\times B$ for each $B \in \mathsf{Obj}(\mathsf{Sets})$.
- 5. By ??, we then have $\otimes_{\mathsf{Sets}} \cong \times$.

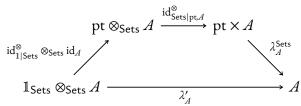
Below, we'll show that if a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \colon A \otimes_{\mathsf{Sets}} B \to A \times B$ from before.

Constructing an Isomorphism $id_1^{\otimes} : \mathbb{1}_{\mathsf{Sets}} \to \mathsf{pt}$: We define an isomorphism $id_1^{\otimes} : \mathbb{1}_{\mathsf{Sets}} \to \mathsf{pt}$ as the composition

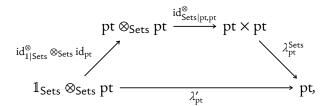
$$\mathbb{1}_{\mathsf{Sets}} \overset{\rho_{\mathsf{1}_{\mathsf{Sets}}^{\mathsf{Sets},-1}}^{\mathsf{Sets},-1}}{\underset{\sim}{\dots}} \mathbb{1}_{\mathsf{Sets}} \times \mathsf{pt} \overset{\mathsf{id}_{\mathsf{Sets}}^{\otimes}|_{\mathsf{Sets}}}{\underset{\sim}{\dots}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathsf{pt} \overset{\lambda'_{\mathsf{pt}}}{\underset{\sim}{\dots}} \mathsf{pt}$$

in Sets.

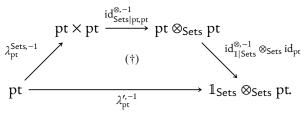
Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

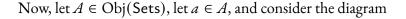


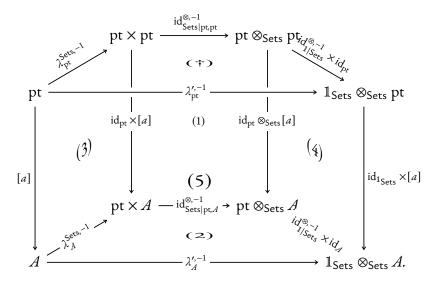
commutes. First, note that the diagram



corresponding to the case A = pt, commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram



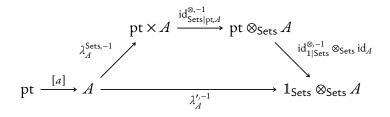




Since:

- Subdiagram (5) commutes by the naturality of λ'^{-1} .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $id_{1|Sets}^{\otimes,-1}$.
- Subdiagram (1) commutes by the naturality of $id_{\mathsf{Sets}}^{\otimes,-1}$.
- Subdiagram (3) commutes by the naturality of $\lambda^{Sets,-1}$.

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\lambda_A^{\prime,-1}(a) = [\lambda_A^{\prime,-1} \circ [a]](\star)$$

$$\begin{split} &= \big[(\mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1} \circ [a] \big] (\star) \\ &= \big[(\mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1} \big] (a) \end{split}$$

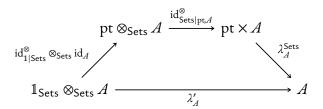
for each $a \in A$, and thus we have

$$\lambda_A^{\prime,-1} = (\mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \times \mathrm{id}_A),$$

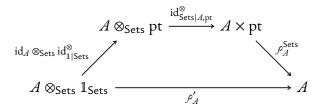
showing that the diagram



indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We can use the same argument we used to prove the monoidal left unity of the isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ above. For completeness, we repeat it below.

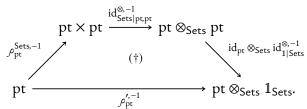
We have to show that the diagram



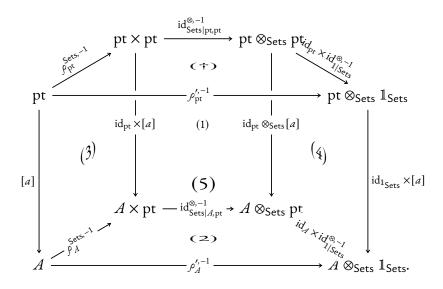
commutes. First, note that the diagram

$$\begin{array}{c} pt \otimes_{\mathsf{Sets}} pt \xrightarrow{id^{\otimes}_{\mathsf{Sets}|pt,pt}} pt \times pt \\ id_{pt} \otimes_{\mathsf{Sets}} id^{\otimes}_{1|\mathsf{Sets}} \end{array} \xrightarrow{p'_{pt}} pt \otimes_{\mathsf{Sets}} 1_{\mathsf{Sets}}$$

corresponding to the case A = pt, commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram



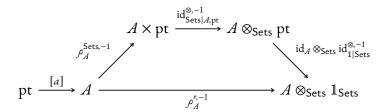
Now, let $A \in \text{Obj}(\mathsf{Sets})$, let $a \in A$, and consider the diagram



Since:

- Subdiagram (5) commutes by the naturality of ρ'^{-1} .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $id_{1|Sets}^{\otimes,-1}$.
- Subdiagram (1) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\text{Sets},-1}$.

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\rho_A^{\prime,-1}(a) = [\rho_A^{\prime,-1} \circ [a]](\star)$$

$$= [(\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1} \circ [a]](\star)$$

$$= [(\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}](a)$$

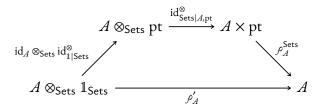
for each $a \in A$, and thus we have

$$\rho_A^{\prime,-1} = (\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

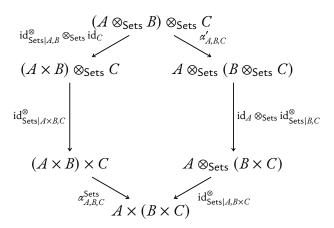
$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes}),$$

showing that the diagram

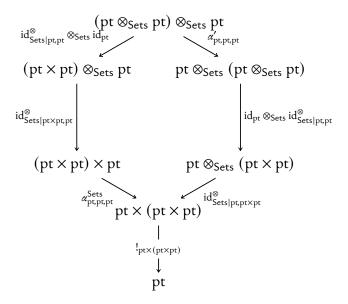


indeed commutes.

Monoidality of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

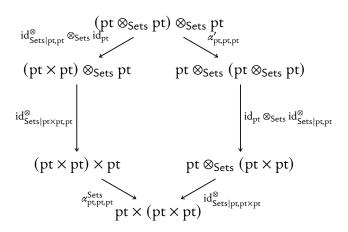


commutes. First, note that the diagram

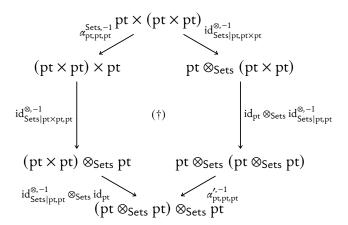


commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map $!_{pt \times (pt \times pt)} : pt \times (pt \times pt) \rightarrow pt$ is an isomorphism (e.g. having

inverse $\lambda_{pt}^{Sets,-1} \circ \lambda_{pt}^{Sets,-1}),$ it follows that the diagram

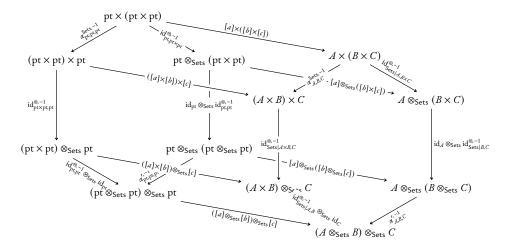


also commutes. Taking inverses, we see that the diagram

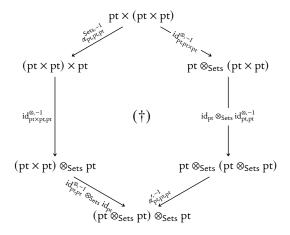


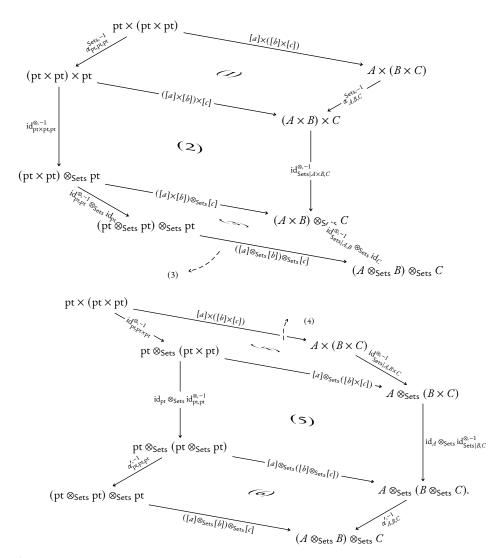
commutes as well. Now, let $A, B, C \in \text{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, let $c \in C$,

and consider the diagram



which we partition into subdiagrams as follows:



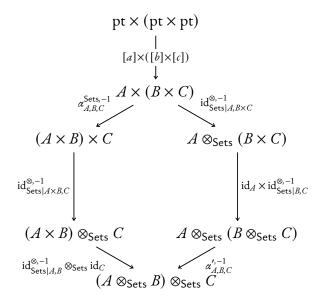


Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{Sets,-1}$.
- Subdiagram (2) commutes by the naturality of $id_{\mathsf{Sets}}^{\otimes,-1}$.
- Subdiagram (3) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (†) commutes, as proved above.

- Subdiagram (4) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (5) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (6) commutes by the naturality of α'^{-1} .

it follows that the diagram



also commutes. We then have

$$\begin{split} \left[(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) &= \left[(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ ([a] \times ([b] \times [c])) \right] (\star,(\star,\star)) \\ &= \left[\alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \\ \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ ([a] \times ([b] \times [c])) \right] (\star,(\star,\star)) \\ &= \left[\alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

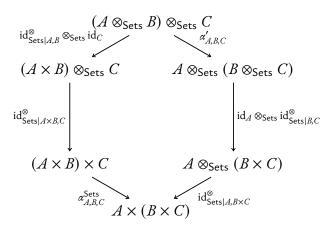
for each $(a, (b, c)) \in A \times (B \times C)$, and thus we have

$$(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1}\otimes_{\mathsf{Sets}}\mathrm{id}_C)\circ\mathrm{id}_{\mathsf{Sets}|A\times B,C}^{\otimes,-1}\circ\alpha_{A,B,C}^{\mathsf{Sets},-1}=\alpha_{A,B,C}'^{,-1}\circ(\mathrm{id}_A\times\mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1})\circ\mathrm{id}_{\mathsf{Sets}|A,B\times C}^{\otimes,-1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ (\mathsf{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathsf{id}_C) = \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ (\mathsf{id}_A \times \mathsf{id}_{\mathsf{Sets}|B,C}^{\otimes}) \circ \alpha_{A,B,C}'$$

showing that the diagram



indeed commutes.

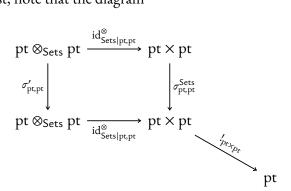
Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\downarrow^{\sigma'_{A,B}} \qquad \qquad \downarrow^{\sigma_{A,B}^{\mathsf{Sets}}}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

commutes. First, note that the diagram



commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map $!_{pt \times pt} : pt \times pt \rightarrow pt$ is invertible (e.g. with inverse $\lambda_{pt}^{Sets,-1}$), the

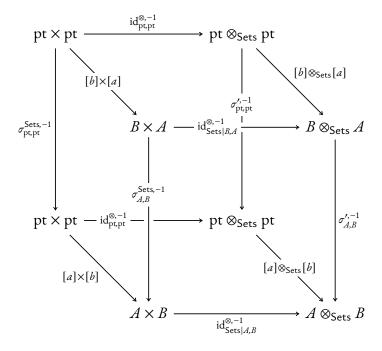
diagram

$$\begin{array}{c|c} pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}}^{\otimes} pt \times pt \\ & & & \downarrow \sigma_{\mathsf{pt,pt}}^{\mathsf{Sets}} \\ pt \otimes_{\mathsf{Sets}} pt & & & \downarrow t \\ pt \otimes_{\mathsf{Sets}} pt & & & \downarrow \tau \end{array}$$

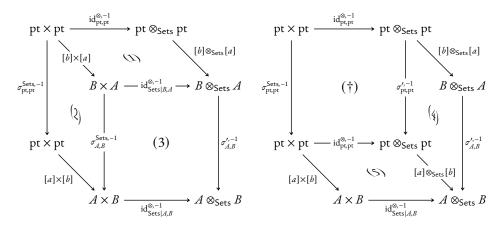
also commutes. Taking inverses, we see that the diagram

$$\begin{array}{c|c} pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} pt \otimes_{\mathsf{Sets}} pt \\ \hline \\ \sigma_{\mathsf{pt,pt}}^{\mathsf{Sets,-1}} & & (\dagger) & & & \\ pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} pt \otimes_{\mathsf{Sets}} pt \end{array}$$

commutes as well. Now, let $A, B \in \text{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, and consider the diagram



which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\text{Sets},-1}$.
- Subdiagram (5) commutes by the naturality of $id^{\otimes,-1}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of σ' ,-1.
- Subdiagram (1) commutes by the naturality of $id^{\otimes,-1}$.

it follows that the diagram

$$B \times A \xrightarrow{id_{\mathsf{Sets}|B,A}^{\otimes}} B \otimes_{\mathsf{Sets}} A$$

$$\sigma_{A,B}^{\mathsf{Sets}} \downarrow \qquad \qquad \qquad \downarrow \sigma_{A,B}'$$

$$A \times B \xrightarrow{id_{\mathsf{Sets}|A,B}^{\otimes}} A \otimes_{\mathsf{Sets}} B$$
then have

commutes. We then have

$$[\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1}](b,a) = [\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a])](\star,\star)$$

$$\begin{split} &= \big[\sigma_{A,B}^{\prime,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a])\big](\star,\star) \\ &= \big[\sigma_{A,B}^{\prime,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}\big](b,a) \end{split}$$

for each $(b, a) \in B \times A$, and thus we have

$$\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}'^{,-1} \circ \operatorname{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}'$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\downarrow^{\sigma'_{A,B}} \qquad \qquad \downarrow^{\sigma_{A,B}^{\mathsf{Sets}}}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Let ϕ , ψ : $-_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to \times and \otimes (as shown above), we have

$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ (\mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y),$$

$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ (\mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y).$$

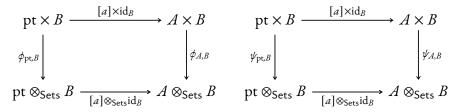
Postcomposing both sides with $\lambda_R^{Sets,-1}$ gives

$$\begin{split} & \lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ (\mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) = \phi_{\mathsf{pt},\mathcal{B}}, \\ & \lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ (\mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) = \psi_{\mathsf{pt},\mathcal{B}}, \end{split}$$

and thus we have

$$\phi_{\text{pt},B} = \psi_{\text{pt},B}$$

for each $B \in \text{Obj}(\mathsf{Sets})$. Now, let $a \in A$ and consider the naturality diagrams



for ϕ and ψ with respect to the morphisms [a] and id_B. Having shown that $\phi_{\text{pt},B} = \psi_{\text{pt},B}$, we have

$$\phi_{A,B}(a,b) = [\phi_{A,B} \circ ([a] \times id_B)](\star, b)$$

$$= [([a] \otimes_{\mathsf{Sets}} id_B) \circ \phi_{\mathsf{pt},B}](\star, b)$$

$$= [([a] \otimes_{\mathsf{Sets}} id_B) \circ \psi_{\mathsf{pt},B}](\star, b)$$

$$= [\psi_{A,B} \circ ([a] \times id_B)](\star, b)$$

$$= \psi_{A,B}(a, b)$$

for each $(a, b) \in A \times B$. Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each $A, B \in \text{Obj}(\mathsf{Sets})$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathsf{Sets}} \cong \mathsf{X}$ to be unique. \square

Corollary 5.1.10.1.2. The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

I. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sets}} : \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

2. The Unit Object Is pt. We have $\mathbb{1}_{Sets} \cong pt$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$ of $\ref{eq:sets}$ spanned by the symmetric monoidal categories (Sets, \otimes_{Sets} , 1_{Sets} , 1_{\mathsf

Proof. Since Sets is locally presentable (??), it follows from ?? that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Definition 5.1.10.1.1. The result then follows from Definition 5.1.10.1.1.

5.2 The Monoidal Category of Sets and Coproducts

5.2.1 Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

5.2.2 The Monoidal Unit

Definition 5.2.2.1.1. The monoidal unit of the coproduct of sets is the functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$0_{\mathsf{Sets}} \stackrel{\mathrm{def}}{=} \emptyset$$

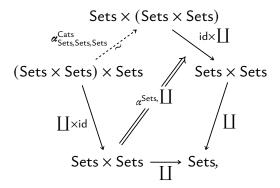
where \emptyset is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

5.2.3 The Associator

Definition 5.2.3.1.1. The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \pmb{\alpha}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}}$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} : (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}(a) \stackrel{\mathsf{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each $a \in (X \coprod Y) \coprod Z$.

Proof. Unwinding the Definitions of $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$: Firstly, we unwind the expressions for $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$. We have

$$(X \coprod Y) \coprod Z \stackrel{\text{def}}{=} \{ (0, a) \in S \mid a \in X \coprod Y \} \cup \{ (1, z) \in S \mid z \in Z \}$$
$$= \{ (0, (0, x)) \in S \mid x \in X \} \cup \{ (0, (1, y)) \in S \mid y \in Y \}$$
$$\cup \{ (1, z) \in S \mid z \in Z \},$$

where $S = \{0, 1\} \times ((X \coprod Y) \cup Z)$ and

$$X \coprod (Y \coprod Z) \stackrel{\text{def}}{=} \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, a) \in S' \mid a \in Y \coprod Z \}$$
$$= \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, (0, y)) \in S' \mid y \in Y \}$$
$$\cup \{ (1, (1, z)) \in S' \mid z \in Z \},$$

where $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$.

Invertibility: The inverse of $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$ is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\text{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each $a \in X \coprod Y(\coprod Z)$. Indeed:

• *Invertibility I.* The map $\alpha_{X,Y,Z}^{\mathsf{Sets}, \coprod, -1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}, \coprod}$ acts on elements as

$$(0, (0, x)) \mapsto (0, x) \mapsto (0, (0, x)),$$

$$(0, (0, y)) \mapsto (1, (0, y)) \mapsto (0, (0, y)),$$

$$(1, z) \mapsto (1, (1, z)) \mapsto (1, z)$$

and hence is equal to the identity map of $(X \coprod Y) \coprod Z$.

• *Invertibility II.* The map $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$ acts on elements as

$$(0,x) \mapsto (0,(0,x)) \mapsto (0,x),$$

$$(1,(0,y)) \mapsto (0,(0,y)) \mapsto (1,(0,y)),$$

$$(1,(1,z)) \mapsto (1,z) \mapsto (1,(1,z))$$

and hence is equal to the identity map of $X \coprod (Y \coprod Z)$.

Therefore $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$ is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to X',$$

 $g: Y \to Y',$
 $h: Z \to Z'$

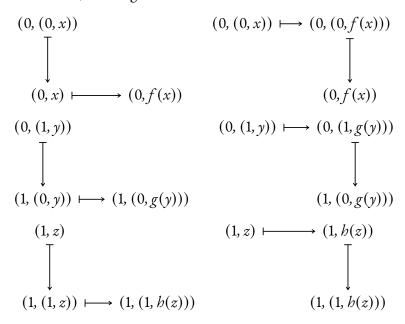
the diagram

$$(X \coprod Y) \coprod Z \xrightarrow{(f \coprod g) \coprod b} (X' \coprod Y') \coprod Z'$$

$$\downarrow^{\text{Sets,} \coprod}_{\alpha_{X',Y',Z'}}$$

$$X \coprod (Y \coprod Z) \xrightarrow{f \coprod (g \coprod b)} X' \coprod (Y' \coprod Z')$$

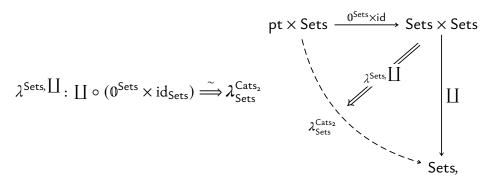
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing $\alpha^{\mathsf{Sets}, \coprod}$ to be a natural transformation. Being a Natural Isomorphism: Since $\alpha^{\mathsf{Sets}, \coprod}$ is natural and $\alpha^{\mathsf{Sets}, \coprod, -1}$ is a componentwise inverse to $\alpha^{\mathsf{Sets}, \coprod}$, it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\lambda^{\mathsf{Sets}, -1}$ is also natural. Thus $\alpha^{\mathsf{Sets}, \coprod}$ is a natural isomorphism.

5.2.4 The Left Unitor

Definition 5.2.4.1.1. The **left unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} : \varnothing \coprod X \xrightarrow{\sim} X$$

at *X* is given by

$$\lambda_X^{\mathsf{Sets},\coprod}((1,x))\stackrel{\mathrm{def}}{=} x$$

for each $(1, x) \in \emptyset \mid X$.

Proof. Unwinding the Definition of $\emptyset \coprod X$: Firstly, we unwind the expressions for $\emptyset \coprod X$. We have

$$\emptyset \coprod X \stackrel{\text{def}}{=} \{ (0, z) \in S \mid z \in \emptyset \} \cup \{ (1, x) \in S \mid x \in X \}$$
$$= \emptyset \cup \{ (1, x) \in S \mid x \in X \}$$
$$= \{ (1, x) \in S \mid x \in X \},$$

where $S = \{0, 1\} \times (\emptyset \cup X)$.

Invertibility: The inverse of $\lambda_X^{\mathsf{Sets},\coprod}$ is the map

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \colon X \to \varnothing \coprod X$$

given by

$$\lambda_X^{\text{Sets}, \coprod, -1}(x) \stackrel{\text{def}}{=} (1, x)$$

for each $x \in X$. Indeed:

• *Invertibility I.* We have

$$\begin{split} [\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod}](1, x) &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (\lambda_X^{\mathsf{Sets}, \coprod} (1, x)) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (1, x) \\ &= [\mathrm{id}_{\varnothing \coprod X}](1, x) \end{split}$$

for each $(1, x) \in \emptyset \mid X$, and therefore we have

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$\begin{split} \big[\lambda_X^{\mathsf{Sets}, \coprod} \circ \lambda_X^{\mathsf{Sets}, \coprod, -1}\big](x) &= \lambda_X^{\mathsf{Sets}, \coprod} (\lambda_X^{\mathsf{Sets}, \coprod, -1}(x)) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1}(1, x) \\ &= x \\ &= \big[\mathrm{id}_X \big](x) \end{split}$$

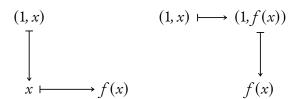
for each $x \in X$, and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore $\lambda_X^{\text{Sets},\coprod}$ is indeed an isomorphism. *Naturality*: We need to show that, given a function $f:X\to Y$, the diagram

$$\begin{array}{c|c}
\varnothing \coprod X & \xrightarrow{\operatorname{id}_{\varnothing} \coprod f} \varnothing \coprod Y \\
\downarrow_{\lambda_{X}^{\operatorname{Sets}}, \coprod} & & \downarrow_{\lambda_{Y}^{\operatorname{Sets}}, \coprod} \\
X & \xrightarrow{f} & Y
\end{array}$$

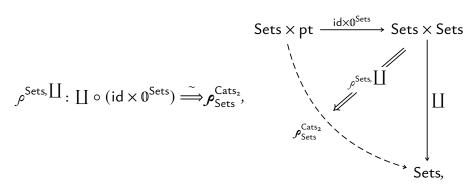
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore $\lambda^{\mathsf{Sets}, \coprod}$ is a natural transformation. Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets}, \coprod}$ is natural and $\lambda^{\mathsf{Sets}, -1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}, \coprod}$, it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\lambda^{\mathsf{Sets}, -1}$ is also natural. Thus $\lambda^{\mathsf{Sets}, \coprod}$ is a natural isomorphism.

5.2.5 The Right Unitor

Definition 5.2.5.1.1. The **right unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}, \coprod} \colon X \coprod \varnothing \stackrel{\sim}{\dashrightarrow} X$$

at X is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x)) \stackrel{\mathrm{def}}{=} x$$

for each $(0, x) \in X \coprod \emptyset$.

Proof. Unwinding the Definition of $X \coprod \emptyset$: Firstly, we unwind the expression for $X \coprod \emptyset$. We have

$$X \coprod \emptyset \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, z) \in S \mid z \in \emptyset\}$$
$$= \{(0, x) \in S \mid x \in X\} \cup \emptyset$$
$$= \{(0, x) \in S \mid x \in X\},$$

where $S = \{0,1\} \times (X \cup \emptyset) = \{0,1\} \times (\emptyset \cup X) = S$. *Invertibility*: The inverse of $\rho_X^{\mathsf{Sets},\coprod}$ is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \emptyset$$

given by

$$\rho_X^{\mathsf{Sets}, \coprod, -1}(x) \stackrel{\mathrm{def}}{=} (0, x)$$

for each $x \in X$. Indeed:

• *Invertibility I.* We have

$$\begin{split} \big[\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} \big] (0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} (\rho_X^{\mathsf{Sets}, \coprod} (0, x)) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (0, x) \\ &= [\mathrm{id}_{X \coprod \emptyset}] (0, x) \end{split}$$

for each $(0, x) \in \emptyset \coprod X$, and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\varnothing \coprod X}.$$

• Invertibility II. We have

$$\begin{split} \left[\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1}\right](x) &= \rho_X^{\mathsf{Sets}, \coprod} \left(\rho_X^{\mathsf{Sets}, \coprod, -1}(x)\right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each $x \in X$, and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1} = \mathrm{id}_X.$$

Therefore $\rho_X^{\text{Sets},\coprod}$ is indeed an isomorphism.

Naturality: We need to show that, given a function $f: X \to Y$, the diagram

$$\begin{array}{c|c} X \coprod \varnothing \xrightarrow{f \coprod \mathrm{id}_{\varnothing}} Y \coprod \varnothing \\ & \downarrow^{\mathsf{Sets}, \coprod} & \downarrow^{\mathsf{Sets}, \coprod} \\ X \xrightarrow{f} Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(0,x) & (0,x) & \longmapsto (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore $\rho^{\text{Sets},\coprod}$ is a natural transformation. Being a Natural Isomorphism: Since $\rho^{\text{Sets},\coprod}$ is natural and $\rho^{\text{Sets},-1}$ is a componentwise inverse to $\rho^{\text{Sets},\coprod}$, it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\rho^{\text{Sets},-1}$ is also natural. Thus $\rho^{\text{Sets},\coprod}$ is a natural isomorphism.

5.2.6 The Symmetry

Definition 5.2.6.1.1. The **symmetry of the coproduct of sets** is the natural isomorphism



whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}: X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at $X, Y \in \text{Obj}(\mathsf{Sets})$ is defined by

$$\sigma_{X,Y}^{\text{Sets},\coprod}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each $(x, y) \in X \times Y$.

Proof. Unwinding the Definitions of $X \coprod Y$ *and* $Y \coprod X$: Firstly, we unwind the expressions for $X \coprod Y$ and $Y \coprod X$. We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},\$$

where $S = \{0, 1\} \times (X \cup Y)$ and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},\$$

where $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$. *Invertibility*: The inverse of $\sigma_{X,Y}^{\mathsf{Sets}, \coprod}$ is the map

$$\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\scriptscriptstyle\mathsf{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\text{Sets},\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = (1,x), \\ (1,y) & \text{if } z = (0,y) \end{cases}$$

for each $z \in Y \coprod X$. Indeed:

• Invertibility I. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}](0, x) &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (\sigma_{X}^{\mathsf{Sets}, \coprod}(0, x)) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (1, x) \\ &= (0, x) \\ &= [\mathrm{id}_{X \coprod Y}](0, x) \end{split}$$

for each $(0, x) \in X \coprod Y$ and

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}](1, y) &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (\sigma_{X}^{\mathsf{Sets}, \coprod}(1, y)) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1}(0, y) \\ &= (1, y) \\ &= [\mathrm{id}_{X \coprod Y}](1, y) \end{split}$$

for each $(1, y) \in X \coprod Y$, and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod} = \mathrm{id}_{X\coprod Y}.$$

• Invertibility II. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}](0, y) &= \sigma_{X}^{\mathsf{Sets}, \coprod} (\sigma_{X}^{\mathsf{Sets}, \coprod, -1}(0, y)) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1}(1, y) \\ &= (0, y) \\ &= [\mathrm{id}_{Y \coprod X}](0, y) \end{split}$$

for each $(0, y) \in Y \coprod X$ and

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}](1, x) &= \sigma_{X}^{\mathsf{Sets}, \coprod} (\sigma_{X}^{\mathsf{Sets}, \coprod, -1}(1, x)) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= (1, x) \\ &= [\mathrm{id}_{Y \coprod X}](1, x) \end{split}$$

for each $(1, x) \in Y \coprod X$, and therefore we have

$$\sigma_X^{\mathsf{Sets}, \coprod} \circ \sigma_X^{\mathsf{Sets}, \coprod, -1} = \mathrm{id}_{Y \coprod X}.$$

Therefore $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$ is indeed an isomorphism. *Naturality*: We need to show that, given functions $f:A\to X$ and $g:B\to Y$, the diagram

$$A \coprod B \xrightarrow{f \coprod g} X \coprod Y$$

$$\downarrow_{\sigma_{A,B}^{\mathsf{Sets}}, \coprod} \qquad \qquad \downarrow_{\sigma_{X,Y}^{\mathsf{Sets}}, \coprod}$$

$$B \coprod A \xrightarrow{g \coprod f} Y \coprod X$$

commutes. Indeed, this diagram acts on elements as

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes. Therefore $\sigma^{\mathsf{Sets}, \coprod}$ is a natural transformation. Being a Natural Isomorphism: Since $\sigma^{\mathsf{Sets}, \coprod}$ is natural and $\sigma^{\mathsf{Sets}, -1}$ is a componentwise inverse to $\sigma^{\mathsf{Sets}, \coprod}$, it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\sigma^{\mathsf{Sets}, -1}$ is also natural. Thus $\sigma^{\mathsf{Sets}, \coprod}$ is a natural isomorphism.

5.2.7 The Monoidal Category of Sets and Coproducts

Proposition 5.2.7.1.1. The category Sets admits a closed symmetric monoidal category structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- The Monoidal Product. The coproduct functor

$$\coprod$$
: Sets \times Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.

• *The Monoidal Unit*. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\text{Sets}, \coprod} : \coprod \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\text{id}_{\text{Sets}} \times \coprod) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$
of Definition 5.2.3.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\text{Sets}, \coprod} : \coprod \circ (\text{id} \times \mathbb{O}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

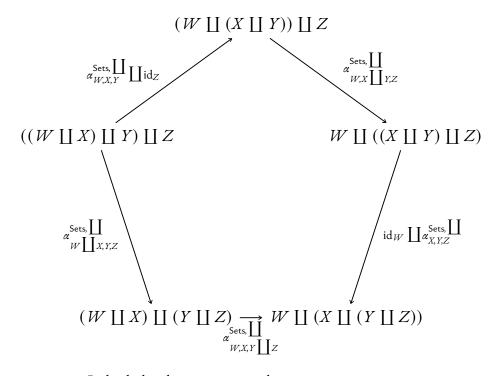
of Definition 5.2.5.1.1.

• *The Symmetry*. The natural isomorphism

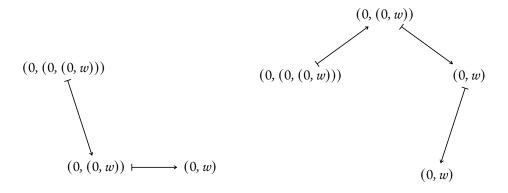
$$\sigma^{\mathsf{Sets},\coprod}: \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

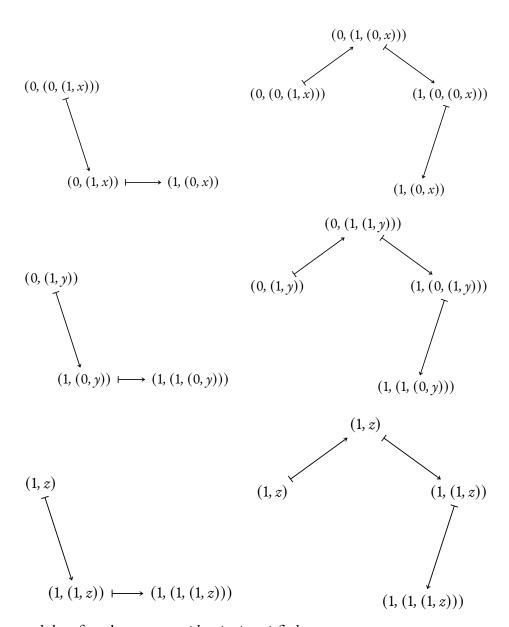
of Definition 5.2.6.1.1.

Proof. The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram



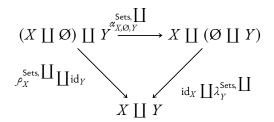
commutes. Indeed, this diagram acts on elements as



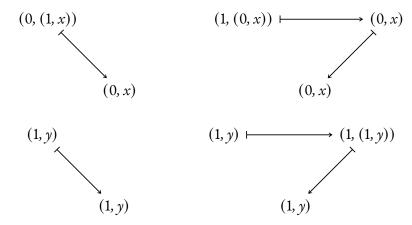


and therefore the pentagon identity is satisfied.

The Triangle Identity: Let *X* and *Y* be sets. We have to show that the diagram

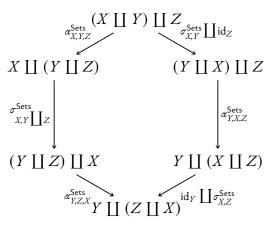


commutes. Indeed, this diagram acts on elements as

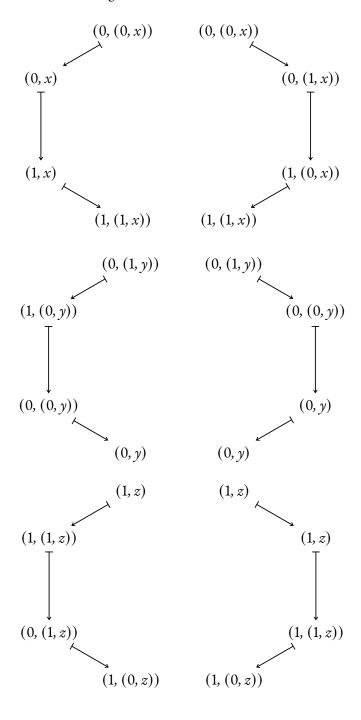


and therefore the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

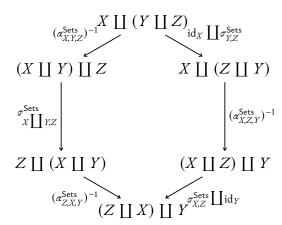


commutes. Indeed, this diagram acts on elements as

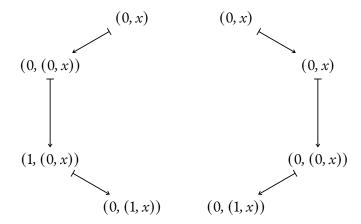


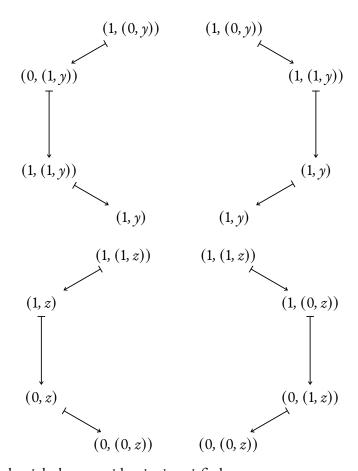
and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as





and thus the right hexagon identity is satisfied.

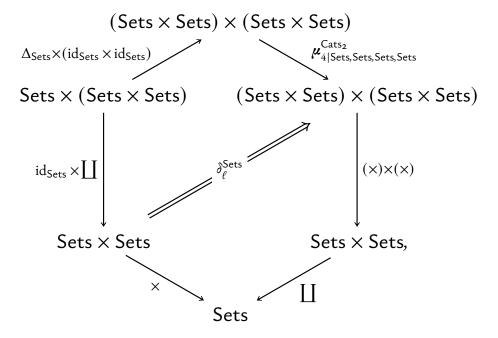
5.3 The Bimonoidal Category of Sets, Products, and Coproducts

5.3.1 The Left Distributor

Definition 5.3.1.1.1. The left distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \overset{\tilde{}}{\Longrightarrow} \coprod \circ ((\mathsf{x}) \times (\mathsf{x})) \circ \pmb{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$$

as in the diagram



whose component

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z} \colon X \times (Y \coprod Z) \xrightarrow{\sim} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets}}(x,a) \stackrel{\text{def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each $(x, a) \in X \times (Y \coprod Z)$.

Proof. Invertibility: The inverse of $\partial^{\mathsf{Sets}}_{\ell|X,Y,Z}$ is the map

$$\delta^{\mathsf{Sets},-1}_{\ell\mid X,Y,Z} \colon (X \times Y) \ {\textstyle\coprod} \ (X \times Z) \stackrel{\sim}{\dashrightarrow} X \times (Y \ {\textstyle\coprod} \ Z)$$

given by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} (x, (0,y)) & \text{if } a = (0, (x,y)), \\ (x, (1,z)) & \text{if } a = (1, (x,z)) \end{cases}$$

for $a \in (X \times Y) \mid | (X \times Z)$. Indeed:

• *Invertibility I.* The map $\delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z} \circ \delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ acts on elements as

$$(x, (0, y)) \mapsto (0, (x, y)) \mapsto (x, (0, y)),$$

 $(x, (1, z)) \mapsto (1, (x, z)) \mapsto (x, (1, z)),$

but these are the two possible cases for elements of $X \times (Y \coprod Z)$. Hence the map is equal to the identity.

• Invertibility II. The map $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z} \circ \delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z}$ acts on elements as

$$(0, (x, y)) \mapsto (x, (0, y)) \mapsto (0, (x, y)),$$

$$(1, (x, z)) \mapsto (x, (1, z)) \mapsto (1, (x, z)),$$

but these are the two possible cases for elements of $(X \times Y) \coprod (X \times Z)$. Hence the map is equal to the identity.

Thus $\delta_{\ell|X,Y,Z}^{\mathsf{Sets}}$ is an isomorphism for all X, Y, Z. *Naturality*: We need to show that, given functions

$$f: X \to X',$$

 $g: Y \to Y',$
 $h: Z \to Z'$

the diagram

$$\begin{array}{c|c} X\times (Y\coprod Z) & \xrightarrow{f\times \left(g\coprod b\right)} & X'\times (Y'\coprod Z') \\ & \downarrow^{\delta \operatorname{Sets}} & & \downarrow^{\delta \operatorname{Sets}} \\ (X\times Y)\coprod (X\times Z) & \xrightarrow{(f\times g)\coprod (f\times b)} & (X'\times Y')\coprod (X'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x, (0, y)) \qquad (x, (0, y)) \longmapsto (f(x), (0, f(y)))$$

$$\downarrow \qquad \qquad \downarrow \qquad$$

$$(x, (1, z)) \qquad (x, (1, z)) \longmapsto (f(x), (1, h(z)))$$

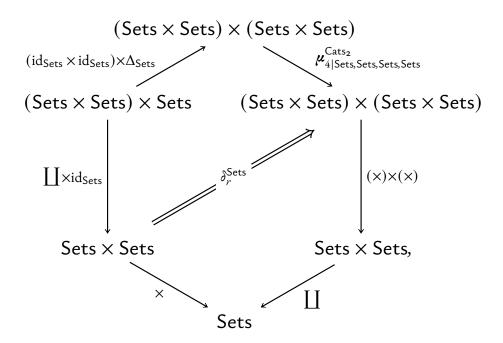
$$\downarrow \qquad \qquad \downarrow \qquad$$

so it commutes, showing $\delta_\ell^{\mathsf{Sets}}$ to be a natural transformation. Being a Natural Isomorphism: Since $\delta_\ell^{\mathsf{Sets}}$ is natural and $\delta_\ell^{\mathsf{Sets},-1}$ is a component-wise inverse to $\delta_\ell^{\mathsf{Sets}}$, it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\delta_\ell^{\mathsf{Sets},-1}$ is also natural. Thus $\delta_\ell^{\mathsf{Sets}}$ is a natural isomorphism.

5.3.2 The Right Distributor

Definition 5.3.2.1.1. The right distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ ((\mathsf{x}) \times (\mathsf{x})) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
 as in the diagram



whose component

$$\delta_{r|X,YZ}^{\mathsf{Sets}} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each $(a, z) \in (X \mid Y) \times Z$.

Proof. Invertibility: The inverse of $\delta_{r|X,Y,Z}^{\mathsf{Sets}}$ is the map

$$\delta_{r|X,YZ}^{\mathsf{Sets},-1} \colon (X \times Z) \coprod (Y \times Z) \xrightarrow{\sim} (X \coprod Y) \times Z$$

given by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} ((0,x),z) & \text{if } a = (0,(x,z)), \\ ((1,y),z) & \text{if } a = (1,(y,z)) \end{cases}$$

for $a \in (X \times Z) \coprod (Y \times Z)$. Indeed:

• *Invertibility I.* The map $\delta_{r|X,Y,Z}^{\mathsf{Sets},-1} \circ \delta_{r|X,Y,Z}^{\mathsf{Sets}}$ acts on elements as

$$((0, x), z) \mapsto (0, (x, z)) \mapsto (0, (x, z)),$$

$$((1, y), z) \mapsto (1, (y, z)) \mapsto (1, (y, z)),$$

but these are the two possible cases for elements of $(X \coprod Y) \times Z$. Hence the map is equal to the identity.

• *Invertibility II.* The map $\partial^{\mathsf{Sets}}_{r|X,Y,Z} \circ \partial^{\mathsf{Sets},-1}_{r|X,Y,Z}$ acts on elements as

$$(0, (x, z)) \mapsto ((0, x), z) \mapsto (0, (x, z)),$$

$$(1, (y, z)) \mapsto ((1, y), z) \mapsto (1, (y, z)),$$

but these are the two possible cases for elements of $(X \times Z) \coprod (Y \times Z)$. Hence the map is equal to the identity.

So $\delta_{r|X,Y,Z}^{\mathsf{Sets}}$ is an isomorphism for all X, Y, Z.

Naturality: We need to show that, given functions

$$f: X \to X',$$

 $g: Y \to Y',$
 $h: Z \to Z'$

the diagram

$$\begin{array}{c|c} (X \coprod Y) \times Z' & \xrightarrow{\qquad (f \coprod g) \times h} & (X' \coprod Y') \times Z' \\ & & \downarrow^{\operatorname{Sets}} & & \downarrow^{\operatorname{Sets}} \\ (X \times Z) \coprod (Y \times Z) & \xrightarrow{\qquad (f \times h) \coprod (g \times h)} & (X' \times Z') \coprod (Y' \times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$((0,x),z) \qquad ((0,x),z) \longmapsto ((0,f(x)),b(z))$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

so it commutes and δ_r^{Sets} is a natural transformation.

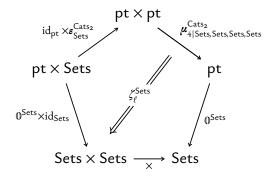
Being a Natural Isomorphism: Since δ_r^{Sets} is natural and $\delta_r^{\mathsf{Sets},-1}$ is a componentwise inverse to δ_r^{Sets} , it follows from Categories, Item 2 of Definition II.9.7.1.2 that $\delta_r^{\mathsf{Sets},-1}$ is also natural. Thus δ_r^{Sets} is a natural isomorphism.

5.3.3 The Left Annihilator

Definition 5.3.3.1.1. The **left annihilator of the product of sets** is the natural isomorphism

$$\mathcal{\zeta}^{\mathsf{Sets}}_{\ell} \colon \mathbb{0}^{\mathsf{Sets}} \circ \mu^{\mathsf{Cats}_2}_{4 \mid \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}} \circ (\mathrm{id}_{\mathsf{pt}} \times \pmb{\varepsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \emptyset \times A \xrightarrow{\sim} \emptyset$$

given by $\zeta_{\ell|A}^{\mathsf{Sets}} \stackrel{\text{def}}{=} \mathsf{pr}_1$.

Proof. Invertibility: The inverse of $\zeta_{\ell|A}^{\mathsf{Sets}}$ is the map

$$\zeta_{\ell|A}^{\mathsf{Sets},-1} \colon \varnothing \xrightarrow{\sim} \varnothing \times A$$

given by

$$\zeta_{\ell|A}^{\mathsf{Sets},-1} \stackrel{\mathrm{def}}{=} \iota_A,$$

where ι_A is as defined in Constructions With Sets, Definition 4.2.1.1.2:

- *Invertibility I.* The map $\zeta_{\ell|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \to \emptyset$ is equal to id_{\emptyset} , as \emptyset is the initial object of Sets .
- *Invertibility II.* The map $\iota_A \circ \zeta_{\ell|A}^{\mathsf{Sets}}$ is equal to the identity on every $(x, a) \in \mathcal{O} \times A$, of which there are none.

Hence $\zeta_{\ell|A}^{\mathsf{Sets}}$ is an isomorphism.

Naturality: We need to show that given a function $f: A \to B$, the diagram

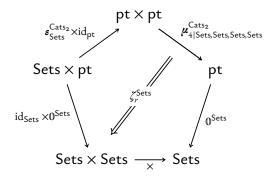
commutes. But since $\varnothing \times A$ has no elements, this is trivially true. Being a Natural Isomorphism: Since $\zeta_\ell^{\mathsf{Sets}}$ is natural and $\zeta_\ell^{\mathsf{Sets},-1}$ is a component-wise inverse to $\zeta_\ell^{\mathsf{Sets}}$, it follows from Categories, Item 2 of Definition II.9.7.I.2 that $\zeta_\ell^{\mathsf{Sets},-1}$ is also natural. Thus $\zeta_\ell^{\mathsf{Sets}}$ is a natural isomorphism.

5.3.4 The Right Annihilator

Definition 5.3.4.1.1. The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{0}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\boldsymbol{\varepsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}) \overset{\sim}{\dashrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{0}^{\mathsf{Sets}})$$

as in the diagram



with components

$$\zeta_{r|A}^{\mathsf{Sets}} \colon A \times \emptyset \xrightarrow{\sim} \emptyset$$

given by $\zeta_{r|A}^{\text{Sets}} \stackrel{\text{def}}{=} \text{pr}_2$.

Proof. Invertibility: The inverse of $\zeta_{r|A}^{\mathsf{Sets}}$ is the map

$$\zeta_{r|A}^{\mathsf{Sets},-1} \colon \varnothing \xrightarrow{\sim} A \times \varnothing$$

given by

$$\zeta_{r|A}^{\text{Sets},-1} \stackrel{\text{def}}{=} \iota_A,$$

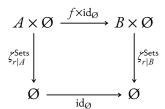
where ι_A is as defined in Constructions With Sets, Definition 4.2.1.1.2:

• *Invertibility I.* The map $\zeta_{r|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \to \emptyset$ is equal to id_{\emptyset} , as \emptyset is the initial object of Sets .

• *Invertibility II.* The map $\iota_A \circ \zeta_{r|A}^{\mathsf{Sets}}$ is equal to the identity on every $(a, x) \in A \times \emptyset$, of which there are none.

Hence $\zeta_{r|A}^{\text{Sets}}$ is an isomorphism.

Naturality: We need to show that given a function $f: A \rightarrow B$, the diagram



commutes. But since $A \times \emptyset$ has no elements, this is trivially true. Being a Natural Isomorphism: Since ζ_r^{Sets} is natural and $\zeta_r^{\mathsf{Sets},-1}$ is a component-wise inverse to ζ_r^{Sets} , it follows from Categories, Item 2 of Definition II.9.7.1.2 that $\zeta_r^{\mathsf{Sets},-1}$ is also natural. Thus ζ_r^{Sets} is a natural isomorphism.

5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

Proposition 5.3.5.1.1. The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category*. The category Sets of pointed sets.
- *The Additive Monoidal Product.* The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.

• The Multiplicative Monoidal Product. The product functor

$$\times$$
: Sets \times Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

• *The Monoidal Unit*. The functor

$$1^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Monoidal Zero. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• *The Internal Hom.* The internal Hom functor

Sets:
$$Sets^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, ?? of ??.

• The Additive Associators. The natural isomorphism

$$\alpha^{\text{Sets}, \coprod} : \coprod \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\text{id}_{\text{Sets}} \times \coprod) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$
of Definition 5.2.3.1.1.

• The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ (0^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Additive Right Unitors. The natural isomorphism

$$\rho^{\text{Sets}, \coprod} : \coprod \circ (\text{id} \times \mathbb{O}^{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of Definition 5.2.5.1.1.

• The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}:\coprod\stackrel{\sim}{\Longrightarrow}\coprod\circ\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

• The Multiplicative Associators. The natural isomorphism

$$\alpha^{\text{Sets}}$$
: $\times \circ (\times \times \text{id}_{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}}_{\text{Sets},\text{Sets},\text{Sets}}$
of Definition 5.I.4.I.I.

• The Multiplicative Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Multiplicative Right Unitors. The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of Definition 5.1.6.1.1.

• *The Multiplicative Symmetry*. The natural isomorphism

$$\sigma^{\text{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\text{Cats}_2}_{\text{Sets,Sets}}$$

of Definition 5.1.7.1.1.

• The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} : \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$$
of Definition 5.3.1.1.1.

• The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.I.I.

• The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \mu_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\mathsf{id}_{\mathsf{pt}} \times \varepsilon_{\mathsf{Sets}}^{\mathsf{Cats}_2}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}})$$
of Definition 5.3.3.1.1.

• The Right Annihilator. The natural isomorphism

$$\zeta_r^{\mathsf{Sets}} : \mathbb{O}^{\mathsf{Sets}} \circ \mu_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\varepsilon_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathsf{id}_{\mathsf{pt}}) \xrightarrow{\sim} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}})$$
of Definition 5.3.4.1.1.

Proof. Omitted.

Appendices

A Other Chapters

Preliminaries

- I. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes