# Monoidal Structures on the Category of Sets

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This chapter contains some material on monoidal structures on Sets.

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## 5.1 The Monoidal Category of Sets and Products

### 5.1.1 Products of Sets

See Constructions With Sets, Section 4.1.3.

### 5.1.2 The Internal Hom of Sets

See Constructions With Sets, Section 4.3.5.

### 5.1.3 The Monoidal Unit

Definition 5.1.3.1.1. The monoidal unit of the product of sets is the functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{1}_{\mathsf{Sets}} \stackrel{\scriptscriptstyle \mathrm{def}}{=} \mathsf{pt},$$

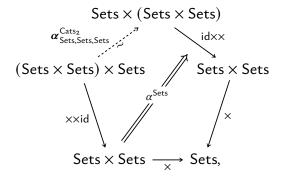
where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.1.

### 5.1.4 The Associator

**Definition 5.1.4.1.1.** The **associator of the product of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}'}$$

as in the diagram



whose component

$$\alpha_{XYZ}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \stackrel{\mathrm{def}}{=} (x,(y,z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

*Proof. Invertibility*: The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is the morphism

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \stackrel{\text{def}}{=} ((x,y),z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

• *Invertibility I*. We have

$$\begin{split} \left[\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}}\right] &((x,y),z) \stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1} \Big(\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z)\Big) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} ((x,y),z) \\ &\stackrel{\mathsf{def}}{=} \left[\mathrm{id}_{(X\times Y)\times Z}\right] &((x,y),z) \end{split}$$

for each  $((x, y), z) \in (X \times Y) \times Z$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{(X \times Y) \times Z} \,.$$

• Invertibility II. We have

$$\begin{split} \left[\alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1}\right] (x,(y,z)) &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} \left(\alpha_{X,Y,Z}^{\mathsf{Sets},-1} (x,(y,z))\right) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} ((x,y),z) \\ &\stackrel{\mathsf{def}}{=} (x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} \left[\mathrm{id}_{(X\times Y)\times Z}\right] (x,(y,z)) \end{split}$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{X \times (Y \times Z)} \;.$$

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

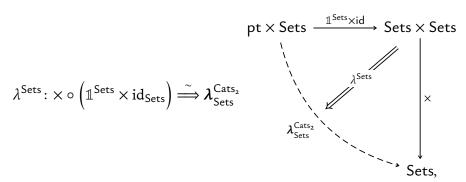
$$\begin{array}{c|c} (X\times Y)\times Z & \xrightarrow{(f\times g)\times h} & (X'\times Y')\times Z' \\ \\ \alpha_{X,Y,Z}^{\mathsf{Sets}} & & & & & & \\ x\times (Y\times Z) & \xrightarrow{f\times (g\times h)} & X'\times (Y'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes, showing  $\alpha^{\mathsf{Sets}}$  to be a natural transformation. Being a Natural Isomorphism: Since  $\alpha^{\mathsf{Sets}}$  is natural and  $\alpha^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\alpha^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\alpha^{\mathsf{Sets},-1}$  is also natural. Thus  $\alpha^{\mathsf{Sets}}$  is a natural isomorphism.

### 5.1.5 The Left Unitor

**Definition 5.1.5.1.1.** The **left unitor of the product of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets}} \colon \mathsf{pt} \times X \xrightarrow{\sim} X$$

at  $X \in Obj(Sets)$  is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\mathsf{def}}{=} x$$

for each  $(\star, x) \in pt \times X$ .

*Proof. Invertibility*: The inverse of  $\lambda_X^{\text{Sets}}$  is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathsf{pt} \times X$$

defined by

$$\lambda_X^{\mathrm{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star,x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}\right] (\mathsf{pt},x) &= \lambda_X^{\mathsf{Sets},-1} \Big(\lambda_X^{\mathsf{Sets}} (\mathsf{pt},x)\Big) \\ &= \lambda_X^{\mathsf{Sets},-1} (x) \\ &= (\mathsf{pt},x) \\ &= \left[\mathrm{id}_{\mathsf{pt} \times X}\right] (\mathsf{pt},x) \end{split}$$

for each  $(pt, x) \in pt \times X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}} = \mathrm{id}_{\mathsf{pt} \times X} \,.$$

• Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}\right](x) &= \lambda_X^{\mathsf{Sets}} \left(\lambda_X^{\mathsf{Sets},-1}(x)\right) \\ &= \lambda_X^{\mathsf{Sets},-1}(\mathsf{pt},x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

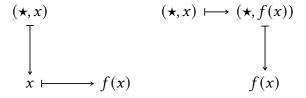
$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X .$$

Therefore  $\lambda_X^{\mathrm{Sets}}$  is indeed an isomorphism.

*Naturality*: We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc}
\operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\
\lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\
X & \xrightarrow{f} & Y
\end{array}$$

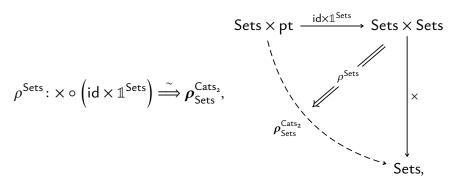
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\lambda^{\text{Sets}}$  is a natural transformation. Being a Natural Isomorphism: Since  $\lambda^{\text{Sets}}$  is natural and  $\lambda^{\text{Sets},-1}$  is a componentwise inverse to  $\lambda^{\text{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\lambda^{\text{Sets},-1}$  is also natural. Thus  $\lambda^{\text{Sets}}$  is a natural isomorphism.

### 5.1.6 The Right Unitor

**Definition 5.1.6.1.1.** The **right unitor of the product of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \mathsf{pt} \stackrel{\sim}{\dashrightarrow} X$$

at  $X \in Obj(Sets)$  is given by

$$\rho_X^{\mathsf{Sets}}(x, \star) \stackrel{\mathsf{def}}{=} x$$

for each  $(x, \star) \in X \times pt$ .

*Proof. Invertibility*: The inverse of  $\rho_X^{\text{Sets}}$  is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (x, \star)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} \right] (x, \star) &= \rho_X^{\mathsf{Sets},-1} \left( \rho_X^{\mathsf{Sets}} (x, \star) \right) \\ &= \rho_X^{\mathsf{Sets},-1} (x) \\ &= (x, \star) \\ &= \left[ \mathrm{id}_{X \times \mathrm{pt}} \right] (x, \star) \end{split}$$

for each  $(x, \star) \in X \times pt$ , and therefore we have

$$ho_X^{\mathsf{Sets},-1} \circ 
ho_X^{\mathsf{Sets}} = \mathrm{id}_{X \times \mathrm{pt}} \,.$$

• Invertibility II. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} \right] (x) &= \rho_X^{\mathsf{Sets}} \left( \rho_X^{\mathsf{Sets},-1} (x) \right) \\ &= \rho_X^{\mathsf{Sets},-1} (x, \bigstar) \\ &= x \\ &= [\mathrm{id}_X] (x) \end{split}$$

for each  $x \in X$ , and therefore we have

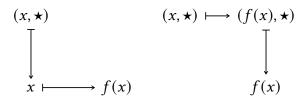
$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X.$$

Therefore  $\rho_X^{\rm Sets}$  is indeed an isomorphism.

*Naturality*: We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{c} X \times \operatorname{pt} \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} Y \times \operatorname{pt} \\ \rho_X^{\operatorname{Sets}} \downarrow & & \downarrow \rho_Y^{\operatorname{Sets}} \\ X \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\rho^{\rm Sets}$  is a natural transformation. Being a Natural Isomorphism: Since  $\rho^{\rm Sets}$  is natural and  $\rho^{\rm Sets,-1}$  is a componentwise inverse to  $\rho^{\rm Sets}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\rho^{\rm Sets,-1}$  is also natural. Thus  $\rho^{\rm Sets}$  is a natural isomorphism.

### 5.1.7 The Symmetry

**Definition 5.1.7.1.1.** The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \xrightarrow{\sim} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{array}{c} \mathsf{Sets} \times \mathsf{Sets} & \xrightarrow{\times} & \mathsf{Sets}, \\ & \downarrow & & \\ & & \downarrow & & \\ & & & \mathsf{Sets} \times \mathsf{Sets} \end{array}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets}} \colon X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in Obj(Sets)$  is defined by

$$\sigma_{X,Y}^{\text{Sets}}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

*Proof. Invertibility*: The inverse of  $\sigma_{X,Y}^{\mathsf{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \colon Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets},-1}(y,x) \stackrel{\text{def}}{=} (x,y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

• Invertibility I. We have

$$\begin{bmatrix} \sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}} \end{bmatrix} (x,y) \stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \left( \sigma_{X,Y}^{\mathsf{Sets}} (x,y) \right) \\
\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} (y,x) \\
\stackrel{\text{def}}{=} (x,y) \\
\stackrel{\text{def}}{=} [\mathrm{id}_{X\times Y}] (x,y)$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets},-1} \circ \sigma_{X,Y}^{\text{Sets}} = \mathrm{id}_{X \times Y}$$
.

• Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1}\right] (y,x) &\stackrel{\scriptscriptstyle{\mathsf{def}}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \Big(\sigma_{X,Y}^{\mathsf{Sets}} (y,x)\Big) \\ &\stackrel{\scriptscriptstyle{\mathsf{def}}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} (x,y) \\ &\stackrel{\scriptscriptstyle{\mathsf{def}}}{=} (y,x) \\ &\stackrel{\scriptscriptstyle{\mathsf{def}}}{=} [\mathrm{id}_{Y \times X}] (y,x) \end{split}$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \mathrm{id}_{Y \times X}$$
.

Therefore  $\sigma_{X,Y}^{\text{Sets}}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to A$$
,  $g: Y \to B$ 

the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & A \times B \\ \sigma_{X,Y}^{\mathsf{Sets}} & & & & & & \\ \sigma_{X,Y}^{\mathsf{Sets}} & & & & & \\ Y \times X & \xrightarrow{g \times f} & B \times A & & & \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x,y) \longmapsto (f(x),g(y))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(y,x) \longmapsto (g(y),f(x)) \qquad \qquad (g(y),f(x))$$

and hence indeed commutes, showing  $\sigma^{\mathsf{Sets}}$  to be a natural transformation. Being a Natural Isomorphism: Since  $\sigma^{\mathsf{Sets}}$  is natural and  $\sigma^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\sigma^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\sigma^{\mathsf{Sets},-1}$  is also natural. Thus  $\sigma^{\mathsf{Sets}}$  is a natural isomorphism.

### 5.1.8 The Diagonal

**Definition 5.1.8.1.1.** The **diagonal of the product of sets** is the natural transformation



whose component

$$\Delta_X \colon X \to X \times X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .

*Proof.* We need to show that, given a function  $f: X \to Y$ , the diagram

$$X \xrightarrow{f} Y$$

$$\Delta_X \downarrow \qquad \qquad \downarrow \Delta_Y$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
x & & x & \longrightarrow f(x) \\
\downarrow & & \downarrow \\
(x,x) & \longmapsto (f(x),f(x)) & & (f(x),f(x))
\end{array}$$

and hence indeed commutes, showing  $\Delta$  to be natural.

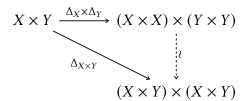
**Proposition 5.1.8.1.2.** Let X be a set.

1. Monoidality. The diagonal map

$$\Delta : id_{\mathsf{Sets}} \Longrightarrow \mathsf{X} \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

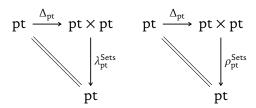
is a monoidal natural transformation:

(a) Compatibility With Strong Monoidality Constraints. For each  $X, Y \in Obj(Sets)$ , the diagram



commutes.

(b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets},-1} \\ &= \rho_{\text{pt}}^{\text{Sets},-1}, \end{split}$$

where we recall that the equalities

$$\begin{split} \lambda_{\mathrm{pt}}^{\mathrm{Sets}} &= \rho_{\mathrm{pt}}^{\mathrm{Sets}}, \\ \lambda_{\mathrm{pt}}^{\mathrm{Sets},-1} &= \rho_{\mathrm{pt}}^{\mathrm{Sets},-1} \end{split}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

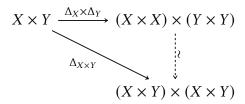
2. *The Diagonal of the Unit*. The component

$$\Delta_{pt} \colon pt \xrightarrow{\sim} pt \times pt$$

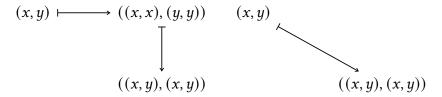
of  $\Delta$  at pt is an isomorphism.

*Proof. Item* 1, *Monoidality*: We claim that  $\Delta$  is indeed monoidal:

1. *Item 1a*: *Compatibility With Strong Monoidality Constraints*: We need to show that the diagram



commutes. Indeed, this diagram acts on elements as



and hence indeed commutes.

2. *Item 1b*: *Compatibility With Strong Unitality Constraints*: As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star,x)$$

for each  $x \in X$ , so when X = pt, we have

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets},-1}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

and also

$$\Delta_{\mathrm{pt}}^{\mathrm{Sets}}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

so we have  $\Delta_{\rm pt} = \lambda_{\rm pt}^{\rm Sets,-1}$ .

This finishes the proof.

*Item 2, The Diagonal of the Unit*: This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to ×, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor. □

### 5.1.9 The Monoidal Category of Sets and Products

**Proposition 5.1.9.1.1.** The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category*. The category Sets of pointed sets.
- The Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Definition 4.3.5.1.2.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{}}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \pmb{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} : \times \circ \left( \mathsf{id} \times \mathbb{1}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• *The Symmetry*. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

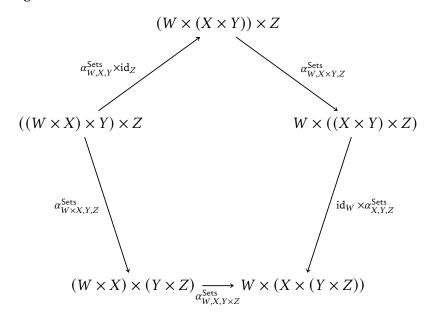
of Definition 5.1.7.1.1.

• The Diagonals. The monoidal natural transformation

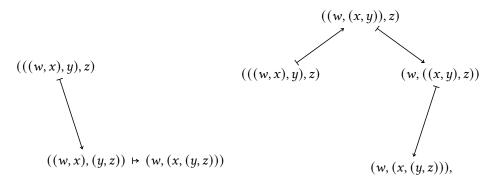
$$\Delta\colon\operatorname{id}_{\mathsf{Sets}} \Longrightarrow \mathsf{X}\circ\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

*Proof.* The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

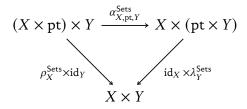


commutes. Indeed, this diagram acts on elements as

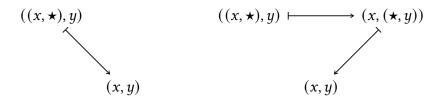


and thus the pentagon identity is satisfied.

The Triangle Identity: Let X and Y be sets. We have to show that the diagram

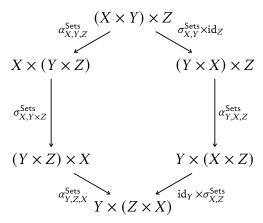


commutes. Indeed, this diagram acts on elements as

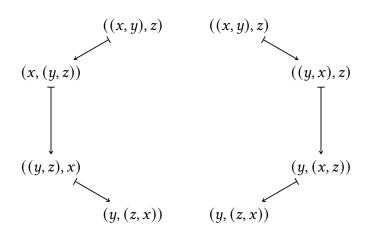


and thus the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

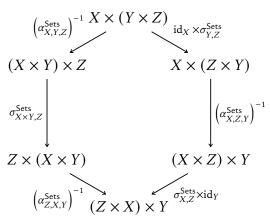


commutes. Indeed, this diagram acts on elements as

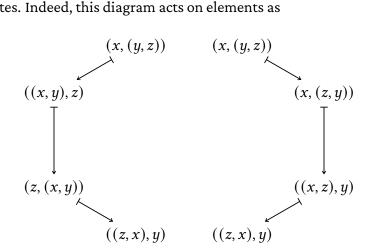


and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the right hexagon identity is satisfied.

Monoidal Closedness: This follows from Constructions With Sets, Item 2 of Definition 4.3.5.1.2

Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 5.1.8.1.2.

#### The Universal Property of (Sets, $\times$ , pt) 5.1.10

**Theorem 5.1.10.1.1.** The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Existence of an Internal Hom. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom  $[-1, -2]_{Sets}$ .

2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

More precisely, the full subcategory of the category  $\mathcal{M}^{cld}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of **??** spanned by the closed symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,  $[-_1, -_2]_{\mathsf{Sets}}$ ,  $\mathbb{1}_{\mathsf{Sets}}$ ,  $\lambda^{\rm Sets}, \, \rho^{\rm Sets}, \, \sigma^{\rm Sets})$  satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

*Proof. Unwinding the Statement:* Let (Sets,  $\otimes_{Sets}$ ,  $[-1, -2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ )

be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$id_{Sets}$$
: Sets  $\rightarrow$  Sets

admits a unique closed symmetric monoidal functor structure

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) to the closed symmetric monoidal category (Sets,  $\times$ , Sets $(-_1, -_2)$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda^{Sets}$ ,  $\rho^{Sets}$ ,  $\sigma^{Sets}$ ) of Definition 5.1.9.1.1.

Constructing an Isomorphism  $[-1, -2]_{Sets} \cong Sets(-1, -2)$ : By ??, we have a natural isomorphism

$$Sets(pt, [-1, -2]_{Sets}) \cong Sets(-1, -2).$$

By Constructions With Sets, Item 3 of Definition 4.3.5.1.2, we also have a natural isomorphism

$$\mathsf{Sets}(\mathsf{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$Sets(-1, -2) \cong [-1, -2]_{Sets}.$$

Given  $A, B \in Obj(Sets)$ , we will write

$$id_{AB}^{Hom} : Sets(A, B) \xrightarrow{\sim} [A, B]_{Sets}$$

for the component of this isomorphism at (A, B).

Constructing an Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : Since  $\otimes_{\mathsf{Sets}}$  is adjoint in each variable to  $[-1, -2]_{\mathsf{Sets}}$  by assumption and  $\times$  is adjoint in each variable to  $\mathsf{Sets}(-1, -2)$  by Constructions With Sets, Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -$$

$$-\otimes_{\mathsf{Sets}} B \cong B \times -.$$

By ??, we then have  $\otimes_{Sets} \cong \times$ . We will write

$$\operatorname{id}_{\operatorname{Sets}|A.B}^{\otimes} \colon A \otimes_{\operatorname{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

Alternative Construction of an Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : Alternatively, we may construct a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  as follows:

- 1. Let  $A \in Obj(Sets)$ .
- 2. Since  $\otimes_{Sets}$  is part of a closed monoidal structure, it preserves colimits in each variable by ??.
- 3. Since  $A \cong \coprod_{a \in A} \operatorname{pt}$  and  $\otimes_{\operatorname{Sets}}$  preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong \left( \bigsqcup_{a \in A} \mathsf{pt} \right) \otimes_{\mathsf{Sets}} B$$

$$\cong \bigsqcup_{a \in A} (\mathsf{pt} \otimes_{\mathsf{Sets}} B)$$

$$\cong \bigsqcup_{a \in A} B$$

$$\cong A \times B.$$

naturally in  $B \in \text{Obj}(\mathsf{Sets})$ , where we have used that pt is the monoidal unit for  $\otimes_{\mathsf{Sets}}$ . Thus  $A \otimes_{\mathsf{Sets}} - \cong A \times -$  for each  $A \in \mathsf{Obj}(\mathsf{Sets})$ .

- 4. Similarly,  $\otimes_{\mathsf{Sets}} B \cong \times B$  for each  $B \in \mathsf{Obj}(\mathsf{Sets})$ .
- 5. By ??, we then have  $\otimes_{Sets} \cong \times$ .

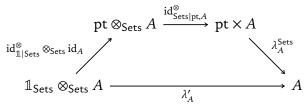
Below, we'll show that if a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \colon A \otimes_{\mathsf{Sets}} B \to A \times B$  from before.

Constructing an Isomorphism  $id_{\mathbb{1}}^{\otimes} : \mathbb{1}_{\mathsf{Sets}} \to pt$ : We define an isomorphism  $id_{\mathbb{1}}^{\otimes} : \mathbb{1}_{\mathsf{Sets}} \to pt$  as the composition

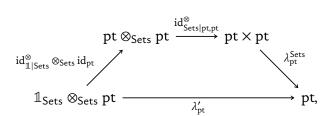
$$\mathbb{1}_{\mathsf{Sets}} \overset{\rho^{\mathsf{Sets},-1}_{\mathbb{1}_{\mathsf{Sets}}}}{\overset{\sim}{\dots}} \mathbb{1}_{\mathsf{Sets}} \times \mathsf{pt} \overset{\mathrm{id}^{\otimes}_{\mathsf{Sets}}|\mathbb{1}_{\mathsf{Sets}}}{\overset{\sim}{\dots}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathsf{pt} \overset{\lambda'_{\mathsf{pt}}}{\overset{\sim}{\dots}} \mathsf{pt}$$

in Sets.

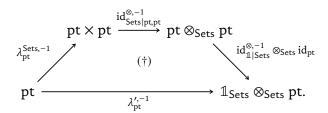
*Monoidal Left Unity of the Isomorphism*  $\otimes_{\mathsf{Sets}} \cong \times$ : We have to show that the diagram

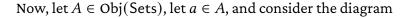


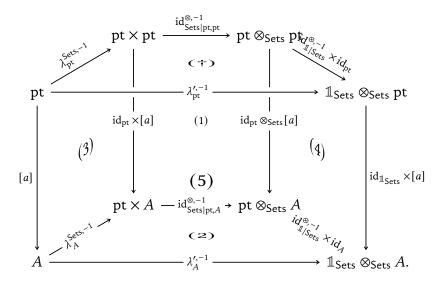
commutes. First, note that the diagram



corresponding to the case  $A = \operatorname{pt}$ , commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram



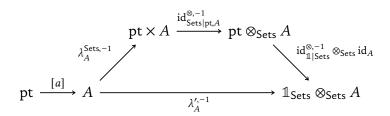




Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'^{-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{1|Sets}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\text{Sets},-1}$ .

#### it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\lambda_A^{\prime,-1}(a) = \left[\lambda_A^{\prime,-1} \circ [a]\right](\star)$$

$$= \left[ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_{A} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_{A}^{\mathsf{Sets},-1} \circ [a] \right] (\star)$$

$$= \left[ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_{A} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_{A}^{\mathsf{Sets},-1} \right] (a)$$

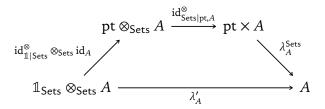
for each  $a \in A$ , and thus we have

$$\lambda_A^{\prime,-1} = \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ \left(\mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \times \mathrm{id}_A\right),$$

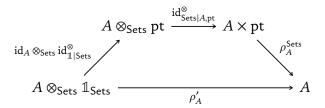
showing that the diagram



indeed commutes.

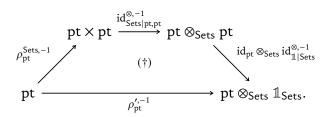
Monoidal Right Unity of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  above. For completeness, we repeat it below.

We have to show that the diagram

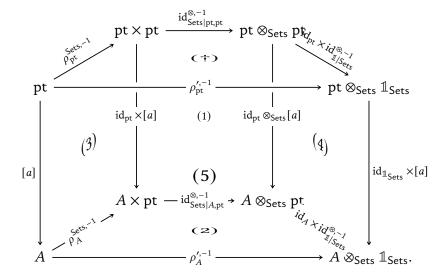


commutes. First, note that the diagram

corresponding to the case  $A = \operatorname{pt}$ , commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram



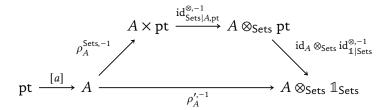
Now, let  $A \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , and consider the diagram



#### Since:

- Subdiagram (5) commutes by the naturality of  $\rho'^{-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{\mathbb{1}|Sets}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\mathrm{Sets},-1}$ .

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\rho_{A}^{\prime,-1}(a) = \left[\rho_{A}^{\prime,-1} \circ [a]\right](\star)$$

$$= \left[\left(\mathrm{id}_{A} \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_{A}^{\mathsf{Sets},-1} \circ [a]\right](\star)$$

$$= \left[\left(\mathrm{id}_{A} \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_{A}^{\mathsf{Sets},-1}\right](a)$$

for each  $a \in A$ , and thus we have

$$\rho_A^{\prime,-1} = \left( \mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ \left(\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes}\right),$$

showing that the diagram

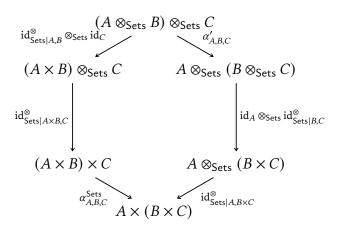
$$A \otimes_{\mathsf{Sets}} \mathsf{pt} \xrightarrow{\mathsf{id}_{\mathsf{Sets}}^{\otimes} |A,\mathsf{pt}|} A \times \mathsf{pt}$$

$$\mathsf{id}_A \otimes_{\mathsf{Sets}} \mathsf{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \xrightarrow{\rho_A'} A$$

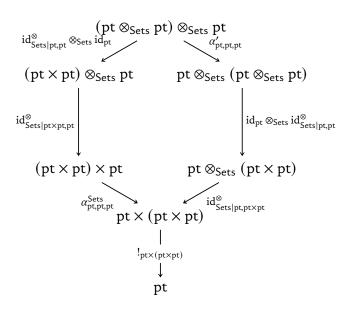
$$A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \xrightarrow{\rho_A'} A$$

indeed commutes.

*Monoidality of the Isomorphism*  $\otimes_{\mathsf{Sets}} \cong \times$ : We have to show that the diagram

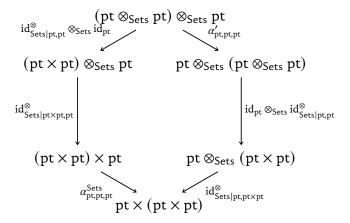


commutes. First, note that the diagram

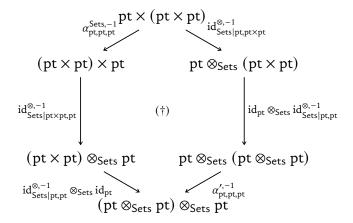


commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map  $!_{pt \times (pt \times pt)} : pt \times (pt \times pt) \rightarrow pt$  is an isomorphism (e.g. having

inverse  $\lambda_{pt}^{\mathsf{Sets},-1} \circ \lambda_{pt}^{\mathsf{Sets},-1})$  , it follows that the diagram

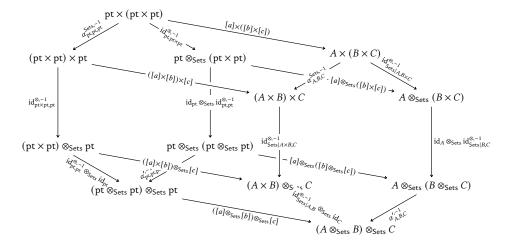


also commutes. Taking inverses, we see that the diagram

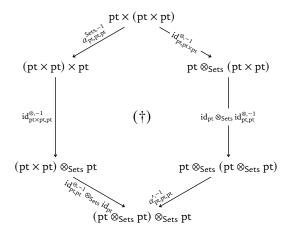


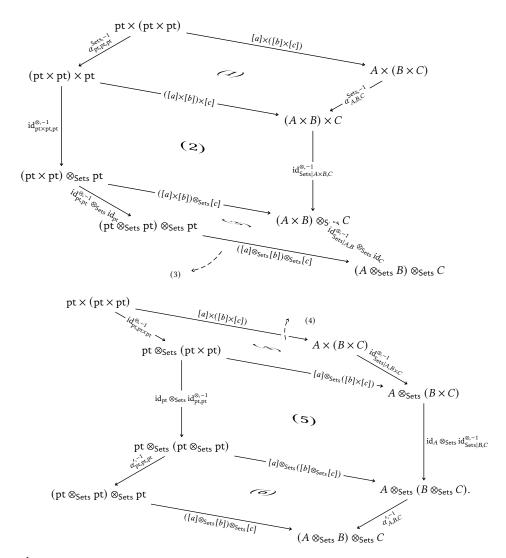
commutes as well. Now, let  $A, B, C \in Obj(Sets)$ , let  $a \in A$ , let  $b \in B$ , let  $c \in C$ ,

### and consider the diagram



### which we partition into subdiagrams as follows:



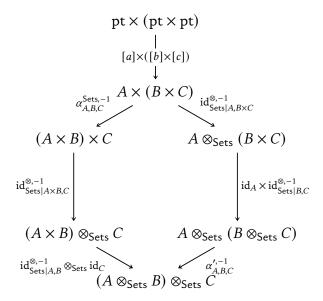


#### Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\rm Sets,-1}.$
- Subdiagram (2) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (†) commutes, as proved above.

- Subdiagram (4) commutes by the naturality of  $id_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (5) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .

#### it follows that the diagram



also commutes. We then have

$$\begin{split} \left[ \left( \operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_{C} \right) \circ \operatorname{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) &= \left[ \left( \operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_{C} \right) \circ \operatorname{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ \left( [a] \times ([b] \times [c]) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \\ \circ \operatorname{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ \left( [a] \times ([b] \times [c]) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \circ \operatorname{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

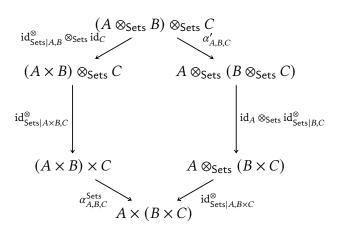
for each  $(a, (b, c)) \in A \times (B \times C)$ , and thus we have

$$\left(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1}\otimes_{\mathsf{Sets}}\mathrm{id}_{C}\right)\circ\mathrm{id}_{\mathsf{Sets}|A\times B,C}^{\otimes,-1}\circ\alpha_{A,B,C}^{\mathsf{Sets},-1}=\alpha_{A,B,C}^{\prime,-1}\circ\left(\mathrm{id}_{A}\times\mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}\right)\circ\mathrm{id}_{\mathsf{Sets}|A,B\times C}^{\otimes,-1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ \left( \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_{C} \right) = \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ \left( \mathrm{id}_{A} \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes} \right) \circ \alpha_{A,B,C}'$$

showing that the diagram



indeed commutes.

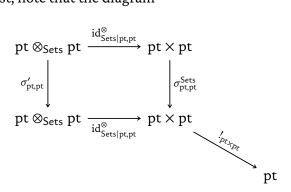
Braidedness of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : We have to show that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\sigma'_{A,B} \downarrow \qquad \qquad \qquad \downarrow \sigma^{\mathsf{Sets}}_{A,B}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

commutes. First, note that the diagram



commutes by the terminality of pt (Constructions With Sets, Definition 4.1.1.1.2). Since the map  $!_{pt \times pt} \colon pt \times pt \to pt$  is invertible (e.g. with inverse  $\lambda_{pt}^{Sets,-1}$ ), the

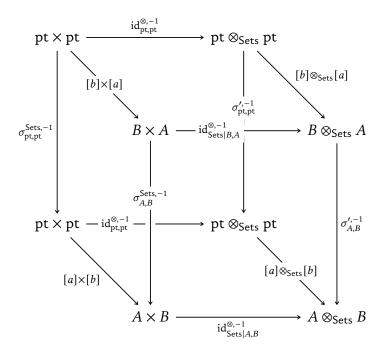
diagram

$$\begin{array}{c|c} pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes}} pt \times pt \\ \\ \sigma'_{\mathsf{pt,pt}} & & & & & \\ \sigma'_{\mathsf{pt,pt}} & & & & \\ pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes}} pt \times pt \end{array}$$

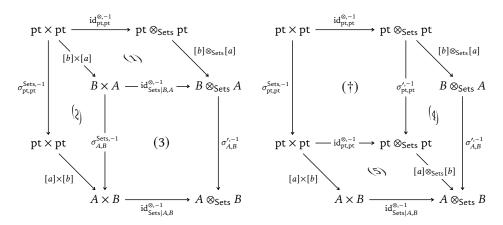
also commutes. Taking inverses, we see that the diagram

$$\begin{array}{ccc} pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} & pt \otimes_{\mathsf{Sets}} pt \\ \\ \sigma_{\mathsf{pt,pt}}^{\mathsf{Sets,-1}} & & (\dagger) & & & \sigma_{\mathsf{pt,pt}}^{\prime,-1} \\ pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} & pt \otimes_{\mathsf{Sets}} pt \end{array}$$

commutes as well. Now, let  $A, B \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , and consider the diagram



which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of  $\sigma^{\mathrm{Sets},-1}$ .
- Subdiagram (5) commutes by the naturality of  $id^{\otimes,-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{-1}$ .
- Subdiagram (1) commutes by the naturality of  $id^{\otimes,-1}$ .

it follows that the diagram

$$\begin{array}{c|c} B \times A & \stackrel{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}}{\longrightarrow} B \otimes_{\mathsf{Sets}} A \\ & \sigma_{A,B}^{\mathsf{Sets}} & & & & & \\ & A \times B & \stackrel{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}}{\longrightarrow} A \otimes_{\mathsf{Sets}} B \end{array}$$

commutes. We then have

$$\left[\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1}\right](b,a) = \left[\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a])\right](\star,\star)$$

$$= \left[\sigma_{A,B}^{\prime,-1} \circ \operatorname{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a])\right](\star, \star)$$
$$= \left[\sigma_{A,B}^{\prime,-1} \circ \operatorname{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}\right](b,a)$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \,.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}',$$

showing that the diagram

$$\begin{array}{ccc} A \otimes_{\mathsf{Sets}} B & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B \\ & & & & & & & & \\ \sigma'_{A,B} & & & & & & & \\ \sigma'_{A,B} & & & & & & & \\ B \otimes_{\mathsf{Sets}} A & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A \end{array}$$

indeed commutes.

Uniqueness of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ : Let  $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\begin{split} \lambda_B' &= \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ \Big( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \Big), \\ \lambda_B' &= \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ \Big( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \Big). \end{split}$$

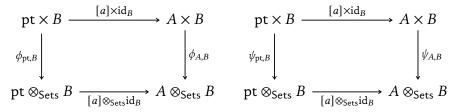
Postcomposing both sides with  $\lambda_B^{\mathsf{Sets},-1}$  gives

$$\begin{split} &\lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right) = \phi_{\mathsf{pt},B}, \\ &\lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right) = \psi_{\mathsf{pt},B}, \end{split}$$

and thus we have

$$\phi_{\mathsf{pt},B} = \psi_{\mathsf{pt},B}$$

for each  $B \in \text{Obj}(\mathsf{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams



for  $\phi$  and  $\psi$  with respect to the morphisms [a] and  $\mathrm{id}_B$ . Having shown that  $\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$ , we have

$$\phi_{A,B}(a,b) = [\phi_{A,B} \circ ([a] \times id_B)](\star,b)$$

$$= [([a] \otimes_{\mathsf{Sets}} id_B) \circ \phi_{\mathsf{pt},B}](\star,b)$$

$$= [([a] \otimes_{\mathsf{Sets}} id_B) \circ \psi_{\mathsf{pt},B}](\star,b)$$

$$= [\psi_{A,B} \circ ([a] \times id_B)](\star,b)$$

$$= \psi_{A,B}(a,b)$$

for each  $(a, b) \in A \times B$ . Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each  $A, B \in \text{Obj}(\mathsf{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\mathsf{Sets}} \cong \mathsf{x}$  to be unique.  $\Box$ 

**Corollary 5.1.10.1.2.** The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:Sets}$  spanned by the symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,  $\mathbb{1}_{\mathsf{Sets}}$ ,  $\lambda^{\mathsf{Sets}}$ ,  $\rho^{\mathsf{Sets}}$ ,  $\sigma^{\mathsf{Sets}}$ ) satisfying Items 1 and 2 is contractible.

*Proof.* Since Sets is locally presentable (??), it follows from ?? that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Definition 5.1.10.1.1.

The result then follows from Definition 5.1.10.1.1. □

# 5.2 The Monoidal Category of Sets and Coproducts

### 5.2.1 Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

### 5.2.2 The Monoidal Unit

**Definition 5.2.2.1.1.** The monoidal unit of the coproduct of sets is the functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{O}_{\mathsf{Sets}} \stackrel{\mathrm{def}}{=} \emptyset$$

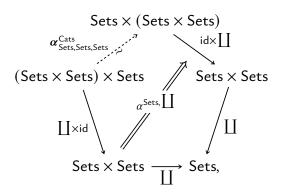
where Ø is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

### 5.2.3 The Associator

**Definition 5.2.3.1.1.** The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{}}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \pmb{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}},$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} : (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each  $a \in (X \coprod Y) \coprod Z$ .

*Proof.* Unwinding the Definitions of  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ : Firstly, we unwind the expressions for  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ . We have

$$\begin{split} (X \coprod Y) \coprod Z &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \coprod Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ &= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ & \cup \{(1, z) \in S \mid z \in Z\}, \end{split}$$

where  $S = \{0, 1\} \times ((X \coprod Y) \cup Z)$  and

$$X \coprod (Y \coprod Z) \stackrel{\text{def}}{=} \{(0, x) \in S' \mid x \in X\} \cup \{(1, a) \in S' \mid a \in Y \coprod Z\}$$
$$= \{(0, x) \in S' \mid x \in X\} \cup \{(1, (0, y)) \in S' \mid y \in Y\}$$
$$\cup \{(1, (1, z)) \in S' \mid z \in Z\},$$

where  $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$ .

*Invertibility*: The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\text{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each  $a \in X \coprod Y(\coprod Z)$ . Indeed:

• *Invertibility I*. The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  acts on elements as

$$(0, (0, x)) \mapsto (0, x) \mapsto (0, (0, x)),$$
  

$$(0, (0, y)) \mapsto (1, (0, y)) \mapsto (0, (0, y)),$$
  

$$(1, z) \mapsto (1, (1, z)) \mapsto (1, z)$$

and hence is equal to the identity map of  $(X \mid \mid Y) \mid \mid Z$ .

• *Invertibility II.* The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$  acts on elements as

$$\begin{array}{ccccc} (0,x) & \mapsto & (0,(0,x)) & \mapsto & (0,x), \\ (1,(0,y)) & \mapsto & (0,(0,y)) & \mapsto & (1,(0,y)), \\ (1,(1,z)) & \mapsto & (1,z) & \mapsto & (1,(1,z)) \end{array}$$

and hence is equal to the identity map of  $X \coprod (Y \coprod Z)$ .

Therefore  $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

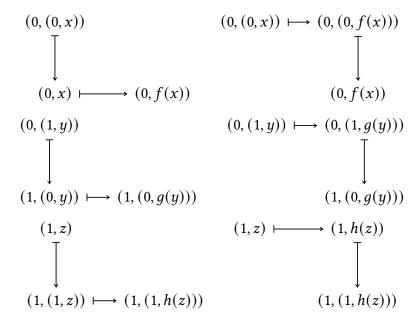
the diagram

$$(X \coprod Y) \coprod Z \xrightarrow{\left(f \coprod g\right) \coprod h} (X' \coprod Y') \coprod Z'$$

$$\downarrow^{\text{Sets,} \coprod}_{\alpha_{X,Y,Z}} \downarrow$$

$$X \coprod (Y \coprod Z) \xrightarrow{f \coprod \left(g \coprod h\right)} X' \coprod (Y' \coprod Z')$$

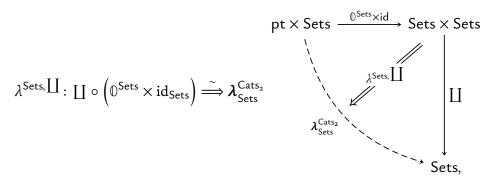
commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing  $\alpha^{\mathsf{Sets}, \coprod}$  to be a natural transformation. Being a Natural Isomorphism: Since  $\alpha^{\mathsf{Sets}, \coprod}$  is natural and  $\alpha^{\mathsf{Sets}, \coprod, -1}$  is a componentwise inverse to  $\alpha^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets}, -1}$  is also natural. Thus  $\alpha^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

# 5.2.4 The Left Unitor

**Definition 5.2.4.1.1.** The **left unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} \colon \mathsf{Ø} \coprod X \stackrel{\sim}{\dashrightarrow} X$$

at X is given by

$$\lambda_X^{\mathsf{Sets},\coprod}((1,x))\stackrel{\mathrm{def}}{=} x$$

for each  $(1, x) \in \emptyset \coprod X$ .

*Proof. Unwinding the Definition of*  $\emptyset \coprod X$ : Firstly, we unwind the expressions for  $\emptyset \coprod X$ . We have

$$\emptyset \coprod X \stackrel{\text{def}}{=} \{(0, z) \in S \mid z \in \emptyset\} \cup \{(1, x) \in S \mid x \in X\}$$
$$= \emptyset \cup \{(1, x) \in S \mid x \in X\}$$
$$= \{(1, x) \in S \mid x \in X\},$$

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

*Invertibility*: The inverse of  $\lambda_X^{\mathsf{Sets}, \coprod}$  is the map

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \colon X \to \emptyset \coprod X$$

given by

$$\lambda_X^{\text{Sets},\coprod,-1}(x) \stackrel{\text{def}}{=} (1,x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod}\right] (1, x) &= \lambda_X^{\mathsf{Sets}, \coprod, -1} \left(\lambda_X^{\mathsf{Sets}, \coprod} (1, x)\right) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (1, x) \\ &= \left[\mathrm{id}_{\emptyset \coprod X}\right] (1, x) \end{split}$$

for each  $(1, x) \in \emptyset \coprod X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}, \coprod} \circ \lambda_X^{\mathsf{Sets}, \coprod, -1}\right](x) &= \lambda_X^{\mathsf{Sets}, \coprod} \left(\lambda_X^{\mathsf{Sets}, \coprod, -1}(x)\right) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1}(1, x) \\ &= x \\ &= [\mathsf{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore  $\lambda_X^{\text{Sets},\coprod}$  is indeed an isomorphism. Naturality: We need to show that, given a function  $f:X\to Y$ , the diagram

$$\begin{array}{c|c}
\emptyset \coprod X & \xrightarrow{\operatorname{id}_{\emptyset} \coprod f} \emptyset \coprod Y \\
\downarrow_{\lambda_{X}^{\operatorname{Sets}}, \coprod} & & \downarrow_{\lambda_{Y}^{\operatorname{Sets}}, \coprod} \\
X & \xrightarrow{f} & Y
\end{array}$$

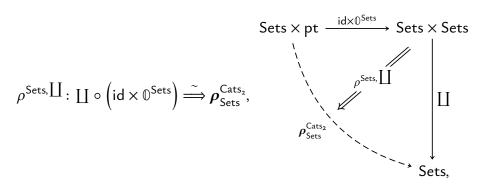
commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(1,x) & (1,x) & \longrightarrow & (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto & f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets}, \coprod}$  is a natural transformation. Being a Natural Isomorphism: Since  $\lambda^{\mathsf{Sets}, \coprod}$  is natural and  $\lambda^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets}, -1}$  is also natural. Thus  $\lambda^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

# 5.2.5 The Right Unitor

**Definition 5.2.5.1.1.** The **right unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets},\coprod} : X \coprod \emptyset \xrightarrow{\sim} X$$

at X is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x))\stackrel{\scriptscriptstyle\mathsf{def}}{=} x$$

for each  $(0, x) \in X \coprod \emptyset$ .

*Proof. Unwinding the Definition of*  $X \coprod \emptyset$ : Firstly, we unwind the expression for  $X \coprod \emptyset$ . We have

$$X \coprod \emptyset \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, z) \in S \mid z \in \emptyset\}$$
$$= \{(0, x) \in S \mid x \in X\} \cup \emptyset$$
$$= \{(0, x) \in S \mid x \in X\},$$

where  $S=\{0,1\}\times (X\cup \emptyset)=\{0,1\}\times (\emptyset\cup X)=S.$  Invertibility: The inverse of  $\rho_X^{\mathrm{Sets},\coprod}$  is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \emptyset$$

given by

$$\rho_X^{\mathsf{Sets}, \coprod, -1}(x) \stackrel{\text{\tiny def}}{=} (0, x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} \right] (0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} \left( \rho_X^{\mathsf{Sets}, \coprod} (0, x) \right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (0, x) \\ &= \left[ \mathrm{id}_{X \coprod \emptyset} \right] (0, x) \end{split}$$

for each  $(0, x) \in \emptyset \mid X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$\begin{split} \left[\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1}\right](x) &= \rho_X^{\mathsf{Sets}, \coprod} \left(\rho_X^{\mathsf{Sets}, \coprod, -1}(x)\right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X \,.$$

Therefore  $\rho_X^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

*Naturality*: We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{c|c} X \coprod \varnothing & \xrightarrow{f \coprod \mathrm{id}_{\varnothing}} Y \coprod \varnothing \\ \\ \rho_X^{\mathsf{Sets}, \coprod} & & & \downarrow \rho_Y^{\mathsf{Sets}, \coprod} \\ X & \xrightarrow{f} & Y \end{array}$$

$$\begin{bmatrix}
(0,x) & (0,x) & \longrightarrow (1,f(x)) \\
\downarrow & & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{bmatrix}$$

and hence indeed commutes. Therefore  $\rho^{\mathsf{Sets}, \coprod}$  is a natural transformation. Being a Natural Isomorphism: Since  $\rho^{\mathsf{Sets}, \coprod}$  is natural and  $\rho^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\rho^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\rho^{\mathsf{Sets}, -1}$  is also natural. Thus  $\rho^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

### 5.2.6 The Symmetry

**Definition 5.2.6.1.1.** The **symmetry of the coproduct of sets** is the natural isomorphism

whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}: X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at  $X, Y \in Obj(Sets)$  is defined by

$$\sigma_{XY}^{\text{Sets,}\coprod}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

*Proof. Unwinding the Definitions of*  $X \coprod Y$  *and*  $Y \coprod X$ : Firstly, we unwind the expressions for  $X \coprod Y$  and  $Y \coprod X$ . We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},\$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},\$$

where  $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$ .

*Invertibility*: The inverse of  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is the map

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\mathrm{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = (1,x), \\ (1,y) & \text{if } z = (0,y) \end{cases}$$

for each  $z \in Y \coprod X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod}\right](0,x) &= \sigma_X^{\mathsf{Sets},\coprod,-1} \left(\sigma_X^{\mathsf{Sets},\coprod}(0,x)\right) \\ &= \sigma_X^{\mathsf{Sets},\coprod,-1}(1,x) \\ &= (0,x) \\ &= \left[\mathrm{id}_{X\coprod Y}\right](0,x) \end{split}$$

for each  $(0, x) \in X \coprod Y$  and

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod}\right] (1,y) &= \sigma_{X}^{\mathsf{Sets},\coprod,-1} \left(\sigma_{X}^{\mathsf{Sets},\coprod} (1,y)\right) \\ &= \sigma_{X}^{\mathsf{Sets},\coprod,-1} (0,y) \\ &= (1,y) \\ &= \left[\mathrm{id}_{X\coprod Y}\right] (1,y) \end{split}$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod} = \mathrm{id}_{X\coprod Y}.$$

• Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}\right](0,y) &= \sigma_X^{\mathsf{Sets},\coprod} \left(\sigma_X^{\mathsf{Sets},\coprod,-1}(0,y)\right) \\ &= \sigma_X^{\mathsf{Sets},\coprod,-1}(1,y) \\ &= (0,y) \end{split}$$

$$= \left[ \mathrm{id}_{Y \coprod X} \right] (0, y)$$

for each  $(0, y) \in Y \coprod X$  and

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}\right] (1,x) &= \sigma_X^{\mathsf{Sets},\coprod} \left(\sigma_X^{\mathsf{Sets},\coprod,-1} (1,x)\right) \\ &= \sigma_X^{\mathsf{Sets},\coprod,-1} (0,x) \\ &= (1,x) \\ &= \left[\mathrm{id}_{Y\coprod X}\right] (1,x) \end{split}$$

for each  $(1, x) \in Y \coprod X$ , and therefore we have

$$\sigma_X^{\mathsf{Sets},\coprod} \circ \sigma_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_{Y\coprod X}$$
 .

Therefore  $\sigma_{X,Y}^{\mathsf{Sets}, \coprod}$  is indeed an isomorphism. *Naturality*: We need to show that, given functions  $f: A \to X$  and  $g: B \to Y$ , the diagram

$$A \coprod B \xrightarrow{f \coprod g} X \coprod Y$$

$$\downarrow_{\sigma_{A,B}^{\mathsf{Sets}, \coprod}} \qquad \qquad \downarrow_{\sigma_{X,Y}^{\mathsf{Sets}, \coprod}}$$

$$B \coprod A \xrightarrow{g \coprod f} Y \coprod X$$

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes. Therefore  $\sigma^{\mathsf{Sets}, \coprod}$  is a natural transformation. Being a Natural Isomorphism: Since  $\sigma^{\mathsf{Sets}, \coprod}$  is natural and  $\sigma^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\sigma^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\sigma^{\mathsf{Sets}, -1}$  is also natural. Thus  $\sigma^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

# 5.2.7 The Monoidal Category of Sets and Coproducts

**Proposition 5.2.7.1.1.** The category Sets admits a closed symmetric monoidal category structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- The Monoidal Product. The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.

• The Monoidal Unit. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \boldsymbol{\alpha}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}}$$
of Definition 5.2.3.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}, \coprod} \colon \coprod \circ \left(\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}\right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

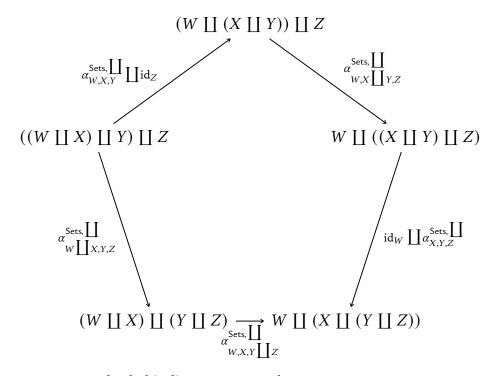
of Definition 5.2.5.1.1.

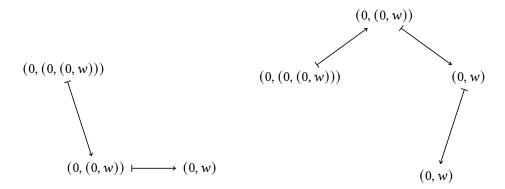
• *The Symmetry*. The natural isomorphism

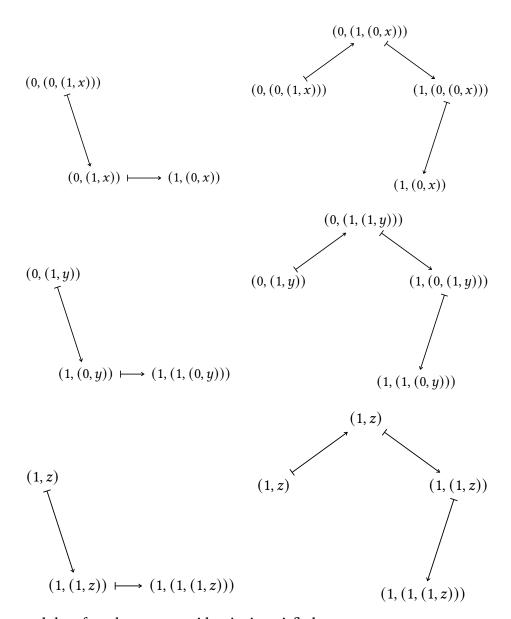
$$\sigma^{\mathsf{Sets},\coprod}: imes \stackrel{\sim}{\Longrightarrow} imes \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

*Proof.* The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

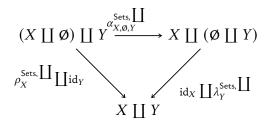




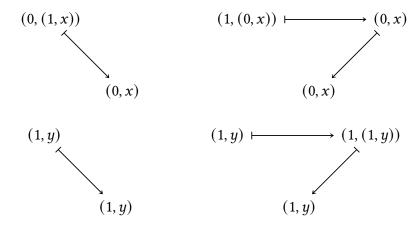


and therefore the pentagon identity is satisfied.

*The Triangle Identity*: Let *X* and *Y* be sets. We have to show that the diagram

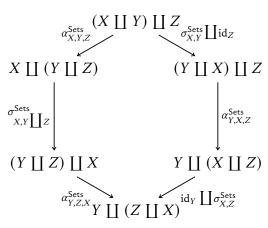


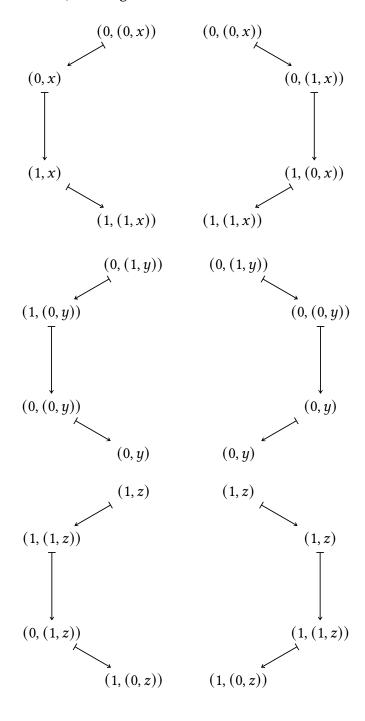
commutes. Indeed, this diagram acts on elements as



and therefore the triangle identity is satisfied.

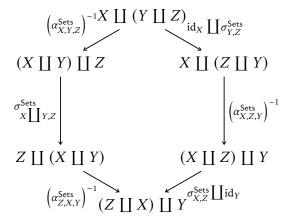
The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

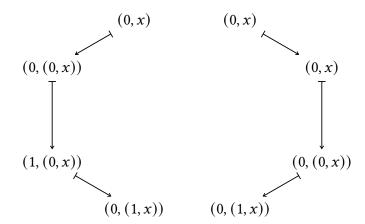


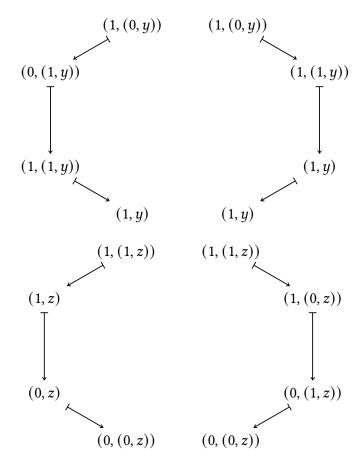


and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram







and thus the right hexagon identity is satisfied.

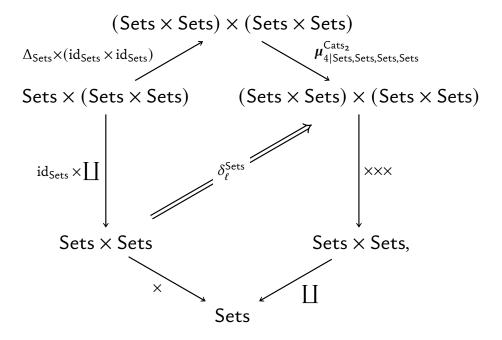
# 5.3 The Bimonoidal Category of Sets, Products, and Coproducts

# 5.3.1 The Left Distributor

Definition 5.3.1.1.1. The left distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$

as in the diagram



whose component

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z} \colon X \times (Y \coprod Z) \stackrel{\sim}{\dashrightarrow} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}(x,a) \stackrel{\text{def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each  $(x, a) \in X \times (Y \mid \mid Z)$ .

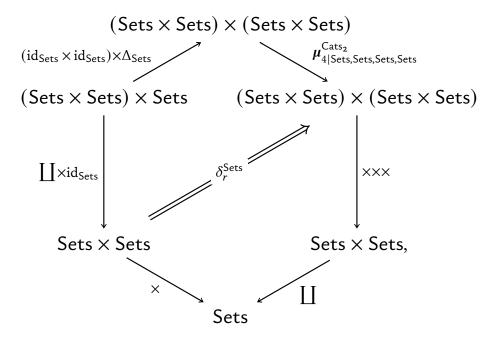
Proof. Omitted.

### 5.3.2 The Right Distributor

Definition 5.3.2.1.1. The right distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$

as in the diagram



whose component

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each  $(a, z) \in (X \coprod Y) \times Z$ .

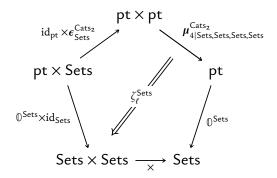
Proof. Omitted.

### 5.3.3 The Left Annihilator

**Definition 5.3.3.1.1.** The **left annihilator of the product of sets** is the natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathrm{id}_{\mathsf{pt}} \times \pmb{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \emptyset \times A \xrightarrow{\sim} \emptyset.$$

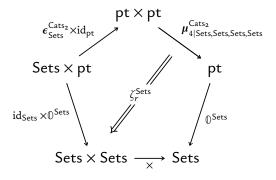
*Proof.* Omitted. For a partial proof, see [Pro25].

# 5.3.4 The Right Annihilator

**Definition 5.3.4.1.1.** The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats_2}} \circ \left(\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats_2}} \times \mathrm{id}_{\mathsf{pt}}\right) \overset{\sim}{\dashrightarrow} \times \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$

as in the diagram



with components

$$\zeta_{r|A}^{\mathsf{Sets}} \colon A \times \emptyset \xrightarrow{\sim} \emptyset.$$

*Proof.* Omitted. For a partial proof, see [Pro25].

# 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

**Proposition 5.3.5.1.1.** The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- The Additive Monoidal Product. The coproduct functor

$$\coprod$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Definition 4.2.3.1.3.

• The Multiplicative Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Monoidal Zero. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, ?? of ??.

• The Additive Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
of Definition 5.2.3.1.1.

• The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Additive Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathsf{id} \times \mathbb{O}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.2.5.1.1.

• The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} : \coprod \stackrel{\widetilde{}}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

• The Multiplicative Associators. The natural isomorphism

$$\alpha^{\text{Sets}}$$
:  $\times \circ (\times \times \text{id}_{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}}_{\text{Sets},\text{Sets},\text{Sets}}$ 
of Definition 5.1.4.1.1.

• The Multiplicative Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Multiplicative Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \times \circ \left( \mathsf{id} \times \mathbb{1}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• The Multiplicative Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} : \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$$
of Definition 5.3.1.1.1.

• The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.1.1.

• The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$
of Definition 5.3.3.1.1.

• The Right Annihilator. The natural isomorphism

$$\zeta_r^{\mathsf{Sets}} : \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}\right) \xrightarrow{\sim} \times \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$
of Definition 5.3.4.1.1.

Proof. Omitted.

# **Appendices**

# **A** Other Chapters

### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

### Sets

3. Sets

- 4. Constructions With Sets
- Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

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### **Relations**

**Categories** 

### **Monoidal Categories**

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations
- 13. Constructions With Monoidal Categories

### **Bicategories**

14. Types of Morphisms in Bicategories

11. Categories

### Extra Part

12. Presheaves and the Yoneda Lemma

15. Notes

# References

[Pro25] Proof Wiki Contributors. Cartesian Product Is Empty Iff Factor Is Empty
— Proof Wiki. 2025. URL: https://proofwiki.org/wiki/Cartesian\_
Product\_is\_Empty\_iff\_Factor\_is\_Empty (cit. on p. 56).