

Tensor Products of Pointed Sets

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00C3 In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

introduced in [Section 7.5.1](#), defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f: X \wedge Y \rightarrow Z$.
- Maps of sets $f: X \times Y \rightarrow Z$ satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \triangleleft: \mathbf{Sets}_* \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ \triangleright: \mathbf{Sets}_* \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to \wedge , which turns out to endow \mathbf{Sets}_* with a monoidal category structure ([Definition 7.5.9.1.1](#)), these do not admit invertible associators and unitors,

but do endow \mathbf{Sets}_* with the structure of a skew monoidal category, however (Definitions 7.3.8.1.1 and 7.4.8.1.1).

Finally, in addition to the tensor products \triangleleft , \triangleright , and \wedge , we also have a “tensor product” of the form

$$\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned} \otimes_{k,\ell} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_\ell}(\mathbf{Sets}) &\rightarrow \mathbf{Mon}_{\mathbb{E}_{k+\ell}}(\mathbf{Sets}), \\ \triangleleft_{i,k} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) &\rightarrow \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}), \\ \triangleright_{i,k} : \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}) &\rightarrow \mathbf{Mon}_{\mathbb{E}_k}(\mathbf{Sets}), \end{aligned}$$

where $k, \ell, i \in \mathbb{N}$ with $i \leq k - 1$. Together with the Cartesian product \times of \mathbf{Sets} , the tensor products studied in this chapter form the cases:

- $(k, \ell) = (-1, -1)$ for the Cartesian product of \mathbf{Sets} ;
- $(k, \ell) = (0, -1)$ and $(-1, 0)$ for the tensor of sets with pointed sets of Definition 7.2.1.1.1;
- $(i, k) = (-1, 0)$ for the left and right tensor products of pointed sets of Sections 7.3 and 7.4;
- $(k, \ell) = (-1, -1)$ for the smash product of pointed sets of Section 7.5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in ??, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k, \ell \leq 2$, and hence *all* cases since \mathbb{E}_k -monoids on \mathbf{Sets} are the same as \mathbb{E}_2 -monoids on \mathbf{Sets} when $k \geq 2$.

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00C4 7.1 Bilinear Morphisms of Pointed Sets

00C5 7.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00C6 **Definition 7.1.1.1.1.** A left bilinear morphism of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is a map of sets

$$f: X \times Y \rightarrow Z$$

satisfying the following condition:^{1,2}

(\star) *Left Unital Bilinearity.* The diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & \text{id}_{\text{pt}} \times \epsilon_Y \nearrow & & \text{---} \sim \text{---} \searrow & \\
 \text{pt} \times Y & & & & \text{pt} \\
 & [x_0] \times \text{id}_Y \searrow & & & \downarrow [z_0] \\
 & X \times Y & \xrightarrow{f} & & Z
 \end{array}$$

¹*Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.

²Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

for each $y \in Y$.

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

00C7 Definition 7.1.1.1.2. The set of left bilinear morphisms of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is the set $\text{Hom}_{\text{Sets}_*}^{\otimes, \text{L}}(X \times Y, Z)$ defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, \text{L}}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

00C8 7.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00C9 Definition 7.1.2.1.1. A right bilinear morphism of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is a map of sets

$$f: X \times Y \rightarrow Z$$

satisfying the following condition.^{3,4}

(\star) *Right Unital Bilinearity.* The diagram

$$\begin{array}{ccc} & \text{pt} \times \text{pt} & \\ \epsilon_X \times \text{id}_{\text{pt}} \nearrow & & \searrow \sim \\ X \times \text{pt} & & \text{pt} \\ \text{id}_X \times [y_0] \searrow & & \nearrow [z_0] \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

00CA Definition 7.1.2.1.2. The set of right bilinear morphisms of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is the set $\text{Hom}_{\text{Sets}_*}^{\otimes, \text{R}}(X \times Y, Z)$ defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, \text{R}}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

³*Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.

⁴Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

00CB 7.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00CC Definition 7.1.3.1.1. A **bilinear morphism of pointed sets** from $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f: X \times Y \rightarrow Z$$

that is both left bilinear and right bilinear.

00CD Remark 7.1.3.1.2. In detail, a **bilinear morphism of pointed sets** from $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

satisfying the following conditions:^{5,6}

025T 1. *Left Unital Bilinearity.* The diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & \text{id}_{\text{pt}} \times \epsilon_Y \nearrow & & \searrow \sim & \\
 \text{pt} \times Y & & & & \text{pt} \\
 & \downarrow [x_0] \times \text{id}_Y & & & \downarrow [z_0] \\
 X \times Y & \xrightarrow{f} & Z & &
 \end{array}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

for each $x \in X$.

⁵*Slogan:* The map f is bilinear if it preserves basepoints in each argument.

⁶Succinctly, f is bilinear if we have

$$\begin{aligned}
 f(x_0, y) &= z_0, \\
 f(x, y_0) &= z_0
 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

025U 2. *Right Unital Bilinearity.* The diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & \nearrow \epsilon_X \times \text{id}_{\text{pt}} & & \searrow \sim & \\
 X \times \text{pt} & & & & \text{pt} \\
 \downarrow \text{id}_X \times [y_0] & & & & \downarrow [z_0] \\
 X \times Y & \xrightarrow{f} & Z & &
 \end{array}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

00CE **Definition 7.1.3.1.3.** The set of bilinear morphisms of pointed sets from $(X \times Y, (x_0, y_0))$ to (Z, z_0) is the set $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$ defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

00CF 7.2 Tensors and Cotensors of Pointed Sets by Sets

00CG 7.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

00CH **Definition 7.2.1.1.1.** The **tensor of** (X, x_0) **by** A ⁷ is the tensor $A \odot (X, x_0)$ ⁸ of (X, x_0) by A as in Limits and Colimits, ??.

00CJ **Remark 7.2.1.1.2.** In detail, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(*) We have a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \text{Obj}(\text{Sets}_*)$.

⁷*Further Terminology:* Also called the **copower of** (X, x_0) **by** A .

⁸*Further Notation:* Often written $A \odot X$ for simplicity.

This universal property is in turn equivalent to the following one:

(\star) We have a bijection

$$\mathbf{Sets}_*(A \odot X, K) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, where $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$ is the set defined by

$$\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Sets}(A \times X, K) \mid \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, x_0) = k_0 \end{array} \right\}.$$

Proof. We claim that we have a bijection

$$\mathbf{Sets}(A, \mathbf{Sets}_*(X, K)) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathbf{Sets}(A, \mathbf{Sets}(X, K)) \cong \mathbf{Sets}(A \times X, K)$$

of **Constructions With Sets, Item 2** of **Definition 4.1.3.1.3**:

- A map

$$\begin{aligned} \xi: A &\longrightarrow \mathbf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a: X \rightarrow K), \end{aligned}$$

in $\mathbf{Sets}(A, \mathbf{Sets}_*(X, K))$ gets sent to the map

$$\xi^\dagger: A \times X \rightarrow K$$

defined by

$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $(a, x) \in A \times X$, which indeed lies in $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$, as we have

$$\begin{aligned} \xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &\stackrel{\text{def}}{=} k_0 \end{aligned}$$

for each $a \in A$, where we have used that $\xi_a \in \mathbf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map

$$\xi: A \times X \rightarrow K$$

in $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$ gets sent to the map

$$\begin{aligned} \xi^\dagger: A &\longrightarrow \mathbf{Sets}_*(X, K), \\ a &\mapsto (\xi_a^\dagger: X \rightarrow K), \end{aligned}$$

where

$$\xi_a^\dagger: X \rightarrow K$$

is the map defined by

$$\xi_a^\dagger(x) \stackrel{\text{def}}{=} \xi(a, x)$$

for each $x \in X$, and indeed lies in $\mathbf{Sets}_*(X, K)$, as we have

$$\begin{aligned} \xi_a^\dagger(x_0) &\stackrel{\text{def}}{=} \xi(a, x_0) \\ &\stackrel{\text{def}}{=} k_0. \end{aligned}$$

This finishes the proof. \square

00CK Construction 7.2.1.1.3. Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0),$$

where $\bigvee_{a \in A} (X, x_0)$ is the wedge product of the A -indexed family $((X, x_0))_{a \in A}$ of **Pointed Sets, Definition 6.3.2.1.1**.

- *The Basepoint.* The point $[(a, x_0)] = [(a', x_0)]$ of $\bigvee_{a \in A} (X, x_0)$.

Proof. (Proven below in a bit.) \square

00CL Notation 7.2.1.1.4. We write $a \odot x$ for the element $[(a, x)]$ of

$$\begin{aligned} A \odot X &\cong \bigvee_{a \in A} (X, x_0) \\ &\stackrel{\text{def}}{=} \left(\prod_{i \in I} X_i \right) / \sim. \end{aligned}$$

00CM Remark 7.2.1.1.5. Taking the tensor of any element of A with the basepoint x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation \sim on

$$\bigvee_{a \in A} (X, x_0) \stackrel{\text{def}}{=} \prod_{a \in A} X / \sim$$

identifying (a, x_0) with (a', x_0) , so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof. We claim we have a bijection

$$\mathbf{Sets}_*(A \odot X, K) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

025V 1. *Map I.* We define a map

$$\Phi_K: \mathbf{Sets}_*(A \odot X, K) \rightarrow \mathbf{Sets}(A, \mathbf{Sets}_*(X, K))$$

by sending a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0)$$

to the map of sets

$$\begin{aligned} \xi^\dagger: A &\longrightarrow \mathbf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a: X \rightarrow K), \end{aligned}$$

where

$$\xi_a: (X, x_0) \rightarrow (K, k_0)$$

is the morphism of pointed sets defined by

$$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

for each $x \in X$. Note that we have

$$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

025W 2. *Map II.* We define a map

$$\Psi_K : \mathbf{Sets}(A, \mathbf{Sets}_*(X, K)) \rightarrow \mathbf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi : A &\longrightarrow \mathbf{Sets}_*(X, K), \\ a &\mapsto (\xi_a : X \rightarrow K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger : (A \odot X, a \odot x_0) \rightarrow (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that ξ^\dagger is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that $\xi(a) \in \mathbf{Sets}_*(X, K)$ is a morphism of pointed sets.

025X 3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \text{id}_{\mathbf{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi : (A \odot X, a \odot x_0) \rightarrow (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \llbracket x \mapsto \xi(a \odot x) \rrbracket \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket) \\ &= \llbracket a \odot x \mapsto \text{ev}_x(\text{ev}_a(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket a \odot x \mapsto \text{ev}_x(\llbracket x' \mapsto \xi(a \odot x') \rrbracket) \rrbracket \\ &= \llbracket a \odot x \mapsto \xi(a \odot x) \rrbracket \\ &= \xi. \end{aligned}$$

025Y 4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \text{id}_{\text{Sets}(A, \text{Sets}_*(X, K))}.$$

Indeed, given a morphism $\xi: A \rightarrow \text{Sets}_*(X, K)$, we have

$$\begin{aligned} [\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket a \odot x \mapsto \xi_a(x) \rrbracket) \\ &= \llbracket a \mapsto \llbracket x \mapsto \xi_a(x) \rrbracket \rrbracket \\ &= \llbracket a \mapsto \xi(a) \rrbracket \\ &= \xi. \end{aligned}$$

025Z 5. *Naturality of Φ .* We need to show that, given a morphism of pointed sets

$$\phi: (K, k_0) \rightarrow (K', k'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}_*(A \odot X, K) & \xrightarrow{\Phi_K} & \text{Sets}(A, \text{Sets}_*(X, K)) \\ \phi_* \downarrow & & \downarrow (\phi_*)_* \\ \text{Sets}_*(A \odot X, K') & \xrightarrow{\Phi_{K'}} & \text{Sets}(A, \text{Sets}_*(X, K')) \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0),$$

we have

$$\begin{aligned} [\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\ &= \Phi_{K'}(\phi \circ \xi) \\ &= (\phi \circ \xi)^\dagger \\ &= \llbracket a \mapsto \phi \circ \xi(a \odot -) \rrbracket \\ &= \llbracket a \mapsto \phi_*(\xi(a \odot -)) \rrbracket \\ &= (\phi_*)_*(\llbracket a \mapsto \xi(a \odot -) \rrbracket) \\ &= (\phi_*)_*(\Phi_K(\xi)) \\ &= [(\phi_*)_* \circ \Phi_K](\xi). \end{aligned}$$

- 0260** 6. *Naturality of Ψ* . Since Φ is natural and Φ is a componentwise inverse to Ψ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that Ψ is also natural.

This finishes the proof. \square

00CN **Proposition 7.2.1.1.6.** Let (X, x_0) be a pointed set and let A be a set.

- 00CP** 1. *Functoriality*. The assignments $A, (X, x_0), (A, (X, x_0))$ define functors

$$\begin{aligned} A \odot - &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ - \odot X &: \text{Sets} \rightarrow \text{Sets}_*, \\ -_1 \odot -_2 &: \text{Sets} \times \text{Sets}_* \rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given:

- A map of sets $f: A \rightarrow B$;
- A pointed map $\phi: (X, x_0) \rightarrow (Y, y_0)$;

the induced map

$$f \odot \phi: A \odot X \rightarrow B \odot Y$$

is given by

$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$

for each $a \odot x \in A \odot X$.

- 00CQ** 2. *Adjointness I*. We have an adjunction

$$(- \odot X \dashv \text{Sets}_*(X, -)): \text{Sets} \begin{array}{c} \xrightarrow{- \odot X} \\ \perp \\ \xleftarrow{\text{Sets}_*(X, -)} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K)),$$

natural in $A \in \text{Obj}(\text{Sets})$ and $X, Y \in \text{Obj}(\text{Sets}_*)$.

- 00CR** 3. *Adjointness II*. We have an adjunctions

$$(A \odot - \dashv A \pitchfork -): \text{Sets}_* \begin{array}{c} \xrightarrow{A \odot -} \\ \perp \\ \xleftarrow{A \pitchfork -} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathbf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathbf{Sets}_*)$.

00CS 4. *As a Weighted Colimit.* We have

$$A \odot X \cong \mathrm{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A: \mathrm{pt} \rightarrow \mathbf{Sets}$ picking $A \in \mathrm{Obj}(\mathbf{Sets})$;
- X for the functor $X: \mathrm{pt} \rightarrow \mathbf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

00CT 5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathbf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

00CU 6. *Interaction With Homs.* We have a natural isomorphism

$$\mathbf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathbf{Sets}_*(X, -).$$

00CV 7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathbf{Sets}_*)$, we have a map

$$\mathrm{ev}_{X,Y}^\odot: \mathbf{Sets}_*(X, Y) \odot X \rightarrow Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathbf{Sets}_*)$, and given by

$$\mathrm{ev}_{X,Y}^\odot(f \odot x) \stackrel{\mathrm{def}}{=} f(x)$$

for each $f \odot x \in \mathbf{Sets}_*(X, Y) \odot X$.

00CW 8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathbf{Sets})$ and each $X \in \mathrm{Obj}(\mathbf{Sets}_*)$, we have a map

$$\mathrm{coev}_{A,X}^\odot: A \rightarrow \mathbf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathbf{Sets})$ and $X \in \mathrm{Obj}(\mathbf{Sets}_*)$, and given by

$$\mathrm{coev}_{A,X}^\odot(a) \stackrel{\mathrm{def}}{=} \llbracket x \mapsto a \odot x \rrbracket$$

for each $a \in A$.

Proof. Item 1, Functoriality: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 2, Adjointness I: This is simply a rephrasing of Definition 7.2.1.1.1.

Item 3, : Adjointness II: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 4, As a Weighted Colimit: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 5, Iterated Tensors: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 6, Interaction With Homs: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 7, The Tensor Evaluation Map: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 8, The Tensor Coevaluation Map: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$. \square

00CX 7.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

00CY **Definition 7.2.2.1.1.** The **cotensor of (X, x_0) by A** ⁹ is the cotensor $A \pitchfork (X, x_0)$ ¹⁰ of (X, x_0) by A as in Limits and Colimits, ??.

00CZ **Remark 7.2.2.1.2.** In detail, the **cotensor of (X, x_0) by A** is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(\star) We have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

This universal property is in turn equivalent to the following one:

(\star) We have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, where $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X)$ is the set defined by

$$\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Sets}(A \times K, X) \mid \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right\}.$$

⁹ *Further Terminology:* Also called the **power of (X, x_0) by A** .

¹⁰ *Further Notation:* Often written $A \pitchfork X$ for simplicity.

Proof. This follows from the bijection

$$\mathbf{Sets}(A, \mathbf{Sets}_*(K, X)) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ constructed in the proof of [Definition 7.2.1.1.2](#). \square

00D0 Construction 7.2.2.1.3. Concretely, the **cotensor of (X, x_0) by A** is the pointed set $A \pitchfork (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

where $\bigwedge_{a \in A} (X, x_0)$ is the smash product of the A -indexed family $((X, x_0))_{a \in A}$ of [Definition 7.6.1.1.1](#).

- *The Basepoint.* The point $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \dots)]$ of $\bigwedge_{a \in A} (X, x_0)$.

Proof. We claim we have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

0261 1. *Map I.* We define a map

$$\Phi_K: \mathbf{Sets}_*(K, A \pitchfork X) \rightarrow \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

by sending a morphism of pointed sets

$$\xi: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

to the map of sets

$$\xi^\dagger: A \rightarrow \mathbf{Sets}_*(K, X),$$

$$a \mapsto (\xi_a: K \rightarrow X),$$

where

$$\xi_a: (K, k_0) \rightarrow (X, x_0)$$

is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x_a^k & \text{if } \xi(k) \neq [(x_0)_{a \in A}] \\ x_0 & \text{if } \xi(k) = [(x_0)_{a \in A}] \end{cases},$$

for each $k \in K$, where x_a^k is the a th component of $\xi(k) = [(x_a^k)_{a \in A}]$.
Note that:

- 0262 (a) The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned}\xi(k) &= \left[(x_a^k)_{a \in A} \right] \\ &= \left[(y_a^k)_{a \in A} \right]\end{aligned}$$

with $x_a^k \neq y_a^k$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x_{a_x}^k = y_{a_y}^k = x_0$. The equivalence relation \sim on $\prod_{a \in A} X$ then forces

$$\begin{aligned}\left[(x_a^k)_{a \in A} \right] &= \left[(x_0)_{a \in A} \right], \\ \left[(y_a^k)_{a \in A} \right] &= \left[(x_0)_{a \in A} \right],\end{aligned}$$

however, and $\xi_a(k)$ is defined to be x_0 in this case.

- 0263 (b) The map ξ_a is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since $\xi(k_0) = \left[(x_0)_{a \in A} \right]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the a th component of $\left[(x_0)_{a \in A} \right]$, is equal to x_0 .

- 0264 2. *Map II.* We define a map

$$\Psi_K: \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)) \rightarrow \mathbf{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned}\xi: A &\longrightarrow \mathbf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a: K \rightarrow X),\end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger: (K, k_0) \rightarrow (A \pitchfork X, \left[(x_0)_{a \in A} \right])$$

defined by

$$\xi^\dagger(k) \stackrel{\text{def}}{=} \left[(\xi_a(k))_{a \in A} \right]$$

for each $k \in K$. Note that ξ^\dagger is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\xi^\dagger(k_0) &\stackrel{\text{def}}{=} [(\xi_a(k_0))_{a \in A}] \\ &= x_0,\end{aligned}$$

where we have used that $\xi_a \in \mathbf{Sets}_*(K, X)$ is a morphism of pointed sets for each $a \in A$.

0265 3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \text{id}_{\mathbf{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \xi_a \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \xi_{a'} \rrbracket) \\ &= \llbracket k \mapsto [(\text{ev}_a(\llbracket a' \mapsto \xi_{a'}(k) \rrbracket))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket.\end{aligned}$$

Now, we have two cases:

0266 (a) If $\xi(k) = [(x_0)_{a \in A}]$, we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(x_0)_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto \xi(k) \rrbracket \\ &= \xi.\end{aligned}$$

0267 (b) If $\xi(k) \neq [(x_0)_{a \in A}]$ and $\xi(k) = [(x_a^k)_{a \in A}]$ instead, we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(x_a^k)_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto \xi(k) \rrbracket \\ &= \xi.\end{aligned}$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

0268 4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \text{id}_{\mathbf{Sets}(A, \mathbf{Sets}_*(K, X))}.$$

Indeed, given a morphism $\xi: A \rightarrow \mathbf{Sets}_*(K, X)$, we have

$$\begin{aligned} [\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket) \\ &= \llbracket a \mapsto \llbracket k \mapsto \xi_a(k) \rrbracket \rrbracket \\ &= \xi \end{aligned}$$

0269 5. *Naturality of Ψ .* We need to show that, given a morphism of pointed sets

$$\phi: (K, k_0) \rightarrow (K', k'_0),$$

the diagram

$$\begin{array}{ccc} \mathbf{Sets}(A, \mathbf{Sets}_*(K', X)) & \xrightarrow{\Psi_{K'}} & \mathbf{Sets}_*(K', A \curvearrowright X) \\ (\phi^*)_* \downarrow & & \downarrow \phi^* \\ \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)) & \xrightarrow{\Psi_K} & \mathbf{Sets}_*(K, A \curvearrowright X) \end{array}$$

commutes. Indeed, given a map of sets

$$\begin{aligned} \xi: A &\rightarrow \mathbf{Sets}_*(K', X), \\ a &\mapsto (\xi_a: K' \rightarrow X), \end{aligned}$$

we have

$$\begin{aligned} [\Psi_K \circ (\phi^*)_*](\xi) &= \Psi_K((\phi^*)_*(\xi)) \\ &= \Psi_K((\phi^*)_*(\llbracket a \mapsto \xi_a \rrbracket)) \\ &= \Psi_K(\llbracket a \mapsto \phi^*(\xi_a) \rrbracket) \\ &= \Psi_K(\llbracket a \mapsto \llbracket k \mapsto \xi_a(\phi(k)) \rrbracket \rrbracket) \\ &= \llbracket k \mapsto [(\xi_a(\phi(k)))_{a \in A}] \rrbracket \\ &= \phi^*(\llbracket k' \mapsto [(\xi_a(k'))_{a \in A}] \rrbracket) \\ &= \phi^*(\Psi_{K'}(\xi)) \\ &= [\phi^* \circ \Psi_{K'}](\xi). \end{aligned}$$

- 026A 6. *Naturality of Φ* . Since Ψ is natural and Ψ is a componentwise inverse to Φ , it follows from **Categories**, **Item 2** of **Definition 11.9.7.1.2** that Φ is also natural.

This finishes the proof. \square

00D1 **Proposition 7.2.2.1.4.** Let (X, x_0) be a pointed set and let A be a set.

- 00D2 1. *Functoriality*. The assignments $A, (X, x_0), (A, (X, x_0))$ define functors

$$\begin{aligned} A \pitchfork - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \pitchfork X &: \mathbf{Sets}^{\text{op}} \rightarrow \mathbf{Sets}_*, \\ -_1 \pitchfork -_2 &: \mathbf{Sets}^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given:

- A map of sets $f: A \rightarrow B$;
- A pointed map $\phi: (X, x_0) \rightarrow (Y, y_0)$;

the induced map

$$f \odot \phi: A \pitchfork X \rightarrow B \pitchfork Y$$

is given by

$$[f \odot \phi]\left(\left[(x_a)_{a \in A}\right]\right) \stackrel{\text{def}}{=} \left[\left(\phi\left(x_{f(a)}\right)\right)_{a \in A}\right]$$

for each $\left[(x_a)_{a \in A}\right] \in A \pitchfork X$.

- 00D3 2. *Adjointness I*. We have an adjunction

$$(- \pitchfork X \dashv \mathbf{Sets}_*(-, X)) : \mathbf{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{- \pitchfork X} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(-, X)} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection

$$\mathbf{Sets}_*^{\text{op}}(A \pitchfork X, K) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

i.e. by a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in $A \in \text{Obj}(\mathbf{Sets})$ and $X, Y \in \text{Obj}(\mathbf{Sets}_*)$.

00D4 3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -): \text{Sets}_* \begin{array}{c} \xrightarrow{A \odot -} \\ \perp \\ \xleftarrow{A \pitchfork -} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets}_*}(A \odot X, Y) \cong \text{Hom}_{\text{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \text{Obj}(\text{Sets})$ and $X, Y \in \text{Obj}(\text{Sets}_*)$.

00D5 4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A: \text{pt} \rightarrow \text{Sets}$ picking $A \in \text{Obj}(\text{Sets})$;
- X for the functor $X: \text{pt} \rightarrow \text{Sets}_*$ picking $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.

00D6 5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in $A, B \in \text{Obj}(\text{Sets})$ and $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.

00D7 6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \text{Sets}_*(X, -) &\cong \text{Sets}_*(A \odot X, -), \\ A \pitchfork \text{Sets}_*(-, Y) &\cong \text{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

00D8 7. *The Cotensor Evaluation Map.* For each $X, Y \in \text{Obj}(\text{Sets}_*)$, we have a map

$$\text{ev}_{X,Y}^{\pitchfork}: X \rightarrow \text{Sets}_*(X, Y) \pitchfork Y,$$

natural in $X, Y \in \text{Obj}(\text{Sets}_*)$, and given by

$$\text{ev}_{X,Y}^{\pitchfork}(x) \stackrel{\text{def}}{=} [(f(x))_{f \in \text{Sets}_*(X,Y)}]$$

for each $x \in X$.

- 00D9 8. *The Cotensor Coevaluation Map.* For each $X \in \text{Obj}(\mathbf{Sets}_*)$ and each $A \in \text{Obj}(\mathbf{Sets})$, we have a map

$$\text{coev}_{A,X}^{\dashv}: A \rightarrow \mathbf{Sets}_*(A \dashv X, X),$$

natural in $X \in \text{Obj}(\mathbf{Sets}_*)$ and $A \in \text{Obj}(\mathbf{Sets})$, and given by

$$\text{coev}_{A,X}^{\dashv}(a) \stackrel{\text{def}}{=} \llbracket (x_b)_{b \in A} \rrbracket \mapsto x_a$$

for each $a \in A$.

Proof. Item 1, Functoriality: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 2, Adjointness I: This is simply a rephrasing of Definition 7.2.2.1.1.

Item 3, : Adjointness II: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 4, As a Weighted Limit: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 5, Iterated Cotensors: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 6, Commutativity With Homs: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 7, The Cotensor Evaluation Map: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$.

Item 8, The Cotensor Coevaluation Map: This is the special case of Limits and Colimits, ?? of ?? for $\mathcal{C} = \mathbf{Sets}_*$. \square

00DA 7.3 The Left Tensor Product of Pointed Sets

00DB 7.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

- 00DC **Definition 7.3.1.1.1.** The left tensor product of pointed sets is the functor¹¹

$$\triangleleft: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_* \times \mathbf{Sets}_* \xrightarrow{\text{id} \times \overline{\text{ev}}} \mathbf{Sets}_* \times \mathbf{Sets} \xrightarrow{\beta_{\mathbf{Sets}_*, \mathbf{Sets}}^{\text{Cats}_2}} \mathbf{Sets} \times \mathbf{Sets}_* \xrightarrow{\odot} \mathbf{Sets}_*,$$

where:

¹¹ *Further Notation:* Also written $\triangleleft_{\mathbf{Sets}_*}$.

- $\omega: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\beta_{\mathbf{Sets}_*, \mathbf{Sets}}^{\mathbf{Cats}_2}: \mathbf{Sets}_* \times \mathbf{Sets} \xrightarrow{\sim} \mathbf{Sets} \times \mathbf{Sets}_*$ is the braiding of \mathbf{Cats}_2 , i.e. the functor witnessing the isomorphism

$$\mathbf{Sets}_* \times \mathbf{Sets} \cong \mathbf{Sets} \times \mathbf{Sets}_*.$$

- $\odot: \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$ is the tensor functor of **Item 1** of **Definition 7.2.1.1.6**.

00DD Remark 7.3.1.1.2. The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}^{\otimes, L}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

- 026B** 1. Pointed maps $f: X \triangleleft Y \rightarrow Z$.
- 026C** 2. Maps of sets $f: X \times Y \rightarrow Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

00DE Remark 7.3.1.1.3. The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \triangleleft Y, x_0 \triangleleft y_0), \iota)$ consisting of
 - A pointed set $(X \triangleleft Y, x_0 \triangleleft y_0)$;
 - A left bilinear morphism of pointed sets $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \triangleleft Y$;

satisfying the following universal property:

- (★) Given another such pair $((Z, z_0), f)$ consisting of
 - * A pointed set (Z, z_0) ;
 - * A left bilinear morphism of pointed sets $f: (X \times Y, (x_0, y_0)) \rightarrow Z$;

there exists a unique morphism of pointed sets $X \triangleleft Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{array}{ccc} & X \triangleleft Y & \\ \iota \nearrow & & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

00DF Construction 7.3.1.1.4. In detail, the **left tensor product of** (X, x_0) and (Y, y_0) is the pointed set $(X \triangleleft Y, [x_0])$ consisting of:

- *The Underlying Set.* The set $X \triangleleft Y$ defined by

$$\begin{aligned} X \triangleleft Y &\stackrel{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0), \end{aligned}$$

where $|Y|$ denotes the underlying set of (Y, y_0) .

- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y \in Y} (X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

Proof. Since $\bigvee_{y \in Y} (X, x_0)$ is defined as the quotient of $\coprod_{y \in Y} X$ by the equivalence relation R generated by declaring $(y, x) \sim (y', x')$ if $x = x' = x_0$, we have, by **Conditions on Relations**, **??**, a natural bijection

$$\text{Sets}_*(X \triangleleft Y, Z) \cong \text{Hom}_{\text{Sets}}^R\left(\coprod_{y \in Y} X, Z\right),$$

where $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$ is the set

$$\text{Hom}_{\text{Sets}}^R\left(\coprod_{y \in Y} X, Z\right) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}\left(\coprod_{y \in Y} X, Z\right) \left| \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (y, x) \sim_R (y', x'), \text{ then} \\ f(y, x) = f(y', x') \end{array} \right. \right\}.$$

However, the condition $(y, x) \sim_R (y', x')$ only holds when:

- 026D** 1. We have $x = x'$ and $y = y'$.
- 026E** 2. We have $x = x' = x_0$.

So, given $f \in \text{Hom}_{\text{Sets}}(\coprod_{y \in Y} X, Z)$ with a corresponding $\bar{f}: X \triangleleft Y \rightarrow Z$, the latter case above implies

$$\begin{aligned} f([(y, x_0)]) &= f([(y', x_0)]) \\ &= f([(y_0, x_0)]), \end{aligned}$$

and since $\bar{f}: X \triangleleft Y \rightarrow Z$ is a pointed map, we have

$$\begin{aligned} f([(y_0, x_0)]) &= \bar{f}([(y_0, x_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$ are precisely those functions $f: X \times Y \rightarrow Z$ satisfying the equality

$$f(x_0, y) = z_0$$

for each $y \in Y$, giving an equality

$$\text{Hom}_{\text{Sets}}^R(X \times Y, Z) = \text{Hom}_{\text{Sets}_*}^{\otimes, L}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\text{Sets}_*(X \triangleleft Y, Z) \cong \text{Hom}_{\text{Sets}}^R(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. \square

00DG Notation 7.3.1.1.5. We write¹² $x \triangleleft y$ for the element $[(y, x)]$ of

$$X \triangleleft Y \cong |Y| \odot X.$$

00DH Remark 7.3.1.1.6. Employing the notation introduced in [Definition 7.3.1.1.5](#), we have

$$x_0 \triangleleft y_0 = x_0 \triangleleft y$$

for each $y \in Y$, and

$$x_0 \triangleleft y = x_0 \triangleleft y'$$

for each $y, y' \in Y$.

00DJ Proposition 7.3.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

00DK 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \triangleleft Y$ define functors

$$\begin{aligned} X \triangleleft -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ - \triangleleft Y: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ -_1 \triangleleft -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$f: (X, x_0) \rightarrow (A, a_0),$$

¹²*Further Notation:* Also written $x \triangleleft_{\text{Sets}_*} y$.

$$g: (Y, y_0) \rightarrow (B, b_0),$$

the induced map

$$f \triangleleft g: X \triangleleft Y \rightarrow A \triangleleft B$$

is given by

$$[f \triangleleft g](x \triangleleft y) \stackrel{\text{def}}{=} f(x) \triangleleft g(y)$$

for each $x \triangleleft y \in X \triangleleft Y$.

00DL 2. *Adjointness I.* We have an adjunction

$$\left(- \triangleleft Y \dashv [Y, -]_{\mathbf{Sets}_*}^{\triangleleft} \right): \quad \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleleft Y} \\ \perp \\ \xleftarrow{[Y, -]_{\mathbf{Sets}_*}^{\triangleleft}} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft})$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$, where $[X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$ is the pointed set of **Definition 7.3.2.1.1**.

00DM 3. *Adjointness II.* The functor

$$X \triangleleft -: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

00DN 4. *Adjointness III.* We have a $\overline{\mathbf{Set}}$ -relative adjunction

$$(X \triangleleft - \dashv \mathbf{Sets}_*(X, -)): \quad \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \triangleleft -} \\ \perp_{\overline{\mathbf{Set}}} \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}}(|Y|, \mathbf{Sets}_*(X, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: This follows from the definition of \triangleleft as a composition of functors (Definition 7.3.1.1.1).

Item 2, Adjointness I: This follows from Item 3 of Definition 7.2.1.1.6.

Item 3, Adjointness II: For $X \triangleleft -$ to admit a right adjoint would require it to preserve colimits by ??, ?? of ??. However, we have

$$\begin{aligned} X \triangleleft \text{pt} &\stackrel{\text{def}}{=} |\text{pt}| \odot X \\ &\cong X \\ &\not\cong \text{pt}, \end{aligned}$$

and thus we see that $X \triangleleft -$ does not have a right adjoint.

Item 4, Adjointness III: This follows from Item 2 of Definition 7.2.1.1.6. \square

00DP Remark 7.3.1.1.8. Here is some intuition on why $X \triangleleft -$ fails to be a left adjoint. Item 4 of Definition 7.3.1.1.7 states that we have a natural bijection

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}}(|Y|, \mathbf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give $X \triangleleft - \dashv \mathbf{Sets}_*(X, -)$. However, such a bijection would require every map

$$f: X \triangleleft Y \rightarrow Z$$

to satisfy

$$f(x \triangleleft y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \triangleleft y$. Thus $\mathbf{Sets}_*(X, -)$ can't be a right adjoint for $X \triangleleft -$, and as shown by Item 3 of Definition 7.3.1.1.7, no functor can.¹³

00DQ 7.3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

¹³The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed sets of Definition 7.5.1.1.1. See Item 2 of Definition 7.5.1.1.10.

00DR Definition 7.3.2.1.1. The left internal Hom¹⁴ of pointed sets is the functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \xrightarrow{\omega \times \text{id}} \mathbf{Sets}^{\text{op}} \times \mathbf{Sets}_* \xrightarrow{\pitchfork} \mathbf{Sets}_*,$$

where:

- $\omega : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork : \mathbf{Sets}^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$ is the cotensor functor of **Item 1** of **Definition 7.2.2.1.4**.

00DS Remark 7.3.2.1.2. The left internal Hom of pointed sets satisfies the following universal property:

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \mathbf{Sets}_*(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft})$$

That is to say, the following data are in bijection:

- 026F** 1. Pointed maps $f : X \triangleleft Y \rightarrow Z$.
- 026G** 2. Pointed maps $f : X \rightarrow [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft}$.

00DT Remark 7.3.2.1.3. In detail, the left internal Hom of (X, x_0) and (Y, y_0) is the pointed set $\left([X, Y]_{\mathbf{Sets}_*}^{\triangleleft}, [(y_0)_{x \in X}]\right)$ consisting of:

- *The Underlying Set.* The set $[X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$ defined by

$$\begin{aligned} [X, Y]_{\mathbf{Sets}_*}^{\triangleleft} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0), \end{aligned}$$

where $|X|$ denotes the underlying set of (X, x_0) .

- *The Underlying Basepoint.* The point $[(y_0)_{x \in X}]$ of $\bigwedge_{x \in X} (Y, y_0)$.

00DU Proposition 7.3.2.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

¹⁴For a proof that $[-, -]_{\mathbf{Sets}_*}^{\triangleleft}$ is indeed the left internal Hom of \mathbf{Sets}_* with respect to the left tensor product of pointed sets, see **Item 2** of **Definition 7.3.1.1.7**.

- 00DV 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]_{\mathbf{Sets}_*}^\triangleleft$ define functors

$$\begin{aligned} [X, -]_{\mathbf{Sets}_*}^\triangleleft &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ [-, Y]_{\mathbf{Sets}_*}^\triangleleft &: \mathbf{Sets}_*^{\text{op}} \rightarrow \mathbf{Sets}_*, \\ [-1, -2]_{\mathbf{Sets}_*}^\triangleleft &: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the induced map

$$[f, g]_{\mathbf{Sets}_*}^\triangleleft : [A, Y]_{\mathbf{Sets}_*}^\triangleleft \rightarrow [X, B]_{\mathbf{Sets}_*}^\triangleleft$$

is given by

$$[f, g]_{\mathbf{Sets}_*}^\triangleleft \left(\left[(y_a)_{a \in A} \right] \right) \stackrel{\text{def}}{=} \left[\left(g(y_{f(x)}) \right)_{x \in X} \right]$$

for each $\left[(y_a)_{a \in A} \right] \in [A, Y]_{\mathbf{Sets}_*}^\triangleleft$.

- 00DW 2. *Adjointness I.* We have an adjunction

$$\left(- \triangleleft Y \dashv [Y, -]_{\mathbf{Sets}_*}^\triangleleft \right) : \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleleft Y} \\ \perp \\ \xleftarrow{[Y, -]_{\mathbf{Sets}_*}^\triangleleft} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, [Y, Z]_{\mathbf{Sets}_*}^\triangleleft)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$

- 00DX 3. *Adjointness II.* The functor

$$X \triangleleft - : \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1, Functoriality:* This follows from the definition of $[-, -]_{\mathbf{Sets}_*}^\triangleleft$ as a composition of functors (Definition 7.3.2.1.1).

Item 2, Adjointness I: This is a repetition of Item 2 of Definition 7.3.1.1.7, and is proved there.

Item 3, Adjointness II: This is a repetition of Item 3 of Definition 7.3.1.1.7, and is proved there. \square

00DY 7.3.3 The Left Skew Unit

00DZ **Definition 7.3.3.1.1.** The left skew unit of the left tensor product of pointed sets is the functor

$$\mathbb{1}^{\text{Sets}_*, \triangleleft}: \text{pt} \rightarrow \text{Sets}_*$$

defined by

$$\mathbb{1}_{\text{Sets}_*}^{\triangleleft} \stackrel{\text{def}}{=} S^0.$$

00E0 7.3.4 The Left Skew Associator

00E1 **Definition 7.3.4.1.1.** The skew associator of the left tensor product of pointed sets is the natural transformation

$$\alpha^{\text{Sets}_*, \triangleleft}: \triangleleft \circ (\triangleleft \times \text{id}_{\text{Sets}_*}) \Longrightarrow \triangleleft \circ (\text{id}_{\text{Sets}_*} \times \triangleleft) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}}$$

as in the diagram

$$\begin{array}{ccc}
 & \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & \\
 \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}} \swarrow \text{dashed} & & \searrow \text{id} \times \triangleleft \\
 (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & \xrightarrow{\alpha^{\text{Sets}_*, \triangleleft}} & \text{Sets}_* \times \text{Sets}_* \\
 \triangleleft \times \text{id} \searrow & & \swarrow \triangleleft \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\triangleleft} & \text{Sets}_*
 \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}: (X \triangleleft Y) \triangleleft Z \rightarrow X \triangleleft (Y \triangleleft Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ is given by

$$\begin{aligned}
 (X \triangleleft Y) \triangleleft Z &\stackrel{\text{def}}{=} |Z| \odot (X \triangleleft Y) \\
 &\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
 &\cong \bigvee_{z \in Z} |Y| \odot X \\
 &\cong \bigvee_{z \in Z} \left(\bigvee_{y \in Y} X \right)
 \end{aligned}$$

$$\begin{aligned}
& \rightarrow \bigvee_{[(z,y)] \in \bigvee_{z \in Z} Y} X \\
& \cong \bigvee_{[(z,y)] \in |Z| \odot Y} X \\
& \cong ||Z| \odot Y| \odot X \\
& \stackrel{\text{def}}{=} |Y \triangleleft Z| \odot X \\
& \stackrel{\text{def}}{=} X \triangleleft (Y \triangleleft Z),
\end{aligned}$$

where the map

$$\bigvee_{z \in Z} \left(\bigvee_{y \in Y} X \right) \rightarrow \bigvee_{(z,y) \in \bigvee_{z \in Z} Y} X$$

is given by $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$.

Proof. (Proven below in a bit.) □

00E2 Remark 7.3.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}
[(z, [(y, x)])] & \stackrel{\text{def}}{=} [(z, x \triangleleft y)] \\
& \stackrel{\text{def}}{=} (x \triangleleft y) \triangleleft z
\end{aligned}$$

and

$$\begin{aligned}
[([(z, y)], x)] & \stackrel{\text{def}}{=} [(y \triangleleft z, x)] \\
& \stackrel{\text{def}}{=} x \triangleleft (y \triangleleft z).
\end{aligned}$$

So, in other words, $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$ acts on elements via

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}((x \triangleleft y) \triangleleft z) \stackrel{\text{def}}{=} x \triangleleft (y \triangleleft z)$$

for each $(x \triangleleft y) \triangleleft z \in (X \triangleleft Y) \triangleleft Z$.

00E3 Remark 7.3.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$ acts on elements as

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}((x \triangleleft y_0) \triangleleft z) \stackrel{\text{def}}{=} x \triangleleft (y_0 \triangleleft z).$$

However, by the definition of \triangleleft , we have $y_0 \triangleleft z = y_0 \triangleleft z'$ for all $z, z' \in Z$, preventing $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$, the map

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleleft}: (X \triangleleft Y) \triangleleft Z \rightarrow X \triangleleft (Y \triangleleft Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleleft}((x_0 \triangleleft y_0) \triangleleft z_0) = x_0 \triangleleft (y_0 \triangleleft z_0).$$

Next, we claim that $\alpha^{\mathbf{Sets}_*, \triangleleft}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} (X \triangleleft Y) \triangleleft Z & \xrightarrow{(f \triangleleft g) \triangleleft h} & (X' \triangleleft Y') \triangleleft Z' \\ \alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleleft} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\mathbf{Sets}_*, \triangleleft} \\ X \triangleleft (Y \triangleleft Z) & \xrightarrow{f \triangleleft (g \triangleleft h)} & X' \triangleleft (Y' \triangleleft Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x \triangleleft y) \triangleleft z & \longmapsto & (f(x) \triangleleft g(y)) \triangleleft h(z) \\ \downarrow & & \downarrow \\ x \triangleleft (y \triangleleft z) & \longmapsto & f(x) \triangleleft (g(y) \triangleleft h(z)) \end{array}$$

and hence indeed commutes, showing $\alpha^{\mathbf{Sets}_*, \triangleleft}$ to be a natural transformation. This finishes the proof. \square

00E4 7.3.5 The Left Skew Left Unitor

00E5 **Definition 7.3.5.1.1.** The skew left unitor of the left tensor product of pointed sets is the natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleleft} : \triangleleft \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

whose component

$$\lambda_X^{\mathbf{Sets}_*, \triangleleft} : S^0 \triangleleft X \rightarrow X$$

at $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ is given by the composition

$$\begin{aligned} S^0 \triangleleft X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X, \end{aligned}$$

where $\bigvee_{x \in X} S^0 \rightarrow X$ is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. (Proven below in a bit.) □

00E6 **Remark 7.3.5.1.2.** In other words, $\lambda_X^{\mathbf{Sets}_*, \triangleleft}$ acts on elements as

$$\begin{aligned} \lambda_X^{\mathbf{Sets}_*, \triangleleft}(0 \triangleleft x) &\stackrel{\text{def}}{=} x_0, \\ \lambda_X^{\mathbf{Sets}_*, \triangleleft}(1 \triangleleft x) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each $1 \triangleleft x \in S^0 \triangleleft X$.

00E7 Remark 7.3.5.1.3. The morphism $\lambda_X^{\mathbf{Sets}_*, \triangleleft}$ is almost invertible, with its would-be-inverse

$$\phi_X: X \rightarrow S^0 \triangleleft X$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} 1 \triangleleft x$$

for each $x \in X$. Indeed, we have

$$\begin{aligned} [\lambda_X^{\mathbf{Sets}_*, \triangleleft} \circ \phi](x) &= \lambda_X^{\mathbf{Sets}_*, \triangleleft}(\phi(x)) \\ &= \lambda_X^{\mathbf{Sets}_*, \triangleleft}(1 \triangleleft x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

so that

$$\lambda_X^{\mathbf{Sets}_*, \triangleleft} \circ \phi = \text{id}_X$$

and

$$\begin{aligned} [\phi \circ \lambda_X^{\mathbf{Sets}_*, \triangleleft}](1 \triangleleft x) &= \phi(\lambda_X^{\mathbf{Sets}_*, \triangleleft}(1 \triangleleft x)) \\ &= \phi(x) \\ &= 1 \triangleleft x \\ &= [\text{id}_{S^0 \triangleleft X}](1 \triangleleft x), \end{aligned}$$

but

$$\begin{aligned} [\phi \circ \lambda_X^{\mathbf{Sets}_*, \triangleleft}](0 \triangleleft x) &= \phi(\lambda_X^{\mathbf{Sets}_*, \triangleleft}(0 \triangleleft x)) \\ &= \phi(x_0) \\ &= 1 \triangleleft x_0, \end{aligned}$$

where $0 \triangleleft x \neq 1 \triangleleft x_0$. Thus

$$\phi \circ \lambda_X^{\mathbf{Sets}_*, \triangleleft} \stackrel{?}{=} \text{id}_{S^0 \triangleleft X}$$

holds for all elements in $S^0 \triangleleft X$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the map

$$\lambda_X^{\mathbf{Sets}_*, \triangleleft}: S^0 \triangleleft X \rightarrow X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\mathbf{Sets}_*, \triangleleft}(0 \triangleleft x_0) = x_0.$$

Next, we claim that $\lambda^{\mathbf{Sets}_*, \triangleleft}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} S^0 \triangleleft X & \xrightarrow{\text{id}_{S^0} \triangleleft f} & S^0 \triangleleft Y \\ \downarrow \lambda_X^{\mathbf{Sets}_*, \triangleleft} & & \downarrow \lambda_Y^{\mathbf{Sets}_*, \triangleleft} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} 0 \triangleleft x & & 0 \triangleleft x \mapsto 0 \triangleleft f(x) \\ \downarrow & & \downarrow \\ x_0 \mapsto & f(x_0) & y_0 \end{array}$$

and

$$\begin{array}{ccc} 1 \triangleleft x \mapsto & 1 \triangleleft f(x) \\ \downarrow & \downarrow \\ x \mapsto & f(x) \end{array}$$

and hence indeed commutes, showing $\lambda^{\mathbf{Sets}_*, \triangleleft}$ to be a natural transformation. This finishes the proof. \square

00E8 7.3.6 The Left Skew Right Unitor

00E9 **Definition 7.3.6.1.1.** The **skew right unitor** of the left tensor product of pointed sets is the natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleleft} : \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2} \xrightarrow{\sim} \triangleleft \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Sets}_*}),$$

whose component

$$\rho_X^{\mathbf{Sets}_*, \triangleleft} : X \rightarrow X \triangleleft S^0$$

at $(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\rightarrow X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \triangleleft S^0, \end{aligned}$$

where $X \rightarrow X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

00EA **Remark 7.3.6.1.2.** In other words, $\rho_X^{\mathbf{Sets}_*, \triangleleft}$ acts on elements as

$$\rho_X^{\mathbf{Sets}_*, \triangleleft}(x) \stackrel{\mathrm{def}}{=} [(1, x)]$$

i.e. by

$$\rho_X^{\mathbf{Sets}_*, \triangleleft}(x) \stackrel{\mathrm{def}}{=} x \triangleleft 1$$

for each $x \in X$.

00EB **Remark 7.3.6.1.3.** The morphism $\rho_X^{\mathbf{Sets}_*, \triangleleft}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \triangleleft 0$ of $X \triangleleft S^0$ with $x \neq x_0$ are outside the image of $\rho_X^{\mathbf{Sets}_*, \triangleleft}$, which sends x to $x \triangleleft 1$.

Proof. Firstly, note that, given $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the map

$$\rho_X^{\mathbf{Sets}_*, \triangleleft}: X \rightarrow X \triangleleft S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho_X^{\mathbf{Sets}_*, \triangleleft}(x_0) &= x_0 \triangleleft 1 \\ &= x_0 \triangleleft 0. \end{aligned}$$

Next, we claim that $\rho^{\mathbf{Sets}_*, \triangleleft}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho_X^{\mathbf{Sets}_*, \triangleleft} \downarrow & & \downarrow \rho_Y^{\mathbf{Sets}_*, \triangleleft} \\ X \triangleleft S^0 & \xrightarrow{f \triangleleft \text{id}_{S^0}} & Y \triangleleft S^0 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleleft 0 & \longmapsto & f(x) \triangleleft 0 \end{array}$$

and hence indeed commutes, showing $\rho^{\mathbf{Sets}_*, \triangleleft}$ to be a natural transformation. This finishes the proof. \square

00EC 7.3.7 The Diagonal

00ED **Definition 7.3.7.1.1.** The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleleft}: \text{id}_{\mathbf{Sets}_*} \Longrightarrow \triangleleft \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

whose component

$$\Delta_X^\triangleleft: (X, x_0) \rightarrow (X \triangleleft X, x_0 \triangleleft x_0)$$

at $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ is given by

$$\Delta_X^\triangleleft(x) \stackrel{\text{def}}{=} x \triangleleft x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta_X^\triangleleft(x_0) \stackrel{\text{def}}{=} x_0 \triangleleft x_0,$$

and thus Δ_X^\triangleleft is a morphism of pointed sets.

Naturality: We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\triangleleft \downarrow & & \downarrow \Delta_Y^\triangleleft \\ X \triangleleft X & \xrightarrow{f \triangleleft f} & Y \triangleleft Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleleft x & \longmapsto & f(x) \triangleleft f(x) \end{array}$$

and hence indeed commutes, showing Δ^\triangleleft to be natural. \square

00EE 7.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to \triangleleft

00EF **Proposition 7.3.8.1.1.** The category \mathbf{Sets}_* admits a left-closed left skew monoidal category structure consisting of:

- *The Underlying Category.* The category \mathbf{Sets}_* of pointed sets.

- *The Left Skew Monoidal Product.* The left tensor product functor

$$\triangleleft: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of Definition 7.3.1.1.1.

- *The Left Internal Skew Hom.* The left internal Hom functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft}: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of Definition 7.3.2.1.1.

- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleleft}: \mathbf{pt} \rightarrow \mathbf{Sets}_*$$

of Definition 7.3.3.1.1.

- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Sets}_*, \triangleleft}: \triangleleft \circ (\triangleleft \times \text{id}_{\mathbf{Sets}_*}) \Longrightarrow \triangleleft \circ (\text{id}_{\mathbf{Sets}_*} \times \triangleleft) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}}$$

of Definition 7.3.4.1.1.

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleleft}: \triangleleft \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

of Definition 7.3.5.1.1.

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleleft}: \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2} \xrightarrow{\sim} \triangleleft \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}_*})$$

of Definition 7.3.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0) , (X, x_0) , (Y, y_0) and (Z, z_0) be

pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \triangleleft (X \triangleleft Y)) \triangleleft Z & \\
 \alpha_{W,X,Y}^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Z \nearrow & & \searrow \alpha_{W,X \triangleleft Y,Z}^{\text{Sets}_*, \triangleleft} \\
 ((W \triangleleft X) \triangleleft Y) \triangleleft Z & & W \triangleleft ((X \triangleleft Y) \triangleleft Z) \\
 \alpha_{W \triangleleft X,Y,Z}^{\text{Sets}_*, \triangleleft} \searrow & & \nearrow \text{id}_W \triangleleft \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} \\
 (W \triangleleft X) \triangleleft (Y \triangleleft Z) & \xrightarrow{\alpha_{W,X,Y \triangleleft Z}^{\text{Sets}_*, \triangleleft}} & W \triangleleft (X \triangleleft (Y \triangleleft Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (w \triangleleft (x \triangleleft y)) \triangleleft z & \\
 \nwarrow & & \searrow \\
 ((w \triangleleft x) \triangleleft y) \triangleleft z & & w \triangleleft ((x \triangleleft y) \triangleleft z) \\
 \nwarrow & & \searrow \\
 (w \triangleleft x) \triangleleft (y \triangleleft z) & \mapsto & w \triangleleft (x \triangleleft (y \triangleleft z))
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

The Left Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets.

We have to show that the diagram

$$\begin{array}{ccc}
 (S^0 \triangleleft X) \triangleleft Y & \xrightarrow{\alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleleft}} & S^0 \triangleleft (X \triangleleft Y) \\
 \searrow \lambda_X^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Y & & \downarrow \lambda_{X \triangleleft Y}^{\text{Sets}_*, \triangleleft} \\
 & & X \triangleleft Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0 \triangleleft x) \triangleleft y & \mapsto & 0 \triangleleft (x \triangleleft y) \\
 \searrow & & \downarrow \\
 & & x_0 \triangleleft y = x_0 \triangleleft y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 (1 \triangleleft x) \triangleleft y & \mapsto & 1 \triangleleft (x \triangleleft y) \\
 \searrow & & \downarrow \\
 & & x \triangleleft y
 \end{array}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Left Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleleft Y & & \\
 \downarrow \rho_{X \triangleleft Y}^{\text{Sets}_*, \triangleleft} & \searrow \text{id}_X \triangleleft \rho_Y^{\text{Sets}_*, \triangleleft} & \\
 (X \triangleleft Y) \triangleleft S^0 & \xrightarrow{\alpha_{X, Y, S^0}^{\text{Sets}_*, \triangleleft}} & X \triangleleft (Y \triangleleft S^0)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleleft y & & \\
 \downarrow & \searrow & \\
 (x \triangleleft y) \triangleleft 1 & \mapsto & x \triangleleft (y \triangleleft 1)
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied. *The Left Skew Middle Triangle Identity:* Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleleft Y & \xlongequal{\quad} & X \triangleleft Y \\
 \downarrow \rho_X^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Y & & \uparrow \text{id}_A \triangleleft \lambda_Y^{\text{Sets}_*, \triangleleft} \\
 (X \triangleleft S^0) \triangleleft Y & \xrightarrow{\alpha_{X, S^0, Y}^{\text{Sets}_*, \triangleleft}} & X \triangleleft (S^0 \triangleleft Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleleft y & \xrightarrow{\quad} & x \triangleleft y \\
 \downarrow & & \uparrow \\
 (x \triangleleft 1) \triangleleft y & \xrightarrow{\quad} & x \triangleleft (1 \triangleleft y)
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity: We have to show that the diagram

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\rho_{S^0}^{\text{Sets}_*, \triangleleft}} & S^0 \triangleleft S^0 \\
 \searrow & & \downarrow \lambda_{S^0}^{\text{Sets}_*, \triangleleft} \\
 & & S^0
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 0 \triangleleft 1 \\
 \swarrow & & \downarrow \\
 & & 0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1 \triangleleft 1 \\
 \swarrow & & \downarrow \\
 & & 1
 \end{array}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

Left Skew Monoidal Left-Closedness: This follows from [Item 2](#) of [Definition 7.3.1.1.7](#). \square

00EG 7.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

00EH **Proposition 7.3.9.1.1.** The category of monoids on $(\mathbf{Sets}_*, \triangleleft, S^0)$ is isomorphic to the category of “monoids with left zero”¹⁵ and morphisms between them.

Proof. Monoids on $(\mathbf{Sets}_*, \triangleleft, S^0)$: A monoid on $(\mathbf{Sets}_*, \triangleleft, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A: A \triangleleft A \rightarrow A,$$

determining a left bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A: S^0 \rightarrow A$$

picking an element 1_A of A .

satisfying the following conditions:

- 026H 1. *Associativity.* The diagram

$$\begin{array}{ccc} & A \triangleleft (A \triangleleft A) & \\ \alpha_{A,A,A}^{\mathbf{Sets}_*, \triangleleft} \nearrow & & \searrow \text{id}_A \triangleleft \mu_A \\ (A \triangleleft A) \triangleleft A & & A \triangleleft A \\ \mu_A \triangleleft \text{id}_A \searrow & & \swarrow \mu_A \\ A \triangleleft A & \xrightarrow{\mu_A} & A \end{array}$$

¹⁵A monoid with left zero is defined similarly as the monoids with zero of ???. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

026J 2. *Left Unitality.* The diagram

$$\begin{array}{ccc}
 S^0 \triangleleft A & \xrightarrow{\eta_A \times \text{id}_A} & A \triangleleft A \\
 & \searrow \lambda_A^{\text{Sets}_*, \triangleleft} & \downarrow \mu_A \\
 & & A
 \end{array}$$

commutes.

026K 3. *Right Unitality.* The diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\rho_A^{\text{Sets}_*, \triangleleft}} & A \triangleleft S^0 \\
 \parallel & & \downarrow \text{id}_A \times \eta_A \\
 A & \xleftarrow{\mu_A} & A \triangleleft A
 \end{array}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

026L 1. *Associativity.* The associativity condition acts as

$$\begin{array}{ccc}
 & & a \triangleleft (b \triangleleft c) \\
 & \nearrow & \searrow \\
 (a \triangleleft b) \triangleleft c & & (a \triangleleft b) \triangleleft c \quad a \triangleleft bc \\
 \searrow & & \searrow \\
 ab \triangleleft c & \mapsto & (ab)c \\
 & & \searrow \\
 & & a(bc)
 \end{array}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

026M 2. *Left Unitality*. The left unitality condition acts:

026N (a) On $0 \triangleleft a$ as

$$\begin{array}{ccc} 0 \triangleleft a & & 0 \triangleleft a \mapsto 0_A \triangleleft a \\ & \searrow & \searrow \downarrow \\ & 0_A & 0_A a. \end{array}$$

026P (b) On $1 \triangleleft a$ as

$$\begin{array}{ccc} 1 \triangleleft a & & 1 \triangleleft a \mapsto 1_A \triangleleft a \\ & \searrow & \searrow \downarrow \\ & a & 1_A a. \end{array}$$

This gives

$$\begin{aligned} 1_A a &= a, \\ 0_A a &= 0_A \end{aligned}$$

for each $a \in A$.

026Q 3. *Right Unitality*. The right unitality condition acts as

$$\begin{array}{ccc} a & & a \mapsto a \triangleleft 1 \\ \downarrow & & \downarrow \\ a & & a 1_A \longleftarrow a \triangleleft 1_A \end{array}$$

This gives

$$a 1_A = a$$

for each $a \in A$.

Thus we see that monoids with respect to \triangleleft are exactly monoids with left zero.

for each $a \in A$.

Morphisms of Monoids on $(\mathbf{Sets}_, \triangleleft, S^0)$* : A morphism of monoids on $(\mathbf{Sets}_*, \triangleleft, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f: (A, 0_A) \rightarrow (B, 0_B)$$

satisfying the following conditions:

- 026R** 1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{array}{ccc} A \triangleleft A & \xrightarrow{f \triangleleft f} & B \triangleleft B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

- 026S** 2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_A} & A \\ & \searrow \eta_B & \downarrow f \\ & & B \end{array}$$

commutes.

These act on elements as

$$\begin{array}{ccc} a \triangleleft b & & a \triangleleft b \mapsto f(a) \triangleleft f(b) \\ \downarrow & & \downarrow \\ ab & \mapsto & f(ab) \end{array} \qquad \begin{array}{ccc} & & \\ & & \downarrow \\ & & f(a)f(b) \end{array}$$

and

$$\begin{array}{ccc} 0 & & 0 \mapsto 0_A \\ & \searrow & \downarrow \\ & & 0_B \end{array} \qquad \begin{array}{ccc} & & \\ & & \downarrow \\ & & f(0_A) \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1_A \\
 \searrow & & \downarrow \\
 & & f(1_A) \\
 & & \uparrow \\
 & & 1_B
 \end{array}$$

giving

$$\begin{aligned}
 f(ab) &= f(a)f(b), \\
 f(0_A) &= 0_B, \\
 f(1_A) &= 1_B,
 \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero.

Identities and Composition: Similarly, the identities and composition of $\mathbf{Mon}(\mathbf{Sets}_*, \triangleleft, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. \square

00EJ 7.4 The Right Tensor Product of Pointed Sets

00EK 7.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00EL **Definition 7.4.1.1.1.** The **right tensor product of pointed sets** is the functor¹⁶

$$\triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_* \times \mathbf{Sets}_* \xrightarrow{\text{忘} \times \text{id}} \mathbf{Sets} \times \mathbf{Sets}_* \xrightarrow{\odot} \mathbf{Sets}_*,$$

where:

- $\text{忘} : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$ is the tensor functor of **Item 1** of **Definition 7.2.1.1.6**.

¹⁶ *Further Notation:* Also written $\triangleright_{\mathbf{Sets}_*}$.

00EM Remark 7.4.1.1.2. The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathbf{Sets}_*(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}^{\otimes, \mathbf{R}}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

026T 1. Pointed maps $f: X \triangleright Y \rightarrow Z$.

026U 2. Maps of sets $f: X \times Y \rightarrow Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

00EN Remark 7.4.1.1.3. The right tensor product of pointed sets may be described as follows:

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \triangleright Y, x_0 \triangleright y_0), \iota)$ consisting of
 - A pointed set $(X \triangleright Y, x_0 \triangleright y_0)$;
 - A right bilinear morphism of pointed sets $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \triangleright Y$;

satisfying the following universal property:

- (★) Given another such pair $((Z, z_0), f)$ consisting of
- * A pointed set (Z, z_0) ;
 - * A right bilinear morphism of pointed sets $f: (X \times Y, (x_0, y_0)) \rightarrow Z$;

there exists a unique morphism of pointed sets $X \triangleright Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{array}{ccc} & & X \triangleright Y \\ & \nearrow \iota & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

00EP Construction 7.4.1.1.4. In detail, the **right tensor product of (X, x_0) and (Y, y_0)** is the pointed set $(X \triangleright Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \triangleright Y$ defined by

$$\begin{aligned} X \triangleright Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

where $|X|$ denotes the underlying set of (X, x_0) .

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x \in X} (Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

Proof. Since $\bigvee_{y \in Y} (X, x_0)$ is defined as the quotient of $\coprod_{x \in X} Y$ by the equivalence relation R generated by declaring $(x, y) \sim (x', y')$ if $y = y' = y_0$, we have, by **Conditions on Relations**, ??, a natural bijection

$$\mathbf{Sets}_*(X \triangleright Y, Z) \cong \text{Hom}_{\mathbf{Sets}}^R\left(\coprod_{x \in X} Y, Z\right),$$

where $\text{Hom}_{\mathbf{Sets}}^R(X \times Y, Z)$ is the set

$$\text{Hom}_{\mathbf{Sets}}^R\left(\coprod_{x \in X} Y, Z\right) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\mathbf{Sets}}\left(\coprod_{x \in X} Y, Z\right) \left| \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right. \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

- 026V 1. We have $x = x'$ and $y = y'$.
- 026W 2. We have $y = y' = y_0$.

So, given $f \in \text{Hom}_{\mathbf{Sets}}(\coprod_{x \in X} Y, Z)$ with a corresponding $\bar{f}: X \triangleright Y \rightarrow Z$, the latter case above implies

$$\begin{aligned} f([(x, y_0)]) &= f([(x', y_0)]) \\ &= f([(x_0, y_0)]), \end{aligned}$$

and since $\bar{f}: X \triangleright Y \rightarrow Z$ is a pointed map, we have

$$\begin{aligned} f([(x_0, y_0)]) &= \bar{f}([(x_0, y_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$ are precisely those functions $f: X \times Y \rightarrow Z$ satisfying the equality

$$f(x, y_0) = z_0$$

for each $y \in Y$, giving an equality

$$\text{Hom}_{\text{Sets}}^R(X \times Y, Z) = \text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\text{Sets}_*(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}}^R(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. \square

00EQ Notation 7.4.1.1.5. We write¹⁷ $x \triangleright y$ for the element $[(x, y)]$ of

$$X \triangleright Y \cong |X| \odot Y.$$

00ER Remark 7.4.1.1.6. Employing the notation introduced in [Definition 7.4.1.1.5](#), we have

$$x_0 \triangleright y_0 = x \triangleright y_0$$

for each $x \in X$, and

$$x \triangleright y_0 = x' \triangleright y_0$$

for each $x, x' \in X$.

00ES Proposition 7.4.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

00ET 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \triangleright Y$ define functors

$$\begin{aligned} X \triangleright - &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ - \triangleright Y &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ -_1 \triangleright -_2 &: \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \triangleright g: X \triangleright Y \rightarrow A \triangleright B$$

is given by

$$[f \triangleright g](x \triangleright y) \stackrel{\text{def}}{=} f(x) \triangleright g(y)$$

for each $x \triangleright y \in X \triangleright Y$.

¹⁷ *Further Notation:* Also written $x \triangleright_{\text{Sets}_*} y$.

00EU 2. *Adjointness I.* We have an adjunction

$$(X \triangleright - \dashv [X, -]_{\mathbf{Sets}_*}^\triangleright): \quad \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \triangleright -} \\ \perp \\ \xleftarrow{[X, -]_{\mathbf{Sets}_*}^\triangleright} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(Y, [X, Z]_{\mathbf{Sets}_*}^\triangleright)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$, where $[X, Y]_{\mathbf{Sets}_*}^\triangleright$ is the pointed set of [Definition 7.4.2.1.1](#).

00EV 3. *Adjointness II.* The functor

$$- \triangleright Y: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

00EW 4. *Adjointness III.* We have a $\overline{\mathbf{Set}}$ -relative adjunction

$$(- \triangleright Y \dashv \mathbf{Sets}_*(Y, -)): \quad \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleright Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|X|, \mathbf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: This follows from the definition of \triangleright as a composition of functors ([Definition 7.4.1.1.1](#)).

Item 2, Adjointness I: This follows from [Item 3](#) of [Definition 7.2.1.1.6](#).

Item 3, Adjointness II: For $- \triangleright Y$ to admit a right adjoint would require it to preserve colimits by [??](#), [??](#) of [??](#). However, we have

$$\begin{aligned} \mathrm{pt} \triangleright X &\stackrel{\mathrm{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\not\cong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \triangleright Y$ does not have a right adjoint.

Item 4, Adjointness III: This follows from [Item 2](#) of [Definition 7.2.1.1.6](#). \square

00EX Remark 7.4.1.1.8. Here is some intuition on why $- \triangleright Y$ fails to be a left adjoint. **Item 4** of **Definition 7.3.1.1.7** states that we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|X|, \mathbf{Sets}_*(Y, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give $- \triangleright Y \dashv \mathbf{Sets}_*(Y, -)$. However, such a bijection would require every map

$$f: X \triangleright Y \rightarrow Z$$

to satisfy

$$f(x_0 \triangleright y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \triangleright y_0$. Thus $\mathbf{Sets}_*(Y, -)$ can't be a right adjoint for $- \triangleright Y$, and as shown by **Item 3** of **Definition 7.4.1.1.7**, no functor can.¹⁸

00EY 7.4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

00EZ Definition 7.4.2.1.1. The **right internal Hom**¹⁹ of pointed sets is the functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\mathbf{f} \times \mathrm{id}} \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\pitchfork} \mathbf{Sets}_*,$$

where:

- $\mathbf{f}: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork: \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$ is the cotensor functor of **Item 1** of **Definition 7.2.2.1.4**.

¹⁸The functor $\mathbf{Sets}_*(Y, -)$ is instead right adjoint to $- \wedge Y$, the smash product of pointed sets of **Definition 7.5.1.1.1**. See **Item 2** of **Definition 7.5.1.1.10**.

¹⁹For a proof that $[-, -]_{\mathbf{Sets}_*}^{\triangleright}$ is indeed the right internal Hom of \mathbf{Sets}_* with respect to

00F0 Remark 7.4.2.1.2. We have

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} = [-, -]_{\mathbf{Sets}_*}^{\triangleright}.$$

00F1 Remark 7.4.2.1.3. The right internal Hom of pointed sets satisfies the following universal property:

$$\mathbf{Sets}_*(X \triangleright Y, Z) \cong \mathbf{Sets}_*(Y, [X, Z]_{\mathbf{Sets}_*}^{\triangleright})$$

That is to say, the following data are in bijection:

- 026X** 1. Pointed maps $f: X \triangleright Y \rightarrow Z$.
- 026Y** 2. Pointed maps $f: Y \rightarrow [X, Z]_{\mathbf{Sets}_*}^{\triangleright}$.

00F2 Remark 7.4.2.1.4. In detail, the **right internal Hom of** (X, x_0) and (Y, y_0) is the pointed set $\left([X, Y]_{\mathbf{Sets}_*}^{\triangleright}, [(y_0)_{x \in X}]\right)$ consisting of:

- *The Underlying Set.* The set $[X, Y]_{\mathbf{Sets}_*}^{\triangleright}$ defined by

$$\begin{aligned} [X, Y]_{\mathbf{Sets}_*}^{\triangleright} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0), \end{aligned}$$

where $|X|$ denotes the underlying set of (X, x_0) .

- *The Underlying Basepoint.* The point $[(y_0)_{x \in X}]$ of $\bigwedge_{x \in X} (Y, y_0)$.

00F3 Proposition 7.4.2.1.5. Let (X, x_0) and (Y, y_0) be pointed sets.

- 00F4** 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]_{\mathbf{Sets}_*}^{\triangleright}$ define functors

$$\begin{aligned} [X, -]_{\mathbf{Sets}_*}^{\triangleright} &: \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ [-, Y]_{\mathbf{Sets}_*}^{\triangleright} &: \mathbf{Sets}_*^{\text{op}} &\rightarrow \mathbf{Sets}_*, \\ [-_1, -_2]_{\mathbf{Sets}_*}^{\triangleright} &: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0), \end{aligned}$$

the right tensor product of pointed sets, see **Item 2** of **Definition 7.4.1.1.7**.

the induced map

$$[f, g]_{\mathbf{Sets}_*}^{\triangleright} : [A, Y]_{\mathbf{Sets}_*}^{\triangleright} \rightarrow [X, B]_{\mathbf{Sets}_*}^{\triangleright}$$

is given by

$$[f, g]_{\mathbf{Sets}_*}^{\triangleright} \left([(y_a)_{a \in A}] \right) \stackrel{\text{def}}{=} \left[\left(g(y_{f(x)}) \right)_{x \in X} \right]$$

for each $[(y_a)_{a \in A}] \in [A, Y]_{\mathbf{Sets}_*}^{\triangleright}$.

00F5 2. *Adjointness I.* We have an adjunction

$$\left(X \triangleright - \dashv [X, -]_{\mathbf{Sets}_*}^{\triangleright} \right) : \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \triangleright -} \\ \perp \\ \xleftarrow{[X, -]_{\mathbf{Sets}_*}^{\triangleright}} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(Y, [X, Z]_{\mathbf{Sets}_*}^{\triangleright})$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$, where $[X, Y]_{\mathbf{Sets}_*}^{\triangleright}$ is the pointed set of [Definition 7.4.2.1.1](#).

00F6 3. *Adjointness II.* The functor

$$- \triangleright Y : \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

Proof. Item 1, Functoriality: This follows from the definition of $[-, -]_{\mathbf{Sets}_*}^{\triangleright}$ as a composition of functors ([Definition 7.4.2.1.1](#)).

Item 2, Adjointness I: This is a repetition of [Item 2](#) of [Definition 7.4.1.1.7](#), and is proved there.

Item 3, Adjointness II: This is a repetition of [Item 3](#) of [Definition 7.4.1.1.7](#), and is proved there. \square

00F7 7.4.3 The Right Skew Unit

00F8 **Definition 7.4.3.1.1.** The right skew unit of the right tensor product of pointed sets is the functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleright} : \text{pt} \rightarrow \mathbf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathbf{Sets}_*}^{\triangleright} \stackrel{\text{def}}{=} S^0.$$

00F9 7.4.4 The Right Skew Associator

00FA **Definition 7.4.4.1.1.** The skew associator of the right tensor product of pointed sets is the natural transformation

$$\alpha^{\text{Sets}_*, \triangleright} : \triangleright \circ (\text{id}_{\text{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \text{id}_{\text{Sets}_*}) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}, -1}$$

as in the diagram

$$\begin{array}{ccccc}
 & & (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & \\
 & \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}, -1} \nearrow & & \searrow \triangleright \times \text{id} & \\
 \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & & & & \text{Sets}_* \times \text{Sets}_* \\
 \downarrow \text{id} \times \triangleright & \nearrow \alpha^{\text{Sets}_*, \triangleright} & & \searrow \triangleright & \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\triangleright} & \text{Sets}_* & &
 \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} : X \triangleright (Y \triangleright Z) \rightarrow (X \triangleright Y) \triangleright Z$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ is given by

$$\begin{aligned}
 X \triangleright (Y \triangleright Z) &\stackrel{\text{def}}{=} |X| \odot (Y \triangleright Z) \\
 &\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
 &\cong \bigvee_{x \in X} (|Y| \odot Z) \\
 &\cong \bigvee_{x \in X} \left(\bigvee_{y \in Y} Z \right) \\
 &\rightarrow \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\
 &\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\
 &\cong ||X| \odot Y| \odot Z \\
 &\stackrel{\text{def}}{=} |X \triangleright Y| \odot Z \\
 &\stackrel{\text{def}}{=} (X \triangleright Y) \triangleright Z,
 \end{aligned}$$

where the map

$$\bigvee_{x \in X} \left(\bigvee_{y \in Y} Z \right) \rightarrow \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z$$

is given by $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$.

Proof. (Proven below in a bit.) □

00FB Remark 7.4.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned} [(x, [(y, z)])] &\stackrel{\text{def}}{=} [(x, y \triangleright z)] \\ &\stackrel{\text{def}}{=} x \triangleright (y \triangleright z) \end{aligned}$$

and

$$\begin{aligned} [([(x, y)], z)] &\stackrel{\text{def}}{=} [(x \triangleright y, z)] \\ &\stackrel{\text{def}}{=} (x \triangleright y) \triangleright z. \end{aligned}$$

So, in other words, $\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright}$ acts on elements via

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright} (x \triangleright (y \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y) \triangleright z$$

for each $x \triangleright (y \triangleright z) \in X \triangleright (Y \triangleright Z)$.

00FC Remark 7.4.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright}$ acts on elements as

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright} (x \triangleright (y_0 \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y_0) \triangleright z.$$

However, by the definition of \triangleright , we have $x \triangleright y_0 = x' \triangleright y_0$ for all $x, x' \in X$, preventing $\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$, the map

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright} : X \triangleright (Y \triangleright Z) \rightarrow (X \triangleright Y) \triangleright Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, \triangleright} (x_0 \triangleright (y_0 \triangleright z_0)) = (x_0 \triangleright y_0) \triangleright z_0.$$

Next, we claim that $\alpha^{\mathbf{Sets}_*, \triangleright}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$f : (X, x_0) \rightarrow (X', x'_0),$$

$$\begin{aligned} g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} X \triangleright (Y \triangleright Z) & \xrightarrow{f \triangleright (g \triangleright h)} & X' \triangleright (Y' \triangleright Z') \\ \alpha_{X, Y, Z}^{\mathbf{Sets}_*, \triangleright} \downarrow & & \downarrow \alpha_{X', Y', Z'}^{\mathbf{Sets}_*, \triangleright} \\ (X \triangleright Y) \triangleright Z & \xrightarrow{(f \triangleright g) \triangleright h} & (X' \triangleright Y') \triangleright Z' \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \triangleright (y \triangleright z) & \longmapsto & f(x) \triangleright (g(y) \triangleright h(z)) \\ \downarrow & & \downarrow \\ (x \triangleright y) \triangleright z & \longmapsto & (f(x) \triangleright g(y)) \triangleright h(z) \end{array}$$

and hence indeed commutes, showing $\alpha^{\mathbf{Sets}_*, \triangleright}$ to be a natural transformation. This finishes the proof. \square

00FD 7.4.5 The Right Skew Left Unitor

00FE **Definition 7.4.5.1.1.** The skew left unitor of the right tensor product of pointed sets is the natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleright} : \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2} \xRightarrow{\sim} \triangleright \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*})$$

The diagram illustrates the naturality of the skew left unitor. It shows a square with vertices $\text{pt} \times \mathbf{Sets}_*$, $\mathbf{Sets}_* \times \mathbf{Sets}_*$, \mathbf{Sets}_* , and \mathbf{Sets}_* . The top arrow is $\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}$, the right arrow is \triangleright , the bottom arrow is $\lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$, and the diagonal arrow is $\lambda^{\mathbf{Sets}_*, \triangleright}$. A dashed curved arrow also points from $\text{pt} \times \mathbf{Sets}_*$ to \mathbf{Sets}_* .

whose component

$$\lambda_X^{\mathbf{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright X$$

at $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$ is given by the composition

$$X \rightarrow X \vee X$$

$$\begin{aligned} &\cong |S^0| \odot X \\ &\cong S^0 \triangleright X, \end{aligned}$$

where $X \rightarrow X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) \square

00FF Remark 7.4.5.1.2. In other words, $\lambda_X^{\mathbf{Sets}_*, \triangleright}$ acts on elements as

$$\lambda_X^{\mathbf{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\lambda_X^{\mathbf{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} 1 \triangleright x$$

for each $x \in X$.

00FG Remark 7.4.5.1.3. The morphism $\lambda_X^{\mathbf{Sets}_*, \triangleright}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0 \triangleright x$ of $S^0 \triangleright X$ with $x \neq x_0$ are outside the image of $\lambda_X^{\mathbf{Sets}_*, \triangleright}$, which sends x to $1 \triangleright x$.

Proof. Firstly, note that, given $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the map

$$\lambda_X^{\mathbf{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\mathbf{Sets}_*, \triangleright}(x_0) &= 1 \triangleright x_0 \\ &= 0 \triangleright x_0. \end{aligned}$$

Next, we claim that $\lambda^{\mathbf{Sets}_*, \triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \lambda_X^{\mathbf{Sets}_*, \triangleright} \downarrow & & \downarrow \lambda_Y^{\mathbf{Sets}_*, \triangleright} \\ S^0 \triangleright X & \xrightarrow{\text{id}_{S^0} \triangleright f} & S^0 \triangleright Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ 1 \triangleright x & \longmapsto & 1 \triangleright f(x) \end{array}$$

and hence indeed commutes, showing $\lambda^{\mathbf{Sets}_*, \triangleright}$ to be a natural transformation. This finishes the proof. \square

00FH 7.4.6 The Right Skew Right Unitor

00FJ **Definition 7.4.6.1.1.** The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xRightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

whose component

$$\rho_X^{\mathbf{Sets}_*, \triangleright} : X \triangleright S^0 \rightarrow X$$

at $(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ is given by the composition

$$\begin{aligned} X \triangleright S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X, \end{aligned}$$

where $\bigvee_{x \in X} S^0 \rightarrow X$ is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. (Proven below in a bit.) □

00FK Remark 7.4.6.1.2. In other words, $\rho_X^{\mathbf{Sets}_*, \triangleright}$ acts on elements as

$$\begin{aligned}\rho_X^{\mathbf{Sets}_*, \triangleright}(x \triangleright 0) &\stackrel{\text{def}}{=} x_0, \\ \rho_X^{\mathbf{Sets}_*, \triangleright}(x \triangleright 1) &\stackrel{\text{def}}{=} x\end{aligned}$$

for each $x \triangleright 1 \in X \triangleright S^0$.

00FL Remark 7.4.6.1.3. The morphism $\rho_X^{\mathbf{Sets}_*, \triangleright}$ is almost invertible, with its would-be-inverse

$$\phi_X: X \rightarrow X \triangleright S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \triangleright 1$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}[\rho_X^{\mathbf{Sets}_*, \triangleright} \circ \phi](x) &= \rho_X^{\mathbf{Sets}_*, \triangleright}(\phi(x)) \\ &= \rho_X^{\mathbf{Sets}_*, \triangleright}(x \triangleright 1) \\ &= x \\ &= [\text{id}_X](x)\end{aligned}$$

so that

$$\rho_X^{\mathbf{Sets}_*, \triangleright} \circ \phi = \text{id}_X$$

and

$$\begin{aligned}[\phi \circ \rho_X^{\mathbf{Sets}_*, \triangleright}](x \triangleright 1) &= \phi(\rho_X^{\mathbf{Sets}_*, \triangleright}(x \triangleright 1)) \\ &= \phi(x) \\ &= x \triangleright 1 \\ &= [\text{id}_{X \triangleright S^0}](x \triangleright 1),\end{aligned}$$

but

$$\begin{aligned}[\phi \circ \rho_X^{\mathbf{Sets}_*, \triangleright}](x \triangleright 0) &= \phi(\rho_X^{\mathbf{Sets}_*, \triangleright}(x \triangleright 0)) \\ &= \phi(x_0) \\ &= 1 \triangleright x_0,\end{aligned}$$

where $x \triangleright 0 \neq 1 \triangleright x_0$. Thus

$$\phi \circ \rho_X^{\mathbf{Sets}_*, \triangleright} \stackrel{?}{=} \text{id}_{X \triangleright S^0}$$

holds for all elements in $X \triangleright S^0$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$, the map

$$\rho_X^{\mathbf{Sets}_*, \triangleright} : X \triangleright S^0 \rightarrow X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\mathbf{Sets}_*, \triangleright}(x_0 \triangleright 0) = x_0.$$

Next, we claim that $\rho^{\mathbf{Sets}_*, \triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X \triangleright S^0 & \xrightarrow{f \triangleright \text{id}_{S^0}} & Y \triangleright S^0 \\ \rho_X^{\mathbf{Sets}_*, \triangleright} \downarrow & & \downarrow \rho_Y^{\mathbf{Sets}_*, \triangleright} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \triangleright 0 & & x \triangleright 0 \longmapsto f(x) \triangleright 0 \\ \downarrow & & \downarrow \\ x_0 \longmapsto f(x_0) & & y_0 \end{array}$$

and

$$\begin{array}{ccc} x \triangleright 1 \longmapsto f(x) \triangleright 1 & & \\ \downarrow & & \downarrow \\ x \longmapsto f(x) & & \end{array}$$

and hence indeed commutes, showing $\rho^{\mathbf{Sets}_*, \triangleright}$ to be a natural transformation. This finishes the proof. \square

00FM 7.4.7 The Diagonal

00FN **Definition 7.4.7.1.1.** The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^\triangleright : \text{id}_{\text{Sets}_*} \Longrightarrow \triangleright \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^\triangleright : (X, x_0) \rightarrow (X \triangleright X, x_0 \triangleright x_0)$$

at $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ is given by

$$\Delta_X^\triangleright(x) \stackrel{\text{def}}{=} x \triangleright x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta_X^\triangleright(x_0) \stackrel{\text{def}}{=} x_0 \triangleright x_0,$$

and thus Δ_X^\triangleright is a morphism of pointed sets.

Naturality: We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\triangleright \downarrow & & \downarrow \Delta_Y^\triangleright \\ X \triangleright X & \xrightarrow{f \triangleright f} & Y \triangleright Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleright x & \longmapsto & f(x) \triangleright f(x) \end{array}$$

and hence indeed commutes, showing Δ^\triangleright to be natural. \square

00FP 7.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to \triangleright

00FQ **Proposition 7.4.8.1.1.** The category \mathbf{Sets}_* admits a right-closed right skew monoidal category structure consisting of:

- *The Underlying Category.* The category \mathbf{Sets}_* of pointed sets.
- *The Right Skew Monoidal Product.* The right tensor product functor

$$\triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of [Definition 7.4.1.1.1](#).

- *The Right Internal Skew Hom.* The right internal Hom functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of [Definition 7.4.2.1.1](#).

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleright} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of [Definition 7.4.3.1.1](#).

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\text{id}_{\mathbf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \text{id}_{\mathbf{Sets}_*}) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}, -1}$$

of [Definition 7.4.4.1.1](#).

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleright} : \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2} \xrightarrow{\sim} \triangleright \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*})$$

of [Definition 7.4.5.1.1](#).

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xrightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

of [Definition 7.4.6.1.1](#).

7.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to \triangleright
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Proof. The Pentagon Identity: Let (W, w_0) , (X, x_0) , (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 & W \triangleright ((X \triangleright Y) \triangleright Z) & \\
 \alpha_{W, X, Y \triangleright Z}^{\text{Sets}_*, \triangleright} \nearrow & & \nwarrow \alpha_{W, X \triangleright Y, Z}^{\text{Sets}_*, \triangleright} \\
 W \triangleright (X \triangleright (Y \triangleright Z)) & & (W \triangleright (X \triangleright Y)) \triangleright Z \\
 \alpha_{W \triangleright X, Y, Z}^{\text{Sets}_*, \triangleright} \searrow & & \searrow \text{id}_W \triangleright \alpha_{X, Y, Z}^{\text{Sets}_*, \triangleright} \\
 (W \triangleright X) \triangleright (Y \triangleright Z) & \xrightarrow[\alpha_{W, X, Y \triangleright Z}^{\text{Sets}_*, \triangleright}]{} & ((W \triangleright X) \triangleright Y) \triangleright Z
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & w \triangleright ((x \triangleright y) \triangleright z) & \\
 \swarrow & & \searrow \\
 w \triangleright (x \triangleright (y \triangleright z)) & & (w \triangleright (x \triangleright y)) \triangleright z \\
 \searrow & & \searrow \\
 (w \triangleright x) \triangleright (y \triangleright z) & \longmapsto & ((w \triangleright x) \triangleright y) \triangleright z
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

The Right Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed

sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright Y & & \\
 \lambda_{X \triangleright Y}^{\text{Sets}_*, \triangleright} \downarrow & \searrow \lambda_X^{\text{Sets}_*, \triangleright} \triangleright \text{id}_Y & \\
 S^0 \triangleright (X \triangleright Y) & \xrightarrow{\alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleright}} & (S^0 \triangleright X) \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright y & & \\
 \downarrow & \searrow & \\
 1 \triangleright (x \triangleright y) & \mapsto & (1 \triangleright x) \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Right Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright (Y \triangleright S^0) & \xrightarrow{\text{id}_X \triangleright \rho_Y^{\text{Sets}_*, \triangleright}} & (X \triangleright Y) \triangleright S^0 \\
 & \searrow \alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleright} & \downarrow \rho_{X \triangleright Y}^{\text{Sets}_*, \triangleright} \\
 & & X \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright (y \triangleright 0) & \mapsto & (x \triangleright y) \triangleright 0 \\
 & \searrow & \downarrow \\
 & & x \triangleright y_0 = x_0 \triangleright y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 x \triangleright (y \triangleright 1) & \mapsto & (x \triangleright y) \triangleright 1 \\
 & \searrow & \downarrow \\
 & & x \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Right Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright Y & \xlongequal{\quad} & X \triangleright Y \\
 \text{id}_X \triangleright \lambda_Y^{\text{Sets}_*, \triangleright} \downarrow & & \uparrow \rho_X^{\text{Sets}_*, \triangleright} \triangleright \text{id}_Y \\
 X \triangleright (S^0 \triangleright Y) & \xrightarrow{\alpha_{X, S^0, Y}^{\text{Sets}_*, \triangleright}} & (X \triangleright S^0) \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright y & \xrightarrow{\quad} & x \triangleright y \\
 \downarrow & & \uparrow \\
 x \triangleright (1 \triangleright y) & \xrightarrow{\quad} & (x \triangleright 1) \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity: We have to show that the diagram

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\lambda_{S^0}^{\text{Sets}_*, \triangleright}} & S^0 \triangleright S^0 \\
 \searrow & & \downarrow \rho_{S^0}^{\text{Sets}_*, \triangleright} \\
 & & S^0
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 1 \triangleright 0 \\
 \swarrow & & \downarrow \\
 & & 0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1 \triangleright 1 \\
 \swarrow & & \downarrow \\
 & & 1
 \end{array}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

Right Skew Monoidal Right-Closedness: This follows from **Item 2** of **Definition 7.4.1.1.7**. \square

00FR 7.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

00FS Proposition 7.4.9.1.1. The category of monoids on $(\mathbf{Sets}_*, \triangleright, S^0)$ is isomorphic to the category of “monoids with right zero”²⁰ and morphisms between them.

Proof. Monoids on $(\mathbf{Sets}_*, \triangleright, S^0)$: A monoid on $(\mathbf{Sets}_*, \triangleright, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A: A \triangleright A \rightarrow A,$$

determining a right bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A: S^0 \rightarrow A$$

picking an element 1_A of A .

satisfying the following conditions:

- 026Z** 1. *Associativity.* The diagram

$$\begin{array}{ccc} & A \triangleright (A \triangleright A) & \\ \alpha_{A,A,A}^{\mathbf{Sets}_*, \triangleright} \nearrow & & \searrow \text{id}_A \triangleright \mu_A \\ (A \triangleright A) \triangleright A & & A \triangleright A \\ \mu_A \triangleright \text{id}_A \searrow & & \swarrow \mu_A \\ A \triangleright A & \xrightarrow{\mu_A} & A \end{array}$$

²⁰A monoid with right zero is defined similarly as the monoids with zero of ???. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

0270 2. *Left Unitality*. The diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\lambda_A^{\text{Sets}_*, \triangleright}} & S^0 \triangleright A \\
 \parallel & & \downarrow \eta_A \times \text{id}_A \\
 A & \xleftarrow{\mu_A} & A \triangleright A
 \end{array}$$

commutes.

0271 3. *Right Unitality*. The diagram

$$\begin{array}{ccc}
 A \triangleright S^0 & \xrightarrow{\text{id}_A \times \eta_A} & A \triangleright A \\
 \searrow \rho_A^{\text{Sets}_*, \triangleright} & & \downarrow \mu_A \\
 & & A
 \end{array}$$

commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

0272 1. *Associativity*. The associativity condition acts as

$$\begin{array}{ccc}
 & & a \triangleright (b \triangleright c) \\
 & \swarrow & \searrow \\
 (a \triangleright b) \triangleright c & & (a \triangleright b) \triangleright c \quad a \triangleright bc \\
 \searrow & & \swarrow \\
 ab \triangleright c & \longmapsto & (ab)c \quad a(bc)
 \end{array}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

0273 2. *Left Unitality.* The left unitality condition acts as

$$\begin{array}{ccc}
 a & \xrightarrow{\quad} & 1 \triangleright a \\
 \downarrow & & \downarrow \\
 a & \xleftarrow{\quad} & 1_A \triangleright a
 \end{array}$$

This gives

$$1_A a = a$$

for each $a \in A$.

0274 3. *Right Unitality.* The right unitality condition acts:

0275 (a) On $1 \triangleright 0$ as

$$\begin{array}{ccc}
 1 \triangleright 0 & \xrightarrow{\quad} & a \triangleright 0 \\
 \searrow & & \searrow \quad \downarrow \\
 0_A & & a 0_A
 \end{array}$$

0276 (b) On $a \triangleright 1$ as

$$\begin{array}{ccc}
 a \triangleright 1 & \xrightarrow{\quad} & a \triangleright 1_A \\
 \searrow & & \searrow \quad \downarrow \\
 a & & a 1_A
 \end{array}$$

This gives

$$a 1_A = a,$$

$$a 0_A = 0_A$$

for each $a \in A$.

Thus we see that monoids with respect to \triangleright are exactly monoids with right zero.

for each $a \in A$.

Morphisms of Monoids on $(\mathbf{Sets}_, \triangleright, S^0)$* : A morphism of monoids on $(\mathbf{Sets}_*, \triangleright, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f: (A, 0_A) \rightarrow (B, 0_B)$$

satisfying the following conditions:

- 0277** 1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{array}{ccc} A \triangleright A & \xrightarrow{f \triangleright f} & B \triangleright B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

- 0278** 2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_A} & A \\ & \searrow \eta_B & \downarrow f \\ & & B \end{array}$$

commutes.

These act on elements as

$$\begin{array}{ccc} a \triangleright b & & a \triangleright b \mapsto f(a) \triangleright f(b) \\ \downarrow & & \downarrow \\ ab & \mapsto & f(ab) \qquad f(a)f(b) \end{array}$$

and

$$\begin{array}{ccc} 0 & \mapsto & 0_A \\ & \searrow & \downarrow \\ & & 0_B \qquad f(0_A) \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1_A \\
 \searrow & & \downarrow \\
 & & f(1_A) \\
 \swarrow & & \\
 1_B & &
 \end{array}$$

giving

$$\begin{aligned}
 f(ab) &= f(a)f(b), \\
 f(0_A) &= 0_B, \\
 f(1_A) &= 1_B,
 \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero. *Identities and Composition:* Similarly, the identities and composition of $\mathbf{Mon}(\mathbf{Sets}_*, \triangleright, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. \square

00FT 7.5 The Smash Product of Pointed Sets

00FU 7.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00FV Definition 7.5.1.1.1. The **smash product** of (X, x_0) and (Y, y_0) ²¹ is the pointed set $X \wedge Y$ ²² satisfying the bijection

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Hom}_{\mathbf{Sets}_*}^{\otimes}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

00FW Remark 7.5.1.1.2. That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f: X \wedge Y \rightarrow Z$.

²¹ *Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the smash product $X \wedge Y$ is also called the **tensor product of \mathbb{F}_1 -modules of (X, x_0) and (Y, y_0)** or the **tensor product of (X, x_0) and (Y, y_0) over \mathbb{F}_1** .

²² *Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the smash product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

- Maps of sets $f: X \times Y \rightarrow Z$ satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

00FX Remark 7.5.1.1.3. The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of
 - A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
 - A bilinear morphism of pointed sets $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \wedge Y$;

satisfying the following universal property:

(\star) Given another such pair $((Z, z_0), f)$ consisting of

- * A pointed set (Z, z_0) ;
- * A bilinear morphism of pointed sets $f: (X \times Y, (x_0, y_0)) \rightarrow Z$;

there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{array}{ccc} & & X \wedge Y \\ & \nearrow \iota & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

00FY Construction 7.5.1.1.4. Concretely, the smash product of (X, x_0) and (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of:

- *The Underlying Set.* The set $X \wedge Y$ defined by

$$X \wedge Y \cong (X \times Y) / \sim_R,$$

where \sim_R is the equivalence relation on $X \times Y$ obtained by declaring

$$\begin{aligned}(x_0, y) &\sim_R (x_0, y'), \\ (x, y_0) &\sim_R (x', y_0)\end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation \sim on $X \times Y$.

Proof. By **Conditions on Relations**, ??, we have a natural bijection

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z),$$

where $\mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z)$ is the set

$$\mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Hom}_{\mathbf{Sets}}(X \times Y, Z) \left| \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right. \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

- 0279 1. We have $x = x'$ and $y = y'$.
- 027A 2. The following conditions are satisfied:
 - 027B (a) We have $x = x_0$ or $y = y_0$.
 - 027C (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathbf{Hom}_{\mathbf{Sets}}(X \times Y, Z)$ with a corresponding $\bar{f}: X \wedge Y \rightarrow Z$, the latter case above implies

$$\begin{aligned}f(x_0, y) &= f(x, y_0) \\ &= f(x_0, y_0),\end{aligned}$$

and since $\bar{f}: X \wedge Y \rightarrow Z$ is a pointed map, we have

$$\begin{aligned}f(x_0, y_0) &= \bar{f}(x_0, y_0) \\ &= z_0.\end{aligned}$$

Thus the elements f in $\mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z)$ are precisely those functions $f: X \times Y \rightarrow Z$ satisfying the equalities

$$f(x_0, y) = z_0,$$

$$f(x, y_0) = z_0$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathrm{Hom}_{\mathbf{Sets}}^R(X \times Y, Z) = \mathrm{Hom}_{\mathbf{Sets}_*}^{\otimes}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}^R(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. \square

00FZ Remark 7.5.1.1.5. It is also somewhat common to write

$$X \wedge Y \stackrel{\mathrm{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

01VX Construction 7.5.1.1.6. Alternatively, the smash product of (X, x_0) and (Y, y_0) may be constructed as the pointed set $X \wedge Y$ given by

$$\begin{aligned} X \wedge Y &\cong \bigvee_{x \in X^-} Y \\ &\cong \bigvee_{y \in Y^-} X. \end{aligned}$$

Proof. Indeed, since $X \cong \bigvee_{x \in X^-} S^0$, we have

$$\begin{aligned} X \wedge Y &\cong \left(\bigvee_{x \in X^-} S^0 \right) \wedge Y \\ &\cong \bigvee_{x \in X^-} S^0 \wedge Y \\ &\cong \bigvee_{x \in X^-} Y, \end{aligned}$$

where we have used that \wedge preserves colimits in both variables via ?? for the second isomorphism above, since it has right adjoints in both variables by [Item 2](#).

A similar proof applies to the isomorphism $X \wedge Y \cong \bigvee_{y \in Y^-} X$. \square

00G0 Notation 7.5.1.1.7. We write $x \wedge y$ for the element $[(x, y)]$ of $X \wedge Y \cong X \times Y / \sim$.

00G1 Remark 7.5.1.1.8. Employing the notation introduced in **Definition 7.5.1.1.7**, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

00G2 Example 7.5.1.1.9. Here are some examples of smash products of pointed sets.

00G3 1. *Smashing With pt.* For any pointed set X , we have isomorphisms of pointed sets

$$\begin{aligned} \text{pt} \wedge X &\cong \text{pt}, \\ X \wedge \text{pt} &\cong \text{pt}. \end{aligned}$$

00G4 2. *Smashing With S^0 .* For any pointed set X , we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

00G5 Proposition 7.5.1.1.10. Let (X, x_0) and (Y, y_0) be pointed sets.

00G6 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ - \wedge Y: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ -_1 \wedge -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$f: (X, x_0) \rightarrow (A, a_0),$$

$$g: (Y, y_0) \rightarrow (B, b_0),$$

the induced map

$$f \wedge g: X \wedge Y \rightarrow A \wedge B$$

is given by

$$[f \wedge g](x \wedge y) \stackrel{\text{def}}{=} f(x) \wedge g(y)$$

for each $x \wedge y \in X \wedge Y$.

00G7 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)), \\ \text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

00G8 3. *Enriched Adjointness.* We have \mathbf{Sets}_* -enriched adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

00G9 4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \text{pt} \coprod_{X \vee Y} (X \times Y),$$

$$\begin{array}{ccc} X \wedge Y & \leftarrow & X \times Y \\ \uparrow \ulcorner & & \uparrow \iota \\ \text{pt} & \xleftarrow{\quad} & X \vee Y \end{array}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets}_*)$, where the pushout is taken in \mathbf{Sets} , and the embedding $\iota: X \vee Y \hookrightarrow X \times Y$ is defined following [Definition 7.5.1.1.5](#).

00GA 5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned} X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z), \\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: The map $f \wedge g$ comes from [Conditions on Relations, Item 4](#) of [Definition 10.6.2.1.3](#) via the map

$$f \wedge g: X \times Y \rightarrow A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where \sim_R is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y) / \sim_R$$

in [Definition 7.5.1.1.4](#). The condition defining \sim is that at least one of the following conditions is satisfied:

027D 1. We have $x = x'$ and $y = y'$;

027E 2. Both of the following conditions are satisfied:

027F (a) We have $x = x_0$ or $y = y_0$.

027G (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

027H 1. In the first case, we clearly have

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

since $x = x'$ and $y = y'$.

027J 2. If $x = x_0$ and $x' = x_0$, we have

$$\begin{aligned} [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$

027K 3. If $x = x_0$ and $y' = y_0$, we have

$$\begin{aligned} [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x') \wedge g(y_0) \\ &\stackrel{\text{def}}{=} [f \wedge g](x', y_0). \end{aligned}$$

027L 4. If $y = y_0$ and $x' = x_0$, we have

$$\begin{aligned} [f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\ &= f(x) \wedge b_0 \\ &= a_0 \wedge b_0 \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$

027M 5. If $y = y_0$ and $y' = y_0$, we have

$$[f \wedge g](x, y_0) \stackrel{\text{def}}{=} f(x) \wedge g(y_0)$$

$$\begin{aligned}
&= f(x) \wedge b_0 \\
&= f(x') \wedge b_0 \\
&= f(x) \wedge g(y_0) \\
&\stackrel{\text{def}}{=} [f \wedge g](x', y_0).
\end{aligned}$$

Thus $f \wedge g$ is well-defined. Next, we claim that \wedge preserves identities and composition:

- *Preservation of Identities.* We have

$$\begin{aligned}
[\text{id}_X \wedge \text{id}_Y](x \wedge y) &\stackrel{\text{def}}{=} \text{id}_X(x) \wedge \text{id}_Y(y) \\
&= x \wedge y \\
&= [\text{id}_{X \wedge Y}](x \wedge y)
\end{aligned}$$

for each $x \wedge y \in X \wedge Y$, and thus

$$\text{id}_X \wedge \text{id}_Y = \text{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

$$\begin{aligned}
f &: (X, x_0) \rightarrow (X', x'_0), \\
h &: (X', x'_0) \rightarrow (X'', x''_0), \\
g &: (Y, y_0) \rightarrow (Y', y'_0), \\
k &: (Y', y'_0) \rightarrow (Y'', y''_0),
\end{aligned}$$

we have

$$\begin{aligned}
[(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\
&\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\
&\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\
&\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y)
\end{aligned}$$

for each $x \wedge y \in X \wedge Y$, and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.

Item 2, Adjointness: We prove only the adjunction $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$,

witnessed by a natural bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathbf{Sets}_*(X, -)$ is similar. We claim we have a bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}^{\otimes}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$, implying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathbf{Sets}(X \times Y, Z) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y, Z))$$

of **Constructions With Sets**, **Item 2** of **Definition 4.1.3.1.3**:

- A map

$$\xi: X \times Y \rightarrow Z$$

in $\mathrm{Hom}_{\mathbf{Sets}_*}^{\otimes}(X \times Y, Z)$ gets sent to the pointed map

$$\begin{aligned} \xi^\dagger: (X, x_0) &\rightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x^\dagger: Y \rightarrow Z), \end{aligned}$$

where $\xi_x^\dagger: Y \rightarrow Z$ is the map defined by

$$\xi_x^\dagger(y) \stackrel{\mathrm{def}}{=} \xi(x, y)$$

for each $y \in Y$, where:

- The map ξ^\dagger is indeed pointed, as we have

$$\begin{aligned} \xi_{x_0}^\dagger(y) &\stackrel{\mathrm{def}}{=} \xi(x_0, y) \\ &\stackrel{\mathrm{def}}{=} z_0 \end{aligned}$$

for each $y \in Y$. Thus $\xi_{x_0}^\dagger = \Delta_{z_0}$ and ξ^\dagger is pointed.

- The map ξ_x^\dagger indeed lies in $\mathbf{Sets}_*(Y, Z)$, as we have

$$\begin{aligned} \xi_x^\dagger(y_0) &\stackrel{\mathrm{def}}{=} \xi(x, y_0) \\ &\stackrel{\mathrm{def}}{=} z_0. \end{aligned}$$

- Conversely, a map

$$\begin{aligned}\xi &: (X, x_0) \rightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x: Y \rightarrow Z),\end{aligned}$$

in $\text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$ gets sent to the map

$$\xi^\dagger: X \times Y \rightarrow Z$$

defined by

$$\xi^\dagger(x, y) \stackrel{\text{def}}{=} \xi_x(y)$$

for each $(x, y) \in X \times Y$, which indeed lies in $\text{Hom}_{\mathbf{Sets}_*}^\otimes(X \times Y, Z)$, as:

- *Left Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x_0, y) &\stackrel{\text{def}}{=} \xi_{x_0}(y) \\ &\stackrel{\text{def}}{=} \Delta_{z_0}(y) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each $y \in Y$, since $\xi_{x_0} = \Delta_{z_0}$ as ξ is assumed to be a pointed map.

- *Right Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x, y_0) &\stackrel{\text{def}}{=} \xi_x(y_0) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each $x \in X$, since $\xi_x \in \mathbf{Sets}_*(Y, Z)$ is a morphism of pointed sets.

This finishes the proof.

Item 3, Enriched Adjointness: This follows from *Item 2* and Monoidal Categories, ?? of ??.

Item 4, As a Pushout: Following the description of *Constructions With Sets*, *Definition 4.2.4.1.3*, we have

$$\text{pt} \amalg_{X \vee Y} (X \times Y) \cong (\text{pt} \times (X \times Y)) / \sim,$$

where \sim identifies the element \star in pt with all elements of the form (x_0, y) and (x, y_0) in $X \times Y$. Thus *Conditions on Relations*, *Item 4* of *Definition 10.6.2.1.3* coupled with *Definition 7.5.1.1.8* then gives us a well-defined map

$$\text{pt} \amalg_{X \vee Y} (X \times Y) \rightarrow X \wedge Y$$

via $[(\star, (x, y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \rightarrow \text{pt} \coprod_{X \vee Y} (X \times Y)$$

given by $x \wedge y \mapsto [(\star, (x, y))]$.

Item 5, Distributivity Over Wedge Sums: This follows from [Definition 7.5.9.1.1](#), Monoidal Categories, ?? of ??, and the fact that \vee is the coproduct in **Sets**_{*} ([Pointed Sets, Definition 6.3.3.1.1](#)). \square

00GB 7.5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

00GC Definition 7.5.2.1.1. The **internal Hom**²³ of pointed sets from (X, x_0) to (Y, y_0) is the pointed set **Sets**_{*} $((X, x_0), (Y, y_0))$ ²⁴ consisting of:

- *The Underlying Set.* The set **Sets**_{*} $((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0) .
- *The Basepoint.* The element

$$\Delta_{y_0} : (X, x_0) \rightarrow (Y, y_0)$$

of **Sets**_{*} $((X, x_0), (Y, y_0))$ given by

$$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

for each $x \in X$.

00GD Proposition 7.5.2.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

00GE 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$ define functors

$$\begin{aligned} \mathbf{Sets}_*(X, -) : \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ \mathbf{Sets}_*(-, Y) : \mathbf{Sets}_*^{\text{op}} &\rightarrow \mathbf{Sets}_*, \\ \mathbf{Sets}_*(-_1, -_2) : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$f : (X, x_0) \rightarrow (A, a_0),$$

²³For a proof that **Sets**_{*} is indeed the internal Hom of **Sets**_{*} with respect to the smash product of pointed sets, see [Item 2 of Definition 7.5.1.1.10](#).

²⁴*Further Notation:* Also written **Hom**_{**Sets**_{*}} (X, Y) .

$$g: (Y, y_0) \rightarrow (B, b_0),$$

the induced map

$$\mathbf{Sets}_*(f, g): \mathbf{Sets}_*(A, Y) \rightarrow \mathbf{Sets}_*(X, B)$$

is given by

$$[\mathbf{Sets}_*(f, g)](\phi) \stackrel{\text{def}}{=} g \circ \phi \circ f$$

for each $\phi \in \mathbf{Sets}_*(A, Y)$.

00GF 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)), \\ \text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

00GG 3. *Enriched Adjointness.* We have \mathbf{Sets}_* -enriched adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) &: \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: This follows from **Constructions With Sets**, **Item 1** of **Definition 4.3.5.1.2** and from the equalities

$$\begin{aligned} g \circ \Delta_{y_0} &= \Delta_{z_0}, \\ \Delta_{y_0} \circ f &= \Delta_{y_0} \end{aligned}$$

for morphisms $f: (K, k_0) \rightarrow (X, x_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2, Adjointness: This is a repetition of **Item 2** of **Definition 7.5.1.1.10**, and is proved there.

Item 3, Enriched Adjointness: This is a repetition of **Item 3** of **Definition 7.5.1.1.10**, and is proved there. \square

00GH 7.5.3 The Monoidal Unit

00GJ Definition 7.5.3.1.1. The monoidal unit of the smash product of pointed sets is the functor

$$\mathbb{1}^{\mathbf{Sets}_*} : \mathbf{pt} \rightarrow \mathbf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathbf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

00GK 7.5.4 The Associator

00GL Definition 7.5.4.1.1. The associator of the smash product of pointed sets is the natural isomorphism

$$\alpha^{\mathbf{Sets}_*} : \wedge \circ (\wedge \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\text{id}_{\mathbf{Sets}_*} \times \wedge) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}},$$

as in the diagram

$$\begin{array}{ccc} & \mathbf{Sets}_* \times (\mathbf{Sets}_* \times \mathbf{Sets}_*) & \\ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}} \swarrow & \searrow \text{id} \times \wedge & \\ (\mathbf{Sets}_* \times \mathbf{Sets}_*) \times \mathbf{Sets}_* & \xrightarrow{\alpha^{\mathbf{Sets}_*}} & \mathbf{Sets}_* \times \mathbf{Sets}_* \\ \wedge \times \text{id} \searrow & & \searrow \wedge \\ \mathbf{Sets}_* \times \mathbf{Sets}_* & \xrightarrow{\wedge} & \mathbf{Sets}_* \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*}: (X \wedge Y) \wedge Z \xrightarrow{\sim} X \wedge (Y \wedge Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ is given by

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*}((x \wedge y) \wedge z) \stackrel{\text{def}}{=} x \wedge (y \wedge z)$$

for each $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$.

Proof. Well-Definedness: Let $[(x, y), z] = [(x', y'), z']$ be an element in $(X \wedge Y) \wedge Z$. Then either:

- 027N 1. We have $x = x'$, $y = y'$, and $z = z'$.
- 027P 2. Both of the following conditions are satisfied:
 - 027Q (a) We have $x = x_0$ or $y = y_0$ or $z = z_0$.
 - 027R (b) We have $x' = x_0$ or $y' = y_0$ or $z' = z_0$.

In the first case, $\alpha_{X,Y,Z}^{\mathbf{Sets}_*}$ clearly sends both elements to the same element in $X \wedge (Y \wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0 \wedge y_0) \wedge z_0$ of $(X \wedge Y) \wedge Z$, which gets sent to the basepoint $x_0 \wedge (y_0 \wedge z_0)$ of $X \wedge (Y \wedge Z)$.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*}((x_0 \wedge y_0) \wedge z_0) \stackrel{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus $\alpha_{X,Y,Z}^{\mathbf{Sets}_*}$ is a morphism of pointed sets.

Invertibility: The inverse of $\alpha_{X,Y,Z}^{\mathbf{Sets}_*}$ is given by the morphism

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, -1}: X \wedge (Y \wedge Z) \xrightarrow{\sim} (X \wedge Y) \wedge Z$$

defined by

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*, -1}(x \wedge (y \wedge z)) \stackrel{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.

Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 (X \wedge Y) \wedge Z & \xrightarrow{(f \wedge g) \wedge h} & (X' \wedge Y') \wedge Z' \\
 \alpha_{X,Y,Z}^{\mathbf{Sets}_*} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\mathbf{Sets}_*} \\
 X \wedge (Y \wedge Z) & \xrightarrow{f \wedge (g \wedge h)} & X' \wedge (Y' \wedge Z')
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x \wedge y) \wedge z & \longmapsto & (f(x) \wedge g(y)) \wedge h(z) \\
 \downarrow & & \downarrow \\
 x \wedge (y \wedge z) & \longmapsto & f(x) \wedge (g(y) \wedge h(z))
 \end{array}$$

and hence indeed commutes, showing $\alpha^{\mathbf{Sets}_*}$ to be a natural transformation. *Being a Natural Isomorphism:* Since $\alpha^{\mathbf{Sets}_*}$ is natural and $\alpha^{\mathbf{Sets}_*, -1}$ is a componentwise inverse to $\alpha^{\mathbf{Sets}_*}$, it follows from **Categories, Item 2 of Definition 11.9.7.1.2** that $\alpha^{\mathbf{Sets}_*, -1}$ is also natural. Thus $\alpha^{\mathbf{Sets}_*}$ is a natural isomorphism. \square

00GM 7.5.5 The Left Unitor

00GN **Definition 7.5.5.1.1.** The left unitor of the smash product of pointed sets is the natural isomorphism

$$\lambda^{\mathbf{Sets}_*} : \wedge \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

whose component

$$\lambda_X^{\mathbf{Sets}_*} : S^0 \wedge X \xrightarrow{\sim} X$$

at $X \in \text{Obj}(\mathbf{Sets}_*)$ is given by

$$0 \wedge x \mapsto x_0,$$

$$1 \wedge x \mapsto x$$

for each $x \in X$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $S^0 \wedge X$. Then either:

- 027S 1. We have $x = x'$ and $y = y'$.
- 027T 2. Both of the following conditions are satisfied:
 - 027U (a) We have $x = 0$ or $y = x_0$.
 - 027V (b) We have $x' = 0$ or $y' = x_0$.

In the first case, $\lambda_X^{\text{Sets}_*}$ clearly sends both elements to the same element in X . Meanwhile, in the latter case both elements are equal to the basepoint $0 \wedge x_0$ of $S^0 \wedge X$, which gets sent to the basepoint x_0 of X .

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\lambda_X^{\text{Sets}_*}(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus $\lambda_X^{\text{Sets}_*}$ is a morphism of pointed sets.

Invertibility: The inverse of $\lambda_X^{\text{Sets}_*}$ is the morphism

$$\lambda_X^{\text{Sets}_*, -1}: X \dashrightarrow S^0 \wedge X$$

defined by

$$\lambda_X^{\text{Sets}_*, -1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$. Indeed:

- 027W 1. *Invertibility I.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*}](0 \wedge x) &= \lambda_X^{\text{Sets}_*, -1}(\lambda_X^{\text{Sets}_*}(0 \wedge x)) \\ &= \lambda_X^{\text{Sets}_*, -1}(x_0) \\ &= 1 \wedge x_0 \\ &= 0 \wedge x, \end{aligned}$$

and

$$\begin{aligned} [\lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*}](1 \wedge x) &= \lambda_X^{\text{Sets}_*, -1}(\lambda_X^{\text{Sets}_*}(1 \wedge x)) \\ &= \lambda_X^{\text{Sets}_*, -1}(x) \end{aligned}$$

$$= 1 \wedge x$$

for each $x \in X$, and thus we have

$$\lambda_X^{\mathbf{Sets}_*, -1} \circ \lambda_X^{\mathbf{Sets}_*} = \text{id}_{S^0 \wedge X}.$$

027X 2. *Invertibility II.* We have

$$\begin{aligned} [\lambda_X^{\mathbf{Sets}_*} \circ \lambda_X^{\mathbf{Sets}_*, -1}](x) &= \lambda_X^{\mathbf{Sets}_*}(\lambda_X^{\mathbf{Sets}_*, -1}(x)) \\ &= \lambda_X^{\mathbf{Sets}_*, -1}(1 \wedge x) \\ &= x \end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda_X^{\mathbf{Sets}_*} \circ \lambda_X^{\mathbf{Sets}_*, -1} = \text{id}_X.$$

This shows $\lambda_X^{\mathbf{Sets}_*}$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} S^0 \wedge X & \xrightarrow{\text{id}_{S^0} \wedge f} & S^0 \wedge Y \\ \lambda_X^{\mathbf{Sets}_*} \downarrow & & \downarrow \lambda_Y^{\mathbf{Sets}_*} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} 0 \wedge x & & 0 \wedge x \mapsto 0 \wedge f(x) \\ \downarrow & & \downarrow \\ x_0 & \mapsto & f(x_0) \\ & & y_0 \end{array}$$

and

$$\begin{array}{ccc} 1 \wedge x & \mapsto & 1 \wedge f(x) \\ \downarrow & & \downarrow \\ x & \mapsto & f(x) \end{array}$$

and hence indeed commutes, showing $\lambda^{\mathbf{Sets}_*}$ to be a natural transformation. *Being a Natural Isomorphism:* Since $\lambda^{\mathbf{Sets}_*}$ is natural and $\lambda^{\mathbf{Sets}_*, -1}$ is a componentwise inverse to $\lambda^{\mathbf{Sets}_*}$, it follows from **Categories, Item 2 of Definition 11.9.7.1.2** that $\lambda^{\mathbf{Sets}_*, -1}$ is also natural. Thus $\lambda^{\mathbf{Sets}_*}$ is a natural isomorphism. \square

00GP 7.5.6 The Right Unitor

00GQ **Definition 7.5.6.1.1.** The **right unitor** of the smash product of pointed sets is the natural isomorphism

$$\rho^{\mathbf{Sets}_*} : \wedge \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xrightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

whose component

$$\rho_X^{\mathbf{Sets}_*} : X \wedge S^0 \xrightarrow{\sim} X$$

at $X \in \mathrm{Obj}(\mathbf{Sets}_*)$ is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge S^0$. Then either:

- 027Y 1. We have $x = x'$ and $y = y'$.
- 027Z 2. Both of the following conditions are satisfied:
 - 0280 (a) We have $x = x_0$ or $y = 0$.
 - 0281 (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho_X^{\text{Sets}*}$ clearly sends both elements to the same element in X . Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge 0$ of $X \wedge S^0$, which gets sent to the basepoint x_0 of X .

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\rho_X^{\text{Sets}*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus $\rho_X^{\text{Sets}*}$ is a morphism of pointed sets.

Invertibility: The inverse of $\rho_X^{\text{Sets}*}$ is the morphism

$$\rho_X^{\text{Sets*},-1}: X \xrightarrow{\sim} X \wedge S^0$$

defined by

$$\rho_X^{\text{Sets*},-1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

0282 1. *Invertibility I.* We have

$$\begin{aligned} [\rho_X^{\text{Sets*},-1} \circ \rho_X^{\text{Sets}*}](x \wedge 0) &= \rho_X^{\text{Sets*},-1}(\rho_X^{\text{Sets}*}(x \wedge 0)) \\ &= \rho_X^{\text{Sets*},-1}(x_0) \\ &= x_0 \wedge 1 \\ &= x \wedge 0, \end{aligned}$$

and

$$\begin{aligned} [\rho_X^{\text{Sets*},-1} \circ \rho_X^{\text{Sets}*}](x \wedge 1) &= \rho_X^{\text{Sets*},-1}(\rho_X^{\text{Sets}*}(x \wedge 1)) \\ &= \rho_X^{\text{Sets*},-1}(x) \\ &= x \wedge 1 \end{aligned}$$

for each $x \in X$, and thus we have

$$\rho_X^{\text{Sets*},-1} \circ \rho_X^{\text{Sets}*} = \text{id}_{X \wedge S^0}.$$

0283 2. *Invertibility II.* We have

$$\begin{aligned} [\rho_X^{\text{Sets}*} \circ \rho_X^{\text{Sets*},-1}](x) &= \rho_X^{\text{Sets}*}(\rho_X^{\text{Sets*},-1}(x)) \\ &= \rho_X^{\text{Sets}*}(x \wedge 1) \\ &= x \end{aligned}$$

for each $x \in X$, and thus we have

$$\rho_X^{\text{Sets}*} \circ \rho_X^{\text{Sets*},-1} = \text{id}_X.$$

This shows $\rho_X^{\mathbf{Sets}_*}$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X \wedge S^0 & \xrightarrow{f \wedge \text{id}_{S^0}} & Y \wedge S^0 \\ \downarrow \rho_X^{\mathbf{Sets}_*} & & \downarrow \rho_Y^{\mathbf{Sets}_*} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \wedge 0 & & x \wedge 0 \mapsto f(x) \wedge 0 \\ \downarrow & & \downarrow \\ x_0 & \mapsto & f(x_0) \\ & & y_0 \end{array}$$

and

$$\begin{array}{ccc} x \wedge 1 & \mapsto & f(x) \wedge 1 \\ \downarrow & & \downarrow \\ x & \mapsto & f(x) \end{array}$$

and hence indeed commutes, showing $\rho^{\mathbf{Sets}_*}$ to be a natural transformation.

Being a Natural Isomorphism: Since $\rho^{\mathbf{Sets}_*}$ is natural and $\rho^{\mathbf{Sets}_*, -1}$ is a componentwise inverse to $\rho^{\mathbf{Sets}_*}$, it follows from **Categories, Item 2 of Definition 11.9.7.1.2** that $\rho^{\mathbf{Sets}_*, -1}$ is also natural. Thus $\rho^{\mathbf{Sets}_*}$ is a natural isomorphism. \square

00GR 7.5.7 The Symmetry

00GS **Definition 7.5.7.1.1.** The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathbf{Sets}_*}: \wedge \xrightarrow{\sim} \wedge \circ \sigma_{\mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}_2}, \quad \begin{array}{ccc} \mathbf{Sets}_* \times \mathbf{Sets}_* & \xrightarrow{\wedge} & \mathbf{Sets}_* \\ \downarrow \sigma_{\mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}_2} & \Downarrow \sigma_{\mathbf{Sets}_*} & \uparrow \wedge \\ & \mathbf{Sets}_* \times \mathbf{Sets}_* & \end{array}$$

whose component

$$\sigma_{X,Y}^{\mathbf{Sets}_*} : X \wedge Y \xrightarrow{\sim} Y \wedge X$$

at $X, Y \in \text{Obj}(\mathbf{Sets}_*)$ is defined by

$$\sigma_{X,Y}^{\mathbf{Sets}_*}(x \wedge y) \stackrel{\text{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge Y$. Then either:

- 0284 1. We have $x = x'$ and $y = y'$.
- 0285 2. Both of the following conditions are satisfied:
 - 0286 (a) We have $x = x_0$ or $y = y_0$.
 - 0287 (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma_X^{\mathbf{Sets}_*}$ clearly sends both elements to the same element in X . Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\sigma_X^{\mathbf{Sets}_*}(x_0 \wedge y_0) \stackrel{\text{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma_X^{\mathbf{Sets}_*}$ is a morphism of pointed sets.

Invertibility: The inverse of $\sigma_{X,Y}^{\mathbf{Sets}_*}$ is given by the morphism

$$\sigma_{X,Y}^{\mathbf{Sets}_*, -1} : Y \wedge X \xrightarrow{\sim} X \wedge Y$$

defined by

$$\sigma_{X,Y}^{\mathbf{Sets}_*, -1}(y \wedge x) \stackrel{\text{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.

Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (A, a_0), \\ g &: (Y, y_0) \rightarrow (B, b_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{f \wedge g} & A \wedge B \\ \sigma_{X,Y}^{\mathbf{Sets}_*} \downarrow & & \downarrow \sigma_{A,B}^{\mathbf{Sets}_*} \\ Y \wedge X & \xrightarrow{g \wedge f} & B \wedge A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \wedge y & \longmapsto & f(x) \wedge g(y) \\ \downarrow & & \downarrow \\ y \wedge x & \longmapsto & g(y) \wedge f(x) \end{array}$$

and hence indeed commutes, showing $\sigma^{\mathbf{Sets}_*}$ to be a natural transformation. *Being a Natural Isomorphism:* Since $\sigma^{\mathbf{Sets}_*}$ is natural and $\sigma^{\mathbf{Sets}_*, -1}$ is a componentwise inverse to $\sigma^{\mathbf{Sets}_*}$, it follows from [Categories, Item 2 of Definition 11.9.7.1.2](#) that $\sigma^{\mathbf{Sets}_*, -1}$ is also natural. Thus $\sigma^{\mathbf{Sets}_*}$ is a natural isomorphism. \square

00GT 7.5.8 The Diagonal

00GU **Definition 7.5.8.1.1.** The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^\wedge : \mathrm{id}_{\mathbf{Sets}_*} \Rightarrow \wedge \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

whose component

$$\Delta_X^\wedge : (X, x_0) \rightarrow (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X^\wedge} (X \times X, (x_0, x_0)) \\ &\longrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\ &\stackrel{\mathrm{def}}{=} (X \wedge X, x_0 \wedge x_0) \end{aligned}$$

in \mathbf{Sets}_* , and thus by

$$\Delta_X^\wedge(x) \stackrel{\mathrm{def}}{=} x \wedge x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta_X^\wedge(x_0) \stackrel{\text{def}}{=} x_0 \wedge x_0,$$

and thus Δ_X^\wedge is a morphism of pointed sets.

Naturality: We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\wedge \downarrow & & \downarrow \Delta_Y^\wedge \\ X \wedge X & \xrightarrow{f \wedge f} & Y \wedge Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \wedge x & \longmapsto & f(x) \wedge f(x) \end{array}$$

and hence indeed commutes, showing Δ^\wedge to be natural. \square

00GV Proposition 7.5.8.1.2. Let $(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)$.

00GW 1. *Monoidality.* The diagonal

$$\Delta^\wedge: \text{id}_{\mathbf{Sets}_*} \Longrightarrow \wedge \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

of the smash product of pointed sets is a monoidal natural transformation:

00GX (a) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \text{Obj}(\mathbf{Sets}_*)$, the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\Delta_X^\wedge \wedge \Delta_Y^\wedge} & (X \wedge X) \wedge (Y \wedge Y) \\ & \searrow \Delta_{X \wedge Y}^\wedge & \downarrow \wr \\ & & (X \wedge Y) \wedge (X \wedge Y) \end{array}$$

commutes.

00GY

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\Delta_{S^0}^\wedge} & S^0 \wedge S^0 \\
 \parallel & \searrow \lambda_{S^0}^{\text{Sets}_*} & \\
 & & S^0
 \end{array}
 \quad
 \begin{array}{ccc}
 S^0 & \xrightarrow{\Delta_{S^0}^\wedge} & S^0 \wedge S^0 \\
 \parallel & \searrow \rho_{S^0}^{\text{Sets}_*} & \\
 & & S^0
 \end{array}$$

commute, i.e. we have

$$\begin{aligned}
 \Delta_{S^0}^\wedge &= \lambda_{S^0}^{\text{Sets}_*, -1} \\
 &= \rho_{S^0}^{\text{Sets}_*, -1},
 \end{aligned}$$

where we recall that the equalities

$$\begin{aligned}
 \lambda_{S^0}^{\text{Sets}_*} &= \rho_{S^0}^{\text{Sets}_*}, \\
 \lambda_{S^0}^{\text{Sets}_*, -1} &= \rho_{S^0}^{\text{Sets}_*, -1}
 \end{aligned}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

00GZ

2. *The Diagonal of the Unit.* The component

$$\Delta_{S^0}^\wedge : S^0 \xrightarrow{\sim} S^0 \wedge S^0$$

of Δ^\wedge at S^0 is an isomorphism.*Proof. Item 1, Monoidality:* We claim that Δ^\wedge is indeed monoidal:

0288

1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{array}{ccc}
 X \wedge Y & \xrightarrow{\Delta_X^\wedge \wedge \Delta_Y^\wedge} & (X \wedge X) \wedge (Y \wedge Y) \\
 \searrow \Delta_{X \wedge Y}^\wedge & & \downarrow ? \\
 & & (X \wedge Y) \wedge (X \wedge Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \wedge y & \xrightarrow{\quad} & (x \wedge x) \wedge (y \wedge y) \\
 \searrow & & \downarrow \\
 & & (x \wedge y) \wedge (x \wedge y)
 \end{array}$$

and hence indeed commutes.

- 0289 2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of [Definition 7.5.5.1.1](#), the inverse of the left unitor of \mathbf{Sets}_* with respect to the smash product of pointed sets at $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ is given by

$$\lambda_X^{\mathbf{Sets}_*, -1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\begin{aligned} \lambda_{S^0}^{\mathbf{Sets}_*, -1}(0) &\stackrel{\text{def}}{=} 1 \wedge 0, \\ \lambda_{S^0}^{\mathbf{Sets}_*, -1}(1) &\stackrel{\text{def}}{=} 1 \wedge 1. \end{aligned}$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\begin{aligned} \Delta_{S^0}^\wedge(0) &\stackrel{\text{def}}{=} 0 \wedge 0, \\ \Delta_{S^0}^\wedge(1) &\stackrel{\text{def}}{=} 1 \wedge 1, \end{aligned}$$

it follows that we indeed have $\Delta_{S^0}^\wedge = \lambda_{S^0}^{\mathbf{Sets}_*, -1}$.

This finishes the proof.

Item 2, The Diagonal of the Unit: This follows from [Item 1](#) and the invertibility of the left/right unitor of \mathbf{Sets}_* with respect to \wedge , proved in the proof of [Definition 7.5.5.1.1](#) for the left unitor or the proof of [Definition 7.5.6.1.1](#) for the right unitor. \square

00H0 7.5.9 The Monoidal Structure on Pointed Sets Associated to \wedge

00H1 **Proposition 7.5.9.1.1.** The category \mathbf{Sets}_* admits a closed monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category \mathbf{Sets}_* of pointed sets.
- *The Monoidal Product.* The smash product functor

$$\wedge: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of [Item 1](#) of [Definition 7.5.1.1.10](#).

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}_*: \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of [Item 1](#) of [Definition 7.5.2.1.2](#).

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\text{Sets}_*} : \text{pt} \rightarrow \text{Sets}_*$$

of [Definition 7.5.3.1.1](#).

- *The Associators.* The natural isomorphism

$$\alpha^{\text{Sets}_*} : \wedge \circ (\wedge \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\text{id}_{\text{Sets}_*} \times \wedge) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}}$$

of [Definition 7.5.4.1.1](#).

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}_*} : \wedge \circ (\mathbb{1}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \lambda_{\text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.5.1.1](#).

- *The Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}_*} : \wedge \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}) \xrightarrow{\sim} \rho_{\text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.6.1.1](#).

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}_*} : \wedge \xrightarrow{\sim} \wedge \circ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.7.1.1](#).

- *The Diagonals.* The monoidal natural transformation

$$\Delta^\wedge : \text{id}_{\text{Sets}_*} \Longrightarrow \wedge \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.8.1.1](#).

Proof. The Pentagon Identity: Let (W, w_0) , (X, x_0) , (Y, y_0) and (Z, z_0) be

pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \wedge (X \wedge Y)) \wedge Z & \\
 \alpha_{W,X,Y}^{\text{Sets}_*} \wedge \text{id}_Z \nearrow & & \searrow \alpha_{W,X \wedge Y,Z}^{\text{Sets}_*} \\
 ((W \wedge X) \wedge Y) \wedge Z & & W \wedge ((X \wedge Y) \wedge Z) \\
 \alpha_{W \wedge X,Y,Z}^{\text{Sets}_*} \searrow & & \swarrow \text{id}_W \wedge \alpha_{X,Y,Z}^{\text{Sets}_*} \\
 (W \wedge X) \wedge (Y \wedge Z) & \xrightarrow{\alpha_{W,X,Y \wedge Z}^{\text{Sets}_*}} & W \wedge (X \wedge (Y \wedge Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (w \wedge (x \wedge y)) \wedge z & \\
 \swarrow & & \searrow \\
 ((w \wedge x) \wedge y) \wedge z & & w \wedge ((x \wedge y) \wedge z) \\
 \searrow & & \swarrow \\
 (w \wedge x) \wedge (y \wedge z) & \longmapsto & w \wedge (x \wedge (y \wedge z))
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

The Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to

show that the diagram

$$\begin{array}{ccc}
 (X \wedge S^0) \wedge Y & \xrightarrow{\alpha_{X,S^0,Y}^{\text{Sets}_*}} & X \wedge (S^0 \wedge Y) \\
 \searrow \rho_X^{\text{Sets}_*} \wedge \text{id}_Y & & \swarrow \text{id}_X \wedge \lambda_Y^{\text{Sets}_*} \\
 & X \wedge Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x \wedge 0) \wedge y & \xrightarrow{\quad} & x \wedge (0 \wedge y) \\
 \searrow & & \swarrow \\
 x_0 \wedge y & & x \wedge y_0
 \end{array}$$

and

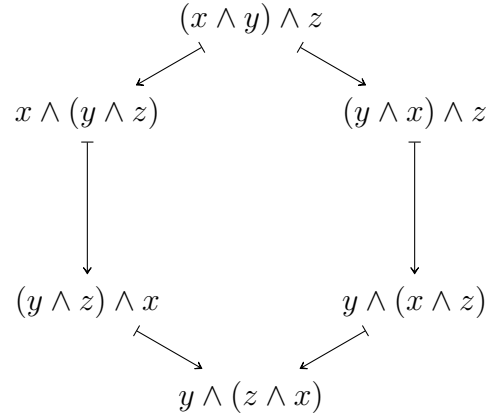
$$\begin{array}{ccc}
 (x \wedge 1) \wedge y & \xrightarrow{\quad} & x \wedge (1 \wedge y) \\
 \searrow & & \swarrow \\
 & x \wedge y &
 \end{array}$$

and thus we see that the triangle identity is satisfied.

The Left Hexagon Identity: Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets. We have to show that the diagram

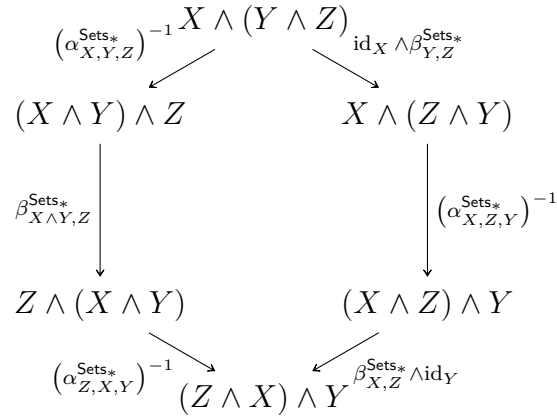
$$\begin{array}{ccc}
 & (X \wedge Y) \wedge Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}_*} \swarrow & & \searrow \beta_{X,Y}^{\text{Sets}_*} \wedge \text{id}_Z \\
 X \wedge (Y \wedge Z) & & (Y \wedge X) \wedge Z \\
 \downarrow \beta_{X,Y \wedge Z}^{\text{Sets}_*} & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}_*} \\
 (Y \wedge Z) \wedge X & & Y \wedge (X \wedge Z) \\
 \alpha_{Y,Z,X}^{\text{Sets}_*} \swarrow & & \swarrow \text{id}_Y \wedge \beta_{X,Z}^{\text{Sets}_*} \\
 & Y \wedge (Z \wedge X) &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

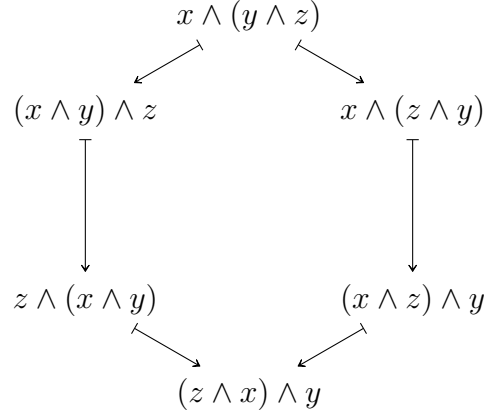


and thus we see that the left hexagon identity is satisfied.

The Right Hexagon Identity: Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus we see that the right hexagon identity is satisfied.

Monoidal Closedness: This follows from **Item 2** of **Definition 7.5.1.1.10**.

Existence of Monoidal Diagonals: This follows from **Items 1** and **2** of **Definition 7.5.8.1.2**. \square

01VY 7.5.10 The Universal Property of $(\mathbf{Sets}_*, \wedge, S^0)$

01VZ **Theorem 7.5.10.1.1.** The symmetric monoidal structure on the category \mathbf{Sets}_* of **Definition 7.5.9.1.1** is uniquely determined by the following requirements:

- 01W0 1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathbf{Sets}_*} : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of \mathbf{Sets}_* admits an internal Hom $[-1, -2]_{\mathbf{Sets}_*}$.

- 01W1 2. *The Unit Object Is S^0 .* We have $\mathbb{1}_{\mathbf{Sets}_*} \cong S^0$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}^{\text{cld}}(\mathbf{Sets}_*)$ of ?? spanned by the closed symmetric monoidal categories $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, [-1, -2]_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$ satisfying **Items 1** and **2** is contractible (i.e. equivalent to the punctual category).

Proof. Unwinding the Statement: Let $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, [-1, -2]_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda', \rho', \sigma')$ be a closed symmetric monoidal category satisfying **Items 1** and **2**. We need to show that the identity functor

$$\text{id}_{\mathbf{Sets}_*} : \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \mathrm{id}_{\mathbf{Sets}_*}^{\otimes} &: X \otimes_{\mathbf{Sets}_*} Y \xrightarrow{\sim} X \wedge Y, \\ \mathrm{id}_{\mathbf{Sets}_*}^{\mathrm{Hom}} &: [X, Y]_{\mathbf{Sets}_*} \xrightarrow{\sim} \mathbf{Sets}_*(X, Y), \\ \mathrm{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes} &: \mathbb{1}_{\mathbf{Sets}_*} \xrightarrow{\sim} S^0, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, [-1, -2]_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda', \rho', \sigma')$ to the closed symmetric monoidal category $(\mathbf{Sets}_*, \times, \mathbf{Sets}_*(-1, -2), \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$ of [Definition 7.5.9.1.1](#).

Constructing an Isomorphism $[-1, -2]_{\mathbf{Sets}_} \cong \mathbf{Sets}_*(-1, -2)$:* By ??, we have a natural isomorphism

$$\mathbf{Sets}_*(S^0, [-1, -2]_{\mathbf{Sets}_*}) \cong \mathbf{Sets}_*(-1, -2).$$

By [Pointed Sets, Item 4](#) of [Definition 6.1.4.1.1](#), we also have a natural isomorphism

$$\mathbf{Sets}_*(S^0, [-1, -2]_{\mathbf{Sets}_*}) \cong [-1, -2]_{\mathbf{Sets}_*}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathbf{Sets}_*(-1, -2) \cong [-1, -2]_{\mathbf{Sets}_*}.$$

Given $X, Y \in \mathrm{Obj}(\mathbf{Sets}_*)$, we will write

$$\mathrm{id}_{X,Y}^{\mathrm{Hom}}: \mathbf{Sets}_*(X, Y) \xrightarrow{\sim} [X, Y]_{\mathbf{Sets}_*}$$

for the component of this isomorphism at (X, Y) .

Constructing an Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* Since $\otimes_{\mathbf{Sets}_*}$ is adjoint in each variable to $[-1, -2]_{\mathbf{Sets}_*}$ by assumption and \wedge is adjoint in each variable to $\mathbf{Sets}_*(-1, -2)$ by [Constructions With Sets, Item 2](#) of [Definition 4.3.5.1.2](#), uniqueness of adjoints (??) gives us natural isomorphisms

$$\begin{aligned} X \otimes_{\mathbf{Sets}_*} - &\cong X \wedge -, \\ - \otimes_{\mathbf{Sets}_*} Y &\cong Y \wedge -. \end{aligned}$$

By ??, we then have $\otimes_{\mathbf{Sets}_*} \cong \wedge$. We will write

$$\mathrm{id}_{\mathbf{Sets}_*|X,Y}^{\otimes}: X \otimes_{\mathbf{Sets}_*} Y \xrightarrow{\sim} X \wedge Y$$

for the component of this isomorphism at (X, Y) .

Alternative Construction of an Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* Alternatively, we may construct a natural isomorphism $\otimes_{\mathbf{Sets}_*} \cong \wedge$ as follows:

- 01W2 1. Let $X \in \mathbf{Obj}(\mathbf{Sets}_*)$.
- 01W3 2. Since $\otimes_{\mathbf{Sets}_*}$ is part of a closed monoidal structure, it preserves colimits in each variable by ??.
- 01W4 3. Since $X \cong \bigvee_{x \in X^-} S^0$ and $\otimes_{\mathbf{Sets}_*}$ preserves colimits in each variable, we have

$$\begin{aligned}
 X \otimes_{\mathbf{Sets}_*} Y &\cong \left(\bigvee_{x \in X^-} S^0 \right) \otimes_{\mathbf{Sets}_*} Y \\
 &\cong \bigvee_{x \in X^-} (S^0 \otimes_{\mathbf{Sets}_*} Y) \\
 &\cong \bigvee_{x \in X^-} Y \\
 &\cong \bigvee_{x \in X^-} S^0 \wedge Y \\
 &\cong \left(\bigvee_{x \in X^-} S^0 \right) \wedge Y \\
 &\cong X \wedge Y,
 \end{aligned}$$

naturally in $Y \in \mathbf{Obj}(\mathbf{Sets}_*)$, where we have used that S^0 is the monoidal unit for $\otimes_{\mathbf{Sets}_*}$. Thus $X \otimes_{\mathbf{Sets}_*} - \cong X \wedge -$ for each $X \in \mathbf{Obj}(\mathbf{Sets}_*)$.

- 01W5 4. Similarly, $- \otimes_{\mathbf{Sets}_*} Y \cong - \wedge Y$ for each $Y \in \mathbf{Obj}(\mathbf{Sets}_*)$.
- 01W6 5. By ??, we then have $\otimes_{\mathbf{Sets}_*} \cong \wedge$.

Below, we'll show that if a natural isomorphism $\otimes_{\mathbf{Sets}_*} \cong \wedge$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}_{\mathbf{Sets}_*|X,Y}^{\otimes}: X \otimes_{\mathbf{Sets}_*} Y \rightarrow X \wedge Y$ from before.

Constructing an Isomorphism $\mathrm{id}_{\mathbb{1}}^{\otimes}: \mathbb{1}_{\mathbf{Sets}_} \rightarrow S^0$:* We define an isomorphism $\mathrm{id}_{\mathbb{1}}^{\otimes}: \mathbb{1}_{\mathbf{Sets}_*} \rightarrow S^0$ as the composition

$$\mathbb{1}_{\mathbf{Sets}_*} \xrightarrow[\sim]{\rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 \xrightarrow[\sim]{\mathrm{id}_{\mathbf{Sets}_*| \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes, -1}} \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 \xrightarrow[\sim]{\lambda'_{S^0}} S^0$$

in \mathbf{Sets}_* .

Monoidal Left Unity of the Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* We have to show that

the diagram

$$\begin{array}{ccc}
 & S^0 \otimes_{\mathbf{Sets}_*} X & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} | S^0, X} S^0 \wedge X \\
 \text{id}_{\mathbf{1}_{\mathbf{Sets}_*}}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_X \nearrow & & \searrow \lambda_X^{\mathbf{Sets}_*} \\
 \mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} X & \xrightarrow{\lambda'_X} & X
 \end{array}$$

commutes. To this end, we will first show that the diagram

$$\begin{array}{ccc}
 & S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} | S^0, S^0} S^0 \wedge S^0 \\
 \text{id}_{\mathbf{1}_{\mathbf{Sets}_*}}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} \nearrow & & \searrow \lambda_{S^0}^{\mathbf{Sets}_*} \\
 \mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\lambda'_{S^0}} & S^0,
 \end{array} \quad (\dagger)$$

corresponding to the case $X = S^0$, commutes. Indeed, consider the diagram

$$\begin{array}{c}
 \mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 \xrightarrow{\rho_{S^0}^{\mathbf{Sets}_*, -1} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0}} (\mathbf{1}_{\mathbf{Sets}_*} \wedge S^0) \otimes_{\mathbf{Sets}_*} S^0 \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes, -1} | \mathbf{1}_{\mathbf{Sets}_*}, S^0 \otimes_{\mathbf{Sets}_*} \text{id}_{S^0}} (\mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 \xrightarrow{\lambda'_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0}} S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \downarrow \text{id}_{\mathbf{Sets}_*}^{\otimes} | \mathbf{1}_{\mathbf{Sets}_*}, S^0 \quad (1) \quad \downarrow \text{id}_{\mathbf{Sets}_*}^{\otimes} | \mathbf{1}_{\mathbf{Sets}_*} \wedge S^0, S^0 \quad (2) \quad \downarrow \text{id}_{\mathbf{Sets}_*}^{\otimes} | \mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0, S^0 \quad (3) \quad \downarrow \text{id}_{\mathbf{Sets}_*}^{\otimes} | S^0, S^0 \\
 \mathbf{1}_{\mathbf{Sets}_*} \wedge S^0 \quad (4) \quad (\mathbf{1}_{\mathbf{Sets}_*} \wedge S^0) \wedge S^0 \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes, -1} | \mathbf{1}_{\mathbf{Sets}_*}, S^0 \wedge S^0} (\mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) \wedge S^0 \xrightarrow{\lambda'_{S^0} \wedge \text{id}_{S^0}} S^0 \wedge S^0 \\
 \downarrow \text{id}_{\mathbf{Sets}_*}^{\otimes, -1} | \mathbf{1}_{\mathbf{Sets}_*}, S^0 \quad (5) \quad \downarrow \rho_{\mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0}^{\mathbf{Sets}_*, -1} \quad (6) \quad \downarrow \rho_{\mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0}^{\mathbf{Sets}_*} \quad (7) \quad \downarrow \rho_{S^0 \wedge S^0}^{\mathbf{Sets}_*} = \wedge_{S^0}^{\mathbf{Sets}_*} \\
 \mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 \xrightarrow{\lambda'_{S^0}} \mathbf{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 \xrightarrow{\lambda'_{S^0}} S^0
 \end{array}$$

whose boundary diagram corresponds to the diagram (\dagger) above. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes}$.
- Subdiagram (4) commutes by ??.
- Subdiagram (5) commutes by the naturality of $\rho^{\mathbf{Sets}_*, -1}$.

- Subdiagram (6) commutes trivially.
- Subdiagram (7) commutes by the naturality of $\rho^{\mathbf{Sets}_*}$, where the equality $\rho_{S^0}^{\mathbf{Sets}_*} = \lambda_{S^0}^{\mathbf{Sets}_*}$ comes from ??.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram (\dagger) above. As a result, the diagram

$$\begin{array}{ccc}
 & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes, -1}} S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \nearrow \lambda_{S^0}^{\mathbf{Sets}_*, -1} & & \searrow \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} \\
 S^0 & \xrightarrow{\lambda_{S^0}'^{-1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0
 \end{array}
 \quad (\dagger)$$

also commutes. Now, let $X \in \text{Obj}(\mathbf{Sets}_*)$, let $x \in X$, and consider the diagram

$$\begin{array}{ccccc}
 & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \wedge \text{id}_{S^0}} S^0 \\
 \nearrow \lambda_{S^0}^{\mathbf{Sets}_*, -1} & & \searrow & & \searrow \\
 S^0 & \xrightarrow{\lambda_{S^0}'^{-1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & & \\
 \downarrow [x] & \downarrow \text{id}_{S^0} \wedge [x] & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} [x] & \downarrow \text{id}_{\mathbb{1}_{\mathbf{Sets}_*}} \wedge [x] & \\
 & S^0 \wedge X & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} X & \\
 \nearrow \lambda_X^{\mathbf{Sets}_*, -1} & & \searrow & & \searrow \\
 X & \xrightarrow{\lambda_X}'^{-1} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} X & &
 \end{array}$$

(\dagger) (1) (4) (5) (2)

Since:

- Subdiagram (5) commutes by the naturality of λ'^{-1} .
- Subdiagram (\dagger) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (1) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes, -1}$.

- Subdiagram (3) commutes by the naturality of $\lambda^{\mathbf{Sets}_*, -1}$.

it follows that the diagram

$$\begin{array}{ccccc}
 & & S^0 \wedge X & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} X \\
 & \nearrow \lambda_X^{\mathbf{Sets}_*, -1} & & & \searrow \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \otimes_{\mathbf{Sets}_*} \text{id}_X \\
 S^0 & \xrightarrow{[x]} & X & \xrightarrow{\lambda_X'^{-1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} X
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}
 \lambda_X'^{-1}(x) &= [\lambda_X'^{-1} \circ [x]](1) \\
 &= [(\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \wedge \text{id}_X) \circ \text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1} \circ \lambda_X^{\mathbf{Sets}_*, -1} \circ [x]](1) \\
 &= [(\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \wedge \text{id}_X) \circ \text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1} \circ \lambda_X^{\mathbf{Sets}_*, -1}](x)
 \end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda_X'^{-1} = (\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \wedge \text{id}_X) \circ \text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1} \circ \lambda_X^{\mathbf{Sets}_*, -1}.$$

Taking inverses then gives

$$\lambda_X' = \lambda_X^{\mathbf{Sets}_*} \circ \text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes} \circ (\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes} \wedge \text{id}_X),$$

showing that the diagram

$$\begin{array}{ccccc}
 & & S^0 \otimes_{\mathbf{Sets}_*} X & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes}} & S^0 \wedge X \\
 & \nearrow \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_X & & & \searrow \lambda_X^{\mathbf{Sets}_*} \\
 \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} X & \xrightarrow{\lambda_X'} & & & X
 \end{array}$$

indeed commutes.

Braidedness of the Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* We have to show that the diagram

$$\begin{array}{ccc}
 X \otimes_{\mathbf{Sets}_*} Y & \xrightarrow{\text{id}_{\mathbf{Sets}_*|X, Y}^{\otimes}} & X \wedge Y \\
 \sigma'_{X, Y} \downarrow & & \downarrow \sigma_{X, Y}^{\mathbf{Sets}_*} \\
 Y \otimes_{\mathbf{Sets}_*} X & \xrightarrow{\text{id}_{\mathbf{Sets}_*|Y, X}^{\otimes}} & Y \wedge X
 \end{array}$$

commutes. To this end, we will first show that the diagram

$$\begin{array}{ccc}
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} |_{S^0, S^0}} & S^0 \wedge S^0 \\
 \sigma'_{S^0, S^0} \downarrow & (\dagger) & \downarrow \sigma_{S^0, S^0}^{\mathbf{Sets}_*} \\
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} |_{S^0, S^0}} & S^0 \wedge S^0
 \end{array}$$

commutes. To that end, we will first show that the diagram

$$\begin{array}{ccc}
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} |_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}} & S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} \\
 \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} \downarrow & (\dagger) & \downarrow \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} \\
 \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} |_{\mathbb{1}_{\mathbf{Sets}_*}, S^0}} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0
 \end{array}$$

commutes, and, to this end, we will first show that the diagram

$$\begin{array}{ccc}
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*}^{\otimes} |_{S^0, S^0}} & S^0 \wedge S^0 \\
 \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*}^{\otimes} |_{\mathbb{1}} \uparrow & (\S) & \downarrow \lambda_{S^0}^{\mathbf{Sets}_*} \\
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho'_{S^0}} & S^0
 \end{array}$$

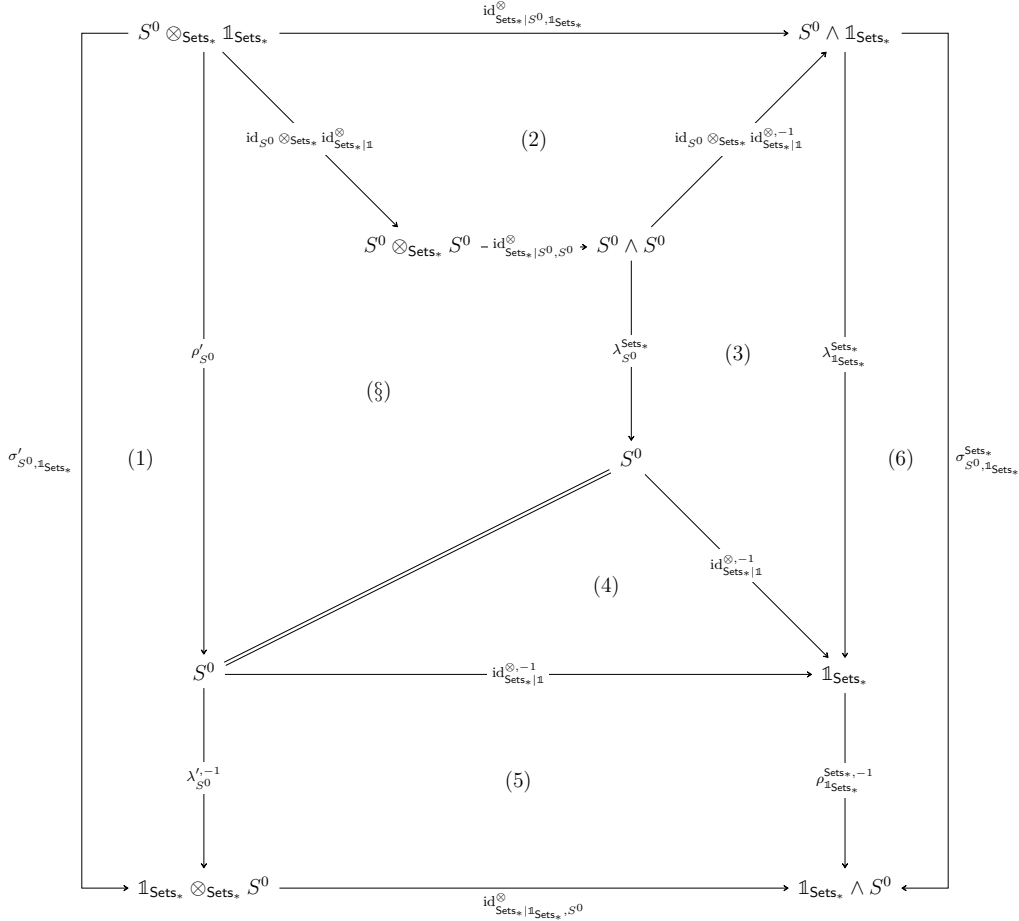
The diagram illustrates the following commutative sub-diagrams:

- (1) A triangle with nodes $S^0 \otimes_{Sets_*} S^0$ (top-left), $S^0 \wedge S^0$ (top-right), and $S^0 \otimes_{Sets_*} \mathbb{1}_{Sets_*}$ (bottom-left). The top arrow is $\text{id}_{Sets_*}^{\otimes} |_{S^0, S^0}$. The right arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} \wedge \text{id}_{S^0}$. The bottom arrow is $\text{id}_{S^0} \otimes_{Sets_*} \text{id}_{\mathbb{1}_{Sets_*}}^{\otimes}$.
- (2) A triangle with nodes $S^0 \otimes_{Sets_*} \mathbb{1}_{Sets_*}$ (bottom-left), $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (middle), and $S^0 \otimes_{Sets_*} S^0$ (top-left). The bottom arrow is $\text{id}_{S^0} \otimes_{Sets_*} \text{id}_{\mathbb{1}_{Sets_*}}^{\otimes}$. The middle arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} \otimes_{Sets_*} \text{id}_{S^0}$. The top arrow is $\text{id}_{S^0 \otimes_{Sets_*} S^0}$.
- (3) A triangle with nodes $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (middle), $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (bottom-right), and $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (top-left). The middle arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} |_{\mathbb{1}_{Sets_*}, S^0}$. The bottom-right arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} \otimes_{Sets_*} \text{id}_{S^0}$. The top-left arrow is $\text{id}_{S^0} \otimes_{Sets_*} \text{id}_{\mathbb{1}_{Sets_*}}^{\otimes}$.
- (4) A triangle with nodes $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (middle), $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (bottom-right), and $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (top-left). The middle arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} |_{\mathbb{1}_{Sets_*}, S^0}$. The bottom-right arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} \otimes_{Sets_*} \text{id}_{S^0}$. The top-left arrow is $\text{id}_{S^0} \otimes_{Sets_*} \text{id}_{\mathbb{1}_{Sets_*}}^{\otimes}$.
- (5) A triangle with nodes $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (middle), $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (bottom-right), and $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (top-left). The middle arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} |_{\mathbb{1}_{Sets_*}, S^0}$. The bottom-right arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} \otimes_{Sets_*} \text{id}_{S^0}$. The top-left arrow is $\text{id}_{S^0} \otimes_{Sets_*} \text{id}_{\mathbb{1}_{Sets_*}}^{\otimes}$.
- (6) A triangle with nodes $S^0 \otimes_{Sets_*} \mathbb{1}_{Sets_*}$ (bottom-left), $\mathbb{1}_{Sets_*} \otimes_{Sets_*} S^0$ (middle), and $S^0 \otimes_{Sets_*} S^0$ (top-left). The bottom-left arrow is $\text{id}_{S^0} \otimes_{Sets_*} \text{id}_{\mathbb{1}_{Sets_*}}^{\otimes}$. The middle arrow is $\text{id}_{\mathbb{1}_{Sets_*}}^{\otimes} \otimes_{Sets_*} \text{id}_{S^0}$. The top-left arrow is $\text{id}_{S^0 \otimes_{Sets_*} S^0}$.

- Subdiagram (1) commutes by the naturality of $\text{id}_{\text{Sets}_*}^\otimes$;
- Subdiagrams (2) and (3) commute by the functoriality of \otimes ;
- Subdiagram (4) commutes by the left monoidal unity of $(\text{id}^\otimes, \text{id}_1^\otimes)$, which we proved above;
- Subdiagram (5) commutes by the naturality of λ' ;
- Subdiagram (6) commutes by the naturality of ρ' , where the equality $\rho'_{1_{\text{Sets}_*}} = \lambda'_{1_{\text{Sets}_*}}$ comes from ??;

it follows that the boundary diagram, i.e. diagram (8), also commutes. Next,

consider the diagram



whose boundary diagram corresponds to the diagram (\ddagger) above. Since:

- Subdiagrams (1) and (6) commute by ??;
- Subdiagram (2) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes}$;
- Subdiagram (§) commutes, as was shown above;
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathbf{Sets}_*}$;
- Subdiagram (4) commutes trivially;
- Subdiagram (5) commutes by **Constructions With Monoidal Categories, Item 2c** of **Item 2** of **Definition 13.1.1.1.4**, whose proof uses only the left monoidal unity of $(\text{id}^{\otimes}, \text{id}_{\mathbb{1}}^{\otimes})$, which has been proven above;

it follows that the boundary diagram, i.e. diagram (\ddagger) , also commutes. Next, consider the diagram

$$\begin{array}{ccccc}
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{S^0, S^0}^\otimes} & S^0 \wedge S^0 & & \\
 \downarrow \sigma'_{S^0, S^0} & \searrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}}^{\otimes, -1} & \swarrow \text{id}_{S^0} \wedge \text{id}_{\mathbb{1}}^{\otimes, -1} & & \downarrow \sigma_{S^0, S^0}^{\mathbf{Sets}_*} \\
 & (1) & & & \\
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^\otimes} & S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} & & \\
 \downarrow \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} & (2) & \downarrow \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} & (3) & \\
 \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbb{1}_{\mathbf{Sets}_*}, S^0}^\otimes} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 & & \\
 \downarrow \text{id}_{\mathbb{1}}^\otimes \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} & (4) & \downarrow \text{id}_{\mathbb{1}}^\otimes \wedge \text{id}_{S^0} & & \\
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{S^0, S^0}^\otimes} & S^0 \wedge S^0, & &
 \end{array}$$

whose boundary diagram corresponds to the diagram (\ddagger) . Since:

- Subdiagram (1) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^\otimes$;
- Subdiagram (2) commutes by the naturality of σ' and the fact that $\text{id}_{\mathbb{1}}^\otimes$ is invertible;
- Subdiagram (\ddagger) commutes as proved above;
- Subdiagram (3) commutes by the naturality of $\sigma^{\mathbf{Sets}_*}$ and the fact that $\text{id}_{\mathbb{1}}^\otimes$ is invertible;
- Subdiagram (4) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^\otimes$;

it follows that the boundary diagram, i.e. diagram (\dagger) also commutes. Taking inverses for the diagram (\dagger) , we see that the diagram

$$\begin{array}{ccc}
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \sigma_{S^0, S^0}^{\mathbf{Sets}_*, -1} \downarrow & (\P) & \downarrow \sigma'_{S^0, S^0, -1} \\
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0
 \end{array}$$

commutes as well. Now, let $X, Y \in \text{Obj}(\mathbf{Sets}_*)$, let $x \in X$, let $y \in Y$, and consider the diagram

$$\begin{array}{ccccc}
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 & & \\
 \downarrow \sigma_{S^0, S^0}^{\mathbf{Sets}_*, -1} & \searrow [y] \wedge [x] & \downarrow \sigma'_{S^0, S^0, -1} & \searrow [y] \otimes_{\mathbf{Sets}_*} [x] & \\
 & Y \wedge X & \xrightarrow{\text{id}_{\mathbf{Sets}_*|Y, A}^{\otimes, -1}} & Y \otimes_{\mathbf{Sets}_*} X & \\
 & \downarrow \sigma_{A, Y}^{\mathbf{Sets}_*, -1} & \downarrow & \downarrow \sigma'_{A, Y, -1} & \\
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 & & \\
 \downarrow \sigma_{S^0, S^0}^{\mathbf{Sets}_*, -1} & \searrow [x] \wedge [y] & \downarrow \sigma'_{S^0, S^0, -1} & \searrow [x] \otimes_{\mathbf{Sets}_*} [y] & \\
 & X \wedge Y & \xrightarrow{\text{id}_{\mathbf{Sets}_*|A, Y}^{\otimes, -1}} & X \otimes_{\mathbf{Sets}_*} Y &
 \end{array}$$

which we partition into subdiagrams as follows:

$$\begin{array}{ccc}
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 \\
 \downarrow \sigma_{S^0, S^0}^{\text{Sets}_*, -1} & \searrow [y] \wedge [x] & \searrow [y] \otimes_{\text{Sets}_*} [x] \\
 & Y \wedge X & \xrightarrow{\text{id}_{\text{Sets}_* | Y, X}^{\otimes, -1}} Y \otimes_{\text{Sets}_*} X \\
 & \downarrow \sigma_{X, Y}^{\text{Sets}_*, -1} & \downarrow \sigma'_{X, Y} \\
 S^0 \wedge S^0 & \xrightarrow{[x] \wedge [y]} & X \wedge Y \\
 & \searrow & \searrow \text{id}_{\text{Sets}_* | X, Y}^{\otimes, -1} \\
 & & X \otimes_{\text{Sets}_*} Y
 \end{array} \quad (2)$$

$$\begin{array}{ccc}
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 \\
 \downarrow \sigma_{S^0, S^0}^{\text{Sets}_*, -1} & \searrow & \searrow [y] \otimes_{\text{Sets}_*} [x] \\
 & & Y \otimes_{\text{Sets}_*} X \\
 & \downarrow \sigma'_{S^0, S^0} & \downarrow \sigma'_{X, Y} \\
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 \\
 \downarrow \sigma_{S^0, S^0}^{\text{Sets}_*, -1} & \searrow [x] \wedge [y] & \searrow [x] \otimes_{\text{Sets}_*} [y] \\
 & X \wedge Y & \xrightarrow{\text{id}_{\text{Sets}_* | X, Y}^{\otimes, -1}} X \otimes_{\text{Sets}_*} Y
 \end{array} \quad (4)$$

Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\text{Sets}_*, -1}$.
- Subdiagram (5) commutes by the naturality of $\text{id}^{\otimes, -1}$.
- Subdiagram (\P) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of σ'^{-1} .
- Subdiagram (1) commutes by the naturality of $\text{id}^{\otimes, -1}$.

it follows that the diagram

$$\begin{array}{ccc}
 S^0 \wedge S^0 & & \\
 \searrow [y] \wedge [x] & & \\
 & Y \wedge X & \xrightarrow{\text{id}_{\text{Sets}_* | Y, X}^{\otimes}} Y \otimes_{\text{Sets}_*} X \\
 & \downarrow \sigma_{X, Y}^{\text{Sets}_*} & \downarrow \sigma'_{X, Y} \\
 & X \wedge Y & \xrightarrow{\text{id}_{\text{Sets}_* | X, Y}^{\otimes}} X \otimes_{\text{Sets}_*} Y
 \end{array}$$

commutes. We then have

$$\begin{aligned}
 [\text{id}_{\text{Sets}_* | X, Y}^{\otimes, -1} \circ \sigma_{X, Y}^{\text{Sets}_*, -1}](y, x) &= [\text{id}_{\text{Sets}_* | X, Y}^{\otimes, -1} \circ \sigma_{X, Y}^{\text{Sets}_*, -1} \circ ([y] \wedge [x])](1, 1) \\
 &= [\sigma'_{X, Y} \circ \text{id}_{\text{Sets}_* | Y, X}^{\otimes, -1} \circ ([y] \wedge [x])](1, 1)
 \end{aligned}$$

$$= [\sigma'_{X,Y}{}^{-1} \circ \text{id}_{\mathbf{Sets}_*|Y,X}^{\otimes,-1}](y, x)$$

for each $(y, x) \in Y \wedge X$, and thus we have

$$\text{id}_{\mathbf{Sets}_*|X,Y}^{\otimes,-1} \circ \sigma_{X,Y}^{\mathbf{Sets}_*, -1} = \sigma'_{X,Y}{}^{-1} \circ \text{id}_{\mathbf{Sets}_*|Y,X}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{X,Y}^{\mathbf{Sets}_*} \circ \text{id}_{\mathbf{Sets}_*|X,Y}^{\otimes} = \text{id}_{\mathbf{Sets}_*|Y,X}^{\otimes} \circ \sigma'_{X,Y},$$

showing that the diagram

$$\begin{array}{ccc} A \otimes_{\mathbf{Sets}_*} B & \xrightarrow{\text{id}_{\mathbf{Sets}_*|A,B}^{\otimes}} & A \wedge B \\ \sigma'_{A,B} \downarrow & & \downarrow \sigma_{A,B}^{\mathbf{Sets}_*} \\ B \otimes_{\mathbf{Sets}_*} A & \xrightarrow{\text{id}_{\mathbf{Sets}_*|B,A}^{\otimes}} & B \wedge A \end{array}$$

indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* We have to show that the diagram

$$\begin{array}{ccc} & X \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|X,S^0}^{\otimes}} X \wedge S^0 \\ \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}_{\mathbf{Sets}_*}}^{\otimes} \nearrow & & \searrow \rho_X^{\mathbf{Sets}_*} \\ X \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho'_X} & X \end{array}$$

commutes. To this end, we will first show that the diagram

$$\begin{array}{ccc} & S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0,S^0}^{\otimes}} S^0 \wedge S^0 \\ \text{id}_{\mathbb{1}_{\mathbf{Sets}_*}}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} \nearrow & & \searrow \rho_{S^0}^{\mathbf{Sets}_*} \\ S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho'_{S^0}} & S^0, \end{array} \quad (\dagger)$$

corresponding to the case $X = S^0$, commutes. First, notice that we may write

$$\sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} : S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} \rightarrow \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0$$

as the composition

$$\begin{aligned}
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} &\xrightarrow{\mathrm{id}_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes}} S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} \\
 &\xrightarrow{\lambda_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*}} \mathbb{1}_{\mathbf{Sets}_*} \\
 &\xrightarrow{\rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 \\
 &\xrightarrow{\mathrm{id}_{\mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes, -1}} \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0.
 \end{aligned}$$

Indeed, we may write this composition as part of the diagram

$$\begin{array}{ccccc}
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\mathrm{id}_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes}} & S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\lambda_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*}} & \mathbb{1}_{\mathbf{Sets}_*} \\
 \downarrow \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} & & \downarrow \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} & (2) & \swarrow \rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1} \\
 \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\mathrm{id}_{\mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes}} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 & \xrightarrow{\mathrm{id}_{\mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes, -1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0,
 \end{array}$$

which commutes since:

- Subdiagram (1) commutes by the braidedness of id^{\otimes} , as proved above.
- Subdiagram (2) commutes by ??.

Next, consider the diagram

$$\begin{array}{ccccccc}
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \rho_{S^0}^{\mathbf{Sets}_*, -1}} & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) & \xrightarrow{\text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_* | \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) & \xrightarrow{\text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \lambda'_{S^0}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \downarrow \text{id}_{\mathbf{Sets}_* | S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes} & (1) & \downarrow \text{id}_{\mathbf{Sets}_* | S^0, \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0}^{\otimes} & (2) & \downarrow \text{id}_{\mathbf{Sets}_* | S^0, \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0}^{\otimes} & (3) & \downarrow \text{id}_{\mathbf{Sets}_* | S^0, S^0}^{\otimes} \\
 S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{S^0} \wedge \rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} & S^0 \wedge (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) & \xrightarrow{\text{id}_{S^0} \wedge \text{id}_{\mathbf{Sets}_* | \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes, -1}} & S^0 \wedge (\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) & \xrightarrow{\text{id}_{S^0} \wedge \lambda'_{S^0}} & S^0 \wedge S^0 \\
 \downarrow \lambda_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} & (4) & \downarrow \lambda_{\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0}^{\mathbf{Sets}_*} & (5) & \downarrow \lambda_{\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0}^{\mathbf{Sets}_*} & (6) & \downarrow \lambda_{S^0}^{\mathbf{Sets}_*} = \rho_{S^0}^{\mathbf{Sets}_*} \\
 \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_* | \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes, -1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\lambda'_{S^0}} & S^0,
 \end{array}$$

whose boundary diagram corresponds to the diagram (\dagger) above, since the composition in **red** is equal to $\sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}$ as proved above, and then the composition in **red** composed with λ'_{S^0} is equal to ρ'_{S^0} by **??**. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes}$.
- Subdiagrams (4), (5), and (6) commute by the naturality of $\lambda_{\mathbf{Sets}_*}^{\mathbf{Sets}_*}$, where the equality $\lambda_{S^0}^{\mathbf{Sets}_*} = \rho_{S^0}^{\mathbf{Sets}_*}$ comes from **??**.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram (\dagger) above. As a result, the diagram

$$\begin{array}{ccc}
 & S^0 \wedge S^0 \xrightarrow{\text{id}_{\mathbf{Sets}_* | S^0, S^0}^{\otimes, -1}} S^0 \otimes_{\mathbf{Sets}_*} S^0 & \\
 \rho_{S^0}^{\mathbf{Sets}_*, -1} \nearrow & (\dagger) & \searrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}_{\mathbf{Sets}_*}}^{\otimes, -1} \\
 S^0 & \xrightarrow{\rho_{S^0}^{\mathbf{Sets}_*, -1}} & S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*}.
 \end{array}$$

also commutes. Now, let $X \in \text{Obj}(\mathbf{Sets}_*)$, let $x \in X$, and consider the

diagram

$$\begin{array}{ccccc}
 & & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 & \nearrow \rho_{S^0}^{\mathbf{Sets}_*, -1} & \downarrow \text{id}_{S^0} \wedge [x] & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} [x] & \searrow \text{id}_{S^0} \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \\
 S^0 & \xrightarrow{\rho_{S^0}^{\mathbf{Sets}_*, -1}} & S^0 \wedge S^0 & \xrightarrow{\rho_{S^0}^{\mathbf{Sets}_*, -1}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \downarrow [x] & & \downarrow \text{id}_{S^0} \wedge [x] & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} [x] & \downarrow \text{id}_{\mathbb{1}|\mathbf{Sets}_*} \wedge [x] \\
 & & X \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|X, S^0}^{\otimes, -1}} & X \otimes_{\mathbf{Sets}_*} S^0 \\
 & \nearrow \rho_X^{\mathbf{Sets}_*, -1} & \downarrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} & \downarrow \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} & \searrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \\
 X & \xrightarrow{\rho_X^{\mathbf{Sets}_*, -1}} & X \wedge S^0 & \xrightarrow{\rho_X^{\mathbf{Sets}_*, -1}} & X \otimes_{\mathbf{Sets}_*} S^0 \\
 & & \downarrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} & \downarrow \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} & \downarrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \\
 & & X \wedge \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho_X^{\mathbf{Sets}_*, -1}} & X \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} \\
 & & \downarrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} & \downarrow \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} & \downarrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \\
 & & X & \xrightarrow{\rho_X^{\mathbf{Sets}_*, -1}} & X
 \end{array}$$

Since:

- Subdiagram (5) commutes by the naturality of $\rho^{\mathbf{Sets}_*, -1}$.
- Subdiagram (\dagger) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (1) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\mathbf{Sets}_*, -1}$.

it follows that the diagram

$$\begin{array}{ccccc}
 & & X \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|X, S^0}^{\otimes, -1}} & X \otimes_{\mathbf{Sets}_*} S^0 \\
 & \nearrow \rho_X^{\mathbf{Sets}_*, -1} & & & \searrow \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \\
 S^0 & \xrightarrow{[x]} & X & \xrightarrow{\rho_X^{\mathbf{Sets}_*, -1}} & X \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*}
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}
 \rho_X^{\mathbf{Sets}_*, -1}(a) &= [\rho_X^{\mathbf{Sets}_*, -1} \circ [x]](1) \\
 &= \left[\left(\text{id}_X \wedge \text{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \right) \circ \text{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1} \circ \rho_X^{\mathbf{Sets}_*, -1} \circ [x] \right](1)
 \end{aligned}$$

$$= \left[\left(\mathrm{id}_X \wedge \mathrm{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \right) \circ \mathrm{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1} \circ \rho_X^{\mathbf{Sets}_*, -1} \right] (a)$$

for each $a \in X$, and thus we have

$$\rho'_X{}^{-1} = \left(\mathrm{id}_X \wedge \mathrm{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes, -1} \right) \circ \mathrm{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes, -1} \circ \rho_X^{\mathbf{Sets}_*, -1}.$$

Taking inverses then gives

$$\rho'_X = \rho_X^{\mathbf{Sets}_*} \circ \mathrm{id}_{\mathbf{Sets}_*|S^0, X}^{\otimes} \circ \left(\mathrm{id}_X \wedge \mathrm{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes} \right),$$

showing that the diagram

$$\begin{array}{ccc} & X \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\mathrm{id}_{\mathbf{Sets}_*|X, S^0}^{\otimes}} X \wedge S^0 \\ \mathrm{id}_X \otimes_{\mathbf{Sets}_*} \mathrm{id}_{\mathbb{1}|\mathbf{Sets}_*}^{\otimes} \nearrow & & \searrow \rho_X^{\mathbf{Sets}_*} \\ X \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho'_X} & X \end{array}$$

indeed commutes.

Monoidality of the Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* We have to show that the diagram

$$\begin{array}{ccc} & (X \otimes_{\mathbf{Sets}_*} Y) \otimes_{\mathbf{Sets}_*} Z & \\ \mathrm{id}_{\mathbf{Sets}_*|X, Y}^{\otimes} \otimes_{\mathbf{Sets}_*} \mathrm{id}_Z \swarrow & & \searrow \alpha'_{X, Y, Z} \\ (X \wedge Y) \otimes_{\mathbf{Sets}_*} Z & & X \otimes_{\mathbf{Sets}_*} (Y \otimes_{\mathbf{Sets}_*} Z) \\ \downarrow \mathrm{id}_{\mathbf{Sets}_*|X \wedge Y, Z}^{\otimes} & & \downarrow \mathrm{id}_X \otimes_{\mathbf{Sets}_*} \mathrm{id}_{\mathbf{Sets}_*|Y, Z}^{\otimes} \\ (X \wedge Y) \wedge Z & & X \otimes_{\mathbf{Sets}_*} (Y \wedge Z) \\ \searrow \alpha_{X, Y, Z}^{\mathbf{Sets}_*} & & \swarrow \mathrm{id}_{\mathbf{Sets}_*|X, Y \wedge Z}^{\otimes} \\ & X \wedge (Y \wedge Z) & \end{array}$$

commutes. To this end, we will first prove that the diagram

$$\begin{array}{ccc}
 & (S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 & \\
 \text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} \swarrow & & \searrow \alpha'_{S^0, S^0, S^0} \\
 (S^0 \wedge S^0) \otimes_{\mathbf{Sets}_*} S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) \\
 \downarrow \text{id}_{\mathbf{Sets}_*|S^0 \wedge S^0, S^0}^{\otimes} & (\dagger) & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes} \\
 (S^0 \wedge S^0) \wedge S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \wedge S^0) \\
 \searrow \alpha_{S^0, S^0, S^0}^{\mathbf{Sets}_*} & & \swarrow \text{id}_{\mathbf{Sets}_*|S^0, S^0 \wedge S^0}^{\otimes} \\
 & S^0 \wedge (S^0 \wedge S^0) &
 \end{array}$$

commutes, and, to that end, we will first show that the diagram

$$\begin{array}{ccc}
 & (S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*}) \otimes_{\mathbf{Sets}_*} S^0 & \\
 \text{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} \swarrow & & \searrow \alpha_{S^0, \mathbb{1}_{\mathbf{Sets}_*}, S^0} \\
 (S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*}) \otimes_{\mathbf{Sets}_*} S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) \\
 \downarrow \text{id}_{\mathbf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes} & (\dagger) & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*|\mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes} \\
 (S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*}) \wedge S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) \\
 \searrow \alpha_{S^0, \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\mathbf{Sets}_*} & & \swarrow \text{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0}^{\otimes} \\
 & S^0 \wedge (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) &
 \end{array}$$

The diagram illustrates the derivation of the identity $S^0 \wedge (\mathbb{1}_{\text{Sets}} \wedge S^0)$ from the initial expression $(S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0$. The nodes and edges are as follows:

- Node 1 (Top):** $(S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0$
- Node 2:** $(S^0 \wedge \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0$ (labeled (1))
- Node 3:** $(S^0 \otimes_{\text{Sets}} S^0) \otimes_{\text{Sets}} S^0$ (labeled (2))
- Node 4:** $(S^0 \wedge S^0) \otimes_{\text{Sets}} S^0$ (labeled (4))
- Node 5:** $S^0 \otimes_{\text{Sets}} S^0$ (labeled (5))
- Node 6:** $(S^0 \wedge S^0) \wedge S^0$ (labeled (6))
- Node 7:** $(S^0 \wedge S^0) \wedge \mathbb{1}_{\text{Sets}}$ (labeled (7))
- Node 8:** $S^0 \wedge S^0$ (labeled (8))
- Node 9:** $S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}$ (labeled (9))
- Node 10:** $S^0 \wedge \mathbb{1}_{\text{Sets}}$ (labeled (10))
- Node 11 (Bottom):** $S^0 \wedge (\mathbb{1}_{\text{Sets}} \wedge S^0)$

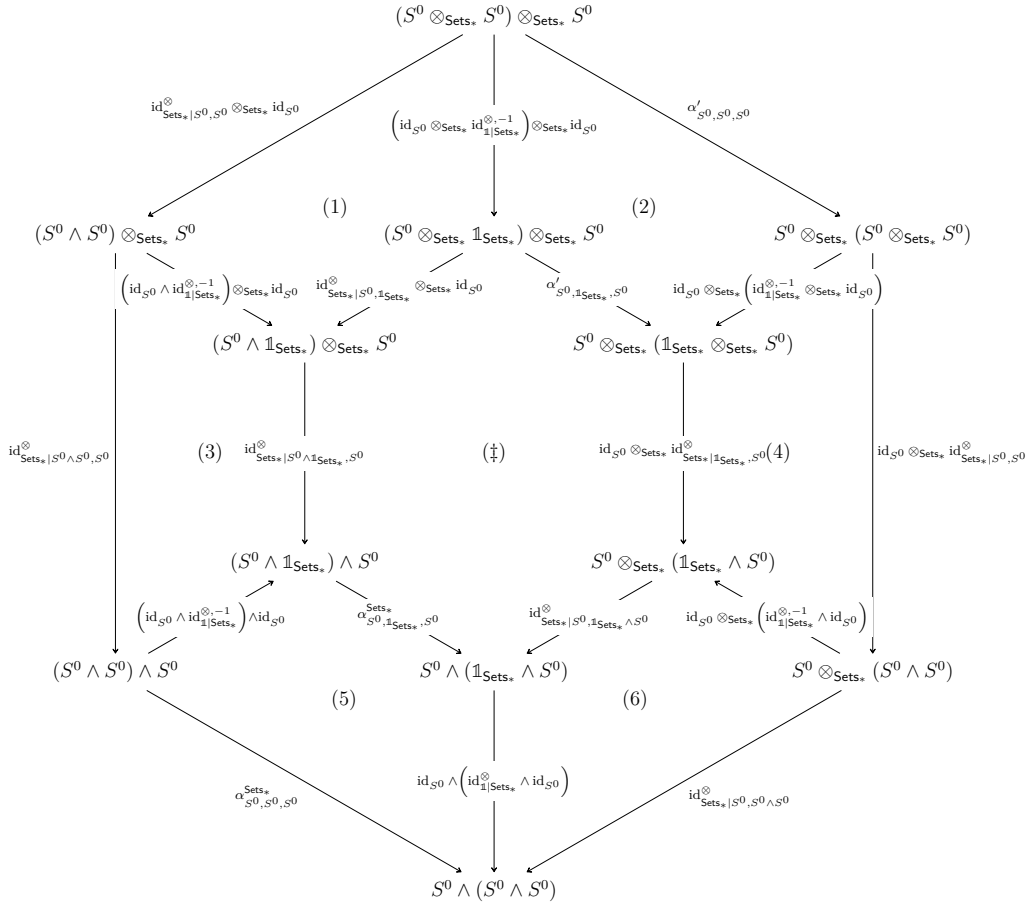
The edges and their labels are:

- $(S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow (S^0 \wedge \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow (S^0 \otimes_{\text{Sets}} S^0) \otimes_{\text{Sets}} S^0$ (labeled $(\text{id}_{S^0} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}}}) \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} S^0$ (labeled $\rho_{S^0}^{\text{Sets}} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} (\mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} S^0)$ (labeled $\alpha_{S^0, \mathbb{1}_{\text{Sets}}, S^0}'$)
- $(S^0 \wedge \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow (S^0 \wedge S^0) \otimes_{\text{Sets}} S^0$ (labeled $(\text{id}_{S^0} \wedge \text{id}_{\mathbb{1}_{\text{Sets}}}) \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \wedge \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow (S^0 \wedge S^0) \otimes_{\text{Sets}} S^0$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \wedge \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} S^0$ (labeled $\rho_{S^0}^{\text{Sets}} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \wedge \mathbb{1}_{\text{Sets}}) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} (\mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} S^0)$ (labeled $\alpha_{S^0, \mathbb{1}_{\text{Sets}}, S^0}'$)
- $(S^0 \wedge S^0) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} S^0$ (labeled $\rho_{S^0}^{\text{Sets}} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \wedge S^0) \otimes_{\text{Sets}} S^0 \rightarrow (S^0 \wedge S^0) \wedge S^0$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \wedge S^0) \otimes_{\text{Sets}} S^0 \rightarrow (S^0 \wedge S^0) \wedge \mathbb{1}_{\text{Sets}}$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{S^0}$)
- $(S^0 \wedge S^0) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \wedge S^0$ (labeled $\rho_{S^0}^{\text{Sets}} \wedge \text{id}_{S^0}$)
- $(S^0 \wedge S^0) \otimes_{\text{Sets}} S^0 \rightarrow S^0 \wedge S^0$ (labeled $\rho_{S^0 \wedge S^0}^{\text{Sets}}$)
- $S^0 \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} (\mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} S^0)$ (labeled $\text{id}_{S^0} \otimes_{\text{Sets}} \lambda_{S^0}'^{-1}$)
- $S^0 \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}}}$)
- $S^0 \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} (\mathbb{1}_{\text{Sets}} \wedge S^0)$ (labeled $\text{id}_{S^0} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} S^0}$)
- $S^0 \otimes_{\text{Sets}} S^0 \rightarrow S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}$ (labeled $\text{id}_{S^0} \otimes_{\text{Sets}} \lambda_{S^0}^{\text{Sets}, -1}$)
- $(S^0 \wedge S^0) \wedge S^0 \rightarrow (S^0 \wedge S^0) \wedge \mathbb{1}_{\text{Sets}}$ (labeled $\text{id}_{S^0} \wedge \text{id}_{\mathbb{1}_{\text{Sets}}}$)
- $(S^0 \wedge S^0) \wedge S^0 \rightarrow S^0 \wedge S^0$ (labeled $\rho_{S^0 \wedge S^0}^{\text{Sets}}$)
- $(S^0 \wedge S^0) \wedge \mathbb{1}_{\text{Sets}} \rightarrow S^0 \wedge \mathbb{1}_{\text{Sets}}$ (labeled $\rho_{S^0 \wedge \mathbb{1}_{\text{Sets}}}^{\text{Sets}}$)
- $(S^0 \wedge S^0) \wedge \mathbb{1}_{\text{Sets}} \rightarrow S^0 \wedge (\mathbb{1}_{\text{Sets}} \wedge S^0)$ (labeled $\alpha_{S^0, \mathbb{1}_{\text{Sets}}, S^0}'$)
- $S^0 \wedge S^0 \rightarrow S^0 \wedge \mathbb{1}_{\text{Sets}}$ (labeled $\text{id}_{S^0} \wedge \text{id}_{\mathbb{1}_{\text{Sets}}}$)
- $S^0 \wedge S^0 \rightarrow S^0 \wedge (\mathbb{1}_{\text{Sets}} \wedge S^0)$ (labeled $\text{id}_{S^0} \wedge \rho_{S^0}^{\text{Sets}}$)
- $S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} \rightarrow S^0 \wedge \mathbb{1}_{\text{Sets}}$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}}}$)
- $S^0 \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} \rightarrow S^0 \wedge (\mathbb{1}_{\text{Sets}} \wedge S^0)$ (labeled $\text{id}_{S^0}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}_{\text{Sets}} \wedge S^0}$)

- Subdiagrams (1), (4), (5), (8), and (11) commute by the naturality of $\text{id}_{\text{Sets}_*}^{\otimes}$;
- Subdiagram (2) commutes by the right monoidal unity of $(\text{id}_{\text{Sets}_*}^{\otimes}, \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes})$;
- Subdiagram (3) commutes by the triangle identity for $(\alpha', \lambda', \rho')$;
- Subdiagram (6) commutes by ??;
- Subdiagram (7) commutes by the naturality of ρ^{Sets_*} ;
- Subdiagram (9) commutes by ??;

- Subdiagram (10) commutes by ??;

it follows that the boundary diagram, i.e. diagram (\dagger) , also commutes. Consider now the diagram



whose boundary corresponds to diagram (\dagger) above. Since:

- Subdiagrams (1), (3), (4), and (6) commute by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes}$;
- Subdiagram (\dagger) commutes, as proved above;
- Subdiagram (2) commutes by the naturality of α' ;
- Subdiagram (5) commutes by the naturality of $\alpha^{\mathbf{Sets}_*}$;

it follows that the boundary diagram, i.e. diagram (\dagger) , also commutes. Taking

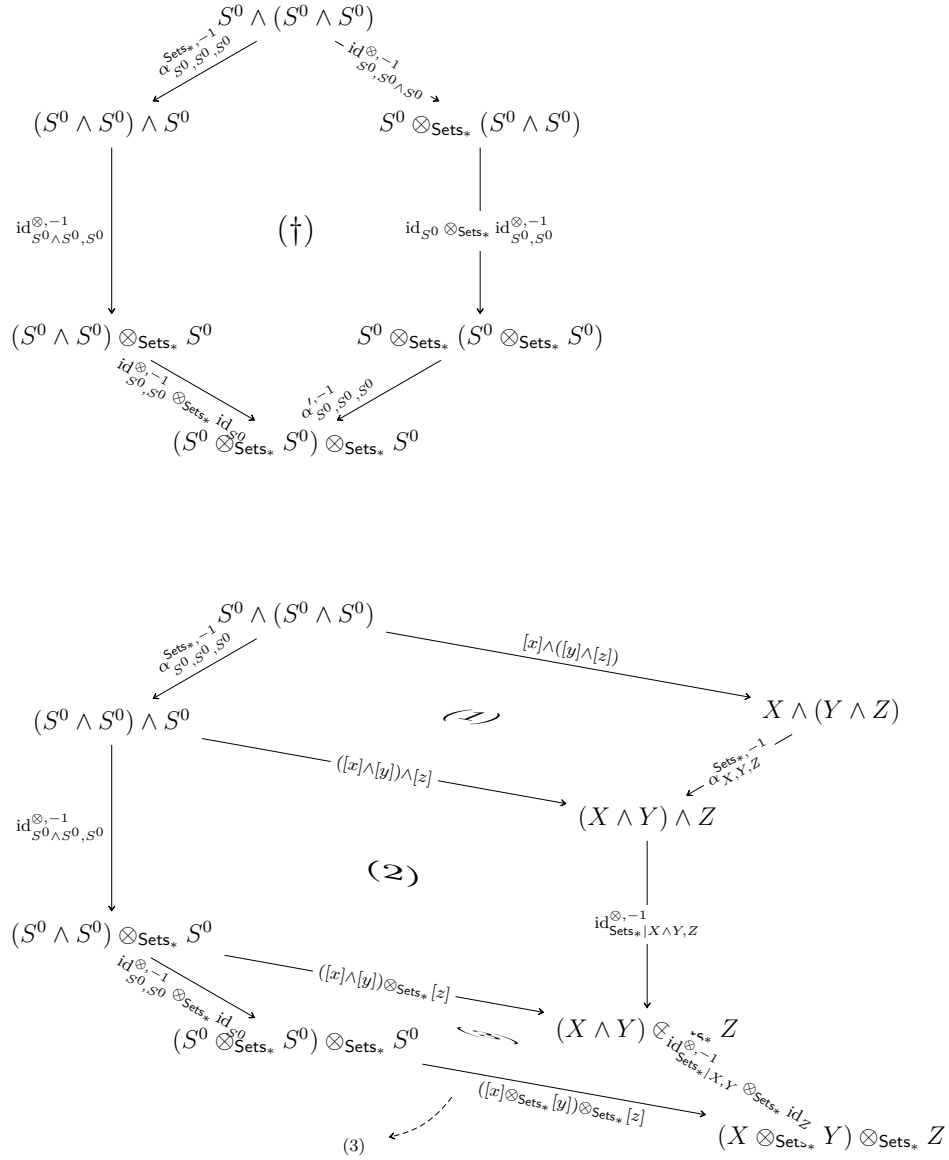
inverses on the diagram (\dagger) , we see that the diagram

$$\begin{array}{ccc}
 & S^0 \wedge (S^0 \wedge S^0) & \\
 \alpha_{S^0, S^0, S^0}^{\mathbf{Sets}_*, -1} \swarrow & & \searrow \mathrm{id}_{\mathbf{Sets}_* | S^0, S^0 \wedge S^0}^{\otimes, -1} \\
 (S^0 \wedge S^0) \wedge S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \wedge S^0) \\
 \downarrow \mathrm{id}_{\mathbf{Sets}_* | S^0 \wedge S^0, S^0}^{\otimes, -1} & (\dagger) & \downarrow \mathrm{id}_{S^0} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{\mathbf{Sets}_* | S^0, S^0}^{\otimes, -1} \\
 (S^0 \wedge S^0) \otimes_{\mathbf{Sets}_*} S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) \\
 \downarrow \mathrm{id}_{\mathbf{Sets}_* | S^0, S^0}^{\otimes, -1} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{S^0} & & \downarrow \alpha_{S^0, S^0, S^0}'^{\otimes, -1} \\
 (S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 & & (S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0
 \end{array}$$

commutes as well. Now, let $X, Y, Z \in \mathrm{Obj}(\mathbf{Sets}_*)$, let $x \in X$, let $y \in Y$, let $z \in Z$, and consider the diagram

$$\begin{array}{ccccccc}
 & S^0 \wedge (S^0 \wedge S^0) & & & & & \\
 \alpha_{S^0, S^0, S^0}^{\mathbf{Sets}_*, -1} \swarrow & \mathrm{id}_{S^0, S^0 \wedge S^0}^{\otimes, -1} & \searrow [x] \wedge ([y] \wedge [z]) & & & & \\
 (S^0 \wedge S^0) \wedge S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \wedge S^0) & & X \wedge (Y \wedge Z) & & \\
 \downarrow \mathrm{id}_{S^0 \wedge S^0, S^0}^{\otimes, -1} & \searrow ([x] \wedge [y]) \wedge [z] & \downarrow \mathrm{id}_{S^0} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{S^0, S^0}^{\otimes, -1} & \searrow \alpha_{X, Y, Z}^{\mathbf{Sets}_*, -1} & \searrow [x] \otimes_{\mathbf{Sets}_*} ([y] \wedge [z]) & \searrow \mathrm{id}_{X \wedge (Y \wedge Z)}^{\otimes, -1} & \\
 (S^0 \wedge S^0) \otimes_{\mathbf{Sets}_*} S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) & & (X \wedge Y) \wedge Z & & X \otimes_{\mathbf{Sets}_*} (Y \wedge Z) \\
 \downarrow \mathrm{id}_{S^0, S^0}^{\otimes, -1} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{S^0} & \searrow ([x] \wedge [y]) \otimes_{\mathbf{Sets}_*} [z] & \downarrow \mathrm{id}_{S^0} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{S^0, S^0}^{\otimes, -1} & \searrow [x] \otimes_{\mathbf{Sets}_*} ([y] \otimes_{\mathbf{Sets}_*} [z]) & \searrow [x] \otimes_{\mathbf{Sets}_*} ([y] \otimes_{\mathbf{Sets}_*} [z]) & \searrow \mathrm{id}_{X \otimes_{\mathbf{Sets}_*} (Y \wedge Z)} & \\
 (S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) & & (X \wedge Y) \otimes_{\mathbf{Sets}_*} Z & & X \otimes_{\mathbf{Sets}_*} (Y \otimes_{\mathbf{Sets}_*} Z) \\
 \downarrow \mathrm{id}_{S^0, S^0}^{\otimes, -1} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{S^0} & \searrow ([x] \otimes_{\mathbf{Sets}_*} [y]) \otimes_{\mathbf{Sets}_*} [z] & \downarrow \mathrm{id}_{S^0} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{S^0, S^0}^{\otimes, -1} & \searrow [x] \otimes_{\mathbf{Sets}_*} ([y] \otimes_{\mathbf{Sets}_*} [z]) & \searrow [x] \otimes_{\mathbf{Sets}_*} ([y] \otimes_{\mathbf{Sets}_*} [z]) & \searrow \mathrm{id}_{(X \wedge Y) \otimes_{\mathbf{Sets}_*} Z} & \\
 (S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) & & (X \otimes_{\mathbf{Sets}_*} Y) \otimes_{\mathbf{Sets}_*} Z & & (X \otimes_{\mathbf{Sets}_*} Y) \otimes_{\mathbf{Sets}_*} Z
 \end{array}$$

which we partition into subdiagrams as follows:



$$\begin{array}{c}
\begin{array}{ccc}
S^0 \wedge (S^0 \wedge S^0) & \xrightarrow{[x] \wedge ([y] \wedge [z])} & X \wedge (Y \wedge Z) \\
\downarrow \alpha_{S^0, S^0 \wedge S^0}^{\otimes, -1} & \nearrow (4) & \downarrow \alpha_{S^0, X \wedge Y \wedge Z}^{\otimes, -1} \\
S^0 \otimes_{\mathbf{Sets}_*} (S^0 \wedge S^0) & \xrightarrow{[x] \otimes_{\mathbf{Sets}_*} ([y] \wedge [z])} & X \otimes_{\mathbf{Sets}_*} (Y \wedge Z) \\
\downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0, S^0}^{\otimes, -1} & \searrow (5) & \downarrow \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{S^0, Y \wedge Z}^{\otimes, -1} \\
S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) & \xrightarrow{[x] \otimes_{\mathbf{Sets}_*} ([y] \otimes_{\mathbf{Sets}_*} [z])} & X \otimes_{\mathbf{Sets}_*} (Y \otimes_{\mathbf{Sets}_*} Z) \\
\downarrow \alpha'_{S^0, S^0 \otimes_{\mathbf{Sets}_*} S^0}^{-1} & \searrow (6) & \downarrow \alpha'_{X, Y \otimes_{\mathbf{Sets}_*} Z}^{-1} \\
(S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{([x] \otimes_{\mathbf{Sets}_*} [y]) \otimes_{\mathbf{Sets}_*} [z]} & (X \otimes_{\mathbf{Sets}_*} Y) \otimes_{\mathbf{Sets}_*} Z
\end{array}
\end{array}$$

Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{\mathbf{Sets}_*, -1}$.
- Subdiagram (2) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (3) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (5) commutes by the naturality of $\text{id}_{\mathbf{Sets}_*}^{\otimes, -1}$.
- Subdiagram (6) commutes by the naturality of α'^{-1} .

it follows that the diagram

$$\begin{array}{ccc}
 & S^0 \wedge (S^0 \wedge S^0) & \\
 & \downarrow [x] \wedge ([y] \wedge [z]) & \\
 & X \wedge (Y \wedge Z) & \\
 \alpha_{X,Y,Z}^{\mathbf{Sets}_*, -1} \swarrow & & \searrow \text{id}_{\mathbf{Sets}_*|X,Y \wedge Z}^{\otimes, -1} \\
 (X \wedge Y) \wedge Z & & X \otimes_{\mathbf{Sets}_*} (Y \wedge Z) \\
 \downarrow \text{id}_{\mathbf{Sets}_*|X \wedge Y, Z}^{\otimes, -1} & & \downarrow \text{id}_X \wedge \text{id}_{\mathbf{Sets}_*|Y, Z}^{\otimes, -1} \\
 (X \wedge Y) \otimes_{\mathbf{Sets}_*} Z & & X \otimes_{\mathbf{Sets}_*} (Y \otimes_{\mathbf{Sets}_*} Z) \\
 \downarrow \text{id}_{\mathbf{Sets}_*|X, Y}^{\otimes, -1} \otimes \text{id}_Z & & \downarrow \alpha'_{X,Y,Z}{}^{-1} \\
 (X \otimes_{\mathbf{Sets}_*} Y) \otimes_{\mathbf{Sets}_*} Z & &
 \end{array}$$

also commutes. We then have

$$\begin{aligned}
 & \left[\left(\text{id}_{\mathbf{Sets}_*|X, Y}^{\otimes, -1} \otimes \text{id}_Z \right) \circ \text{id}_{\mathbf{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{X,Y,Z}^{\mathbf{Sets}_*, -1} \right] (x, (y, z)) = \left[\left(\text{id}_{\mathbf{Sets}_*|X, Y}^{\otimes, -1} \otimes \text{id}_Z \right) \circ \text{id}_{\mathbf{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{X,Y,Z}^{\mathbf{Sets}_*, -1} \circ ([x] \wedge ([y] \wedge [z])) \right] (1, (1, 1)) \\
 & = \left[\alpha'_{X,Y,Z}{}^{-1} \circ \left(\text{id}_X \wedge \text{id}_{\mathbf{Sets}_*|Y, Z}^{\otimes, -1} \right) \right. \\
 & \quad \left. \circ \text{id}_{\mathbf{Sets}_*|X, Y \wedge Z}^{\otimes, -1} \circ ([x] \wedge ([y] \wedge [z])) \right] (1, (1, 1)) \\
 & = \left[\alpha'_{X,Y,Z}{}^{-1} \circ \left(\text{id}_X \wedge \text{id}_{\mathbf{Sets}_*|Y, Z}^{\otimes, -1} \right) \circ \text{id}_{\mathbf{Sets}_*|X, Y \wedge Z}^{\otimes, -1} \right] (x, (y, z))
 \end{aligned}$$

for each $(x, (y, z)) \in X \wedge (Y \wedge Z)$, and thus we have

$$\left(\text{id}_{\mathbf{Sets}_*|X, Y}^{\otimes, -1} \otimes \text{id}_Z \right) \circ \text{id}_{\mathbf{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \circ \alpha_{X,Y,Z}^{\mathbf{Sets}_*, -1} = \alpha'_{X,Y,Z}{}^{-1} \circ \left(\text{id}_X \wedge \text{id}_{\mathbf{Sets}_*|Y, Z}^{\otimes, -1} \right) \circ \text{id}_{\mathbf{Sets}_*|X, Y \wedge Z}^{\otimes, -1}.$$

Taking inverses then gives

$$\alpha_{X,Y,Z}^{\mathbf{Sets}_*} \circ \text{id}_{\mathbf{Sets}_*|X \wedge Y, Z}^{\otimes} \circ \left(\text{id}_{\mathbf{Sets}_*|X, Y}^{\otimes} \otimes \text{id}_Z \right) = \text{id}_{\mathbf{Sets}_*|X, Y \wedge Z}^{\otimes} \circ \left(\text{id}_X \wedge \text{id}_{\mathbf{Sets}_*|Y, Z}^{\otimes} \right) \circ \alpha'_{X,Y,Z},$$

showing that the diagram

$$\begin{array}{ccc}
 & (X \otimes_{\mathbf{Sets}_*} Y) \otimes_{\mathbf{Sets}_*} Z & \\
 \text{id}_{\mathbf{Sets}_*}^{\otimes} \downarrow \text{id}_{X,Y} \otimes_{\mathbf{Sets}_*} \text{id}_Z \swarrow & & \searrow \alpha'_{X,Y,Z} \\
 (X \wedge Y) \otimes_{\mathbf{Sets}_*} Z & & X \otimes_{\mathbf{Sets}_*} (Y \otimes_{\mathbf{Sets}_*} Z) \\
 \downarrow \text{id}_{\mathbf{Sets}_*}^{\otimes} \text{id}_{X \wedge Y, Z} & & \downarrow \text{id}_X \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*}^{\otimes} \text{id}_{Y,Z} \\
 (X \wedge Y) \wedge Z & & X \otimes_{\mathbf{Sets}_*} (Y \wedge Z) \\
 \swarrow \alpha_{X,Y,Z}^{\mathbf{Sets}_*} & & \nwarrow \text{id}_{\mathbf{Sets}_*}^{\otimes} \text{id}_{X,Y \wedge Z} \\
 & X \wedge (Y \wedge Z) &
 \end{array}$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\mathbf{Sets}_} \cong \wedge$:* Let $\phi, \psi: -_1 \otimes_{\mathbf{Sets}_*} -_2 \Rightarrow -_1 \wedge -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of \mathbf{Sets}_* with respect to \wedge and \otimes (as shown above), we have

$$\begin{aligned}
 \lambda'_Y &= \lambda_Y^{\mathbf{Sets}_*} \circ \phi_{S^0, Y} \circ (\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_Y), \\
 \lambda'_Y &= \lambda_Y^{\mathbf{Sets}_*} \circ \psi_{S^0, Y} \circ (\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_Y).
 \end{aligned}$$

Postcomposing both sides with $\lambda_Y^{\mathbf{Sets}_*, -1}$ and then precomposing both sides with $\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_Y$ gives

$$\begin{aligned}
 \lambda_Y^{\mathbf{Sets}_*, -1} \circ \lambda'_Y \circ (\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_Y) &= \phi_{S^0, Y}, \\
 \lambda_Y^{\mathbf{Sets}_*, -1} \circ \lambda'_Y \circ (\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_Y) &= \psi_{S^0, Y},
 \end{aligned}$$

and thus we have

$$\phi_{S^0, Y} = \psi_{S^0, Y}$$

for each $Y \in \text{Obj}(\mathbf{Sets}_*)$. Now, let $x \in X$ and consider the naturality diagrams

$$\begin{array}{ccc}
 S^0 \wedge Y & \xrightarrow{[x] \wedge \text{id}_Y} & X \wedge Y \\
 \downarrow \phi_{S^0, Y} & & \downarrow \phi_{X, Y} \\
 S^0 \otimes_{\mathbf{Sets}_*} Y & \xrightarrow{[x] \otimes_{\mathbf{Sets}_*} \text{id}_Y} & X \otimes_{\mathbf{Sets}_*} Y
 \end{array}
 \quad
 \begin{array}{ccc}
 S^0 \wedge Y & \xrightarrow{[x] \wedge \text{id}_Y} & X \wedge Y \\
 \downarrow \psi_{S^0, Y} & & \downarrow \psi_{X, Y} \\
 S^0 \otimes_{\mathbf{Sets}_*} Y & \xrightarrow{[x] \otimes_{\mathbf{Sets}_*} \text{id}_Y} & X \otimes_{\mathbf{Sets}_*} Y
 \end{array}$$

for ϕ and ψ with respect to the morphisms $[x]$ and id_Y . Having shown that $\phi_{S^0, Y} = \psi_{S^0, Y}$, we have

$$\begin{aligned}\phi_{X, Y}(x, y) &= [\phi_{X, Y} \circ ([x] \wedge \text{id}_Y)](1, y) \\ &= [[x] \otimes_{\mathbf{Sets}_*} \text{id}_Y] \circ \phi_{S^0, Y}(1, y) \\ &= [[x] \otimes_{\mathbf{Sets}_*} \text{id}_Y] \circ \psi_{S^0, Y}(1, y) \\ &= [\psi_{X, Y} \circ ([x] \wedge \text{id}_Y)](1, y) \\ &= \psi_{X, Y}(x, y)\end{aligned}$$

for each $(x, y) \in X \wedge Y$. Therefore we have

$$\phi_{X, Y} = \psi_{X, Y}$$

for each $X, Y \in \text{Obj}(\mathbf{Sets}_*)$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathbf{Sets}_*} \cong \times$ to be unique. \square

01W7 Corollary 7.5.10.1.2. The symmetric monoidal structure on the category \mathbf{Sets}_* of [Definition 7.5.9.1.1](#) is uniquely determined by the following requirements:

01W8 1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\mathbf{Sets}_*} : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of \mathbf{Sets}_* preserves colimits separately in each variable.

01W9 2. *The Unit Object Is S^0 .* We have $\mathbb{1}_{\mathbf{Sets}_*} \cong S^0$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathbf{Sets}_*)$ of ?? spanned by the symmetric monoidal categories $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$ satisfying [Items 1](#) and [2](#) is contractible.

Proof. Since \mathbf{Sets}_* is locally presentable (??), it follows from ?? that [Definition 7.5.10.1.2](#) is equivalent to the existence of an internal Hom as in [Item 1](#) of [Definition 7.5.10.1.1](#). The result then follows from [Definition 7.5.10.1.1](#). \square

01WA Corollary 7.5.10.1.3. The symmetric monoidal structure on the category \mathbf{Sets}_* is the unique symmetric monoidal structure on \mathbf{Sets}_* such that the free pointed set functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*$$

admits a symmetric monoidal structure, i.e. the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathbf{Sets}_*)$ of ?? spanned by the symmetric monoidal categories $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$ with respect to which $(-)^+$ admits a symmetric monoidal structure is contractible.

Proof. Let $(\otimes_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$ be a symmetric monoidal structure on \mathbf{Sets}_* such that $(-)^+$ admits a symmetric monoidal structure with respect to $\otimes_{\mathbf{Sets}_*}$ and \wedge . We have isomorphisms

$$\begin{aligned} X \otimes_{\mathbf{Sets}_*} Y &\cong (X^-)^+ \otimes_{\mathbf{Sets}_*} (Y^-)^+ \\ &\cong (X^- \times Y^-)^+ \\ &\cong (X^-)^+ \wedge (Y^-)^+ \\ &\cong X \wedge Y, \end{aligned}$$

all natural in X and Y . Now, since \wedge preserves colimits in both variables and $\otimes_{\mathbf{Sets}_*} \cong \wedge$, it follows that $\otimes_{\mathbf{Sets}_*}$ also preserves colimits in both variables, so the result then follows from [Definition 7.5.10.1.2](#). \square

00H6 7.5.11 Monoids With Respect to the Smash Product of Pointed Sets

00H7 **Proposition 7.5.11.1.1.** The category of monoids on $(\mathbf{Sets}_*, \wedge, S^0)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof. See ??, in particular ??, ??, ??, and ??. \square

00H8 7.5.12 Comonoids With Respect to the Smash Product of Pointed Sets

00H9 **Proposition 7.5.12.1.1.** The symmetric monoidal functor

$$((-)^+, (-)^{+, \times}, (-)^{+, \times}_{\mathbb{1}}): (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

of [Pointed Sets](#), [Item 4](#) of [Definition 6.4.1.1.2](#) lifts to an equivalence of categories

$$\begin{aligned} \text{CoMon}(\mathbf{Sets}_*, \wedge, S^0) &\stackrel{\text{eq.}}{\cong} \text{CoMon}(\mathbf{Sets}, \times, \text{pt}) \\ &\cong \mathbf{Sets}. \end{aligned}$$

Proof. We follow [\[PS19, Lemma 2.4\]](#).

Faithfulness: Given morphisms $f, g: X \rightarrow Y$, if $f^+ = g^+$, then we have

$$f(x) \stackrel{\text{def}}{=} f^+(x)$$

$$\begin{aligned}
 &= g^+(x) \\
 &\stackrel{\text{def}}{=} g(x)
 \end{aligned}$$

for each $x \in X^+$, and thus $f = g$, showing $(-)^+$ to be faithful.

Fullness: Let $f: X^+ \rightarrow Y^+$ be a morphism of comonoids in \mathbf{Sets}_* . By counitality, the diagram

$$\begin{array}{ccc}
 X^+ & \xrightarrow{f} & Y^+ \\
 \searrow \epsilon_X^+ & & \swarrow \epsilon_Y^+ \\
 & S^0 &
 \end{array}$$

commutes. If $f(x) = \star_Y$ for $x \neq \star_X$, the commutativity of this diagram then gives

$$\begin{aligned}
 1 &= \epsilon_X^+(x) \\
 &= \epsilon_Y^+(f(x)) \\
 &= \epsilon_Y^+(\star_Y) \\
 &= 0,
 \end{aligned}$$

which is a contradiction. Thus f is an active morphism of pointed sets, so there exists a map f^- such that $(f^-)^+ = f$ (**Pointed Sets, Item 1 of Definition 6.4.2.1.2**).

Essential Surjectivity: Let $(X, \Delta_X, \epsilon_X)$ be a comonoid in \mathbf{Sets}_* . We claim that

$$\Delta_X(x) = x \wedge x$$

for each $x \in X$ with $x \neq \star_X$. Indeed:

- Suppose that $x \neq \star_X$ and write $\Delta_X(x) = x_1 \wedge x_2$.
- Since $\text{id}_X \wedge \epsilon_X$ is pointed, we have

$$[\text{id}_X \wedge \epsilon_X](x_1 \wedge x_2) = \star_{X \wedge S^0}.$$

- The counitality condition for Δ_X , corresponding to the commutativity of the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\Delta_X} & X \wedge X \\
 \searrow \rho_X^{\mathbf{Sets}_*, -1} & & \downarrow \text{id}_X \wedge \epsilon_X \\
 & & X \wedge S^0
 \end{array}$$

gives

$$\begin{aligned}
 x \wedge 1 &= \rho_X^{\mathbf{Sets}_*, -1}(x) \\
 &= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\
 &= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x)) \\
 &= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\
 &= \star_{X \wedge S^0},
 \end{aligned}$$

which is a contradiction. Thus $x_1 \neq \star_X$.

- Similarly, if $x \neq \star_X$, then $x_2 \neq \star_X$.
- Next, we claim that $\epsilon_X(x_2) = 1$, as otherwise we would have

$$\begin{aligned}
 \star_{X \wedge S^0} &= x_1 \wedge 0 \\
 &= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\
 &= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x)) \\
 &= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\
 &= \rho_X^{\mathbf{Sets}_*, -1}(x) \\
 &= x \wedge 1,
 \end{aligned}$$

a contradiction. Thus $\epsilon_X(x_2) = 1$.

- Similarly, if $x \neq \star_X$, then $\epsilon_X(x_1) = 1$.
- Now, since Δ_X is counital, we have

$$\begin{aligned}
 x \wedge 1 &= \rho_X^{\mathbf{Sets}_*, -1}(x) \\
 &= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\
 &= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x)) \\
 &= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\
 &= x_1 \wedge 1,
 \end{aligned}$$

so $x = x_1$.

- Similarly, $x = x_2$, and we are done.

Next, notice that $X \cong \epsilon_X^{-1}(0) \amalg \epsilon_X^{-1}(1)$, and let $x \in \epsilon_X^{-1}(0)$. We then have

$$[(\mathrm{id}_X \wedge \epsilon_X) \circ \Delta_X](x) = [\mathrm{id}_X \wedge \epsilon_X](x \wedge x)$$

$$\begin{aligned}
&= x \wedge 0 \\
&= \star_{X \wedge S^0}.
\end{aligned}$$

The counitality condition for Δ_X then gives $x = \star_X$, so $\epsilon_X^{-1}(0) = \{\star_X\}$. Thus we have $(\epsilon_X^{-1}(1))^+ \cong X$, and this isomorphism is compatible with the comonoid structures when equipping $\epsilon_X^{-1}(1)$ with its unique comonoid structure. This shows that $(-)^+$ is essentially surjective.

Equivalence: Since $(-)^+$ is fully faithful and essentially surjective, it is an equivalence by [Categories](#), [Item 1b](#) of [Item 1](#) of [Definition 11.6.7.1.2](#). \square

00HA 7.6 Miscellany

00HB 7.6.1 The Smash Product of a Family of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00HC Definition 7.6.1.1.1. The **smash product of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pointed set $\bigwedge_{i \in I} X_i$ consisting of:

- *The Underlying Set.* The set $\bigwedge_{i \in I} X_i$ defined by

$$\bigwedge_{i \in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\prod_{i \in I} X_i$ obtained by declaring

$$(x_i)_{i \in I} \sim (y_i)_{i \in I}$$

if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i$.

- *The Basepoint.* The element $[(x_0)_{i \in I}]$ of $\bigwedge_{i \in I} X_i$.

Appendices

A Other Chapters

Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

Relations

8. [Relations](#)
9. [Constructions With Relations](#)

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Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

Monoidal Categories

13. [Constructions With Monoidal Categories](#)

Bicategories

14. [Types of Morphisms in Bicategories](#)

Extra Part

15. [Notes](#)

References

- [PS19] Maximilien Péroux and Brooke Shipley. “Coalgebras in Symmetric Monoidal Categories of Spectra”. In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. ISSN: 1532-0073. DOI: [10.4310/HHA.2019.v21.n1.a1](https://doi.org/10.4310/HHA.2019.v21.n1.a1). URL: <https://doi.org/10.4310/HHA.2019.v21.n1.a1> (cit. on p. 127).