# Constructions With Monoidal Categories

# The Clowder Project Authors

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This chapter contains some material on constructions with monoidal categories.

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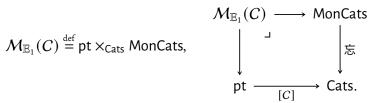
Let *C* be a category.

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# DEFINITION 13.1.1.1.1 ➤ THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATE-

The **moduli category of monoidal structures on** C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{B}_1}(C)\stackrel{ ext{def}}{=}\mathsf{pt} imes_{\mathsf{Cats}}\mathsf{MonCats},$$



#### REMARK 13.1.1.1.2 ► Unwinding Definition 13.1.1.1.1, I

In detail, **the moduli category of monoidal structures on** *C* is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- Objects. The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is C.
- *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \alpha^C, \alpha^C, \lambda^C, \rho^C)$  $\mathbb{1}'_C$ ,  $\alpha^{C,\prime}$ ,  $\lambda^{C,\prime}$ ,  $\rho^{C,\prime}$ ) is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C : C \to C$  of C.

• *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{R}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \stackrel{\operatorname{def}}{=} (\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes}),$$

where  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  is the identity monoidal functor of C of ??.

• *Composition.* For each  $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\begin{split} \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,N) &\to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_1}(C)}(M,P) \\ & \text{of } \mathcal{M}_{\mathbb{B}_1}(C) \text{ at } (M,N,P) \text{ is defined by} \\ & \left( \operatorname{id}_{C}^{\otimes,\prime}, \operatorname{id}_{\mathbb{A}|C}^{\otimes,\prime} \right) \circ^{\mathcal{M}_{\mathbb{B}_1}(C)}_{M,N,P} \left( \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{A}|C}^{\otimes} \right) \overset{\text{def}}{=} \left( \operatorname{id}_{C}^{\otimes,\prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{A}|C}^{\otimes,\prime} \circ \operatorname{id}_{\mathbb{A}|C}^{\otimes} \right). \end{split}$$

#### REMARK 13.1.1.1.3 ► Unwinding Definition 13.1.1.1.1, II

In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

1. *Naturality*. For each pair  $f: A \to X$  and  $g: B \to Y$  of morphisms of C, the diagram

$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

commutes.

2. Monoidality. For each  $A, B, C \in \text{Obj}(C)$ , the diagram

$$(A \boxtimes_{C} B) \boxtimes_{C} C$$

$$(A \otimes_{C} B) \boxtimes_{C} C$$

$$(A \otimes_{C} B) \boxtimes_{C} C$$

$$id^{\otimes}_{A \otimes_{C} B, C}$$

$$(A \otimes_{C} B) \otimes_{C} C$$

$$id^{\otimes}_{A \otimes_{C} B, C}$$

$$(A \otimes_{C} B) \otimes_{C} C$$

$$A \boxtimes_{C} (B \otimes_{C} C)$$

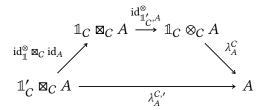
$$id^{\otimes}_{A, B, C}$$

$$(A \otimes_{C} B) \otimes_{C} C$$

$$A \boxtimes_{C} (B \otimes_{C} C)$$

commutes.

3. Left Monoidal Unity. For each  $A \in Obj(C)$ , the diagram



commutes.

4. Right Monoidal Unity. For each  $A \in Obj(C)$ , the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A$$

commutes.

#### PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUC-TURES ON A CATEGORY

Let *C* be a category.

- 1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B\boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
  - (a) The diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C}$$

$$\downarrow^{C} \qquad \qquad \downarrow^{\rho_{\mathbb{1}_{C}}^{C,'}}$$

$$\mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes}} \mathbb{1}_{C}$$

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. Mixed Associators. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \longrightarrow -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} \colon (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A \otimes_{C} B,C} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\operatorname{id}_{A,B}^{\otimes} \boxtimes_{C} \operatorname{id}_{C} \downarrow \qquad \qquad \qquad \operatorname{id}_{A,B\boxtimes_{C} C}$$

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_C (B \otimes_C C) \\ \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & & & \operatorname{id}_{A,B \otimes_C C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A\boxtimes_C B)\boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_C (B\boxtimes_C C) \\ \mathrm{id}_{A\boxtimes_C B,C}^\otimes & & & & & & & \\ (A\boxtimes_C B)\otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} A\boxtimes_C (B\otimes_C C) \end{array}$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow \operatorname{id}_{A,B} \otimes_{C} \operatorname{id}_{C} \qquad \qquad \qquad \downarrow \operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

#### PROOF 13.1.1.1.5 ► PROOF OF PROPOSITION 13.1.1.1.4

## Item 1: Extra Monoidality Conditions

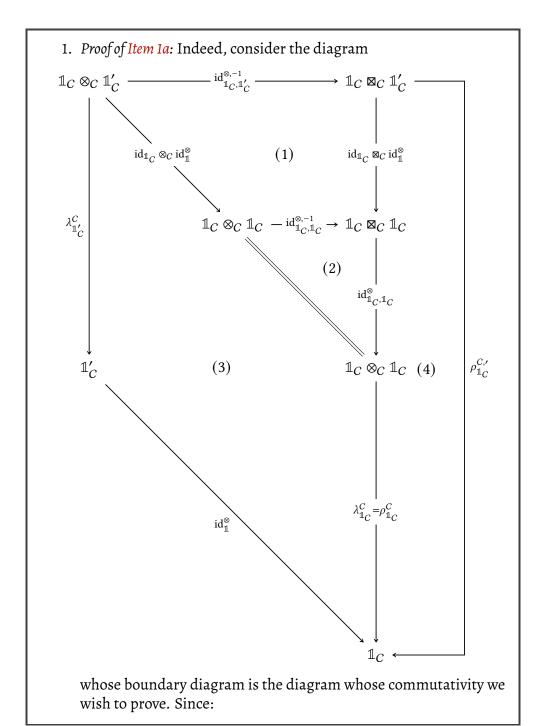
We claim that Items 1a and 1b are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{AB}^{\otimes}$  and  $id_{C}$ .
- 2. *Proof of Item 1b*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

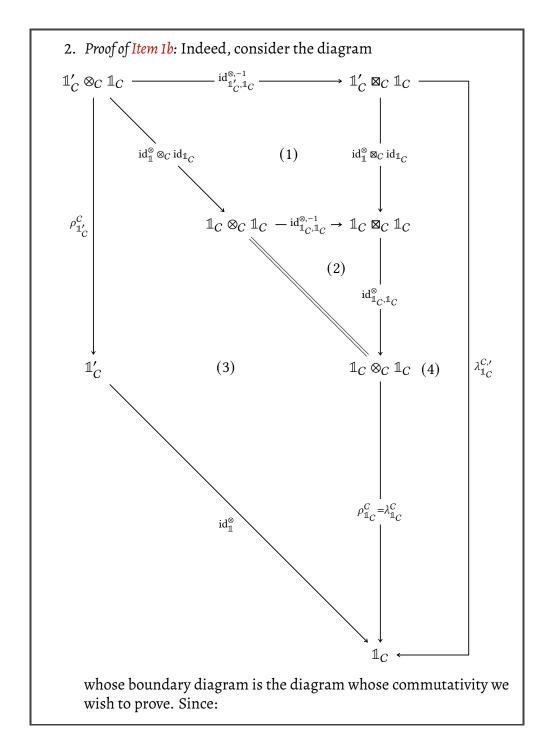
### Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:



- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$  comes from  $\ref{eq:composition}$ ;
- Subdiagram (4) commutes by the right monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.



- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from  $\ref{eq:composition}$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes});$

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{C,'}_{\mathbb{1}_{C}} \qquad \qquad (\dagger) \qquad \qquad \downarrow^{\rho_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes. But since  $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \\
\rho_{\mathbb{1}_{C}}^{C,'} \qquad (\dagger) \qquad \downarrow_{\lambda_{\mathbb{1}'_{C}}^{C}}$$

$$\downarrow \\
\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}_{C}$$

commutes. But since  $id_1^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

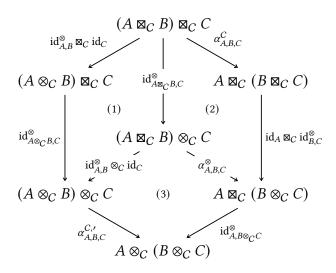
This finishes the proof.

#### Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

1. *Proof of Item 3a*: We may partition the monoidality diagram for  $id^{\otimes}$ 

#### of Item 2 of Remark 13.1.1.1.3 as follows:



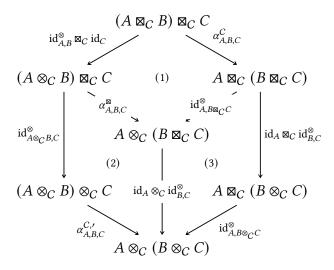
#### Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. *Proof of Item 3b*: We may partition the monoidality diagram for  $id^{\otimes}$ 

#### of Item 2 of Remark 13.1.1.1.3 as follows:



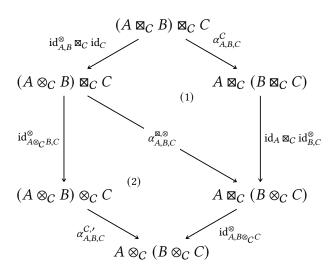
#### Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for  $id^{\otimes}$ 

#### of Item 2 of Remark 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# **Appendices**

## A Other Chapters

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- 2. A Guide to the Literature

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- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
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- 11. Categories
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#### **Monoidal Categories**

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Categories

## Bicategories

### Extra Part

14. Types of Morphisms in Bicate- 15. Notes