Conditions on Relations

The Clowder Project Authors

July 22, 2025

00TJ This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

Contents

02D0 10.1 Functional and Total Relations

00JC 10.1.1 Functional Relations

Let A and B be sets.

00JD **Definition 10.1.1.1.1.** A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

00JE **Proposition 10.1.1.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JF 1. *Characterisations.* The following conditions are equivalent:

00JG (a) The relation R is functional.

00JH (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b:* Let $(b, b') \in B \times B$. We need to show that

$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\Longrightarrow$ *Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  - Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  - Since $R \diamond R^{\dagger} \subset \chi_B$, we have

$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

    and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^{\dagger}\right](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

00JJ 10.1.2 Total Relations

Let A and B be sets.

00JK **Definition 10.1.2.1.1.** A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

00JL **Proposition 10.1.2.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JM 1. *Characterisations.* The following conditions are equivalent:

00JN (a) The relation R is total.

00JP (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* We have to show that, for each $(a, a') \in A$, we have
$$\chi_A(a, a') \preceq_{\{\mathsf{t},\mathsf{f}\}} \left[R^\dagger \diamond R\right](a, a'),$$
i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b $\implies$ Item 1a:* Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have
$$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$
implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

00TK 10.2 Reflexive Relations

00TL 10.2.1 Foundations

Let A be a set.

00TM **Definition 10.2.1.1.1.** A **reflexive relation** is equivalently:[1]

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)),\chi_A)$.
- A pointed object in $(\mathbf{Rel}(A,A),\chi_A)$.

00TN **Remark 10.2.1.1.2.** In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R\colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a\in A$, we have $a\sim_R a$.

00TP **Definition 10.2.1.1.3.** Let A be a set.

00TQ 1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A,A)$ of $\mathrm{Rel}(A,A)$ spanned by the reflexive relations.

00TR 2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{refl}}(A,A)$ of $\mathbf{Rel}(A,A)$ spanned by the reflexive relations.

00TS **Proposition 10.2.1.1.4.** Let R and S be relations on A.

00TT 1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.

00TU 2. *Interaction With Composition.* If R and S are reflexive, then so is $S\diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

00TV 10.2.2 The Reflexive Closure of a Relation

Let R be a relation on A.

00TW **Definition 10.2.2.1.1.** The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[2] satisfying the following universal property:[3]

(⋆) Given another reflexive relation $\sim_S$ on A such that $R\subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}}\subset\sim_S$.

[2] *Further Notation:* Also written R^{refl}.

[3] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

00TX **Construction 10.2.2.1.2.** Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A,A),\chi_A)$[4], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \textstyle\coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

Proof. Clear. □

00TY **Proposition 10.2.2.1.3.** Let R be a relation on A.

00TZ 1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathsf{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00U0 2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

00U1 3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

00U2 4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

[4] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A,A)),\chi_A)$.

00U3 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. Item 1, Adjointness: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 10.2.2.1.1.
Item 2, The Reflexive Closure of a Reflexive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.2.1.1.4. □

00U4 10.3 Symmetric Relations

00U5 10.3.1 Foundations

Let A be a set.

00U6 **Definition 10.3.1.1.1.** A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

00U7 **Remark 10.3.1.1.2.** In detail, a relation R is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

00U8 **Definition 10.3.1.1.3.** Let A be a set.

00U9 1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

00UA 2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UB **Proposition 10.3.1.1.4.** Let R and S be relations on A.

00UC 1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.

00UD 2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. Item 1, Interaction With Inverses: Clear.
Item 2, Interaction With Composition: Clear. □

00UE 10.3.2 The Symmetric Closure of a Relation

Let R be a relation on A.

00UF **Definition 10.3.2.1.1.** The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5] satisfying the following universal property:[6]

($\star$) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

00UG **Construction 10.3.2.1.2.** Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof. Clear. □

00UH **Proposition 10.3.2.1.3.** Let R be a relation on A.

00UJ 1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{symm}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00UK 2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

00UL 3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

[5] *Further Notation:* Also written R^{symm}.
[6] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

00UM 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

00UN 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 10.3.2.1.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.3.1.1.4. □

00UP 10.4 Transitive Relations

00UQ 10.4.1 Foundations

Let A be a set.

00UR **Definition 10.4.1.1.1.** A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A,A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

[7] Note that since $\mathbf{Rel}(A,A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

00US **Remark 10.4.1.1.2.** In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00UT **Definition 10.4.1.1.3.** Let A be a set.

00UU 1. The **set of transitive relations from A to B** is the subset $\mathrm{Rel}^{\mathsf{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.

00UV 2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathsf{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UW **Proposition 10.4.1.1.4.** Let R and S be relations on A.

00UX 1. *Interaction With Inverses.* If R is transitive, then so is $R^\dagger$.

00UY 2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: See [MSE 2096272].[8] □

00UZ 10.4.2 The Transitive Closure of a Relation

Let R be a relation on A.

[8] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
  - There is some $b \in A$ such that:
    - $a \sim_R b$;
    - $b \sim_S c$;
  - There is some $d \in A$ such that:
    - $c \sim_R d$;
    - $d \sim_S e$.

00V0 **Definition 10.4.2.1.1.** The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[9] satisfying the following universal property:[10]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

00V1 **Construction 10.4.2.1.2.** Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[11], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

Proof. Clear. □

00V2 **Proposition 10.4.2.1.3.** Let R be a relation on A.

00V3 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right) \colon \quad \begin{tikzcd} \mathbf{Rel}(A, A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{trans}}(A, A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{trans}}\left(R^{\mathrm{trans}}, S\right) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathsf{trans}}(A, A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

00V4 2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

00V5 3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

[9] *Further Notation:* Also written R^{trans}.
[10] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.
[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

00V6 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

00V7 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 10.4.2.1.1.
Item 2, The Transitive Closure of a Transitive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: We have

$$\begin{aligned} \left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty} \left(R^{\dagger}\right)^{\diamond n} \\ &= \bigcup_{n=1}^{\infty} \left(R^{\diamond n}\right)^{\dagger} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} \\ &= \left(R^{\mathrm{trans}}\right)^{\dagger}, \end{aligned}$$

where we have used, respectively:

- Definition 10.4.2.1.2.
- Constructions With Relations, **??** of **??**.
- Constructions With Relations, **??** of Definition 9.2.3.1.2.
- Definition 10.4.2.1.2.

This finishes the proof.

Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.4.1.1.4. □

00V8 10.5 Equivalence Relations

00V9 10.5.1 Foundations

Let A be a set.

00VA **Definition 10.5.1.1.1.** A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

00VB **Example 10.5.1.1.2.** The **kernel of a function** $f\colon A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

00VC **Definition 10.5.1.1.3.** Let A and B be sets.

00VD 1. The **set of equivalence relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

00VE 2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

00VF 10.5.2 The Equivalence Closure of a Relation

Let R be a relation on A.

00VG **Definition 10.5.2.1.1.** The **equivalence closure**[14] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[15] satisfying the following universal property:[16]

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

---

[12] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Constructions With Relations, **??** of **??**.

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written R^{eq}.

[16] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

00VH **Construction 10.5.2.1.2.** Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1,\ldots,x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

00VJ 1. The symmetric closure of a reflexive relation is still reflexive.

00VK 2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. ☐

00VL **Proposition 10.5.2.1.3.** Let R be a relation on A.

00VM 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

00VN 2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

00VP 3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 10.5.2.1.1.
Item 2, The Equivalence Closure of an Equivalence Relation: Clear.
Item 3, Idempotency: This follows from Item 2. □

00VQ 10.6 Quotients by Equivalence Relations

00VR 10.6.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

02B1 **Definition 10.6.1.1.1.** The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned}[a] &\overset{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)}\end{aligned}$$

02B2 10.6.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

02B3 **Definition 10.6.2.1.1.** The **quotient of** X **by** R is the set $X/\sim_R$ defined by

$$X/\sim_R \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00VV **Remark 10.6.2.1.2.** The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.

- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

$$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[17]

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

00VW **Proposition 10.6.2.1.3.** Let $f \colon X \to Y$ be a function and let R be a relation on X.

02B4 1. *As a Coequaliser.* We have an isomorphism of sets

$$X/{\sim_R^{\mathrm{eq}}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} X \right),$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

02B5 2. *As a Pushout.* We have an isomorphism of sets[18]

$$X/{\sim_R^{\mathrm{eq}}} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/{\sim_R^{\mathrm{eq}}} & X \arrow[l] \\ X \arrow[u] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2). \arrow[l] \arrow[u] \end{tikzcd}$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

---

[17] When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

[18] Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/{\sim_R^{\mathrm{eq}}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d] \\ X \arrow[r] & X/{\sim_R^{\mathrm{eq}}}. \end{tikzcd}$$

02B6 3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]

$$X/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

00W0 4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

02B7 (a) There exists a map

$$\bar{f} \colon X/\sim_R \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "q"'] & Y \\ X/\sim_R \arrow[ur, dotted, "\exists"', "\bar{f}"] & \end{tikzcd}$$

commute.

02B8 (b) We have $R \subset \mathrm{Ker}(f)$.

02B9 (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

00W1 5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\bar{f}$ is the *unique*

---

[19] *Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{CoIm}(f)$.

[20] In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f) \colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Constructions With Relations, **??** of **??**.

map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"', two heads] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\exists!", "\bar{f}"'] & \end{tikzcd}$$

commute.

00W2 6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/{\sim_R}, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \bar{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

02BA 7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BB (a) The map $\bar{f}$ is an injection.

02BC (b) We have $R = \mathrm{Ker}(f)$.

02BD (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

02BE 8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BF (a) The map $f \colon X \to Y$ is surjective.

02BG (b) The map $\bar{f} \colon X/{\sim_R} \to Y$ is surjective.

02BH 9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim^{\mathrm{eq}}_R$ be the equivalence relation associated to R. The following conditions are equivalent:

02BJ (a) The map f satisfies the equivalent conditions of Item 4:

- There exists a map

$$\bar{f}\colon X/{\sim_R^{\mathrm{eq}}} \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R^{\mathrm{eq}}} \arrow[ur, dotted, "\exists"', "\bar{f}"] & \end{tikzcd}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

02BK (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. *Item 1, As a Coequaliser*: Omitted.
Item 2, As a Pushout: Omitted.
Item 3, The First Isomorphism Theorem for Sets: Clear.
Item 4, Descending Functions to Quotient Sets, I: See [Pro25c].
Item 5, Descending Functions to Quotient Sets, II: See [Pro25d].
Item 6, Descending Functions to Quotient Sets, III: This follows from Items 5 and 6.
Item 7, Descending Functions to Quotient Sets, IV: See [Pro25b].
Item 8, Descending Functions to Quotient Sets, V: See [Pro25a].
Item 9, Descending Functions to Quotient Sets, VI: The implication Item 8a $\Longrightarrow$ Item 8b is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

- The following conditions are satisfied:
  - We have $x \sim_R x_1$ or $x_1 \sim_R x$;
  - We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
  - We have $y \sim_R x_n$ or $x_n \sim_R y$;

– We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. $\square$

Appendices

A Other Chapters

References

[MSE 2096272] Akiva Weinberger. *Is composition of two transitive relations transitive? If not, can you give me a counterexample?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2096272 (cit. on p. 9).

[Pro25a] Proof Wiki Contributors. *Condition For Mapping from Quotient Set To Be A Surjection — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Surjection (cit. on p. 18).

[Pro25b] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be An Injection— Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Injection (cit. on p. 18).

[Pro25c] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be Well-Defined — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Well-Defined (cit. on p. 18).

[Pro25d] Proof Wiki Contributors. *Mapping From Quotient Set When Defined Is Unique — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Mapping_from_Quotient_Set_when_Defined_is_Unique (cit. on p. 18).