

# Conditions on Relations

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00TJ This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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## 02D0 10.1 Functional and Total Relations

### 00JC 10.1.1 Functional Relations

Let  $A$  and  $B$  be sets.

00JD **Definition 10.1.1.1.** A relation  $R: A \rightarrow B$  is **functional** if, for each  $a \in A$ , the set  $R(a)$  is either empty or a singleton.

00JE **Proposition 10.1.1.2.** Let  $R: A \rightarrow B$  be a relation.

00JF 1. *Characterisations.* The following conditions are equivalent:

00JG (a) The relation  $R$  is functional.

00JH (b) We have  $R \diamond R^\dagger \subset \chi_B$ .

*Proof.* **Item 1, Characterisations:** We claim that **Items 1a** and **1b** are indeed equivalent:

- **Item 1a**  $\implies$  **Item 1b:** Let  $(b, b') \in B \times B$ . We need to show that

$$[R \diamond R^\dagger](b, b') \preceq_{\{t, f\}} \chi_B(b, b'),$$

i.e. that if there exists some  $a \in A$  such that  $b \sim_{R^\dagger} a$  and  $a \sim_R b'$ , then  $b = b'$ . But since  $b \sim_{R^\dagger} a$  is the same as  $a \sim_R b$ , we have both  $a \sim_R b$  and  $a \sim_R b'$  at the same time, which implies  $b = b'$  since  $R$  is functional.

- **Item 1b**  $\implies$  **Item 1a:** Suppose that we have  $a \sim_R b$  and  $a \sim_R b'$  for  $b, b' \in B$ . We claim that  $b = b'$ :

- Since  $a \sim_R b$ , we have  $b \sim_{R^\dagger} a$ .

- Since  $R \diamond R^\dagger \subset \chi_B$ , we have

$$[R \diamond R^\dagger](b, b') \preceq_{\{t, f\}} \chi_B(b, b'),$$

and since  $b \sim_{R^\dagger} a$  and  $a \sim_R b'$ , it follows that  $[R \diamond R^\dagger](b, b') = \text{true}$ , and thus  $\chi_B(b, b') = \text{true}$  as well, i.e.  $b = b'$ .

This finishes the proof. □

## 00JJ 10.1.2 Total Relations

Let  $A$  and  $B$  be sets.

**00JK Definition 10.1.2.1.1.** A relation  $R: A \rightarrow B$  is **total** if, for each  $a \in A$ , we have  $R(a) \neq \emptyset$ .

**00JL Proposition 10.1.2.1.2.** Let  $R: A \rightarrow B$  be a relation.

**00JM** 1. *Characterisations.* The following conditions are equivalent:

**00JN** (a) The relation  $R$  is total.

**00JP** (b) We have  $\chi_A \subset R^\dagger \diamond R$ .

*Proof.* **Item 1, Characterisations:** We claim that **Items 1a** and **1b** are indeed equivalent:

- **Item 1a**  $\implies$  **Item 1b:** We have to show that, for each  $(a, a') \in A$ , we have

$$\chi_A(a, a') \preceq_{\{t, f\}} [R^\dagger \diamond R](a, a'),$$

i.e. that if  $a = a'$ , then there exists some  $b \in B$  such that  $a \sim_R b$  and  $b \sim_{R^\dagger} a'$  (i.e.  $a \sim_R b$  again), which follows from the totality of  $R$ .

- **Item 1b**  $\implies$  **Item 1a:** Given  $a \in A$ , since  $\chi_A \subset R^\dagger \diamond R$ , we must have

$$\{a\} \subset [R^\dagger \diamond R](a),$$

implying that there must exist some  $b \in B$  such that  $a \sim_R b$  and  $b \sim_{R^\dagger} a$  (i.e.  $a \sim_R b$ ) and thus  $R(a) \neq \emptyset$ , as  $b \in R(a)$ .

This finishes the proof. □

## 00TK 10.2 Reflexive Relations

### 00TL 10.2.1 Foundations

Let  $A$  be a set.

**00TM Definition 10.2.1.1.1.** A **reflexive relation** is equivalently:<sup>1</sup>

- An  $\mathbb{E}_0$ -monoid in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$ .
- A pointed object in  $(\mathbf{Rel}(A, A), \chi_A)$ .

**00TN Remark 10.2.1.1.2.** In detail, a relation  $R$  on  $A$  is **reflexive** if we have an inclusion

$$\eta_R: \chi_A \subset R$$

of relations in  $\mathbf{Rel}(A, A)$ , i.e. if, for each  $a \in A$ , we have  $a \sim_R a$ .

**00TP Definition 10.2.1.1.3.** Let  $A$  be a set.

**00TQ** 1. The **set of reflexive relations on  $A$**  is the subset  $\mathbf{Rel}^{\text{refl}}(A, A)$  of  $\mathbf{Rel}(A, A)$  spanned by the reflexive relations.

**00TR** 2. The **poset of relations on  $A$**  is the subposet  $\mathbf{Rel}^{\text{refl}}(A, A)$  of  $\mathbf{Rel}(A, A)$  spanned by the reflexive relations.

**00TS Proposition 10.2.1.1.4.** Let  $R$  and  $S$  be relations on  $A$ .

**00TT** 1. *Interaction With Inverses.* If  $R$  is reflexive, then so is  $R^\dagger$ .

**00TU** 2. *Interaction With Composition.* If  $R$  and  $S$  are reflexive, then so is  $S \diamond R$ .

*Proof.* **Item 1, Interaction With Inverses:** Clear.

**Item 2, Interaction With Composition:** Clear. □

## **00TV 10.2.2 The Reflexive Closure of a Relation**

Let  $R$  be a relation on  $A$ .

**00TW Definition 10.2.2.1.1.** The **reflexive closure** of  $\sim_R$  is the relation  $\sim_R^{\text{refl}}$ <sup>2</sup> satisfying the following universal property:<sup>3</sup>

- (★) Given another reflexive relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{refl}} \subset \sim_S$ .

<sup>1</sup>Note that since  $\mathbf{Rel}(A, A)$  is posetal, reflexivity is a property of a relation, rather than extra structure.

<sup>2</sup>*Further Notation:* Also written  $R^{\text{refl}}$ .

<sup>3</sup>*Slogan:* The reflexive closure of  $R$  is the smallest reflexive relation containing  $R$ .

**00TX Construction 10.2.2.1.2.** Concretely,  $\sim_R^{\text{refl}}$  is the free pointed object on  $R$  in  $(\mathbf{Rel}(A, A), \chi_A)^4$ , being given by

$$\begin{aligned} R^{\text{refl}} &\stackrel{\text{def}}{=} R \coprod^{\mathbf{Rel}(A, A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

*Proof.* Clear. □

**00TY Proposition 10.2.2.1.3.** Let  $R$  be a relation on  $A$ .

**00TZ** 1. *Adjointness.* We have an adjunction

$$\left( (-)^{\text{refl}} \dashv \overset{\circlearrowleft}{\sim} \right): \mathbf{Rel}(A, A) \overset{(-)^{\text{refl}}}{\underset{\overset{\circlearrowleft}{\sim}}{\rightleftarrows}} \mathbf{Rel}^{\text{refl}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{refl}}(R^{\text{refl}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{refl}}(A, A))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, A))$ .

**00U0** 2. *The Reflexive Closure of a Reflexive Relation.* If  $R$  is reflexive, then  $R^{\text{refl}} = R$ .

**00U1** 3. *Idempotency.* We have

$$(R^{\text{refl}})^{\text{refl}} = R^{\text{refl}}.$$

**00U2** 4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} & \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} \mathbf{Rel}(A, A) \\ \left( R^\dagger \right)^{\text{refl}} = \left( R^{\text{refl}} \right)^\dagger, & \downarrow (-)^\dagger & \downarrow (-)^\dagger \\ & \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} \mathbf{Rel}(A, A). \end{array}$$

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<sup>4</sup>Or, equivalently, the free  $\mathbb{E}_0$ -monoid on  $R$  in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$ .

00U3 5. *Interaction With Composition.* We have

$$\begin{array}{ccc}
 \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A) \\
 (S \diamond R)^{\text{refl}} = S^{\text{refl}} \diamond R^{\text{refl}}, & \begin{array}{c} (-)^{\text{refl}} \times (-)^{\text{refl}} \downarrow \\ \text{Rel}(A, A) \times \text{Rel}(A, A) \end{array} & \begin{array}{c} \downarrow (-)^{\text{refl}} \\ \text{Rel}(A, A) \end{array} \\
 & \xrightarrow{\diamond} & \text{Rel}(A, A).
 \end{array}$$

*Proof.* **Item 1, Adjointness:** This is a rephrasing of the universal property of the reflexive closure of a relation, stated in **Definition 10.2.2.1.1**.

**Item 2, The Reflexive Closure of a Reflexive Relation:** Clear.

**Item 3, Idempotency:** This follows from **Item 2**.

**Item 4, Interaction With Inverses:** Clear.

**Item 5, Interaction With Composition:** This follows from **Item 2** of **Definition 10.2.1.1.4**.

□

## 00U4 10.3 Symmetric Relations

### 00U5 10.3.1 Foundations

Let  $A$  be a set.

00U6 **Definition 10.3.1.1.1.** A relation  $R$  on  $A$  is **symmetric** if we have  $R^\dagger = R$ .

00U7 **Remark 10.3.1.1.2.** In detail, a relation  $R$  is symmetric if it satisfies the following condition:

(★) For each  $a, b \in A$ , if  $a \sim_R b$ , then  $b \sim_R a$ .

00U8 **Definition 10.3.1.1.3.** Let  $A$  be a set.

00U9 1. The **set of symmetric relations on  $A$**  is the subset  $\text{Rel}^{\text{symm}}(A, A)$  of  $\text{Rel}(A, A)$  spanned by the symmetric relations.

00UA 2. The **poset of relations on  $A$**  is the subposet  $\text{Rel}^{\text{symm}}(A, A)$  of  $\text{Rel}(A, A)$  spanned by the symmetric relations.

00UB **Proposition 10.3.1.1.4.** Let  $R$  and  $S$  be relations on  $A$ .

00UC 1. *Interaction With Inverses.* If  $R$  is symmetric, then so is  $R^\dagger$ .

00UD 2. *Interaction With Composition.* If  $R$  and  $S$  are symmetric, then so is  $S \diamond R$ .

*Proof.* **Item 1, Interaction With Inverses:** Clear.

**Item 2, Interaction With Composition:** Clear.

□

### 00UE 10.3.2 The Symmetric Closure of a Relation

Let  $R$  be a relation on  $A$ .

00UF **Definition 10.3.2.1.1.** The **symmetric closure** of  $\sim_R$  is the relation  $\sim_R^{\text{symm}}$ <sup>5</sup> satisfying the following universal property:<sup>6</sup>

(★) Given another symmetric relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{symm}} \subset \sim_S$ .

00UG **Construction 10.3.2.1.2.** Concretely,  $\sim_R^{\text{symm}}$  is the symmetric relation on  $A$  defined by

$$\begin{aligned} R^{\text{symm}} &\stackrel{\text{def}}{=} R \cup R^\dagger \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

*Proof.* Clear. □

00UH **Proposition 10.3.2.1.3.** Let  $R$  be a relation on  $A$ .

00UJ 1. *Adjointness.* We have an adjunction

$$((-)^{\text{symm}} \dashv \overline{\phantom{x}}): \mathbf{Rel}(A, A) \begin{array}{c} \xrightarrow{(-)^{\text{symm}}} \\ \perp \\ \xleftarrow{\overline{\phantom{x}}} \end{array} \mathbf{Rel}^{\text{symm}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{symm}}(R^{\text{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{symm}}(A, A))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, A))$ .

00UK 2. *The Symmetric Closure of a Symmetric Relation.* If  $R$  is symmetric, then  $R^{\text{symm}} = R$ .

00UL 3. *Idempotency.* We have

$$(R^{\text{symm}})^{\text{symm}} = R^{\text{symm}}.$$

<sup>5</sup>Further Notation: Also written  $R^{\text{symm}}$ .

<sup>6</sup>Slogan: The symmetric closure of  $R$  is the smallest symmetric relation containing  $R$ .

00UM 4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} & \text{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} \text{Rel}(A, A) \\ \left(R^{\dagger}\right)^{\text{symm}} = \left(R^{\text{symm}}\right)^{\dagger}, & \downarrow (-)^{\dagger} & \downarrow (-)^{\dagger} \\ & \text{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} \text{Rel}(A, A). \end{array}$$

00UN 5. *Interaction With Composition.* We have

$$\begin{array}{ccc} & \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} \text{Rel}(A, A) \\ (S \diamond R)^{\text{symm}} = S^{\text{symm}} \diamond R^{\text{symm}}, & \downarrow (-)^{\text{symm}} \times (-)^{\text{symm}} & \downarrow (-)^{\text{symm}} \\ & \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} \text{Rel}(A, A). \end{array}$$

*Proof.* **Item 1, Adjointness:** This is a rephrasing of the universal property of the symmetric closure of a relation, stated in **Definition 10.3.2.1.1**.

**Item 2, The Symmetric Closure of a Symmetric Relation:** Clear.

**Item 3, Idempotency:** This follows from **Item 2**.

**Item 4, Interaction With Inverses:** Clear.

**Item 5, Interaction With Composition:** This follows from **Item 2** of **Definition 10.3.1.1.4**.

□

## 00UP 10.4 Transitive Relations

### 00UQ 10.4.1 Foundations

Let  $A$  be a set.

00UR **Definition 10.4.1.1.1.** A **transitive relation** is equivalently:<sup>7</sup>

- A non-unital  $\mathbb{E}_1$ -monoid in  $(\mathbf{N}_{\bullet}(\mathbf{Rel}(A, A)), \diamond)$ .
- A non-unital monoid in  $(\mathbf{Rel}(A, A), \diamond)$ .

<sup>7</sup>Note that since  $\mathbf{Rel}(A, A)$  is posetal, transitivity is a property of a relation, rather than extra structure.



**00US Remark 10.4.1.1.2.** In detail, a relation  $R$  on  $A$  is **transitive** if we have an inclusion

$$\mu_R: R \diamond R \subset R$$

of relations in  $\text{Rel}(A, A)$ , i.e. if, for each  $a, c \in A$ , the following condition is satisfied:

(★) If there exists some  $b \in A$  such that  $a \sim_R b$  and  $b \sim_R c$ , then  $a \sim_R c$ .

**00UT Definition 10.4.1.1.3.** Let  $A$  be a set.

- 00UU** 1. The **set of transitive relations from  $A$  to  $B$**  is the subset  $\text{Rel}^{\text{trans}}(A)$  of  $\text{Rel}(A, A)$  spanned by the transitive relations.
- 00UV** 2. The **poset of relations from  $A$  to  $B$**  is the subposet  $\text{Rel}^{\text{trans}}(A)$  of  $\text{Rel}(A, A)$  spanned by the transitive relations.

**00UW Proposition 10.4.1.1.4.** Let  $R$  and  $S$  be relations on  $A$ .

- 00UX** 1. *Interaction With Inverses.* If  $R$  is transitive, then so is  $R^\dagger$ .
- 00UY** 2. *Interaction With Composition.* If  $R$  and  $S$  are transitive, then  $S \diamond R$  **may fail to be transitive**.

*Proof.* **Item 1, Interaction With Inverses:** Clear.

**Item 2, Interaction With Composition:** See [MSE 2096272].<sup>8</sup>

□

## **00UZ 10.4.2 The Transitive Closure of a Relation**

Let  $R$  be a relation on  $A$ .

<sup>8</sup>*Intuition:* Transitivity for  $R$  and  $S$  fails to imply that of  $S \diamond R$  because the composition operation for relations intertwines  $R$  and  $S$  in an incompatible way:

- If  $a \sim_{S \diamond R} c$  and  $c \sim_{S \diamond R} e$ , then:
  - There is some  $b \in A$  such that:
    - \*  $a \sim_R b$ ;
    - \*  $b \sim_S c$ ;
  - There is some  $d \in A$  such that:
    - \*  $c \sim_R d$ ;
    - \*  $d \sim_S e$ .

**00V0 Definition 10.4.2.1.1.** The **transitive closure** of  $\sim_R$  is the relation  $\sim_R^{\text{trans}}$ <sup>9</sup> satisfying the following universal property:<sup>10</sup>

(★) Given another transitive relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{trans}} \subset \sim_S$ .

**00V1 Construction 10.4.2.1.2.** Concretely,  $\sim_R^{\text{trans}}$  is the free non-unital monoid on  $R$  in  $(\mathbf{Rel}(A, A), \diamond)$ <sup>11</sup>, being given by

$$\begin{aligned} R^{\text{trans}} &\stackrel{\text{def}}{=} \prod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists some } (x_1, \dots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \dots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

*Proof.* Clear. □

**00V2 Proposition 10.4.2.1.3.** Let  $R$  be a relation on  $A$ .

**00V3** 1. *Adjointness.* We have an adjunction

$$((-)^{\text{trans}} \dashv \overline{\phantom{x}}): \quad \mathbf{Rel}(A, A) \begin{array}{c} \xrightarrow{(-)^{\text{trans}}} \\ \perp \\ \xleftarrow{\overline{\phantom{x}}} \end{array} \mathbf{Rel}^{\text{trans}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{trans}}(R^{\text{trans}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{trans}}(A, A))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, B))$ .

**00V4** 2. *The Transitive Closure of a Transitive Relation.* If  $R$  is transitive, then  $R^{\text{trans}} = R$ .

**00V5** 3. *Idempotency.* We have

$$(R^{\text{trans}})^{\text{trans}} = R^{\text{trans}}.$$

<sup>9</sup>Further Notation: Also written  $R^{\text{trans}}$ .

<sup>10</sup>Slogan: The transitive closure of  $R$  is the smallest transitive relation containing  $R$ .

<sup>11</sup>Or, equivalently, the free non-unital  $\mathbb{B}_1$ -monoid on  $R$  in  $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$ .

00V6 4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \text{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \text{Rel}(A, A) \\ \downarrow (-)^{\dagger} & & \downarrow (-)^{\dagger} \\ \text{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \text{Rel}(A, A). \end{array}$$

$$(R^{\dagger})^{\text{trans}} = (R^{\text{trans}})^{\dagger},$$

00V7 5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A) \\ \downarrow (-)^{\text{trans}} \times (-)^{\text{trans}} & \text{X} & \downarrow (-)^{\text{trans}} \\ \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A). \end{array}$$

$$(S \diamond R)^{\text{trans}} \stackrel{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}},$$

*Proof.* **Item 1, Adjointness:** This is a rephrasing of the universal property of the transitive closure of a relation, stated in **Definition 10.4.2.1.1.**

**Item 2, The Transitive Closure of a Transitive Relation:** Clear.

**Item 3, Idempotency:** This follows from **Item 2.**

**Item 4, Interaction With Inverses:** We have

$$\begin{aligned} (R^{\dagger})^{\text{trans}} &= \bigcup_{n=1}^{\infty} (R^{\dagger})^{\diamond n} \\ &= \bigcup_{n=1}^{\infty} (R^{\diamond n})^{\dagger} \\ &= \left( \bigcup_{n=1}^{\infty} R^{\diamond n} \right)^{\dagger} \\ &= (R^{\text{trans}})^{\dagger}, \end{aligned}$$

where we have used, respectively:

- **Definition 10.4.2.1.2.**
- **Constructions With Relations**, ?? of ??.
- **Constructions With Relations**, ?? of **Definition 9.2.3.1.2.**

- **Definition 10.4.2.1.2.**

This finishes the proof.

*Item 5, Interaction With Composition:* This follows from **Item 2** of **Definition 10.4.1.1.4.**

□

## 00V8 10.5 Equivalence Relations

### 00V9 10.5.1 Foundations

Let  $A$  be a set.

00VA **Definition 10.5.1.1.1.** A relation  $R$  is an **equivalence relation** if it is reflexive, symmetric, and transitive.<sup>12</sup>

00VB **Example 10.5.1.1.2.** The **kernel of a function**  $f: A \rightarrow B$  is the equivalence relation  $\sim_{\text{Ker}(f)}$  on  $A$  obtained by declaring  $a \sim_{\text{Ker}(f)} b$  iff  $f(a) = f(b)$ .<sup>13</sup>

00VC **Definition 10.5.1.1.3.** Let  $A$  and  $B$  be sets.

00VD 1. The **set of equivalence relations from  $A$  to  $B$**  is the subset  $\text{Rel}^{\text{eq}}(A, B)$  of  $\text{Rel}(A, B)$  spanned by the equivalence relations.

00VE 2. The **poset of relations from  $A$  to  $B$**  is the subposet  $\text{Rel}^{\text{eq}}(A, B)$  of  $\text{Rel}(A, B)$  spanned by the equivalence relations.

### 00VF 10.5.2 The Equivalence Closure of a Relation

Let  $R$  be a relation on  $A$ .

00VG **Definition 10.5.2.1.1.** The **equivalence closure**<sup>14</sup> of  $\sim_R$  is the relation  $\sim_R^{\text{eq}}$ <sup>15</sup> satisfying the following universal property:<sup>16</sup>

<sup>12</sup>*Further Terminology:* If instead  $R$  is just symmetric and transitive, then it is called a **partial equivalence relation**.

<sup>13</sup>The kernel  $\text{Ker}(f): A \dashv A$  of  $f$  is the underlying functor of the monad induced by the adjunction  $\text{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$  in **Rel** of **Constructions With Relations**, ?? of ??.

<sup>14</sup>*Further Terminology:* Also called the **equivalence relation associated to  $\sim_R$** .

<sup>15</sup>*Further Notation:* Also written  $R^{\text{eq}}$ .

<sup>16</sup>*Slogan:* The equivalence closure of  $R$  is the smallest equivalence relation containing  $R$ .

- (★) Given another equivalence relation  $\sim_S$  on  $A$  such that  $R \subset S$ , there exists an inclusion  $\sim_R^{\text{eq}} \subset \sim_S$ .

**00VH Construction 10.5.2.1.2.** Concretely,  $\sim_R^{\text{eq}}$  is the equivalence relation on  $A$  defined by

$$\begin{aligned} R^{\text{eq}} &\stackrel{\text{def}}{=} ((R^{\text{refl}})^{\text{symm}})^{\text{trans}} \\ &= ((R^{\text{symm}})^{\text{trans}})^{\text{refl}} \\ &= \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists } (x_1, \dots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \\ 1. \text{ The following conditions are satisfied:} \\ \\ \quad (a) \text{ We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \quad (b) \text{ We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \quad \quad \text{for each } 1 \leq i \leq n-1; \\ \quad (c) \text{ We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \\ 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

*Proof.* From the universal properties of the reflexive, symmetric, and transitive closures of a relation (**Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1**), we see that it suffices to prove that:

- 00VJ** 1. The symmetric closure of a reflexive relation is still reflexive.  
**00VK** 2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

**00VL Proposition 10.5.2.1.3.** Let  $R$  be a relation on  $A$ .

- 00VM** 1. *Adjointness.* We have an adjunction

$$((-)^{\text{eq}} \dashv \overline{\phantom{x}}): \quad \mathbf{Rel}(A, B) \begin{array}{c} \xrightarrow{(-)^{\text{eq}}} \\ \perp \\ \xleftarrow{\overline{\phantom{x}}} \end{array} \mathbf{Rel}^{\text{eq}}(A, B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{eq}}(R^{\text{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in  $R \in \text{Obj}(\mathbf{Rel}^{\text{eq}}(A, B))$  and  $S \in \text{Obj}(\mathbf{Rel}(A, B))$ .

00VN 2. *The Equivalence Closure of an Equivalence Relation.* If  $R$  is an equivalence relation, then  $R^{\text{eq}} = R$ .

00VP 3. *Idempotency.* We have

$$(R^{\text{eq}})^{\text{eq}} = R^{\text{eq}}.$$

*Proof.* **Item 1, Adjointness:** This is a rephrasing of the universal property of the equivalence closure of a relation, stated in **Definition 10.5.2.1.1**.

**Item 2, The Equivalence Closure of an Equivalence Relation:** Clear.

**Item 3, Idempotency:** This follows from **Item 2**. □

## 00VQ 10.6 Quotients by Equivalence Relations

### 00VR 10.6.1 Equivalence Classes

Let  $A$  be a set, let  $R$  be a relation on  $A$ , and let  $a \in A$ .

02B1 **Definition 10.6.1.1.1.** The **equivalence class associated to**  $a$  is the set  $[a]$  defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \end{aligned} \quad (\text{since } R \text{ is symmetric})$$

### 02B2 10.6.2 Quotients of Sets by Equivalence Relations

Let  $A$  be a set and let  $R$  be a relation on  $A$ .

02B3 **Definition 10.6.2.1.1.** The **quotient of  $X$  by  $R$**  is the set  $X/\sim_R$  defined by

$$X/\sim_R \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00VV **Remark 10.6.2.1.2.** The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalence classes  $[a]$  of  $X$  under  $R$  are well-behaved:

- *Reflexivity.* If  $R$  is reflexive, then, for each  $a \in X$ , we have  $a \in [a]$ .

- *Symmetry.* The equivalence class  $[a]$  of an element  $a$  of  $X$  is defined by

$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when  $R$  is symmetric, as we then have  $[a] = [a]'$ .<sup>17</sup>

- *Transitivity.* If  $R$  is transitive, then  $[a]$  and  $[b]$  are disjoint iff  $a \not\sim_R b$ , and equal otherwise.

**00VW Proposition 10.6.2.1.3.** Let  $f: X \rightarrow Y$  be a function and let  $R$  be a relation on  $X$ .

- 02B4** 1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\text{eq}} \cong \text{CoEq}(R \hookrightarrow X \times X \xrightarrow{\text{pr}_1} X \xrightarrow{\text{pr}_2} X),$$

where  $\sim_R^{\text{eq}}$  is the equivalence relation generated by  $\sim_R$ .

- 02B5** 2. *As a Pushout.* We have an isomorphism of sets<sup>18</sup>

$$X/\sim_R^{\text{eq}} \cong X \amalg_{\text{Eq}(\text{pr}_1, \text{pr}_2)} X, \quad \begin{array}{ccc} X/\sim_R^{\text{eq}} & \longleftarrow & X \\ \uparrow \lrcorner & & \uparrow \\ X & \longleftarrow & \text{Eq}(\text{pr}_1, \text{pr}_2) \end{array}$$

where  $\sim_R^{\text{eq}}$  is the equivalence relation generated by  $\sim_R$ .

<sup>17</sup>When categorifying equivalence relations, one finds that  $[a]$  and  $[a]'$  correspond to presheaves and copresheaves; see Constructions With Categories, ??.

<sup>18</sup>Dually, we also have an isomorphism of sets

$$\text{Eq}(\text{pr}_1, \text{pr}_2) \cong X \times_{X/\sim_R^{\text{eq}}} X, \quad \begin{array}{ccc} \text{Eq}(\text{pr}_1, \text{pr}_2) & \longrightarrow & X \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & X/\sim_R^{\text{eq}} \end{array}$$

02B6 3. *The First Isomorphism Theorem for Sets*. We have an isomorphism of sets<sup>19,20</sup>

$$X/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

00W0 4. *Descending Functions to Quotient Sets, I*. Let  $R$  be an equivalence relation on  $X$ . The following conditions are equivalent:

02B7 (a) There exists a map

$$\bar{f}: X/\sim_R \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists \nearrow \bar{f} & \\ X/\sim_R & & \end{array}$$

commute.

02B8 (b) We have  $R \subset \text{Ker}(f)$ .

02B9 (c) For each  $x, y \in X$ , if  $x \sim_R y$ , then  $f(x) = f(y)$ .

<sup>19</sup>*Further Terminology*: The set  $X/\sim_{\text{Ker}(f)}$  is often called the **coimage** of  $f$ , and denoted by  $\text{CoIm}(f)$ .

<sup>20</sup>In a sense this is a result relating the monad in **Rel** induced by  $f$  with the comonad in **Rel** induced by  $f$ , as the kernel and image

$$\begin{aligned} \text{Ker}(f) &: X \rightharpoonup X, \\ \text{Im}(f) &\subset Y \end{aligned}$$

of  $f$  are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$(\text{Gr}(f) \dashv f^{-1}): \begin{array}{ccc} & \text{Gr}(f) & \\ & \downarrow \perp & \\ A & \xrightarrow{\quad} & B \\ & \uparrow \perp & \\ & f^{-1} & \end{array}$$

of **Constructions With Relations**, ?? of ??.



- 00W1 5. *Descending Functions to Quotient Sets, II.* Let  $R$  be an equivalence relation on  $X$ . If the conditions of **Item 4** hold, then  $\bar{f}$  is the *unique* map making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \nearrow \exists! \bar{f} & \\ X/\sim_R & & \end{array}$$

commute.

- 00W2 6. *Descending Functions to Quotient Sets, III.* Let  $R$  be an equivalence relation on  $X$ . We have a bijection

$$\text{Hom}_{\text{Sets}}(X/\sim_R, Y) \cong \text{Hom}_{\text{Sets}}^R(X, Y),$$

natural in  $X, Y \in \text{Obj}(\text{Sets})$ , given by the assignment  $f \mapsto \bar{f}$  of **Items 4** and **5**, where  $\text{Hom}_{\text{Sets}}^R(X, Y)$  is the set defined by

$$\text{Hom}_{\text{Sets}}^R(X, Y) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}(X, Y) \left| \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right. \right\}.$$

- 02BA 7. *Descending Functions to Quotient Sets, IV.* Let  $R$  be an equivalence relation on  $X$ . If the conditions of **Item 4** hold, then the following conditions are equivalent:

02BB (a) The map  $\bar{f}$  is an injection.

02BC (b) We have  $R = \text{Ker}(f)$ .

02BD (c) For each  $x, y \in X$ , we have  $x \sim_R y$  iff  $f(x) = f(y)$ .

- 02BE 8. *Descending Functions to Quotient Sets, V.* Let  $R$  be an equivalence relation on  $X$ . If the conditions of **Item 4** hold, then the following conditions are equivalent:

02BF (a) The map  $f: X \rightarrow Y$  is surjective.

02BG (b) The map  $\bar{f}: X/\sim_R \rightarrow Y$  is surjective.

02BH 9. *Descending Functions to Quotient Sets, VI.* Let  $R$  be a relation on  $X$  and let  $\sim_R^{\text{eq}}$  be the equivalence relation associated to  $R$ . The following conditions are equivalent:

02BJ (a) The map  $f$  satisfies the equivalent conditions of **Item 4**:

- There exists a map

$$\bar{f}: X/\sim_R^{\text{eq}} \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \searrow \bar{f} & \uparrow \exists \\ X/\sim_R^{\text{eq}} & & \end{array}$$

commute.

- For each  $x, y \in X$ , if  $x \sim_R^{\text{eq}} y$ , then  $f(x) = f(y)$ .

02BK (b) For each  $x, y \in X$ , if  $x \sim_R y$ , then  $f(x) = f(y)$ .

*Proof.* **Item 1**, As a Coequaliser: Omitted.

**Item 2**, As a Pushout: Omitted.

**Item 3**, The First Isomorphism Theorem for Sets: Clear.

**Item 4**, Descending Functions to Quotient Sets, I: See [Pro25c].

**Item 5**, Descending Functions to Quotient Sets, II: See [Pro25d].

**Item 6**, Descending Functions to Quotient Sets, III: This follows from **Items 5** and **6**.

**Item 7**, Descending Functions to Quotient Sets, IV: See [Pro25b].

**Item 8**, Descending Functions to Quotient Sets, V: See [Pro25a].

**Item 9**, Descending Functions to Quotient Sets, VI: The implication **Item 8a**  $\implies$  **Item 8b** is clear.

Conversely, suppose that, for each  $x, y \in X$ , if  $x \sim_R y$ , then  $f(x) = f(y)$ . Spelling out the definition of the equivalence closure of  $R$ , we see that the condition  $x \sim_R^{\text{eq}} y$  unwinds to the following:

- (★) There exist  $(x_1, \dots, x_n) \in R^{\times n}$  satisfying at least one of the following conditions:
  - The following conditions are satisfied:

- \* We have  $x \sim_R x_1$  or  $x_1 \sim_R x$ ;
- \* We have  $x_i \sim_R x_{i+1}$  or  $x_{i+1} \sim_R x_i$  for each  $1 \leq i \leq n - 1$ ;
- \* We have  $y \sim_R x_n$  or  $x_n \sim_R y$ ;
- We have  $x = y$ .

Now, if  $x = y$ , then  $f(x) = f(y)$  trivially; otherwise, we have

$$\begin{aligned}
 f(x) &= f(x_1), \\
 f(x_1) &= f(x_2), \\
 &\vdots \\
 f(x_{n-1}) &= f(x_n), \\
 f(x_n) &= f(y),
 \end{aligned}$$

and  $f(x) = f(y)$ , as we wanted to show. □

# Appendices

## A Other Chapters

### Preliminaries

1. Introduction
2. A Guide to the Literature

### Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

### Relations

8. Relations
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10. Conditions on Relations

### Categories

11. Categories
12. Presheaves and the Yoneda Lemma

### Monoidal Categories

13. Constructions With Monoidal Categories

**Bicategories**

14. **Types of Morphisms in Bicat-  
egories**

**Extra Part**

15. **Notes**

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