Constructions With Sets

The Clowder Project Authors

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 4.2.4.1.1, 4.2.4.1.3, 4.2.5.1.1 and 4.2.5.1.3).

2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:

   (a) A discussion of the internal Hom of a powerset (Section 4.4.7).

   (b) A 0-categorical version of the Yoneda lemma (Presheaves and the Yoneda Lemma, Definition 12.1.5.1.1), which we term the *Yoneda lemma for sets* (Definition 4.5.5.1.1).

   (c) A characterisation of powersets as free cocompletions (Section 4.4.5), mimicking the corresponding statement for categories of presheaves (**??**).

   (d) A characterisation of powersets as free completions (Section 4.4.6), mimicking the corresponding statement for categories of copresheaves (**??**).

   (e) A (-1)-categorical version of un/straightening (Item 2 of Definition 4.5.1.1.4 and Definition 4.5.1.1.5).

   (f) A 0-categorical form of Isbell duality internal to powersets (Section 4.4.8).

3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* \colon \mathcal{P}(A) \overset{\rightrightarrows}{\to} \mathcal{P}(B)$$

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of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \to B$, including in particular:

(a) How f^{-1} can be described as a precomposition while $f_!$ and f_* can be described as Kan extensions (Definitions 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).

(b) An extensive list of the properties of $f_!$, f^{-1}, and f_* (Definitions 4.6.1.1.5, 4.6.1.1.6, 4.6.2.1.3, 4.6.2.1.4, 4.6.3.1.7 and 4.6.3.1.8).

(c) How the functors $f_!$, f^{-1}, f_*, along with the functors

$$-_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X),$$
$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

may be viewed as a six-functor formalism with the empty set $\varnothing$ as the dualising object (Section 4.6.4).

Contents

4.1 Limits of Sets

4.1.1 The Terminal Set

Definition 4.1.1.1.1. The **terminal set** is the terminal object of Sets as in Limits and Colimits, **??**.

Construction 4.1.1.1.2. Concretely, the terminal set is the pair $\left(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

1. *The Limit.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.

2. *The Cone.* The collection of maps

$$\{!_A\colon A \to \mathrm{pt}\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

   defined by

$$!_A(a) \overset{\mathrm{def}}{=} \star$$

   for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi\colon A \to \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. $\qquad\square$

4.1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 4.1.2.1.1. The **product**[1] of $\{A_i\}_{i\in I}$ is the product of $\{A_i\}_{i\in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.1.2.1.2. Concretely, the product of $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of:

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[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

1. *The Limit.* The set $\prod_{i \in I} A_i$ defined by

$$\prod_{i \in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i \in I} A_i\right) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \text{ we} \\ \text{have } f(i) \in A_i \end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i \right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i(f) \overset{\text{def}}{=} f(i)$$

for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i \in I} A_i$ is the categorical product of $\{A_i\}_{i \in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{c}
P \\
\quad \searrow {\scriptstyle p_i} \\
\prod_{i \in I} A_i \xrightarrow[\mathrm{pr}_i]{} A_i
\end{array}
$$

in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i \in I} A_i$ making the diagram

$$
\begin{array}{c}
P \\
{\scriptstyle \phi} \downarrow {\scriptstyle \exists!} \quad \searrow {\scriptstyle p_i} \\
\prod_{i \in I} A_i \xrightarrow[\mathrm{pr}_i]{} A_i
\end{array}
$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$.   □

Remark 4.1.2.1.3. Less formally, we may think of Cartesian products and projection maps as follows:

1. We think of $\prod_{i \in I} A_i$ as the set whose elements are I-indexed collections $(a_i)_{i \in I}$ with $a_i \in A_i$ for each $i \in I$.

2. We view the projection maps

$$\left\{ \mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i \right\}_{i \in I}$$

   as being given by

$$\mathrm{pr}_i \big((a_j)_{j \in I} \big) \overset{\mathrm{def}}{=} a_i$$

   for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

Proposition 4.1.2.1.4. Let $\{A_i\}_{i \in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

   where

   - *Action on Objects.* For each $(A_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i \in I} \right] \big((A_i)_{i \in I} \big) \overset{\mathrm{def}}{=} \prod_{i \in I} A_i$$

   - *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I} \right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} \colon \mathrm{Nat}\big((A_i)_{i \in I}, (B_i)_{i \in I} \big) \to \mathsf{Sets}\left(\prod_{i \in I} A_i, \prod_{i \in I} B_i \right)$$

   of $\prod_{i \in I}$ at $\big((A_i)_{i \in I}, (B_i)_{i \in I} \big)$ is defined by sending a map

$$\{ f_i \colon A_i \to B_i \}_{i \in I}$$

   in $\mathrm{Nat}\big((A_i)_{i \in I}, (B_i)_{i \in I} \big)$ to the map of sets

$$\prod_{i \in I} f_i \colon \prod_{i \in I} A_i \to \prod_{i \in I} B_i$$

   defined by

$$\left[\prod_{i \in I} f_i \right] \big((a_i)_{i \in I} \big) \overset{\mathrm{def}}{=} (f_i(a_i))_{i \in I}$$

   for each $(a_i)_{i \in I} \in \prod_{i \in I} A_i$.

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

4.1.3 Binary Products of Sets

Let A and B be sets.

Definition 4.1.3.1.1. The **product of A and B**[2] is the product of A and B in Sets as in Limits and Colimits, **??**.

Construction 4.1.3.1.2. Concretely, the product of A and B is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times B$ defined by

$$
\begin{aligned}
A \times B &\overset{\text{def}}{=} \prod_{z \in \{A,B\}} z \\
&\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\
&\cong \{\{\{a\}, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\
&\cong \left\{ \begin{array}{l} \text{ordered pairs } (a,b) \text{ with} \\ a \in A \text{ and } b \in B \end{array} \right\}.
\end{aligned}
$$

2. *The Cone.* The maps

$$
\begin{aligned}
\mathrm{pr}_1 &: A \times B \to A, \\
\mathrm{pr}_2 &: A \times B \to B
\end{aligned}
$$

   defined by

$$
\begin{aligned}
\mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a, \\
\mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b
\end{aligned}
$$

   for each $(a,b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in the category of sets. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccc}
& P & \\
{}^{p_1}\swarrow & & \searrow^{p_2} \\
A \overset{\mathrm{pr}_1}{\longleftarrow} & A \times B & \overset{\mathrm{pr}_2}{\longrightarrow} B
\end{array}
$$

---

[2] *Further Terminology:* Also called the **Cartesian product of A and B**.

in Sets. Then there exists a unique map $\phi\colon P \to A \times B$ making the diagram

$$
\begin{array}{ccc}
& P & \\
\overset{p_1}{\swarrow} & \phi \Big\downarrow \exists! & \overset{p_2}{\searrow} \\
A \underset{\mathrm{pr}_1}{\longleftarrow} & A \times B & \underset{\mathrm{pr}_2}{\longrightarrow} B
\end{array}
$$

commute, being uniquely determined by the conditions

$$
\mathrm{pr}_1 \circ \phi = p_1,
$$
$$
\mathrm{pr}_2 \circ \phi = p_2
$$

via

$$
\phi(x) = (p_1(x), p_2(x))
$$

for each $x \in P$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.1.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$
\begin{aligned}
A \times - \colon & \quad \mathsf{Sets} & \to \mathsf{Sets}, \\
- \times B \colon & \quad \mathsf{Sets} & \to \mathsf{Sets}, \\
-_1 \times -_2 \colon & \ \mathsf{Sets} \times \mathsf{Sets} & \to \mathsf{Sets},
\end{aligned}
$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$
[-_1 \times -_2](A, B) \overset{\mathrm{def}}{=} A \times B.
$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$
\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)
$$

of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$
f \times g \colon A \times B \to X \times Y
$$

defined by

$$
[f \times g](a, b) \overset{\mathrm{def}}{=} (f(a), g(b))
$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness I.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -))\colon \quad \mathsf{Sets} \overset{A \times -}{\underset{\mathsf{Sets}(A,-)}{\rightleftarrows}} \perp \mathsf{Sets},$$

$$(- \times B \dashv \mathsf{Sets}(B, -))\colon \quad \mathsf{Sets} \overset{- \times B}{\underset{\mathsf{Sets}(B,-)}{\rightleftarrows}} \perp \mathsf{Sets},$$

witnessed by bijections

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$
$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\mathsf{Sets}} \dashv -_1 \times -_2)\colon \quad \mathsf{Sets} \overset{\Delta_{\mathsf{Sets}}}{\underset{-_1 \times -_2}{\rightleftarrows}} \perp \mathsf{Sets} \times \mathsf{Sets},$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A, A), (B, C)) \cong \mathsf{Sets}(A, B \times C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$.

4. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets}}_{A,B,C}\colon (A \times B) \times C \overset{\sim}{\dashrightarrow} A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Unitality.* We have isomorphisms of sets

$$\lambda^{\mathsf{Sets}}_A\colon \mathrm{pt} \times A \overset{\sim}{\dashrightarrow} A,$$
$$\rho^{\mathsf{Sets}}_A\colon A \times \mathrm{pt} \overset{\sim}{\dashrightarrow} A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Commutativity.* We have an isomorphism of sets

$$\sigma_{A,B}^{\mathsf{Sets}} \colon A \times B \dashrightarrow^{\sim} B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\delta_{\ell}^{\mathsf{Sets}} \colon A \times (B \amalg C) \dashrightarrow^{\sim} (A \times B) \amalg (A \times C),$$
$$\delta_{r}^{\mathsf{Sets}} \colon (A \amalg B) \times C \dashrightarrow^{\sim} (A \times C) \amalg (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \emptyset \times A \dashrightarrow^{\sim} \emptyset,$$
$$\zeta_{r}^{\mathsf{Sets}} \colon A \times \emptyset \dashrightarrow^{\sim} \emptyset,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Unions.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cup W) = (U \times V) \cup (U \times W),$$
$$(U \cup V) \times W = (U \times W) \cup (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

10. *Distributivity Over Intersections.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cap W) = (U \times V) \cap (U \times W),$$
$$(U \cap V) \times W = (U \times W) \cap (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

11. *Distributivity Over Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \setminus W) = (U \times V) \setminus (U \times W),$$
$$(U \setminus V) \times W = (U \times W) \setminus (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

12. *Distributivity Over Symmetric Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \triangle W) = (U \times V) \triangle (U \times W),$$
$$(U \triangle V) \times W = (U \times W) \triangle (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \xrightarrow{\cap \times \cap} \mathcal{P}(X) \times \mathcal{P}(X)$$

$$\downarrow {\scriptstyle \mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}} \qquad\qquad\qquad\qquad \downarrow {\scriptstyle \mathcal{P}^{\times}_{X,X}}$$

$$\mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \xrightarrow{\quad\cap\quad} \mathcal{P}(X \times X)$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).$$

for each $U, V, W, T \in \mathcal{P}(X)$.

14. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}, \times, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}},\right.$ $\left.\lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\right)$ is a closed symmetric monoidal category.

15. *Symmetric Bimonoidality.* The 18-tuple

$$\left(\mathsf{Sets}, \amalg, \times, \varnothing, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}, \right.$$
$$\left. \alpha^{\mathsf{Sets},\amalg}, \lambda^{\mathsf{Sets},\amalg}, \rho^{\mathsf{Sets},\amalg}, \sigma^{\mathsf{Sets},\amalg}, \delta_{\ell}^{\mathsf{Sets}}, \delta_{r}^{\mathsf{Sets}}, \zeta_{\ell}^{\mathsf{Sets}}, \zeta_{r}^{\mathsf{Sets}}\right),$$

is a symmetric closed bimonoidal category, where $\alpha^{\mathsf{Sets},\amalg}$, $\lambda^{\mathsf{Sets},\amalg}$, $\rho^{\mathsf{Sets},\amalg}$, and $\sigma^{\mathsf{Sets},\amalg}$ are the natural transformations from Items 3 to 5 of Definition 4.2.3.1.3.

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.

Item 2, Adjointness: We prove only that there's an adjunction $- \times B \dashv \mathsf{Sets}(B, -)$, witnessed by a bijection

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathsf{Sets}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

$$\Phi_{B,C}\colon \mathsf{Sets}(A\times B, C) \to \mathsf{Sets}(A, \mathsf{Sets}(B,C)),$$

  by sending a function

$$\xi\colon A\times B \to C$$

  to the function

$$\xi^{\dagger}\colon A \longrightarrow \mathsf{Sets}(B,C),$$
$$a \mapsto \left(\xi_a^{\dagger}\colon B \to C\right),$$

  where we define

$$\xi_a^{\dagger}(b) \stackrel{\mathrm{def}}{=} \xi(a,b)$$

  for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$\xi^{\dagger} \stackrel{\mathrm{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a,b)]\!]]\!].$$

- *Map II.* We define a map

$$\Psi_{B,C}\colon \mathsf{Sets}(A, \mathsf{Sets}(B,C)), \to \mathsf{Sets}(A\times B, C)$$

  given by sending a function

$$\xi\colon A \longrightarrow \mathsf{Sets}(B,C),$$
$$a \mapsto (\xi_a\colon B \to C),$$

  to the function

$$\xi^{\dagger}\colon A\times B \to C$$

  defined by

$$\xi^{\dagger}(a,b) \stackrel{\mathrm{def}}{=} \mathrm{ev}_b(\mathrm{ev}_a(\xi))$$
$$\stackrel{\mathrm{def}}{=} \mathrm{ev}_b(\xi_a)$$
$$\stackrel{\mathrm{def}}{=} \xi_a(b)$$

  for each $(a,b) \in A\times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A \times B, C)}.$$

  Indeed, given a function $\xi \colon A \times B \to C$, we have

$$\begin{aligned}
[\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\
&= \Psi_{A,B}(\Phi_{A,B}([\![(a,b) \mapsto \xi(a,b)]\!])) \\
&= \Psi_{A,B}([\![a \mapsto [\![b \mapsto \xi(a,b)]\!]]\!]) \\
&= \Psi_{A,B}([\![a' \mapsto [\![b' \mapsto \xi(a',b')]\!]]\!]) \\
&= [\![(a,b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a([\![a' \mapsto [\![b' \mapsto \xi(a',b')]\!]]\!]))]\!] \\
&= [\![(a,b) \mapsto \mathrm{ev}_b([\![b' \mapsto \xi(a,b')]\!])]\!] \\
&= [\![(a,b) \mapsto \xi(a,b)]\!] \\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A, \mathsf{Sets}(B,C))}.$$

  Indeed, given a function

$$\xi \colon A \longrightarrow \mathsf{Sets}(B, C),$$

$$a \longmapsto (\xi_a \colon B \to C),$$

  we have

$$\begin{aligned}
[\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\overset{\mathrm{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\
&\overset{\mathrm{def}}{=} \Phi_{A,B}([\![(a,b) \mapsto \xi_a(b)]\!]) \\
&\overset{\mathrm{def}}{=} \Phi_{A,B}([\![(a',b') \mapsto \xi_{a'}(b')]\!]) \\
&\overset{\mathrm{def}}{=} [\![a \mapsto [\![b \mapsto \mathrm{ev}_{(a,b)}([\![(a',b') \mapsto \xi_{a'}(b')]\!])]\!]]\!] \\
&\overset{\mathrm{def}}{=} [\![a \mapsto [\![b \mapsto \xi_a(b)]\!]]\!] \\
&\overset{\mathrm{def}}{=} [\![a \mapsto \xi_a]\!] \\
&\overset{\mathrm{def}}{=} \xi.
\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g \colon B \to B'$, the diagram

$$\begin{array}{ccc}
\mathsf{Sets}(A \times B', C) & \xrightarrow{\Phi_{B',C}} & \mathsf{Sets}(A, \mathsf{Sets}(B', C)), \\
{\scriptstyle \mathrm{id}_A \times g^*} \downarrow & & \downarrow {\scriptstyle (g^*)_!} \\
\mathsf{Sets}(A \times B, C) & \xrightarrow[\Phi_{B,C}]{} & \mathsf{Sets}(A, \mathsf{Sets}(B, C))
\end{array}$$

commutes. Indeed, given a function

$$\xi\colon A\times B'\to C,$$

we have

$$
\begin{aligned}
[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)](\xi)&=\Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&=\Phi_{B,C}(\xi(-_1,g(-_2)))\\
&=[\xi(-_1,g(-_2))]^{\dagger}\\
&=\xi^{\dagger}_{-_1}(g(-_2))\\
&=(g^*)_!\big(\xi^{\dagger}\big)\\
&=(g^*)_!(\Phi_{B',C}(\xi))\\
&=[(g^*)_!\circ\Phi_{B',C}](\xi).
\end{aligned}
$$

Alternatively, using the $[\![a\mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$
\begin{aligned}
[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)](\xi)&=\Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\
&=\Phi_{B,C}([\mathrm{id}_A\times g^*]([\![(a,b')\mapsto\xi(a,b')]\!]))\\
&=\Phi_{B,C}([\![(a,b)\mapsto\xi(a,g(b))]\!])\\
&=[\![a\mapsto[\![b\mapsto\xi(a,g(b))]\!]]\!]\\
&=[\![a\mapsto g^*([\![b'\mapsto\xi(a,b')]\!])]\!]\\
&=(g^*)_!([\![a\mapsto[\![b'\mapsto\xi(a,b')]\!]]\!])\\
&=(g^*)_!(\Phi_{B',C}([\![(a,b')\mapsto\xi(a,b')]\!]))\\
&=(g^*)_!(\Phi_{B',C}(\xi))\\
&=[(g^*)_!\circ\Phi_{B',C}](\xi).
\end{aligned}
$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C\to C'$, the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(A\times B,C) & \xrightarrow{\ \Phi_{B,C}\ } & \mathsf{Sets}(A,\mathsf{Sets}(B,C)),\\
{\scriptstyle h_!}\downarrow & & \downarrow{\scriptstyle (h_!)_!}\\
\mathsf{Sets}(A\times B,C') & \xrightarrow[\ \Phi_{B,C'}\ ]{} & \mathsf{Sets}(A,\mathsf{Sets}(B,C'))
\end{array}
$$

commutes. Indeed, given a function

$$\xi\colon A\times B\to C,$$

we have

$$
\begin{aligned}
[\Phi_{B,C} \circ h_!](\xi) &= \Phi_{B,C}(h_!(\xi)) \\
&= \Phi_{B,C}(h_!([\![(a,b) \mapsto \xi(a,b)]\!])) \\
&= \Phi_{B,C}([\![(a,b) \mapsto h(\xi(a,b))]\!]) \\
&= [\![a \mapsto [\![b \mapsto h(\xi(a,b))]\!]]\!] \\
&= [\![a \mapsto h_!([\![b \mapsto \xi(a,b)]\!])]\!] \\
&= (h_!)_!([\![a \mapsto [\![b \mapsto \xi(a,b)]\!]]\!]) \\
&= (h_!)_!(\Phi_{B,C}([\![(a,b) \mapsto \xi(a,b)]\!])) \\
&= (h_!)_!(\Phi_{B,C}(\xi)) \\
&= [(h_!)_! \circ \Phi_{B,C}](\xi).
\end{aligned}
$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Adjointness II: This follows from the universal property of the product.

Item 4, Associativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.4.1.1.

Item 5, Unitality: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.1.5.1.1 and 5.1.6.1.1.

Item 6, Commutativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.7.1.1.

Item 7, Distributivity Over Coproducts: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.1.1.1 and 5.3.2.1.1.

Item 8, Annihilation With the Empty Set: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.3.1.1 and 5.3.4.1.1.

Item 9, Distributivity Over Unions: See [Pro25c].

Item 10, Distributivity Over Intersections: See [Pro25d, Corollary 1].

Item 11, Distributivity Over Differences: See [Pro25a].

Item 12, Distributivity Over Symmetric Differences: See [Pro25b].

Item 13, Middle-Four Exchange With Respect to Intersections: See [Pro25d, Corollary 1].

Item 14, Symmetric Monoidality: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.1.9.1.1, and is proved there.

Item 15, Symmetric Bimonoidality: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.3.5.1.1, and is proved there. $\square$

Remark 4.1.3.1.4. As shown in Item 1 of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

$$-_1 \times -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}.$$

This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k,\ell} \colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets})$$

of tensor products of $\mathbb{E}_k$-monoid objects on Sets with $\mathbb{E}_\ell$-monoid objects on Sets; see **??**.

Remark 4.1.3.1.5. We may state the equalities in Items 9 to 12 of Definition 4.1.3.1.3 as the commutativity of the following diagrams:

$$(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X))$$

$$\Delta_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \qquad \mu_4^{\mathsf{Sets}}$$

$$\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \qquad (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X))$$

$$\mathrm{id}_{\mathcal{P}(X)} \times \Delta \qquad\qquad \mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}$$

$$\mathcal{P}(X) \times \mathcal{P}(X) \qquad \mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$$

$$\mathcal{P}^{\times}_{X,X} \qquad \Delta$$

$$\mathcal{P}(X \times X)$$

$$(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X))$$

$$\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \times \Delta_{\mathcal{P}(X)} \qquad \mu_4^{\mathsf{Sets}}$$

$$(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \qquad (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X))$$

$$\Delta \times \mathrm{id}_{\mathcal{P}(X)} \qquad\qquad \mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}$$

$$\mathcal{P}(X) \times \mathcal{P}(X) \qquad \mathcal{P}(X \times X) \times \mathcal{P}(X \times X)$$

$$\mathcal{P}^{\times}_{X,X} \qquad \Delta$$

$$\mathcal{P}(X \times X)$$

4.1.4 Pullbacks

Let A, B, and C be sets and let $f \colon A \to C$ and $g \colon B \to C$ be functions.

Definition 4.1.4.1.1. The **pullback of A and B over C along f and g**[3] is the pullback of A and B over C along f and g in Sets as in Limits and Colimits, **??**.

Construction 4.1.4.1.2. Concretely, the pullback of A and B over C along f and g is the pair $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\mathrm{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

2. *The Cone.* The maps[4]

$$\mathrm{pr}_1 \colon A \times_C B \to A,$$
$$\mathrm{pr}_2 \colon A \times_C B \to B$$

defined by

$$\mathrm{pr}_1(a, b) \overset{\mathrm{def}}{=} a,$$
$$\mathrm{pr}_2(a, b) \overset{\mathrm{def}}{=} b$$

for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over

---

[3]*Further Terminology:* Also called the **fibre product of A and B over C along f and g.

[4]*Further Notation:* Also written $\mathrm{pr}_1^{A \times_C B}$ and $\mathrm{pr}_2^{A \times_C B}$.

C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,$$

$$
\begin{array}{ccc}
A \times_C B & \xrightarrow{\mathrm{pr}_2} & B \\
{\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle g} \\
A & \xrightarrow{f} & C.
\end{array}
$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$
\begin{aligned}
[f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b)) \\
&= f(a) \\
&= g(b) \\
&= g(\mathrm{pr}_2(a, b)) \\
&= [g \circ \mathrm{pr}_2](a, b),
\end{aligned}
$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$ making the diagram

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. $\square$

Remark 4.1.4.1.3. It is common practice to write $A \times_C B$ for the pullback of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \times_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

Example 4.1.4.1.4. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let X be a set. We have

$$A \cap B \cong A \times_{A \cup B} B,$$

$$
\begin{array}{ccc}
A \cap B & \longrightarrow\!\!\!\rightarrow & B \\
\downarrow & \lrcorner & \uparrow{\scriptstyle \iota_B} \\
A & \underset{\iota_A}{\lhook\joinrel\longrightarrow} & A \cup B
\end{array}
$$

for each $A, B \in \mathcal{P}(X)$.

Proof. *Item 1, Unions via Intersections*: Indeed, we have

$$A \times_{A \cup B} B \cong \{(x, y) \in A \times B \mid x = y\}$$
$$\cong A \cap B.$$

This finishes the proof. $\square$

Proposition 4.1.4.1.5. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor
$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$
where $\mathcal{P}$ is the category that looks like this:

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet. \end{array}$$
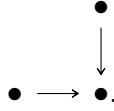
In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism
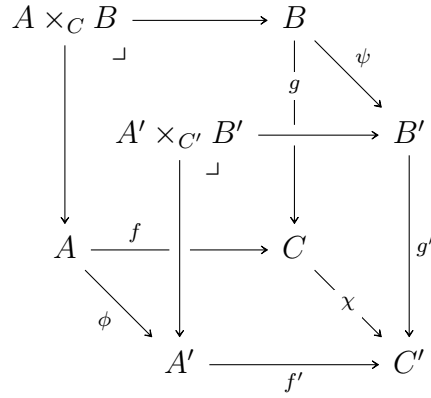
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists !} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\mathrm{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

commute.

2. *Adjointness I.* We have adjunctions

$$\left(A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)\right)\colon \quad \mathsf{Sets}_{/X} \overset{A \times_X -}{\underset{\mathbf{Sets}_{/X}(A,-)}{\rightleftarrows}} {\scriptstyle\perp} \, \mathsf{Sets}_{/X},$$

$$\left(- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)\right)\colon \quad \mathsf{Sets}_{/X} \overset{- \times_X B}{\underset{\mathbf{Sets}_{/X}(B,-)}{\rightleftarrows}} {\scriptstyle\perp} \, \mathsf{Sets}_{/X},$$

witnessed by bijections

$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}\left(A, \mathbf{Sets}_{/X}(B, C)\right),$$
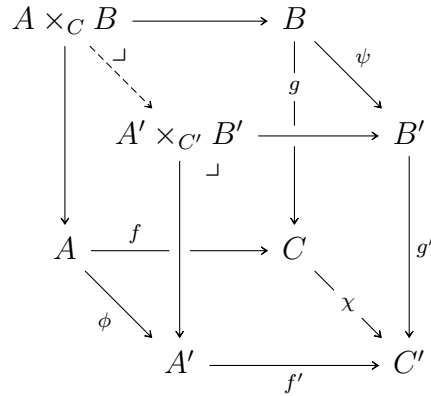
$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}\left(B, \mathbf{Sets}_{/X}(A, C)\right),$$

natural in $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \mathrm{Obj}\left(\mathsf{Sets}_{/X}\right)$, where $\mathbf{Sets}_{/X}(A, B)$ is the object of $\mathsf{Sets}_{/X}$ consisting of (see Fibred Sets, **??**):

- *The Set.* The set $\mathbf{Sets}_{/X}(A, B)$ defined by

$$\mathbf{Sets}_{/X}(A, B) \overset{\mathrm{def}}{=} \coprod_{x \in X} \mathsf{Sets}\left(\phi_A^{-1}(x), \phi_Y^{-1}(x)\right)$$

- *The Map to X.* The map

$$\phi_{\mathbf{Sets}_{/X}(A,B)}\colon \mathbf{Sets}_{/X}(A, B) \to X$$

  defined by
$$\phi_{\mathbf{Sets}_{/X}(A,B)}(x, f) \overset{\mathrm{def}}{=} x$$
  for each $(x, f) \in \mathbf{Sets}_{/X}(A, B)$.

3. *Adjointness II.* We have an adjunction

$$\left(\Delta_{\mathsf{Sets}_{/X}} \dashv -_1 \times -_2\right)\colon \quad \mathsf{Sets}_{/X} \overset{\Delta_{\mathsf{Sets}_{/X}}}{\underset{-_1 \times -_2}{\rightleftarrows}} {\scriptstyle\perp} \, \mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X},$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}}((A, A), (B, C)) \cong \mathsf{Sets}_{/X}(A, B \times_X C),$$

natural in $A \in \mathrm{Obj}\left(\mathsf{Sets}_{/X}\right)$ and in $(B, C) \in \mathrm{Obj}\left(\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}\right)$.

4. *Associativity.* Given a diagram
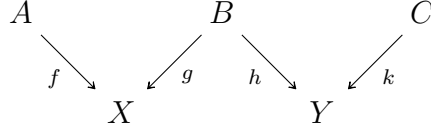
$$
\begin{array}{ccccccc}
A & & B & & C \\
& \searrow f & \swarrow g & h \searrow & \swarrow k \\
& X & & Y &
\end{array}
$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$
\begin{array}{c}
(A \times_X B) \times_Y C \\
\end{array}
\qquad
\begin{array}{c}
(A \times_X B) \times_B (B \times_Y C) \\
\end{array}
\qquad
\begin{array}{c}
A \times_X (B \times_Y C) \\
\end{array}
$$

5. *Interaction With Composition.* Given a diagram

$$
\begin{array}{ccccc}
X & & & & Y \\
& \phi \searrow & & \psi \swarrow & \\
& & A \qquad B & & \\
& & f \searrow \quad \swarrow g & & \\
& & K & &
\end{array}
$$

in Sets, we have isomorphisms of sets

$$
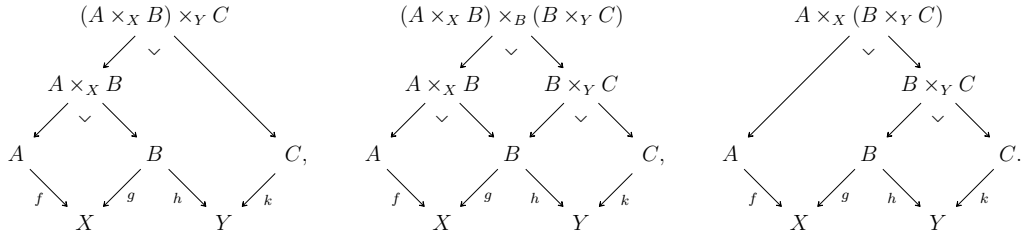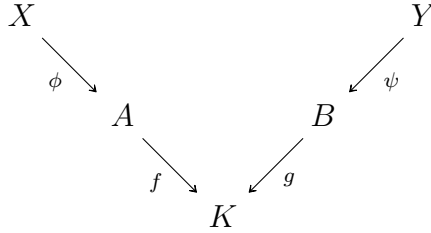X \times_K^{f \circ \phi, g \circ \psi} Y \cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f;g} B \right) \right) \times_{A \times_K^{f;g} B}^{p_2, p_1} \left(\left(A \times_K^{f;g} B \right) \times_B^{q_2, \psi} Y \right)
$$

$$
\cong X \times_A^{\phi, p} \left(\left(A \times_K^{f;g} B \right) \times_B^{q_2, \psi} Y \right)
$$

$$
\cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f;g} B \right) \right) \times_B^{q, \psi} Y
$$

where

$$
q_1 = \mathrm{pr}_1^{A \times_K^{f;g} B}, \qquad\qquad q_2 = \mathrm{pr}_2^{A \times_K^{f;g} B},
$$

$$
p_1 = \mathrm{pr}_1^{\left(A \times_K^{f;g} B \right) \times_Y^{q_2, \psi}}, \qquad\qquad p_2 = \mathrm{pr}_2^{X \times_{A \times_K^{f;g} B}^{\phi, q_1} \left(A \times_K^{f;g} B \right)},
$$

$$
p = q_1 \circ \mathrm{pr}_1^{\left(A \times_K^{f;g} B \right) \times_B^{q_2, \psi} Y}, \qquad q = q_2 \circ \mathrm{pr}_2^{X \times_A^{\phi, q_1} \left(A \times_K^{f;g} B \right)},
$$

and where these pullbacks are built as in the following diagrams:

$$
\begin{array}{c}
(X \times_A (A \times_K B)) \times_{A\times_K B} ((A \times_K B) \times_B Y) \\
\swarrow \qquad \searrow \\
X \times_A (A \times_K B) \qquad (A \times_K B) \times_B Y \\
\swarrow \qquad \searrow \quad \swarrow \qquad \searrow \\
X \qquad A \times_K B \qquad Y, \\
{}_\phi\searrow \quad \swarrow \qquad \searrow \quad \swarrow_\psi \\
A \qquad\qquad B \\
{}_f\searrow \qquad \swarrow_g \\
K
\end{array}
\qquad
\begin{array}{c}
X \times_A ((A \times_K B) \times_B Y) \\
\swarrow \qquad \searrow \\
(A \times_K B) \times_B Y \\
\swarrow \qquad \swarrow \qquad \searrow \\
X \qquad A \times_K B \qquad Y, \\
{}_\phi\searrow \quad \swarrow \qquad \searrow \quad \swarrow_\psi \\
A \qquad\qquad B \\
{}_f\searrow \qquad \swarrow_g \\
K
\end{array}
$$

$$
\begin{array}{c}
(X \times_A (A \times_K B)) \times_B Y \\
\swarrow \qquad \searrow \\
X \times_A (A \times_K B) \\
\swarrow \qquad \searrow \qquad \searrow \\
X \qquad A \times_K B \qquad Y, \\
{}_\phi\searrow \quad \swarrow \qquad \searrow \quad \swarrow_\psi \\
A \qquad\qquad B \\
{}_f\searrow \qquad \swarrow_g \\
K
\end{array}
\qquad
\begin{array}{c}
X \times_K Y \\
\swarrow \qquad\qquad \searrow \\
X \qquad\qquad\qquad Y. \\
{}_\phi\searrow \qquad\qquad \swarrow_\psi \\
A \qquad\qquad B \\
{}_f\searrow \qquad \swarrow_g \\
K
\end{array}
$$

6. *Unitality.* We have isomorphisms of sets

$$
\begin{array}{ccc}
A & = & A \\
{\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
X & = & X
\end{array}
\qquad
\begin{aligned}
\lambda_A^{\mathsf{Sets}/X} &: X \times_X A \overset{\sim}{\dashrightarrow} A, \\
\rho_A^{\mathsf{Sets}/X} &: A \times_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & X \\
\| & \lrcorner & \| \\
X & \underset{f}{\longrightarrow} & X,
\end{array}
$$

natural in $(A, f) \in \mathrm{Obj}\big(\mathsf{Sets}_{/X}\big)$.

7. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
A \times_X B & \longrightarrow & B \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\
A & \xrightarrow{\ f\ } & X,
\end{array}
\qquad
\sigma^{\mathsf{Sets}/X}_{A,B} : A \times_X B \dashrightarrow^{\sim} B \times_X A
\qquad
\begin{array}{ccc}
B \times_X A & \longrightarrow & A \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\
B & \xrightarrow{\ g\ } & X,
\end{array}
$$

natural in $(A, f), (B, g) \in \mathrm{Obj}\big(\mathsf{Sets}_{/X}\big)$.

8. *Distributivity Over Coproducts.* Let A, B, and C be sets and let $\phi_A \colon A \to X$, $\phi_B \colon B \to X$, and $\phi_C \colon C \to X$ be morphisms of sets. We have isomorphisms of sets

$$
\delta^{\mathsf{Sets}/X}_{\ell} : A \times_X (B \coprod C) \dashrightarrow^{\sim} (A \times_X B) \coprod (A \times_X C),
$$
$$
\delta^{\mathsf{Sets}/X}_{r} : (A \coprod B) \times_X C \dashrightarrow^{\sim} (A \times_X C) \coprod (B \times_X C),
$$

as in the diagrams

$$
\begin{array}{ccc}
(A \times_X B) \coprod (A \times_X C) & \twoheadrightarrow & B \coprod C \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \phi_B \coprod \phi_C} \\
A & \xrightarrow{\ \phi_A\ } & X
\end{array}
\qquad
\begin{array}{ccc}
(A \times_X C) \coprod (B \times_X C) & \twoheadrightarrow & C \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \phi_C} \\
A \coprod B & \xrightarrow{\ \phi_A \coprod \phi_B\ } & X
\end{array}
$$

natural in $A, B, C \in \mathrm{Obj}\big(\mathsf{Sets}_{/X}\big)$.

9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & \emptyset \\
\big\downarrow & \lrcorner & \big\downarrow \\
A & \xrightarrow{\ f\ } & X,
\end{array}
\qquad
\begin{array}{c}
\zeta^{\mathsf{Sets}/X}_{\ell} : A \times_X \emptyset \dashrightarrow^{\sim} \emptyset, \\[4pt]
\zeta^{\mathsf{Sets}/X}_{r} : \emptyset \times_X A \dashrightarrow^{\sim} \emptyset,
\end{array}
\qquad
\begin{array}{ccc}
\emptyset & \longrightarrow & A \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\
\emptyset & \longrightarrow & X,
\end{array}
$$

natural in $(A, f) \in \mathrm{Obj}\big(\mathsf{Sets}_{/X}\big)$.

10. *Interaction With Products.* We have an isomorphism of sets

$$
A \times_{\mathrm{pt}} B \cong A \times B,
\qquad
\begin{array}{ccc}
A \times B & \longrightarrow & B \\
\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle !_B} \\
A & \xrightarrow{\ !_A\ } & \mathrm{pt}.
\end{array}
$$

11. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}_{/X},\ \times_X,\ X,\ \mathbf{Sets}_{/X},\ \alpha^{\mathsf{Sets}_{/X}},\right.$ $\left. \lambda^{\mathsf{Sets}_{/X}},\ \rho^{\mathsf{Sets}_{/X}},\ \sigma^{\mathsf{Sets}_{/X}}\right)$ is a symmetric closed monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Adjointness I: This is a repetition of Fibred Sets, **??** of **??**, and is proved there.

Item 3, Adjointness II: This follows from the universal property of the product (pullbacks are products in $\mathsf{Sets}_{/X}$).

Item 4, Associativity: We have

$$
\begin{aligned}
(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\
&\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\
&\cong A \times_X (B \times_Y C)
\end{aligned}
$$

and

$$
\begin{aligned}
(A \times_X B) \times_B (B \times_Y C) &\cong \left\{ \left((a,b),(b',c)\right) \in (A \times_X B) \times (B \times_Y C) \,\middle|\, b = b' \right\} \\
&\cong \left\{ \left((a,b),(b',c)\right) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\
&\cong \left\{ \left(a,(b,(b',c))\right) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\
&\cong \left\{ \left(a,((b,b'),c)\right) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h(b') = k(c) \end{array} \right\} \\
&\cong \left\{ \left(a,((b,b'),c)\right) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array} \right\} \\
&\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\
&\cong A \times_X (B \times_Y C),
\end{aligned}
$$

where we have used Item 6 for the isomorphism $B \times_B B \cong B$.

Item 5, Interaction With Composition: By Item 4, it suffices to construct only the isomorphism

$$
X \times_K^{f \circ \phi, g \circ \psi} Y \cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f,g} B \right) \right) \times_{A \times_K^{f,g} B}^{p_2, p_1} \left(\left(A \times_K^{f,g} B \right) \times_B^{q_2, \psi} Y \right).
$$

We have

$$
\left(X \times_A^{\phi, q_1} \left(A \times_K^{f,g} B \right) \right) \overset{\text{def}}{=} \left\{ (x, (a,b)) \in X \times \left(A \times_K^{f,g} B \right) \,\middle|\, \phi(x) = q_1(a,b) \right\}
$$

$$\overset{\text{def}}{=} \left\{ (x, (a, b)) \in X \times \left(A \times_K^{f;g} B \right) \,\middle|\, \phi(x) = a \right\}$$
$$\cong \{ (x, (a, b)) \in X \times (A \times B) \mid \phi(x) = a \text{ and } f(a) = g(b) \},$$
$$\left(\left(A \times_K^{f;g} B \right) \times_B^{q_2, \psi} Y \right) \overset{\text{def}}{=} \left\{ ((a, b), y) \in \left(A \times_K^{f;g} B \right) \times Y \,\middle|\, q_2(a, b) = \psi(y) \right\}$$
$$\overset{\text{def}}{=} \left\{ ((a, b), y) \in \left(A \times_K^{f;g} B \right) \times Y \,\middle|\, b = \psi(y) \right\}$$
$$\cong \{ ((a, b), y) \in (A \times B) \times Y \mid b = \psi(y) \text{ and } f(a) = g(b) \},$$

so writing

$$S = \left(X \times_A^{\phi, q_1} \left(A \times_K^{f;g} B \right) \right)$$
$$S' = \left(\left(A \times_K^{f;g} B \right) \times_B^{q_2, \psi} Y \right),$$

we have

$$S \times_{A \times_K^{f;g} B}^{p_2, p_1} S' \overset{\text{def}}{=} \{ ((x, (a, b)), ((a', b'), y)) \in S \times S' \mid p_1(x, (a, b)) = p_2((a', b'), y) \}$$
$$\overset{\text{def}}{=} \{ ((x, (a, b)), ((a', b'), y)) \in S \times S' \mid (a, b) = (a', b') \}$$
$$\cong \{ ((x, a, b, y)) \in X \times A \times B \times Y \mid \phi(x) = a, \, \psi(y) = b, \text{ and } f(a) = g(b) \}$$
$$\cong \{ ((x, a, b, y)) \in X \times A \times B \times Y \mid f(\phi(x)) = g(\psi(y)) \}$$
$$\overset{\text{def}}{=} X \times_K Y.$$

This finishes the proof.
Item 6, Unitality: We have

$$X \times_X A \cong \{ (x, a) \in X \times A \mid f(a) = x \},$$
$$A \times_X X \cong \{ (a, x) \in X \times A \mid f(a) = x \},$$

which are isomorphic to A via the maps $(x, a) \mapsto a$ and $(a, x) \mapsto a$. The proof of the naturality of $\lambda^{\mathsf{Sets}/X}$ and $\rho^{\mathsf{Sets}/X}$ is omitted.
Item 7, Commutativity: We have

$$A \times_C B \overset{\text{def}}{=} \{ (a, b) \in A \times B \mid f(a) = g(b) \}$$
$$= \{ (a, b) \in A \times B \mid g(b) = f(a) \}$$
$$\cong \{ (b, a) \in B \times A \mid g(b) = f(a) \}$$
$$\overset{\text{def}}{=} B \times_C A.$$

The proof of the naturality of $\sigma^{\mathsf{Sets}/X}$ is omitted.
Item 8, Distributivity Over Coproducts: We have

$$A \times_X (B \coprod C) \overset{\text{def}}{=} \left\{ (a, z) \in A \times (B \coprod C) \,\middle|\, \phi_A(a) = \phi_{B \coprod C}(z) \right\}$$

$$= \Big\{(a, z) \in A \times (B \textstyle\coprod C) \Bigm| z = (0, b) \text{ and } \phi_A(a) = \phi_{B\coprod C}(z)\Big\}$$
$$\cup \Big\{(a, z) \in A \times (B \textstyle\coprod C) \Bigm| z = (1, c) \text{ and } \phi_A(a) = \phi_{B\coprod C}(z)\Big\}$$
$$= \{(a, z) \in A \times (B \textstyle\coprod C) \mid z = (0, b) \text{ and } \phi_A(a) = \phi_B(b)\}$$
$$\cup \{(a, z) \in A \times (B \textstyle\coprod C) \mid z = (1, c) \text{ and } \phi_A(a) = \phi_C(c)\}$$
$$\cong \{(a, b) \in A \times B \mid \phi_A(a) = \phi_B(b)\}$$
$$\cup \{(a, c) \in A \times C \mid \phi_A(a) = \phi_C(c)\}$$
$$\stackrel{\text{def}}{=} (A \times_X B) \cup (A \times_X C)$$
$$\cong (A \times_X B) \textstyle\coprod (A \times_X C),$$

with the construction of the isomorphism

$$\delta_r^{\mathsf{Sets}/X} \colon (A \textstyle\coprod B) \times_X C \dashrightarrow^{\sim} (A \times_X C) \textstyle\coprod (B \times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\mathsf{Sets}/X}$ and $\delta_r^{\mathsf{Sets}/X}$ is omitted.
Item 9, Annihilation With the Empty Set: We have

$$A \times_X \emptyset \stackrel{\text{def}}{=} \{(a, b) \in A \times \emptyset \mid f(a) = g(b)\}$$
$$= \{k \in \emptyset \mid f(a) = g(b)\}$$
$$= \emptyset,$$

and similarly for $\emptyset \times_X A$, where we have used Item 8 of Definition 4.1.3.1.3.
The proof of the naturality of $\zeta_\ell^{\mathsf{Sets}/X}$ and $\zeta_r^{\mathsf{Sets}/X}$ is omitted.
Item 10, Interaction With Products: We have

$$A \times_{\mathrm{pt}} B \stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid {!}_A(a) = {!}_B(b)\}$$
$$\stackrel{\text{def}}{=} \{(a, b) \in A \times B \mid \star = \star\}$$
$$= \{(a, b) \in A \times B\}$$
$$= A \times B.$$

Item 11, Symmetric Monoidality: Omitted. $\square$

4.1.5   Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 4.1.5.1.1. The **equaliser of f and g** is the equaliser of f and g in Sets as in Limits and Colimits, **??**.

Construction 4.1.5.1.2. Concretely, the equaliser of f and g is the pair $(\mathrm{Eq}(f, g), \mathrm{eq}(f, g))$ consisting of:

1. *The Limit.* The set $\mathrm{Eq}(f, g)$ defined by

$$\mathrm{Eq}(f, g) \overset{\mathrm{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

2. *The Cone.* The inclusion map

$$\mathrm{eq}(f, g) \colon \mathrm{Eq}(f, g) \hookrightarrow A.$$

Proof. We claim that $\mathrm{Eq}(f, g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\mathrm{Eq}(f, g) \xrightarrow{\ \mathrm{eq}(f,g)\ } A \underset{g}{\overset{f}{\rightrightarrows}} B$$

$$\nearrow{\scriptstyle e}$$

$$E$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f, g)$ making the diagram

$$\mathrm{Eq}(f, g) \xrightarrow{\ \mathrm{eq}(f,g)\ } A \underset{g}{\overset{f}{\rightrightarrows}} B$$

$$\phi \uparrow \exists! \qquad \nearrow{\scriptstyle e}$$

$$E$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. $\square$

Proposition 4.1.5.1.3. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[5]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

---

[5]That is, the following three ways of forming "the" equaliser of (f, g, h) agree:

1. Take the equaliser of (f, g, h), i.e. the limit of the diagram

$$A \overset{f}{\underset{h}{\overset{\longrightarrow}{-g\to}}} B$$

   in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

   and then take the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \overset{f}{\underset{h}{\rightrightarrows}} B,$$

   obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))$$

   of $\mathrm{Eq}(f, g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \overset{g}{\underset{h}{\rightrightarrows}} B$$

   and then take the equaliser of the composition

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \overset{f}{\underset{g}{\rightrightarrows}} B,$$

   obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))$$

   of $\mathrm{Eq}(g, h)$.

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \xrightarrow[\substack{- g \to \\ h}]{f} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f, g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1, Associativity*: We first prove that $\mathrm{Eq}(f, g, h)$ is indeed given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\mathrm{Eq}(f, g, h) \xrightarrow{\mathrm{eq}(f, g, h)} A \xrightarrow[\substack{- g \to \\ h}]{f} B$$
$$\overset{e}{\nearrow}$$
$$E$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f, g, h)$, uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g, h)$.

We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) \cong \mathrm{Eq}(f, g, h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)).$$

Indeed, we have

$$
\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) &\cong \{x \in \mathrm{Eq}(g, h) \mid [f \circ \mathrm{eq}(g, h)](a) = [g \circ \mathrm{eq}(g, h)](a)\} \\
&\cong \{x \in \mathrm{Eq}(g, h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f, g, h).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) &\cong \{x \in \mathrm{Eq}(f, g) \mid [f \circ \mathrm{eq}(f, g)](a) = [h \circ \mathrm{eq}(f, g)](a)\} \\
&\cong \{x \in \mathrm{Eq}(f, g) \mid f(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f, g, h).
\end{aligned}
$$

Item 4, Unitality: Indeed, we have

$$
\begin{aligned}
\mathrm{Eq}(f, f) &\overset{\mathrm{def}}{=} \{a \in A \mid f(a) = f(a)\} \\
&= A.
\end{aligned}
$$

Item 5, Commutativity: Indeed, we have

$$\mathrm{Eq}(f, g) \overset{\mathrm{def}}{=} \{a \in A \mid f(a) = g(a)\}$$

$$= \{a \in A \mid g(a) = f(a)\}$$
$$\overset{\text{def}}{=} \text{Eq}(g, f).$$

Item 6, Interaction With Composition: Indeed, we have

$$\text{Eq}(h \circ f \circ \text{eq}(f,g), k \circ g \circ \text{eq}(f,g)) \cong \{a \in \text{Eq}(f,g) \mid h(f(a)) = k(g(a))\}$$
$$\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.$$

and

$$\text{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\text{Eq}(h \circ f \circ \text{eq}(f,g), k \circ g \circ \text{eq}(f,g))$ to $\text{Eq}(h \circ f, k \circ g)$. $\qquad\square$

4.1.6   Inverse Limits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \to \mathsf{Sets}$ be an inverse system of sets.

Definition 4.1.6.1.1. The **inverse limit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.1.6.1.2. Concretely, the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the pair $\left(\varprojlim_{\alpha \in I}(X_\alpha), \{\text{pr}_\alpha\}_{\alpha \in I} \right)$ consisting of:

1. *The Limit.* The set $\varprojlim_{\alpha \in I}(X_\alpha)$ defined by

$$\varprojlim_{\alpha \in I}(X_\alpha) \overset{\text{def}}{=} \left\{ (x_\alpha)_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha \ \middle| \ \begin{array}{l} \text{for each } \alpha, \beta \in I, \text{ if } \alpha \preceq \beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta) \end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \text{pr}_\gamma \colon \varprojlim_{\alpha \in I}(X_\alpha) \to X_\gamma \right\}_{\gamma \in I}$$

   of maps of sets defined as the restriction of the maps

$$\left\{ \text{pr}_\gamma \colon \prod_{\alpha \in I} X_\alpha \to X_\gamma \right\}_{\gamma \in I}$$

   of Item 2 of Definition 4.1.2.1.2 to $\varprojlim_{\alpha \in I}(X_\alpha)$ and hence given by

$$\text{pr}_\gamma\left((x_\alpha)_{\alpha \in I} \right) \overset{\text{def}}{=} x_\gamma$$

   for each $\gamma \in I$ and each $(x_\alpha)_{\alpha \in I} \in \varprojlim_{\alpha \in I}(X_\alpha)$.

Proof. We claim that $\varprojlim_{\alpha \in I}(X_\alpha)$ is the limit of the inverse system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$. First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta} \circ \mathrm{pr}_\alpha = \mathrm{pr}_\beta,$$

$$
\begin{array}{ccc}
 & \varprojlim_{\alpha \in I}(X_\alpha) & \\
\mathrm{pr}_\alpha \swarrow & & \searrow \mathrm{pr}_\beta \\
X_\alpha & \xrightarrow{\;\;f_{\alpha\beta}\;\;} & X_\beta
\end{array}
$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma \in I} \in \varprojlim_{\gamma \in I}(X_\gamma)$, we have

$$
\begin{aligned}
[f_{\alpha\beta} \circ \mathrm{pr}_\alpha]\big((x_\gamma)_{\gamma \in I}\big) &\overset{\mathrm{def}}{=} f_{\alpha\beta}\Big(\mathrm{pr}_\alpha\big((x_\gamma)_{\gamma \in I}\big)\Big) \\
&\overset{\mathrm{def}}{=} f_{\alpha\beta}(x_\alpha) \\
&= x_\beta \\
&\overset{\mathrm{def}}{=} \mathrm{pr}_\beta\Big((x_\gamma)_{\gamma \in I}\Big),
\end{aligned}
$$

where the third equality comes from the definition of $\varprojlim_{\alpha \in I}(X_\alpha)$. Next, we prove that $\varprojlim_{\alpha \in I}(X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$
\begin{array}{ccc}
 & L & \\
p_\alpha \swarrow & & \searrow p_\beta \\
 & \varprojlim_{\alpha \in I}(X_\alpha) & \\
\mathrm{pr}_\alpha \swarrow & & \searrow \mathrm{pr}_\beta \\
X_\alpha & \xrightarrow{\;\;f_{\alpha\beta}\;\;} & X_\beta
\end{array}
$$

in Sets. Then there indeed exists a unique map $\phi \colon L \xrightarrow{\exists!} \varprojlim_{\alpha \in I}(X_\alpha)$ making

the diagram

commute, being uniquely determined by the family of conditions

$$\{p_\alpha = \mathrm{pr}_\alpha \circ \phi\}_{\alpha \in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha \in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha \in I} \in \prod_{\alpha \in I} X_\alpha$ indeed lies in $\varprojlim_{\alpha \in I}(X_\alpha)$, as we have

$$f_{\alpha\beta}(p_\alpha(\ell)) \overset{\mathrm{def}}{=} [f_{\alpha\beta} \circ p_\alpha](\ell)$$
$$\overset{\mathrm{def}}{=} p_\beta(\ell)$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $\left(L, \{p_\alpha\}_{\alpha \in I}\right)$. $\square$

Example 4.1.6.1.3. Here are some examples of inverse limits of sets.

1. *The p-Adic Integers.* The ring of p-adic integers $\mathbb{Z}_p$ of **??** is the inverse limit

    $$\mathbb{Z}_p \cong \varprojlim_{n \in \mathbb{N}}\left(\mathbb{Z}_{/p^n}\right);$$

    see **??**.

2. *Rings of Formal Power Series.* The ring $R[\![t]\!]$ of formal power series in a variable t is the inverse limit

    $$R[\![t]\!] \cong \varprojlim_{n \in \mathbb{N}}(R[t]/t^n R[t]);$$

    see **??**.

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see **??**.

4.2 Colimits of Sets

4.2.1 The Initial Set

Definition 4.2.1.1.1. The **initial set** is the initial object of Sets as in Limits and Colimits, **??**.

Construction 4.2.1.1.2. Concretely, the initial set is the pair $\left(\emptyset, \{\iota_A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

1. *The Colimit.* The empty set $\emptyset$ of Definition 4.3.1.1.1.

2. *The Cocone.* The collection of maps

$$\{\iota_A \colon \emptyset \to A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

   given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi \colon \emptyset \to A$ making the diagram

$$\emptyset \xdashrightarrow[\exists!]{\phi} A$$

commute, namely the inclusion map ι_A. $\qquad\square$

4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

Definition 4.2.2.1.1. The **coproduct of** $\{A_i\}_{i \in I}$[6] is the coproduct of $\{A_i\}_{i \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.2.2.1.2. Concretely, the disjoint union of $\{A_i\}_{i \in I}$ is the pair $\left(\coprod_{i \in I} A_i, \{\mathrm{inj}_i\}_{i \in I}\right)$ consisting of:

1. *The Colimit.* The set $\coprod_{i \in I} A_i$ defined by

$$\coprod_{i \in I} A_i \overset{\mathrm{def}}{=} \left\{(i, x) \in I \times \left(\bigcup_{i \in I} A_i\right) \,\middle|\, x \in A_i\right\}.$$

---

[6] *Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i \in I}$.

2. *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon A_i \to \coprod_{i \in I} A_i \right\}_{i \in I}$$

of maps given by

$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i, x)$$

for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i \in I} A_i$ is the categorical coproduct of $\{A_i\}_{i \in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{ccc}
& & C \\
& \nearrow & \\
\iota_i & & \\
& & \\
A_i & \xrightarrow[\mathrm{inj}_i]{} & \coprod_{i \in I} A_i
\end{array}
$$

in Sets. Then there exists a unique map $\phi \colon \coprod_{i \in I} A_i \to C$ making the diagram

$$
\begin{array}{ccc}
& & C \\
& \nearrow & \uparrow \\
\iota_i & & \phi \mid \exists! \\
& & \\
A_i & \xrightarrow[\mathrm{inj}_i]{} & \coprod_{i \in I} A_i
\end{array}
$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i, x)) = \iota_i(x)$$

for each $(i, x) \in \coprod_{i \in I} A_i$. $\square$

Proposition 4.2.2.1.3. Let $\{A_i\}_{i \in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \coprod_{i \in I} A_i$ defines a functor

$$\coprod_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]\left((A_i)_{i\in I}\right) \overset{\mathrm{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I}, (B_i)_{i\in I}} : \mathrm{Nat}\left((A_i)_{i\in I}, (B_i)_{i\in I}\right) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $\left((A_i)_{i\in I}, (B_i)_{i\in I}\right)$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}\left((A_i)_{i\in I}, (B_i)_{i\in I}\right)$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i, a) \overset{\mathrm{def}}{=} f_i(a)$$

for each $(i, a) \in \coprod_{i\in I} A_i$.

Proof. Item 1, Functoriality: This follows from Limits and Colimits, **??** of **??**. $\qquad\square$

4.2.3   Binary Coproducts

Let A and B be sets.

Definition 4.2.3.1.1. The **coproduct of A and B**[7] is the coproduct of A and B in Sets as in Limits and Colimits, **??**.

Construction 4.2.3.1.2. Concretely, the coproduct of A and B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

---

[7] *Further Terminology:* Also called the **disjoint union of A and B**.

1. *The Colimit.* The set $A \amalg B$ defined by

$$A \amalg B \overset{\text{def}}{=} \coprod_{z \in \{A,B\}} z$$
$$\overset{\text{def}}{=} \{(0,a) \in S \mid a \in A\} \cup \{(1,b) \in S \mid b \in B\},$$

where $S = \{0,1\} \times (A \cup B)$.

2. *The Cocone.* The maps

$$\mathrm{inj}_1 \colon A \to A \amalg B,$$
$$\mathrm{inj}_2 \colon B \to A \amalg B,$$

given by

$$\mathrm{inj}_1(a) \overset{\text{def}}{=} (0,a),$$
$$\mathrm{inj}_2(b) \overset{\text{def}}{=} (1,b),$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \amalg B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccc}
 & C & \\
\iota_1 \nearrow & & \nwarrow \iota_2 \\
A \xrightarrow[\mathrm{inj}_1]{} & A \amalg B & \xleftarrow[\mathrm{inj}_2]{} B
\end{array}
$$

in Sets. Then there exists a unique map $\phi \colon A \amalg B \to C$ making the diagram

$$
\begin{array}{ccc}
 & C & \\
\iota_1 \nearrow & \uparrow \phi \,\, \exists! & \nwarrow \iota_2 \\
A \xrightarrow[\mathrm{inj}_1]{} & A \amalg B & \xleftarrow[\mathrm{inj}_2]{} B
\end{array}
$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_A = \iota_A,$$
$$\phi \circ \mathrm{inj}_B = \iota_B$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0,a), \\ \iota_B(b) & \text{if } x = (1,b) \end{cases}$$

for each $x \in A \amalg B$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4.2.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \amalg B$ defines functors

$$
\begin{aligned}
A \amalg - : & \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\
- \amalg B : & \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\
-_1 \amalg -_2 : & \ \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},
\end{aligned}
$$

where $-_1 \amalg -_2$ is the functor where

* *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$
[-_1 \amalg -_2](A, B) \overset{\mathrm{def}}{=} A \amalg B.
$$

* *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$
\amalg_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \amalg B, X \amalg Y)
$$

of $\amalg$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$
f \amalg g \colon A \amalg B \to X \amalg Y
$$

defined by

$$
[f \amalg g](x) \overset{\mathrm{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}
$$

for each $x \in A \amalg B$.

and where $A \amalg -$ and $- \amalg B$ are the partial functors of $-_1 \amalg -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have an adjunction

$$
(-_1 \amalg -_2 \dashv \Delta_{\mathsf{Sets}}) \colon \quad \mathsf{Sets} \times \mathsf{Sets} \underset{\Delta_{\mathsf{Sets}}}{\overset{-_1 \amalg -_2}{\rightleftarrows}} \bot \ \mathsf{Sets},
$$

witnessed by a bijection

$$
\mathsf{Sets}(A \amalg B, C), \cong \mathrm{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A, B), (C, C))
$$

natural in $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$ and in $C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \colon (X \coprod Y) \coprod Z \overset{\sim}{\dashrightarrow} X \coprod (Y \coprod Z),$$

natural in $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\lambda_X^{\mathsf{Sets},\coprod} \colon \varnothing \coprod X \overset{\sim}{\dashrightarrow} X,$$
$$\rho_X^{\mathsf{Sets},\coprod} \colon X \coprod \varnothing \overset{\sim}{\dashrightarrow} X,$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod} \colon X \coprod Y \overset{\sim}{\dashrightarrow} Y \coprod X,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Symmetric Monoidality.* The 7-tuple $\left(\mathsf{Sets},\ \coprod,\ \varnothing,\ \alpha_{\coprod}^{\mathsf{Sets}},\ \lambda_{\coprod}^{\mathsf{Sets}},\ \rho_{\coprod}^{\mathsf{Sets}},\right.$ $\left.\sigma^{\mathsf{Sets}}\right)$ is a symmetric monoidal category.

Proof. Item 1, Functoriality: This follows from Limits and Colimits, **??** of **??**.
Item 2, Adjointness: This follows from the universal property of the coproduct.
Item 3, Associativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.3.1.1.
Item 4, Unitality: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.2.4.1.1 and 5.2.5.1.1.
Item 5, Commutativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.6.1.1.
Item 6, Symmetric Monoidality: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.2.7.1.1, and is proved there. □

4.2.4 Pushouts

Let A, B, and C be sets and let $f \colon C \to A$ and $g \colon C \to B$ be functions.

Definition 4.2.4.1.1. The **pushout of A and B over C along f and**

g[8] is the pushout of A and B over C along f and g in Sets as in Limits and Colimits, **??**.

Construction 4.2.4.1.2. Concretely, the pushout of A and B over C along f and g is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

1. *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

   where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

2. *The Cocone.* The maps

$$\mathrm{inj}_1 \colon A \to A \coprod_C B,$$
$$\mathrm{inj}_2 \colon B \to A \coprod_C B$$

   given by

$$\mathrm{inj}_1(a) \overset{\text{def}}{=} [(0, a)]$$
$$\mathrm{inj}_2(b) \overset{\text{def}}{=} [(1, b)]$$

   for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g,$$

$$
\begin{array}{ccc}
A \coprod_C B & \xleftarrow{\ \mathrm{inj}_2\ } & B \\
{\scriptstyle \mathrm{inj}_1}\uparrow & & \uparrow{\scriptstyle g} \\
A & \xleftarrow{\ f\ } & C.
\end{array}
$$

Indeed, given $c \in C$, we have

$$[\mathrm{inj}_1 \circ f](c) = \mathrm{inj}_1(f(c))$$
$$= [(0, f(c))]$$

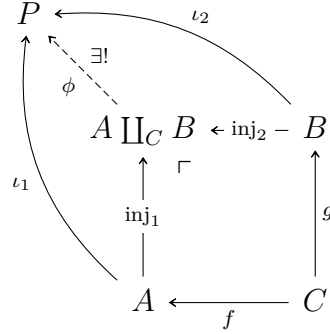[8] *Further Terminology:* Also called the **fibre coproduct of A and B over C along**

$$= [(1, g(c))]$$
$$= \mathrm{inj}_2(g(c))$$
$$= [\mathrm{inj}_2 \circ g](c),$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \amalg B$. Next, we prove that $A \amalg_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form
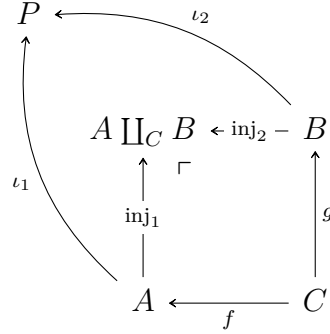
in Sets. Then there exists a unique map $\phi \colon A \amalg_C B \to P$ making the diagram

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_1 = \iota_1,$$
$$\phi \circ \mathrm{inj}_2 = \iota_2$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

f **and** g.

for each $x \in A \amalg_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \amalg B$ as follows:

1. *Case 1:* Suppose we have $x = [(0,a)] = [(0,a')]$ for some $a, a' \in A$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

2. *Case 2:* Suppose we have $x = [(1,b)] = [(1,b')]$ for some $b, b' \in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

3. *Case 3:* Suppose we have $x = [(0,a)] = [(1,b)]$ for some $a \in A$ and $b \in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned}
\phi([x]) &= \phi([(0, f(c))]) \\
&\overset{\text{def}}{=} \iota_1(f(c)) \\
&= \iota_2(g(c)) \\
&\overset{\text{def}}{=} \phi([(1, g(c))]) \\
&= \phi([y]),
\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}
(0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\
(1, b) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\
(0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)
\end{aligned}$$

gives

$$\begin{aligned}
\phi([(0, a)]) &= \phi([(0, a')]), \\
\phi([(1, b)]) &= \phi([(1, b')]), \\
\phi([(0, a)]) &= \phi([(1, b)]),
\end{aligned}$$

showing ϕ to be well-defined. $\square$

Remark 4.2.4.1.3. In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a, b \in A$ and $a = b$.

2. We have $a, b \in B$ and $a = b$.

3. There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

   (a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.

   (b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

   In other words, there exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

   (c) There exists $c_0 \in C$ satisfying one of the following conditions:

       i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.

       ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.

   (d) For each $1 \leq i \leq n - 1$, there exists $c_i \in C$ satisfying one of the following conditions:

       i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.

       ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.

   (e) There exists $c_n \in C$ satisfying one of the following conditions:

       i. We have $x_n = f(c_n)$ and $b = g(c_n)$.

       ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

Remark 4.2.4.1.4. It is common practice to write $A \coprod_C B$ for the pushout of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \coprod_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \coprod_{f,C,g} B$ or $A \coprod_C^{f,g} B$ for $A \coprod_C B$.

Example 4.2.4.1.5. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 6.3.3.1.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let X be a set. We have

$$A \cup B \cong A \coprod_{A \cap B} B,$$

$$
\begin{array}{ccc}
A \cup B & \longleftarrow & B \\
\uparrow & & \uparrow \\
\\
A & \longleftarrow & A \cap B
\end{array}
$$

for each $A, B \in \mathcal{P}(X)$.

Proof. Item 1, Wedge Sums of Pointed Sets: This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.

Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. $\qquad\square$

Proposition 4.2.4.1.6. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$
\begin{array}{c}
\bullet \\
\uparrow \\
\bullet \longleftarrow \bullet.
\end{array}
$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$
\begin{array}{ccc}
A \coprod_C B & \longleftarrow & B \\
\uparrow & & \uparrow \psi \\
& A' \coprod_{C'} B' \longleftarrow B' & \\
& \uparrow \quad g \uparrow & \\
A \longleftarrow & \overset{f}{\quad} C & g' \\
\phi & & \chi \\
& A' \longleftarrow C' & \\
& f' &
\end{array}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \amalg_C B \xrightarrow{\exists!} A' \amalg_{C'} B'$ given by

$$\xi(x) \overset{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0,a)], \\ \psi(b) & \text{if } x = [(1,b)] \end{cases}$$

for each $x \in A \amalg_C B$, which is the unique map making the diagram

commute.

2. *Adjointness.* We have an adjunction

$$\left(-_1 \amalg X -_2 \dashv \Delta_{\mathsf{Sets}_{X/}} \right) \colon \quad \mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/} \overset{\overset{-_1 \amalg X -_2}{\overbrace{}}}{\underset{\underset{\Delta_{\mathsf{Sets}_{X/}}}{\underbrace{}}}{\overset{\bot}{\rightleftarrows}}} \mathsf{Sets}_{X/},$$

witnessed by a bijection

$$\mathsf{Sets}_{X/}(A \amalg_X B, C), \cong \mathsf{Hom}_{\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/}}((A,B),(C,C))$$

natural in $(A,B) \in \mathrm{Obj}\Big(\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/}\Big)$ and in $C \in \mathrm{Obj}\Big(\mathsf{Sets}_{X/}\Big)$.
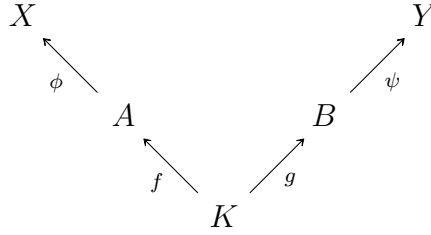
3. *Associativity.* Given a diagram

in Sets, we have isomorphisms of sets

$$(A \amalg_X B) \amalg_Y C \cong (A \amalg_X B) \amalg_B (B \amalg_Y C) \cong A \amalg_X (B \amalg_Y C)$$

where these pullbacks are built as in the diagrams

$$(A \amalg_X B) \amalg_Y C \qquad\qquad (A \amalg_X B) \amalg_B (B \amalg_Y C) \qquad\qquad A \amalg_X (B \amalg_Y C)$$
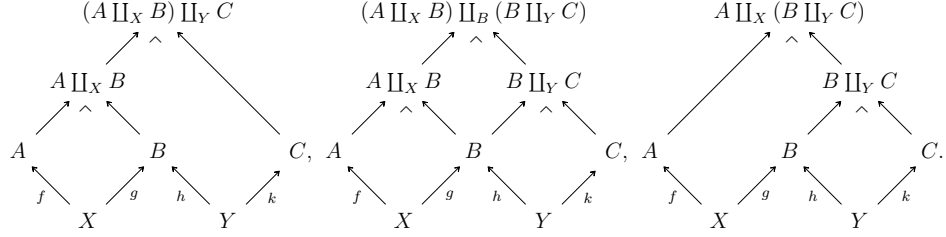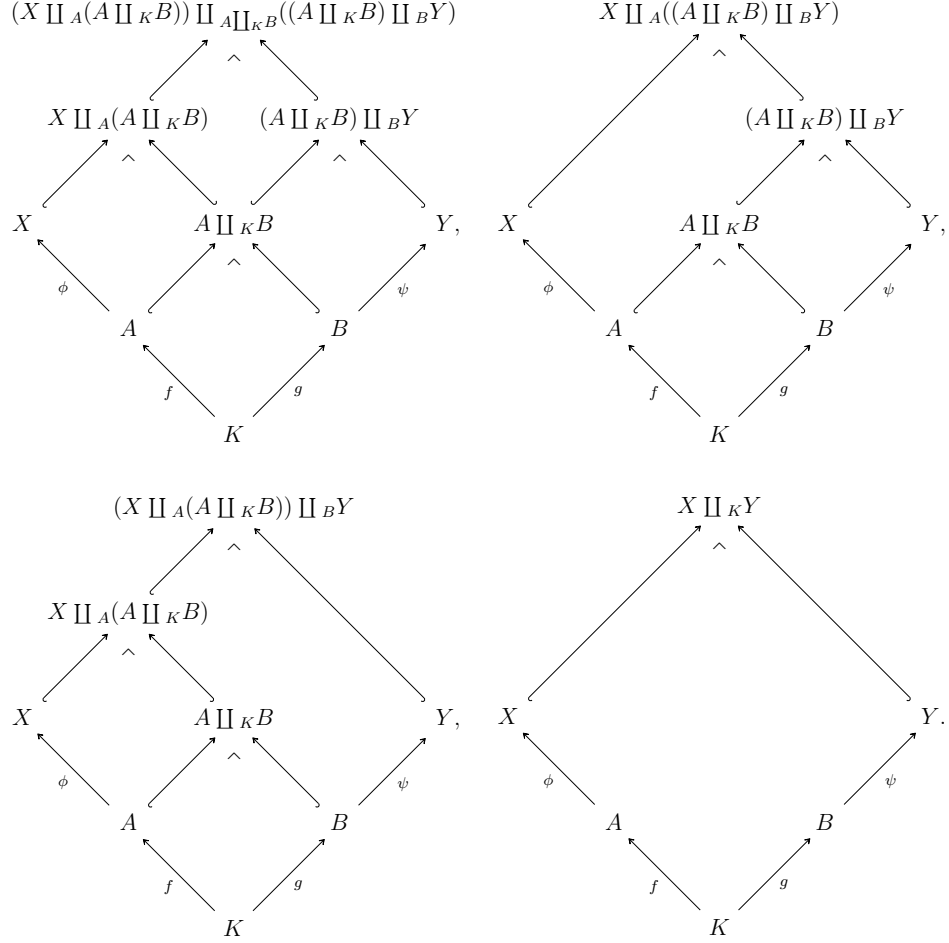
4. *Interaction With Composition.* Given a diagram

in Sets, we have isomorphisms of sets

$$X \amalg_K^{\phi \circ f, \psi \circ g} Y \cong \left(X \amalg_A^{\phi, j_1}\left(A \amalg_K^{f,g} B\right)\right) \amalg_{A \amalg_K^{f,g} B}^{i_2, i_1} \left(\left(A \amalg_K^{f,g} B\right) \amalg_B^{j_2, \psi} Y\right)$$
$$\cong X \amalg_A^{\phi, i}\left(\left(A \amalg_K^{f,g} B\right) \amalg_B^{j_2, \psi} Y\right)$$
$$\cong \left(X \amalg_A^{\phi, i_1}\left(A \amalg_K^{f,g} B\right)\right) \amalg_B^{j, \psi} Y$$

where

$$j_1 = \mathrm{inj}_1^{A \times_K^{f,g} B}, \qquad\qquad\qquad j_2 = \mathrm{inj}_2^{A \times_K^{f,g} B},$$
$$i_1 = \mathrm{inj}_1^{\left(A \times_K^{f,g} B\right) \times_Y^{q_2, \psi}}, \qquad\qquad i_2 = \mathrm{inj}_2^{X \times_{A \times_K^{f,g} B}^{\phi, q_1}\left(A \times_K^{f,g} B\right)},$$
$$i = j_1 \circ \mathrm{inj}_1^{\left(A \times_K^{f,g} B\right) \times_B^{q_2, \psi} Y}, \qquad j = j_2 \circ \mathrm{inj}_2^{X \times_A^{\phi, q_1}\left(A \times_K^{f,g} B\right)},$$

and where these pullbacks are built as in the diagrams

$$(X \coprod_A (A \coprod_K B)) \coprod_{A \coprod_K B} ((A \coprod_K B) \coprod_B Y)$$

$$X \coprod_A (A \coprod_K B) \qquad (A \coprod_K B) \coprod_B Y$$

$$X \qquad A \coprod_K B \qquad Y,$$

$$A \qquad B$$

$$\phi \qquad \psi$$

$$f \qquad g$$

$$K$$

$$X \coprod_A ((A \coprod_K B) \coprod_B Y)$$

$$(A \coprod_K B) \coprod_B Y$$

$$X \qquad A \coprod_K B \qquad Y,$$

$$A \qquad B$$

$$\phi \qquad \psi$$

$$f \qquad g$$

$$K$$

$$(X \coprod_A (A \coprod_K B)) \coprod_B Y$$

$$X \coprod_A (A \coprod_K B)$$

$$X \qquad A \coprod_K B \qquad Y,$$

$$A \qquad B$$

$$\phi \qquad \psi$$

$$f \qquad g$$

$$K$$

$$X \coprod_K Y$$

$$X \qquad\qquad Y.$$

$$A \qquad B$$

$$\phi \qquad \psi$$

$$f \qquad g$$

$$K$$

5. *Unitality.* We have isomorphisms of sets

$$
\begin{array}{ccc}
A & =\!=\!= & A \\
\uparrow_f & \ulcorner & \uparrow_f \\
X & =\!=\!= & X
\end{array}
\qquad
\begin{array}{l}
\lambda_A^{\mathsf{Sets}_{X/}} : X \coprod_X A \overset{\sim}{\dashrightarrow} A, \\[4pt]
\rho_A^{\mathsf{Sets}_{X/}} : A \coprod_X X \overset{\sim}{\dashrightarrow} A,
\end{array}
\qquad
\begin{array}{ccc}
A & \overset{f}{\longleftarrow} & X \\
\| & \ulcorner & \| \\
X & \underset{f}{\longleftarrow} & X,
\end{array}
$$

natural in $(A, f) \in \mathrm{Obj}\big(\mathsf{Sets}_{X/}\big)$.

6. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
A \coprod_X B & \longleftarrow & B \\
\uparrow^{\ulcorner} & & \uparrow g \\
\\
A & \longleftarrow_{f} & X,
\end{array}
\qquad
\sigma_A^{\mathsf{Sets}_{X/}} : A \coprod_X B \overset{\sim}{\dashrightarrow} B \coprod_X A
\qquad
\begin{array}{ccc}
B \coprod_X A & \longleftarrow & A \\
\uparrow^{\ulcorner} & & \uparrow f \\
\\
B & \longleftarrow_{g} & X.
\end{array}
$$

   natural in $(A, f), (B, g) \in \mathrm{Obj}\big(\mathsf{Sets}_{X/}\big)$.

7. *Interaction With Coproducts.* We have

$$
A \coprod_{\varnothing} B \cong A \coprod B,
\qquad
\begin{array}{ccc}
A \coprod B & \longleftarrow & B \\
\uparrow^{\ulcorner} & & \downarrow \iota_B \\
\\
A & \longrightarrow_{\iota_A} & \varnothing.
\end{array}
$$

8. *Symmetric Monoidality.* The triple $\big(\mathsf{Sets}_{X/}, \coprod_X, X\big)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, : Adjointness: This follows from the universal property of the coproduct (pushouts are coproducts in $\mathsf{Sets}_{X/}$).
Item 3, Associativity: Omitted.
Item 4, Interaction With Composition: Omitted.
Item 5, Unitality: Omitted.
Item 6, Commutativity: Omitted.
Item 7, Interaction With Coproducts: Omitted.
Item 8, Symmetric Monoidality: Omitted.   □

4.2.5   Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 4.2.5.1.1. The **coequaliser of f and g** is the coequaliser of f and g in Sets as in Limits and Colimits, **??**.

Construction 4.2.5.1.2. Concretely, the coequaliser of f and g is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

1. *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \overset{\mathrm{def}}{=} B/{\sim},$$

   where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

2. *The Cocone.* The map

$$\mathrm{coeq}(f,g)\colon B \twoheadrightarrow \mathrm{CoEq}(f,g)$$

   given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$
\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\overset{\mathrm{def}}{=} [f(a)] \\
&= [g(a)] \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}
$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\mathrm{coeq}(f,g)} \mathrm{CoEq}(f,g)$$

$$\overset{c}{\searrow}$$

$$C$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\mathrm{coeq}(f,g)} \mathrm{CoEq}(f,g)$$

$$\overset{c}{\searrow} \qquad \vdots{\exists!}$$

$$C$$

commute. □

Remark 4.2.5.1.3. In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a = b$;

2. There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

   (a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.

   (b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

   In other words, there exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

   (a) There exists $z_0 \in A$ satisfying one of the following conditions:

      i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.
      ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.

   (b) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:

      i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
      ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.

   (c) There exists $z_n \in A$ satisfying one of the following conditions:

      i. We have $x_n = f(z_n)$ and $b = g(z_n)$.
      ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 4.2.5.1.4. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/\!\sim_R \,\cong\, \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof. Item 1, *Quotients by Equivalence Relations*: See [Pro25z]. □

Proposition 4.2.5.1.5. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[9]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$A \xrightarrow[\substack{-g\to \\ h}]{f} B$$

---

[9]That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$A \xrightarrow[\substack{-g\to \\ h}]{f} B$$

   in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

   and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

   obtaining a quotient

   $$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

   of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

   and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

   obtaining a quotient

   $$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

   of $\mathrm{CoEq}(g,h)$.

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1, Associativity*: Omitted.
Item 4, Unitality: Omitted.
Item 5, Commutativity: Omitted.
Item 6, Interaction With Composition: Omitted. $\square$

4.2.6   Direct Colimits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \to \mathsf{Top}$ be a direct system of sets.

Definition 4.2.6.1.1. The **direct colimit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.2.6.1.2. Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the pair $\left(\underset{\alpha \in I}{\mathrm{colim}}(X_\alpha), \{\mathrm{inj}_\alpha\}_{\alpha \in I} \right)$ consisting of:

1. *The Colimit.* The set $\underset{\alpha \in I}{\mathrm{colim}}(X_\alpha)$ defined by

$$\underset{\alpha \in I}{\mathrm{colim}}(X_\alpha) \overset{\mathrm{def}}{=} \left(\coprod_{\alpha \in I} X_\alpha \right) \Big/ {\sim},$$

where $\sim$ is the equivalence relation on $\coprod_{\alpha \in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

(a) We have $\alpha \preceq \gamma$.

(b) We have $\beta \preceq \gamma$.

(c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

2. *The Cocone.*The collection

$$\left\{ \mathrm{inj}_\gamma \colon X_\gamma \to \underset{\alpha \in I}{\mathrm{colim}}(X_\alpha) \right\}_{\gamma \in I}$$

of maps of sets defined by

$$\mathrm{inj}_\gamma(x) \overset{\mathrm{def}}{=} [(\gamma, x)]$$

for each $\gamma \in I$ and each $x \in X_\gamma$.

Proof. We will prove Definition 4.2.6.1.2 below in a bit, but first we need a lemma (which is interesting in its own right). □

Lemma 4.2.6.1.3. For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have

$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in $\underset{\alpha \in I}{\mathrm{colim}}(X_\alpha)$.

Proof. Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \overset{\mathrm{def}}{=} \mathrm{id}_{X_\beta}$, and we have

$$\begin{aligned}
f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\
&\overset{\mathrm{def}}{=} \mathrm{id}_{X_\beta}(f_{\alpha\beta}(x)), \\
&= f_{\alpha\beta}(x).
\end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, Items 1a to 1c of Definition 4.2.6.1.2 are met. Thus we have $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$. □

We can now prove Definition 4.2.6.1.2:

Proof. We claim that $\underset{\alpha \in I}{\mathrm{colim}}(X_\alpha)$ is the colimit of the direct system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$.
Commutativity of the Colimit Diagram: First, we need to check that the

colimit diagram defined by $\varinjlim_{\alpha \in I}(X_\alpha)$ commutes, i.e. that we have

$$\mathrm{inj}_\alpha = \mathrm{inj}_\beta \circ f_{\alpha\beta},$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$\begin{aligned}
\left[\mathrm{inj}_\beta \circ f_{\alpha\beta}\right](x) &\overset{\text{def}}{=} \mathrm{inj}_\beta(f_{\alpha\beta}(x)) \\
&\overset{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\
&= [(\alpha, x)] \\
&\overset{\text{def}}{=} \mathrm{inj}_\alpha(x),
\end{aligned}$$

where we have used Definition 4.2.6.1.3 for the third equality.

Proof of the Universal Property of the Colimit: Next, we prove that $\varinjlim_{\alpha \in I}(X_\alpha)$ as constructed in Definition 4.2.6.1.2 satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

in Sets. We claim that there exists a unique map $\phi \colon \varinjlim_{\alpha \in I}(X_\alpha) \overset{\exists!}{\longrightarrow} C$ making

the diagram

$$
\begin{array}{c}
C \\
\phi \mid \exists! \\
i_\alpha \quad \operatorname*{colim}_{\alpha \in I}(X_\alpha) \quad i_\beta \\
\operatorname{inj}_\alpha \qquad \operatorname{inj}_\beta \\
X_\alpha \xrightarrow{\ f_{\alpha\beta}\ } X_\beta
\end{array}
$$

commute. To this end, first consider the diagram

$$
\coprod_{\alpha \in I} X_\alpha \xrightarrow{\ \mathrm{pr}\ } \operatorname*{colim}_{\alpha \in I}(X_\alpha) \\
\coprod_{\alpha \in I} i_\alpha \searrow \\
C.
$$

Lemma. If $(\alpha, x) \sim (\beta, y)$, then we have

$$
\left[\coprod_{\alpha \in I} i_\alpha \right](x) = \left[\coprod_{\alpha \in I} i_\alpha \right](y).
$$

Proof. Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

1. We have $\alpha \preceq \gamma$.

2. We have $\beta \preceq \gamma$.

3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$
\begin{aligned}
\left[\coprod_{\alpha \in I} i_\alpha \right](x) &\overset{\mathrm{def}}{=} i_\alpha(x) \\
&\overset{\mathrm{def}}{=} [i_\gamma \circ f_{\alpha\gamma}](x) \\
&\overset{\mathrm{def}}{=} i_\gamma(f_{\alpha\gamma}(x)) \\
&= i_\gamma(f_{\beta\gamma}(x))
\end{aligned}
$$

$$\overset{\text{def}}{=} [i_\gamma \circ f_{\beta\gamma}](x)$$

$$= i_\beta(y)$$

$$\overset{\text{def}}{=} \left[\coprod_{\alpha\in I} i_\alpha\right](y).$$

This finishes the proof of the lemma. Continuing, by Conditions on Relations, **??** of Definition 10.6.2.1.3, there then exists a map $\phi\colon \varinjlim_{\alpha\in I}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\coprod_{\alpha\in I} X_\alpha \xrightarrow{\;\mathrm{pr}\;} \varinjlim_{\alpha\in I}(X_\alpha)$$

$$\coprod_{\alpha\in I} i_\alpha \searrow \qquad \downarrow \phi$$

$$C$$

commute. In particular, this implies that the diagram

$$X_\alpha \xrightarrow{\;\mathrm{inj}_\alpha\;} \varinjlim_{\alpha\in I}(X_\alpha)$$

$$i_\alpha \searrow \qquad \downarrow \phi$$

$$C$$

also commutes, and thus so does the diagram

$$C$$

$$\phi \;\vdots\; \exists!$$

$$i_\alpha \qquad \varinjlim_{\alpha\in I}(X_\alpha) \qquad i_\beta$$

$$\mathrm{inj}_\alpha \qquad \mathrm{inj}_\beta$$

$$X_\alpha \xrightarrow{\quad f_{\alpha\beta}\quad} X_\beta.$$

This finishes the proof.[10] □

Example 4.2.6.1.4. Here are some examples of direct colimits of sets.

1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \stackrel{\text{def}}{=} \operatorname*{colim}_{n\in\mathbb{N}}\left(\mathbb{Z}_{/p^n}\right);$$

see **??**.

4.3 Operations With Sets

4.3.1 The Empty Set

Definition 4.3.1.1.1. The **empty set** is the set $\varnothing$ defined by

$$\varnothing \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where X is the set in the set existence axiom, **??** of **??**.

4.3.2 Singleton Sets

Let X be a set.

Definition 4.3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself of Definition 4.3.3.1.1.

---

[10]Incidentally, the conditions

$$\{i_\alpha = \phi \circ \operatorname{inj}_\alpha\}_{\alpha\in I}$$

show that ϕ must be given by

$$\phi([(\alpha, x)]) = (i_\alpha(x))_{\alpha\in I}$$

for each $[(\alpha, x)] \in \operatorname*{colim}_{\alpha\in I}(X_\alpha)$, although we would need to show that this assignment is well-defined were we to prove Definition 4.2.6.1.2 in this way. Instead, invoking Conditions on Relations, **??** of Definition 10.6.2.1.3 gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

4.3.3   Pairings of Sets

Let X and Y be sets.

Definition 4.3.3.1.1. The **pairing of X and Y** is the set $\{X, Y\}$ defined by

$$\{X, Y\} \stackrel{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

4.3.4   Ordered Pairs

Let A and B be sets.

Definition 4.3.4.1.1. The **ordered pair associated to A and B** is the set (A, B) defined by

$$(A, B) \stackrel{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 4.3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3]. □

4.3.5   Sets of Maps

Let A and B be sets.

Definition 4.3.5.1.1. The **set of maps from A to B**[11] is the set $\mathsf{Sets}(A, B)$[12] whose elements are the functions from A to B.

Proposition 4.3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors
$$\begin{aligned}
\mathsf{Sets}(X, -)\colon &&\mathsf{Sets} &\to \mathsf{Sets}, \\
\mathsf{Sets}(-, Y)\colon &&\mathsf{Sets}^{\mathsf{op}} &\to \mathsf{Sets}, \\
\mathsf{Sets}(-_1, -_2)\colon &&\mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} &\to \mathsf{Sets}.
\end{aligned}$$

---

[11] *Further Terminology:* Also called the **Hom set from A to B**.

[12] *Further Notation:* Also written $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$.

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A,-))\colon \; \mathsf{Sets} \overset{A\times-}{\underset{\mathsf{Sets}(A,-)}{\rightleftarrows}} \bot \; \mathsf{Sets},$$

$$(- \times B \dashv \mathsf{Sets}(B,-))\colon \; \mathsf{Sets} \overset{-\times B}{\underset{\mathsf{Sets}(B,-)}{\rightleftarrows}} \bot \; \mathsf{Sets},$$

witnessed by bijections

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$
$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Maps From the Punctual Set.* We have a bijection

$$\mathsf{Sets}(\mathrm{pt}, A) \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Maps to the Punctual Set.* We have a bijection

$$\mathsf{Sets}(A, \mathrm{pt}) \cong \mathrm{pt},$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: This follows from Categories, Items 2 and 5 of Definition 11.1.4.1.2.
Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.1.3.1.3 and is proved there.
Item 3, Maps From the Punctual Set: The bijection

$$\Phi_A \colon \mathsf{Sets}(\mathrm{pt}, A) \overset{\sim}{\dashrightarrow} A$$

is given by

$$\Phi_A(f) \overset{\mathrm{def}}{=} f(\star)$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$, admitting an inverse

$$\Phi_A^{-1} \colon A \overset{\sim}{\dashrightarrow} \mathsf{Sets}(\mathrm{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \overset{\text{def}}{=} [\![\star \mapsto a]\!]$$

for each $a \in A$. Indeed, we have

$$
\begin{aligned}
\left[\Phi_A^{-1} \circ \Phi_A\right](f) &\overset{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\
&\overset{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\
&\overset{\text{def}}{=} [\![\star \mapsto f(\star)]\!] \\
&\overset{\text{def}}{=} f \\
&\overset{\text{def}}{=} \left[\text{id}_{\mathsf{Sets}(\text{pt},A)}\right](f)
\end{aligned}
$$

for each $f \in \mathsf{Sets}(\text{pt}, A)$ and

$$
\begin{aligned}
\left[\Phi_A \circ \Phi_A^{-1}\right](a) &\overset{\text{def}}{=} \Phi_A\left(\Phi_A^{-1}(a)\right) \\
&\overset{\text{def}}{=} \Phi_A([\![\star \mapsto a]\!]) \\
&\overset{\text{def}}{=} \text{ev}_\star([\![\star \mapsto a]\!]) \\
&\overset{\text{def}}{=} a \\
&\overset{\text{def}}{=} [\text{id}_A](a)
\end{aligned}
$$

for each $a \in A$, and thus we have

$$
\begin{aligned}
\Phi_A^{-1} \circ \Phi_A &= \text{id}_{\mathsf{Sets}(\text{pt},A)} \\
\Phi_A \circ \Phi_A^{-1} &= \text{id}_A \, .
\end{aligned}
$$

To prove naturality, we need to show that the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(\text{pt}, A) & \xrightarrow{\ f_! \ } & \mathsf{Sets}(\text{pt}, B) \\
\Phi_A \downarrow \wr & & \wr \downarrow \Phi_B \\
A & \xrightarrow[\ f \ ]{} & B
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[f \circ \Phi_A](\phi) &\overset{\text{def}}{=} f(\Phi_A(\phi)) \\
&\overset{\text{def}}{=} f(\phi(\star)) \\
&\overset{\text{def}}{=} [f \circ \phi](\star) \\
&\overset{\text{def}}{=} \Phi_B(f \circ \phi)
\end{aligned}
$$

$$\overset{\mathrm{def}}{=} \Phi_B(f_!(\phi))$$
$$\overset{\mathrm{def}}{=} [\Phi_B \circ f_!](\phi)$$

for each $\phi \in \mathsf{Sets}(\mathrm{pt}, A)$. This finishes the proof.

Item 4, Maps to the Punctual Set: This follows from the universal property of pt as the terminal set, Definition 4.1.1.1.1. $\square$

4.3.6   Unions of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.6.1.1. The **union of** $\mathcal{U}$ is the set $\bigcup_{U \in \mathcal{U}} U$ defined by

$$\bigcup_{U \in \mathcal{U}} U \overset{\mathrm{def}}{=} \left\{ x \in X \; \middle| \; \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}.$$

Proposition 4.3.6.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$ defines a functor

$$\bigcup \colon (\mathcal{P}(\mathcal{P}(X)), \subset) \to (\mathcal{P}(X), \subset).$$

   In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

   $(\star)$ If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V$.

2. *Associativity.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\;\mathrm{id}_{\mathcal{P}(X)} \star \bigcup\;} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcup \star \, \mathrm{id}_{\mathcal{P}(X)}} \big\downarrow & & \big\downarrow {\scriptstyle \bigcup} \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\;\bigcup\;]{} & \mathcal{P}(X)
\end{array}
$$

   commutes, i.e. we have

$$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

   for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(X) & \\
{\scriptstyle\chi_{\mathcal{P}(X)}}\downarrow & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \underset{\bigcup}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{V\in\{U\}} V = U
$$

for each $U \in \mathcal{P}(X)$.

4. *Right Unitality.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(X) & \\
{\scriptstyle\mathcal{P}(\chi_X)}\downarrow & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \underset{\bigcup}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{\{u\}\in\chi_X(U)} \{u\} = U
$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Unions I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \overset{\cup}{\longrightarrow} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\bigcup\times\bigcup}\downarrow & & \downarrow{\scriptstyle\bigcup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \underset{\cup}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{W\in\mathcal{U}\cup\mathcal{V}} W = \left(\bigcup_{U\in\mathcal{U}} U\right) \cup \left(\bigcup_{V\in\mathcal{V}} V\right)
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)\ } \mathcal{P}(\mathcal{P}(X)) & \qquad & \mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)\ } \mathcal{P}(\mathcal{P}(X)) \\
\ \ \bigcup\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow\bigcup & & \ \ \bigcup\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow\bigcup \\
\mathcal{P}(X) \xrightarrow[\ \ U\cup-\ \ ]{} \mathcal{P}(X) & & \mathcal{P}(X) \xrightarrow[\ \ -\cup V\ \ ]{} \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$
U \cup \left(\bigcup_{V\in\mathcal{V}} V \right) = \bigcup_{V\in\mathcal{V}} (U\cup V),
$$

$$
\left(\bigcup_{U\in\mathcal{U}} U \right) \cup V = \bigcup_{U\in\mathcal{U}} (U\cup V)
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \cap\ } \mathcal{P}(\mathcal{P}(X)) \\
\bigcup\times\bigcup \Big\downarrow \qquad\qquad \supset \qquad\qquad \Big\downarrow\bigcup \\
\mathcal{P}(X) \times \mathcal{P}(X) \xrightarrow[\ \ \cap\ \ ]{} \mathcal{P}(X),
\end{array}
$$

with components

$$
\bigcup_{W\in\mathcal{U}\cap\mathcal{V}} W \subset \left(\bigcup_{U\in\mathcal{U}} U \right) \cap \left(\bigcup_{V\in\mathcal{V}} V \right)
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cap-)\ } \mathcal{P}(\mathcal{P}(X)) & \qquad & \mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cap V)\ } \mathcal{P}(\mathcal{P}(X)) \\
\ \ \bigcup\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow\bigcup & & \ \ \bigcup\Big\downarrow \qquad\qquad\qquad\qquad \Big\downarrow\bigcup \\
\mathcal{P}(X) \xrightarrow[\ \ U\cap-\ \ ]{} \mathcal{P}(X) & & \mathcal{P}(X) \xrightarrow[\ \ -\cap V\ \ ]{} \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$
U \cup \left(\bigcup_{V\in\mathcal{V}} V \right) = \bigcup_{V\in\mathcal{V}} (U\cup V),
$$

$$\left(\bigcup_{U\in\mathcal{U}} U\right) \cup V = \bigcup_{U\in\mathcal{U}} (U\cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\;\setminus\;} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\bigcup\times\bigcup}\Big\downarrow & {\color{orange}\times} & \Big\downarrow{\scriptstyle\bigcup} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\;\setminus\;]{} & \mathcal{P}(X),
\end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{W\in\mathcal{U}\setminus\mathcal{V}} W \neq \left(\bigcup_{U\in\mathcal{U}} U\right) \setminus \left(\bigcup_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \xrightarrow{\;(-)^{\mathsf{c}}\;} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\bigcup^{\mathsf{op}}}\Big\downarrow & {\color{orange}\times} & \Big\downarrow{\scriptstyle\bigcup} \\
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow[\;(-)^{\mathsf{c}}\;]{} & \mathcal{P}(X),
\end{array}$$

does not commute in general, i.e. we may have

$$\bigcup_{U\in\mathcal{U}^{\mathsf{c}}} U \neq \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$\begin{array}{ccc}
& \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\
{\scriptstyle\mathrm{id}_{\mathcal{P}(X)}\star(-)^{\mathsf{c}}}\nearrow & & \overset{\sim}{\dashrightarrow} \\
\mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\
{\scriptstyle\bigcup}\searrow & & \nearrow{\scriptstyle\bigcap^{\mathsf{op}}} \\
& \mathcal{P}(X) \xrightarrow[\;(-)^{\mathsf{c}}\;]{} \mathcal{P}(X)^{\mathsf{op}}, &
\end{array}$$

commutes, i.e. we have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

commutes, i.e. we have

$$\left(\bigcap_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

does not commute in general, i.e. we may have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \xrightarrow{[-_1,-_2]_{\mathcal{P}(X)}} \mathcal{P}(\mathcal{P}(X))$$

$$\bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}} \Big\downarrow \qquad\qquad \textcolor{orange}{\times} \qquad\qquad \Big\downarrow \bigcup$$

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \xrightarrow{\qquad [-_1,-_2]_X \qquad} \mathcal{P}(X),$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}}$$

$$\overset{\cong}{\dashrightarrow} \qquad\qquad \searrow \bigcup^{\mathsf{op}}$$

$$\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \qquad\qquad \mathcal{P}(X)^{\mathsf{op}}$$

$$\mathsf{id}_{\mathcal{P}(X)} \star [-,V]_X \searrow \qquad\qquad \swarrow [-,V]_X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\bigcap} \mathcal{P}(X)$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V \right]_X = \bigcap_{U \in \mathcal{U}} [U,V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\bigcup} \mathcal{P}(X)$$

$$\mathsf{id}_{\mathcal{P}(X)} \star [U,-]_X \Big\downarrow \qquad\qquad \Big\downarrow [U,-]_X$$

$$\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\bigcup} \mathcal{P}(X)$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V\right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\
\Big\downarrow{\scriptstyle\cup} & & \Big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\mathrm{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{\left(f^{-1}\right)^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
\Big\downarrow{\scriptstyle\cup} & & \Big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\mathrm{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\
\Big\downarrow{\scriptstyle\cup} & & \Big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Intersections of Families I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\;\text{id}_{\mathcal{P}(X)} \star \bigcap\;} & \mathcal{P}(\mathcal{P}(x)) \\
{\scriptstyle \bigcup \star \text{id}_{\mathcal{P}(X)}} \Big\downarrow & & \Big\downarrow {\scriptstyle \bigcap} \\
\mathcal{P}(X) & \xrightarrow[\;\bigcap\;]{} & X
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Intersections of Families II.* Let X be a set and consider the compositions

given by

$$\mathcal{A} \mapsto \bigcup_{\substack{U \in \bigcap_{A \in \mathcal{A}} A}} U, \qquad \mathcal{A} \mapsto \bigcap_{\substack{U \in \bigcup_{A \in \mathcal{A}} A}} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\bigcup_{\substack{U \in \bigcap_{A \in \mathcal{A}} A}} U$$

$$\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \qquad\qquad \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

$$\bigcap_{\substack{U \in \bigcup_{A \in \mathcal{A}} A}} U$$

All other possible inclusions fail to hold in general.

Proof. Item 1, Functoriality: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition $(\star)$. So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

Item 2, Associativity: We have

$$\bigcup_{\substack{U \in \bigcup_{A \in \mathcal{A}} A}} U \stackrel{\mathrm{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } U \in \bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\}$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right\}
$$

$$
\overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right).
$$

This finishes the proof.

Item 3, Left Unitality: We have

$$
\bigcup_{V \in \{U\}} V \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in U \end{array} \right\}
$$

$$
= \{ x \in X \mid x \in U \}
$$

$$
= U.
$$

This finishes the proof.

Item 4, Right Unitality: We have

$$
\bigcup_{\{u\} \in \chi_X(U)} \{u\} \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right\}
$$

$$
= \{ x \in X \mid x \in U \}
$$

$$
= U.
$$

This finishes the proof.

Item 5, Interaction With Unions I: We have

$$
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right\}
$$

$$
\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \\ \text{such that we have } x \in W \end{array} \right\}
$$

$$
\cup \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\}
$$

$$
\overset{\text{def}}{=} \left(\bigcup_{W \in \mathcal{U}} W \right) \cup \left(\bigcup_{W \in \mathcal{V}} W \right)
$$

$$
= \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right).
$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$
\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cap \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\}
$$

$$
\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } U \in \mathcal{U} \text{ and some } V \in \mathcal{V} \\ \text{such that we have } x \in U \text{ and } x \in V \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } U \in \mathcal{U} \\ \text{such that we have } x \in U \end{array} \right\}
$$

$$
\cup \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that we have } x \in V \end{array} \right\}
$$

$$
\overset{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right).
$$

This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$
\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} U = \bigcup_{W \in \{\{0,1\}\}} W
$$

$$
= \{0, 1\},
$$

whereas

$$
\left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right) = \{0, 1\} \setminus \{0\}
$$

$$= \{1\}.$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \{1\} = \left(\bigcup_{U \in \mathcal{U}} U\right) \setminus \left(\bigcup_{V \in \mathcal{V}} V\right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0, 1\}$ and let $\mathcal{U} = \{0\}$. We have

$$\bigcup_{U \in \mathcal{U}^{\mathrm{c}}} U = \bigcup_{U \in \{\emptyset, \{1\}, \{0,1\}\}} U$$
$$= \{0, 1\},$$

whereas

$$\bigcup_{U \in \mathcal{U}} U^{\mathrm{c}} = \{0\}^{\mathrm{c}}$$
$$= \{1\}.$$

Thus we have

$$\bigcup_{U \in \mathcal{U}^{\mathrm{c}}} U = \{0, 1\} \neq \{1\} = \bigcup_{U \in \mathcal{U}} U^{\mathrm{c}}.$$

This finishes the proof.

Item 11, Interaction With Complements II: Omitted.

Item 12, Interaction With Complements III: Omitted.

Item 13, Interaction With Symmetric Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\bigcup_{W \in \mathcal{U} \triangle \mathcal{V}} W = \bigcup_{W \in \{\{0\}\}} W$$
$$= \{0\},$$

whereas

$$\left(\bigcup_{U \in \mathcal{U}} U\right) \triangle \left(\bigcup_{V \in \mathcal{V}} V\right) = \{0, 1\} \triangle \{0, 1\}$$
$$= \emptyset,$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \triangle \mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U \in \mathcal{U}} U\right) \triangle \left(\bigcup_{V \in \mathcal{V}} V\right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of Item 7 of Definition 4.4.7.1.3 and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of Item 8 of Definition 4.4.7.1.3 and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of Item 9 of Definition 4.4.7.1.3 and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of Item 3 of Definition 4.6.1.1.5 and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of Item 3 of Definition 4.6.2.1.3 and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of Item 3 of Definition 4.6.3.1.7 and is proved there.

Item 20, Interaction With Intersections of Families I: We have

$$
\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \overset{\mathrm{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\}
$$

$$
\overset{\mathrm{def}}{=} \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right).
$$

This finishes the proof.

Item 21, Interaction With Intersections of Families II: Omitted. □

4.3.7   Intersections of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.7.1.1. The **intersection of $\mathcal{U}$** is the set $\bigcap_{U \in \mathcal{U}} U$ defined by

$$
\bigcap_{U \in \mathcal{U}} U \overset{\mathrm{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\}.
$$

Proposition 4.3.7.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcap_{U \in \mathcal{U}} U$ defines a functor

$$
\bigcap \colon (\mathcal{P}(\mathcal{P}(X)), \supset) \to (\mathcal{P}(X), \subset).
$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

($\star$) If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U$.

2. *Oplax Associativity.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) & \xrightarrow{\mathrm{id}_{\mathcal{P}(X)} \star \bigcap} & \mathcal{P}(\mathcal{P}(X)) \\
\bigcap \star \mathrm{id}_{\mathcal{P}(X)} \Big\downarrow & \quad\supset\quad & \Big\downarrow \bigcap \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcap]{} & \mathcal{P}(X)
\end{array}
$$

with components

$$
\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \subset \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U
$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(X) & \\
\chi_{\mathcal{P}(X)} \Big\downarrow & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcap]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcap_{V \in \{U\}} V = U.
$$

for each $U \in \mathcal{P}(X)$.

4. *Oplax Right Unitality.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(X) & \\
\mathcal{P}(\chi_X) \Big\downarrow & \textcolor{orange}{\times} \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcap]{} & \mathcal{P}(X)
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcap_{\{x\}\in\chi_X(U)} \{x\} \neq U$$

in general, where $U \in \mathcal{P}(X)$. However, when U is nonempty, we have

$$\bigcap_{\{x\}\in\chi_X(U)} \{x\} \subset U.$$

5. *Interaction With Unions I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \cup\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \cap\times\cap}\Big\downarrow & & \Big\downarrow{\scriptstyle \cap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \cap\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{W\in\mathcal{U}\cup\mathcal{V}} W = \left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)\ } \mathcal{P}(\mathcal{P}(X)) & \qquad & \mathcal{P}(\mathcal{P}(X)) \xrightarrow{\ \mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)\ } \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap}\Big\downarrow \qquad\qquad\qquad \Big\downarrow{\scriptstyle\cap} & & {\scriptstyle\cap}\Big\downarrow \qquad\qquad\qquad \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) \xrightarrow[\ U\cup-\ ]{} \mathcal{P}(X) & & \mathcal{P}(X) \xrightarrow[\ -\cup V\ ]{} \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V\in\mathcal{V}} V\right) = \bigcap_{V\in\mathcal{V}} (U\cup V),$$

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \cup V = \bigcap_{U\in\mathcal{U}} (U\cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\;\cap\;} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap\times\cap}\Big\downarrow & \supset & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\;\cap\;]{} & \mathcal{P}(X),
\end{array}
$$

with components

$$
\left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right) \subset \bigcap_{W\in\mathcal{U}\cap\mathcal{V}} W
$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cap-)} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap}\Big\downarrow & & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) & \xrightarrow[U\cap-]{} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cap V)} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap}\Big\downarrow & & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) & \xrightarrow[-\cap V]{} & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have

$$
U \cup \left(\bigcap_{V\in\mathcal{V}} V\right) = \bigcap_{V\in\mathcal{V}} (U\cup V),
$$

$$
\left(\bigcap_{U\in\mathcal{U}} U\right) \cup V = \bigcap_{U\in\mathcal{U}} (U\cup V)
$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U,V\in\mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\;\setminus\;} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\cap\times\cap}\Big\downarrow & \times & \Big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\;\setminus\;]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$
\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} W \neq \left(\bigcap_{U\in\mathcal{U}} U\right) \setminus \left(\bigcap_{V\in\mathcal{V}} V\right)
$$

in general, where $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \xrightarrow{\ (-)^{\mathsf{c}}\ } & \mathcal{P}(\mathcal{P}(X)) \\ {\scriptstyle\bigcap^{\mathsf{op}}}\Big\downarrow & \textcolor{orange}{\times} & \Big\downarrow{\scriptstyle\bigcap} \\ \mathcal{P}(X)^{\mathsf{op}} & \xrightarrow[\ (-)^{\mathsf{c}}\ ]{} & \mathcal{P}(X), \end{array}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U}^{\mathsf{c}}} W \neq \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\ {\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}}\nearrow & & \overset{\sim}{\dashrightarrow} \\ \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\ {\scriptstyle\bigcap}\searrow & & \nearrow{\scriptstyle\bigcup^{\mathsf{op}}} \\ & \mathcal{P}(X) \xrightarrow[\ (-)^{\mathsf{c}}\ ]{} \mathcal{P}(X)^{\mathsf{op}}, & \end{array}$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U \right)^{\mathsf{c}} = \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$\begin{array}{ccc} & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & \\ {\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}}\nearrow & & \overset{\sim}{\dashrightarrow} \\ \mathcal{P}(\mathcal{P}(X)) & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \\ {\scriptstyle\bigcup}\searrow & & \nearrow{\scriptstyle\bigcap^{\mathsf{op}}} \\ & \mathcal{P}(X) \xrightarrow[\ (-)^{\mathsf{c}}\ ]{} \mathcal{P}(X)^{\mathsf{op}}, & \end{array}$$

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U \right)^{\mathsf{c}} = \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\triangle} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcap \times \bigcap} \downarrow & \textcolor{orange}{\times} & \downarrow {\scriptstyle \bigcap} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\triangle]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \triangle \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \triangle \left(\bigcap_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-1,-2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}}} \downarrow & \textcolor{orange}{\times} & \downarrow {\scriptstyle \bigcap} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[{[-1,-2]_X}]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \\
& & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & & \mathcal{P}(X)^{\mathsf{op}} \\
& & \\
& \mathcal{P}(\mathcal{P}(X)) \xrightarrow[\bigcup]{} \mathcal{P}(X) &
\end{array}
$$

with $\cong$ (dashed), $\bigcap^{\mathsf{op}}$, $\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X$, $[-,V]_X$

commutes, i.e. we have

$$
\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X
$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\bigcap} & \mathcal{P}(X) \\
\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X \downarrow & & \downarrow [U,-]_X \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcap]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\left[U, \bigcap_{V \in \mathcal{V}} V \right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X
$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_!)_!} & \mathcal{P}(\mathcal{P}(Y)) \\
\bigcap \downarrow & & \downarrow \bigcap \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{\left(f^{-1}\right)^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{\quad f_* \quad} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Unions of Families I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\text{id}_{\mathcal{P}(X)} \star \bigcap} & \mathcal{P}(\mathcal{P}(x)) \\
\bigcup \star \text{id}_{\mathcal{P}(X)} \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{\qquad \bigcap \qquad} & X
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U\in\bigcup_{A\in\mathcal{A}}A}U=\bigcap_{A\in\mathcal{A}}\left(\bigcap_{U\in A}U\right)$$

for each $\mathcal{A}\in\mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Unions of Families II.* Let X be a set and consider the compositions

given by

$$\mathcal{A}\mapsto\bigcup_{U\in\bigcap_{A\in\mathcal{A}}A}U,\qquad\mathcal{A}\mapsto\bigcap_{U\in\bigcup_{A\in\mathcal{A}}A}U,$$

$$\mathcal{A}\mapsto\bigcup_{A\in\mathcal{A}}\left(\bigcap_{U\in A}U\right),\quad\mathcal{A}\mapsto\bigcap_{A\in\mathcal{A}}\left(\bigcup_{U\in A}U\right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

$$\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \qquad\qquad \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U$$

All other possible inclusions fail to hold in general.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition ($\star$). So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion.

Item 2, Oplax Associativity: We have

$$\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \overset{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \bigcap_{U \in A} U \end{array} \right\}$$

$$\overset{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\}$$

$$= \left\{ x \in X \,\middle|\, \begin{array}{l} \text{for each } U \in \bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}$$

$$\subset \left\{ x \in X \,\middle|\, \begin{array}{l} \text{for each } U \in \bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}$$

$$\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 3, Left Unitality: We have

$$\bigcap_{V \in \{U\}} V \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in U \end{array} \right\}$$
$$= \{x \in X \mid x \in U\}$$
$$= U.$$

This finishes the proof.

Item 4, Oplax Right Unitality: If $U = \varnothing$, then we have

$$\bigcap_{\{u\} \in \chi_X(U)} \{u\} = \bigcap_{\{u\} \in \varnothing} \{u\}$$
$$= X,$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \varnothing = U$. When U is nonempty, we have two cases:

1. If U is a singleton, say $U = \{u\}$, we have

$$\bigcap_{\{u\} \in \chi_X(U)} \{u\} = \{u\}$$
$$\stackrel{\text{def}}{=} U.$$

2. If U contains at least two elements, we have

$$\bigcap_{\{u\} \in \chi_X(U)} \{u\} = \varnothing$$
$$\subset U.$$

This finishes the proof.

Item 5, Interaction With Unions I: We have

$$\bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\}$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \text{ and each} \\ W \in \mathcal{V}, \text{ we have } x \in W \end{array} \right\}
$$

$$
\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U}, \\ \text{we have } x \in W \end{array} \right\}
$$

$$
\cap \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\}
$$

$$
\stackrel{\text{def}}{=} \left(\bigcap_{W \in \mathcal{U}} W \right) \cap \left(\bigcap_{W \in \mathcal{V}} W \right)
$$

$$
= \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right).
$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$
\left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\}
$$

$$
\cup \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } V \in \mathcal{V}, \\ \text{we have } x \in V \end{array} \right\}
$$

$$
= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \cap \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\}
$$

$$
\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\}
$$

$$
\stackrel{\text{def}}{=} \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W.
$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0\}, \{0,1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$
\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} U = \bigcap_{W \in \{\{0,1\}\}} W
$$

$$
= \{0, 1\},
$$

whereas

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \setminus \left(\bigcap_{V\in\mathcal{V}} V\right) = \{0\} \setminus \{0\}$$
$$= \emptyset.$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} W = \{0,1\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right) \setminus \left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0,1\}$ and let $\mathcal{U} = \{\{0\}\}$. We have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U = \bigcap_{W\in\{\emptyset,\{1\},\{0,1\}\}} W$$
$$= \emptyset,$$

whereas

$$\bigcap_{U\in\mathcal{U}} U^{\mathsf{c}} = \{0\}^{\mathsf{c}}$$
$$= \{1\}.$$

Thus we have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U = \emptyset \neq \{1\} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.

Item 11, Interaction With Complements II: This is a repetition of Item 12 of Definition 4.3.6.1.2 and is proved there.

Item 12, Interaction With Complements III: This is a repetition of Item 11 of Definition 4.3.6.1.2 and is proved there.

Item 13, Interaction With Symmetric Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0,1\}\}$. We have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W = \bigcap_{W\in\{\{0\}\}} W$$
$$= \{0\},$$

whereas

$$\left(\bigcap_{U \in \mathcal{U}} U\right) \triangle \left(\bigcap_{V \in \mathcal{V}} V\right) = \{0, 1\} \triangle \{0\}$$
$$= \varnothing,$$

Thus we have

$$\bigcap_{W \in \mathcal{U} \triangle \mathcal{V}} W = \{0\} \neq \varnothing = \left(\bigcap_{U \in \mathcal{U}} U\right) \triangle \left(\bigcap_{V \in \mathcal{V}} V\right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of Item 10 of Definition 4.4.7.1.3 and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of Item 11 of Definition 4.4.7.1.3 and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of Item 12 of Definition 4.4.7.1.3 and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of Item 4 of Definition 4.6.1.1.5 and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of Item 4 of Definition 4.6.2.1.3 and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of Item 4 of Definition 4.6.3.1.7 and is proved there.

Item 20, Interaction With Unions of Families I: This is a repetition of Item 20 of Definition 4.3.6.1.2 and is proved there.

Item 21, Interaction With Unions of Families II: This is a repetition of Item 21 of Definition 4.3.6.1.2 and is proved there.   □

4.3.8   Binary Unions

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.8.1.1. The **union of U and V** is the set $U \cup V$ defined by

$$U \cup V \overset{\text{def}}{=} \bigcup_{z \in \{U, V\}} z$$
$$\overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}.$$

Proposition 4.3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$
\begin{aligned}
U \cup - \colon && (\mathcal{P}(X), \subset) && &\to (\mathcal{P}(X), \subset), \\
- \cup V \colon && (\mathcal{P}(X), \subset) && &\to (\mathcal{P}(X), \subset), \\
-_1 \cup -_2 \colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &&&\to (\mathcal{P}(X), \subset).
\end{aligned}
$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

(a) If $U \subset A$, then $U \cup V \subset A \cup V$.

(b) If $V \subset B$, then $U \cup V \subset U \cup B$.

(c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

2. *Associativity.* The diagram

$$
\begin{array}{c}
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \\[2mm]
\overset{\alpha^{\mathsf{Sets}}_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}}{\nearrow} \qquad \overset{\mathrm{id}_{\mathcal{P}(X)} \times \cup}{\searrow} \\[2mm]
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \qquad \mathcal{P}(X) \times \mathcal{P}(X) \\[2mm]
{\scriptstyle \cup \times \mathrm{id}_{\mathcal{P}(X)}} \searrow \qquad\qquad \searrow {\scriptstyle \cup} \\[2mm]
\mathcal{P}(X) \times \mathcal{P}(X) \underset{\cup}{\to} \mathcal{P}(X),
\end{array}
$$

commutes, i.e. we have an equality of sets

$$
(U \cup V) \cup W = U \cup (V \cup W)
$$

for each $U, V, W \in \mathcal{P}(X)$.

3. *Unitality.* The diagrams

$$
\begin{array}{ccc}
\mathrm{pt} \times \mathcal{P}(X) \xrightarrow{[\varnothing] \times \mathrm{id}_{\mathcal{P}(X)}} \mathcal{P}(X) \times \mathcal{P}(X) & \qquad & \mathcal{P}(X) \times \mathrm{pt} \xrightarrow{\mathrm{id}_{\mathcal{P}(X)} \times [\varnothing]} \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle \lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}} \searrow \quad \downarrow {\scriptstyle \cup} & & {\scriptstyle \rho^{\mathsf{Sets}}_{\mathcal{P}(X)}} \searrow \quad \downarrow {\scriptstyle \cup} \\
\mathcal{P}(X) & & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
\varnothing \cup U &= U, \\
U \cup \varnothing &= U
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

4. *Commutativity.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)\times\mathcal{P}(X) & \xrightarrow{\;\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}\;} & \mathcal{P}(X)\times\mathcal{P}(X)\\
& \searrow{\scriptstyle\cup} & \big\downarrow{\scriptstyle\cup}\\
& & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$U\cup V = V\cup U$$

for each $U,V\in\mathcal{P}(X)$.

5. *Annihilation With X.* The diagrams

$$
\begin{array}{ccc}
& \mathrm{pt}\times\mathrm{pt} & \\
{\scriptstyle\mathrm{id}_{\mathrm{pt}}\times\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}}\nearrow & & \searrow{\scriptstyle\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}}\\
\mathrm{pt}\times\mathcal{P}(X) & & \mathrm{pt}\\
{\scriptstyle[X]\times\mathrm{id}_{\mathcal{P}(X)}}\big\downarrow & & \big\downarrow{\scriptstyle[X]}\\
\mathcal{P}(X)\times\mathcal{P}(X) & \underset{\cup}{\to} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
& \mathrm{pt}\times\mathrm{pt} & \\
{\scriptstyle\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathrm{pt}}}\nearrow & & \searrow{\scriptstyle\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}}\\
\mathcal{P}(X)\times\mathrm{pt} & & \mathrm{pt}\\
{\scriptstyle\mathrm{id}_{\mathcal{P}(X)}\times[X]}\big\downarrow & & \big\downarrow{\scriptstyle[X]}\\
\mathcal{P}(X)\times\mathcal{P}(X) & \underset{\cup}{\to} & \mathcal{P}(X)
\end{array}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
U\cup X &= X,\\
X\cup V &= X
\end{aligned}
$$

for each $U,V\in\mathcal{P}(X)$.

6. *Distributivity of Unions Over Intersections.* The diagrams

$$
\begin{array}{ccc}
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) & \\
{\scriptstyle\Delta_{\mathcal{P}(X)}\times\left(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\right)}\nearrow & & \searrow{\scriptstyle\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}}\\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) & \\
{\scriptstyle\mathrm{id}_{\mathcal{P}(X)}\times\cap}\big\downarrow & & \big\downarrow{\scriptstyle\cup\times\cup}\\
\mathcal{P}(X)\times\mathcal{P}(X) & \mathcal{P}(X)\times\mathcal{P}(X), & \\
& \searrow{\scriptstyle\cup}\quad\swarrow{\scriptstyle\cap} & \\
& \mathcal{P}(X) &
\end{array}
$$

$$
\begin{array}{ccc}
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) & \\
{\scriptstyle\left(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\right)\times\Delta_{\mathcal{P}(X)}}\nearrow & & \searrow{\scriptstyle\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}}\\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) & \\
{\scriptstyle\cap\times\mathrm{id}_{\mathcal{P}(X)}}\big\downarrow & & \big\downarrow{\scriptstyle\cup\times\cup}\\
\mathcal{P}(X)\times\mathcal{P}(X) & \mathcal{P}(X)\times\mathcal{P}(X), & \\
& \searrow{\scriptstyle\cup}\quad\swarrow{\scriptstyle\cap} & \\
& \mathcal{P}(X) &
\end{array}
$$

commute, i.e. we have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Distributivity of Intersections Over Unions.* The diagrams
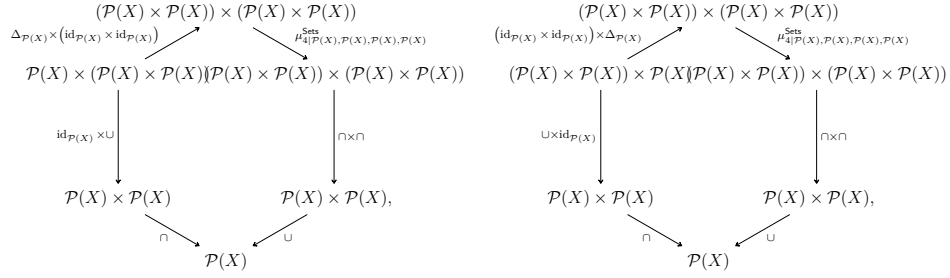
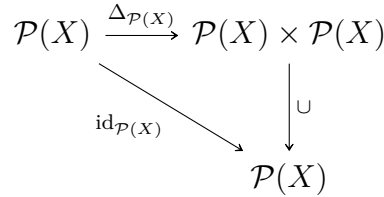commute, i.e. we have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

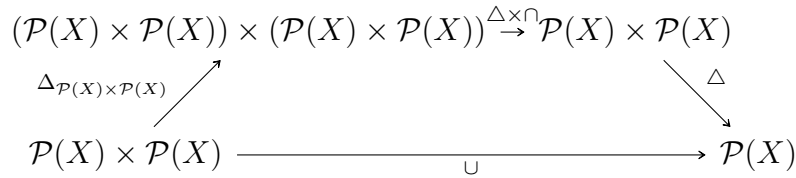for each $U, V, W \in \mathcal{P}(X)$.

8. *Idempotency.* The diagram

commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each $U \in \mathcal{P}(X)$.

9. *Via Intersections and Symmetric Differences.* The diagram

commutes, i.e. we have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cup \downarrow & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
\cup \downarrow & & \downarrow \cup \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle\cup}\big\downarrow & \subset & \big\downarrow{\scriptstyle\cup} \\
\mathcal{P}(X) & \xrightarrow[\;f_*\;]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_*(U) \cup f_*(V) \subset f_*(U \cup V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. Item 1, Functoriality: See [Pro25an].
Item 2, Associativity: See [Pro25ba].
Item 3, Unitality: This follows from [Pro25bd] and Item 4.
Item 4, Commutativity: See [Pro25bb].
Item 5, Annihilation With X: We have

$$
\begin{aligned}
U \cup X &\overset{\mathrm{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\
&= \{x \in X \mid x \in X\}, \\
&= X
\end{aligned}
$$

and

$$
\begin{aligned}
X \cup V &\overset{\mathrm{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\
&= \{x \in X \mid x \in X\} \\
&= X.
\end{aligned}
$$

This finishes the proof.
Item 6, Distributivity of Unions Over Intersections: See [Pro25az].
Item 7, Distributivity of Intersections Over Unions: See [Pro25aj].
Item 8, Idempotency: See [Pro25am].
Item 9, Via Intersections and Symmetric Differences: See [Pro25ay].
Item 10, Interaction With Characteristic Functions I: See [Pro25h].
Item 11, Interaction With Characteristic Functions II: See [Pro25h].

Item 12, Interaction With Direct Images: See [Pro25p].
Item 13, Interaction With Inverse Images: See [Pro25y].
Item 14, Interaction With Codirect Images: This is a repetition of Item 5 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 of this propostition and Items 3 to 6 and 8 of Definition 4.3.9.1.2.

$\square$

4.3.9   Binary Intersections

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.9.1.1. The **intersection of U and V** is the set $U \cap V$ defined by

$$U \cap V \overset{\mathrm{def}}{=} \bigcap_{z \in \{U,V\}} z$$
$$\overset{\mathrm{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}.$$

Proposition 4.3.9.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{array}{lll} U \cap - : & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\ - \cap V : & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 : & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) & \to (\mathcal{P}(X), \subset). \end{array}$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \cap V \subset A \cap V$.

   (b) If $V \subset B$, then $U \cap V \subset U \cap B$.

   (c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X) : \quad \mathcal{P}(X) \overset{U \cap -}{\underset{[U,-]_X}{\rightleftarrows}} \perp \mathcal{P}(X),$$

$$(- \cap V \dashv [V, -]_X) : \quad \mathcal{P}(X) \overset{- \cap V}{\underset{[V,-]_X}{\rightleftarrows}} \perp \mathcal{P}(X),$$

witnessed by bijections

$$\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) \cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X),$$

natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

is the bifunctor of Section 4.4.7. In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

  (a) The following conditions are equivalent:
     i. We have $U \cap V \subset W$.
     ii. We have $U \subset [V, W]_X$.
  (b) The following conditions are equivalent:
     i. We have $U \cap V \subset W$.
     ii. We have $V \subset [U, W]_X$.

3. *Associativity.* The diagram

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* The diagrams

commute, i.e. we have equalities of sets

$$X \cap U = U,$$
$$U \cap X = U$$

for each $U \in \mathcal{P}(X)$.

5. *Commutativity.* The diagram

$$\mathcal{P}(X) \times \mathcal{P}(X) \xrightarrow{\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}} \mathcal{P}(X) \times \mathcal{P}(X)$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each $U, V \in \mathcal{P}(X)$.
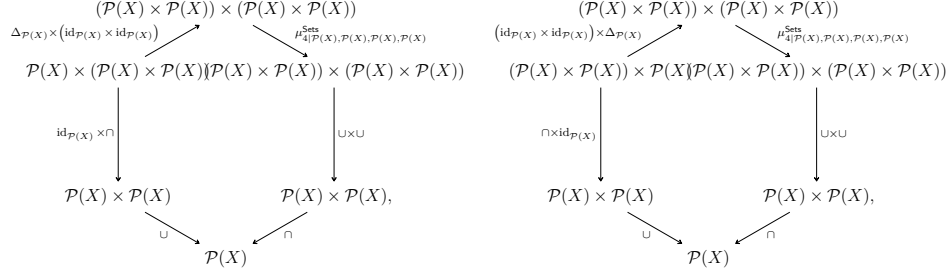
6. *Annihilation With the Empty Set.* The diagrams

commute, i.e. we have equalities of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $U \in \mathcal{P}(X)$.

7. *Distributivity of Unions Over Intersections.* The diagrams

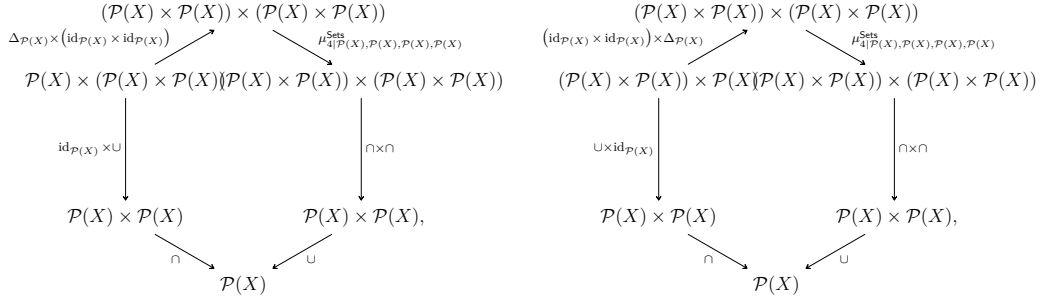commute, i.e. we have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

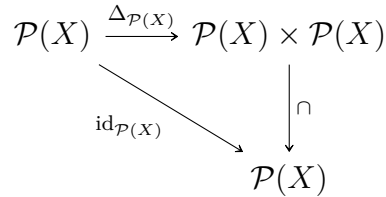8. *Distributivity of Intersections Over Unions.* The diagrams

commute, i.e. we have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Idempotency.* The diagram

commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cap \downarrow & \subseteq & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{\quad f_! \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{f^{-1} \times f^{-1}} & \mathcal{P}(X) \times \mathcal{P}(X) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\ f_* \times f_*\ } & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle\cap}\big\downarrow & & \big\downarrow{\scriptstyle\cap} \\
\mathcal{P}(X) & \xrightarrow[\ f_*\ ]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
f_*(U) \cap f_*(V) = f_*(U \cap V)
$$

for each $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. Item 1, Functoriality: See [Pro25al].
Item 2, Adjointness: See [MSE 267469].
Item 3, Associativity: See [Pro25r].
Item 4, Unitality: This follows from [Pro25v] and Item 5.
Item 5, Commutativity: See [Pro25s].
Item 6, Annihilation With the Empty Set: This follows from [Pro25t] and Item 5.
Item 7, Distributivity of Unions Over Intersections: See [Pro25az].
Item 8, Distributivity of Intersections Over Unions: See [Pro25aj].
Item 9, Idempotency: See [Pro25ak].
Item 10, Interaction With Characteristic Functions I: See [Pro25e].
Item 11, Interaction With Characteristic Functions II: See [Pro25e].
Item 12, Interaction With Direct Images: See [Pro25n].
Item 13, Interaction With Inverse Images: See [Pro25w].
Item 14, Interaction With Codirect Images: This is a repetition of Item 6 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 6.
Item 16, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 and Items 3 to 6 and 8 of Definition 4.3.9.1.2. $\square$

4.3.10   Differences

Let X and Y be sets.

Definition 4.3.10.1.1. The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\mathrm{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 4.3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$
\begin{aligned}
U \setminus - &: & (\mathcal{P}(X), \supset) & \to (\mathcal{P}(X), \subset), \\
- \setminus V &: & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\
-_1 \setminus -_2 &: (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) & & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.

   (b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.

   (c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

2. *De Morgan's Laws.* We have equalities of sets

$$
\begin{aligned}
X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\
X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V)
\end{aligned}
$$

   for each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

   for each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

   for each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$U \setminus (V \cup W) = (U \cup W) \setminus (V \cup W)$$
$$= (U \setminus V) \setminus W$$
$$= (U \setminus W) \setminus V$$

for each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$(U \setminus V) \cap W = (U \cap W) \setminus V$$
$$= U \cap (W \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \bigtriangleup (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

13. *Right Annihilation.* We have

$$U \setminus X = U$$

for each $U \in \mathcal{P}(X)$.

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

15. *Interaction With Containment.* The following conditions are equivalent:

   (a) We have $V \setminus U \subset W$.
   (b) We have $V \setminus W \subset U$.

16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Direct Images.* We have a natural transformation

$$
\begin{CD}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) @>{f_!^{\mathsf{op}} \times f_!}>> \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
@V{\setminus}VV \quad\supset\quad @VV{\setminus}V \\
\mathcal{P}(X) @>>{f_!}> \mathcal{P}(Y)
\end{CD}
$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

18. *Interaction With Inverse Images.* The diagram

$$\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \xrightarrow{f^{\mathsf{op},-1} \times f^{-1}} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)$$

$$\mathcal{P}(Y) \xrightarrow{\quad f^{-1} \quad} \mathcal{P}(X)$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

19. *Interaction With Codirect Images.* We have a natural transformation

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \xrightarrow{f_!^{\mathsf{op}} \times f_!} \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y)$$

$$\mathcal{P}(X) \xrightarrow{\quad f_! \quad} \mathcal{P}(Y)$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: See [Pro25ad] and [Pro25ah].
Item 2, De Morgan's Laws: See [Pro25k].
Item 3, Interaction With Unions I: See [Pro25l].
Item 4, Interaction With Unions II: Omitted.
Item 5, Interaction With Unions III: See [Pro25ai].
Item 6, Interaction With Unions IV: See [Pro25ac].
Item 7, Interaction With Intersections: See [Pro25u].
Item 8, Interaction With Complements: See [Pro25aa].
Item 9, Interaction With Symmetric Differences: See [Pro25ab].
Item 10, Triple Differences: See [Pro25ag].
Item 11, Left Annihilation: Omitted.
Item 12, Right Unitality: See [Pro25ae].
Item 13, Right Annihilation: Omitted.

Item 14, Invertibility: See [Pro25af].
Item 15, Interaction With Containment: Omitted.
Item 16, Interaction With Characteristic Functions: See [Pro25f].
Item 17, Interaction With Direct Images: See [Pro25o].
Item 18, Interaction With Inverse Images: See [Pro25x].   □

4.3.11   Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.3.11.1.1. The **complement of** U is the set U^{c} defined by

$$U^{\mathsf{c}} \overset{\mathrm{def}}{=} X \setminus U$$
$$\overset{\mathrm{def}}{=} \{a \in X \mid a \notin U\}.$$

Proposition 4.3.11.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X).$$

   In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

   ($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* The diagrams

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} & \overset{\cup^{\mathsf{op}}}{\longrightarrow} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}} \times (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \underset{\cap}{\longrightarrow} & \mathcal{P}(X)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} & \overset{\cap^{\mathsf{op}}}{\longrightarrow} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}} \times (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \underset{\cup}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

   commute, i.e. we have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

   for each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* The diagram

$$
\begin{CD}
\mathcal{P}(X)^{\mathsf{op}} @>{(-)^{\mathsf{c}}}>> \mathcal{P}(X) \\
@V{\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}}VV @VV{(-)^{\mathsf{c},\mathsf{op}}}V \\
& & \mathcal{P}(X)^{\mathsf{op}}
\end{CD}
$$

commutes, i.e. we have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} \equiv 1 - \chi_U \pmod 2$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{CD}
\mathcal{P}(X)^{\mathsf{op}} @>{f_*^{\mathsf{op}}}>> \mathcal{P}(Y)^{\mathsf{op}} \\
@V{(-)^{\mathsf{c}}}VV @VV{(-)^{\mathsf{c}}}V \\
\mathcal{P}(X) @>>{f_!}> \mathcal{P}(Y)
\end{CD}
$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

6. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{CD}
\mathcal{P}(Y)^{\mathsf{op}} @>{f^{-1,\mathsf{op}}}>> \mathcal{P}(X)^{\mathsf{op}} \\
@V{(-)^{\mathsf{c}}}VV @VV{(-)^{\mathsf{c}}}V \\
\mathcal{P}(Y) @>>{f^{-1}}> \mathcal{P}(X)
\end{CD}
$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_!^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) & \xrightarrow[f_*]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: This follows from Item 1 of Definition 4.3.10.1.2.
Item 2, De Morgan's Laws: See [Pro25k].
Item 3, Involutority: See [Pro25i].
Item 4, Interaction With Characteristic Functions: Omitted.
Item 5, Interaction With Direct Images: This is a repetition of Item 8 of Definition 4.6.1.1.5 and is proved there.
Item 6, Interaction With Inverse Images: This is a repetition of Item 8 of Definition 4.6.2.1.3 and is proved there.
Item 7, Interaction With Codirect Images: This is a repetition of Item 7 of Definition 4.6.3.1.7 and is proved there.   □

4.3.12   Symmetric Differences

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.12.1.1. The **symmetric difference of U and V** is the set

$U \triangle V$ defined by[13]

$$U \triangle V \overset{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

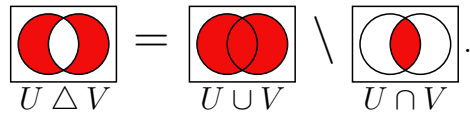**Proposition 4.3.12.1.2.** Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ **does not** in general define functors

$$
\begin{aligned}
U \triangle - &: & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\
- \triangle V &: & (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\
-_1 \triangle -_2 &: & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

2. *Via Unions and Intersections.* We have

$$U \triangle V = (U \cup V) \setminus (U \cap V)$$

   for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram

3. *Symmetric Differences of Disjoint Sets.* If U and V are disjoint, then we have

$$U \triangle V = U \cup V.$$
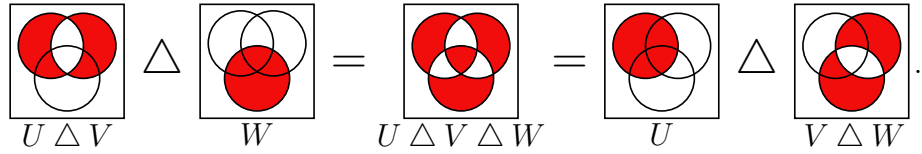
4. *Associativity.* The diagram

---

[13] *Illustration:*

commutes, i.e. we have

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram

5. *Unitality.* The diagrams

commute, i.e. we have

$$U \triangle \emptyset = U,$$
$$\emptyset \triangle U = U$$

for each $U \in \mathcal{P}(X)$.

6. *Commutativity.* The diagram

commutes, i.e. we have

$$U \triangle V = V \triangle U$$

for each $U, V \in \mathcal{P}(X)$.

7. *Invertibility.* We have

$$U \bigtriangleup U = \varnothing$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Unions.* We have

$$(U \bigtriangleup V) \cup (V \bigtriangleup T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Interaction With Complements I.* We have

$$U \bigtriangleup U^{\mathsf{c}} = X$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements II.* We have

$$U \bigtriangleup X = U^{\mathsf{c}},$$
$$X \bigtriangleup U = U^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

11. *Interaction With Complements III.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\ \bigtriangleup\ } & \mathcal{P}(X) \\
{\scriptstyle (-)^{\mathsf{c}} \times (-)^{\mathsf{c}}}\big\downarrow & & \big\downarrow{\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow[\ \bigtriangleup\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$U^{\mathsf{c}} \bigtriangleup V^{\mathsf{c}} = U \bigtriangleup V$$

for each $U, V \in \mathcal{P}(X)$.

12. *"Transitivity".* We have

$$(U \bigtriangleup V) \bigtriangleup (V \bigtriangleup W) = U \bigtriangleup W$$

for each $U, V, W \in \mathcal{P}(X)$.

13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Distributivity Over Intersections.* We have

$$U \cap (V \triangle W) = (U \cap V) \triangle (U \cap W),$$
$$(U \triangle V) \cap W = (U \cap W) \triangle (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

16. *Bijectivity.* Given $U, V \in \mathcal{P}(X)$, the maps

$$U \triangle - \colon \mathcal{P}(X) \to \mathcal{P}(X),$$
$$- \triangle V \colon \mathcal{P}(X) \to \mathcal{P}(X)$$

are bijections with inverses given by

$$(U \triangle -)^{-1} = - \cup (U \cap -),$$
$$(- \triangle V)^{-1} = - \cup (V \cap -).$$

Moreover, the map

$$\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$
$$C \longmapsto C \triangle (U \triangle V)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending U to V and V to U.

17. *Interaction With Powersets and Groups.* Let X be a set.

(a) The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[14]

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

- The group $\mathcal{P}(X)$ of Item 17;
- The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$
\begin{aligned}
0 \cdot U &\overset{\mathrm{def}}{=} \emptyset, \\
1 \cdot U &\overset{\mathrm{def}}{=} U;
\end{aligned}
$$

is an $\mathbb{F}_2$-vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

(a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 4.

(b) We have
$$\dim(\mathcal{P}(X)) = \#X.$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[15]

---

[14] Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0,1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0,1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[15] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25aw] for a proof.

7. *Interaction With Direct Images.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathsf{op}} \times f_!} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \triangle}\Big\downarrow & \circlearrowright & \Big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow{\quad f_! \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_!(U) \mathbin{\triangle} f_!(V) \subset f_!(U \mathbin{\triangle} V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Inverse Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\mathsf{op},-1} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle \triangle}\Big\downarrow & & \Big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

i.e. we have

$$
f^{-1}(U) \mathbin{\triangle} f^{-1}(V) = f^{-1}(U \mathbin{\triangle} V)
$$

for each $U, V \in \mathcal{P}(Y)$.

9. *Interaction With Codirect Images.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathsf{op}} \times f_*} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \triangle}\Big\downarrow & \circlearrowright & \Big\downarrow{\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow{\quad f_* \quad} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
f_*(U \mathbin{\triangle} V) \subset f_*(U) \mathbin{\triangle} f_*(V)
$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: See [Pro25m].
Item 3, Symmetric Differences of Disjoint Sets: Since U and V are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$\begin{aligned} U \triangle V &= (U \cup V) \setminus (U \cap V) \\ &= (U \cup V) \setminus \emptyset \\ &= U \cup V, \end{aligned}$$

where we've used Item 2 and Item 12 of Definition 4.3.10.1.2.
Item 4, Associativity: See [Pro25ao].
Item 5, Unitality: This follows from Item 6 and [Pro25at].
Item 6, Commutativity: See [Pro25ap].
Item 7, Invertibility: See [Pro25av].
Item 8, Interaction With Unions: See [Pro25bc].
Item 9, Interaction With Complements I: See [Pro25as].
Item 10, Interaction With Complements II: This follows from Item 6 and [Pro25ax].
Item 11, Interaction With Complements III: See [Pro25aq].
Item 12, "Transitivity": We have

$$\begin{aligned} (U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) & \text{(by Item 4)} \\ &= U \triangle ((V \triangle V) \triangle W) & \text{(by Item 4)} \\ &= U \triangle (\emptyset \triangle W) & \text{(by Item 7)} \\ &= U \triangle W. & \text{(by Item 5)} \end{aligned}$$

This finishes the proof.
Item 13, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 12.
Item 14, Distributivity Over Intersections: See [Pro25q].
Item 15, Interaction With Characteristic Functions: See [Pro25g].
Item 16, Bijectivity: Omitted.
Item 17, Interaction With Powersets and Groups: Item 17a follows from Items 4 to 7, while Item 3b follows from Item 7.[16]
Item 4, Interaction With Powersets and Vector Spaces I: See [MSE 2719059].
Item 5, Interaction With Powersets and Vector Spaces II: See [MSE 2719059].
Item 6, Interaction With Powersets and Rings: This follows from Items 6 and 15 of Definition 4.3.9.1.2 and Items 14 and 17.[17]

---

[16] *Reference:* [Pro25ar].
[17] *Reference:* [Pro25au].

Item 7, Interaction With Direct Images: This is a repetition of Item 9 of Definition 4.6.1.1.5 and is proved there.

Item 8, Interaction With Inverse Images: This is a repetition of Item 9 of Definition 4.6.2.1.3 and is proved there.

Item 9, Interaction With Codirect Images: This is a repetition of Item 8 of Definition 4.6.3.1.7 and is proved there. □

4.4 Powersets

4.4.1 Foundations

Let X be a set.

Definition 4.4.1.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.4.1.1.2. Under the analogy that $\{\mathsf{t}, \mathsf{f}\}$ should be the (-1)-categorical analogue of Sets, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set X is equivalently (Item 2 of Definition 4.5.1.1.4) the set
$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$
of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category $\mathcal{C}$ is the category
$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$
of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

Notation 4.4.1.1.3. Let X be a set.

1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of X.

2. We write $\mathcal{P}_{\text{fin}}(X)$ for the set of finite subsets of X.

Proposition 4.4.1.1.4. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

    (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

    (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

    (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.

3. *Powersets as Sets of Relations.* We have bijections

$$\mathcal{P}(X) \cong \mathrm{Rel}(\mathrm{pt}, X),$$
$$\mathcal{P}(X) \cong \mathrm{Rel}(X, \mathrm{pt}),$$

    natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Interaction With Products I.* The map

$$\mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \amalg Y)$$
$$(U, V) \longmapsto U \cup V$$

    is an isomorphism of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

5. *Interaction With Products II.* The map

$$\mathcal{P}(X) \times \mathcal{P}(Y) \longrightarrow \mathcal{P}(X \amalg Y)$$
$$(U, V) \longmapsto U \boxtimes_{X \times Y} V,$$

    where[18]

$$U \boxtimes_{X \times Y} V \overset{\mathrm{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

---

[18]The set $U \boxtimes_{X \times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this

is an inclusion of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

6. *Interaction With Products III.* We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1, where $\otimes$ denotes the tensor product of suplattices of **??**. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

Proof. Item 1, Co/Completeness: Omitted.
Item 2, Cartesian Closedness: See Section 4.4.7.
Item 3, Powersets as Sets of Relations: Indeed, we have

$$\mathrm{Rel}(\mathrm{pt}, X) \overset{\mathrm{def}}{=} \mathcal{P}(\mathrm{pt} \times X)$$
$$\cong \mathcal{P}(X)$$

and

$$\mathrm{Rel}(X, \mathrm{pt}) \overset{\mathrm{def}}{=} \mathcal{P}(X \times \mathrm{pt})$$
$$\cong \mathcal{P}(X),$$

where we have used Item 5 of Definition 4.1.3.1.3.
Item 4, Interaction With Products I: The inverse of the map in the statement is the map
$$\Phi \colon \mathcal{P}(X \amalg Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$$
defined by
$$\Phi(S) \overset{\mathrm{def}}{=} (S_X, S_Y)$$
for each $S \in \mathcal{P}(X \amalg Y)$, where

$$S_X \overset{\mathrm{def}}{=} \{x \in X \mid (0, x) \in S\}$$
$$S_Y \overset{\mathrm{def}}{=} \{y \in Y \mid (1, y) \in S\}.$$

The rest of the proof is omitted.
Item 5, Interaction With Products II: Omitted.
Item 6, Interaction With Products III: Omitted. $\square$

4.4.2   Functoriality of Powersets

Proposition 4.4.2.1.1. Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets},$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$\mathcal{P}_!(A) \overset{\mathrm{def}}{=} \mathcal{P}(A).$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

   $$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

   of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

   $$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   defined by
   $$\mathcal{P}_!(f) \overset{\mathrm{def}}{=} f_!,$$

   as in Definition 4.6.1.1.1.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$\mathcal{P}^{-1}(A) \overset{\mathrm{def}}{=} \mathcal{P}(A).$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on

---

somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see also Section 4.6.4).

morphisms

$$\mathcal{P}^{-1}_{A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(B),\mathcal{P}(A))$$

of $\mathcal{P}^{-1}$ at (A,B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}^{-1}(f)\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by
$$\mathcal{P}^{-1}(f) \stackrel{\mathrm{def}}{=} f^{-1},$$

as in Definition 4.6.2.1.1.

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\mathrm{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B}\colon \mathsf{Sets}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B))$$

of $\mathcal{P}_*$ at (A,B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}_*(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by
$$\mathcal{P}_*(f) \stackrel{\mathrm{def}}{=} f_*,$$

as in Definition 4.6.3.1.1.

Proof. Item 1, Functoriality I: This follows from Items 3 and 4 of Definition 4.6.1.1.6.
Item 2, Functoriality II: This follows from Items 3 and 4 of Definition 4.6.2.1.4.
Item 3, Functoriality III: This follows from Items 3 and 4 of Definition 4.6.3.1.8.

$\square$

4.4.3   Adjointness of Powersets I

Proposition 4.4.3.1.1. We have an adjunction

$$
\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \mathsf{Sets}^{\mathsf{op}} \overset{\mathcal{P}^{-1}}{\underset{\mathcal{P}^{-1,\mathsf{op}}}{\rightleftarrows}} \perp \; \mathsf{Sets},
$$

witnessed by a bijection

$$
\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\overset{\mathrm{def}}{=}\,\mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),
$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

Proof. We have

$$
\begin{aligned}
\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) \quad &\overset{\mathrm{def}}{=}\quad \mathsf{Sets}(B, \mathcal{P}(A)) \\
&\cong\quad \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) \quad &&(\text{by Item 2 of Definition 4.5.1.1.4}) \\
&\cong\quad \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \quad &&(\text{by Item 2 of Definition 4.1.3.1.3}) \\
&\cong\quad \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \quad &&(\text{by Item 2 of Definition 4.1.3.1.3}) \\
&\cong\quad \mathsf{Sets}(A, \mathcal{P}(B)), \quad &&(\text{by Item 2 of Definition 4.5.1.1.4})
\end{aligned}
$$

where all bijections are natural in A and B.[19] $\square$

4.4.4   Adjointness of Powersets II

Proposition 4.4.4.1.1. We have an adjunction

$$
(\mathrm{Gr} \dashv \mathcal{P}_!)\colon \quad \mathsf{Sets} \overset{\mathrm{Gr}}{\underset{\mathcal{P}_!}{\rightleftarrows}} \perp \; \mathrm{Rel},
$$

witnessed by a bijection of sets

$$
\mathrm{Rel}(\mathrm{Gr}(X), Y) \cong \mathsf{Sets}(X, \mathcal{P}(Y))
$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, Item 1 of Definition 8.2.2.1.2 and $\mathcal{P}_!$ is the functor of Relations, Definition 8.7.5.1.1.

---

[19]Here we are using Item 3 of Definition 4.5.1.1.4.

Proof. We have

$$
\begin{array}{rcll}
\mathrm{Rel}(\mathrm{Gr}(A),B) & \cong & \mathcal{P}(A \times B) & \\
& \cong & \mathsf{Sets}(A \times B, \{\mathsf{t},\mathsf{f}\}) & \text{(by Item 2 of Definition 4.5.1.1.4)} \\
& \cong & \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t},\mathsf{f}\})) & \text{(by Item 2 of Definition 4.1.3.1.3)} \\
& \cong & \mathsf{Sets}(A, \mathcal{P}(B)), & \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{array}
$$

where all bijections are natural in A, (where we are using Item 3 of Definition 4.5.1.1.4). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Relations, Definition 8.1.1.1.1.

Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$
\begin{array}{ccc}
\mathrm{Rel}(\mathrm{Gr}(A),B) & \xrightarrow{R\diamond-} & \mathrm{Rel}(\mathrm{Gr}(A),B') \\
\Big\downarrow{\scriptstyle\wr} & & \Big\downarrow{\scriptstyle\wr} \\
\mathsf{Sets}(A,\mathcal{P}(B)) & \xrightarrow[R_!]{} & \mathsf{Sets}(A,\mathcal{P}(B'))
\end{array}
$$

commutes, which follows from Relations, Definition 8.7.1.1.3. $\qquad\square$

4.4.5 Powersets as Free Cocompletions

Let X be a set.

Proposition 4.4.5.1.1. The pair $\big(\mathcal{P}(X), \chi_{(-)}\big)$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of X of Definition 4.4.1.1.1;

- The characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

($\star$) Given another pair (Y, f) consisting of

　　– A suplattice $(Y, \preceq)$;

　　– A function $f\colon X \to Y$;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$
\begin{array}{ccc}
 & & \mathcal{P}(X) \\
 & \chi_X \nearrow & \Big\downarrow \exists! \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

commute.

Proof. This is a rephrasing of Definition 4.4.5.1.2, which we prove below.[20]

$\square$

Proposition 4.4.5.1.2. We have an adjunction

$$\left(\mathcal{P} \dashv 忘\right): \quad \mathsf{Sets} \underset{忘}{\overset{\mathcal{P}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{SupLat},$$

witnessed by a bijection

$$\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{SupLat})$, where:

- The category SupLat is the category of suplattices of **??**.

- The map

$$\chi_X^* \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

witnessing the above bijection is defined by

$$\chi_X^*(f) \overset{\mathrm{def}}{=} f \circ \chi_X,$$

i.e. by sending a morphism of suplattices $f \colon \mathcal{P}(X) \to Y$ to the composition

$$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$

---

[20]Here we only remark that the unique morphism of suplattices in the statement is

- The map

$$\mathrm{Lan}_{\chi_X}\colon \mathsf{Sets}(X,Y) \to \mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))$$

witnessing the above bijection is given by sending a function $f\colon X \to Y$ to its left Kan extension along χ_X,

$$\mathrm{Lan}_{\chi_X}(f)\colon \mathcal{P}(X) \to Y,$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f)](U) &= \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)\\
&= \int^{x\in X} \chi_U(x) \odot f(x)\\
&= \bigvee_{x\in X} (\chi_U(x) \odot f(x))\\
&= \left(\bigvee_{x\in U} (\chi_U(x) \odot f(x))\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} (\chi_U(x) \odot f(x))\right)\\
&= \left(\bigvee_{x\in U} f(x)\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} \varnothing_Y\right)\\
&= \bigvee_{x\in U} f(x)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigvee$ denotes the join in $(Y,\preceq)$.
- The symbol $\odot$ denotes the tensor of an element of Y by a truth value as in **??**. In particular, we have

$$\mathsf{true} \odot f(x) \overset{\mathrm{def}}{=} f(x),$$

$$\mathsf{false} \odot f(x) \overset{\text{def}}{=} \varnothing_Y,$$

where $\varnothing_Y$ is the bottom element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathrm{Lan}_{\chi_X}(f)$ is given by

$$[\mathrm{Lan}_{\chi_X}(f)](U) = \bigvee_{x \in U} f(x)$$
$$= \bigcup_{x \in U} f(x)$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y} \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \overset{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.
Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f),$$

for each $f \in \mathsf{Sets}(X, Y)$.
Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))} .$$

We have

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) \overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f))$$

$$\overset{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X)$$
$$\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f \circ \chi_X)$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. Indeed, we have

$$[\mathrm{Lan}_{\chi_X}(f \circ \chi_X)](U) = \bigvee_{x \in U} f(\chi_X(x))$$
$$= f\left(\bigvee_{x \in U} \chi_X(x)\right)$$
$$= f\left(\bigcup_{x \in U} \{x\}\right)$$
$$= f(U)$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))}$ of $\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. *Invertibility II*: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

We have

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) \overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))$$
$$\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Lan}_{\chi_X}(f))$$
$$\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f) \circ \chi_X$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$[\mathrm{Lan}_{\chi_X}(f) \circ \chi_X](x) = \bigvee_{y \in \{x\}} f(y)$$
$$= f(x)$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, Part I: We need to show that, given a function $f \colon X \to X'$, the diagram

$$
\begin{array}{ccc}
\mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) & \xrightarrow{\Phi_{X',Y}} & \mathsf{Sets}(X', Y) \\
\big\downarrow{\scriptstyle \mathcal{P}_!(f)^*} & & \big\downarrow{\scriptstyle f^*} \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) & \xrightarrow[\Phi_{X,Y}]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
&\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
&= \xi \circ (f_! \circ \chi_X) \\
&\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\overset{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}
$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, Part II: We need to show that, given a morphism of suplattices

$$g \colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq)) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X,Y)$$

$$g_! \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow g_!$$

$$\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y',\preceq)) \xrightarrow[\Phi_{X,Y'}]{} \mathsf{Sets}(X,Y')$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_!](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi)) \\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} g_!(\Phi_{X,Y}(\xi)) \\
&\overset{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))$.

Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. $\square$

Warning 4.4.5.1.3. Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion of X*, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.

4.4.6   Powersets as Free Completions

Let X be a set.

Proposition 4.4.6.1.1. The pair $\left(\mathcal{P}(X), \chi_{(-)}\right)$ consisting of

- The powerset of X together with reverse inclusion $\mathcal{P}(X)^{\mathsf{op}} = (\mathcal{P}(X), \supset)$ of Definition 4.4.1.1.1;

- The characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

($\star$) Given another pair (Y, f) consisting of

- An inflattice $(Y, \preceq)$;
- A function $f \colon X \to Y$;

there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(X)^{\mathsf{op}} & \\
\chi_X \nearrow & \vdots & \exists! \\
X \xrightarrow{\quad f \quad} & Y &
\end{array}
$$

commute.

Proof. This is a rephrasing of Definition 4.4.6.1.2, which we prove below.[21]

$\square$

Proposition 4.4.6.1.2. We have an adjunction

$$\left(\mathcal{P} \dashv \text{忘}\right) \colon \quad \mathsf{Sets} \underset{\text{忘}}{\overset{\mathcal{P}}{\rightleftarrows}} \bot \quad \mathsf{InfLat},$$

witnessed by a bijection

$$\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{InfLat})$, where:

- The category InfLat is the category of inflattices of **??**.

---

given by the left Kan extension $\mathrm{Lan}_{\chi_X}(f)$ of f along χ_X.

[21] Here we only remark that the unique morphism of inflattices in the statement is given by the right Kan extension $\mathrm{Ran}_{\chi_X}(f)$ of f along χ_X.

- The map

$$\chi_X^* \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  witnessing the above bijection is defined by
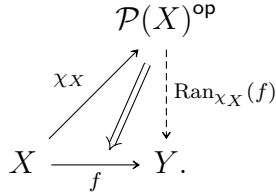
$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of inflattices $f \colon \mathcal{P}(X)^{\mathsf{op}} \to Y$ to the composition

$$X \stackrel{\chi_X}{\Longrightarrow} \mathcal{P}(X)^{\mathsf{op}} \stackrel{f}{\longrightarrow} Y.$$

- The map

$$\mathrm{Ran}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

  witnessing the above bijection is given by sending a function $f \colon X \to Y$ to its right Kan extension along χ_X,

$$\mathrm{Ran}_{\chi_X}(f) \colon \mathcal{P}(X)^{\mathsf{op}} \to Y,$$

  Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$
\begin{aligned}
[\mathrm{Ran}_{\chi_X}(f)](U) &= \int_{x \in X} \chi_{\mathcal{P}(X)^{\mathsf{op}}}(\chi_x, U) \pitchfork f(x) \\
&= \int_{x \in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \pitchfork f(x) \\
&= \int_{x \in X} \chi_U(x) \pitchfork f(x) \\
&= \bigwedge_{x \in X} \chi_U(x) \pitchfork f(x) \\
&= \left(\bigwedge_{x \in U} \chi_U(x) \pitchfork f(x) \right) \wedge \left(\bigwedge_{x \in U^{\mathsf{c}}} \chi_U(x) \pitchfork f(x) \right) \\
&= \left(\bigwedge_{x \in U} f(x) \right) \wedge \left(\bigwedge_{x \in U^{\mathsf{c}}} \infty_Y \right)
\end{aligned}
$$

$$= \left(\bigwedge_{x \in U} f(x) \right) \wedge \infty_Y$$

$$= \bigwedge_{x \in U} f(x)$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigwedge$ denotes the meet in $(Y, \preceq)$.
- The symbol $\pitchfork$ denotes the cotensor of an element of Y by a truth value as in **??**. In particular, we have

$$\mathsf{true} \pitchfork f(x) \overset{\mathrm{def}}{=} f(x),$$
$$\mathsf{false} \pitchfork f(x) \overset{\mathrm{def}}{=} \infty_Y,$$

  where ∞_Y is the top element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathrm{Ran}_{\chi_X}(f)$ is given by

$$[\mathrm{Ran}_{\chi_X}(f)](U) = \bigwedge_{x \in U} f(x)$$
$$= \bigcap_{x \in U} f(x)$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y} \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by
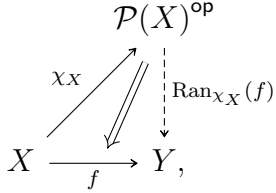
$$\Phi_{X,Y}(f) \overset{\mathrm{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f),$$

for each $f \in \mathsf{Sets}(X, Y)$.

Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))} .$$

We have

$$\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\
&\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\
&\stackrel{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f \circ \chi_X)
\end{aligned}$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned}
[\mathrm{Ran}_{\chi_X}(f \circ \chi_X)](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\
&= f\left(\bigwedge_{x \in U} \chi_X(x) \right) \\
&= f\left(\bigcup_{x \in U} \{x\} \right) \\
&= f(U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}$ of $\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)} \,.$$

We have

$$
\begin{aligned}
{[\Phi_{X,Y} \circ \Psi_{X,Y}](f)} &\overset{\mathrm{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f)) \\
&\overset{\mathrm{def}}{=} \Phi_{X,Y}(\mathrm{Ran}_{\chi_X}(f)) \\
&\overset{\mathrm{def}}{=} \mathrm{Ran}_{\chi_X}(f) \circ \chi_X
\end{aligned}
$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$
\begin{aligned}
{[\mathrm{Ran}_{\chi_X}(f) \circ \chi_X](x)} &= \bigwedge_{y \in \{x\}} f(y) \\
&= f(x)
\end{aligned}
$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, Part I: We need to show that, given a function $f \colon X \to X'$, the diagram

$$
\begin{array}{ccc}
\mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) & \xrightarrow{\;\Phi_{X',Y}\;} & \mathsf{Sets}(X', Y) \\
\Big\downarrow{\scriptstyle \mathcal{P}_!(f)^*} & & \Big\downarrow{\scriptstyle f^*} \\
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow[\;\Phi_{X,Y}\;]{} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, we have

$$
\begin{aligned}
{[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi)} &\overset{\mathrm{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\overset{\mathrm{def}}{=} \Phi_{X,Y}(\xi \circ f_!)
\end{aligned}
$$

$$\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X$$
$$= \xi \circ (f_! \circ \chi_X)$$
$$\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f)$$
$$= (\xi \circ \chi_{X'}) \circ f$$
$$\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f$$
$$\overset{\text{def}}{=} f^*(\Phi_{X',Y}(\xi))$$
$$\overset{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, Part II: We need to show that, given a cocontinuous morphism of posets

$$g \colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) & \xrightarrow{\ \Phi_{X,Y}\ } & \mathsf{Sets}(X, Y) \\
\Big\downarrow{\scriptstyle g_!} & & \Big\downarrow{\scriptstyle g_!} \\
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) & \xrightarrow[\ \Phi_{X,Y'}\ ]{} & \mathsf{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, we have

$$[\Phi_{X,Y'} \circ g_!](\xi) \overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi))$$
$$\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)$$
$$\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X$$
$$= g \circ (\xi \circ \chi_X)$$
$$\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))$$
$$\overset{\text{def}}{=} g_!(\Phi_{X,Y}(\xi))$$
$$\overset{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. $\square$

Warning 4.4.6.1.3. Although the assignment $X \mapsto \mathcal{P}(X)^{\mathsf{op}}$ is called the *free completion of X*, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}})^{\mathsf{op}} \neq \mathcal{P}(X)^{\mathsf{op}}$.

4.4.7   The Internal Hom of a Powerset

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Proposition 4.4.7.1.1. The **internal Hom of $\mathcal{P}(X)$ from U to V** is the subset $[U, V]_X{}^{22}$ of X given by

$$[U, V]_X = U^{\mathsf{c}} \cup V$$
$$= (U \setminus V)^{\mathsf{c}}$$

where U^{c} is the complement of U of Definition 4.3.11.1.1.

Proof. Proof of the Equality $U^{\mathsf{c}} \cup V = (U \setminus V)^{\mathsf{c}}$: We have

$$(U \setminus V)^{\mathsf{c}} \overset{\text{def}}{=} X \setminus (U \setminus V)$$
$$= (X \cap V) \cup (X \setminus U)$$
$$= V \cup (X \setminus U)$$
$$\overset{\text{def}}{=} V \cup U^{\mathsf{c}}$$
$$= U^{\mathsf{c}} \cup V,$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the second equality.

2. Item 4 of Definition 4.3.9.1.2 for the third equality.

3. Item 4 of Definition 4.3.8.1.2 for the last equality.

This finishes the proof.
Proof that $U^{\mathsf{c}} \cup V$ Is Indeed the Internal Hom: This follows from Item 2 of Definition 4.3.9.1.2. □

Remark 4.4.7.1.2. Henning Makholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from U to V ([MSE 267365]):

1. Since products in $\mathcal{P}(X)$ are given by binary intersections (Item 1 of Definition 4.4.1.1.4), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.

2. Under the Curry–Howard correspondence (**??**), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.

---

[22]*Further Notation:* Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$.

3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.

4. The expression $\neg U \vee V$ then corresponds to the set $U^{\mathsf{c}} \cup V$ in $\mathcal{P}(X)$.

5. The set $U^{\mathsf{c}} \vee V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

Proposition 4.4.7.1.3. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$
\begin{aligned}
[U, -]_X\colon && (\mathcal{P}(X), \supset) && \to (\mathcal{P}(X), \subset), \\
[-, V]_X\colon && (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\
[-_1, -_2]_X\colon & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) & \to (\mathcal{P}(X), \subset).
\end{aligned}
$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

(a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.

(b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.

(c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

2. *Adjointness.* We have adjunctions

$$
(U \cap - \dashv [U, -]_X)\colon \quad \mathcal{P}(X) \overset{U \cap -}{\underset{[U, -]_X}{\rightleftarrows}} \perp \mathcal{P}(X),
$$

$$
(- \cap V \dashv [V, -]_X)\colon \quad \mathcal{P}(X) \overset{- \cap V}{\underset{[V, -]_X}{\rightleftarrows}} \perp \mathcal{P}(X),
$$

witnessed by bijections

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X).
\end{aligned}
$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

  i. We have $U \cap V \subset W$.

  ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

      i. We have $U \cap V \subset W$.

     ii. We have $V \subset [U, W]_X$.

3. *Interaction With the Empty Set I.* We have

$$[U, \varnothing]_X = U^{\mathsf{c}},$$
$$[\varnothing, V]_X = X,$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Interaction With X.* We have

$$[U, X]_X = X,$$
$$[X, V]_X = V,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Interaction With the Empty Set II.* The functor

$$D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

defined by

$$D_X \stackrel{\mathrm{def}}{=} [-, \varnothing]_X$$
$$= (-)^{\mathsf{c}}$$

is an involutory isomorphism of categories, making $\varnothing$ into a dualising object for $(\mathcal{P}(X), \cap, X, [-, -]_X)$ in the sense of **??**. In particular:

(a) The diagram

$$\mathcal{P}(X)^{\mathsf{op}} \xrightarrow{D_X} \mathcal{P}(X)$$

with $\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}$ and D_X to $\mathcal{P}(X)^{\mathsf{op}}$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\stackrel{\mathrm{def}}{=}\left[[U,\varnothing]_X,\varnothing\right]_X} = U$$

for each $U \in \mathcal{P}(X)$.

(b) The diagram

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \overset{\cap^{\mathsf{op}}}{\rightarrow} \mathcal{P}(X)^{\mathsf{op}}$$

$$\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X \nearrow \qquad\qquad \searrow D_X$$

$$\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \xrightarrow[{[-_1,-_2]_X}]{} \mathcal{P}(X)$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\mathrm{def}}{=}\left[U \cap [V,\emptyset]_X,\emptyset\right]_X} = [U,V]_X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With the Empty Set III.* Let $f \colon X \to Y$ be a function.

(a) *Interaction With Direct Images.* The diagram

$$\mathcal{P}(X)^{\mathsf{op}} \xrightarrow{f_*^{\mathsf{op}}} \mathcal{P}(Y)^{\mathsf{op}}$$

$$D_X \downarrow \qquad\qquad \downarrow D_Y$$

$$\mathcal{P}(X) \xrightarrow[{f_!}]{} \mathcal{P}(Y)$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(b) *Interaction With Inverse Images.* The diagram

$$\mathcal{P}(Y)^{\mathsf{op}} \xrightarrow{f^{-1,\mathsf{op}}} \mathcal{P}(X)^{\mathsf{op}}$$

$$D_Y \downarrow \qquad\qquad \downarrow D_X$$

$$\mathcal{P}(Y) \xrightarrow[{f^{-1}}]{} \mathcal{P}(X)$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X\left(f^{-1}(U)\right)$$

for each $U \in \mathcal{P}(X)$.

(c) *Interaction With Codirect Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_!^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle D_X}\downarrow & & \downarrow{\scriptstyle D_Y} \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$
f_*(D_X(U)) = D_Y(f_!(U))
$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{[-_1,-_2]_{\mathcal{P}(X)}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}}}\downarrow & \textcolor{orange}{\times} & \downarrow{\scriptstyle \bigcup} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[{[-_1,-_2]_X}]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$
\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V \right]_X
$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \\
{\scriptstyle ?}\nearrow & & \searrow{\scriptstyle \bigcup^{\mathsf{op}}} \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}\downarrow & & \downarrow{\scriptstyle [-,V]_X} \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\bigcap]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\left[\bigcup_{U\in\mathcal{U}} U, V\right]_X = \bigcap_{U\in\mathcal{U}} [U,V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ \bigcup\ } & \mathcal{P}(X) \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)}\,\star\,[U,-]_X}\Big\downarrow & & \Big\downarrow{\scriptstyle [U,-]_X} \\
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow[\ \bigcup\ ]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\left[U, \bigcup_{V\in\mathcal{V}} V\right]_X = \bigcup_{V\in\mathcal{V}} [U,V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) & \xrightarrow{\ [-_1,-_2]_{\mathcal{P}(X)}\ } & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle \bigcap^{\mathsf{op}}\times\bigcap^{\mathsf{op}}}\Big\downarrow & {\color{orange}\times} & \Big\downarrow{\scriptstyle \bigcap} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[\ [-_1,-_2]_X\ ]{} & \mathcal{P}(X),
\end{array}
$$

does not commute in general, i.e. we may have

$$\bigcap_{W\in[\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U\in\mathcal{U}} U, \bigcap_{V\in\mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} & \\
& \overset{\cong}{\nearrow} \qquad \searrow {\textstyle\bigcap}^{\mathsf{op}} & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) & & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)}\,\star[-,V]_X} \downarrow & & \downarrow {\scriptstyle [-,V]_X} \\
\mathcal{P}(\mathcal{P}(X)) & \underset{\bigcup}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\left[\bigcap_{U\in\mathcal{U}} U, V\right]_X = \bigcup_{U\in\mathcal{U}} [U,V]_X
$$

for each $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$ and each $V\in\mathcal{P}(X)$.

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \overset{\bigcap}{\longrightarrow} & \mathcal{P}(X) \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)}\,\star[U,-]_X} \downarrow & & \downarrow {\scriptstyle [U,-]_X} \\
\mathcal{P}(\mathcal{P}(X)) & \underset{\bigcap}{\longrightarrow} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\left[U, \bigcap_{V\in\mathcal{V}} V\right]_X = \bigcap_{V\in\mathcal{V}} [U,V]_X
$$

for each $U\in\mathcal{P}(X)$ and each $\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Binary Unions.* We have equalities of sets

$$
\begin{aligned}
[U\cap V, W]_X &= [U,W]_X \cup [V,W]_X, \\
[U, V\cap W]_X &= [U,V]_X \cap [U,W]_X
\end{aligned}
$$

for each $U,V,W\in\mathcal{P}(X)$.

14. *Interaction With Binary Intersections.* We have equalities of sets

$$[U \cup V, W]_X = [U, W]_X \cap [V, W]_X,$$
$$[U, V \cup W]_X = [U, V]_X \cup [U, W]_X$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Differences.* We have equalities of sets

$$[U \setminus V, W]_X = [U, W]_X \cup [V^{\mathsf{c}}, W]_X$$
$$= [U, W]_X \cup [U, V]_X,$$
$$[U, V \setminus W]_X = [U, V]_X \setminus (U \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

16. *Interaction With Complements.* We have equalities of sets

$$[U^{\mathsf{c}}, V]_X = U \cup V,$$
$$[U, V^{\mathsf{c}}]_X = U \cap V,$$
$$[U, V]_X^{\mathsf{c}} = U \setminus V$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

18. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathsf{op}} \times f_!} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle [-1,-2]_X} \downarrow & & \downarrow {\scriptstyle [-1,-2]_Y} \\
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y)
\end{array}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

19. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{\;f^{-1,\mathsf{op}} \times f^{-1}\;} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle [-_1,-_2]_Y} \Big\downarrow & & \Big\downarrow {\scriptstyle [-_1,-_2]_X} \\
\mathcal{P}(Y) & \xrightarrow[\;f^{-1}\;]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$
f^{-1}([U,V]_Y) = \left[f^{-1}(U), f^{-1}(V)\right]_X,
$$

natural in $U,V \in \mathcal{P}(X)$.

20. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{\;f_!^{\mathsf{op}} \times f_*\;} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle [-_1,-_2]_X} \Big\downarrow & \quad\subset & \Big\downarrow {\scriptstyle [-_1,-_2]_Y} \\
\mathcal{P}(X) & \xrightarrow[\;f_*\;]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$
[f_!(U), f_*(V)]_Y \subset f_*([U,V]_X)
$$

indexed by $U,V \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: Since $\mathcal{P}(X)$ is posetal, it suffices to prove Items 1a to 1c.

1. *Proof of Item 1a:* We have

$$
\begin{aligned}
[A,V]_X &\stackrel{\mathrm{def}}{=} A^{\mathsf{c}} \cup V \\
&\subset U^{\mathsf{c}} \cup V \\
&\stackrel{\mathrm{def}}{=} [U,V]_X,
\end{aligned}
$$

where we have used:

(a) Item 1 of Definition 4.3.11.1.2, which states that if $U \subset A$, then $A^\mathsf{c} \subset U^\mathsf{c}$.

(b) Item 1a of Item 1 of Definition 4.3.11.1.2, which states that if $A^\mathsf{c} \subset U^\mathsf{c}$, then $A^\mathsf{c} \cup K \subset U^\mathsf{c} \cup K$ for any $K \in \mathcal{P}(X)$.

2. *Proof of Item 1b:* We have

$$[U,V]_X \overset{\mathrm{def}}{=} U^\mathsf{c} \cup V$$
$$\subset U^\mathsf{c} \cup B$$
$$\overset{\mathrm{def}}{=} [U,B]_X,$$

where we have used Item 1b of Item 1 of Definition 4.3.11.1.2, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

3. *Proof of Item 1c:* We have

$$[A,V]_X \subset [U,V]_X$$
$$\subset [U,B]_X,$$

where we have used Items 1a and 1b.

This finishes the proof.

Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.3.9.1.2 and is proved there.

Item 3, Interaction With the Empty Set I: We have

$$[U,\emptyset]_X \overset{\mathrm{def}}{=} U^\mathsf{c} \cup \emptyset$$
$$= U^\mathsf{c},$$

where we have used Item 3 of Definition 4.3.8.1.2, and we have

$$[\emptyset,V]_X \overset{\mathrm{def}}{=} \emptyset^\mathsf{c} \cup V$$
$$\overset{\mathrm{def}}{=} (X \setminus \emptyset) \cup V$$
$$= X \cup V$$
$$= X,$$

where we have used:

1. Item 12 of Definition 4.3.10.1.2 for the first equality.

2. Item 5 of Definition 4.3.8.1.2 for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

Item 4, Interaction With X: We have

$$[U, X]_X \overset{\text{def}}{=} U^{\mathsf{c}} \cup X$$
$$= X,$$

where we have used Item 5 of Definition 4.3.8.1.2, and we have

$$[X, V]_X \overset{\text{def}}{=} X^{\mathsf{c}} \cup V$$
$$\overset{\text{def}}{=} (X \setminus X) \cup V$$
$$= \emptyset \cup V$$
$$= V,$$

where we have used Item 3 of Definition 4.3.8.1.2 for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

Item 5, Interaction With the Empty Set II: We have

$$D_X(D_X(U)) \overset{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X$$
$$= [U^{\mathsf{c}}, \emptyset]_X$$
$$= (U^{\mathsf{c}})^{\mathsf{c}}$$
$$= U,$$

where we have used:

1. Item 3 for the second and third equalities.

2. Item 3 of Definition 4.3.11.1.2 for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and thus we have

$$[[-, \emptyset]_X, \emptyset]_X \cong \text{id}_{\mathcal{P}(X)}$$

This finishes the proof.

Item 6, Interaction With the Empty Set III: Since $D_X = (-)^{\mathsf{c}}$, this is essentially a repetition of the corresponding results for $(-)^{\mathsf{c}}$, namely Items 5 to 7 of Definition 4.3.11.1.2.

Item 7, Interaction With Unions of Families of Subsets I: By Item 3 of

Definition 4.4.7.1.3, we have

$$[\mathcal{U}, \varnothing]_{\mathcal{P}(X)} = \mathcal{U}^{\mathsf{c}},$$
$$[U, \varnothing]_X = U^{\mathsf{c}}.$$

With this, the counterexample given in the proof of Item 10 of Definition 4.3.6.1.2 then applies.

Item 8, Interaction With Unions of Families of Subsets II: We have

$$\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X \overset{\mathrm{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U\right)^{\mathsf{c}} \cup V$$
$$= \left(\bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}\right) \cup V$$
$$= \bigcap_{U \in \mathcal{U}} (U^{\mathsf{c}} \cup V)$$
$$\overset{\mathrm{def}}{=} \bigcap_{U \in \mathcal{U}} [U, V]_X,$$

where we have used:

1. Item 11 of Definition 4.3.6.1.2 for the second equality.

2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.

Item 9, Interaction With Unions of Families of Subsets III: We have

$$\bigcup_{V \in \mathcal{V}} [U, V]_X \overset{\mathrm{def}}{=} \bigcup_{V \in \mathcal{V}} (U^{\mathsf{c}} \cup V)$$
$$= U^{\mathsf{c}} \cup \left(\bigcup_{V \in \mathcal{V}} V\right)$$
$$\overset{\mathrm{def}}{=} \left[U, \bigcup_{V \in \mathcal{V}} V\right]_X.$$

where we have used Item 6. This finishes the proof.

Item 10, Interaction With Intersections of Families of Subsets I: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \bigcap_{W \in \mathcal{P}(X)} W$$

$$= \{0, 1\},$$

whereas

$$\left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X = [\{0, 1\}, \{0\}]$$

$$= \{0\},$$

Thus we have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V \right]_X.$$

This finishes the proof.

Item 11, *Interaction With Intersections of Families of Subsets II*: We have

$$\left[\bigcap_{U \in \mathcal{U}} U, V \right]_X \overset{\text{def}}{=} \left(\bigcap_{U \in \mathcal{U}} U \right)^{\mathsf{c}} \cup V$$

$$= \left(\bigcup_{U \in \mathcal{U}} U^{\mathsf{c}} \right) \cup V$$

$$= \bigcup_{U \in \mathcal{U}} (U^{\mathsf{c}} \cup V)$$

$$\overset{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X,$$

where we have used:

1. Item 12 of Definition 4.3.6.1.2 for the second equality.

2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.

Item 12, *Interaction With Intersections of Families of Subsets III*: We have

$$\bigcap_{V \in \mathcal{V}} [U, V]_X \overset{\text{def}}{=} \bigcap_{V \in \mathcal{V}} (U^{\mathsf{c}} \cup V)$$

$$= U^{\mathsf{c}} \cup \left(\bigcap_{V \in \mathcal{V}} V \right)$$

$$\overset{\text{def}}{=} \left[U, \bigcap_{V \in \mathcal{V}} V \right]_X.$$

where we have used Item 6. This finishes the proof.
Item 13, Interaction With Binary Unions: We have

$$
\begin{aligned}
[U \cap V, W]_X &\overset{\text{def}}{=} (U \cap V)^{\mathsf{c}} \cup W \\
&= (U^{\mathsf{c}} \cup V^{\mathsf{c}}) \cup W \\
&= (U^{\mathsf{c}} \cup V^{\mathsf{c}}) \cup (W \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup (V^{\mathsf{c}} \cup W) \\
&\overset{\text{def}}{=} [U, W]_X \cup [V, W]_X,
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.

2. Item 8 of Definition 4.3.8.1.2 for the third equality.

3. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the fourth equality.

For the second equality in the statement, we have

$$
\begin{aligned}
[U, V \cap W]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup (V \cap W) \\
&= (U^{\mathsf{c}} \cup V) \cap (U^{\mathsf{c}} \cap W) \\
&\overset{\text{def}}{=} [U, V]_X \cap [U, W]_X,
\end{aligned}
$$

where we have used Item 6 of Definition 4.3.8.1.2 for the second equality.
Item 14, Interaction With Binary Intersections: We have

$$
\begin{aligned}
[U \cup V, W]_X &\overset{\text{def}}{=} (U \cup V)^{\mathsf{c}} \cup W \\
&= (U^{\mathsf{c}} \cap V^{\mathsf{c}}) \cup W \\
&= (U^{\mathsf{c}} \cup W) \cap (V^{\mathsf{c}} \cup W) \\
&\overset{\text{def}}{=} [U, W]_X \cap [V, W]_X,
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.

2. Item 6 of Definition 4.3.8.1.2 for the third equality.

Now, for the second equality in the statement, we have

$$
[U, V \cup W]_X \overset{\text{def}}{=} U^{\mathsf{c}} \cup (V \cup W)
$$

$$= (U^{\mathsf{c}} \cup U^{\mathsf{c}}) \cup (V \cup W)$$
$$= (U^{\mathsf{c}} \cup V) \cup (U^{\mathsf{c}} \cup W)$$
$$\overset{\text{def}}{=} [U, V]_X \cup [U, W]_X,$$

where we have used:

1. Item 8 of Definition 4.3.8.1.2 for the second equality.

2. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the third equality.

This finishes the proof.
Item 15, Interaction With Differences: We have

$$
\begin{aligned}
[U \setminus V, W]_X &\overset{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\overset{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W \\
&\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\
&= (V \cup (U^{\mathsf{c}} \cup U^{\mathsf{c}})) \cup W \\
&= (U^{\mathsf{c}} \cup W) \cup (U^{\mathsf{c}} \cup V) \\
&\overset{\text{def}}{=} [U, W]_X \cup [U, V]_X,
\end{aligned}
$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.

2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.

3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.

4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

We also have

$$
\begin{aligned}
[U \setminus V, W]_X &\overset{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\overset{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W
\end{aligned}
$$

$$\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W$$
$$= (V \cup U^{\mathsf{c}}) \cup (W \cup W)$$
$$= (U^{\mathsf{c}} \cup W) \cup (V \cup W)$$
$$= (U^{\mathsf{c}} \cup W) \cup ((V^{\mathsf{c}})^{\mathsf{c}} \cup W)$$
$$\overset{\text{def}}{=} [U, W]_X \cup [V^{\mathsf{c}}, W]_X,$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.

2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.

3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.

4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

5. Item 3 of Definition 4.3.11.1.2 for the eighth equality.

Now, for the second equality in the statement, we have

$$[U, V \setminus W]_X \overset{\text{def}}{=} U^{\mathsf{c}} \cup (V \setminus W)$$
$$= (V \setminus W) \cup U^{\mathsf{c}}$$
$$= (V \cup U^{\mathsf{c}}) \setminus (W \setminus U^{\mathsf{c}})$$
$$\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \setminus (W \setminus (X \setminus U))$$
$$= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup (W \setminus X))$$
$$= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup \emptyset)$$
$$= (V \cup U^{\mathsf{c}}) \setminus (W \cap U)$$
$$= (V \cup U^{\mathsf{c}}) \setminus (U \cap W)$$
$$\overset{\text{def}}{=} [U, V]_X \setminus (U \cap W)$$

where we have used:

1. Item 4 of Definition 4.3.8.1.2 for the second equality.

2. Item 4 of Definition 4.3.10.1.2 for the third equality.

3. Item 10 of Definition 4.3.10.1.2 for the fifth equality.

4. Item 13 of Definition 4.3.10.1.2 for the sixth equality.

5. Item 3 of Definition 4.3.8.1.2 for the seventh equality.

6. Item 5 of Definition 4.3.9.1.2 for the eighth equality.

This finishes the proof.
Item 16, Interaction With Complements: We have

$$[U^{\mathsf{c}}, V]_X \overset{\text{def}}{=} (U^{\mathsf{c}})^{\mathsf{c}} \cup V,$$
$$= U \cup V,$$

where we have used Item 3 of Definition 4.3.11.1.2. We also have

$$[U, V^{\mathsf{c}}]_X \overset{\text{def}}{=} U^{\mathsf{c}} \cup V^{\mathsf{c}}$$
$$= U \cap V$$

where we have used Item 2 of Definition 4.3.11.1.2. Finally, we have

$$[U, V]_X^{\mathsf{c}} = ((U \setminus V)^{\mathsf{c}})^{\mathsf{c}}$$
$$= U \setminus V,$$

where we have used Item 2 of Definition 4.3.11.1.2.
Item 17, Interaction With Characteristic Functions: We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) \overset{\text{def}}{=} \chi_{U^{\mathsf{c}} \cup V}(x)$$
$$= \max(\chi_{U^{\mathsf{c}}}, \chi_V)$$
$$= \max(1 - \chi_U \ (\text{mod } 2), \chi_V),$$

where we have used:

1. Item 10 of Definition 4.3.8.1.2 for the second equality.

2. Item 4 of Definition 4.3.11.1.2 for the third equality.

This finishes the proof.
Item 18, Interaction With Direct Images: This is a repetition of Item 10 of Definition 4.6.1.1.5 and is proved there.
Item 19, Interaction With Inverse Images: This is a repetition of Item 10 of Definition 4.6.2.1.3 and is proved there.
Item 20, Interaction With Codirect Images: This is a repetition of Item 9 of Definition 4.6.3.1.7 and is proved there.   □

4.4.8   Isbell Duality for Sets

Let X be a set.

Definition 4.4.8.1.1. The **Isbell function** of X is the map

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

defined by

$$\mathsf{I}(U) \overset{\mathrm{def}}{=} [\![x \mapsto [U, \{x\}]_X]\!]$$

for each $U \in \mathcal{P}(X)$.

Remark 4.4.8.1.2. Recall from Definition 4.4.1.1.2 that we may view the powerset $\mathcal{P}(X)$ of a set X as the decategorification of the category of presheaves $\mathsf{PSh}(\mathcal{C})$ of a category $\mathcal{C}$. Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\mathsf{Spec}(\mathcal{F}) \overset{\mathrm{def}}{=} \mathrm{Nat}\Big(\mathcal{F}, h_{(-)}\Big)$$

for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$. To this end, we could define

$$\mathsf{I}(U) \overset{\mathrm{def}}{=} \Big[U, \chi_{(-)}\Big]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ with the characteristic embedding $x \mapsto \chi_x$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1.

- The internal Hom Nat of $\mathsf{PSh}(\mathcal{C})$ with the internal Hom $[-, -]_X$ of $\mathcal{P}(X)$ of Definition 4.4.7.1.1.

However, since $[U, \chi_x]_X$ is a subset of U instead of a truth value, we get a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathcal{P}(X).$$

This makes some of the properties involving I a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $\mathsf{I}_! \circ \mathsf{I}$ evaluates to $\mathrm{id}_{\mathcal{P}(X)}$ in the sense of Definition 4.4.8.1.3.

Proposition 4.4.8.1.3. The diagram

$$\mathcal{P}(X) \xrightarrow{\;\mathsf{I}\;} \mathsf{Sets}(X, \mathcal{P}(X))$$

$$\Delta_{\Delta_{\mathrm{id}_{\mathcal{P}(X)}}} \qquad\qquad \Big\downarrow{\mathsf{I}_!}$$

$$\mathsf{Sets}(X, \mathsf{Sets}(X, \mathcal{P}(X)))$$

commutes, i.e. we have

$$\mathsf{I}_!(\mathsf{I}(U)) = [\![x \mapsto [\![y \mapsto U]\!]]\!]$$

for each $U \in \mathcal{P}(X)$.

Proof. We have

$$\begin{aligned}
\mathsf{I}_!(\mathsf{I}(U)) &\overset{\mathrm{def}}{=} \mathsf{I}_!([\![x \mapsto U^{\mathsf{c}} \cup \{x\}]\!]) \\
&\overset{\mathrm{def}}{=} [\![x \mapsto \mathsf{I}(U^{\mathsf{c}} \cup \{x\})]\!] \\
&\overset{\mathrm{def}}{=} [\![x \mapsto [\![y \mapsto (U^{\mathsf{c}} \cup \{x\})^{\mathsf{c}} \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \setminus \{x\}) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto U]\!]]\!],
\end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2 for the fourth equality above. □

4.5 Characteristic Functions

4.5.1 The Characteristic Function of a Subset

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.5.1.1.1. The **characteristic function of U**[23] is the function $\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$[24] defined by

$$\chi_U(x) \overset{\mathrm{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

---

[23]*Further Terminology:* Also called the **indicator function of U**.

[24]*Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

Remark 4.5.1.1.2. Under the analogy that $\{t, f\}$ should be the (-1)-categorical analogue of Sets, we may view a function

$$f \colon X \to \{t, f\}$$

as a decategorification of presheaves and copresheaves

$$\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
$$F \colon \mathcal{C} \to \mathsf{Sets}.$$

The characteristic functions χ_U of the subsets of X are then the primordial examples of such functions (and, in fact, all of them).

Notation 4.5.1.1.3. We will often employ the bijection $\{t, f\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when disucssing characteristic functions.

  Examples of this include Items 4 to 11 of Definition 4.5.1.1.4 below.

Proposition 4.5.1.1.4. Let X be a set.

1. *Functionality.* The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)} \colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{t, f\}).$$

2. *Bijectivity.* The function $\chi_{(-)}$ from Item 1 is bijective.

3. *Naturality.* The collection

$$\left\{ \chi_{(-)} \colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{t, f\}) \right\}_{X \in \mathrm{Obj}(\mathsf{Sets})}$$

defines a natural isomorphism between $\mathcal{P}^{-1}$ and $\mathsf{Sets}(-, \{t, f\})$. In particular, given a function $f \colon X \to Y$, the diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) & \xrightarrow{\ f^{-1}\ } & \mathcal{P}(X) \\[4pt]
{\scriptstyle \chi_{(-)}} \Big\downarrow {\scriptstyle \wr} & & {\scriptstyle \wr} \Big\downarrow {\scriptstyle \chi_{(-)}} \\[4pt]
\mathsf{Sets}(Y, \{t, f\}) & \xrightarrow[\ f^*\ ]{} & \mathsf{Sets}(X, \{t, f\})
\end{array}
$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Complements.* We have

$$\chi_{U^c} \equiv 1 - \chi_U \pmod{2}$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod{2}$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Internal Homs.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

Proof. Item 1, Functionality: There is nothing to prove.
Item 2, Bijectivity: We proceed in three steps:

1. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi \colon \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \dashrightarrow^{\sim} \mathcal{P}(X),$$

defined by

$$\begin{aligned}
\Phi(f) &\overset{\mathrm{def}}{=} U_f \\
&\overset{\mathrm{def}}{=} f^{-1}(\mathsf{true}) \\
&\overset{\mathrm{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\}
\end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$.

2. *Invertibility I.* We have

$$\begin{aligned}
\left[\Phi \circ \chi_{(-)}\right](U) &\overset{\mathrm{def}}{=} \Phi(\chi_U) \\
&\overset{\mathrm{def}}{=} \chi_U^{-1}(\mathsf{true}) \\
&\overset{\mathrm{def}}{=} \{x \in X \mid \chi_U(x) = \mathsf{true}\} \\
&\overset{\mathrm{def}}{=} \{x \in X \mid x \in U\} \\
&= U \\
&\overset{\mathrm{def}}{=} \left[\mathrm{id}_{\mathcal{P}(X)}\right](U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus, we have

$$\Phi \circ \chi_{(-)} = \mathrm{id}_{\mathcal{P}(X)}.$$

3. *Invertibility II.* We have

$$\begin{aligned}
\left[\chi_{(-)} \circ \Phi\right](U) &\overset{\mathrm{def}}{=} \chi_{\Phi(f)} \\
&\overset{\mathrm{def}}{=} \chi_{f^{-1}(\mathsf{true})} \\
&\overset{\mathrm{def}}{=} [\![x \mapsto \begin{cases} \mathsf{true} & \text{if } x \in f^{-1}(\mathsf{true}) \\ \mathsf{false} & \text{otherwise} \end{cases}]\!]
\end{aligned}$$

$$= [\![x \mapsto f(x)]\!]$$
$$= f$$
$$\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})}\right](f)$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$. Thus, we have

$$\chi_{(-)} \circ \Phi = \mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})} \,.$$

This finishes the proof.

Item 3, Naturality: We proceed in two steps:

1. *Naturality of $\chi_{(-)}$.* We have

$$[\chi_V \circ f](v) \overset{\text{def}}{=} \chi_V(f(v))$$
$$= \begin{cases} \mathsf{true} & \text{if } f(v) \in V, \\ \mathsf{false} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases}$$
$$\overset{\text{def}}{=} \chi_{f^{-1}(V)}(v)$$

   for each $v \in V$.

2. *Naturality of Φ.* Since $\chi_{(-)}$ is natural and a componentwise inverse to Φ, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Φ is also natural in each argument.

This finishes the proof.

Item 4, Interaction With Unions I: This is a repetition of Item 10 of Definition 4.3.8.1.2 and is proved there.

Item 5, Interaction With Unions II: This is a repetition of Item 11 of Definition 4.3.8.1.2 and is proved there.

Item 6, Interaction With Intersections I: This is a repetition of Item 10 of Definition 4.3.9.1.2 and is proved there.

Item 7, Interaction With Intersections II: This is a repetition of Item 11 of Definition 4.3.9.1.2 and is proved there.

Item 8, Interaction With Differences: This is a repetition of Item 16 of Definition 4.3.10.1.2 and is proved there.

Item 9, Interaction With Complements: This is a repetition of Item 4 of Definition 4.3.11.1.2 and is proved there.

Item 10, Interaction With Symmetric Differences: This is a repetition of Item 15 of Definition 4.3.12.1.2 and is proved there.

Item 11, Interaction With Internal Homs: This is a repetition of Item 17 of Definition 4.4.7.1.3 and is proved there.   □

Remark 4.5.1.1.5. The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 2 of Definition 4.5.1.1.4, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U \colon X \to \{\mathsf{true}, \mathsf{false}\}$;

- Takes a function $f \colon X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\mathsf{FibSets}_X}_{\stackrel{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}_X}_{\stackrel{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}}, \mathsf{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, **??**. Here we view:

- Subsets $U \hookrightarrow X$ as being analogous to X-fibred sets $\phi_X \colon A \to X$.

- Functions $f \colon X \to \{\mathsf{t}, \mathsf{f}\}$ as being analogous to X-indexed sets $A \colon X_{\mathsf{disc}} \to \mathsf{Sets}$.

4.5.2   The Characteristic Function of a Point

Let X be a set and let $x \in X$.

Definition 4.5.2.1.1. The **characteristic function of** x is the function[25]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

---

[25] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

Remark 4.5.2.1.2. Expanding upon Definition 4.5.1.1.2, we may think of the characteristic function

$$\chi_x \colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an *element* x of X as a decategorification of the representable presheaf and of the representable copresheaf

$$h_X \colon C^{\mathsf{op}} \to \mathsf{Sets},$$
$$h^X \colon C \to \mathsf{Sets}$$

associated of an *object* X of a category C.

4.5.3   The Characteristic Relation of a Set

Let X be a set.

Definition 4.5.3.1.1. The **characteristic relation on** X[26] is the relation[27]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t},\mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

Remark 4.5.3.1.2. Expanding upon Definitions 4.5.1.1.2 and 4.5.2.1.2, we may view the characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t},\mathsf{f}\}$$

of X as a decategorification of the Hom profunctor

$$\mathrm{Hom}_C(-_1, -_2) \colon C^{\mathsf{op}} \times C \to \mathsf{Sets}$$

of a category C.

---

[26] *Further Terminology:* Also called the **identity relation on** X.

[27] *Further Notation:* Also written $\chi_{-_2}^{-1}$, or $\sim_{\mathrm{id}}$ in the context of relations.

[28] Under the bijection $\mathsf{Sets}(X \times X, \{\mathsf{t},\mathsf{f}\}) \cong \mathcal{P}(X \times X)$ of Item 2 of Definition 4.5.1.1.4, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

Proposition 4.5.3.1.3. Let $f\colon X \to Y$ be a function.

1. *The Inclusion of Characteristic Relations Associated to a Function.*
   Let $f\colon A \to B$ be a function. We have an inclusion[29]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{array}{c} A \times A \xrightarrow{\;f\times f\;} B \times B \\ {}^{\chi_A}\searrow \quad \circlearrowright \quad \swarrow^{\chi_B} \\ \{\mathsf{t},\mathsf{f}\}. \end{array}$$

Proof. Item 1, The Inclusion of Characteristic Relations Associated to a Function: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. $\qquad\square$

4.5.4   The Characteristic Embedding of a Set

Let X be a set.

Definition 4.5.4.1.1. The **characteristic embedding**[30] **of X into $\mathcal{P}(X)$** is the function

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

defined by[31]

$$\chi_{(-)}(x) \overset{\mathrm{def}}{=} \chi_x$$
$$= \{x\}$$

for each $x \in X$.

Remark 4.5.4.1.2. Expanding upon Definitions 4.5.1.1.2, 4.5.2.1.2 and 4.5.3.1.2, we may view the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$\text{よ}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

---

[29]*Note:* This is the 0-categorical version of Categories, Definition 11.5.4.1.1.

[30]The name "characteristic *embedding*" is justified by Definition 4.5.5.1.2, which gives an analogue of fully faithfulness for $\chi_{(-)}$.

[31]Here we are identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t},\mathsf{f}\})$ as per Item 2 of Definition 4.5.1.1.4.

Proposition 4.5.4.1.3. Let $f\colon X \to Y$ be a map of sets.

1. *Interaction With Functions.* We have

$$f_! \circ \chi_X = \chi_Y \circ f,$$

$$\begin{CD} X @>f>> Y \\ @V\chi_X VV @VV\chi_Y V \\ \mathcal{P}(X) @>>f_!> \mathcal{P}(Y). \end{CD}$$

Proof. *Item 1, Interaction With Functions*: Indeed, we have

$$\begin{aligned}
[f_! \circ \chi_X](x) &\overset{\text{def}}{=} f_!(\chi_X(x)) \\
&\overset{\text{def}}{=} f_!(\{x\}) \\
&= \{f(x)\} \\
&\overset{\text{def}}{=} \chi_{X'}(f(x)) \\
&\overset{\text{def}}{=} [\chi_{X'} \circ f](x),
\end{aligned}$$

for each $x \in X$, showing the desired equality. $\square$

4.5.5   The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 4.5.5.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } U \subset V, \\ \text{false} & \text{otherwise.} \end{cases}$$

Proof. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } \{x\} \subset U, \\ \text{false} & \text{otherwise} \end{cases}$$

$$= \begin{cases} \text{true} & \text{if } x \in U \\ \text{false} & \text{otherwise} \end{cases}$$
$$\overset{\text{def}}{=} \chi_U(x).$$

This finishes the proof. $\qquad\qquad\square$

Corollary 4.5.5.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) \cong \chi_X(x, y)$$

for each $x, y \in X$.

Proof. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_y(x)$$
$$\overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in \{y\} \\ \text{false} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \text{true} & \text{if } x = y \\ \text{false} & \text{otherwise} \end{cases}$$
$$\overset{\text{def}}{=} \chi_X(x, y).$$

where we have used Definition 4.5.5.1.1 for the first equality. $\qquad\square$

4.6 The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

4.6.1 Direct Images

Let $f \colon X \to Y$ be a function.

Definition 4.6.1.1.1. The **direct image function associated to** f is the function[32]

$$f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[33]

$$f_!(U) \overset{\text{def}}{=} \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\}$$

---

[32] *Further Notation:* Also written simply $f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$.
[33] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

$$= \{f(x) \in Y \mid x \in U\}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.1.1.2. Sometimes one finds the notation

$$\exists_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \exists_f(U)$.

- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.1.1.3. Notation for direct images between powersets is tricky:

1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:

   (a) *Powersets.* Given a function $f \colon X \to Y$, we have an inverse image functor
   $$f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X).$$
   The *left* adjoint of this functor is the usual direct image, defined above in Definition 4.6.1.1.1.

   (b) *Presheaves.* Given a morphism of topological spaces $f \colon X \to Y$, we have an inverse image functor
   $$f^{-1} \colon \mathsf{PSh}(Y) \to \mathsf{PSh}(X).$$
   The *right* adjoint of this functor is the direct image functor of presheaves, defined in **??**.

2. The presheaf direct image functor is denoted f_*, but the direct image functor for powersets is denoted $f_!$ (as it's a left adjoint).

3. Adding to the confusion, it's somewhat common for $f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ to be denoted f_*.

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

Situation	Adjoint String
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*)\colon \mathcal{P}(X) \overset{\rightarrow}{\rightrightarrows} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*)\colon \mathsf{PSh}(X) \overset{\rightarrow}{\rightrightarrows} \mathsf{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*)\colon \mathcal{C}_{/X} \overset{\rightarrow}{\rightrightarrows} \mathcal{C}_{/Y}$
Kan Extensions	$(F_! \dashv F^* \dashv F_*)\colon \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \overset{\rightarrow}{\rightrightarrows} \mathsf{Fun}(\mathcal{D}, \mathcal{E})$

Remark 4.6.1.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the direct image function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$\begin{aligned}
f_!(\chi_U) &\overset{\mathrm{def}}{=} \mathrm{Lan}_f(\chi_U) \\
&= \mathrm{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)} \right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \overset{\chi_U}{\longrightarrow} \{\mathsf{t}, \mathsf{f}\} \right) \\
&= \underset{\substack{x \in X \\ f(x) = -_1}}{\mathrm{colim}} (\chi_U(x)) \\
&= \bigvee_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)),
\end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned}
[f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $y \in Y$.

Proposition 4.6.1.1.5. Let $f \colon X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

   $(\star)$ If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_* \right) \colon \quad \mathcal{P}(X) \overset{f_!}{\underset{f_*}{\overset{\perp}{\underset{\perp}{\longleftarrow f^{-1} \longrightarrow}}}} \mathcal{P}(Y),$$

   witnessed by:

   (a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \hookrightarrow f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \hookrightarrow f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \hookrightarrow \mathrm{id}_{\mathcal{P}(X)},$$

   having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*\left(f^{-1}(V) \right),$$
$$f_!\left(f^{-1}(V) \right) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

   (b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}\left(U, f^{-1}(V) \right),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}\left(f^{-1}(U), V \right) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

   natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

   i. The following conditions are equivalent:
      A. We have $f_!(U) \subset V$.

B. We have $U \subset f^{-1}(V)$.

    ii. The following conditions are equivalent:

      A. We have $f^{-1}(U) \subset V$.

      B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$
\begin{CD}
\mathcal{P}(\mathcal{P}(X)) @>(f_!)_!>> \mathcal{P}(\mathcal{P}(Y)) \\
@V\bigcup VV @VV\bigcup V \\
\mathcal{P}(X) @>f_!>> \mathcal{P}(Y)
\end{CD}
$$

commutes, i.e. we have

$$
\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V
$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\mathrm{def}}{=} (f_!)_!(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$
\begin{CD}
\mathcal{P}(\mathcal{P}(X)) @>(f_!)_!>> \mathcal{P}(\mathcal{P}(Y)) \\
@V\bigcap VV @VV\bigcap V \\
\mathcal{P}(X) @>f_!>> \mathcal{P}(Y)
\end{CD}
$$

commutes, i.e. we have

$$
\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V
$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\mathrm{def}}{=} (f_!)_!(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$
\begin{CD}
\mathcal{P}(X) \times \mathcal{P}(X) @>f_! \times f_!>> \mathcal{P}(Y) \times \mathcal{P}(Y) \\
@V\cup VV @VV\cup V \\
\mathcal{P}(X) @>f_!>> \mathcal{P}(Y)
\end{CD}
$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_! \times f_!} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
{\scriptstyle \cap} \downarrow & \subseteq & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

7. *Interaction With Differences.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathsf{op}} \times f_!} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \setminus} \downarrow & \supset & \downarrow {\scriptstyle \setminus} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_*^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}}} \downarrow & & \downarrow {\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathsf{op}} \times f_!} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle \triangle} \downarrow & \supset & \downarrow {\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_!(U) \triangle f_!(V) \subset f_!(U \triangle V)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathsf{op}} \times f_!} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
{\scriptstyle [-1,-2]_X} \downarrow & & \downarrow {\scriptstyle [-1,-2]_Y} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i \right) = \bigcup_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$
\begin{aligned}
f_!(U) \cup f_!(V) &= f_!(U \cup V), \\
f_!(\varnothing) &= \varnothing,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V),$$
$$f_!(X) \subset Y,$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(X), \cup, \varnothing) \to (\mathcal{P}(Y), \cup, \varnothing),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \stackrel{=}{\to} f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \varnothing \stackrel{=}{\to} \varnothing,$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \hookrightarrow f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(X) \hookrightarrow Y,$$

natural in $U, V \in \mathcal{P}(X)$.

15. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \amalg g)_!(U \amalg V) = f_!(U) \amalg g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Interaction With Products.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X\times Y} g)_!(U \boxtimes_{X\times Y} V) = f_!(U) \boxtimes_{X'\times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

17. *Relation to Codirect Images.* We have

$$f_!(U) = f_*(U^{\mathsf{c}})^{\mathsf{c}}$$
$$\stackrel{\mathrm{def}}{=} Y \setminus f_*(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: Omitted.
Item 2, Triple Adjointness: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
Item 3, Interaction With Unions of Families of Subsets: We have

$$\bigcup_{V \in f_!(\mathcal{U})} V = \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \;\mid\; U \in \mathcal{U}\}} V$$
$$= \bigcup_{U \in \mathcal{U}} f_!(U).$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\bigcap_{V \in f_!(\mathcal{U})} V = \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \;\mid\; U \in \mathcal{U}\}} V$$
$$= \bigcap_{U \in \mathcal{U}} f_!(U).$$

This finishes the proof.
Item 5, Interaction With Binary Unions: See [Pro25p].
Item 6, Interaction With Binary Intersections: See [Pro25n].
Item 7, Interaction With Differences: See [Pro25o].
Item 8, Interaction With Complements: Applying Item 17 to $X \setminus U$, we have

$$f_!(U^{\mathsf{c}}) = f_!(X \setminus U)$$
$$= Y \setminus f_*(X \setminus (X \setminus U))$$
$$= Y \setminus f_*(U)$$
$$= f_*(U)^{\mathsf{c}}.$$

This finishes the proof.

Item 9, Interaction With Symmetric Differences: We have

$$
\begin{aligned}
f_!(U) \bigtriangleup f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\
&\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\
&= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\
&\subset f_!((U \cup V) \setminus (U \cap V)) \\
&= f_!(U \bigtriangleup V),
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.

2. Item 6 of this proposition together with Item 1 of Definition 4.3.10.1.2 for the first inclusion.

3. Item 5 for the second equality.

4. Item 7 for the second inclusion.

5. Item 2 of Definition 4.3.12.1.2 for the tchird equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 10, Interaction With Internal Homs of Powersets: We have

$$
\begin{aligned}
f_!([U,V]_X) &\overset{\text{def}}{=} f_!(U^{\mathsf{c}} \cup V) \\
&= f_!(U^{\mathsf{c}}) \cup f_!(V) \\
&= f_*(U)^{\mathsf{c}} \cup f_!(V) \\
&\overset{\text{def}}{=} [f_*(U), f_!(V)]_Y,
\end{aligned}
$$

where we have used:

1. Item 5 for the second equality.

2. Item 17 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 11, Preservation of Colimits: This follows from Item 2 and **??**, **??** of

??.[34]

Item 12, Oplax Preservation of Limits: The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25n] for the other inclusions.

Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.

Item 14, Symmetric Oplax Monoidality With Respect to Intersections: The inclusions in the statement follow from Item 12. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

Item 15, Interaction With Coproducts: Omitted.

Item 16, Interaction With Products: Omitted.

Item 17, Relation to Codirect Images: Applying Item 16 of Definition 4.6.3.1.7 to $X \setminus U$, we have

$$f_*(X \setminus U) = B \setminus f_!(X \setminus (X \setminus U))$$
$$= B \setminus f_!(U).$$

Taking complements, we then obtain

$$f_!(U) = B \setminus (B \setminus f_!(U)),$$
$$= B \setminus f_*(X \setminus U),$$

which finishes the proof. $\square$

Proposition 4.6.1.1.6. Let $f \colon X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

---

[34]*Reference:* [Pro25p].

4. *Interaction With Composition.* For each pair of composable functions $f\colon X \to Y$ and $g\colon Y \to Z$, we have

$$(g \circ f)_! = g_! \circ f_!,$$

$$
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{f_!} & \mathcal{P}(Y) \\
& {\scriptstyle (g \circ f)_!} \searrow & \downarrow {\scriptstyle g_!} \\
& & \mathcal{P}(Z).
\end{array}
$$

Proof. Item 1, Functionality I: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.1.1.5.
Item 3, Interaction With Identities: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**.
Item 4, Interaction With Composition: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**. $\qquad\square$

4.6.2   Inverse Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.2.1.1. The **inverse image function associated to f** is the function[35]

$$f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[36]

$$f^{-1}(V) \overset{\mathrm{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 4.6.2.1.2. Identifying $\mathcal{P}(Y)$ with $\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by

$$f^*(\chi_V) \overset{\mathrm{def}}{=} \chi_V \circ f$$

---

[35]*Further Notation:* Also written $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$.
[36]*Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of V by f**.

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 4.6.2.1.3. Let $f \colon X \to Y$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} \colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

   ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_*\right) \colon \quad \mathcal{P}(X) \xleftarrow{\overset{f_!}{\underset{\perp}{}} \underset{f_*}{\underset{\perp}{}}} \mathcal{P}(Y),$$

   witnessed by:

   (a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \hookrightarrow f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \hookrightarrow f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \hookrightarrow \mathrm{id}_{\mathcal{P}(X)},$$

   having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*\left(f^{-1}(V)\right),$$
$$f_!\left(f^{-1}(V)\right) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

   (b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

   natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

    i. The following conditions are equivalent:

      A. We have $f_!(U) \subset V$.

      B. We have $U \subset f^{-1}(V)$.

   ii. The following conditions are equivalent:

      A. We have $f^{-1}(U) \subset V$.

      B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{\left(f^{-1}\right)^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\bigcup}\downarrow & & \downarrow{\scriptstyle\bigcup} \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U
$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} \left(f^{-1}\right)^{-1}(\mathcal{V})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(Y)) & \xrightarrow{\left(f^{-1}\right)^{-1}} & \mathcal{P}(\mathcal{P}(X)) \\
{\scriptstyle\bigcap}\downarrow & & \downarrow{\scriptstyle\bigcap} \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U
$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} \left(f^{-1}\right)^{-1}(\mathcal{V})$.

5. *Interaction With Binary Unions.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\;f^{-1} \times f^{-1}\;} & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle \cup} \downarrow & & \downarrow {\scriptstyle \cup} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

6. *Interaction With Binary Intersections.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y) \times \mathcal{P}(Y) & \xrightarrow{\;f^{-1} \times f^{-1}\;} & \mathcal{P}(X) \times \mathcal{P}(X) \\
{\scriptstyle \cap} \downarrow & & \downarrow {\scriptstyle \cap} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{\;f^{\mathsf{op},-1} \times f^{-1}\;} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle \backslash} \downarrow & & \downarrow {\scriptstyle \backslash} \\
\mathcal{P}(Y) & \xrightarrow{\quad f^{-1} \quad} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} & \xrightarrow{f^{-1,\mathsf{op}}} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle (-)^{\mathsf{c}}}\downarrow & & \downarrow{\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$
f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}
$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{\mathsf{op},-1} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle \triangle}\downarrow & & \downarrow{\scriptstyle \triangle} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

i.e. we have

$$
f^{-1}(U) \mathbin{\triangle} f^{-1}(V) = f^{-1}(U \mathbin{\triangle} V)
$$

for each $U, V \in \mathcal{P}(Y)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1,\mathsf{op}} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
{\scriptstyle [-_1,-_2]_Y}\downarrow & & \downarrow{\scriptstyle [-_1,-_2]_X} \\
\mathcal{P}(Y) & \xrightarrow{f^{-1}} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$
f^{-1}([U,V]_Y) = \left[f^{-1}(U), f^{-1}(V) \right]_X,
$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V),$$
$$f^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(Y)$.

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V),$$
$$f^{-1}(Y) = X,$$

natural in $U, V \in \mathcal{P}(Y)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(Y), \cup, \emptyset) \to (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) \stackrel{=}{\Rightarrow} f^{-1}(U \cup V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon \emptyset \stackrel{=}{\Rightarrow} f^{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(Y)$.

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(Y), \cap, Y) \to (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \overset{=}{\to} f^{-1}(U \cap V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon X \overset{=}{\to} f^{-1}(Y),$$

natural in $U, V \in \mathcal{P}(Y)$.

15. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \amalg g)^{-1}(U' \amalg V') = f^{-1}(U') \amalg g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

16. *Interaction With Products.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

Proof. Item 1, Functoriality: Omitted.
Item 2, Triple Adjointness: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
Item 3, Interaction With Unions of Families of Subsets: We have

$$\bigcup_{U \in f^{-1}(\mathcal{V})} U = \bigcup_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \ | \ V \in \mathcal{V}\}} U$$
$$= \bigcup_{V \in \mathcal{V}} f^{-1}(V).$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\bigcap_{U \in f^{-1}(\mathcal{V})} U = \bigcap_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \ | \ V \in \mathcal{V}\}} U$$
$$= \bigcap_{V \in \mathcal{V}} f^{-1}(V).$$

This finishes the proof.
Item 5, Interaction With Binary Unions: See [Pro25y].
Item 6, Interaction With Binary Intersections: See [Pro25w].
Item 7, Interaction With Differences: See [Pro25x].

Item 8, Interaction With Complements: See [Pro25j].
Item 9, Interaction With Symmetric Differences: We have

$$
\begin{aligned}
f^{-1}(U \bigtriangleup V) &= f^{-1}((U \cup V) \setminus (U \cap V)) \\
&= f^{-1}(U \cup V) \setminus f^{-1}(U \cap V) \\
&= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U \cap V) \\
&= f^{-1}(U) \cup f^{-1}(V) \setminus f^{-1}(U) \cap f^{-1}(V) \\
&= f^{-1}(U) \bigtriangleup f^{-1}(V),
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.

2. Item 7 for the second equality.

3. Item 5 for the third equality.

4. Item 6 for the fourth equality.

5. Item 2 of Definition 4.3.12.1.2 for the fifth equality.

This finishes the proof.
Item 10, Interaction With Internal Homs of Powersets: We have

$$
\begin{aligned}
f^{-1}([U, V]_Y) &\overset{\text{def}}{=} f^{-1}(U^{\mathsf{c}} \cup V) \\
&= f^{-1}(U^{\mathsf{c}}) \cup f^{-1}(V) \\
&= f^{-1}(U)^{\mathsf{c}} \cup f^{-1}(V) \\
&\overset{\text{def}}{=} \left[f^{-1}(U), f^{-1}(V) \right]_X,
\end{aligned}
$$

where we have used:

1. Item 8 for the second equality.

2. Item 5 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
Item 11, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.[37]

---

[37] *Reference:* [Pro25y].

Item 12, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.[38]
Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.
Item 14, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 12.
Item 15, Interaction With Coproducts: Omitted.
Item 16, Interaction With Products: Omitted.   □

Proposition 4.6.2.1.4. Let $f\colon X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{X,Y}\colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}(\mathcal{P}(Y),\mathcal{P}(X)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{X,Y}\colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}((\mathcal{P}(Y),\subset),(\mathcal{P}(X),\subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_X^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon X \to Y$ and $g\colon Y \to Z$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1},$$

$$
\begin{array}{ccc}
\mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y) \\
& {\scriptstyle (g\circ f)^{-1}}\searrow & \downarrow {\scriptstyle f^{-1}} \\
& & \mathcal{P}(X).
\end{array}
$$

Proof. Item 1, Functionality I: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.2.1.3.
Item 3, Interaction With Identities: This follows from Definition 4.6.2.1.2 and Categories, Item 5 of Definition 11.1.4.1.2.
Item 4, Interaction With Composition: This follows from Definition 4.6.2.1.2 and Categories, Item 2 of Definition 11.1.4.1.2.   □

---

[38] *Reference:* [Pro25w].

4.6.3   Codirect Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.3.1.1. The **codirect image function associated to f** is the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[39,40]

$$f_*(U) \overset{\text{def}}{=} \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right\}$$
$$= \left\{ y \in Y \,\middle|\, \text{we have } f^{-1}(y) \subset U \right\}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.3.1.2. Sometimes one finds the notation

$$\forall_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.

- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.3.1.3. See Definition 4.6.1.1.3.

Remark 4.6.3.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the codirect image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

---

[39] *Further Terminology:* The set $f_*(U)$ is called the **codirect image of U by f**.

[40] We also have

$$f_*(U) = f_!(U^{\mathsf{c}})^{\mathsf{c}}$$
$$\overset{\text{def}}{=} Y \setminus f_!(X \setminus U);$$

see Item 16 of Definition 4.6.3.1.7.

defined by

$$
\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\
&= \lim\left(\left(\underline{(-1)} \overset{\rightarrow}{\times} f \right) \overset{\text{pr}}{\twoheadrightarrow} X \xrightarrow{\chi_U} \{\text{true}, \text{false}\} \right) \\
&= \lim_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)) \\
&= \bigwedge_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)).
\end{aligned}
$$

where we have used **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_*(\chi_U)](y) &= \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\
&= \begin{cases} \text{true} & \text{if, for each } x \in X \text{ such that} \\ & \quad f(x) = y, \text{ we have } x \in U, \\ \text{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \text{true} & \text{if } f^{-1}(y) \subset U \\ \text{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $y \in Y$.

Definition 4.6.3.1.5. Let U be a subset of X.[41,42]

---

[41]Note that we have

$$
f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U),
$$

as

$$
\begin{aligned}
f_*(U) &= f_*(U) \cap Y \\
&= f_*(U) \cap (\mathrm{Im}(f) \cup (Y \setminus \mathrm{Im}(f))) \\
&= (f_*(U) \cap \mathrm{Im}(f)) \cup (f_*(U) \cap (Y \setminus \mathrm{Im}(f))) \\
&\overset{\text{def}}{=} f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U).
\end{aligned}
$$

[42]In terms of the meet computation of $f_*(U)$ of Definition 4.6.3.1.4, namely

$$
f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)),
$$

1. The **image part of the codirect image** $f_*(U)$ **of** U is the set $f_{*,\mathrm{im}}(U)$ defined by

$$f_{*,\mathrm{im}}(U) \overset{\mathrm{def}}{=} f_*(U) \cap \mathrm{Im}(f)$$
$$= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \varnothing. \end{array} \right\}.$$

2. The **complement part of the codirect image** $f_*(U)$ **of** U is the set $f_{*,\mathrm{cp}}(U)$ defined by

$$f_{*,\mathrm{cp}}(U) \overset{\mathrm{def}}{=} f_*(U) \cap (Y \setminus \mathrm{Im}(f))$$
$$= Y \setminus \mathrm{Im}(f)$$
$$= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \varnothing. \end{array} \right\}$$
$$= \left\{ y \in Y \mid f^{-1}(y) = \varnothing \right\}.$$

Example 4.6.3.1.6. Here are some examples of codirect images.

1. *Multiplication by Two.* Consider the function $f \colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\mathrm{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$f_{*,\mathrm{im}}(U) = f_!(U)$$
$$f_{*,\mathrm{cp}}(U) = \{\text{odd natural numbers}\}$$

for any $U \subset \mathbb{N}$. In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

2. *Parabolas.* Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\mathrm{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$f_{*,\mathrm{im}}([0,1]) = \{0\},$$
$$f_{*,\mathrm{im}}([-1,1]) = [0,1],$$
$$f_{*,\mathrm{im}}([1,2]) = \emptyset,$$
$$f_{*,\mathrm{im}}([-2,-1] \cup [1,2]) = [1,4].$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\mathrm{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$f_{*,\mathrm{im}}([-1,1] \times [-1,1]) = [0,1],$$
$$f_{*,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) = \emptyset.$$

Proposition 4.6.3.1.7. Let $f\colon X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

---

we see that $f_{*,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_*\right): \quad \mathcal{P}(X) \xleftarrow{\;\; f^{-1} \;\;} \mathcal{P}(Y),$$

with $f_!$ on top ($\perp$) and f_* on bottom ($\perp$)

witnessed by:

(a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \hookrightarrow f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \hookrightarrow f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \hookrightarrow \mathrm{id}_{\mathcal{P}(X)},$$

having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*\left(f^{-1}(V)\right),$$
$$f_!\left(f^{-1}(V)\right) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(X)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

   i. The following conditions are equivalent:

     A. We have $f_!(U) \subset V$.

     B. We have $U \subset f^{-1}(V)$.

   ii. The following conditions are equivalent:

     A. We have $f^{-1}(U) \subset V$.

     B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\
\cup \downarrow & & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{\;\;f_*\;\;} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\mathrm{def}}{=} (f_*)_*(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(\mathcal{P}(X)) & \xrightarrow{(f_*)_*} & \mathcal{P}(\mathcal{P}(Y)) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\mathrm{def}}{=} (f_*)_*(\mathcal{U})$.

5. *Interaction With Binary Unions.* Let $f \colon X \to Y$ be a function. We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cup \downarrow & \subset & \downarrow \cup \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{f_* \times f_*} & \mathcal{P}(Y) \times \mathcal{P}(Y) \\
\cap \downarrow & & \downarrow \cap \\
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_!^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
\downarrow{\scriptstyle (-)^{\mathsf{c}}} & & \downarrow{\scriptstyle (-)^{\mathsf{c}}} \\
\mathcal{P}(X) & \xrightarrow[f_*]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_*^{\mathsf{op}} \times f_*} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
\downarrow{\scriptstyle \triangle} & \raise1ex{\scriptstyle \supset} & \downarrow{\scriptstyle \triangle} \\
\mathcal{P}(X) & \xrightarrow[f_*]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$f_*(U \triangle V) \subset f_*(U) \triangle f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow{f_!^{\mathsf{op}} \times f_*} & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \\
\downarrow{\scriptstyle [-1,-2]_X} & \raise1ex{\scriptstyle \supset} & \downarrow{\scriptstyle [-1,-2]_Y} \\
\mathcal{P}(X) & \xrightarrow[f_*]{} & \mathcal{P}(Y)
\end{array}
$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V),$$
$$\varnothing \hookrightarrow f_*(\varnothing),$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Limits.* We have an equality of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_*(U) \cap f^{-1}(V),$$
$$f_*(X) = Y,$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$\left(f_*, f_*^\otimes, f_{*|\mathbb{1}}^\otimes\right) \colon (\mathcal{P}(X), \cup, \varnothing) \to (\mathcal{P}(Y), \cup, \varnothing),$$

being equipped with inclusions

$$f_{*|U,V}^\otimes \colon f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^\otimes \colon \varnothing \hookrightarrow f_*(\varnothing),$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of [Item 1] has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right)\colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes}\colon f_*(U \cap V) \overset{=}{\Rightarrow} f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes}\colon f_*(X) \overset{=}{\Rightarrow} Y,$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Interaction With Coproducts.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \amalg g)_*(U \amalg V) = f_*(U) \amalg g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

15. *Interaction With Products.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_*(U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Relation to Direct Images.* We have

$$f_*(U) = f_!(U^{\mathsf{c}})^{\mathsf{c}}$$
$$= Y \setminus f_!(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

17. *Interaction With Injections.* If f is injective, then we have

$$f_{*,\mathrm{im}}(U) = f_!(U),$$
$$f_{*,\mathrm{cp}}(U) = Y \setminus \mathrm{Im}(f),$$

and so

$$f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U)$$
$$= f_!(U) \cup (Y \setminus \mathrm{Im}(f))$$

for each $U \in \mathcal{P}(X)$.

18. *Interaction With Surjections.* If f is surjective, then we have

$$f_{*,\mathrm{im}}(U) \subset f_!(U),$$
$$f_{*,\mathrm{cp}}(U) = \emptyset,$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Triple Adjointness: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
Item 3, Interaction With Unions of Families of Subsets: We have

$$\bigcup_{V \in f_*(\mathcal{U})} V = \bigcup_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$
$$= \bigcup_{U \in \mathcal{U}} f_*(U).$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\bigcap_{V \in f_*(\mathcal{U})} V = \bigcap_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$
$$= \bigcap_{U \in \mathcal{U}} f_*(U).$$

This finishes the proof.
Item 5, Interaction With Binary Unions: We have

$$f_*(U) \cup f_*(V) = f_!(U^{\mathsf{c}})^{\mathsf{c}} \cup f_!(V^{\mathsf{c}})^{\mathsf{c}}$$
$$= (f_!(U^{\mathsf{c}}) \cap f_!(V^{\mathsf{c}}))^{\mathsf{c}}$$
$$\subset (f_!(U^{\mathsf{c}} \cap V^{\mathsf{c}}))^{\mathsf{c}}$$
$$= f_!((U \cup V)^{\mathsf{c}})^{\mathsf{c}}$$
$$= f_*(U \cup V),$$

where:

1. We have used Item 16 for the first equality.

2. We have used Item 2 of Definition 4.3.11.1.2 for the second equality.

3. We have used Item 6 of Definition 4.6.1.1.5 for the third equality.

4. We have used Item 2 of Definition 4.3.11.1.2 for the fourth equality.

5. We have used Item 16 for the last equality.

This finishes the proof.

Item 6, Interaction With Binary Intersections: This follows from Item 11.

Item 7, Interaction With Complements: Omitted.

Item 8, Interaction With Symmetric Differences: Omitted.

Item 9, Interaction With Internal Homs of Powersets: We have

$$
\begin{aligned}
\left[f_!(U), f^!(V) \right]_X &\overset{\mathrm{def}}{=} f_!(U)^{\mathsf{c}} \cup f_*(V) \\
&= f_*(U^{\mathsf{c}}) \cup f_*(V) \\
&\subset f_*(U^{\mathsf{c}} \cup V) \\
&\overset{\mathrm{def}}{=} f_*([U,V]_X),
\end{aligned}
$$

where we have used:

1. Item 7 of Definition 4.6.3.1.7 for the second equality.

2. Item 5 of Definition 4.6.3.1.7 for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 10, Lax Preservation of Colimits: Omitted.

Item 11, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.

Item 12, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 10.

Item 13, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 11.

Item 14, Interaction With Coproducts: Omitted.

Item 15, Interaction With Products: Omitted.

Item 16, Relation to Direct Images: We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication.* We claim that

$$
f_*(U) \subset Y \setminus f_!(X \setminus U).
$$

  Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.

This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).

Thus $y \in Y \setminus f_!(X \setminus U)$.

- *The Second Implication.* We claim that

$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$

Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.

Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.

Thus $y \in f_*(U)$.

This finishes the proof of Item 16.
Item 17, Interaction With Injections: Omitted.
Item 18, Interaction With Surjections: Omitted.   □

Proposition 4.6.3.1.8. Let $f \colon X \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}(\mathcal{P}(X),\mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{!|X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}((\mathcal{P}(X),\subset),(\mathcal{P}(Y),\subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon X \to Y$ and $g \colon Y \to Z$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{f_*} & \mathcal{P}(Y) \\
 & \searrow{\scriptstyle (g\circ f)_*} & \downarrow{\scriptstyle g_*} \\
 & & \mathcal{P}(Z).
\end{array}$$

Proof. *Item 1, Functionality I*: There is nothing to prove.

Item 2, Functionality II: This follows from Item 1 of Definition 4.6.3.1.7.

Item 3, Interaction With Identities: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**.

Item 4, Interaction With Composition: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**. □

4.6.4 A Six-Functor Formalism for Sets

Remark 4.6.4.1.1. The assignment $X \mapsto \mathcal{P}(X)$ together with the functors f_*, f^{-1}, and $f_!$ of Item 1 of Definition 4.6.1.1.5, Item 1 of Definition 4.6.2.1.3, and Item 1 of Definition 4.6.3.1.7, and the functors

$$-_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X),$$
$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

of Item 1 of Definition 4.3.9.1.2 and Item 1 of Definition 4.4.7.1.3 satisfy several properties reminiscent of a six functor formalism in the sense of **??**.

We collect these properties in Definition 4.6.4.1.2 below.[43]

Proposition 4.6.4.1.2. Let X be a set.

1. *The Beck–Chevalley Condition.* Let

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\mathrm{pr}_2} & Y \\
{\scriptstyle \mathrm{pr}_1} \downarrow & \lrcorner & \downarrow {\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}
$$

be a pullback diagram in Sets. We have

$$
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\mathrm{pr}_1^{-1}} & \mathcal{P}(X \times_Z Y) \\
{\scriptstyle f_!} \downarrow & & \downarrow {\scriptstyle (\mathrm{pr}_2)_!} \\
\mathcal{P}(Z) & \xrightarrow{g^{-1}} & \mathcal{P}(Y),
\end{array}
\qquad
\begin{array}{l}
g^{-1} \circ f_! = (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}, \\
f^{-1} \circ g_! = (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1},
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\mathrm{pr}_2^{-1}} & \mathcal{P}(X \times_Z Y) \\
{\scriptstyle g_!} \downarrow & & \downarrow {\scriptstyle (\mathrm{pr}_1)_!} \\
\mathcal{P}(Z) & \xrightarrow{f^{-1}} & \mathcal{P}(Y).
\end{array}
$$

---

[43]See also [nLa25].

2. *The Projection Formula I.* The diagram

$$
\begin{array}{ccc}
& \mathcal{P}(X) \times \mathcal{P}(X) & \\
\overset{\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}}{\nearrow} & & \overset{\cap}{\searrow} \\
\mathcal{P}(X) \times \mathcal{P}(Y) & & \mathcal{P}(X) \\
\downarrow{\scriptstyle f_! \times \mathrm{id}_{\mathcal{P}(Y)}} & & \downarrow{\scriptstyle f_!} \\
& \mathcal{P}(Y) \times \mathcal{P}(Y) \underset{\cap}{\rightrightarrows} \mathcal{P}(Y), &
\end{array}
$$

commutes, i.e. we have

$$
f_!\Big(U \cap f^{-1}(V)\Big) = f_!(U) \cap V
$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

3. *The Projection Formula II.* We have a natural transformation

$$
\begin{array}{ccc}
& \mathcal{P}(X) \times \mathcal{P}(X) & \\
\overset{\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}}{\nearrow} & & \overset{\cap}{\searrow} \\
\mathcal{P}(X) \times \mathcal{P}(Y) & \cup & \mathcal{P}(X) \\
\downarrow{\scriptstyle f_* \times \mathrm{id}_{\mathcal{P}(Y)}} & & \downarrow{\scriptstyle f_*} \\
& \mathcal{P}(Y) \times \mathcal{P}(Y) \underset{\cap}{\rightrightarrows} \mathcal{P}(Y), &
\end{array}
$$

with components

$$
f_*(U) \cap V \subset f_*\Big(U \cap f^{-1}(V)\Big)
$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

4. *Strong Closed Monoidality.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) & \xrightarrow{f^{-1,\mathsf{op}} \times f^{-1}} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \\
\downarrow{\scriptstyle [-_1,-_2]_Y} & & \downarrow{\scriptstyle [-_1,-_2]_X} \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = \left[f^{-1}(U), f^{-1}(V)\right]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *The External Tensor Product.* We have an external tensor product

$$-_1 \boxtimes_{X \times Y} -_2 \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$$

given by

$$U \boxtimes_{X \times Y} V \overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)$$
$$= \{(u,v) \in X \times Y \mid u \in U \text{ and } v \in V\}.$$

This is the same map as the one in Item 5 of Definition 4.4.1.1.4. Moreover, the following conditions are satisfied:

(a) *Interaction With Direct Images.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be functions. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_! \times g_!} & \mathcal{P}(X') \times \mathcal{P}(Y') \\
\downarrow{\scriptstyle \boxtimes_{X \times Y}} & & \downarrow{\scriptstyle \boxtimes_{X' \times Y'}} \\
\mathcal{P}(X \times Y) & \xrightarrow[f_! \times g_!]{} & \mathcal{P}(X' \times Y')
\end{array}
$$

commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(b) *Interaction With Inverse Images.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be functions. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X') \times \mathcal{P}(Y') & \xrightarrow{f^{-1} \times g^{-1}} & \mathcal{P}(X) \times \mathcal{P}(Y) \\
\downarrow{\scriptstyle \boxtimes_{X' \times Y'}} & & \downarrow{\scriptstyle \boxtimes_{X \times Y}} \\
\mathcal{P}(X' \times Y') & \xrightarrow[f^{-1} \times g^{-1}]{} & \mathcal{P}(X \times Y)
\end{array}
$$

commutes, i.e. we have

$$\left[f^{-1} \times g^{-1}\right](U \boxtimes_{X' \times Y'} V) = f^{-1}(U) \boxtimes_{X \times Y} g^{-1}(V)$$

for each $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$.

(c) *Interaction With Codirect Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be functions. The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(Y) & \xrightarrow{f_* \times g_*} & \mathcal{P}(X') \times \mathcal{P}(Y') \\
\downarrow{\scriptstyle \boxtimes_{X \times Y}} & & \downarrow{\scriptstyle \boxtimes_{X' \times Y'}} \\
\mathcal{P}(X \times Y) & \xrightarrow[f_* \times g_*]{} & \mathcal{P}(X' \times Y')
\end{array}
$$

commutes, i.e. we have

$$
[f_* \times g_*](U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)
$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(d) *Interaction With Diagonals.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X) \times \mathcal{P}(X) & \xrightarrow{\boxtimes_{X \times X}} & \mathcal{P}(X \times X) \\
& \searrow{\scriptstyle \cap} & \downarrow{\scriptstyle \Delta_X^{-1}} \\
& & \mathcal{P}(X),
\end{array}
$$

i.e. we have

$$
U \cap V = \Delta_X^{-1}(U \boxtimes_{X \times X} V)
$$

for each $U, V \in \mathcal{P}(X)$.

6. *The Dualisation Functor.* We have a functor

$$
D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)
$$

given by

$$
\begin{aligned}
D_X(U) &\overset{\text{def}}{=} [U, \varnothing]_X \\
&\overset{\text{def}}{=} U^{\mathsf{c}}
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, as in Item 5 of Definition 4.4.7.1.3, satisfying the following conditions:

(a) *Duality.* We have

$$D_X(D_X(U)) = U,$$

$$
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{D_X} & \mathcal{P}(X) \\
 & \searrow^{\mathrm{id}_{\mathcal{P}(X)}} & \downarrow^{D_X} \\
 & & \mathcal{P}(X).
\end{array}
$$

(b) *Duality.* The diagram

$$
\begin{array}{ccc}
 & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \xrightarrow{\cap^{\mathsf{op}}} \mathcal{P}(X)^{\mathsf{op}} & \\
{\scriptstyle \mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X} \nearrow & & \searrow {\scriptstyle D_X} \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) & \xrightarrow[{[-_1,-_2]_X}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\mathrm{def}}{=} \left[U \cap [V, \varnothing]_X, \varnothing \right]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

(c) *Interaction With Direct Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{f_*^{\mathsf{op}}} & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle D_X} \downarrow & & \downarrow {\scriptstyle D_Y} \\
\mathcal{P}(X) & \xrightarrow[f_!]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(d) *Interaction With Inverse Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(Y)^{\mathsf{op}} & \xrightarrow{f^{-1,\mathsf{op}}} & \mathcal{P}(X)^{\mathsf{op}} \\
{\scriptstyle D_Y} \downarrow & & \downarrow {\scriptstyle D_X} \\
\mathcal{P}(Y) & \xrightarrow[f^{-1}]{} & \mathcal{P}(X)
\end{array}
$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X\Big(f^{-1}(U)\Big)$$

for each $U \in \mathcal{P}(X)$.

(e) *Interaction With Codirect Images.* The diagram

$$
\begin{array}{ccc}
\mathcal{P}(X)^{\mathsf{op}} & \xrightarrow{\ f_!^{\mathsf{op}}\ } & \mathcal{P}(Y)^{\mathsf{op}} \\
{\scriptstyle D_X}\Big\downarrow & & \Big\downarrow{\scriptstyle D_Y} \\
\mathcal{P}(X) & \xrightarrow[\ f_*\ ]{} & \mathcal{P}(Y)
\end{array}
$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, The Beck–Chevalley Condition*: We have

$$
\begin{aligned}
\big[g^{-1} \circ f_!\big](U) &\overset{\mathrm{def}}{=} g^{-1}(f_!(U)) \\
&\overset{\mathrm{def}}{=} \{y \in Y \mid g(y) \in f_!(U)\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid x \in U\} \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } y = y \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } \mathrm{pr}_2(x,y) = y \end{array} \right\} \\
&\overset{\mathrm{def}}{=} (\mathrm{pr}_2)_!(\{(x,y) \in X \times_Z Y \mid x \in U\}) \\
&= (\mathrm{pr}_2)_!(\{(x,y) \in X \times_Z Y \mid \mathrm{pr}_1(x,y) \in U\}) \\
&\overset{\mathrm{def}}{=} (\mathrm{pr}_2)_!\Big(\mathrm{pr}_1^{-1}(U)\Big)
\end{aligned}
$$

$$\overset{\text{def}}{=} \left[(\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1} \right](U)$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$g^{-1} \circ f_! = (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}.$$

For the second equality, we have

$$
\begin{aligned}
\left[f^{-1} \circ g_! \right](U) &\overset{\text{def}}{=} f^{-1}(g_!(U)) \\
&\overset{\text{def}}{=} \{ x \in X \mid f(x) \in g_!(V) \} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \{ (x, y) \in X \times_Z Y \mid y \in V \} \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \{ (x, y) \in X \times_Z Y \mid y \in V \} \\ \text{such that } x = x \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \{ (x, y) \in X \times_Z Y \mid y \in V \} \\ \text{such that } \mathrm{pr}_1(x, y) = x \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\{ (x, y) \in X \times_Z Y \mid y \in V \}) \\
&= (\mathrm{pr}_1)_!(\{ (x, y) \in X \times_Z Y \mid \mathrm{pr}_2(x, y) \in V \}) \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_! \left(\mathrm{pr}_2^{-1}(V) \right) \\
&\overset{\text{def}}{=} \left[(\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1} \right](V)
\end{aligned}
$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$f^{-1} \circ g_! = (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}.$$

This finishes the proof.

Item 2, The Projection Formula I: We claim that

$$f_!(U) \cap V \subset f_!\left(U \cap f^{-1}(V) \right).$$

Indeed, we have

$$
\begin{aligned}
f_!(U) \cap V &\subset f_!(U) \cap f_!\left(f^{-1}(V) \right) \\
&= f_!\left(U \cap f^{-1}(V) \right),
\end{aligned}
$$

where we have used:

1. Item 2 of Definition 4.6.1.1.5 for the inclusion.

2. Item 6 of Definition 4.6.1.1.5 for the equality.

Conversely, we claim that

$$f_!\Big(U \cap f^{-1}(V)\Big) \subset f_!(U) \cap V.$$

Indeed:

1. Let $y \in f_!(U \cap f^{-1}(V))$.

2. Since $y \in f_!(U \cap f^{-1}(V))$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.

3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.

4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.

5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.

6. But $y = f(x)$, so $y \in f_!(U) \cap V$.

7. Thus $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$.

This finishes the proof.
Item 3, The Projection Formula II: We have

$$f_*(U) \cap V \subset f_*(U) \cap f_*\Big(f^{-1}(V)\Big)$$
$$= f_*\Big(U \cap f^{-1}(V)\Big),$$

where we have used:

1. Item 2 of Definition 4.6.3.1.7 for the inclusion.

2. Item 6 of Definition 4.6.3.1.7 for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
Item 4, Strong Closed Monoidality: This is a repetition of Item 19 of Definition 4.4.7.1.3 and is proved there.
Item 5, The External Tensor Product: We have

$$U \boxtimes_{X \times Y} V \overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \{(x,y) \in X \times Y \mid \mathrm{pr}_1(x,y) \in U\} \\
&\quad \cup \{(x,y) \in X \times Y \mid \mathrm{pr}_2(x,y) \in V\} \\
&= \{(x,y) \in X \times Y \mid x \in U\} \\
&\quad \cup \{(x,y) \in X \times Y \mid y \in V\} \\
&= \{(x,y) \in X \times Y \mid x \in U \text{ and } y \in V\} \\
&\overset{\text{def}}{=} U \times V.
\end{aligned}
$$

Next, we claim that Items 5a to 5d are indeed true:

1. *Proof of Item 5a:* This is a repetition of Item 16 of Definition 4.6.1.1.5 and is proved there.

2. *Proof of Item 5b:* This is a repetition of Item 16 of Definition 4.6.2.1.3 and is proved there.

3. *Proof of Item 5c:* This is a repetition of Item 15 of Definition 4.6.3.1.7 and is proved there.

4. *Proof of Item 5d:* We have

$$
\begin{aligned}
\Delta_X^{-1}(U \boxtimes_{X \times X} V) &\overset{\text{def}}{=} \{x \in X \mid (x,x) \in U \boxtimes_{X \times X} V\} \\
&= \{x \in X \mid (x,x) \in \{(u,v) \in X \times X \mid u \in U \text{ and } v \in V\}\} \\
&= U \cap V.
\end{aligned}
$$

This finishes the proof.

Item 6, The Dualisation Functor: This is a repetition of Items 5 and 6 of Definition 4.4.7.1.3 and is proved there. □

Appendices

A Other Chapters

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