

Types of Morphisms in Bicategories

The Clowder Project Authors

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In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories* ([Sections 14.1 and 14.2](#)). There is a large number of different notions capturing the idea of a “monomorphism” or of an “epimorphism” in a bicategory.

Arguably, the notion that best captures these concepts is that of a *pseudomononic morphism* ([Definition 14.1.10.1.1](#)) and of a *pseudoepic morphism* ([Definition 14.2.10.1.1](#)), although the other notions introduced in [Sections 14.1](#) and [14.2](#) are also interesting on their own.

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14.1 Monomorphisms in Bicategories

14.1.1 Representably Faithful Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.1.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably faithful**¹ if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is faithful.

Remark 14.1.1.1.2. In detail, f is representably faithful if, for all diagrams in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 14.1.1.1.3. Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in \mathbf{Cats}_2 .* The representably faithful morphisms in \mathbf{Cats}_2 are precisely the faithful functors; see [Categories, Item 2](#) of [Definition 11.6.1.1.2](#).

¹*Further Terminology:* Also called simply a **faithful morphism**, based on [Item 1](#) of

2. *Representably Faithful Morphisms in **Rel***. Every morphism of **Rel** is representably faithful; see [Relations](#), [Item 1](#) of [Definition 8.5.11.1.1](#).

14.1.2 Representably Full Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.2.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably full**² if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

Remark 14.1.2.1.2. In detail, f is representably full if, for each $X \in \text{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of \mathcal{C} , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of \mathcal{C} such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

Example 14.1.2.1.3. Here are some examples of representably full morphisms.

Definition 14.1.1.1.3.

²*Further Terminology:* Also called simply a **full morphism**, based on [Item 1](#) of

1. *Representably Full Morphisms in \mathbf{Cats}_2* . The representably full morphisms in \mathbf{Cats}_2 are precisely the full functors; see [Categories](#), ?? of [Definition 11.6.2.1.2](#).
2. *Representably Full Morphisms in \mathbf{Rel}* . The representably full morphisms in \mathbf{Rel} are characterised in [Relations](#), [Item 2](#) of [Definition 8.5.11.1.1](#).

14.1.3 Representably Fully Faithful Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.3.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably fully faithful**³ if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful ([Definition 14.1.1.1.1](#)) and representably full ([Definition 14.1.2.1.1](#)).
2. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is fully faithful.

Remark 14.1.3.1.2. In detail, f is representably fully faithful if the conditions in [Definition 14.1.1.1.2](#) and [Definition 14.1.2.1.2](#) hold:

1. For all diagrams in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

Definition 14.1.2.1.3.

³*Further Terminology:* Also called simply a **fully faithful morphism**, based on [Item 1](#) of [Definition 14.1.3.1.3](#).

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta: f \circ \phi \Rightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of \mathcal{C} , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of \mathcal{C} such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

Example 14.1.3.1.3. Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in \mathbf{Cats}_2 .* The representably fully faithful morphisms in \mathbf{Cats}_2 are precisely the fully faithful functors; see [Categories, Item 6](#) of [Definition 11.6.3.1.2](#).
2. *Representably Fully Faithful Morphisms in \mathbf{Rel} .* The representably fully faithful morphisms of \mathbf{Rel} coincide ([Relations, Item 3](#) of [Definition 8.5.11.1.1](#)) with the representably full morphisms in \mathbf{Rel} , which are characterised in [Relations, Item 2](#) of [Definition 8.5.11.1.1](#).

14.1.4 Morphisms Representably Faithful on Cores

Let \mathcal{C} be a bicategory.

Definition 14.1.4.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably faithful on cores** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is faithful.

Remark 14.1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

14.1.5 Morphisms Representably Full on Cores

Let \mathcal{C} be a bicategory.

Definition 14.1.5.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*: \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \rightarrow \mathrm{Core}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is full.

Remark 14.1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta: f \circ \phi \xRightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of \mathcal{C} , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of \mathcal{C} such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B = X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.6 Morphisms Representably Fully Faithful on Cores

Let \mathcal{C} be a bicategory.

Definition 14.1.6.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores ([Definition 14.1.5.1.1](#)) and representably full on cores ([Definition 14.1.4.1.1](#)).
2. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Core}(\text{Hom}_{\mathcal{C}}(X, A)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is fully faithful.

Remark 14.1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in [Definition 14.1.4.1.2](#) and [Definition 14.1.5.1.2](#) hold:

1. For all diagrams in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of \mathcal{C} , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of \mathcal{C} such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B \quad = \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

14.1.7 Representably Essentially Injective Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.7.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably essentially injective** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is essentially injective.

Remark 14.1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ of \mathcal{C} , the following condition is satisfied:

(\star) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

14.1.8 Representably Conservative Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.8.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **representably conservative** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is conservative.

Remark 14.1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi: X \rightrightarrows A$ and each 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of \mathcal{C} , if the 2-morphism

$$\mathrm{id}_f \star \alpha: f \circ \phi \Longrightarrow f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \parallel \\ \mathrm{id}_f \star \alpha \\ \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

is a 2-isomorphism, then so is α .

14.1.9 Strict Monomorphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.9.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*: \mathrm{Hom}_{\mathcal{C}}(X, A) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*: \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, A)) \rightarrow \mathrm{Obj}(\mathrm{Hom}_{\mathcal{C}}(X, B))$$

is injective.

Remark 14.1.9.1.2. In detail, f is a strict monomorphism in \mathcal{C} if, for each diagram in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \Downarrow \psi \end{array} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 14.1.9.1.3. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in \mathbf{Cats}_2 .* The strict monomorphisms in \mathbf{Cats}_2 are precisely the functors which are injective on objects and injective on morphisms; see [Categories](#), [Item 1](#) of [Definition 11.7.2.1.2](#).
2. *Strict Monomorphisms in \mathbf{Rel} .* The strict monomorphisms in \mathbf{Rel} are characterised in [Relations](#), [Definition 8.5.10.1.1](#).

14.1.10 Pseudomononic Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.1.10.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **pseudomononic** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f_*: \text{Hom}_{\mathcal{C}}(X, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is pseudomononic.

Remark 14.1.10.1.2. In detail, a 1-morphism $f: A \rightarrow B$ of \mathcal{C} is pseudomononic if it satisfies the following conditions:

1. For all diagrams in \mathcal{C} of the form

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B,$$

if we have

$$\text{id}_f \star \alpha = \text{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta: f \circ \phi \xrightarrow{\sim} f \circ \psi, \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of \mathcal{C} , there exists a 2-isomorphism

$$\alpha: \phi \xrightarrow{\sim} \psi, \quad X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A$$

of \mathcal{C} such that we have an equality

$$X \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} A \xrightarrow{f} B \quad = \quad X \begin{array}{c} \xrightarrow{f \circ \phi} \\ \beta \Downarrow \\ \xrightarrow{f \circ \psi} \end{array} B$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \text{id}_f \star \alpha.$$

Proposition 14.1.10.1.3. Let $f: A \rightarrow B$ be a 1-morphism of \mathcal{C} .

1. *Characterisations.* The following conditions are equivalent:
 - (a) The morphism f is pseudomonic.
 - (b) The morphism f is representably full on cores and representably faithful.
 - (c) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_B A, \quad \begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \text{id}_A \downarrow & \nearrow \text{dashed} & \downarrow F \\ A & \xrightarrow{F} & B \end{array}$$

in \mathcal{C} up to equivalence.

2. *Interaction With Cotensors.* If \mathcal{C} has cotensors with $\mathbb{1}$, then the following conditions are equivalent:
 - (a) The morphism f is pseudomonic.
 - (b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \times_{\mathbb{1} \pitchfork F} B, \quad \begin{array}{ccc} A & \longrightarrow & \mathbb{1} \pitchfork A \\ F \downarrow & \nearrow \text{dashed} & \downarrow \mathbb{1} \pitchfork F \\ B & \longrightarrow & \mathbb{1} \pitchfork B \end{array}$$

in \mathcal{C} up to equivalence.

Proof. **Item 1, Characterisations:** Omitted.

Item 2, Interaction With Cotensors: Omitted. □

14.2 Epimorphisms in Bicategories

14.2.1 Corepresentably Faithful Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.1.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably faithful** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is faithful.

Remark 14.2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

Example 14.2.1.1.3. Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in \mathbf{Cats}_2 .* The corepresentably faithful morphisms in \mathbf{Cats}_2 are characterised in [Categories, Item 5 of Definition 11.6.1.1.2](#).
2. *Corepresentably Faithful Morphisms in \mathbf{Rel} .* Every morphism of \mathbf{Rel} is corepresentably faithful; see [Relations, Item 1 of Definition 8.5.13.1.1](#).

14.2.2 Corepresentably Full Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.2.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably full** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

Remark 14.2.2.1.2. In detail, f is corepresentably full if, for each $X \in$

$\text{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of \mathcal{C} , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of \mathcal{C} such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

Example 14.2.2.1.3. Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in \mathbf{Cats}_2 .* The corepresentably full morphisms in \mathbf{Cats}_2 are characterised in [Categories, Item 7 of Definition 11.6.2.1.2](#).
2. *Corepresentably Full Morphisms in \mathbf{Rel} .* The corepresentably full morphisms in \mathbf{Rel} are characterised in [Relations, Item 2 of Definition 8.5.13.1.1](#).

14.2.3 Corepresentably Fully Faithful Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.3.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably fully faithful**⁴ if the following equivalent conditions are satisfied:

⁴*Further Terminology:* Corepresentably fully faithful morphisms have also been called

1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).
2. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is fully faithful.

Remark 14.2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Definition 14.2.1.1.2 and Definition 14.2.2.1.2 hold:

1. For all diagrams in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta: \phi \circ f \Rightarrow \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of \mathcal{C} , there exists a 2-morphism

$$\alpha: \phi \Rightarrow \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of \mathcal{C} such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

Example 14.2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in \mathbf{Cats}_2 .* The fully faithful epimorphisms in \mathbf{Cats}_2 are characterised in [Categories, Item 10](#) of [Definition 11.6.3.1.2](#).
2. *Corepresentably Fully Faithful Morphisms in \mathbf{Rel} .* The corepresentably fully faithful morphisms of \mathbf{Rel} coincide ([Relations, Item 3](#) of [Definition 8.5.13.1.1](#)) with the corepresentably full morphisms in \mathbf{Rel} , which are characterised in [Relations, Item 2](#) of [Definition 8.5.13.1.1](#).

14.2.4 Morphisms Corepresentably Faithful on Cores

Let \mathcal{C} be a bicategory.

Definition 14.2.4.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably faithful on cores** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is faithful.

Remark 14.2.4.1.2. In detail, f is corepresentably faithful on cores if, for all diagrams in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

14.2.5 Morphisms Corepresentably Full on Cores

Let \mathcal{C} be a bicategory.

lax epimorphisms in the literature (e.g. in [\[Ad +01\]](#)), though we will always use the name “corepresentably fully faithful morphism” instead in this work.

Definition 14.2.5.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably full on cores** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is full.

Remark 14.2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \text{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of \mathcal{C} , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of \mathcal{C} such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

14.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let \mathcal{C} be a bicategory.

Definition 14.2.6.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores ([Definition 14.2.5.1.1](#)) and corepresentably faithful on cores ([Definition 14.2.1.1.1](#)).

2. For each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^* : \text{Core}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Core}(\text{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is fully faithful.

Remark 14.2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in [Definition 14.2.4.1.2](#) and [Definition 14.2.5.1.2](#) hold:

1. For all diagrams in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta : \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of \mathcal{C} , there exists a 2-isomorphism

$$\alpha : \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of \mathcal{C} such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X \quad = \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

14.2.7 Corepresentably Essentially Injective Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.7.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably essentially injective** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is essentially injective.

Remark 14.2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi: B \rightrightarrows X$ of \mathcal{C} , the following condition is satisfied:

$$(\star) \text{ If } \phi \circ f \cong \psi \circ f, \text{ then } \phi \cong \psi.$$

14.2.8 Corepresentably Conservative Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.8.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **corepresentably conservative** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is conservative.

Remark 14.2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi: B \rightrightarrows X$ and each 2-morphism

$$\alpha: \phi \rightrightarrows \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of \mathcal{C} , if the 2-morphism

$$\alpha \star \text{id}_f: \phi \circ f \rightrightarrows \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \parallel \\ \alpha \star \text{id}_f \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

is a 2-isomorphism, then so is α .

14.2.9 Strict Epimorphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.9.1.1. A 1-morphism $f: A \rightarrow B$ is a **strict epimorphism** in \mathcal{C} if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_*: \text{Obj}(\text{Hom}_{\mathcal{C}}(B, X)) \rightarrow \text{Obj}(\text{Hom}_{\mathcal{C}}(A, X))$$

is injective.

Remark 14.2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \xrightarrow{\psi} \end{array} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 14.2.9.1.3. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in \mathbf{Cats}_2 .* The strict epimorphisms in \mathbf{Cats}_2 are characterised in [Categories, Item 1](#) of [Definition 11.7.3.1.2](#).
2. *Strict Epimorphisms in \mathbf{Rel} .* The strict epimorphisms in \mathbf{Rel} are characterised in [Relations, Definition 8.5.12.1.1](#).

14.2.10 Pseudoepic Morphisms

Let \mathcal{C} be a bicategory.

Definition 14.2.10.1.1. A 1-morphism $f: A \rightarrow B$ of \mathcal{C} is **pseudoepic** if, for each $X \in \text{Obj}(\mathcal{C})$, the functor

$$f^*: \text{Hom}_{\mathcal{C}}(B, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is pseudomononic.

Remark 14.2.10.1.2. In detail, a 1-morphism $f: A \rightarrow B$ of \mathcal{C} is pseudoepic if it satisfies the following conditions:

1. For all diagrams in \mathcal{C} of the form

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \Downarrow \beta \\ \xrightarrow{\psi} \end{array} X,$$

if we have

$$\alpha \star \text{id}_f = \beta \star \text{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \text{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta: \phi \circ f \xRightarrow{\sim} \psi \circ f, \quad A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of \mathcal{C} , there exists a 2-isomorphism

$$\alpha: \phi \xRightarrow{\sim} \psi, \quad B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X$$

of \mathcal{C} such that we have an equality

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{\phi} \\ \alpha \Downarrow \\ \xrightarrow{\psi} \end{array} X = A \begin{array}{c} \xrightarrow{\phi \circ f} \\ \beta \Downarrow \\ \xrightarrow{\psi \circ f} \end{array} X$$

of pasting diagrams in \mathcal{C} , i.e. such that we have

$$\beta = \alpha \star \text{id}_f.$$

Proposition 14.2.10.1.3. Let $f: A \rightarrow B$ be a 1-morphism of \mathcal{C} .

1. *Characterisations.* The following conditions are equivalent:
 - (a) The morphism f is pseudoepic.
 - (b) The morphism f is corepresentably full on cores and corepresentably faithful.

(c) We have an isococcomma square of the form

$$B \stackrel{\text{eq.}}{\cong} B \coprod_A B, \quad \begin{array}{ccc} B & \xleftarrow{\text{id}_B} & B \\ \text{id}_B \uparrow & \nearrow \text{dashed} & \uparrow F \\ B & \xleftarrow{F} & A \end{array}$$

in \mathcal{C} up to equivalence.

Proof. *Item 1, Characterisations:* Omitted. □

Appendices

A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations

10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

References

- [Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil.
“On Functors Which Are Lax Epimorphisms”. In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 15).