Monoidal Structures on the Category of Sets

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01NK This chapter contains some material on monoidal structures on Sets.

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01NM 5.1.1 Products of Sets

See Constructions With Sets, Section 4.1.3.

01NN 5.1.2 The Internal Hom of Sets

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01NP 5.1.3 The Monoidal Unit

01NQ **Definition 5.1.3.1.1.** The **monoidal unit of the product of sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}$$

defined by

$$\mathbb{1}_{\mathsf{Sets}} \overset{\mathrm{def}}{=} \mathrm{pt},$$

where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.1.

01NR 5.1.4 The Associator

01NS **Definition 5.1.4.1.1.** The **associator of the product of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}\times(\mathsf{Sets}\times\mathsf{Sets}) \arrow[dr, "\mathsf{id}\times\times"] & \\
(\mathsf{Sets}\times\mathsf{Sets})\times\mathsf{Sets} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}", "\sim"'] \arrow[d, "\times\times\mathsf{id}"'] & & \mathsf{Sets}\times\mathsf{Sets} \arrow[d, "\times"] \\
\mathsf{Sets}\times\mathsf{Sets} \arrow[rr, "\times"'] \arrow[urr, Rightarrow, "\alpha^{\mathsf{Sets}}"] & & \mathsf{Sets},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}}_{X,Y,Z}\colon (X\times Y)\times Z \overset{\sim}{\dashrightarrow} X\times(Y\times Z)$$

at (X, Y, Z) is given by

$$\alpha^{\mathsf{Sets}}_{X,Y,Z}((x,y),z) \overset{\text{def}}{=} (x,(y,z))$$

for each $((x,y),z) \in (X\times Y)\times Z$.

Proof. Invertibility: The inverse of $\alpha^{\mathsf{Sets}}_{X,Y,Z}$ is the morphism

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z}\colon X\times(Y\times Z) \overset{\sim}{\dashrightarrow} (X\times Y)\times Z$$

defined by

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z}(x,(y,z)) \overset{\text{def}}{=} ((x,y),z)$$

for each $(x,(y,z)) \in X\times(Y\times Z)$. Indeed:

- *Invertibility I*. We have

$$\begin{aligned}
[\alpha^{\mathsf{Sets},-1}_{X,Y,Z}\circ\alpha^{\mathsf{Sets}}_{X,Y,Z}]((x,y),z) &\overset{\text{def}}{=} \alpha^{\mathsf{Sets},-1}_{X,Y,Z}(\alpha^{\mathsf{Sets}}_{X,Y,Z}((x,y),z))\\
&\overset{\text{def}}{=} \alpha^{\mathsf{Sets},-1}_{X,Y,Z}(x,(y,z))\\
&\overset{\text{def}}{=} ((x,y),z)\\
&\overset{\text{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}]((x,y),z)
\end{aligned}$$

  for each $((x,y),z) \in (X\times Y)\times Z$, and therefore we have

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z}\circ\alpha^{\mathsf{Sets}}_{X,Y,Z} = \mathrm{id}_{(X\times Y)\times Z}\,.$$

- *Invertibility II*. We have
$$
\begin{aligned}
[\alpha^{\mathsf{Sets}}_{X,Y,Z} \circ \alpha^{\mathsf{Sets},-1}_{X,Y,Z}](x,(y,z)) &\overset{\text{def}}{=} \alpha^{\mathsf{Sets}}_{X,Y,Z}(\alpha^{\mathsf{Sets},-1}_{X,Y,Z}(x,(y,z)))\\
&\overset{\text{def}}{=} \alpha^{\mathsf{Sets}}_{X,Y,Z}((x,y),z)\\
&\overset{\text{def}}{=} (x,(y,z))\\
&\overset{\text{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}](x,(y,z))
\end{aligned}
$$
for each $(x,(y,z)) \in X\times(Y\times Z)$, and therefore we have
$$
\alpha^{\mathsf{Sets},-1}_{X,Y,Z} \circ \alpha^{\mathsf{Sets}}_{X,Y,Z} = \mathrm{id}_{X\times(Y\times Z)}\,.
$$

Therefore $\alpha^{\mathsf{Sets}}_{X,Y,Z}$ is indeed an isomorphism.
Naturality: We need to show that, given functions
$$
\begin{aligned}
f &\colon X \to X',\\
g &\colon Y \to Y',\\
h &\colon Z \to Z'
\end{aligned}
$$
the diagram
$$
\begin{tikzcd}
(X\times Y)\times Z \arrow[r, "(f\times g)\times h"] \arrow[d, "\alpha^{\mathsf{Sets}}_{X,Y,Z}"'] & (X'\times Y')\times Z' \arrow[d, "\alpha^{\mathsf{Sets}}_{X',Y',Z'}"] \\
X\times(Y\times Z) \arrow[r, "f\times(g\times h)"'] & X'\times(Y'\times Z')
\end{tikzcd}
$$
commutes. Indeed, this diagram acts on elements as
$$
\begin{tikzcd}
((x,y),z) \arrow[d, mapsto] & & ((x,y),z) \arrow[r, mapsto] & ((f(x),g(y)),h(z)) \arrow[d, mapsto] \\
(x,(y,z)) \arrow[r, mapsto] & (f(x),(g(y),h(z))) & & (f(x),(g(y),h(z)))
\end{tikzcd}
$$
and hence indeed commutes, showing α^{Sets} to be a natural transformation.
Being a Natural Isomorphism: Since α^{Sets} is natural and $\alpha^{\mathsf{Sets},-1}$ is a componentwise inverse to α^{Sets}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\alpha^{\mathsf{Sets},-1}$ is also natural. Thus α^{Sets} is a natural isomorphism. □

01NT 5.1.5 The Left Unitor

01NU **Definition 5.1.5.1.1.** The **left unitor of the product of sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}}\colon \times \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}} \qquad \begin{tikzcd} \mathrm{pt} \times \mathsf{Sets} \arrow[r, "\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Sets}}"] \\ & \mathsf{Sets}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}}_X\colon \mathrm{pt} \times X \overset{\sim}{\dashrightarrow} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\lambda^{\mathsf{Sets}}_X(\star, x) \overset{\text{def}}{=} x$$

for each $(\star, x) \in \mathrm{pt} \times X$.

Proof. Invertibility: The inverse of $\lambda^{\mathsf{Sets}}_X$ is the morphism

$$\lambda^{\mathsf{Sets},-1}_X\colon X \overset{\sim}{\dashrightarrow} \mathrm{pt} \times X$$

defined by

$$\lambda^{\mathsf{Sets},-1}_X(x) \overset{\text{def}}{=} (\star, x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} [\lambda^{\mathsf{Sets},-1}_X \circ \lambda^{\mathsf{Sets}}_X](\mathrm{pt}, x) &= \lambda^{\mathsf{Sets},-1}_X(\lambda^{\mathsf{Sets}}_X(\mathrm{pt}, x)) \\ &= \lambda^{\mathsf{Sets},-1}_X(x) \\ &= (\mathrm{pt}, x) \\ &= [\mathrm{id}_{\mathrm{pt}\times X}](\mathrm{pt}, x) \end{aligned}$$

  for each $(\mathrm{pt}, x) \in \mathrm{pt} \times X$, and therefore we have

$$\lambda^{\mathsf{Sets},-1}_X \circ \lambda^{\mathsf{Sets}}_X = \mathrm{id}_{\mathrm{pt}\times X}.$$

- *Invertibility II*. We have

$$\begin{aligned}[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}](x) &= \lambda_X^{\mathsf{Sets}}(\lambda_X^{\mathsf{Sets},-1}(x))\\ &= \lambda_X^{\mathsf{Sets},-1}(\mathrm{pt}, x)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

  for each $x \in X$, and therefore we have

$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X .$$

Therefore $\lambda_X^{\mathsf{Sets}}$ is indeed an isomorphism.
Naturality: We need to show that, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd} \mathrm{pt} \times X \arrow[r, "\mathrm{id}_{\mathrm{pt}} \times f"] \arrow[d, "\lambda_X^{\mathsf{Sets}}"'] & \mathrm{pt} \times Y \arrow[d, "\lambda_Y^{\mathsf{Sets}}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (\star, x) \arrow[d, mapsto] & \\ x \arrow[r, mapsto] & f(x) \end{tikzcd} \qquad \begin{tikzcd} (\star, x) \arrow[r, mapsto] & (\star, f(x)) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$

and hence indeed commutes. Therefore λ^{Sets} is a natural transformation.
Being a Natural Isomorphism: Since λ^{Sets} is natural and $\lambda^{\mathsf{Sets},-1}$ is a componentwise inverse to λ^{Sets}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus λ^{Sets} is a natural isomorphism. □

01NV 5.1.6 The Right Unitor

01NW **Definition 5.1.6.1.1.** The **right unitor of the product of sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}}\colon \times \circ (\mathrm{id}\times \mathbb{1}^{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets}\times\mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets}\times\mathsf{Sets} \arrow[d, "\times"] \arrow[d, phantom, shift right=4ex, "\Leftarrow \rho^{\mathsf{Sets}}"'] \\ & \mathsf{Sets}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}}_X\colon X\times\mathrm{pt} \overset{\sim}{\dashrightarrow} X$$

at $X\in\mathrm{Obj}(\mathsf{Sets})$ is given by

$$\rho^{\mathsf{Sets}}_X(x,\star) \overset{\text{def}}{=} x$$

for each $(x,\star)\in X\times\mathrm{pt}$.

Proof. *Invertibility*: The inverse of ρ^{Sets}_X is the morphism

$$\rho^{\mathsf{Sets},-1}_X\colon X \overset{\sim}{\dashrightarrow} X\times\mathrm{pt}$$

defined by

$$\rho^{\mathsf{Sets},-1}_X(x) \overset{\text{def}}{=} (x,\star)$$

for each $x\in X$. Indeed:

- *Invertibility I*. We have

$$\begin{aligned} [\rho^{\mathsf{Sets},-1}_X \circ \rho^{\mathsf{Sets}}_X](x,\star) &= \rho^{\mathsf{Sets},-1}_X(\rho^{\mathsf{Sets}}_X(x,\star)) \\ &= \rho^{\mathsf{Sets},-1}_X(x) \\ &= (x,\star) \\ &= [\mathrm{id}_{X\times\mathrm{pt}}](x,\star) \end{aligned}$$

  for each $(x,\star)\in X\times\mathrm{pt}$, and therefore we have

$$\rho^{\mathsf{Sets},-1}_X \circ \rho^{\mathsf{Sets}}_X = \mathrm{id}_{X\times\mathrm{pt}}.$$

- *Invertibility II*. We have

$$\begin{aligned}[\rho^{\mathsf{Sets}}_X \circ \rho^{\mathsf{Sets},-1}_X](x) &= \rho^{\mathsf{Sets}}_X(\rho^{\mathsf{Sets},-1}_X(x))\\ &= \rho^{\mathsf{Sets},-1}_X(x,\star)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

  for each $x \in X$, and therefore we have

$$\rho^{\mathsf{Sets}}_X \circ \rho^{\mathsf{Sets},-1}_X = \mathrm{id}_X .$$

Therefore ρ^{Sets}_X is indeed an isomorphism.
Naturality: We need to show that, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd} X \times \mathrm{pt} \arrow[r, "f\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\rho^{\mathsf{Sets}}_X"'] & Y \times \mathrm{pt} \arrow[d, "\rho^{\mathsf{Sets}}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x,\star) \arrow[d, mapsto] & \\ x \arrow[r, mapsto] & f(x) \end{tikzcd} \qquad \begin{tikzcd} (x,\star) \arrow[r, mapsto] & (f(x),\star) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$

and hence indeed commutes. Therefore ρ^{Sets} is a natural transformation.
Being a Natural Isomorphism: Since ρ^{Sets} is natural and $\rho^{\mathsf{Sets},-1}$ is a componentwise inverse to ρ^{Sets}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\rho^{\mathsf{Sets},-1}$ is also natural. Thus ρ^{Sets} is a natural isomorphism. □

01NX 5.1.7 The Symmetry

01NY **Definition 5.1.7.1.1.** The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets}\times\mathsf{Sets} \arrow[rr, "\times"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}"'] & \arrow[d, phantom, "\overset{\Vert}{\underset{\Downarrow}{\sigma^{\mathsf{Sets}}}}"] & \mathsf{Sets}, \\ & \mathsf{Sets}\times\mathsf{Sets} \arrow[ur, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}}_{X,Y}\colon X\times Y \overset{\sim}{\dashrightarrow} Y\times X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ is defined by

$$\sigma^{\mathsf{Sets}}_{X,Y}(x,y) \overset{\text{def}}{=} (y,x)$$

for each $(x,y)\in X\times Y$.

Proof. Invertibility: The inverse of $\sigma^{\mathsf{Sets}}_{X,Y}$ is the morphism

$$\sigma^{\mathsf{Sets},-1}_{X,Y}\colon Y\times X \overset{\sim}{\dashrightarrow} X\times Y$$

defined by

$$\sigma^{\mathsf{Sets},-1}_{X,Y}(y,x) \overset{\text{def}}{=} (x,y)$$

for each $(y,x)\in Y\times X$. Indeed:

- *Invertibility I*. We have

$$\begin{aligned} [\sigma^{\mathsf{Sets},-1}_{X,Y}\circ\sigma^{\mathsf{Sets}}_{X,Y}](x,y) &\overset{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}(\sigma^{\mathsf{Sets}}_{X,Y}(x,y)) \\ &\overset{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}(y,x) \\ &\overset{\text{def}}{=} (x,y) \\ &\overset{\text{def}}{=} [\mathrm{id}_{X\times Y}](x,y) \end{aligned}$$

  for each $(x,y)\in X\times Y$, and therefore we have

$$\sigma^{\mathsf{Sets},-1}_{X,Y}\circ\sigma^{\mathsf{Sets}}_{X,Y} = \mathrm{id}_{X\times Y}.$$

- *Invertibility II.* We have

$$
\begin{aligned}
[\sigma^{\mathsf{Sets}}_{X,Y} \circ \sigma^{\mathsf{Sets},-1}_{X,Y}](y,x) &\stackrel{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}(\sigma^{\mathsf{Sets}}_{X,Y}(y,x))\\
&\stackrel{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}(x,y)\\
&\stackrel{\text{def}}{=} (y,x)\\
&\stackrel{\text{def}}{=} [\mathrm{id}_{Y\times X}](y,x)
\end{aligned}
$$

  for each $(y,x) \in Y \times X$, and therefore we have

$$\sigma^{\mathsf{Sets}}_{X,Y} \circ \sigma^{\mathsf{Sets},-1}_{X,Y} = \mathrm{id}_{Y\times X}\,.$$

Therefore $\sigma^{\mathsf{Sets}}_{X,Y}$ is indeed an isomorphism.
Naturality: We need to show that, given functions

$$
\begin{aligned}
f\colon X &\to A,\\
g\colon Y &\to B
\end{aligned}
$$

the diagram

$$
\begin{tikzcd}
X \times Y \arrow[r, "f\times g"] \arrow[d, "\sigma^{\mathsf{Sets}}_{X,Y}"'] & A \times B \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"] \\
Y \times X \arrow[r, "g\times f"'] & B \times A
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(x,y) \arrow[d, mapsto] & \\
(y,x) \arrow[r, mapsto] & (g(y),f(x))
\end{tikzcd}
\qquad
\begin{tikzcd}
(x,y) \arrow[r, mapsto] & (f(x),g(y)) \arrow[d, mapsto] \\
 & (g(y),f(x))
\end{tikzcd}
$$

and hence indeed commutes, showing σ^{Sets} to be a natural transformation.
Being a Natural Isomorphism: Since σ^{Sets} is natural and $\sigma^{\mathsf{Sets},-1}$ is a componentwise inverse to σ^{Sets}, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\sigma^{\mathsf{Sets},-1}$ is also natural. Thus σ^{Sets} is a natural isomorphism. □

01NZ 5.1.8 The Diagonal

01P0 **Definition 5.1.8.1.1.** The **diagonal of the product of sets** is the natural transformation

$$\Delta\colon \mathrm{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets} \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \arrow[d, phantom, "\Downarrow\Delta"] & \mathsf{Sets} \\ & \mathsf{Sets} \times \mathsf{Sets}, \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\Delta_X\colon X \longrightarrow X \times X$$

at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\Delta_X(x) \overset{\mathrm{def}}{=} (x, x)$$

for each $x \in X$.

Proof. We need to show that, given a function $f\colon X \longrightarrow Y$, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[d, mapsto] & \\ (x, x) \arrow[r, mapsto] & (f(x), f(x)) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] & f(x) \arrow[d, mapsto] \\ & (f(x), f(x)) \end{tikzcd}$$

and hence indeed commutes, showing Δ to be natural. □

01P1 **Proposition 5.1.8.1.2.** Let X be a set.

01P2 1. *Monoidality.* The diagonal map

$$\Delta\colon \mathrm{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

is a monoidal natural transformation:

01P3 (a) *Compatibility With Strong Monoidality Constraints.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\Delta_X \times \Delta_Y"] \arrow[dr, "\Delta_{X\times Y}"'] & (X \times X) \times (Y \times Y) \arrow[d, dashed, "\wr"] \\ & (X \times Y) \times (X \times Y) \end{tikzcd}$$

commutes.

01P4 (b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\Delta_{\mathrm{pt}}"] \arrow[dr, equal] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\lambda^{\mathsf{Sets}}_{\mathrm{pt}}"] \\ & \mathrm{pt} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "\Delta_{\mathrm{pt}}"] \arrow[dr, equal] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\rho^{\mathsf{Sets}}_{\mathrm{pt}}"] \\ & \mathrm{pt} \end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned} \Delta_{\mathrm{pt}} &= \lambda^{\mathsf{Sets},-1}_{\mathrm{pt}} \\ &= \rho^{\mathsf{Sets},-1}_{\mathrm{pt}}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned} \lambda^{\mathsf{Sets}}_{\mathrm{pt}} &= \rho^{\mathsf{Sets}}_{\mathrm{pt}}, \\ \lambda^{\mathsf{Sets},-1}_{\mathrm{pt}} &= \rho^{\mathsf{Sets},-1}_{\mathrm{pt}} \end{aligned}$$

are always true in any monoidal category by Monoidal Categories, **??** of **??**.

01P5 2. *The Diagonal of the Unit.* The component

$$\Delta_{\mathrm{pt}}\colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \mathrm{pt} \times \mathrm{pt}$$

of Δ at pt is an isomorphism.

Proof. *Item 1, Monoidality*: We claim that Δ is indeed monoidal:

024S 1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\Delta_X \times \Delta_Y"] \arrow[dr, "\Delta_{X\times Y}"'] & (X \times X) \times (Y \times Y) \arrow[d, dashed, "\wr"] \\ & (X \times Y) \times (X \times Y) \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x, y) \arrow[r, mapsto] & ((x, x), (y, y)) \arrow[d, mapsto] \\ & ((x, y), (x, y)) \end{tikzcd} \qquad \begin{tikzcd} (x, y) \arrow[dr, mapsto] & \\ & ((x, y), (x, y)) \end{tikzcd}$$

and hence indeed commutes.

024T 2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \overset{\text{def}}{=} (\star, x)$$

for each $x \in X$, so when $X = \mathrm{pt}$, we have

$$\lambda_{\mathrm{pt}}^{\mathsf{Sets},-1}(\star) \overset{\text{def}}{=} (\star, \star),$$

and also

$$\Delta_{\mathrm{pt}}^{\mathsf{Sets}}(\star) \overset{\text{def}}{=} (\star, \star),$$

so we have $\Delta_{\mathrm{pt}} = \lambda_{\mathrm{pt}}^{\mathsf{Sets},-1}$.

This finishes the proof.

Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to $\times$, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor. □

01P6 5.1.9 The Monoidal Category of Sets and Products

01P7 **Proposition 5.1.9.1.1.** The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Monoidal Product.* The product functor

$$\times \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.
- *The Internal Hom.* The internal Hom functor

$$\mathsf{Sets} \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Constructions With Sets, Item 1 of Definition 4.3.5.1.2.
- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

  of Definition 5.1.3.1.1.
- *The Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.1.4.1.1.
- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \times \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

  of Definition 5.1.5.1.1.
- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \times \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

  of Definition 5.1.6.1.1.

- *The Symmetry.* The natural isomorphism
$$\sigma^{\mathsf{Sets}}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$
of Definition 5.1.7.1.1.
- *The Diagonals.* The monoidal natural transformation
$$\Delta\colon \ \mathrm{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$
of Definition 5.1.8.1.1.

Proof. The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \times (X \times Y)) \times Z \arrow[dr, "\alpha^{\mathsf{Sets}}_{W,X\times Y,Z}"] & \\
((W \times X) \times Y) \times Z \arrow[ur, "\alpha^{\mathsf{Sets}}_{W,X,Y} \times \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}}_{W\times X,Y,Z}"'] & & W \times ((X \times Y) \times Z) \arrow[d, "\mathrm{id}_W \times \alpha^{\mathsf{Sets}}_{X,Y,Z}"] \\
(W \times X) \times (Y \times Z) \arrow[rr, "\alpha^{\mathsf{Sets}}_{W,X,Y\times Z}"'] & & W \times (X \times (Y \times Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(((w,x),y),z) \arrow[d, mapsto] \\
((w,x),(y,z)) \arrow[r, mapsto] & (w,(x,(y,z)))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & ((w,(x,y)),z) \arrow[dr, mapsto] & \\
(((w,x),y),z) \arrow[ur, mapsto] & & (w,((x,y),z)) \arrow[d, mapsto] \\
 & & (w,(x,(y,z))),
\end{tikzcd}$$

and thus the pentagon identity is satisfied.

The Triangle Identity: Let X and Y be sets. We have to show that the diagram

$$\begin{tikzcd}
(X \times \mathrm{pt}) \times Y \arrow[rr, "\alpha^{\mathsf{Sets}}_{X,\mathrm{pt},Y}"] \arrow[dr, "\rho^{\mathsf{Sets}}_X \times \mathrm{id}_Y"'] & & X \times (\mathrm{pt} \times Y) \arrow[dl, "\mathrm{id}_X \times \lambda^{\mathsf{Sets}}_Y"] \\
& X \times Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
((x,\star),y) \arrow[dr, mapsto] & & & ((x,\star),y) \arrow[rr, mapsto] & & (x,(\star,y)) \arrow[dl, mapsto] \\
& (x,y) & & & (x,y) &
\end{tikzcd}$$

and thus the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
& (X \times Y) \times Z \arrow[dl, "\alpha^{\mathsf{Sets}}_{X,Y,Z}"'] \arrow[dr, "\sigma^{\mathsf{Sets}}_{X,Y} \times \mathrm{id}_Z"] & \\
X \times (Y \times Z) \arrow[d, "\sigma^{\mathsf{Sets}}_{X,Y\times Z}"'] & & (Y \times X) \times Z \arrow[d, "\alpha^{\mathsf{Sets}}_{Y,X,Z}"] \\
(Y \times Z) \times X \arrow[dr, "\alpha^{\mathsf{Sets}}_{Y,Z,X}"'] & & Y \times (X \times Z) \arrow[dl, "\mathrm{id}_Y \times \sigma^{\mathsf{Sets}}_{X,Z}"] \\
& Y \times (Z \times X) &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
& ((x,y),z) \arrow[dl, mapsto] & ((x,y),z) \arrow[dr, mapsto] & \\
(x,(y,z)) \arrow[d, mapsto] & & & ((y,x),z) \arrow[d, mapsto] \\
((y,z),x) \arrow[dr, mapsto] & & & (y,(x,z)) \arrow[dl, mapsto] \\
& (y,(z,x)) & (y,(z,x)) &
\end{tikzcd}$$

and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
& X \times (Y \times Z) \arrow[dl, "(\alpha^{\mathsf{Sets}}_{X,Y,Z})^{-1}"'] \arrow[dr, "\mathrm{id}_X \times \sigma^{\mathsf{Sets}}_{Y,Z}"] & \\
(X \times Y) \times Z \arrow[d, "\sigma^{\mathsf{Sets}}_{X \times Y, Z}"'] & & X \times (Z \times Y) \arrow[d, "(\alpha^{\mathsf{Sets}}_{X,Z,Y})^{-1}"] \\
Z \times (X \times Y) \arrow[dr, "(\alpha^{\mathsf{Sets}}_{Z,X,Y})^{-1}"'] & & (X \times Z) \times Y \arrow[dl, "\sigma^{\mathsf{Sets}}_{X,Z} \times \mathrm{id}_Y"] \\
& (Z \times X) \times Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (x,(y,z)) \arrow[dl, mapsto] & (x,(y,z)) \arrow[dr, mapsto] & \\
((x,y),z) \arrow[d, mapsto] & & & (x,(z,y)) \arrow[d, mapsto] \\
(z,(x,y)) \arrow[dr, mapsto] & & & ((x,z),y) \arrow[dl, mapsto] \\
 & ((z,x),y) & ((z,x),y) &
\end{tikzcd}$$

and thus the right hexagon identity is satisfied.

Monoidal Closedness: This follows from Constructions With Sets, Item 2 of Definition 4.3.5.1.2.

Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 5.1.8.1.2. □

01P8 5.1.10 The Universal Property of $(\mathsf{Sets}, \times, \mathbf{pt})$

01P9 **Theorem 5.1.10.1.1.** The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

01PA 1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

of Sets admits an internal Hom $[-_1, -_2]_{\mathsf{Sets}}$.

01PB 2. *The Unit Object Is pt.* We have $\mathbb{1}_{\mathsf{Sets}} \cong \mathrm{pt}$.

More precisely, the full subcategory of the category $\mathcal{M}^{\mathrm{cld}}_{\mathbb{E}_\infty}(\mathsf{Sets})$ of **??** spanned by the closed symmetric monoidal categories $\left(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, [-_1, -_2]_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\right)$ satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

Proof. Unwinding the Statement: Let $(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, [-_1, -_2]_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda', \rho', \sigma')$ be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$\mathrm{id}_{\mathsf{Sets}} \colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \mathrm{id}^{\otimes}_{\mathsf{Sets}} \colon \quad A \otimes_{\mathsf{Sets}} B &\overset{\sim}{\dashrightarrow} A \times B, \\ \mathrm{id}^{\mathrm{Hom}}_{\mathsf{Sets}} \colon \quad [A, B]_{\mathsf{Sets}} &\overset{\sim}{\dashrightarrow} \mathsf{Sets}(A, B), \\ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \colon \quad \mathbb{1}_{\mathsf{Sets}} &\overset{\sim}{\dashrightarrow} \mathrm{pt}, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from $(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, [-_1, -_2]_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda', \rho', \sigma')$ to the closed symmetric monoidal category $\left(\mathsf{Sets}, \times, \mathsf{Sets}(-_1, -_2), \mathbb{1}_{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\right)$ of Definition 5.1.9.1.1.

Constructing an Isomorphism $[-_1, -_2]_{\mathsf{Sets}} \cong \mathsf{Sets}(-_1, -_2)$: By **??**, we have a natural isomorphism

$$\mathsf{Sets}(\mathrm{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong \mathsf{Sets}(-_1, -_2).$$

By Constructions With Sets, Item 3 of Definition 4.3.5.1.2, we also have a natural isomorphism

$$\mathsf{Sets}(\mathrm{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}(-_1, -_2) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Given $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we will write

$$\mathrm{id}^{\mathrm{Hom}}_{A,B} \colon \mathsf{Sets}(A, B) \overset{\sim}{\dashrightarrow} [A, B]_{\mathsf{Sets}}$$

for the component of this isomorphism at (A, B).

Constructing an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Since $\otimes_{\mathsf{Sets}}$ is adjoint in each variable to $[-_1, -_2]_{\mathsf{Sets}}$ by assumption and $\times$ is adjoint in each variable to $\mathsf{Sets}(-_1, -_2)$ by Constructions With Sets, Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (**??**) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -,$$

$$- \otimes_{\mathsf{Sets}} B \cong B \times -.$$

By **??**, we then have $\otimes_{\mathsf{Sets}} \cong \times$. We will write

$$\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \colon A \otimes_{\mathsf{Sets}} B \overset{\sim}{\dashrightarrow} A \times B$$

for the component of this isomorphism at (A, B).

Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Alternatively, we may construct a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ as follows:

01PC 1. Let $A \in \mathrm{Obj}(\mathsf{Sets})$.

01PD 2. Since $\otimes_{\mathsf{Sets}}$ is part of a closed monoidal structure, it preserves colimits in each variable by **??**.

01PE 3. Since $A \cong \coprod_{a\in A} \mathrm{pt}$ and $\otimes_{\mathsf{Sets}}$ preserves colimits in each variable, we have

$$\begin{aligned} A \otimes_{\mathsf{Sets}} B &\cong \Big(\coprod_{a\in A} \mathrm{pt}\Big) \otimes_{\mathsf{Sets}} B \\ &\cong \coprod_{a\in A} (\mathrm{pt} \otimes_{\mathsf{Sets}} B) \\ &\cong \coprod_{a\in A} B \\ &\cong A \times B, \end{aligned}$$

naturally in $B \in \mathrm{Obj}(\mathsf{Sets})$, where we have used that pt is the monoidal unit for $\otimes_{\mathsf{Sets}}$. Thus $A \otimes_{\mathsf{Sets}} - \cong A \times -$ for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

01PF 4. Similarly, $- \otimes_{\mathsf{Sets}} B \cong - \times B$ for each $B \in \mathrm{Obj}(\mathsf{Sets})$.

01PG 5. By **??**, we then have $\otimes_{\mathsf{Sets}} \cong \times$.

Below, we'll show that if a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \colon A \otimes_{\mathsf{Sets}} B \longrightarrow A \times B$ from before.

Constructing an Isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}} \longrightarrow \mathrm{pt}$: We define an isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}} \longrightarrow \mathrm{pt}$ as the composition

$$\mathbb{1}_{\mathsf{Sets}} \xrightarrow[\sim]{\rho^{\mathsf{Sets},-1}_{\mathbb{1}_{\mathsf{Sets}}}} \mathbb{1}_{\mathsf{Sets}} \times \mathrm{pt} \xrightarrow[\sim]{\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathbb{1}_{\mathsf{Sets}}}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt} \xrightarrow[\sim]{\lambda'_{\mathrm{pt}}} \mathrm{pt}$$

in Sets.

Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$\begin{tikzcd}
& \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A}"] & \mathrm{pt} \times A \arrow[dr, "\lambda^{\mathsf{Sets}}_{A}"] & \\
\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_{A}"] \arrow[rrr, "\lambda'_{A}"'] & & & A
\end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd}
& \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\lambda^{\mathsf{Sets}}_{\mathrm{pt}}"] & \\
\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"] \arrow[rrr, "\lambda'_{\mathrm{pt}}"'] & & & \mathrm{pt},
\end{tikzcd}$$

corresponding to the case $A = \mathrm{pt}$, commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"] & \\
\mathrm{pt} \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrr, "\lambda'^{,-1}_{\mathrm{pt}}"'] & & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt}.
\end{tikzcd}$$

Now, let $A \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, and consider the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \times [a]"] \arrow[dr, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_{\mathrm{pt}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} [a]"] & \\
\mathrm{pt} \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrr, "\lambda^{\prime,-1}_{\mathrm{pt}}"] \arrow[dd, "{[a]}"'] \arrow[dr, phantom, "(3)"] & \arrow[dr, phantom, "(1)", pos=0.3] \arrow[dr, phantom, "(5)", pos=0.8] & \arrow[dr, phantom, "(4)"] & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}}} \times [a]"] \\
 & \mathrm{pt} \times A \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A}"] \arrow[dr, phantom, "(2)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A"] & \\
A \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{A}"] \arrow[rrr, "\lambda^{\prime,-1}_{A}"] & & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A.
\end{tikzcd}$$

Since:

- Subdiagram (5) commutes by the naturality of $\lambda^{\prime,-1}$.
- Subdiagram $(\dagger)$ commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathsf{Sets},-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & & \mathrm{pt} \times A \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A}"] & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_A"] & \\
\mathrm{pt} \arrow[r, "{[a]}"] & A \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{A}"] \arrow[rrr, "\lambda^{\prime,-1}_{A}"'] & & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A
\end{tikzcd}$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\lambda^{\prime,-1}_{A}(a) = [\lambda^{\prime,-1}_{A} \circ [a]](\star)$$

$$\begin{aligned}&= [(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \lambda^{\mathsf{Sets},-1}_A \circ [a]](\star)\\ &= [(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \lambda^{\mathsf{Sets},-1}_A](a)\end{aligned}$$

for each $a \in A$, and thus we have

$$\lambda'^{,-1}_A = (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \lambda^{\mathsf{Sets},-1}_A.$$

Taking inverses then gives

$$\lambda'_A = \lambda^{\mathsf{Sets}}_A \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A} \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A),$$

showing that the diagram

$$\begin{tikzcd} & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A}"] & \mathrm{pt} \times A \arrow[dr, "\lambda^{\mathsf{Sets}}_A"] & \\ \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_A"] \arrow[rrr, "\lambda'_A"'] & & & A \end{tikzcd}$$

indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We can use the same argument we used to prove the monoidal left unity of the isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ above. For completeness, we repeat it below.

We have to show that the diagram

$$\begin{tikzcd} & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,\mathrm{pt}}"] & A \times \mathrm{pt} \arrow[dr, "\rho^{\mathsf{Sets}}_A"] & \\ A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[ur, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[rrr, "\rho'_A"'] & & & A \end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd} & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\rho^{\mathsf{Sets}}_{\mathrm{pt}}"] & \\ \mathrm{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[rrr, "\rho'_{\mathrm{pt}}"'] & & & \mathrm{pt}, \end{tikzcd}$$

corresponding to the case $A = \mathrm{pt}$, commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] & \\
\mathrm{pt} \arrow[ur, "\rho^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrr, "\rho'^{,-1}_{\mathrm{pt}}"'] & & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}}.
\end{tikzcd}$$

Now, let $A \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, and consider the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \times [a]"] \arrow[rr, phantom, shift right=4, "(\dagger)"] & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_{\mathrm{pt}} \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} [a]"] & \\
\mathrm{pt} \arrow[ur, "\rho^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrrr, "\rho'^{,-1}_{\mathrm{pt}}"] \arrow[dd, "{[a]}"'] & \arrow[d, phantom, "(3)"] & (1) & \arrow[d, phantom, "(4)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}}} \times [a]"] \\
 & A \times \mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,\mathrm{pt}}"] \arrow[rr, phantom, shift left=6, "(5)"] \arrow[rr, phantom, shift right=4, "(2)"] & & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] & \\
A \arrow[ur, "\rho^{\mathsf{Sets},-1}_{A}"] \arrow[rrrr, "\rho'^{,-1}_{A}"] & & & & A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}}.
\end{tikzcd}$$

Since:

- Subdiagram (5) commutes by the naturality of $\rho'^{,-1}$.
- Subdiagram $(\dagger)$ commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\mathsf{Sets},-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & & A \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,\mathrm{pt}}"] & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] & \\
\mathrm{pt} \arrow[r, "{[a]}"] & A \arrow[ur, "\rho^{\mathsf{Sets},-1}_A"] \arrow[rrr, "\rho'^{,-1}_A"'] & & & A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}}
\end{tikzcd}$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{aligned}
\rho'^{,-1}_A(a) &= [\rho'^{,-1}_A \circ [a]](\star) \\
&= [(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \rho^{\mathsf{Sets},-1}_A \circ [a]](\star) \\
&= [(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \rho^{\mathsf{Sets},-1}_A](a)
\end{aligned}$$

for each $a \in A$, and thus we have

$$\rho'^{,-1}_A = (\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \rho^{\mathsf{Sets},-1}_A.$$

Taking inverses then gives

$$\rho'_A = \rho^{\mathsf{Sets}}_A \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A} \circ (\mathrm{id}_A \times \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}),$$

showing that the diagram

$$\begin{tikzcd}
 & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,\mathrm{pt}}"] & A \times \mathrm{pt} \arrow[dr, "\rho^{\mathsf{Sets}}_A"] & \\
A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[ur, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[rrr, "\rho'_A"'] & & & A
\end{tikzcd}$$

indeed commutes.

Monoidality of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$\begin{tikzcd}[column sep=tiny]
 & (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] \arrow[dr, "\alpha'_{A,B,C}"] & \\
(A \times B) \otimes_{\mathsf{Sets}} C \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A \times B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[d, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,C}"] \\
(A \times B) \times C \arrow[dr, "\alpha^{\mathsf{Sets}}_{A,B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B \times C}"] \\
 & A \times (B \times C) &
\end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd}[column sep=tiny]
 & (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"'] \arrow[dr, "\alpha'_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] & \\
(\mathrm{pt} \times \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt} \times \mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt} \times \mathrm{pt}) \times \mathrm{pt} \arrow[dr, "\alpha^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \times \mathrm{pt}) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt} \times \mathrm{pt}}"] \\
 & \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \arrow[d, "!_{\mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt})}" description] & \\
 & \mathrm{pt} &
\end{tikzcd}$$

commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map $!_{\mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt})} \colon \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \to \mathrm{pt}$ is an isomorphism (e.g. having inverse $\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}} \circ \lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}$), it follows that the diagram

$$\begin{tikzcd}
 & (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"'] \arrow[dr, "\alpha'_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] & \\
(\mathrm{pt} \times \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt} \times \mathrm{pt}) \times \mathrm{pt} \arrow[dr, "\alpha^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \times \mathrm{pt}) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] \\
 & \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) &
\end{tikzcd}$$

also commutes. Taking inverses, we see that the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] & \\
(\mathrm{pt} \times \mathrm{pt}) \times \mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & (\dagger) & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \times \mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt} \times \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \arrow[dl, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] \\
 & (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} &
\end{tikzcd}$$

commutes as well. Now, let $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, let $c \in C$, and consider the diagram

$$
\begin{tikzcd}
& \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] \arrow[rr, "{[a]\times([b]\times[c])}"] & & A \times (B \times C) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{A,B,C}"'] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}"] \\
(\mathrm{pt} \times \mathrm{pt}) \times \mathrm{pt} \arrow[rr, "{([a]\times[b])\times[c]}"] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \times \mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[rr, "{[a]\otimes_{\mathsf{Sets}}([b]\times[c])}"] & (A \times B) \times C \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C}"] & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[d, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}"] \\
(\mathrm{pt} \times \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"'] \arrow[rr, "{([a]\times[b])\otimes_{\mathsf{Sets}}[c]}"] & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \arrow[d, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] \arrow[rr, "{[a]\otimes_{\mathsf{Sets}}([b]\otimes_{\mathsf{Sets}}[c])}"] & (A \times B) \otimes_{\mathsf{Sets}} C \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[d, "\alpha'^{,-1}_{A,B,C}"] \\
& (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[rr, "{([a]\otimes_{\mathsf{Sets}}[b])\otimes_{\mathsf{Sets}}[c]}"'] & & (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C
\end{tikzcd}
$$

which we partition into subdiagrams as follows:

$$
\begin{tikzcd}
& \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] & \\
(\mathrm{pt} \times \mathrm{pt}) \times \mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & (\dagger) & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \times \mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt} \times \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \arrow[dl, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] \\
& (\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}) \otimes_{\mathsf{Sets}} \mathrm{pt} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[rr, "{[a]\times([b]\times[c])}"] \arrow[dr, phantom, "(1)"] & & A\times(B\times C) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{A,B,C}"] \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[rr, "{([a]\times[b])\times[c]}"] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] \arrow[drr, phantom, "(2)"] & & (A\times B)\times C \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C}"] & \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rr, "{([a]\times[b])\otimes_{\mathsf{Sets}}[c]}"] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] & & (A\times B)\otimes_{\mathsf{Sets}} C \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}\otimes_{\mathsf{Sets}}\mathrm{id}_C"] \arrow[dl, phantom, "(3)"] & \\
& (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rr, "{([a]\otimes_{\mathsf{Sets}}[b])\otimes_{\mathsf{Sets}}[c]}"'] & & (A\otimes_{\mathsf{Sets}} B)\otimes_{\mathsf{Sets}} C
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[rr, "{[a]\times([b]\times[c])}"] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"'] \arrow[drr, phantom, "(4)"] & & A\times(B\times C) \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}"] & \\
& \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[rr, "{[a]\otimes_{\mathsf{Sets}}([b]\times[c])}"] \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"'] \arrow[drr, phantom, "(5)"] & & A\otimes_{\mathsf{Sets}}(B\times C) \arrow[d, "\mathrm{id}_A\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}"] \\
& \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[rr, "{[a]\otimes_{\mathsf{Sets}}([b]\otimes_{\mathsf{Sets}}[c])}"] \arrow[dl, "\alpha'^{-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, phantom, "(6)"] & & A\otimes_{\mathsf{Sets}}(B\otimes_{\mathsf{Sets}} C). \arrow[dl, "\alpha'^{-1}_{A,B,C}"] \\
(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rr, "{([a]\otimes_{\mathsf{Sets}}[b])\otimes_{\mathsf{Sets}}[c]}"'] & & (A\otimes_{\mathsf{Sets}} B)\otimes_{\mathsf{Sets}} C &
\end{tikzcd}
$$

Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{\mathsf{Sets},-1}$.
- Subdiagram (2) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (3) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (†) commutes, as proved above.

- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (6) commutes by the naturality of $\alpha'^{,-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \arrow[d, "{[a]\times([b]\times[c])}"] & \\
 & A \times (B \times C) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{A,B,C}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}"] & \\
(A \times B) \times C \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[d, "\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}"] \\
(A \times B) \otimes_{\mathsf{Sets}} C \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] & & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[dl, "\alpha'^{,-1}_{A,B,C}"] \\
 & (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C &
\end{tikzcd}$$

also commutes. We then have

$$\begin{aligned}
\Big[(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C} & \\
\circ\, \alpha^{\mathsf{Sets},-1}_{A,B,C}\Big](a,(b,c)) &= \Big[(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C} \\
&\qquad \circ\, \alpha^{\mathsf{Sets},-1}_{A,B,C} \circ ([a] \times ([b] \times [c]))\Big](\star,(\star,\star)) \\
&= \Big[\alpha'^{,-1}_{A,B,C} \circ (\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}) \\
&\qquad \circ\, \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C} \circ ([a] \times ([b] \times [c]))\Big](\star,(\star,\star)) \\
&= [\alpha'^{,-1}_{A,B,C} \circ (\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}](a,(b,c))
\end{aligned}$$

for each $(a,(b,c)) \in A \times (B \times C)$, and thus we have

$$(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C} \circ \alpha^{\mathsf{Sets},-1}_{A,B,C} = \alpha'^{,-1}_{A,B,C} \circ (\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}.$$

Taking inverses then gives

$$\alpha^{\mathsf{Sets}}_{A,B,C} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|A\times B,C} \circ (\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C) = \mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B\times C} \circ (\mathrm{id}_A \times \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,C}) \circ \alpha'_{A,B,C},$$

showing that the diagram

$$\begin{tikzcd}
 & (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] \arrow[dr, "\alpha'_{A,B,C}"] & \\
(A \times B) \otimes_{\mathsf{Sets}} C \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A\times B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[d, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,C}"] \\
(A \times B) \times C \arrow[dr, "\alpha^{\mathsf{Sets}}_{A,B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B\times C}"] \\
 & A \times (B \times C) &
\end{tikzcd}$$

indeed commutes.

Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$\begin{tikzcd}
A \otimes_{\mathsf{Sets}} B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B}"] \arrow[d, "\sigma'_{A,B}"'] & A \times B \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"] \\
B \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A}"'] & B \times A
\end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd}
\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, "\sigma'_{\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\sigma^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt}}"] & \\
\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "!_{\mathrm{pt}\times\mathrm{pt}}"] & \\
 & & \mathrm{pt}
\end{tikzcd}$$

commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map $!_{\mathrm{pt}\times\mathrm{pt}} \colon \mathrm{pt} \times \mathrm{pt} \longrightarrow \mathrm{pt}$ is invertible (e.g. with inverse $\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}$), the diagram

$$\begin{tikzcd}
\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, "\sigma'_{\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\sigma^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt}}"] \\
\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \times \mathrm{pt}
\end{tikzcd}$$

also commutes. Taking inverses, we see that the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, "\sigma^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[d, "\sigma'^{,-1}_{\mathrm{pt},\mathrm{pt}}"] \\
\mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt}
\end{tikzcd}$$

commutes as well. Now, let $A, B \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, and consider the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dr, "{[b]\times[a]}"] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "{[b]\otimes_{\mathsf{Sets}}[a]}"] \arrow[dd, "\sigma'^{,-1}_{\mathrm{pt},\mathrm{pt}}" near start] & \\
& B \times A \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A}" near start] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{A,B}" near start] & & B \otimes_{\mathsf{Sets}} A \arrow[dd, "\sigma'^{,-1}_{A,B}"] \\
\mathrm{pt} \times \mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}" near start] \arrow[dr, "{[a]\times[b]}"'] & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "{[a]\otimes_{\mathsf{Sets}}[b]}"] & \\
& A \times B \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}"'] & & A \otimes_{\mathsf{Sets}} B
\end{tikzcd}$$

which we partition into subdiagrams as follows:

$$
\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dr, "{[b]\times[a]}"] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] & \arrow[d, phantom, "(1)"] & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "{[b]\otimes_{\mathsf{Sets}}[a]}"] & \\
\arrow[r, phantom, "(2)"] & B\times A \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A}"] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{A,B}"'] \arrow[ddrr, phantom, "(3)"] & & B\otimes_{\mathsf{Sets}}A \arrow[dd, "\sigma'^{,-1}_{A,B}"'] \\
\mathrm{pt}\times\mathrm{pt} \arrow[dr, "{[a]\times[b]}"'] & & & \\
& A\times B \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}"'] & & A\otimes_{\mathsf{Sets}}B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] \arrow[ddrr, phantom, "(\dagger)"] & & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "{[b]\otimes_{\mathsf{Sets}}[a]}"] \arrow[dd, "\sigma'^{,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[ddr, phantom, "(4)"] & \\
& & & B\otimes_{\mathsf{Sets}}A \arrow[dd, "\sigma'^{,-1}_{A,B}"] \\
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dr, "{[a]\times[b]}"'] \arrow[drrr, phantom, "(5)"] & & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "{[a]\otimes_{\mathsf{Sets}}[b]}"] & \\
& A\times B \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}"'] & & A\otimes_{\mathsf{Sets}}B
\end{tikzcd}
$$

Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\mathsf{Sets},-1}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.
- Subdiagram $(\dagger)$ commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\sigma'^{,-1}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.

it follows that the diagram

$$
\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[dr, "{[b]\times[a]}"] & & \\
& B\times A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A}"] \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"'] & B\otimes_{\mathsf{Sets}}A \arrow[d, "\sigma'_{A,B}"] \\
& A\times B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B}"'] & A\otimes_{\mathsf{Sets}}B
\end{tikzcd}
$$

commutes. We then have

$$[\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}\circ\sigma^{\mathsf{Sets},-1}_{A,B}](b,a) = [\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}\circ\sigma^{\mathsf{Sets},-1}_{A,B}\circ([b]\times[a])](\star,\star)$$

$$
\begin{aligned}
&= [\sigma'^{,-1}_{A,B} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A} \circ ([b] \times [a])](\star, \star) \\
&= [\sigma'^{,-1}_{A,B} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A}](b, a)
\end{aligned}
$$

for each $(b, a) \in B \times A$, and thus we have

$$\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \circ \sigma^{\mathsf{Sets},-1}_{A,B} = \sigma'^{,-1}_{A,B} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A}.$$

Taking inverses then gives

$$\sigma^{\mathsf{Sets}}_{A,B} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} = \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A} \circ \sigma'_{A,B},$$

showing that the diagram

$$
\begin{tikzcd}
A \otimes_{\mathsf{Sets}} B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B}"] \arrow[d, "\sigma'_{A,B}"'] & A \times B \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"] \\
B \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A}"'] & B \times A
\end{tikzcd}
$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Let $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to $\times$ and $\otimes$ (as shown above), we have

$$
\begin{aligned}
\lambda'_B &= \lambda^{\mathsf{Sets}}_B \circ \phi_{\mathrm{pt},B} \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y), \\
\lambda'_B &= \lambda^{\mathsf{Sets}}_B \circ \psi_{\mathrm{pt},B} \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y).
\end{aligned}
$$

Postcomposing both sides with $\lambda^{\mathsf{Sets},-1}_B$ gives

$$
\begin{aligned}
\lambda^{\mathsf{Sets},-1}_B \circ \lambda'_B \circ (\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) &= \phi_{\mathrm{pt},B}, \\
\lambda^{\mathsf{Sets},-1}_B \circ \lambda'_B \circ (\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) &= \psi_{\mathrm{pt},B},
\end{aligned}
$$

and thus we have

$$\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$$

for each $B \in \mathrm{Obj}(\mathsf{Sets})$. Now, let $a \in A$ and consider the naturality diagrams

$$\begin{tikzcd} \mathrm{pt} \times B \arrow[r, "{[a] \times \mathrm{id}_B}"] \arrow[d, "\phi_{\mathrm{pt},B}"'] & A \times B \arrow[d, "\phi_{A,B}"] \\ \mathrm{pt} \otimes_{\mathsf{Sets}} B \arrow[r, "{[a] \otimes_{\mathsf{Sets}} \mathrm{id}_B}"'] & A \otimes_{\mathsf{Sets}} B \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pt} \times B \arrow[r, "{[a] \times \mathrm{id}_B}"] \arrow[d, "\psi_{\mathrm{pt},B}"'] & A \times B \arrow[d, "\psi_{A,B}"] \\ \mathrm{pt} \otimes_{\mathsf{Sets}} B \arrow[r, "{[a] \otimes_{\mathsf{Sets}} \mathrm{id}_B}"'] & A \otimes_{\mathsf{Sets}} B \end{tikzcd}$$

for ϕ and ψ with respect to the morphisms $[a]$ and id_B. Having shown that $\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$, we have

$$\begin{aligned} \phi_{A,B}(a,b) &= [\phi_{A,B} \circ ([a] \times \mathrm{id}_B)](\star, b) \\ &= [([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \phi_{\mathrm{pt},B}](\star, b) \\ &= [([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \psi_{\mathrm{pt},B}](\star, b) \\ &= [\psi_{A,B} \circ ([a] \times \mathrm{id}_B)](\star, b) \\ &= \psi_{A,B}(a,b) \end{aligned}$$

for each $(a, b) \in A \times B$. Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ to be unique. □

01PH **Corollary 5.1.10.1.2.** The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

01PJ 1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

01PK 2. *The Unit Object Is pt.* We have $\mathbb{1}_{\mathsf{Sets}} \cong \mathrm{pt}$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathsf{Sets})$ of **??** spanned by the symmetric monoidal categories $(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}})$ satisfying Items 1 and 2 is contractible.

Proof. Since Sets is locally presentable (**??**), it follows from **??** that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Definition 5.1.10.1.1. The result then follows from Definition 5.1.10.1.1. □

01PL 5.2 The Monoidal Category of Sets and Coproducts

01PM 5.2.1 Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

01PN 5.2.2 The Monoidal Unit

01PP **Definition 5.2.2.1.1.** The **monoidal unit of the coproduct of sets** is the functor

$$\mathbb{0}^{\mathsf{Sets}}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}$$

defined by

$$\mathbb{0}_{\mathsf{Sets}} \stackrel{\text{def}}{=} \emptyset,$$

where $\emptyset$ is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

01PQ 5.2.3 The Associator

01PR **Definition 5.2.3.1.1.** The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets},\coprod}\colon \coprod \circ \left(\coprod \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \coprod \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \coprod\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets} \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[dr, "\mathrm{id}\times\coprod"] & \\
(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}", "\sim"'] \arrow[d, "\coprod\times\mathrm{id}"'] & & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\coprod"] \\
\mathsf{Sets} \times \mathsf{Sets} \arrow[rr, "\coprod"'] \arrow[from=3-1, to=2-3, phantom, "\Nearrow \alpha^{\mathsf{Sets},\coprod}"] & & \mathsf{Sets},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}\colon \left(X \coprod Y\right) \coprod Z \overset{\sim}{\dashrightarrow} X \coprod \left(Y \coprod Z\right)$$

at (X, Y, Z) is given by

$$\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}(a) \overset{\text{def}}{=} \begin{cases} (0, x) & \text{if } a = (0, (0, x)), \\ (1, (0, y)) & \text{if } a = (0, (1, y)), \\ (1, (1, a)) & \text{if } a = (1, z) \end{cases}$$

for each $a \in (X \coprod Y) \coprod Z$.

Proof. Unwinding the Definitions of $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$: Firstly, we unwind the expressions for $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$. We have

$$\begin{aligned} (X \coprod Y) \coprod Z &\overset{\text{def}}{=} \{(0, a) \in S \mid a \in X \coprod Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ &= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ &\quad \cup \{(1, z) \in S \mid z \in Z\}, \end{aligned}$$

where $S = \{0, 1\} \times ((X \coprod Y) \cup Z)$ and

$$\begin{aligned} X \coprod (Y \coprod Z) &\overset{\text{def}}{=} \{(0, x) \in S' \mid x \in X\} \cup \{(1, a) \in S' \mid a \in Y \coprod Z\} \\ &= \{(0, x) \in S' \mid x \in X\} \cup \{(1, (0, y)) \in S' \mid y \in Y\} \\ &\quad \cup \{(1, (1, z)) \in S' \mid z \in Z\}, \end{aligned}$$

where $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$.

Invertibility: The inverse of $\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}$ is the map

$$\alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z}(a) \overset{\text{def}}{=} \begin{cases} (0, (0, x)) & \text{if } a = (0, x), \\ (0, (1, y)) & \text{if } a = (1, (0, y)), \\ (1, z) & \text{if } a = (1, (1, z)) \end{cases}$$

for each $a \in X \coprod Y (\coprod Z)$. Indeed:

- *Invertibility I*. The map $\alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z} \circ \alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}$ acts on elements as

$$\begin{array}{ccccc} (0, (0, x)) & \mapsto & (0, x) & \mapsto & (0, (0, x)), \\ (0, (0, y)) & \mapsto & (1, (0, y)) & \mapsto & (0, (0, y)), \\ (1, z) & \mapsto & (1, (1, z)) & \mapsto & (1, z) \end{array}$$

  and hence is equal to the identity map of $(X \coprod Y) \coprod Z$.

- *Invertibility II.* The map $\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z} \circ \alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z}$ acts on elements as

$$\begin{aligned}
(0,x) &\mapsto (0,(0,x)) \mapsto (0,x),\\
(1,(0,y)) &\mapsto (0,(0,y)) \mapsto (1,(0,y)),\\
(1,(1,z)) &\mapsto (1,z) \mapsto (1,(1,z))
\end{aligned}$$

  and hence is equal to the identity map of $X \coprod (Y \coprod Z)$.

Therefore $\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}$ is indeed an isomorphism.
Naturality: We need to show that, given functions

$$\begin{aligned}
f &\colon X \to X',\\
g &\colon Y \to Y',\\
h &\colon Z \to Z'
\end{aligned}$$

the diagram

$$\begin{tikzcd}
(X \coprod Y) \coprod Z \arrow[r, "(f \coprod g) \coprod h"] \arrow[d, "\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}"'] & (X' \coprod Y') \coprod Z' \arrow[d, "\alpha^{\mathsf{Sets},\coprod}_{X',Y',Z'}"] \\
X \coprod (Y \coprod Z) \arrow[r, "f \coprod (g \coprod h)"'] & X' \coprod (Y' \coprod Z')
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(0,(0,x)) \arrow[d, mapsto] & & & (0,(0,x)) \arrow[r, mapsto] & (0,(0,f(x))) \arrow[d, mapsto] \\
(0,x) \arrow[r, mapsto] & (0,f(x)) & & & (0,f(x))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,(1,y)) \arrow[d, mapsto] & & & (0,(1,y)) \arrow[r, mapsto] & (0,(1,g(y))) \arrow[d, mapsto] \\
(1,(0,y)) \arrow[r, mapsto] & (1,(0,g(y))) & & & (1,(0,g(y)))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(1,z) \arrow[d, mapsto] & & & (1,z) \arrow[r, mapsto] & (1,h(z)) \arrow[d, mapsto] \\
(1,(1,z)) \arrow[r, mapsto] & (1,(1,h(z))) & & & (1,(1,h(z)))
\end{tikzcd}
$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets},\coprod}$ to be a natural transformation. *Being a Natural Isomorphism*: Since $\alpha^{\mathsf{Sets},\coprod}$ is natural and $\alpha^{\mathsf{Sets},\coprod,-1}$ is a componentwise inverse to $\alpha^{\mathsf{Sets},\coprod}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus $\alpha^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

01PS 5.2.4 The Left Unitor

01PT **Definition 5.2.4.1.1.** The **left unitor of the coproduct of sets** is the natural isomorphism

$$
\lambda^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}
\qquad
\begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets} \arrow[r, "\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}"] \arrow[rd, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\coprod"] \arrow[ld, phantom, "\Swarrow \lambda^{\mathsf{Sets},\coprod}"] \\
 & \mathsf{Sets},
\end{tikzcd}
$$

whose component

$$\lambda^{\mathsf{Sets},\coprod}_X \colon \emptyset \coprod X \overset{\sim}{\dashrightarrow} X$$

at X is given by

$$\lambda^{\mathsf{Sets},\coprod}_X((1,x)) \overset{\text{def}}{=} x$$

for each $(1,x) \in \emptyset \coprod X$.

Proof. Unwinding the Definition of $\emptyset \coprod X$: Firstly, we unwind the expressions for $\emptyset \coprod X$. We have

$$\begin{aligned}
\emptyset \coprod X &\overset{\text{def}}{=} \{(0,z) \in S \mid z \in \emptyset\} \cup \{(1,x) \in S \mid x \in X\} \\
&= \emptyset \cup \{(1,x) \in S \mid x \in X\} \\
&= \{(1,x) \in S \mid x \in X\},
\end{aligned}$$

where $S = \{0,1\} \times (\emptyset \cup X)$.
Invertibility: The inverse of $\lambda^{\mathsf{Sets},\coprod}_X$ is the map

$$\lambda^{\mathsf{Sets},\coprod,-1}_X \colon X \to \emptyset \coprod X$$

given by

$$\lambda^{\mathsf{Sets},\coprod,-1}_X(x) \overset{\text{def}}{=} (1,x)$$

for each $x \in X$. Indeed:

- *Invertibility I*. We have

$$\begin{aligned}
[\lambda^{\mathsf{Sets},\coprod,-1}_X \circ \lambda^{\mathsf{Sets},\coprod}_X](1,x) &= \lambda^{\mathsf{Sets},\coprod,-1}_X(\lambda^{\mathsf{Sets},\coprod}_X(1,x)) \\
&= \lambda^{\mathsf{Sets},\coprod,-1}_X(x) \\
&= (1,x) \\
&= [\mathrm{id}_{\emptyset \coprod X}](1,x)
\end{aligned}$$

for each $(1,x) \in \emptyset \coprod X$, and therefore we have

$$\lambda^{\mathsf{Sets},\coprod,-1}_X \circ \lambda^{\mathsf{Sets},\coprod}_X = \mathrm{id}_{\emptyset \coprod X}.$$

- *Invertibility II*. We have

$$\begin{aligned}[\lambda^{\mathsf{Sets},\coprod}_X \circ \lambda^{\mathsf{Sets},\coprod,-1}_X](x) &= \lambda^{\mathsf{Sets},\coprod}_X(\lambda^{\mathsf{Sets},\coprod,-1}_X(x)) \\ &= \lambda^{\mathsf{Sets},\coprod,-1}_X(1,x) \\ &= x \\ &= [\mathrm{id}_X](x)\end{aligned}$$

  for each $x \in X$, and therefore we have

$$\lambda^{\mathsf{Sets},\coprod}_X \circ \lambda^{\mathsf{Sets},\coprod,-1}_X = \mathrm{id}_X .$$

Therefore $\lambda^{\mathsf{Sets},\coprod}_X$ is indeed an isomorphism.

Naturality: We need to show that, given a function $f\colon X \longrightarrow Y$, the diagram

$$\begin{tikzcd} \emptyset \coprod X \arrow[r, "\mathrm{id}_\emptyset \coprod f"] \arrow[d, "\lambda^{\mathsf{Sets},\coprod}_X"'] & \emptyset \coprod Y \arrow[d, "\lambda^{\mathsf{Sets},\coprod}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (1,x) \arrow[d, mapsto] & \\ x \arrow[r, mapsto] & f(x) \end{tikzcd} \qquad \begin{tikzcd} (1,x) \arrow[r, mapsto] & (1,f(x)) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$

and hence indeed commutes. Therefore $\lambda^{\mathsf{Sets},\coprod}$ is a natural transformation.

Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets},\coprod}$ is natural and $\lambda^{\mathsf{Sets},-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets},\coprod}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus $\lambda^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

5.2.5 The Right Unitor

01PU

01PV **Definition 5.2.5.1.1.** The **right unitor of the coproduct of sets** is the natural isomorphism

$$\rho^{\mathsf{Sets},\coprod}\colon \coprod\circ\left(\mathrm{id}\times\mathbb{0}^{\mathsf{Sets}}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}},\qquad \begin{tikzcd} \mathsf{Sets}\times\mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{0}^{\mathsf{Sets}}"] \arrow[rd, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets}\times\mathsf{Sets} \arrow[d, "\coprod"] \arrow[ld, phantom, "\Swarrow\rho^{\mathsf{Sets},\coprod}"] \\ & \mathsf{Sets}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets},\coprod}_X\colon X\coprod\emptyset\overset{\sim}{\dashrightarrow} X$$

at X is given by

$$\rho^{\mathsf{Sets},\coprod}_X((0,x))\overset{\mathrm{def}}{=}x$$

for each $(0,x)\in X\coprod\emptyset$.

Proof. *Unwinding the Definition of $X\coprod\emptyset$*: Firstly, we unwind the expression for $X\coprod\emptyset$. We have

$$\begin{aligned} X\coprod\emptyset &\overset{\mathrm{def}}{=}\{(0,x)\in S\mid x\in X\}\cup\{(1,z)\in S\mid z\in\emptyset\}\\ &=\{(0,x)\in S\mid x\in X\}\cup\emptyset\\ &=\{(0,x)\in S\mid x\in X\}, \end{aligned}$$

where $S=\{0,1\}\times(X\cup\emptyset)=\{0,1\}\times(\emptyset\cup X)=S$.

Invertibility: The inverse of $\rho^{\mathsf{Sets},\coprod}_X$ is the map

$$\rho^{\mathsf{Sets},\coprod,-1}_X\colon X\to X\coprod\emptyset$$

given by

$$\rho^{\mathsf{Sets},\coprod,-1}_X(x)\overset{\mathrm{def}}{=}(0,x)$$

for each $x\in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned}[\rho_X^{\mathsf{Sets},\coprod,-1} \circ \rho_X^{\mathsf{Sets},\coprod}](0,x) &= \rho_X^{\mathsf{Sets},\coprod,-1}(\rho_X^{\mathsf{Sets},\coprod}(0,x)) \\ &= \rho_X^{\mathsf{Sets},\coprod,-1}(x) \\ &= (0,x) \\ &= [\mathrm{id}_{X\coprod\emptyset}](0,x)\end{aligned}$$

  for each $(0,x) \in \emptyset \coprod X$, and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod,-1} \circ \rho_X^{\mathsf{Sets},\coprod} = \mathrm{id}_{\emptyset\coprod X}.$$

- *Invertibility II.* We have

$$\begin{aligned}[\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1}](x) &= \rho_X^{\mathsf{Sets},\coprod}(\rho_X^{\mathsf{Sets},\coprod,-1}(x)) \\ &= \rho_X^{\mathsf{Sets},\coprod,-1}(0,x) \\ &= x \\ &= [\mathrm{id}_X](x)\end{aligned}$$

  for each $x \in X$, and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore $\rho_X^{\mathsf{Sets},\coprod}$ is indeed an isomorphism.
Naturality: We need to show that, given a function $f : X \longrightarrow Y$, the diagram

$$\begin{tikzcd} X \coprod \emptyset \arrow[r, "f\coprod \mathrm{id}_\emptyset"] \arrow[d, "\rho_X^{\mathsf{Sets},\coprod}"'] & Y \coprod \emptyset \arrow[d, "\rho_Y^{\mathsf{Sets},\coprod}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (0,x) \arrow[d, mapsto] & \\ x \arrow[r, mapsto] & f(x) \end{tikzcd} \qquad \begin{tikzcd} (0,x) \arrow[r, mapsto] & (1, f(x)) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$

and hence indeed commutes. Therefore $\rho^{\mathsf{Sets},\coprod}$ is a natural transformation.
Being a Natural Isomorphism: Since $\rho^{\mathsf{Sets},\coprod}$ is natural and $\rho^{\mathsf{Sets},-1}$ is a componentwise inverse to $\rho^{\mathsf{Sets},\coprod}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\rho^{\mathsf{Sets},-1}$ is also natural. Thus $\rho^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

01PW 5.2.6 The Symmetry

01PX **Definition 5.2.6.1.1.** The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}\colon \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets}\times\mathsf{Sets} \arrow[rr, "\coprod"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}"'] & \arrow[d, phantom, "\overset{\Vert}{\underset{\Downarrow}{\sigma^{\mathsf{Sets},\coprod}}}"] & \mathsf{Sets}, \\ & \mathsf{Sets}\times\mathsf{Sets} \arrow[ur, "\coprod"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets},\coprod}_{X,Y}\colon X \coprod Y \overset{\sim}{\dashrightarrow} Y \coprod X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ is defined by

$$\sigma^{\mathsf{Sets},\coprod}_{X,Y}(x, y) \overset{\text{def}}{=} (y, x)$$

for each $(x, y) \in X \times Y$.

Proof. Unwinding the Definitions of $X \coprod Y$ and $Y \coprod X$: Firstly, we unwind the expressions for $X \coprod Y$ and $Y \coprod X$. We have

$$X \coprod Y \overset{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},$$

where $S = \{0, 1\} \times (X \cup Y)$ and

$$Y \coprod X \overset{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},$$

where $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$.
Invertibility: The inverse of $\sigma^{\mathsf{Sets},\coprod}_{X,Y}$ is the map

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}\colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y} \overset{\text{def}}{=} \sigma^{\mathsf{Sets},\coprod}_{Y,X}$$

and hence given by

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} (0,x) & \text{if } z=(1,x), \\ (1,y) & \text{if } z=(0,y) \end{cases}$$

for each $z \in Y \coprod X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} [\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y} \circ \sigma^{\mathsf{Sets},\coprod}_{X,Y}](0,x) &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(\sigma^{\mathsf{Sets},\coprod}_{X}(0,x)) \\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(1,x) \\ &= (0,x) \\ &= [\mathrm{id}_{X\coprod Y}](0,x) \end{aligned}$$

  for each $(0,x) \in X \coprod Y$ and

$$\begin{aligned} [\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y} \circ \sigma^{\mathsf{Sets},\coprod}_{X,Y}](1,y) &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(\sigma^{\mathsf{Sets},\coprod}_{X}(1,y)) \\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(0,y) \\ &= (1,y) \\ &= [\mathrm{id}_{X\coprod Y}](1,y) \end{aligned}$$

  for each $(1,y) \in X \coprod Y$, and therefore we have

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y} \circ \sigma^{\mathsf{Sets},\coprod}_{X,Y} = \mathrm{id}_{X\coprod Y}.$$

- *Invertibility II.* We have

$$\begin{aligned} [\sigma^{\mathsf{Sets},\coprod}_{X,Y} \circ \sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}](0,y) &= \sigma^{\mathsf{Sets},\coprod}_{X}(\sigma^{\mathsf{Sets},\coprod,-1}_{X}(0,y)) \\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(1,y) \\ &= (0,y) \\ &= [\mathrm{id}_{Y\coprod X}](0,y) \end{aligned}$$

for each $(0,y) \in Y \coprod X$ and

$$\begin{aligned}[\sigma^{\mathsf{Sets},\coprod}_{X,Y} \circ \sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}](1,x) &= \sigma^{\mathsf{Sets},\coprod}_{X}(\sigma^{\mathsf{Sets},\coprod,-1}_{X}(1,x)) \\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(0,x) \\ &= (1,x) \\ &= [\mathrm{id}_{Y \coprod X}](1,x)\end{aligned}$$

for each $(1,x) \in Y \coprod X$, and therefore we have

$$\sigma^{\mathsf{Sets},\coprod}_{X} \circ \sigma^{\mathsf{Sets},\coprod,-1}_{X} = \mathrm{id}_{Y \coprod X}.$$

Therefore $\sigma^{\mathsf{Sets},\coprod}_{X,Y}$ is indeed an isomorphism.
Naturality: We need to show that, given functions $f\colon A \to X$ and $g\colon B \to Y$, the diagram

$$\begin{tikzcd} A \coprod B \arrow[r, "f \coprod g"] \arrow[d, "\sigma^{\mathsf{Sets},\coprod}_{A,B}"'] & X \coprod Y \arrow[d, "\sigma^{\mathsf{Sets},\coprod}_{X,Y}"] \\ B \coprod A \arrow[r, "g \coprod f"'] & Y \coprod X \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (0,a) \arrow[d, mapsto] & & (0,a) \arrow[r, mapsto] & (0,f(a)) \arrow[d, mapsto] \\ (1,a) \arrow[r, mapsto] & (1,f(a)) & & (1,f(a)) \end{tikzcd}$$

$$\begin{tikzcd} (1,b) \arrow[d, mapsto] & & (1,b) \arrow[r, mapsto] & (1,g(b)) \arrow[d, mapsto] \\ (0,b) \arrow[r, mapsto] & (0,g(b)) & & (0,g(b)) \end{tikzcd}$$

and hence indeed commutes. Therefore $\sigma^{\mathsf{Sets},\coprod}$ is a natural transformation.

Being a Natural Isomorphism: Since $\sigma^{\mathsf{Sets},\coprod}$ is natural and $\sigma^{\mathsf{Sets},-1}$ is a componentwise inverse to $\sigma^{\mathsf{Sets},\coprod}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\sigma^{\mathsf{Sets},-1}$ is also natural. Thus $\sigma^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

01PY 5.2.7 The Monoidal Category of Sets and Coproducts

01PZ **Proposition 5.2.7.1.1.** The category Sets admits a closed symmetric monoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Monoidal Product.* The coproduct functor

  $$\coprod\colon \mathsf{Sets}\times\mathsf{Sets}\longrightarrow\mathsf{Sets}$$

  of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.
- *The Monoidal Unit.* The functor

  $$\mathbb{0}^{\mathsf{Sets}}\colon \mathsf{pt}\longrightarrow\mathsf{Sets}$$

  of Definition 5.2.2.1.1.
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets},\coprod}\colon \coprod\circ\left(\coprod\times\mathrm{id}_{\mathsf{Sets}}\right)\overset{\sim}{\Longrightarrow}\coprod\circ\left(\mathrm{id}_{\mathsf{Sets}}\times\coprod\right)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.2.3.1.1.
- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathsf{Sets},\coprod}\colon \coprod\circ\left(\mathbb{0}^{\mathsf{Sets}}\times\mathrm{id}_{\mathsf{Sets}}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

  of Definition 5.2.4.1.1.
- *The Right Unitors.* The natural isomorphism

  $$\rho^{\mathsf{Sets},\coprod}\colon \coprod\circ\left(\mathrm{id}\times\mathbb{0}^{\mathsf{Sets}}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

  of Definition 5.2.5.1.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.2.6.1.1.

Proof. The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \coprod (X \coprod Y)) \coprod Z \arrow[dr, "\alpha^{\mathsf{Sets},\coprod}_{W,X\coprod Y,Z}"] & \\
((W \coprod X) \coprod Y) \coprod Z \arrow[ur, "\alpha^{\mathsf{Sets},\coprod}_{W,X,Y} \coprod \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets},\coprod}_{W\coprod X,Y,Z}"'] & & W \coprod ((X \coprod Y) \coprod Z) \arrow[d, "\mathrm{id}_W \coprod \alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}"] \\
(W \coprod X) \coprod (Y \coprod Z) \arrow[rr, "\alpha^{\mathsf{Sets},\coprod}_{W,X,Y\coprod Z}"'] & & W \coprod (X \coprod (Y \coprod Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0,(0,(0,w))) \arrow[d, mapsto] & \\
(0,(0,w)) \arrow[r, mapsto] & (0,w)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (0,(0,w)) \arrow[dr, mapsto] & \\
(0,(0,(0,w))) \arrow[ur, mapsto] & & (0,w) \arrow[d, mapsto] \\
 & & (0,w)
\end{tikzcd}$$

$$\begin{tikzcd}
(0,(0,(1,x))) \arrow[d, mapsto] & \\
(0,(1,x)) \arrow[r, mapsto] & (1,(0,x))
\end{tikzcd}
\qquad
\begin{tikzcd}
& (0,(1,(0,x))) \arrow[dr, mapsto] & \\
(0,(0,(1,x))) \arrow[ur, mapsto] & & (1,(0,(0,x))) \arrow[d, mapsto] \\
& & (1,(0,x))
\end{tikzcd}$$

$$\begin{tikzcd}
(0,(1,y)) \arrow[d, mapsto] & \\
(1,(0,y)) \arrow[r, mapsto] & (1,(1,(0,y)))
\end{tikzcd}
\qquad
\begin{tikzcd}
& (0,(1,(1,y))) \arrow[dr, mapsto] & \\
(0,(1,y)) \arrow[ur, mapsto] & & (1,(0,(1,y))) \arrow[d, mapsto] \\
& & (1,(1,(0,y)))
\end{tikzcd}$$

$$\begin{tikzcd}
(1,z) \arrow[d, mapsto] & \\
(1,(1,z)) \arrow[r, mapsto] & (1,(1,(1,z)))
\end{tikzcd}
\qquad
\begin{tikzcd}
& (1,z) \arrow[dr, mapsto] & \\
(1,z) \arrow[ur, mapsto] & & (1,(1,z)) \arrow[d, mapsto] \\
& & (1,(1,(1,z)))
\end{tikzcd}$$

and therefore the pentagon identity is satisfied.

The Triangle Identity: Let X and Y be sets. We have to show that the diagram

$$\begin{tikzcd}
(X \coprod \emptyset) \coprod Y \arrow[rr, "\alpha^{\mathsf{Sets},\coprod}_{X,\emptyset,Y}"] \arrow[dr, "\rho^{\mathsf{Sets},\coprod}_X \coprod \mathrm{id}_Y"'] & & X \coprod (\emptyset \coprod Y) \arrow[dl, "\mathrm{id}_X \coprod \lambda^{\mathsf{Sets},\coprod}_Y"] \\
& X \coprod Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0,(1,x)) \arrow[dr, mapsto] & & (1,(0,x)) \arrow[rr, mapsto] & & (0,x) \arrow[dl, mapsto] \\
& (0,x) & & (0,x) &
\end{tikzcd}$$

$$\begin{tikzcd}
(1,y) \arrow[dr, mapsto] & & (1,y) \arrow[rr, mapsto] & & (1,(1,y)) \arrow[dl, mapsto] \\
& (1,y) & & (1,y) &
\end{tikzcd}$$

and therefore the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
& (X \coprod Y) \coprod Z \arrow[dl, "\alpha^{\mathsf{Sets}}_{X,Y,Z}"'] \arrow[dr, "\sigma^{\mathsf{Sets}}_{X,Y} \coprod \mathrm{id}_Z"] & \\
X \coprod (Y \coprod Z) \arrow[d, "\sigma^{\mathsf{Sets}}_{X,Y \coprod Z}"'] & & (Y \coprod X) \coprod Z \arrow[d, "\alpha^{\mathsf{Sets}}_{Y,X,Z}"] \\
(Y \coprod Z) \coprod X \arrow[dr, "\alpha^{\mathsf{Sets}}_{Y,Z,X}"'] & & Y \coprod (X \coprod Z) \arrow[dl, "\mathrm{id}_Y \coprod \sigma^{\mathsf{Sets}}_{X,Z}"] \\
& Y \coprod (Z \coprod X) &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (0,(0,x)) \arrow[dl, mapsto] & (0,(0,x)) \arrow[dr, mapsto] & \\
(0,x) \arrow[d, mapsto] & & & (0,(1,x)) \arrow[d, mapsto] \\
(1,x) \arrow[dr, mapsto] & & & (1,(0,x)) \arrow[dl, mapsto] \\
 & (1,(1,x)) & (1,(1,x)) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (0,(1,y)) \arrow[dl, mapsto] & (0,(1,y)) \arrow[dr, mapsto] & \\
(1,(0,y)) \arrow[d, mapsto] & & & (0,(0,y)) \arrow[d, mapsto] \\
(0,(0,y)) \arrow[dr, mapsto] & & & (0,y) \arrow[dl, mapsto] \\
 & (0,y) & (0,y) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (1,z) \arrow[dl, mapsto] & (1,z) \arrow[dr, mapsto] & \\
(1,(1,z)) \arrow[d, mapsto] & & & (1,z) \arrow[d, mapsto] \\
(0,(1,z)) \arrow[dr, mapsto] & & & (1,(1,z)) \arrow[dl, mapsto] \\
 & (1,(0,z)) & (1,(0,z)) &
\end{tikzcd}$$

and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$
\begin{tikzcd}
& X \coprod (Y \coprod Z) \arrow[dl, "(\alpha^{\mathsf{Sets}}_{X,Y,Z})^{-1}"'] \arrow[dr, "\mathrm{id}_X \coprod \sigma^{\mathsf{Sets}}_{Y,Z}"] & \\
(X \coprod Y) \coprod Z \arrow[d, "\sigma^{\mathsf{Sets}}_{X \coprod Y, Z}"'] & & X \coprod (Z \coprod Y) \arrow[d, "(\alpha^{\mathsf{Sets}}_{X,Z,Y})^{-1}"] \\
Z \coprod (X \coprod Y) \arrow[dr, "(\alpha^{\mathsf{Sets}}_{Z,X,Y})^{-1}"'] & & (X \coprod Z) \coprod Y \arrow[dl, "\sigma^{\mathsf{Sets}}_{X,Z} \coprod \mathrm{id}_Y"] \\
& (Z \coprod X) \coprod Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
& (0, x) \arrow[dl, mapsto] & (0, x) \arrow[dr, mapsto] & \\
(0, (0, x)) \arrow[d, mapsto] & & & (0, x) \arrow[d, mapsto] \\
(1, (0, x)) \arrow[dr, mapsto] & & & (0, (0, x)) \arrow[dl, mapsto] \\
& (0, (1, x)) & (0, (1, x)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (1,(0,y)) \arrow[dl, mapsto] & (1,(0,y)) \arrow[dr, mapsto] & \\
(0,(1,y)) \arrow[d, mapsto] & & & (1,(1,y)) \arrow[d, mapsto] \\
(1,(1,y)) \arrow[dr, mapsto] & & & (1,y) \arrow[dl, mapsto] \\
 & (1,y) & (1,y) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (1,(1,z)) \arrow[dl, mapsto] & (1,(1,z)) \arrow[dr, mapsto] & \\
(1,z) \arrow[d, mapsto] & & & (1,(0,z)) \arrow[d, mapsto] \\
(0,z) \arrow[dr, mapsto] & & & (0,(1,z)) \arrow[dl, mapsto] \\
 & (0,(0,z)) & (0,(0,z)) &
\end{tikzcd}
$$

and thus the right hexagon identity is satisfied. □

01Q0 5.3 The Bimonoidal Category of Sets, Products, and Coproducts

01Q1 5.3.1 The Left Distributor

01Q2 **Definition 5.3.1.1.1.** The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_\ell^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$

as in the diagram

$$
\begin{tikzcd}
 & & (\mathsf{Sets}\times\mathsf{Sets})\times(\mathsf{Sets}\times\mathsf{Sets}) \arrow[dr, "\boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] & & \\
 & \mathsf{Sets}\times(\mathsf{Sets}\times\mathsf{Sets}) \arrow[ur, "\Delta_{\mathsf{Sets}}\times(\mathrm{id}_{\mathsf{Sets}}\times\mathrm{id}_{\mathsf{Sets}})"] \arrow[d, "\mathrm{id}_{\mathsf{Sets}}\times\coprod"'] & & (\mathsf{Sets}\times\mathsf{Sets})\times(\mathsf{Sets}\times\mathsf{Sets}) \arrow[d, "(\times)\times(\times)"] & \\
 & \mathsf{Sets}\times\mathsf{Sets} \arrow[dr, "\times"'] \arrow[urr, Rightarrow, shorten=10mm, "\delta^{\mathsf{Sets}}_{\ell}"] & & \mathsf{Sets}\times\mathsf{Sets}, \arrow[dl, "\coprod"] & \\
 & & \mathsf{Sets} & &
\end{tikzcd}
$$

whose component

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}\colon X\times\left(Y\coprod Z\right)\overset{\sim}{\dashrightarrow}(X\times Y)\coprod(X\times Z)$$

at (X,Y,Z) is defined by

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}(x,a)\overset{\text{def}}{=}\begin{cases}(0,(x,y)) & \text{if } a=(0,y),\\ (1,(x,z)) & \text{if } a=(1,z)\end{cases}$$

for each $(x,a)\in X\times\left(Y\coprod Z\right)$.

Proof. Invertibility: The inverse of $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ is the map

$$\delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z}\colon (X\times Y)\coprod(X\times Z)\overset{\sim}{\dashrightarrow}X\times\left(Y\coprod Z\right)$$

given by

$$\delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z}(a)\overset{\text{def}}{=}\begin{cases}(x,(0,y)) & \text{if } a=(0,(x,y)),\\ (x,(1,z)) & \text{if } a=(1,(x,z))\end{cases}$$

for $a\in(X\times Y)\coprod(X\times Z)$. Indeed:

- *Invertibility I.* The map $\delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z} \circ \delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ acts on elements as

$$\begin{aligned} (x,(0,y)) &\mapsto (0,(x,y)) \mapsto (x,(0,y)), \\ (x,(1,z)) &\mapsto (1,(x,z)) \mapsto (x,(1,z)), \end{aligned}$$

  but these are the two possible cases for elements of $X \times (Y \coprod Z)$. Hence the map is equal to the identity.

- *Invertibility II.* The map $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z} \circ \delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z}$ acts on elements as

$$\begin{aligned} (0,(x,y)) &\mapsto (x,(0,y)) \mapsto (0,(x,y)), \\ (1,(x,z)) &\mapsto (x,(1,z)) \mapsto (1,(x,z)), \end{aligned}$$

  but these are the two possible cases for elements of $(X \times Y) \coprod (X \times Z)$. Hence the map is equal to the identity.

Thus $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ is an isomorphism for all X, Y, Z.
Naturality: We need to show that, given functions

$$\begin{aligned} f &\colon X \to X', \\ g &\colon Y \to Y', \\ h &\colon Z \to Z' \end{aligned}$$

the diagram

$$\begin{tikzcd} X \times (Y \coprod Z) \arrow[r, "f\times\left(g\coprod h\right)"] \arrow[d, "\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}"'] & X' \times (Y' \coprod Z') \arrow[d, "\delta^{\mathsf{Sets}}_{\ell|X',Y',Z'}"] \\ (X \times Y) \coprod (X \times Z) \arrow[r, "(f\times g)\coprod(f\times h)"'] & (X' \times Y') \coprod (X' \times Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x,(0,y)) \arrow[d, mapsto] & \\ (0,(x,y)) \arrow[r, mapsto] & (0,(f(x),g(y))) \end{tikzcd} \qquad \begin{tikzcd} (x,(0,y)) \arrow[r, mapsto] & (f(x),(0,f(y))) \arrow[d, mapsto] \\ & (0,(f(x),g(y))) \end{tikzcd}$$

$$
\begin{tikzcd}
(x,(1,z)) \arrow[d, mapsto] & & (x,(1,z)) \arrow[r, mapsto] & (f(x),(1,h(z))) \arrow[d, mapsto] \\
(1,(x,z)) \arrow[r, mapsto] & (1,(f(x),h(z))) & & (1,(f(x),h(z))),
\end{tikzcd}
$$

so it commutes, showing $\delta_{\ell}^{\mathsf{Sets}}$ to be a natural transformation.

Being a Natural Isomorphism: Since $\delta_{\ell}^{\mathsf{Sets}}$ is natural and $\delta_{\ell}^{\mathsf{Sets},-1}$ is a componentwise inverse to $\delta_{\ell}^{\mathsf{Sets}}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\delta_{\ell}^{\mathsf{Sets},-1}$ is also natural. Thus $\delta_{\ell}^{\mathsf{Sets}}$ is a natural isomorphism. □

01Q3 5.3.2 The Right Distributor

01Q4 **Definition 5.3.2.1.1.** The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ (\coprod \times \mathrm{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ ((\times) \times (\times)) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ ((\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$

as in the diagram

$$
\begin{tikzcd}
& (\mathsf{Sets}\times\mathsf{Sets})\times(\mathsf{Sets}\times\mathsf{Sets}) \arrow[dr, "\boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] & \\
(\mathsf{Sets}\times\mathsf{Sets})\times\mathsf{Sets} \arrow[ur, "(\mathrm{id}_{\mathsf{Sets}}\times\mathrm{id}_{\mathsf{Sets}})\times\Delta_{\mathsf{Sets}}"] \arrow[d, "\coprod\times\mathrm{id}_{\mathsf{Sets}}"'] & & (\mathsf{Sets}\times\mathsf{Sets})\times(\mathsf{Sets}\times\mathsf{Sets}) \arrow[d, "(\times)\times(\times)"] \\
\mathsf{Sets}\times\mathsf{Sets} \arrow[dr, "\times"'] \arrow[urr, Rightarrow, "\delta_r^{\mathsf{Sets}}"] & & \mathsf{Sets}\times\mathsf{Sets}, \arrow[dl, "\coprod"] \\
& \mathsf{Sets} &
\end{tikzcd}
$$

whose component

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z}\colon (X \coprod Y) \times Z \overset{\sim}{\dashrightarrow} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z}(a, z) \overset{\text{def}}{=} \begin{cases} (0, (x, z)) & \text{if } a = (0, x), \\ (1, (y, z)) & \text{if } a = (1, y) \end{cases}$$

for each $(a, z) \in (X \coprod Y) \times Z$.

Proof. *Invertibility*: The inverse of $\delta^{\mathsf{Sets}}_{r|X,Y,Z}$ is the map

$$\delta^{\mathsf{Sets},-1}_{r|X,Y,Z}\colon (X \times Z) \coprod (Y \times Z) \overset{\sim}{\dashrightarrow} (X \coprod Y) \times Z$$

given by

$$\delta^{\mathsf{Sets},-1}_{r|X,Y,Z}(a) \overset{\text{def}}{=} \begin{cases} ((0, x), z) & \text{if } a = (0, (x, z)), \\ ((1, y), z) & \text{if } a = (1, (y, z)) \end{cases}$$

for $a \in (X \times Z) \coprod (Y \times Z)$. Indeed:

- *Invertibility I*. The map $\delta^{\mathsf{Sets},-1}_{r|X,Y,Z} \circ \delta^{\mathsf{Sets}}_{r|X,Y,Z}$ acts on elements as

$$\begin{aligned} ((0, x), z) &\mapsto (0, (x, z)) \mapsto (0, (x, z)), \\ ((1, y), z) &\mapsto (1, (y, z)) \mapsto (1, (y, z)), \end{aligned}$$

  but these are the two possible cases for elements of $(X \coprod Y) \times Z$. Hence the map is equal to the identity.

- *Invertibility II*. The map $\delta^{\mathsf{Sets}}_{r|X,Y,Z} \circ \delta^{\mathsf{Sets},-1}_{r|X,Y,Z}$ acts on elements as

$$\begin{aligned} (0, (x, z)) &\mapsto ((0, x), z) \mapsto (0, (x, z)), \\ (1, (y, z)) &\mapsto ((1, y), z) \mapsto (1, (y, z)), \end{aligned}$$

  but these are the two possible cases for elements of $(X \times Z) \coprod (Y \times Z)$. Hence the map is equal to the identity.

So $\delta^{\mathsf{Sets}}_{r|X,Y,Z}$ is an isomorphism for all X, Y, Z.

Naturality: We need to show that, given functions

$$\begin{aligned} f &\colon X \to X', \\ g &\colon Y \to Y', \\ h &\colon Z \to Z' \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \coprod Y) \times Z' \arrow[r, "(f \coprod g) \times h"] \arrow[d, "\delta^{\mathsf{Sets}}_{r|X,Y,Z}"'] & (X' \coprod Y') \times Z' \arrow[d, "\delta^{\mathsf{Sets}}_{r|X',Y',Z'}"] \\ (X \times Z) \coprod (Y \times Z) \arrow[r, "(f \times h) \coprod (g \times h)"'] & (X' \times Z') \coprod (Y' \times Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} ((0,x),z) \arrow[d, mapsto] & ((0,x),z) \arrow[r, mapsto] & ((0,f(x)),h(z)) \arrow[d, mapsto] \\ (0,(x,z)) \arrow[r, mapsto] & (0,(f(x),h(z))) & (0,(f(x),h(z))) \end{tikzcd}$$

$$\begin{tikzcd} ((1,y),z) \arrow[d, mapsto] & ((1,y),z) \arrow[r, mapsto] & ((1,g(y)),h(z)) \arrow[d, mapsto] \\ (1,(y,z)) \arrow[r, mapsto] & (1,(g(y),h(z))) & (1,(g(y),h(z))) \end{tikzcd}$$

so it commutes and $\delta^{\mathsf{Sets}}_{r}$ is a natural transformation.
Being a Natural Isomorphism: Since $\delta^{\mathsf{Sets}}_{r}$ is natural and $\delta^{\mathsf{Sets},-1}_{r}$ is a componentwise inverse to $\delta^{\mathsf{Sets}}_{r}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\delta^{\mathsf{Sets},-1}_{r}$ is also natural. Thus $\delta^{\mathsf{Sets}}_{r}$ is a natural isomorphism. □

01Q5 5.3.3 The Left Annihilator

01Q6 **Definition 5.3.3.1.1.** The **left annihilator of the product of sets** is the natural isomorphism

$$\zeta^{\mathsf{Sets}}_{\ell} \colon \mathbb{0}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\mathrm{id}_{\mathrm{pt}} \times \epsilon^{\mathsf{Cats}_2}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$

as in the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] & \\
\mathrm{pt} \times \mathsf{Sets} \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon^{\mathsf{Cats}_2}_{\mathsf{Sets}}"] \arrow[d, "\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}"'] & \arrow[d, phantom, "\Swarrow \zeta^{\mathsf{Sets}}_{\ell}"] & \mathrm{pt} \arrow[d, "\mathbb{0}^{\mathsf{Sets}}"] \\
\mathsf{Sets} \times \mathsf{Sets} \arrow[rr, "\times"'] & & \mathsf{Sets}
\end{tikzcd}$$

with components

$$\zeta^{\mathsf{Sets}}_{\ell|A} \colon \emptyset \times A \overset{\sim}{\dashrightarrow} \emptyset$$

given by $\zeta^{\mathsf{Sets}}_{\ell|A} \overset{\mathrm{def}}{=} \mathrm{pr}_1$.

Proof. *Invertibility*: The inverse of $\zeta^{\mathsf{Sets}}_{\ell|A}$ is the map

$$\zeta^{\mathsf{Sets},-1}_{\ell|A} \colon \emptyset \overset{\sim}{\dashrightarrow} \emptyset \times A$$

given by

$$\zeta^{\mathsf{Sets},-1}_{\ell|A} \overset{\mathrm{def}}{=} \iota_A,$$

where ι_A is as defined in Constructions With Sets, Definition 4.2.1.1.2:

- *Invertibility I*. The map $\zeta^{\mathsf{Sets}}_{\ell|A} \circ \iota_A \colon \emptyset \longrightarrow \emptyset$ is equal to $\mathrm{id}_{\emptyset}$, as $\emptyset$ is the initial object of Sets.
- *Invertibility II*. The map $\iota_A \circ \zeta^{\mathsf{Sets}}_{\ell|A}$ is equal to the identity on every $(x, a) \in \emptyset \times A$, of which there are none.

Hence $\zeta^{\mathsf{Sets}}_{\ell|A}$ is an isomorphism.
Naturality: We need to show that given a function $f \colon A \longrightarrow B$, the diagram

$$\begin{tikzcd}
\emptyset \times A \arrow[r, "\mathrm{id}_{\emptyset} \times f"] \arrow[d, "\zeta^{\mathsf{Sets}}_{\ell|A}"'] & \emptyset \times B \arrow[d, "\zeta^{\mathsf{Sets}}_{\ell|B}"] \\
\emptyset \arrow[r, "\mathrm{id}_{\emptyset}"'] & \emptyset
\end{tikzcd}$$

commutes. But since $\emptyset \times A$ has no elements, this is trivially true.
Being a Natural Isomorphism: Since $\zeta_\ell^{\mathsf{Sets}}$ is natural and $\zeta_\ell^{\mathsf{Sets},-1}$ is a componentwise inverse to $\zeta_\ell^{\mathsf{Sets}}$, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\zeta_\ell^{\mathsf{Sets},-1}$ is also natural. Thus $\zeta_\ell^{\mathsf{Sets}}$ is a natural isomorphism. □

01Q7 5.3.4 The Right Annihilator

01Q8 **Definition 5.3.4.1.1.** The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{0}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}} \times \mathrm{id}_{\mathrm{pt}}) \overset{\sim}{\dashrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{0}^{\mathsf{Sets}})$$

as in the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] & \\
\mathsf{Sets} \times \mathrm{pt} \arrow[ur, "\boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}} \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_{\mathsf{Sets}} \times \mathbb{0}^{\mathsf{Sets}}"'] \arrow[r, phantom, "\Swarrow \zeta_r^{\mathsf{Sets}}"] & {} & \mathrm{pt} \arrow[d, "\mathbb{0}^{\mathsf{Sets}}"] \\
\mathsf{Sets} \times \mathsf{Sets} \arrow[rr, "\times"'] & & \mathsf{Sets}
\end{tikzcd}$$

with components

$$\zeta_{r|A}^{\mathsf{Sets}} \colon A \times \emptyset \overset{\sim}{\dashrightarrow} \emptyset$$

given by $\zeta_{r|A}^{\mathsf{Sets}} \overset{\mathrm{def}}{=} \mathrm{pr}_2$.

Proof. *Invertibility*: The inverse of $\zeta_{r|A}^{\mathsf{Sets}}$ is the map

$$\zeta_{r|A}^{\mathsf{Sets},-1} \colon \emptyset \overset{\sim}{\dashrightarrow} A \times \emptyset$$

given by

$$\zeta_{r|A}^{\mathsf{Sets},-1} \overset{\mathrm{def}}{=} \iota_A,$$

where ι_A is as defined in Constructions With Sets, Definition 4.2.1.1.2:

- *Invertibility I.* The map $\zeta_{r|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \longrightarrow \emptyset$ is equal to $\mathrm{id}_\emptyset$, as $\emptyset$ is the initial object of Sets.

- *Invertibility II.* The map $\iota_A \circ \zeta^{\mathsf{Sets}}_{r|A}$ is equal to the identity on every $(a, x) \in A \times \emptyset$, of which there are none.

Hence $\zeta^{\mathsf{Sets}}_{r|A}$ is an isomorphism.
Naturality: We need to show that given a function $f \colon A \to B$, the diagram

$$\begin{tikzcd}
A \times \emptyset \arrow[r, "f \times \mathrm{id}_\emptyset"] \arrow[d, "\zeta^{\mathsf{Sets}}_{r|A}"'] & B \times \emptyset \arrow[d, "\zeta^{\mathsf{Sets}}_{r|B}"] \\
\emptyset \arrow[r, "\mathrm{id}_\emptyset"'] & \emptyset
\end{tikzcd}$$

commutes. But since $A \times \emptyset$ has no elements, this is trivially true.
Being a Natural Isomorphism: Since ζ^{Sets}_r is natural and $\zeta^{\mathsf{Sets},-1}_r$ is a componentwise inverse to ζ^{Sets}_r, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that $\zeta^{\mathsf{Sets},-1}_r$ is also natural. Thus ζ^{Sets}_r is a natural isomorphism. □

01Q9 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

01QA **Proposition 5.3.5.1.1.** The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Additive Monoidal Product.* The coproduct functor

$$\coprod \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.
- *The Multiplicative Monoidal Product.* The product functor

$$\times \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.
- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

  of Definition 5.1.3.1.1.

- *The Monoidal Zero.* The functor

$$\mathbb{0}^{\mathsf{Sets}}\colon \mathrm{pt} \longrightarrow \mathsf{Sets}$$

of Definition 5.1.3.1.1.

- *The Internal Hom.* The internal Hom functor

$$\mathsf{Sets}\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

of Constructions With Sets, **??** of **??**.

- *The Additive Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets},\coprod}\colon \coprod \circ \left(\coprod \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \coprod \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \coprod\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.3.1.1.

- *The Additive Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets},\coprod}\colon \coprod \circ \left(\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

- *The Additive Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod}\colon \coprod \circ \left(\mathrm{id} \times \mathbb{0}^{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.5.1.1.

- *The Additive Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}\colon \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

- *The Multiplicative Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}}\colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

- *The Multiplicative Left Unitors.* The natural isomorphism
$$\lambda^{\mathsf{Sets}}\colon \times \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$
of Definition 5.1.5.1.1.

- *The Multiplicative Right Unitors.* The natural isomorphism
$$\rho^{\mathsf{Sets}}\colon \times \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$
of Definition 5.1.6.1.1.

- *The Multiplicative Symmetry.* The natural isomorphism
$$\sigma^{\mathsf{Sets}}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$
of Definition 5.1.7.1.1.

- *The Left Distributor.* The natural isomorphism
$$\delta^{\mathsf{Sets}}_{\ell}\colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \textstyle\coprod) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$
of Definition 5.3.1.1.1.

- *The Right Distributor.* The natural isomorphism
$$\delta^{\mathsf{Sets}}_{r}\colon \times \circ (\textstyle\coprod \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ ((\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.1.1.

- *The Left Annihilator.* The natural isomorphism
$$\zeta^{\mathsf{Sets}}_{\ell}\colon \mathbb{0}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\mathrm{id}_{\mathrm{pt}} \times \boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$
of Definition 5.3.3.1.1.

- *The Right Annihilator.* The natural isomorphism
$$\zeta^{\mathsf{Sets}}_{r}\colon \mathbb{0}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}} \times \mathrm{id}_{\mathrm{pt}}) \overset{\sim}{\dashrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{0}^{\mathsf{Sets}})$$
of Definition 5.3.4.1.1.

Proof. Omitted. □

Appendices

A Other Chapters