Constructions With Monoidal Categories

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O1UF This chapter contains some material on constructions with monoidal categories.

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Moduli Categories of Monoidal Structures 01UG **13.1**

The Moduli Category of Monoidal Structures on a 01UH 13.1.1 **Category**

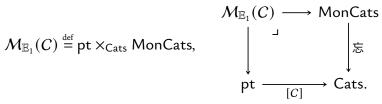
Let *C* be a category.

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DEFINITION 13.1.1.1.1 ➤ THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATE-

The **moduli category of monoidal structures on** C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{B}_1}(\mathcal{C})\stackrel{\scriptscriptstyle
m def}{=}\operatorname{pt} imes_{\operatorname{\mathsf{Cats}}}\operatorname{\mathsf{MonCats}},$$



01UK

REMARK 13.1.1.1.2 ► UNWINDING DEFINITION 13.1.1.1.1, I

In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \otimes_C, \otimes_C)$ $\mathbb{1}_C$, α^C , λ^C , ρ^C) whose underlying category is C.
- *Morphisms*. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ $\mathbb{1}'_{C},\alpha^{C,\prime},\lambda^{C,\prime},\,\rho^{C,\prime}\big)$ is a strong monoidal functor structure

$$\operatorname{id}_C^{\otimes} \colon A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$
$$\operatorname{id}_{\mathbb{I}|C}^{\otimes} \colon \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor $id_C : C \to C$ of C.

• *Identities.* For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{B}_1}(C)),$ the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\mathrm{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\mathrm{def}}{=}(\mathrm{id}_{C}^{\otimes},\mathrm{id}_{\mathbb{1}|C}^{\otimes}),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of ??.

• *Composition*. For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,P)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at (M, N, P) is defined by

$$\left(\operatorname{id}_{C}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{H}_{1}}(C)}\left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right)\stackrel{\scriptscriptstyle \mathrm{def}}{=}\left(\operatorname{id}_{C}^{\otimes,\prime}\circ\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{H}|C}^{\otimes}\right).$$

01UL REMARK 13.1.1.1.3 ➤ UNWINDING DEFINITION 13.1.1.1.1, II

In particular, a morphism in $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

1. *Naturality*. For each pair $f:A\to X$ and $g:B\to Y$ of morphisms of C, the diagram

$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow_{\mathrm{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow_{\mathrm{id}_{X,Y}^{\otimes}}$$

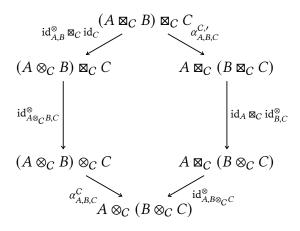
$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

commutes.

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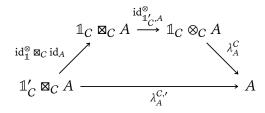
2. *Monoidality*. For each $A, B, C \in Obj(C)$, the diagram



commutes.

01UP

3. *Left Monoidal Unity*. For each $A \in Obj(C)$, the diagram



commutes.

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4. Right Monoidal Unity. For each $A \in Obj(C)$, the diagram

$$A \boxtimes_C \mathbb{1}_C \xrightarrow{\operatorname{id}_{A,\mathbb{1}'_C}^{\circ}} A \otimes_C \mathbb{1}_C$$

$$\operatorname{id}_A \boxtimes_C \operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_C \mathbb{1}'_C \xrightarrow{\rho_A^{C,\prime}} A$$

commutes.

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PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

Let C be a category.

- 01US
- 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.
- 01UT
- (a) The diagram

commutes.

01UU

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B\boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

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2. Extra Monoidal Unity Constraints. Let $(id_C^{\otimes}, id_{1|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

01WC

(a) The diagram

commutes.

01WD (b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

01WE

01WF

01UV

3. *Mixed Associators*. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

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(a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} \colon (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ id_{A \otimes_C B,C}^{\otimes} & & \downarrow id_A \otimes_C id_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_C (B \boxtimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C & & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ & (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id[®] satisfies the monoidality condition of Item 2 of Remark 13,1,1,1,3.

01UX

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} \colon (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B \otimes_{C}}^{\otimes} C}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{C}B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A}\boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation id[⊗] satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

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PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

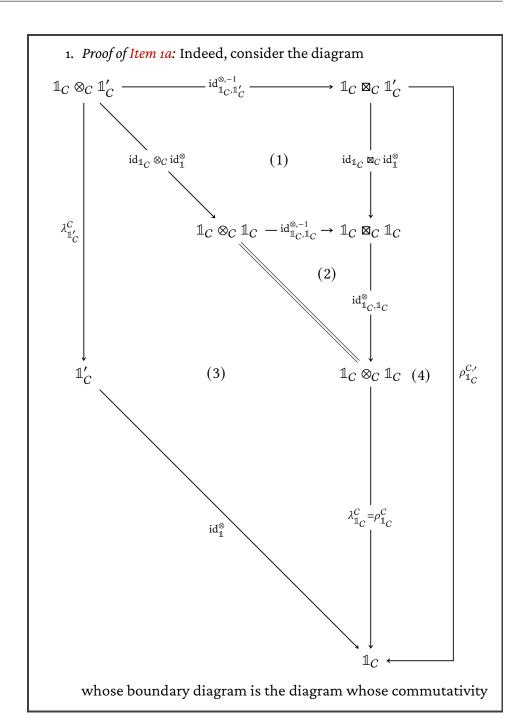
We claim that Items 1a and 1b are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of id^{\otimes} with respect to the morphisms $id_{A,B}^{\otimes}$ and id_{C} .
- 2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and $id_{B,C}^{\otimes}$.

This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

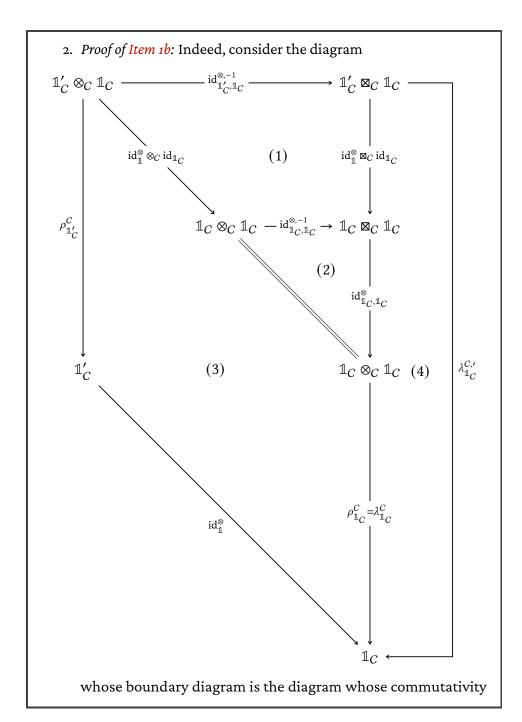
We claim that Items 2a and 2b are indeed true:



we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.



we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $id_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|\mathbb{I}}^{\otimes});$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{\rho_{\mathbb{1}'_{C}}^{C}} \qquad \qquad \downarrow^{\rho_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow^{C}_{\mathbb{1}'_{C}} \qquad (\dagger) \qquad \downarrow^{\lambda^{C}_{\mathbb{1}'_{C}}}_{\mathbb{1}'_{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}_{C}$$

commutes. But since $id_{1}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

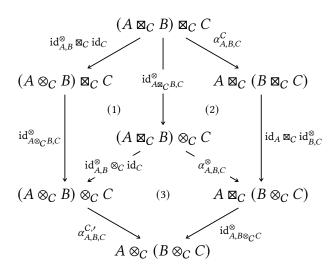
This finishes the proof.

Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

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1. *Proof of Item 3a*: We may partition the monoidality diagram for id[⊗] of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

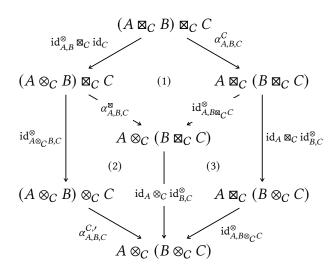
- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

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2. *Proof of Item 3b*: We may partition the monoidality diagram for id^{\otimes}

of Item 2 of Remark 13.1.1.1.3 as follows:



Since:

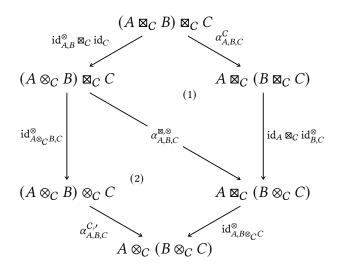
- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for id^{\otimes}

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of Item 2 of Remark 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- 01V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets

7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal gories
Categories

Bicategories

Extra Part

14. Types of Morphisms in Bicate- 15. Notes