Monoidal Structures on the Category of Sets

The Clowder Project Authors

July 29, 2025

O1NK This chapter contains some material on monoidal structures on Sets.

Contents

5.1	The Monoidal Category of Sets and Products			
	5.1.1	Products of Sets	2	
	5.1.2	The Internal Hom of Sets	2	
	5.1.3	The Monoidal Unit	2	
	5.1.4	The Associator	3	
	5.1.5	The Left Unitor	5	
	5.1.6	The Right Unitor	8	
	5.1.7	The Symmetry	10	
	5.1.8	The Diagonal	13	
	5.1.9	The Monoidal Category of Sets and Products	16	
	5.1.10	The Universal Property of (Sets, \times , pt)	21	
5.2	The Mo	onoidal Category of Sets and Coproducts	41	
	5.2.1	Coproducts of Sets	41	
	5.2.2	The Monoidal Unit	41	
	5.2.3	The Associator	41	
	5.2.4	The Left Unitor	45	
	5.2.5	The Right Unitor	48	
	5.2.6	The Symmetry	50	
	5.2.6 5.2.7	The Symmetry	50 54	

	5.3	The Bi	monoidal Category of Sets, Products, and Coproducts	63
		5.3.1	The Left Distributor	63
		5.3.2	The Right Distributor	67
		5.3.3	The Left Annihilator	70
		5.3.4	The Right Annihilator	71
		5.3.5	The Bimonoidal Category of Sets, Products, and Coproducts	73
	A	Other	Chapters	76
01NL	5.1	ı Ti	ne Monoidal Category of Sets and Products	
01NM	5.1	.1 P	roducts of Sets	
	See	Const	ructions With Sets, Section 4.1.3.	
01NN	5.1	.2 T	he Internal Hom of Sets	
	_	_	ructions With Sets, Section 4.3.5.	

01NP 5.1.3 The Monoidal Unit

01NQ DEFINITION 5.1.3.1.1 ► THE MONOIDAL UNIT OF ×

The **monoidal unit of the product of sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{1}_{\mathsf{Sets}} \stackrel{\mathsf{def}}{=} \mathsf{pt},$$

where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.1.

5.1.4 The Associator

01NR **5.1.4** The Associator

01NS

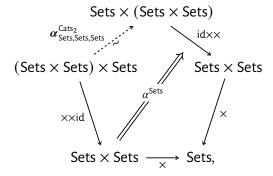
DEFINITION 5.1.4.1.1 ► THE ASSOCIATOR OF ×

The associator of the product of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

3

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\text{Sets}}((x,y),z) \stackrel{\text{def}}{=} (x,(y,z))$$

for each $((x, y), z) \in (X \times Y) \times Z$.

PROOF 5.1.4.1.2 ➤ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.4.1.1

Invertibility

The inverse of $\alpha_{X,Y,Z}^{\mathsf{Sets}}$ is the morphism

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

5.1.4 The Associator

defined by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \stackrel{\text{def}}{=} ((x,y),z)$$

for each $(x, (y, z)) \in X \times (Y \times Z)$. Indeed:

• Invertibility I. We have

$$\begin{split} [\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}}]((x,y),z) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z)) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \\ &\stackrel{\text{def}}{=} ((x,y),z) \\ &\stackrel{\text{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}]((x,y),z) \end{split}$$

for each $((x, y), z) \in (X \times Y) \times Z$, and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{(X \times Y) \times Z}$$
.

• Invertibility II. We have

$$\begin{split} [\alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1}](x,(y,z)) &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}}(\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z))) \\ &\stackrel{\mathsf{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \\ &\stackrel{\mathsf{def}}{=} (x,(y,z)) \\ &\stackrel{\mathsf{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}](x,(y,z)) \end{split}$$

for each $(x, (y, z)) \in X \times (Y \times Z)$, and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{X \times (Y \times Z)}$$
.

Therefore $\alpha_{X,Y,Z}^{\mathsf{Sets}}$ is indeed an isomorphism.

Naturality

We need to show that, given functions

$$f: X \to X',$$

 $g: Y \to Y',$
 $h: Z \to Z'$

the diagram

$$\begin{array}{c|c} (X\times Y)\times Z & \xrightarrow{(f\times g)\times h} & (X'\times Y')\times Z' \\ \\ \alpha^{\mathsf{Sets}}_{X,Y,Z} & & & \downarrow \alpha^{\mathsf{Sets}}_{X',Y',Z'} \\ X\times (Y\times Z) & \xrightarrow{f\times (g\times h)} & X'\times (Y'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes, showing $\alpha^{\rm Sets}$ to be a natural transformation.

Being a Natural Isomorphism

Since α^{Sets} is natural and $\alpha^{\mathsf{Sets},-1}$ is a componentwise inverse to α^{Sets} , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\alpha^{\mathsf{Sets},-1}$ is also natural. Thus α^{Sets} is a natural isomorphism.

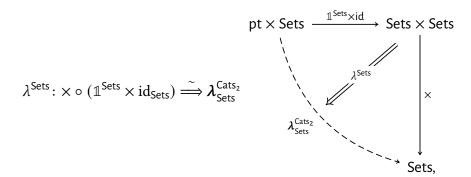
01NT 5.1.5 The Left Unitor

01NU DEFINITION 5.1.5.1.1 ► THE LEFT UNITOR OF ×

5.1.5 The Left Unitor

The ${\it left\ unitor\ of\ the\ product\ of\ sets}$ is the natural isomorphism

6



whose component

$$\lambda_X^{\mathsf{Sets}} \colon \mathsf{pt} \times X \xrightarrow{\sim} X$$

at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\mathsf{def}}{=} x$$

for each $(\star, x) \in \text{pt} \times X$.

PROOF 5.1.5.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.5.1.1

Invertibility

The inverse of $\lambda_X^{\mathsf{Sets}}$ is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathsf{pt} \times X$$

defined by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star,x)$$

for each $x \in X$. Indeed:

• Invertibility I. We have

$$\begin{split} \big[\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}\big](\mathsf{pt},x) &= \lambda_X^{\mathsf{Sets},-1}(\lambda_X^{\mathsf{Sets}}(\mathsf{pt},x)) \\ &= \lambda_X^{\mathsf{Sets},-1}(x) \end{split}$$

5.1.5 The Left Unitor

$$= (pt, x)$$
$$= [id_{pt \times X}](pt, x)$$

for each $(pt, x) \in pt \times X$, and therefore we have

$$\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}} = \mathrm{id}_{\mathsf{pt} \times X}$$
.

• Invertibility II. We have

$$[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}](x) = \lambda_X^{\mathsf{Sets}}(\lambda_X^{\mathsf{Sets},-1}(x))$$

$$= \lambda_X^{\mathsf{Sets},-1}(\mathsf{pt},x)$$

$$= x$$

$$= [\mathrm{id}_X](x)$$

for each $x \in X$, and therefore we have

$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X$$
.

Therefore $\lambda_X^{\mathrm{Sets}}$ is indeed an isomorphism.

Naturality

We need to show that, given a function $f \colon X \to Y$, the diagram

$$\begin{array}{ccc} \operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\ \lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(\star, x) \qquad (\star, x) \longmapsto (\star, f(x))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$x \longmapsto f(x) \qquad \qquad f(x)$$

and hence indeed commutes. Therefore λ^{Sets} is a natural transformation.

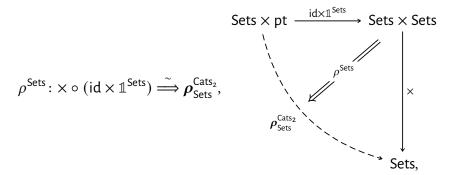
Being a Natural Isomorphism

Since λ^{Sets} is natural and $\lambda^{\text{Sets},-1}$ is a componentwise inverse to λ^{Sets} , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\lambda^{\text{Sets},-1}$ is also natural. Thus λ^{Sets} is a natural isomorphism.

01NV 5.1.6 The Right Unitor

O1NW DEFINITION 5.1.6.1.1 ► THE RIGHT UNITOR OF ×

The **right unitor of the product of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \mathsf{pt} \xrightarrow{\sim} X$$

at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\rho_X^{\mathsf{Sets}}(x, \star) \stackrel{\mathsf{def}}{=} x$$

for each $(x, \star) \in X \times pt$.

PROOF 5.1.6.1.2 ▶ Proof of the Claims Made in Definition 5.1.6.1.1

Invertibility

The inverse of $\rho_X^{\rm Sets}$ is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (x,\star)$$

for each $x \in X$. Indeed:

• Invertibility I. We have

$$\begin{aligned} [\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}}](x, \star) &= \rho_X^{\mathsf{Sets},-1}(\rho_X^{\mathsf{Sets}}(x, \star)) \\ &= \rho_X^{\mathsf{Sets},-1}(x) \\ &= (x, \star) \\ &= [\mathrm{id}_{X \times \mathrm{pt}}](x, \star) \end{aligned}$$

for each $(x, \star) \in X \times pt$, and therefore we have

$$\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} = \mathrm{id}_{X \times \mathrm{pt}}.$$

• Invertibility II. We have

$$\begin{split} [\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1}](x) &= \rho_X^{\mathsf{Sets}}(\rho_X^{\mathsf{Sets},-1}(x)) \\ &= \rho_X^{\mathsf{Sets},-1}(x, \bigstar) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each $x \in X$, and therefore we have

$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X.$$

Therefore ρ_X^{Sets} is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \to Y$, the diagram

$$\begin{array}{c|c} X \times \operatorname{pt} & \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} & Y \times \operatorname{pt} \\ \rho_X^{\operatorname{Sets}} & & & \downarrow \rho_Y^{\operatorname{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x, \star) \qquad (x, \star) \longmapsto (f(x), \star)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$x \longmapsto f(x) \qquad f(x)$$

and hence indeed commutes. Therefore ρ^{Sets} is a natural transformation.

Being a Natural Isomorphism

Since ρ^{Sets} is natural and $\rho^{\text{Sets},-1}$ is a componentwise inverse to ρ^{Sets} , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\rho^{\text{Sets},-1}$ is also natural. Thus ρ^{Sets} is a natural isomorphism.

01NX 5.1.7 The Symmetry

01NY

DEFINITION 5.1.7.1.1 ► THE SYMMETRY OF ×

The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{array}{c} \times \\ \longrightarrow \\ \sigma^{\mathsf{Cats}_2} \\ \sigma^{\mathsf{Cats}_2} \\ \end{array} \qquad \begin{array}{c} \times \\ \longrightarrow \\ \sigma^{\mathsf{Sets}} \\ \times \\ \mathsf{Sets} \times \mathsf{Sets} \end{array}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets}} \colon X \times Y \xrightarrow{\sim} Y \times X$$

at $X, Y \in \text{Obj}(\mathsf{Sets})$ is defined by

$$\sigma_{X,Y}^{\text{Sets}}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each $(x, y) \in X \times Y$.

PROOF 5.1.7.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.7.1.1

Invertibility

The inverse of $\sigma_{X,Y}^{\mathsf{Sets}}$ is the morphism

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \colon Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets},-1}(y,x) \stackrel{\text{def}}{=} (x,y)$$

for each $(y, x) \in Y \times X$. Indeed:

• *Invertibility I.* We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}}](x,y) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(\sigma_{X,Y}^{\mathsf{Sets}}(x,y)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(y,x) \\ &\stackrel{\text{def}}{=} (x,y) \\ &\stackrel{\text{def}}{=} [\mathrm{id}_{X\times Y}](x,y) \end{split}$$

for each $(x, y) \in X \times Y$, and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}} = \mathrm{id}_{X \times Y} .$$

• Invertibility II. We have

$$\begin{split} \big[\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1}\big](y,x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(\sigma_{X,Y}^{\mathsf{Sets}}(y,x)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(x,y) \end{split}$$

$$\stackrel{\text{def}}{=} (y, x)$$

$$\stackrel{\text{def}}{=} [\text{id}_{Y \times X}](y, x)$$

for each $(y, x) \in Y \times X$, and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \mathrm{id}_{Y \times X}$$
.

Therefore $\sigma_{XY}^{\mathsf{Sets}}$ is indeed an isomorphism.

Naturality

We need to show that, given functions

$$f: X \to A$$
, $g: Y \to B$

the diagram

$$\begin{array}{c|c} X \times Y & \xrightarrow{f \times g} & A \times B \\ \\ \sigma_{X,Y}^{\mathsf{Sets}} & & & & \\ & & & \\ Y \times X & \xrightarrow{g \times f} & B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x,y) \qquad (x,y) \longmapsto (f(x),g(y))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(y,x) \longmapsto (g(y),f(x)) \qquad (g(y),f(x))$$

and hence indeed commutes, showing $\sigma^{\rm Sets}$ to be a natural transformation.

Being a Natural Isomorphism

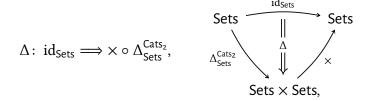
Since σ^{Sets} is natural and $\sigma^{\text{Sets},-1}$ is a componentwise inverse to σ^{Sets} , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\sigma^{\text{Sets},-1}$ is also natural. Thus σ^{Sets} is a natural isomorphism.

01NZ 5.1.8 The Diagonal

01P0

DEFINITION 5.1.8.1.1 ► THE DIAGONAL OF ×

The diagonal of the product of sets is the natural transformation



whose component

$$\Delta_X \colon X \to X \times X$$

at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each $x \in X$.

PROOF 5.1.8.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.8.1.1

We need to show that, given a function $f: X \to Y$, the diagram

$$X \xrightarrow{f} Y$$

$$\Delta_X \downarrow \qquad \qquad \downarrow \Delta_Y$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{cccc}
x & & & x & \longrightarrow f(x) \\
\downarrow & & & \downarrow \\
(x,x) & \longmapsto (f(x),f(x)) & & & (f(x),f(x))
\end{array}$$

and hence indeed commutes, showing Δ to be natural.

PROPOSITION 5.1.8.1.3 ► PROPERTIES OF THE DIAGONAL MAP

Let *X* be a set.

01P3

01P4

01P2 1. *Monoidality*. The diagonal map

$$\Delta\colon \operatorname{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

is a monoidal natural transformation:

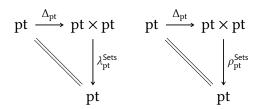
(a) Compatibility With Strong Monoidality Constraints. For each $X, Y \in \text{Obj}(\mathsf{Sets})$, the diagram

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} (X \times X) \times (Y \times Y)$$

$$\downarrow (X \times Y) \times (X \times Y)$$

commutes.

(b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\mathrm{pt}} &= \lambda_{\mathrm{pt}}^{\mathrm{Sets},-1} \\ &= \rho_{\mathrm{pt}}^{\mathrm{Sets},-1}, \end{split}$$

where we recall that the equalities

$$\begin{split} \lambda_{\mathrm{pt}}^{\mathrm{Sets}} &= \rho_{\mathrm{pt}}^{\mathrm{Sets}}, \\ \lambda_{\mathrm{pt}}^{\mathrm{Sets},-1} &= \rho_{\mathrm{pt}}^{\mathrm{Sets},-1} \end{split}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

01P5

2. The Diagonal of the Unit. The component

$$\Delta_{pt} \colon pt \xrightarrow{\sim} pt \times pt$$

of Δ at pt is an isomorphism.

PROOF 5.1.8.1.4 ▶ PROOF OF PROPOSITION 5.1.8.1.3

Item 1: Monoidality

We claim that Δ is indeed monoidal:

024S

1. <u>Item 1a</u>: Compatibility With Strong Monoidality Constraints: We need to show that the diagram

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} (X \times X) \times (Y \times Y)$$

$$\downarrow X \times Y \times (X \times Y) \times (X \times Y)$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes.

024T

2. *Item 1b*: *Compatibility With Strong Unitality Constraints*: As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at $X \in \text{Obj}(\mathsf{Sets})$ is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star, x)$$

for each $x \in X$, so when X = pt, we have

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets},-1}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

and also

$$\Delta_{\mathrm{pt}}^{\mathrm{Sets}}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

so we have $\Delta_{\text{pt}} = \lambda_{\text{pt}}^{\text{Sets},-1}$.

This finishes the proof.

Item 2: The Diagonal of the Unit

This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to ×, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor.

01P6 5.1.9 The Monoidal Category of Sets and Products

PROPOSITION 5.1.9.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO THE PROD-01P7

The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- *The Monoidal Product.* The product functor

$$\times$$
: Sets \times Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

• The Internal Hom. The internal Hom functor

Sets: Sets^{op}
$$\times$$
 Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Proposition 4.3.5.1.2.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• *The Associators*. The natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \mathsf{X} \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{^{\sim}}{\Longrightarrow} \pmb{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \mathsf{X} \circ (\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}}) \stackrel{\widetilde{\longrightarrow}}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.6.1.1.

• The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Diagonals. The monoidal natural transformation

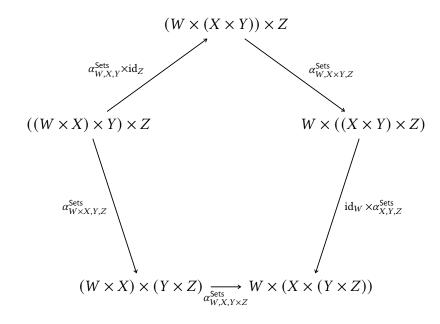
$$\Delta\colon\operatorname{id}_{\mathsf{Sets}}\Longrightarrow\times\circ\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

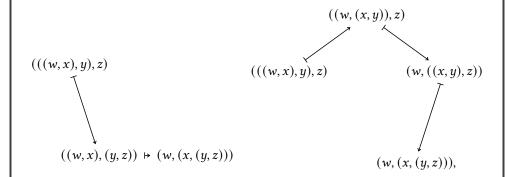
PROOF 5.1.9.1.2 ▶ PROOF OF PROPOSITION 5.1.9.1.1

The Pentagon Identity

Let W, X, Y and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the pentagon identity is satisfied.

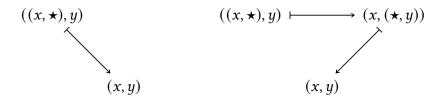
The Triangle Identity

Let *X* and *Y* be sets. We have to show that the diagram

$$(X \times \mathrm{pt}) \times Y \xrightarrow{\alpha_{X,\mathrm{pt},Y}^{\mathsf{Sets}}} X \times (\mathrm{pt} \times Y)$$

$$\rho_X^{\mathsf{Sets}} \times \mathrm{id}_Y \xrightarrow{\mathrm{id}_X \times \lambda_Y^{\mathsf{Sets}}} X \times Y$$

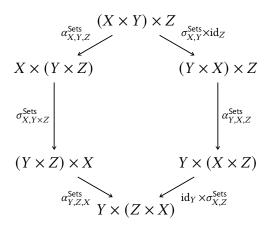
commutes. Indeed, this diagram acts on elements as



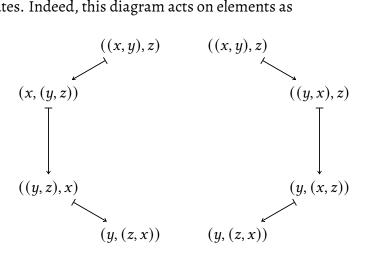
and thus the triangle identity is satisfied.

The Left Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram



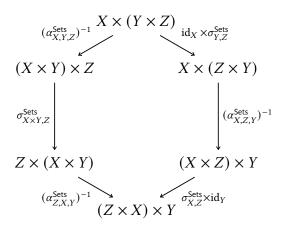
commutes. Indeed, this diagram acts on elements as

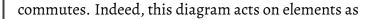


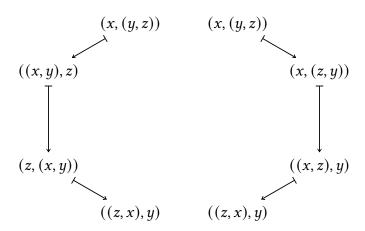
and thus the left hexagon identity is satisfied.

The Right Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram







and thus the right hexagon identity is satisfied.

Monoidal Closedness

01PA

01PB

This follows from Constructions With Sets, Item 2 of Proposition 4.3.5.1.2

Existence of Monoidal Diagonals

This follows from Items 1 and 2 of Proposition 5.1.8.1.3.

O1P8 5.1.10 The Universal Property of (Sets, \times , pt)

THEOREM 5.1.10.1.1 ► THE UNIVERSAL PROPERTY OF (Sets, ×, pt)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Existence of an Internal Hom. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom [-1, -2] Sets.

2. The Unit Object Is pt. We have $\mathbb{1}_{Sets} \cong pt$.

More precisely, the full subcategory of the category $\mathcal{M}^{cld}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$ of $\ref{eq:sets}$ spanned by the closed symmetric monoidal categories (Sets, \otimes_{Sets} ,

 $[-1, -2]_{Sets}$, $\mathbb{1}_{Sets}$, λ^{Sets} , ρ^{Sets} , σ^{Sets}) satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

PROOF 5.1.10.1.2 ▶ PROOF OF THEOREM 5.1.10.1.1

Unwinding the Statement

Let (Sets, \otimes_{Sets} , $[-1, -2]_{Sets}$, $\mathbb{1}_{Sets}$, λ' , ρ' , σ') be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$id_{Sets} : Sets \rightarrow Sets$$

admits a unique closed symmetric monoidal functor structure

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets, \otimes_{Sets} , $[-_1, -_2]_{Sets}$, $\mathbb{1}_{Sets}$, λ' , ρ' , σ') to the closed symmetric monoidal category (Sets, \times , Sets $(-_1, -_2)$, $\mathbb{1}_{Sets}$, λ^{Sets} , ρ^{Sets} , σ^{Sets}) of Proposition 5.1.9.1.1.

Constructing an Isomorphism $[-1, -2]_{Sets} \cong Sets(-1, -2)$

By ??, we have a natural isomorphism

$$Sets(pt, [-1, -2]_{Sets}) \cong Sets(-1, -2).$$

By Constructions With Sets, Item 3 of Proposition 4.3.5.1.2, we also have a natural isomorphism

$$\mathsf{Sets}(\mathsf{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}(-_1,-_2) \cong [-_1,-_2]_{\mathsf{Sets}}.$$

Given $A, B \in \text{Obj}(\mathsf{Sets})$, we will write

$$id_{A,B}^{Hom} : Sets(A, B) \xrightarrow{\sim} [A, B]_{Sets}$$

for the component of this isomorphism at (A, B).

Constructing an Isomorphism $\otimes_{Sets} \cong \times$

Since \otimes_{Sets} is adjoint in each variable to $[-1, -2]_{Sets}$ by assumption and \times is adjoint in each variable to Sets(-1, -2) by Constructions With Sets, Item 2 of Proposition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -,$$

 $- \otimes_{\mathsf{Sets}} B \cong B \times -.$

By **??**, we then have $\otimes_{Sets} \cong \times$. We will write

$$\operatorname{id}_{\operatorname{\mathsf{Sets}}|A,B}^{\otimes} \colon A \otimes_{\operatorname{\mathsf{Sets}}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

Alternative Construction of an Isomorphism $\otimes_{Sets} \cong \times$

Alternatively, we may construct a natural isomorphism $\otimes_{Sets} \cong \times$ as follows:

01PC 1. Let $A \in \text{Obj}(\mathsf{Sets})$.

> 2. Since \otimes_{Sets} is part of a closed monoidal structure, it preserves colimits in each variable by ??.

> 3. Since $A \cong \coprod_{a \in A} \operatorname{pt}$ and $\otimes_{\operatorname{Sets}}$ preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong (\coprod_{a \in A} \mathsf{pt}) \otimes_{\mathsf{Sets}} B$$

$$\cong \coprod_{a \in A} (\mathsf{pt} \otimes_{\mathsf{Sets}} B)$$

$$\cong \coprod_{a \in A} B$$

$$\cong A \times B,$$

01PD

01PE

naturally in $B \in \text{Obj}(\mathsf{Sets})$, where we have used that pt is the monoidal unit for \otimes_{Sets} . Thus $A \otimes_{\mathsf{Sets}} - \cong A \times -$ for each $A \in \mathsf{Obj}(\mathsf{Sets})$.

01PF

- 4. Similarly, ⊗_{Sets} $B \cong × B$ for each B ∈ Obj(Sets).
- 01PG
- 5. By **??**, we then have $\otimes_{Sets} \cong \times$.

Below, we'll show that if a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \colon A \otimes_{\mathsf{Sets}} B \to A \times B$ from before.

Constructing an Isomorphism $id_1^{\otimes} : \mathbb{1}_{Sets} \to pt$

We define an isomorphism $id_{\mathbb{1}}^{\otimes} \colon \mathbb{1}_{\mathsf{Sets}} \to \mathsf{pt}$ as the composition

$$\mathbb{1}_{\mathsf{Sets}} \overset{\rho^{\mathsf{Sets},-1}_{\mathbb{1}_{\mathsf{Sets}}}}{\overset{\circ}{\underset{\sim}{\longrightarrow}}} \mathbb{1}_{\mathsf{Sets}} \times \mathsf{pt} \overset{\mathrm{id}^{\otimes}_{\mathsf{Sets} \mid \mathbb{1}_{\mathsf{Sets}}}}{\overset{\circ}{\underset{\sim}{\longrightarrow}}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathsf{pt} \overset{\lambda'_{\mathsf{pt}}}{\overset{\circ}{\underset{\sim}{\longrightarrow}}} \mathsf{pt}$$

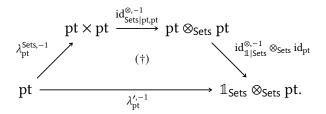
in Sets.

Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

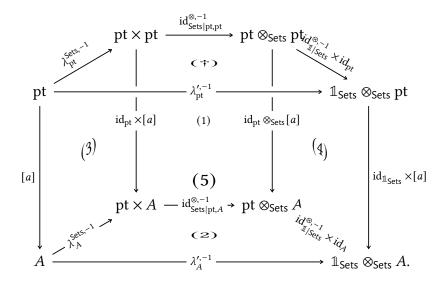
We have to show that the diagram

commutes. First, note that the diagram

corresponding to the case $A = \operatorname{pt}$, commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so does the diagram



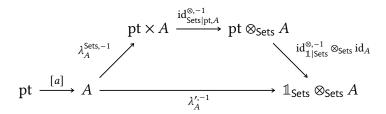
Now, let $A \in \text{Obj}(\mathsf{Sets})$, let $a \in A$, and consider the diagram



Since:

- Subdiagram (5) commutes by the naturality of λ'^{-1} .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $id_{1|Sets}^{\otimes,-1}$.
- Subdiagram (1) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.

• Subdiagram (3) commutes by the naturality of $\lambda^{\text{Sets},-1}$. it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{split} \lambda_A^{\prime,-1}(a) &= [\lambda_A^{\prime,-1} \circ [a]](\star) \\ &= [(\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1} \circ [a]](\star) \\ &= [(\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}](a) \end{split}$$

for each $a \in A$, and thus we have

$$\lambda_A^{\prime,-1} = (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \times \mathrm{id}_A)$$

showing that the diagram

$$\operatorname{pt} \otimes_{\operatorname{Sets}} A \xrightarrow{\operatorname{id}_{\operatorname{Sets}|\operatorname{pt},A}^{\otimes}} \operatorname{pt} \times A$$

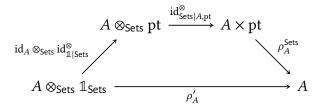
$$\operatorname{id}_{\mathbb{1}|\operatorname{Sets}}^{\otimes} \otimes_{\operatorname{Sets}} \operatorname{id}_{A} \xrightarrow{\lambda_{A}^{\wedge}} A$$

$$\operatorname{1}_{\operatorname{Sets}} \otimes_{\operatorname{Sets}} A \xrightarrow{\lambda_{A}^{\wedge}} A$$

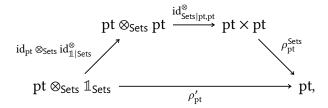
indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{Sets} \cong \times$

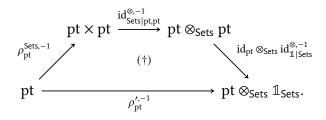
We can use the same argument we used to prove the monoidal left unity of the isomorphism $\otimes_{Sets} \cong \times$ above. For completeness, we repeat it below. We have to show that the diagram



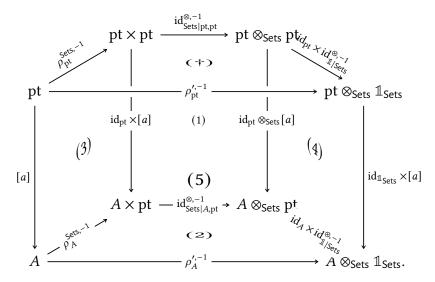
commutes. First, note that the diagram



corresponding to the case $A = \operatorname{pt}$, commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so does the diagram



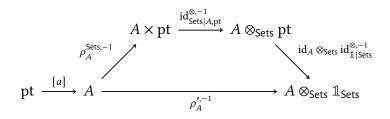




Since:

- Subdiagram (5) commutes by the naturality of ρ'^{-1} .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $id_{\mathbb{1}|Sets}^{\otimes,-1}$.
- Subdiagram (1) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\rm Sets,-1}.$

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{split} \rho_A^{\prime,-1}(a) &= [\rho_A^{\prime,-1} \circ [a]](\star) \\ &= [(\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1} \circ [a]](\star) \\ &= [(\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}](a) \end{split}$$

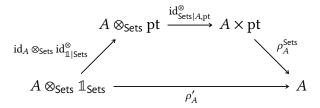
for each $a \in A$, and thus we have

$$\rho_A^{\prime,-1} = (\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes}),$$

showing that the diagram



indeed commutes.

Monoidality of the Isomorphism $\otimes_{Sets} \cong \times$

We have to show that the diagram

$$(A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C$$

$$(A \times B) \otimes_{\mathsf{Sets}} C$$

$$A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C)$$

$$(A \times B) \times C$$

$$A \otimes_{\mathsf{Sets}} (B \times C)$$

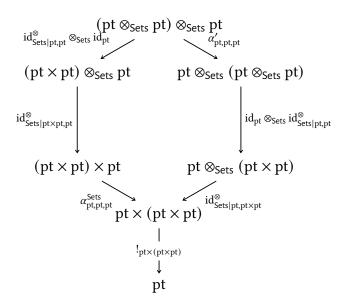
$$(A \times B) \times C$$

$$A \otimes_{\mathsf{Sets}} (B \times C)$$

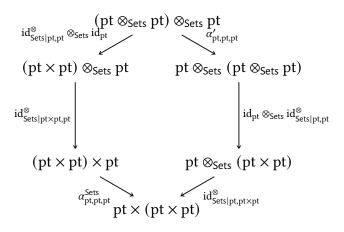
$$(A \times B) \times C$$

$$A \otimes_{\mathsf{Sets}} (B \times C)$$

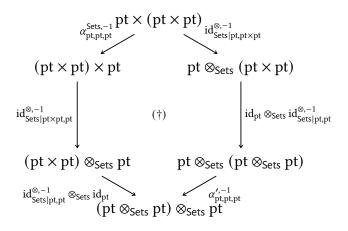
commutes. First, note that the diagram



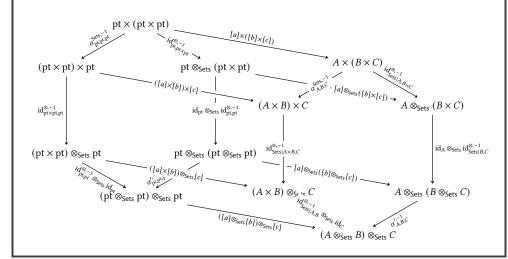
commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.2). Since the map $!_{pt \times (pt \times pt)} : pt \times (pt \times pt) \rightarrow pt$ is an isomorphism (e.g. having inverse $\lambda_{pt}^{Sets,-1} \circ \lambda_{pt}^{Sets,-1}$), it follows that the diagram

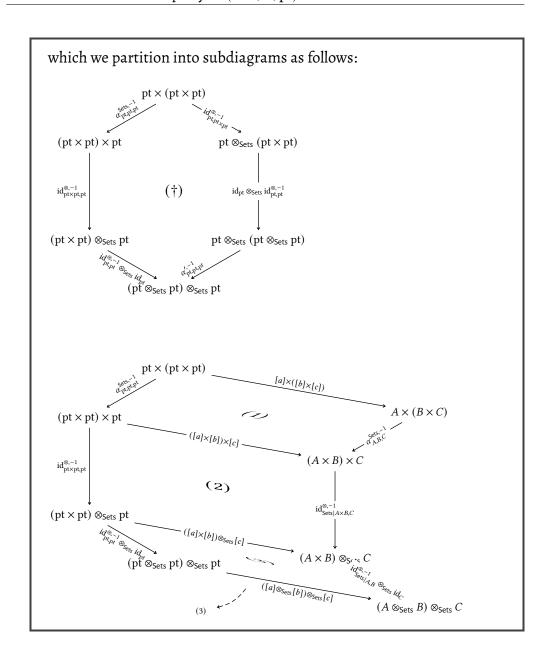


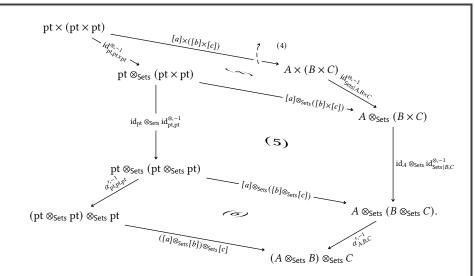
also commutes. Taking inverses, we see that the diagram



commutes as well. Now, let $A, B, C \in \text{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, let $c \in C$, and consider the diagram



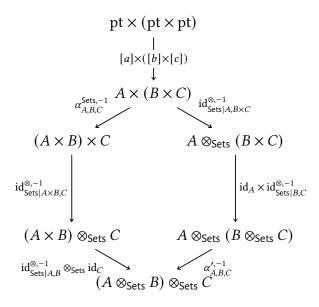




Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{\text{Sets},-1}$.
- Subdiagram (2) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (3) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $id_{\mathsf{Sets}}^{\otimes,-1}.$
- Subdiagram (5) commutes by the naturality of $id_{Sets}^{\otimes,-1}$.
- Subdiagram (6) commutes by the naturality of α'^{-1} .

it follows that the diagram



also commutes. We then have

$$\begin{split} \left[(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) &= \left[(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ ([a] \times ([b] \times [c])) \right] (\star,(\star,\star)) \\ &= \left[\alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \\ \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ ([a] \times ([b] \times [c])) \right] (\star,(\star,\star)) \\ &= \left[\alpha_{A,B,C}^{\prime,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

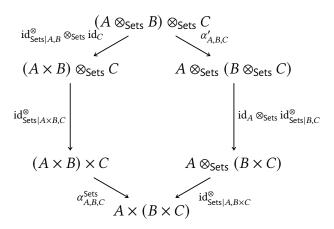
for each $(a, (b, c)) \in A \times (B \times C)$, and thus we have

$$(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_C) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} = \alpha_{A,B,C}'^{,-1} \circ (\mathrm{id}_A \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}) \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \,.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ (\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_{C}) = \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ (\mathrm{id}_{A} \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes}) \circ \alpha_{A,B,C}',$$

showing that the diagram



indeed commutes.

Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\sigma'_{A,B} \downarrow \qquad \qquad \qquad \downarrow \sigma_{A,B}^{\mathsf{Sets}}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

commutes. First, note that the diagram

commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since the map $!_{pt \times pt} : pt \times pt \to pt$ is invertible (e.g. with inverse $\lambda_{pt}^{\mathsf{Sets},-1}$), the diagram

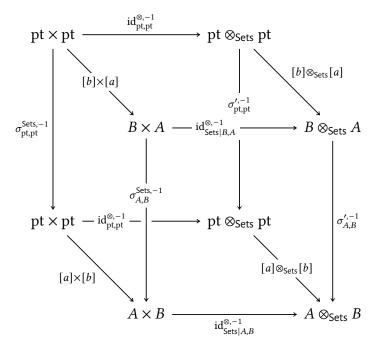
$$\begin{array}{c|c} pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes}} pt \times pt \\ \\ \sigma'_{\mathsf{pt},\mathsf{pt}} \downarrow & & \downarrow \sigma^{\mathsf{Sets}}_{\mathsf{pt},\mathsf{pt}} \\ pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes}} pt \times pt \end{array}$$

also commutes. Taking inverses, we see that the diagram

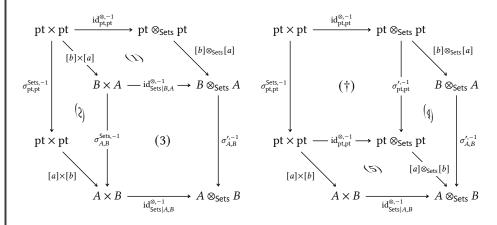
$$\begin{array}{c|c} pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} & pt \otimes_{\mathsf{Sets}} pt \\ \\ \sigma_{\mathsf{pt},\mathsf{pt}}^{\mathsf{Sets},-1} & & (\dagger) & & & \\ pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} & pt \otimes_{\mathsf{Sets}} pt \end{array}$$

commutes as well. Now, let $A, B \in Obj(Sets)$, let $a \in A$, let $b \in B$, and

consider the diagram



which we partition into subdiagrams as follows:

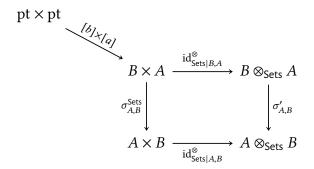


Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\rm Sets,-1}.$

- Subdiagram (5) commutes by the naturality of $id^{\otimes,-1}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of σ'^{-1} .
- Subdiagram (1) commutes by the naturality of $id^{\otimes,-1}$.

it follows that the diagram



commutes. We then have

$$\begin{split} [\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1}](b,a) &= [\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a])](\star,\star) \\ &= [\sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a])](\star,\star) \\ &= [\sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}](b,a) \end{split}$$

for each $(b, a) \in B \times A$, and thus we have

$$\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \,.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}',$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathsf{id}^{\otimes}_{\mathsf{Sets}|A,B}} A \times B$$

$$\downarrow \sigma'_{A,B} \qquad \qquad \downarrow \sigma^{\mathsf{Sets}}_{A,B}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathsf{id}^{\otimes}_{\mathsf{Sets}|B,A}} B \times A$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{Sets} \cong \times$

Let $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to \times and \otimes (as shown above), we have

$$\begin{split} \lambda_B' &= \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y), \\ \lambda_B' &= \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y). \end{split}$$

Postcomposing both sides with $\lambda_B^{\text{Sets},-1}$ gives

$$\begin{split} &\lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) = \phi_{\mathsf{pt},B}, \\ &\lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y) = \psi_{\mathsf{pt},B}, \end{split}$$

and thus we have

$$\phi_{\text{pt},B} = \psi_{\text{pt},B}$$

for each $B \in \text{Obj}(\mathsf{Sets})$. Now, let $a \in A$ and consider the naturality diagrams

for ϕ and ψ with respect to the morphisms [a] and id_B . Having shown that $\phi_{\mathrm{pt},B}=\psi_{\mathrm{pt},B}$, we have

$$\begin{aligned} \phi_{A,B}(a,b) &= [\phi_{A,B} \circ ([a] \times \mathrm{id}_B)](\star,b) \\ &= [([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \phi_{\mathsf{pt},B}](\star,b) \\ &= [([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \psi_{\mathsf{pt},B}](\star,b) \\ &= [\psi_{A,B} \circ ([a] \times \mathrm{id}_B)](\star,b) \\ &= \psi_{A,B}(a,b) \end{aligned}$$

for each $(a, b) \in A \times B$. Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each $A, B \in \text{Obj}(\mathsf{Sets})$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ to be unique.

O1PH COROLLARY 5.1.10.1.3 ► A SECOND UNIVERSAL PROPERTY FOR (Sets, ×, pt)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

2. The Unit Object Is pt. We have $\mathbb{1}_{Sets} \cong pt$.

01PJ

01PK

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$ of $\ref{eq:Sets}$ spanned by the symmetric monoidal categories (Sets, \otimes_{Sets} , $\mathbb{1}_{\mathsf{Sets}}$, λ^{Sets} , ρ^{Sets} , σ^{Sets}) satisfying Items 1 and 2 is contractible.

PROOF 5.1.10.1.4 ▶ PROOF OF COROLLARY 5.1.10.1.3

Since Sets is locally presentable (??), it follows from ?? that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Theorem 5.1.10.1.1.

The result then follows from Theorem 5.1.10.1.1.

01PL 5.2 The Monoidal Category of Sets and Coproducts

01PM 5.2.1 Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

01PN 5.2.2 The Monoidal Unit

01PP DEFINITION 5.2.2.1.1 ► THE MONOIDAL UNIT OF [

The monoidal unit of the coproduct of sets is the functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{O}_{\mathsf{Sets}} \stackrel{\mathrm{def}}{=} \emptyset$$
,

where \emptyset is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

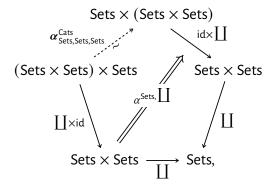
01P0 5.2.3 The Associator

01PR DEFINITION 5.2.3.1.1 ► THE ASSOCIATOR OF [

The associator of the coproduct of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{\longrightarrow}}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \pmb{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}},$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \colon (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each $a \in (X \coprod Y) \coprod Z$.

PROOF 5.2.3.1.2 ▶ Proof of the Claims Made in Definition 5.2.3.1.1

Unwinding the Definitions of $(X \mid \mid Y) \mid \mid Z$ and $X \mid \mid (Y \mid \mid Z)$

Firstly, we unwind the expressions for $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$. We have

$$(X \coprod Y) \coprod Z \stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \coprod Y\} \cup \{(1, z) \in S \mid z \in Z\}$$
$$= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\}$$
$$\cup \{(1, z) \in S \mid z \in Z\},$$

where
$$S = \{0, 1\} \times ((X \coprod Y) \cup Z)$$
 and

$$X \coprod (Y \coprod Z) \stackrel{\text{def}}{=} \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, a) \in S' \mid a \in Y \coprod Z \}$$

$$= \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, (0, y)) \in S' \mid y \in Y \}$$

$$\cup \{ (1, (1, z)) \in S' \mid z \in Z \},$$

where $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$.

Invertibility

The inverse of $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$ is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\text{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each $a \in X \coprod Y(\coprod Z)$. Indeed:

• *Invertibility I.* The map $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$ acts on elements as

$$(0,(0,x)) \mapsto (0,x) \mapsto (0,(0,x)),$$

$$(0,(0,y)) \mapsto (1,(0,y)) \mapsto (0,(0,y)),$$

$$(1,z) \mapsto (1,(1,z)) \mapsto (1,z)$$

and hence is equal to the identity map of $(X \coprod Y) \coprod Z$.

• Invertibility II. The map $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$ acts on elements as

$$(0,x) \mapsto (0,(0,x)) \mapsto (0,x),$$

$$(1,(0,y)) \mapsto (0,(0,y)) \mapsto (1,(0,y)),$$

$$(1,(1,z)) \mapsto (1,z) \mapsto (1,(1,z))$$

and hence is equal to the identity map of $X \coprod (Y \coprod Z)$.

Therefore $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$ is indeed an isomorphism.

Naturality

We need to show that, given functions

$$f: X \to X',$$

 $g: Y \to Y',$
 $h: Z \to Z'$

the diagram

$$\begin{array}{c|c} (X \coprod Y) \coprod Z & \xrightarrow{(f \coprod g) \coprod h} & (X' \coprod Y') \coprod Z' \\ \\ \alpha_{X,Y,Z}^{\mathsf{Sets}, \coprod} & & & & \\ X \coprod (Y \coprod Z) & \xrightarrow{f \coprod (g \coprod h)} & X' \coprod (Y' \coprod Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(0, (0, x)) \qquad (0, (0, x)) \longmapsto (0, (0, f(x)))$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

and hence indeed commutes, showing $\alpha^{\text{Sets},\coprod}$ to be a natural transformation.

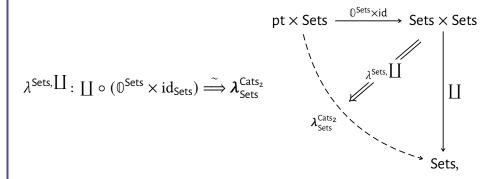
Being a Natural Isomorphism

Since $\alpha^{\mathsf{Sets}, \coprod}$ is natural and $\alpha^{\mathsf{Sets}, \coprod, -1}$ is a componentwise inverse to $\alpha^{\mathsf{Sets}, \coprod}$, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\lambda^{\mathsf{Sets}, -1}$ is also natural. Thus $\alpha^{\mathsf{Sets}, \coprod}$ is a natural isomorphism.

01PS 5.2.4 The Left Unitor

01PT DEFINITION 5.2.4.1.1 ► THE LEFT UNITOR OF [

The **left unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} \colon \emptyset \coprod X \stackrel{\sim}{\dashrightarrow} X$$

at X is given by

$$\lambda_X^{\mathsf{Sets},\coprod}((1,x))\stackrel{\mathrm{def}}{=} x$$

for each $(1, x) \in \emptyset \coprod X$.

PROOF 5.2.4.1.2 ▶ Proof of the Claims Made in Definition 5.2.4.1.1

Unwinding the Definition of $\emptyset \mid \mid X$

Firstly, we unwind the expressions for $\emptyset \coprod X$. We have

$$\emptyset \coprod X \stackrel{\text{def}}{=} \{(0, z) \in S \mid z \in \emptyset\} \cup \{(1, x) \in S \mid x \in X\}$$
$$= \emptyset \cup \{(1, x) \in S \mid x \in X\}$$
$$= \{(1, x) \in S \mid x \in X\},$$

where $S = \{0, 1\} \times (\emptyset \cup X)$.

Invertibility |

The inverse of $\lambda_X^{\mathsf{Sets}, \coprod}$ is the map

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \colon X \to \emptyset \coprod X$$

given by

$$\lambda_X^{\mathsf{Sets}, \coprod, -1}(x) \stackrel{\mathrm{def}}{=} (1, x)$$

for each $x \in X$. Indeed:

• Invertibility I. We have

$$\begin{split} [\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod}](1, x) &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (\lambda_X^{\mathsf{Sets}, \coprod}(1, x)) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (1, x) \\ &= [\mathrm{id}_{\emptyset \coprod X}](1, x) \end{split}$$

for each $(1, x) \in \emptyset \coprod X$, and therefore we have

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$[\lambda_X^{\mathsf{Sets}, \coprod} \circ \lambda_X^{\mathsf{Sets}, \coprod, -1}](x) = \lambda_X^{\mathsf{Sets}, \coprod} (\lambda_X^{\mathsf{Sets}, \coprod, -1}(x))$$

$$= \lambda_X^{\text{Sets}, \coprod, -1}(1, x)$$

$$= x$$

$$= [\text{id}_X](x)$$

for each $x \in X$, and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X$$
.

Therefore $\lambda_X^{\text{Sets}, \coprod}$ is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \to Y$, the diagram

$$\begin{array}{c|c}
\emptyset \coprod X & \xrightarrow{\operatorname{id}_{\emptyset} \coprod f} \emptyset \coprod Y \\
\downarrow^{\operatorname{Sets}, \coprod} & & \downarrow^{\lambda_{Y}^{\operatorname{Sets}, \coprod}} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes. Therefore $\lambda^{\text{Sets},\coprod}$ is a natural transformation.

Being a Natural Isomorphism

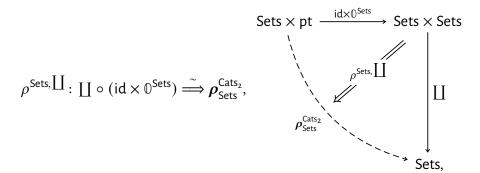
Since $\lambda^{\text{Sets}, \coprod}$ is natural and $\lambda^{\text{Sets}, -1}$ is a componentwise inverse to $\lambda^{\text{Sets}, \coprod}$, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\lambda^{\text{Sets}, -1}$ is also natural. Thus $\lambda^{\text{Sets}, \coprod}$ is a natural isomorphism.

01PU 5.2.5 The Right Unitor

01PV

DEFINITION 5.2.5.1.1 ► THE RIGHT UNITOR OF ∐

The **right unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets},\coprod}: X \coprod \emptyset \xrightarrow{\sim} X$$

at *X* is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x)) \stackrel{\mathrm{def}}{=} x$$

for each $(0, x) \in X \coprod \emptyset$.

PROOF 5.2.5.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.5.1.1

Unwinding the Definition of $X \coprod \emptyset$

Firstly, we unwind the expression for $X \coprod \emptyset$. We have

$$X \coprod \emptyset \stackrel{\text{def}}{=} \{ (0, x) \in S \mid x \in X \} \cup \{ (1, z) \in S \mid z \in \emptyset \}$$
$$= \{ (0, x) \in S \mid x \in X \} \cup \emptyset$$
$$= \{ (0, x) \in S \mid x \in X \},$$

where $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$.

Invertibility

The inverse of $\rho_X^{\mathrm{Sets},\coprod}$ is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \emptyset$$

given by

$$\rho_X^{\mathsf{Sets},\coprod,-1}(x)\stackrel{\mathrm{def}}{=} (0,x)$$

for each $x \in X$. Indeed:

• Invertibility I. We have

$$\begin{split} [\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod}](0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} (\rho_X^{\mathsf{Sets}, \coprod}(0, x)) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1}(x) \\ &= (0, x) \\ &= [\mathrm{id}_{X \coprod \emptyset}](0, x) \end{split}$$

for each $(0, x) \in \emptyset \coprod X$, and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X} \,.$$

• Invertibility II. We have

$$[\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1}](x) = \rho_X^{\mathsf{Sets}, \coprod} (\rho_X^{\mathsf{Sets}, \coprod, -1}(x))$$

$$= \rho_X^{\mathsf{Sets}, \coprod, -1}(0, x)$$

$$= x$$

$$= [\mathrm{id}_X](x)$$

for each $x \in X$, and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1} = \mathrm{id}_X \,.$$

Therefore $\rho_X^{\mathrm{Sets},\coprod}$ is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \to Y$, the diagram

$$\begin{array}{ccc} X \coprod \varnothing & \xrightarrow{f \coprod \mathrm{id}_{\varnothing}} & Y \coprod \varnothing \\ \\ \rho_X^{\mathsf{Sets}, \coprod} & & & \downarrow \rho_Y^{\mathsf{Sets}, \coprod} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(0,x) & (0,x) & \longmapsto (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore $\rho^{\text{Sets},\coprod}$ is a natural transformation.

Being a Natural Isomorphism

Since $\rho^{\text{Sets}, \coprod}$ is natural and $\rho^{\text{Sets}, -1}$ is a componentwise inverse to $\rho^{\text{Sets}, \coprod}$, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\rho^{\text{Sets}, -1}$ is also natural. Thus $\rho^{\text{Sets}, \coprod}$ is a natural isomorphism.

01PW 5.2.6 The Symmetry

01PX DEFINITION 5.2.6.1.1 ► THE SYMMETRY OF [

The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}, \coprod} : \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}, \qquad \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}} \xrightarrow{\mathsf{Sets}} \overset{\coprod}{\Longrightarrow} \overset{\mathsf{Sets}}{\coprod} \overset{\mathsf{Sets}}{\Longrightarrow} \overset{\mathsf{Sets}}{\coprod}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}: X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at $X, Y \in Obj(Sets)$ is defined by

$$\sigma_{X,Y}^{\text{Sets},\coprod}(x,y)\stackrel{\text{def}}{=}(y,x)$$

for each $(x, y) \in X \times Y$.

PROOF 5.2.6.1.2 ▶ Proof of the Claims Made in Definition 5.2.6.1.1

Unwinding the Definitions of $X \coprod Y$ and $Y \coprod X$

Firstly, we unwind the expressions for $X \coprod Y$ and $Y \coprod X$. We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},\$$

where $S = \{0, 1\} \times (X \cup Y)$ and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},\$$

where
$$S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$$
.

Invertibility

The inverse of $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$ is the map

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\mathrm{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\text{Sets},\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = (1,x), \\ (1,y) & \text{if } z = (0,y) \end{cases}$$

for each $z \in Y \coprod X$. Indeed:

• Invertibility I. We have

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}](0, x) &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (\sigma_X^{\mathsf{Sets}, \coprod}(0, x)) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (1, x) \\ &= (0, x) \\ &= [\mathrm{id}_{X \coprod Y}](0, x) \end{split}$$

for each $(0, x) \in X \coprod Y$ and

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}](1,y) &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (\sigma_X^{\mathsf{Sets}, \coprod}(1,y)) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1} (0,y) \\ &= (1,y) \\ &= [\mathrm{id}_{X \coprod Y}](1,y) \end{split}$$

for each $(1, y) \in X \coprod Y$, and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod} = \mathrm{id}_{X\coprod Y} \,.$$

• Invertibility II. We have

$$\begin{split} \big[\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}\big](0,y) &= \sigma_X^{\mathsf{Sets}, \coprod} (\sigma_X^{\mathsf{Sets}, \coprod, -1}(0,y)) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1}(1,y) \end{split}$$

=
$$(0, y)$$

= $[id_{Y}][X](0, y)$

for each $(0, y) \in Y \coprod X$ and

$$\begin{split} [\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}](1, x) &= \sigma_X^{\mathsf{Sets}, \coprod} (\sigma_X^{\mathsf{Sets}, \coprod, -1}(1, x)) \\ &= \sigma_X^{\mathsf{Sets}, \coprod, -1}(0, x) \\ &= (1, x) \\ &= [\mathrm{id}_{Y \coprod X}](1, x) \end{split}$$

for each $(1, x) \in Y \coprod X$, and therefore we have

$$\sigma_X^{\mathsf{Sets},\coprod} \circ \sigma_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_{Y\coprod X}$$
 .

Therefore $\sigma_{X,Y}^{\mathrm{Sets},\coprod}$ is indeed an isomorphism.

Naturality

We need to show that, given functions $f\colon A\to X$ and $g\colon B\to Y$, the diagram

$$A \coprod B \xrightarrow{f \coprod g} X \coprod Y$$

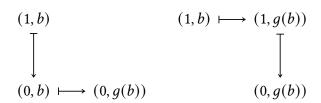
$$\downarrow_{\sigma_{A,B}}^{\text{Sets,}} \coprod \qquad \qquad \downarrow_{\sigma_{X,Y}}^{\text{Sets,}} \coprod B \coprod A \xrightarrow{g \coprod f} Y \coprod X$$

commutes. Indeed, this diagram acts on elements as

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(1,a) \longmapsto (1,f(a)) \qquad (1,f(a))$$



and hence indeed commutes. Therefore $\sigma^{\rm Sets, \coprod}$ is a natural transformation.

Being a Natural Isomorphism

Since $\sigma^{Sets, \coprod}$ is natural and $\sigma^{Sets, -1}$ is a componentwise inverse to $\sigma^{Sets, \coprod}$, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\sigma^{Sets, -1}$ is also natural. Thus $\sigma^{Sets, \coprod}$ is a natural isomorphism.

01PY 5.2.7 The Monoidal Category of Sets and Coproducts

01PZ PROPOSITION 5.2.7.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO

The category Sets admits a closed symmetric monoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- The Monoidal Product. The coproduct functor

$$[]: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

• The Monoidal Unit. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
 of Definition 5.2.3.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} \colon \coprod \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{^{\sim}}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ (\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \boldsymbol{\rho}_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.2.5.1.1.

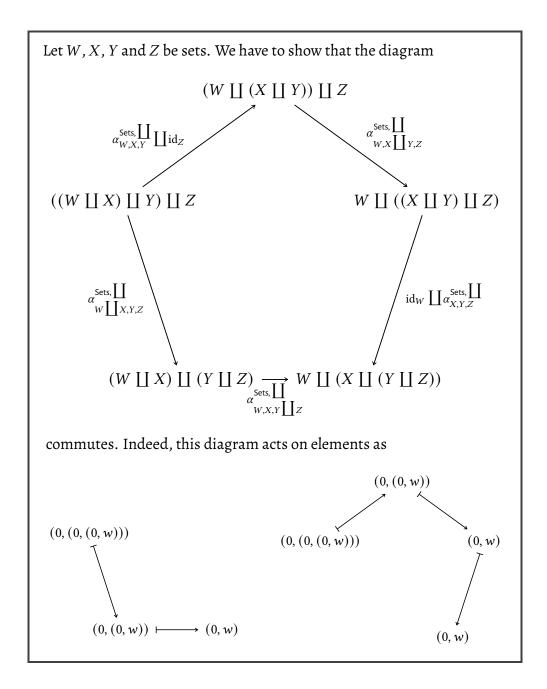
• The Symmetry. The natural isomorphism

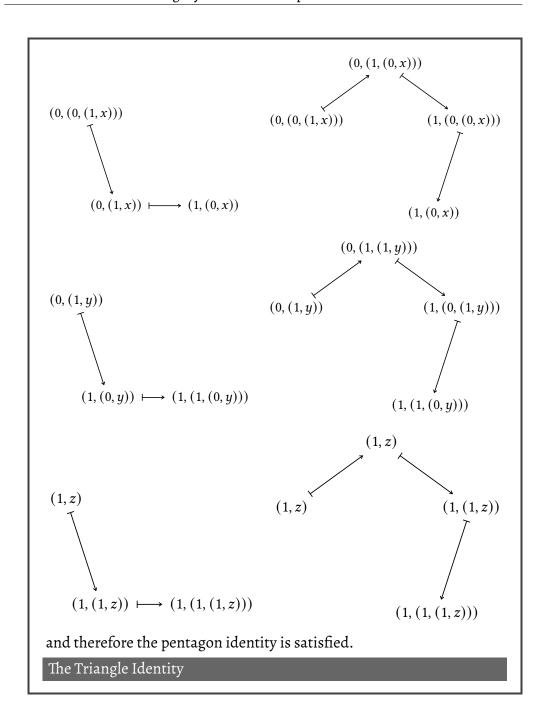
$$\sigma^{\mathsf{Sets}, \coprod} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}$$

of Definition 5.2.6.1.1.

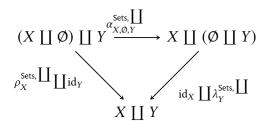
PROOF 5.2.7.1.2 ► PROOF OF PROPOSITION 5.2.7.1.1

The Pentagon Identity

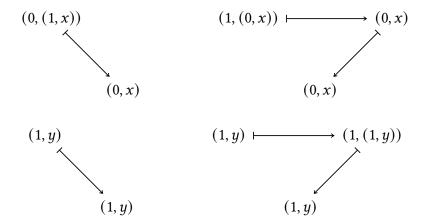




Let *X* and *Y* be sets. We have to show that the diagram

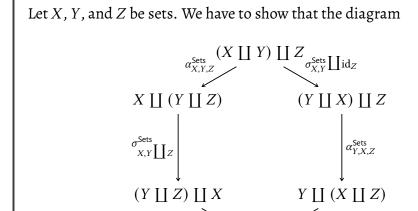


commutes. Indeed, this diagram acts on elements as

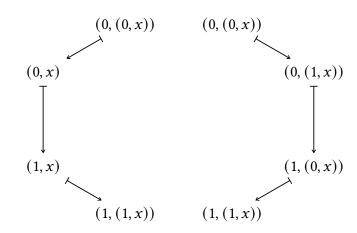


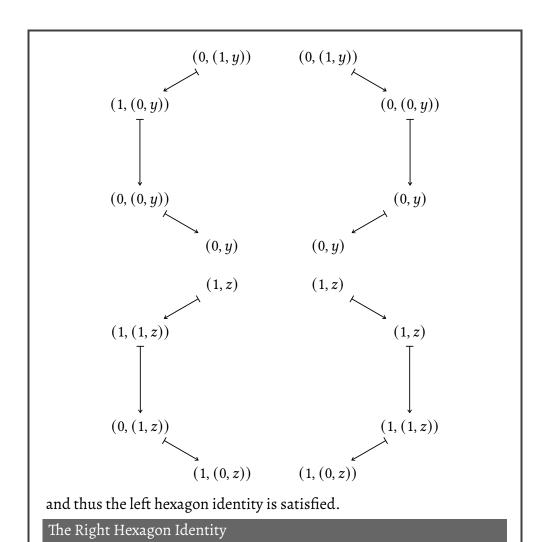
and therefore the triangle identity is satisfied.

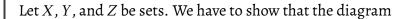
The Left Hexagon Identity

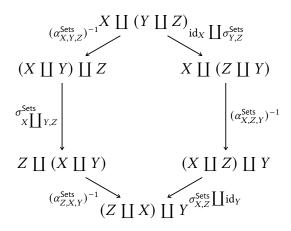


commutes. Indeed, this diagram acts on elements as

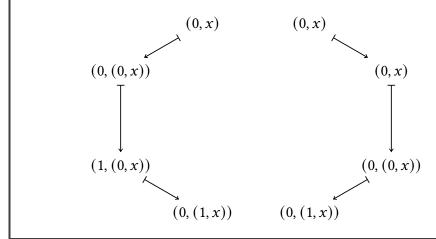


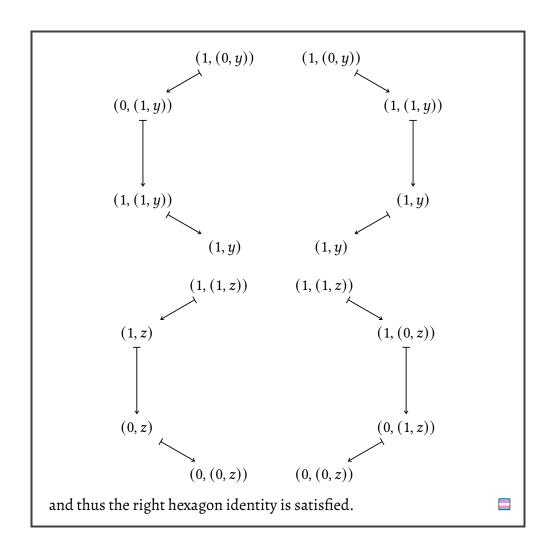






commutes. Indeed, this diagram acts on elements as



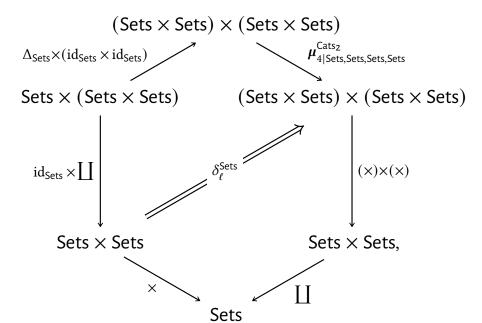


- o100 5.3 The Bimonoidal Category of Sets, Products, and Coproducts
- 01Q1 5.3.1 The Left Distributor

01Q2 DEFINITION 5.3.1.1.1 ► THE LEFT DISTRIBUTOR OF × OVER [

The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

 $\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \stackrel{\widetilde{}}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$ as in the diagram



whose component

$$\delta^{\mathsf{Sets}}_{\ell \mid X,Y,Z} \colon X \times (Y \coprod Z) \xrightarrow{\sim} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets}}(x,a) \stackrel{\text{def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each $(x, a) \in X \times (Y \coprod Z)$.

PROOF 5.3.1.1.2 ▶ Proof of the Claims Made in Definition 5.3.1.1.1

Invertibility

The inverse of $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ is the map

$$\delta^{\mathsf{Sets},-1}_{\ell\mid X,Y,Z} \colon (X\times Y) \coprod (X\times Z) \xrightarrow{\sim} X\times (Y\coprod Z)$$

given by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} (x, (0,y)) & \text{if } a = (0, (x,y)), \\ (x, (1,z)) & \text{if } a = (1, (x,z)) \end{cases}$$

for $a \in (X \times Y) \coprod (X \times Z)$. Indeed:

• *Invertibility I.* The map $\delta^{\mathsf{Sets},-1}_{\ell|X,Y,Z}\circ\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ acts on elements as

$$(x, (0, y)) \mapsto (0, (x, y)) \mapsto (x, (0, y)),$$

 $(x, (1, z)) \mapsto (1, (x, z)) \mapsto (x, (1, z)),$

but these are the two possible cases for elements of $X \times (Y \coprod Z)$. Hence the map is equal to the identity.

• Invertibility II. The map $\delta^{\sf Sets}_{\ell|X,Y,Z} \circ \delta^{\sf Sets,-1}_{\ell|X,Y,Z}$ acts on elements as

$$(0,(x,y)) \mapsto (x,(0,y)) \mapsto (0,(x,y)), (1,(x,z)) \mapsto (x,(1,z)) \mapsto (1,(x,z)),$$

but these are the two possible cases for elements of $(X \times Y) \coprod (X \times Z)$. Hence the map is equal to the identity.

Thus $\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}$ is an isomorphism for all X,Y,Z.

Naturality

We need to show that, given functions

$$f: X \to X'$$
, $g: Y \to Y'$,

$$h: Z \to Z'$$

the diagram

$$\begin{array}{c|c} X\times (Y\coprod Z) & \xrightarrow{f\times \left(g\coprod h\right)} & X'\times (Y'\coprod Z') \\ \delta^{\mathsf{Sets}}_{\ell\mid X,Y,Z} & & & & & \\ (X\times Y)\coprod (X\times Z) & \xrightarrow{(f\times g)\coprod (f\times h)} & (X'\times Y')\coprod (X'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x, (0, y)) \qquad (x, (0, y)) \longmapsto (f(x), (0, f(y)))$$

$$\downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$

so it commutes, showing $\delta_\ell^{\mathsf{Sets}}$ to be a natural transformation.

Being a Natural Isomorphism

Since $\delta_\ell^{\mathsf{Sets}}$ is natural and $\delta_\ell^{\mathsf{Sets},-1}$ is a componentwise inverse to $\delta_\ell^{\mathsf{Sets}}$, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\delta_\ell^{\mathsf{Sets},-1}$ is also natural. Thus $\delta_\ell^{\mathsf{Sets}}$ is a natural isomorphism.

01Q3 5.3.2 The Right Distributor

01Q4 DEFINITION 5.3.2.1.1 ► THE RIGHT DISTRIBUTOR OF × OVER \[\]

The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{-}}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \pmb{\mu}_{\mathsf{4} \mid \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$

as in the diagram

$$(\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets})$$

$$(\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}}$$

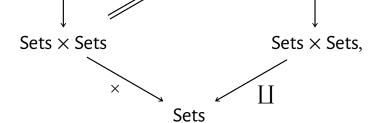
$$(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets}$$

$$(\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets})$$

$$\sqcup \times \mathsf{id}_{\mathsf{Sets}}$$

$$\delta_r^{\mathsf{Sets}}$$

$$(\times) \times (\times)$$



whose component

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each $(a, z) \in (X \coprod Y) \times Z$.

PROOF 5.3.2.1.2 ▶ Proof of the Claims Made in Definition 5.3.2.1.1

Invertibility

The inverse of $\delta_{r|X,Y,Z}^{\mathsf{Sets}}$ is the map

$$\delta_{r|X,Y,Z}^{\mathsf{Sets},-1} \colon (X \times Z) \coprod (Y \times Z) \xrightarrow{\sim} (X \coprod Y) \times Z$$

given by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} ((0,x),z) & \text{if } a = (0,(x,z)), \\ ((1,y),z) & \text{if } a = (1,(y,z)) \end{cases}$$

for $a \in (X \times Z) \coprod (Y \times Z)$. Indeed:

• *Invertibility I.* The map $\delta^{\mathsf{Sets},-1}_{r|X,Y,Z} \circ \delta^{\mathsf{Sets}}_{r|X,Y,Z}$ acts on elements as

$$((0,x),z) \mapsto (0,(x,z)) \mapsto (0,(x,z)),$$

$$((1,y),z) \mapsto (1,(y,z)) \mapsto (1,(y,z)),$$

but these are the two possible cases for elements of $(X \coprod Y) \times Z$. Hence the map is equal to the identity.

• Invertibility II. The map $\delta_{r|X,Y,Z}^{\mathsf{Sets}} \circ \delta_{r|X,Y,Z}^{\mathsf{Sets},-1}$ acts on elements as

$$(0,(x,z)) \mapsto ((0,x),z) \mapsto (0,(x,z)), (1,(y,z)) \mapsto ((1,y),z) \mapsto (1,(y,z)),$$

but these are the two possible cases for elements of $(X \times Z) \coprod (Y \times Z)$. Hence the map is equal to the identity.

So $\delta^{\mathsf{Sets}}_{r|X,Y,Z}$ is an isomorphism for all X,Y,Z.

Naturality

We need to show that, given functions

$$f: X \to X'$$
, $g: Y \to Y'$,

$$h: Z \to Z'$$

the diagram

$$(X \coprod Y) \times Z' \xrightarrow{(f \coprod g) \times h} (X' \coprod Y') \times Z'$$

$$\delta_{r|X,Y,Z}^{\mathsf{Setts}} \qquad \qquad \qquad \downarrow \delta_{r|X',Y',Z'}^{\mathsf{Setts}}$$

$$(X \times Z) \coprod (Y \times Z) \xrightarrow{(f \times h) \coprod (g \times h)} (X' \times Z') \coprod (Y' \times Z')$$

commutes. Indeed, this diagram acts on elements as

so it commutes and $\delta_r^{\rm Sets}$ is a natural transformation.

Being a Natural Isomorphism

Since δ_r^{Sets} is natural and $\delta_r^{\text{Sets},-1}$ is a componentwise inverse to δ_r^{Sets} , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\delta_r^{\text{Sets},-1}$ is also natural. Thus δ_r^{Sets} is a natural isomorphism.

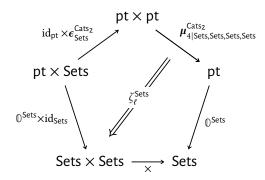
0105 5.3.3 The Left Annihilator

01Q6 DEFINITION 5.3.3.1.1 ► THE LEFT ANNIHILATOR OF ×

The left annihilator of the product of sets is the natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\mathrm{id}_{\mathsf{pt}} \times \pmb{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \mathscr{O} \times A \xrightarrow{\sim} \mathscr{O}$$

given by $\zeta_{\ell|A}^{\mathsf{Sets}} \stackrel{\text{def}}{=} \mathrm{pr}_1$.

PROOF 5.3.3.1.2 ▶ Proof of the Claims Made in Definition 5.3.3.1.1

Invertibility

The inverse of $\zeta_{\ell|A}^{\mathsf{Sets}}$ is the map

$$\zeta^{\mathsf{Sets},-1}_{\ell|A} \colon \varnothing \xrightarrow{\sim} \varnothing \times A$$

given by

$$\zeta_{\ell|A}^{\mathsf{Sets},-1} \stackrel{\mathsf{def}}{=} \iota_A,$$

where ι_A is as defined in Constructions With Sets, Construction 4.2.1.1.2:

- *Invertibility I*. The map $\zeta_{\ell|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \to \emptyset$ is equal to id_{\emptyset} , as \emptyset is the initial object of Sets.
- *Invertibility II*. The map $\iota_A \circ \zeta_{\ell|A}^{\mathsf{Sets}}$ is equal to the identity on every $(x, a) \in \emptyset \times A$, of which there are none.

Hence $\zeta_{\ell|A}^{\mathsf{Sets}}$ is an isomorphism.

Naturality

We need to show that given a function $f: A \rightarrow B$, the diagram

$$\emptyset \times A \xrightarrow{\operatorname{id}_{\mathcal{O}} \times f} \emptyset \times B$$

$$\zeta_{\ell|A}^{\operatorname{Sets}} \qquad \qquad \qquad \qquad \qquad \downarrow \zeta_{\ell|B}^{\operatorname{Sets}}$$

$$\emptyset \xrightarrow{\operatorname{id}_{\mathcal{O}}} \emptyset$$

commutes. But since $\emptyset \times A$ has no elements, this is trivially true.

Being a Natural Isomorphism

Since $\zeta_{\ell}^{\text{Sets}}$ is natural and $\zeta_{\ell}^{\text{Sets},-1}$ is a componentwise inverse to $\zeta_{\ell}^{\text{Sets}}$, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\zeta_{\ell}^{\text{Sets},-1}$ is also natural. Thus $\zeta_{\ell}^{\text{Sets}}$ is a natural isomorphism.

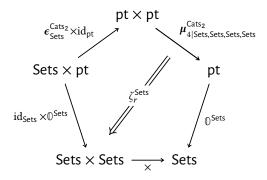
01Q7 5.3.4 The Right Annihilator

01Q8 DEFINITION 5.3.4.1.1 ► THE RIGHT ANNIHILATOR OF ×

The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats_2}} \circ (\pmb{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats_2}} \times \mathrm{id}_{\mathsf{pt}}) \xrightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}})$$

as in the diagram



with components

$$\zeta_{r|A}^{\mathsf{Sets}} : A \times \emptyset \xrightarrow{\sim} \emptyset$$

given by $\zeta_{r|A}^{\mathsf{Sets}} \stackrel{\mathsf{def}}{=} \mathrm{pr}_2$.

PROOF 5.3.4.1.2 ▶ Proof of the Claims Made in Definition 5.3.4.1.1

Invertibility

The inverse of $\zeta_{r|A}^{\rm Sets}$ is the map

$$\zeta^{\mathsf{Sets},-1}_{r|A} \colon \varnothing \xrightarrow{\sim} A \times \varnothing$$

given by

$$\zeta_{r|A}^{\text{Sets},-1} \stackrel{\text{def}}{=} \iota_A,$$

where ι_A is as defined in Constructions With Sets, Construction 4.2.1.1.2:

- *Invertibility I.* The map $\zeta_{r|A}^{\mathsf{Sets}} \circ \iota_A \colon \emptyset \to \emptyset$ is equal to id_{\emptyset} , as \emptyset is the initial object of Sets.
- *Invertibility II*. The map $\iota_A \circ \zeta_{r|A}^{\mathsf{Sets}}$ is equal to the identity on every $(a, x) \in A \times \emptyset$, of which there are none.

Hence $\zeta_{r|A}^{\mathsf{Sets}}$ is an isomorphism.

Naturality

We need to show that given a function $f: A \rightarrow B$, the diagram

$$\begin{array}{ccc}
A \times \emptyset & \xrightarrow{f \times \mathrm{id}_{\emptyset}} & B \times \emptyset \\
\downarrow^{\zeta_{r|A}} & & & \downarrow^{\zeta_{r|B}} \\
\emptyset & \xrightarrow{\mathrm{id}_{\emptyset}} & \emptyset
\end{array}$$

commutes. But since $A \times \emptyset$ has no elements, this is trivially true.

Being a Natural Isomorphism

Since ζ_r^{Sets} is natural and $\zeta_r^{\mathsf{Sets},-1}$ is a componentwise inverse to ζ_r^{Sets} , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that $\zeta_r^{\mathsf{Sets},-1}$ is also natural. Thus ζ_r^{Sets} is a natural isomorphism.

01Q9 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

010A PROPOSITION 5.3.5.1.1 ➤ THE BIMONOIDAL STRUCTURE ON SETS ASSOCIATED TO × AND []

The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category*. The category Sets of pointed sets.
- The Additive Monoidal Product. The coproduct functor

$$\coprod$$
: Sets \times Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

• The Multiplicative Monoidal Product. The product functor

$$\times$$
: Sets \times Sets \rightarrow Sets

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Monoidal Zero. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, ?? of ??.

• The Additive Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
of Definition 5.2.3.1.1.

• The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\tilde{}}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Additive Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ (\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.2.5.1.1.

• The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}:\coprod\stackrel{\widetilde{}}{\Longrightarrow}\coprod\circ\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

• The Multiplicative Associators. The natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}}_{\text{Sets,Sets,Sets}}$$
of Definition 5.1.4.1.1.

• The Multiplicative Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \mathsf{\times} \circ (\mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Multiplicative Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} : \times \circ (\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• The Multiplicative Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} : \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$
of Definition 5.3.1.1.1.

• The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \coprod \circ ((\times) \times (\times)) \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.1.1.

• The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} : \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}) \xrightarrow{\sim} \times \circ (\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})$$
of Definition 5.3.3.1.1.

• The Right Annihilator. The natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}) \xrightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}})$$
of Definition 5.3.4.1.1.



Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

Relations

- 8. Relations
- 9. Constructions With Relations

10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes