

# Constructions With Monoidal Categories

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**01UF** This chapter contains some material on constructions with monoidal categories.

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## **01UG 13.1 Moduli Categories of Monoidal Structures**

### **01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category**

Let  $\mathcal{C}$  be a category.

**01UJ Definition 13.1.1.1.1.** The **moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow \text{忘} \\ \text{pt} & \xrightarrow{[C]} & \text{Cats.} \end{array}$$

**01UK Remark 13.1.1.1.2.** In detail, **the moduli category of monoidal structures on  $C$**  is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- *Objects.* The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is  $C$ .
- *Morphisms.* A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  is a strong monoidal functor structure

$$\text{id}_C^\otimes: A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$

$$\text{id}_{\mathbb{1}|C}^\otimes: \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor  $\text{id}_C: C \rightarrow C$  of  $C$ .

- *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{pt} \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $M$  is defined by

$$\text{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \stackrel{\text{def}}{=} (\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes),$$

where  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  is the identity monoidal functor of  $C$  of ??.

- *Composition.* For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N, P) \times \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, N) \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, P)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at  $(M, N, P)$  is defined by

$$(\text{id}_C^{\otimes'}, \text{id}_{\mathbb{1}|C}^{\otimes'}) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} (\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes) \stackrel{\text{def}}{=} (\text{id}_C^{\otimes'} \circ \text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^{\otimes'} \circ \text{id}_{\mathbb{1}|C}^\otimes).$$

**01UL Remark 13.1.1.3.** In particular, a morphism in  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  satisfies the following conditions:

**01UM** 1. *Naturality.* For each pair  $f: A \rightarrow X$  and  $g: B \rightarrow Y$  of morphisms of  $C$ , the diagram

$$\begin{array}{ccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \text{id}_{A,B}^\otimes \downarrow & & \downarrow \text{id}_{X,Y}^\otimes \\ A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y \end{array}$$

commutes.

**01UN** 2. *Monoidality.* For each  $A, B, C \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc} & (A \boxtimes_C B) \boxtimes_C C & \\ \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^{C'} \\ (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \otimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\ (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\ \alpha_{A,B,C}^C \searrow & & \swarrow \text{id}_{A,B \otimes_C C}^\otimes \\ & A \otimes_C (B \otimes_C C) & \end{array}$$

commutes.

**01UP** 3. *Left Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc} & \mathbb{1}_C \boxtimes_C A & \xrightarrow{\text{id}_{\mathbb{1}'_C, A}^\otimes} \mathbb{1}_C \otimes_C A \\ \text{id}_{\mathbb{1}}^\otimes \boxtimes_C \text{id}_A \nearrow & & \searrow \lambda_A^C \\ \mathbb{1}'_C \boxtimes_C A & \xrightarrow{\lambda_A^{C'}} & A \end{array}$$

commutes.

01UQ 4. *Right Monoidal Unity.* For each  $A \in \text{Obj}(C)$ , the diagram

$$\begin{array}{ccc}
 & A \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{A, \mathbb{1}'_C}^\otimes} A \otimes_C \mathbb{1}_C \\
 \text{id}_A \boxtimes_C \text{id}_{\mathbb{1}}^\otimes \nearrow & & \searrow \rho_A^C \\
 A \boxtimes_C \mathbb{1}'_C & \xrightarrow{\rho_A^{C, '}} & A
 \end{array}$$

commutes.

01UR **Proposition 13.1.1.4.** Let  $C$  be a category.

01US 1. *Extra Monoidality Conditions.* Let  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, '}, \lambda^{C, '}, \rho^{C, '})$ .

01UT (a) The diagram

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\text{id}_{A, B}^\otimes \boxtimes_C \text{id}_C} & (A \otimes_C B) \boxtimes_C C \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A \otimes_C B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\text{id}_{A, B}^\otimes \otimes_C \text{id}_C} & (A \otimes_C B) \otimes_C C
 \end{array}$$

commutes.

01UU (b) The diagram

$$\begin{array}{ccc}
 A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \boxtimes_C \text{id}_{B, C}^\otimes} & A \boxtimes_C (B \otimes_C C) \\
 \text{id}_{A, B \boxtimes_C C}^\otimes \downarrow & & \downarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 A \otimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \otimes_C \text{id}_{B, C}^\otimes} & A \otimes_C (B \otimes_C C)
 \end{array}$$

commutes.

01WB 2. *Extra Monoidal Unity Constraints.* Let  $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C, '}, \lambda^{C, '}, \rho^{C, '})$ .

01WC

(a) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & & \downarrow \rho_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

01WD

(b) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\
 \rho_{\mathbb{1}'_C}^C \downarrow & & \downarrow \lambda_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

01WE

(c) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \lambda_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

01WF

(d) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \rho_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

- 01UV 3. *Mixed Associators.* Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$  be monoidal structures on  $C$  and let

$$\mathrm{id}_{-1, -2}^{\otimes} : -_1 \boxtimes_C -_2 \rightarrow -_1 \otimes_C -_2$$

be a natural transformation.

- 01UW (a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} : (A \otimes_C B) \boxtimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \mathrm{id}_{A \otimes_C B, C}^{\otimes} \downarrow & & \downarrow \mathrm{id}_A \otimes \mathrm{id}_{B, C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C'}} & A \boxtimes_C (B \boxtimes_C C) \\ \mathrm{id}_{A, B}^{\otimes} \boxtimes \mathrm{id}_C \downarrow & & \downarrow \mathrm{id}_{A, B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation  $\mathrm{id}^{\otimes}$  satisfies the monoidal-ity condition of **Item 2** of **Definition 13.1.1.3**.

- 01UX (b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \\ \mathrm{id}_{A, B}^{\otimes} \otimes \mathrm{id}_C \downarrow & & \downarrow \mathrm{id}_{A, B \otimes_C C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \boxtimes \text{id}_{B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} & A \boxtimes_C (B \otimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.3**.

01UY

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes, \otimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C) \\
 \text{id}_{A, B}^\otimes \otimes \text{id}_C \downarrow & & \downarrow \text{id}_A \otimes \text{id}_{B, C}^\otimes \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A, B \boxtimes_C C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.3**.

*Proof.* **Item 1, Extra Monoidality Conditions:** We claim that **Items 1a** and **1b** are indeed true:

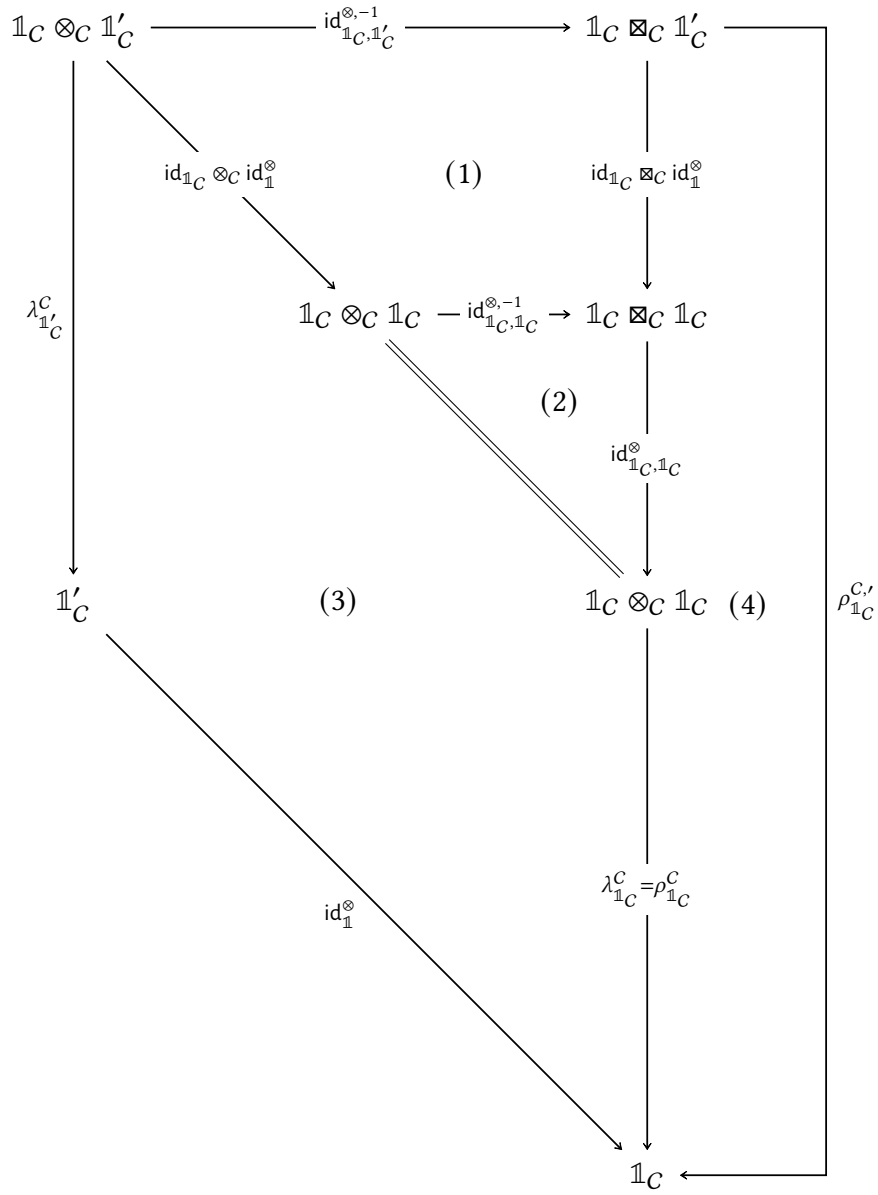
1. *Proof of Item 1a:* This follows from the naturality of  $\text{id}^\otimes$  with respect to the morphisms  $\text{id}_{A, B}^\otimes$  and  $\text{id}_C$ .

2. *Proof of Item 1b*: This follows from the naturality of  $\text{id}^\otimes$  with respect to the morphisms  $\text{id}_A$  and  $\text{id}_{B,C}^\otimes$ .

This finishes the proof.

*Item 2, Extra Monoidal Unity Constraints*: We claim that *Items 2a* and *2b* are indeed true:

1. *Proof of Item 1a*: Indeed, consider the diagram



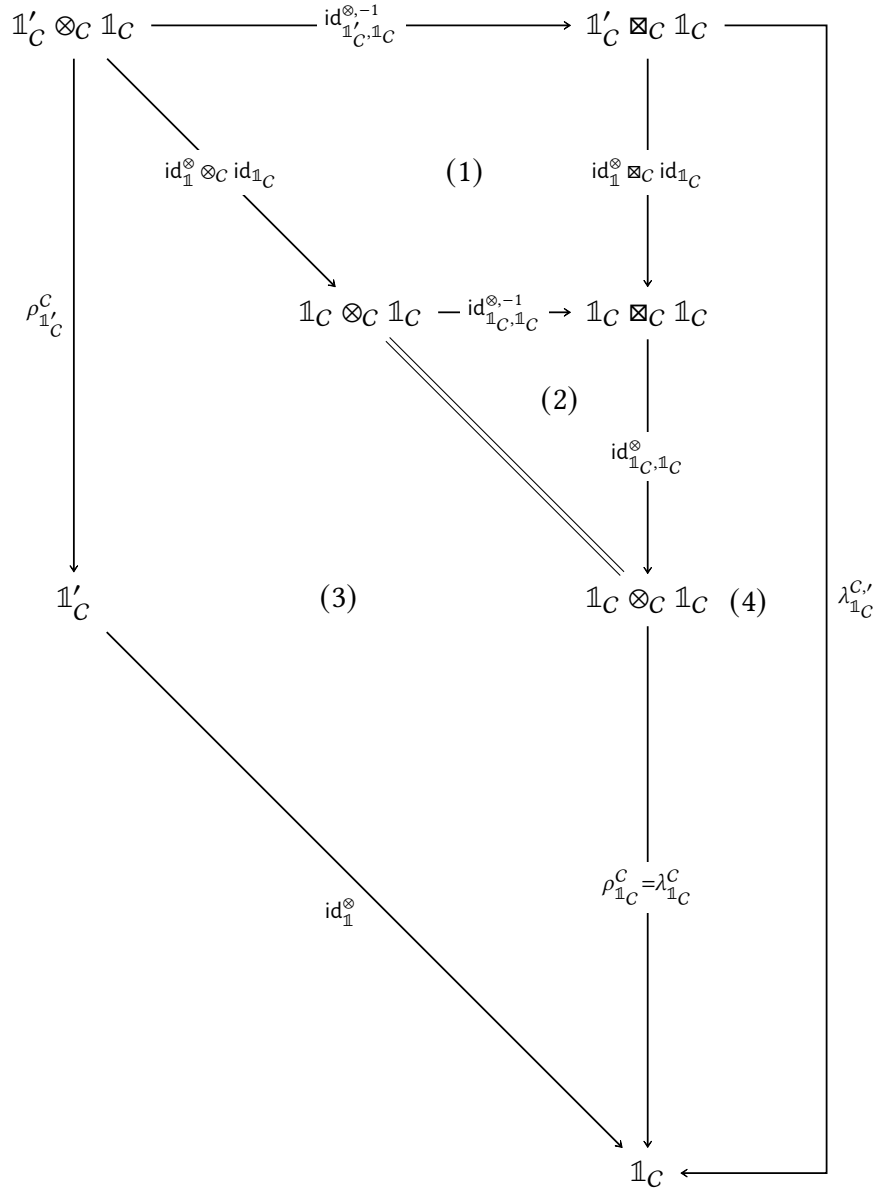


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of  $(\text{id}_C, \text{id}_C^{\otimes}, \text{id}_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_C^{\otimes, -1}$ ;
- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of  $(\text{id}_C, \text{id}_C^\otimes, \text{id}_{C|1}^\otimes)$ ;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^\otimes} & 1'_C \otimes_C 1_C \\
 \rho_{1'_C}^C \downarrow & & \downarrow \lambda_{1_C}^{C, '}& & \downarrow \rho_{1'_C}^C \\
 1'_C & \xrightarrow{\text{id}_1^\otimes} & 1_C & \xrightarrow{\text{id}_1^{\otimes, -1}} & 1'_C.
 \end{array}
 \quad \begin{array}{c} (1) \end{array} \quad \begin{array}{c} (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1b**;

it follows that the diagram

$$\begin{array}{ccccc}
 1'_C \otimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1'_C \boxtimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1'_C}^\otimes} & 1'_C \otimes_C 1_C \\
 & & \downarrow \lambda_{1_C}^{C, '}& & \downarrow \rho_{1'_C}^C \\
 & & 1_C & \xrightarrow{\text{id}_1^{\otimes, -1}} & 1'_C
 \end{array}
 \quad \begin{array}{c} (\dagger)$$

commutes. But since  $\text{id}_{1_C, 1'_C}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

4. *Proof of Item 2d*: Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & (1) & \rho_{\mathbb{1}_C}^{C, ' } \downarrow & (\dagger) & \lambda_{\mathbb{1}'_C}^C \downarrow \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by *Item 1a*;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 & & \rho_{\mathbb{1}_C}^{C, ' } \downarrow & (\dagger) & \lambda_{\mathbb{1}'_C}^C \downarrow \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}$$

commutes. But since  $\text{id}_{\mathbb{1}}^{\otimes, -1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

*Item 3, Mixed Associators*: We claim that *Items 3a* to *3c* are indeed true:

- 01UZ      1. *Proof of Item 3a*: We may partition the monoidality diagram for  $\text{id}^{\otimes}$  of *Item 2*

of **Definition 13.1.1.3** as follows:

$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C & \downarrow & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) \downarrow & \downarrow & (2) \downarrow & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 & \swarrow \text{id}_{A,B}^\otimes \otimes_C \text{id}_C & \searrow \alpha_{A,B,C}^\otimes & & \\
 & (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) & \\
 & \searrow \alpha_{A,B,C}^{C'} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & \\
 & A \otimes_C (B \otimes_C C) & & & 
 \end{array}$$

Since:

- Subdiagram (1) commutes by **Item 1a** of **Item 1**.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.3**.

**01V0** 2. *Proof of Item 3b:* We may partition the monoidality diagram for  $\text{id}^\otimes$  of **Item 2** of **Definition 13.1.1.3** as follows:

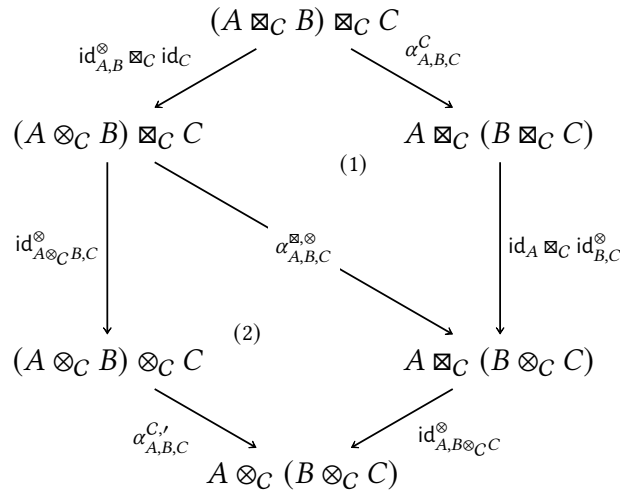
$$\begin{array}{ccccc}
 & & (A \boxtimes_C B) \boxtimes_C C & & \\
 & \swarrow \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C & \downarrow & \searrow \alpha_{A,B,C}^C & \\
 (A \otimes_C B) \boxtimes_C C & & (A \boxtimes_C B) \otimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \searrow \alpha_{A,B,C}^\otimes & \downarrow & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \otimes_C (B \boxtimes_C C) & & A \boxtimes_C (B \otimes_C C) \\
 & (2) \downarrow & \downarrow & (3) \downarrow & \\
 (A \otimes_C B) \otimes_C C & & A \otimes_C (B \otimes_C C) & & A \boxtimes_C (B \otimes_C C) \\
 & \searrow \alpha_{A,B,C}^{C'} & \downarrow \text{id}_A \otimes_C \text{id}_{B,C}^\otimes & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & \\
 & A \otimes_C (B \otimes_C C) & & & 
 \end{array}$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by **Item 1b** of **Item 1**.

it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

- 01V1 3. *Proof of Item 3c:* We may partition the monoidality diagram for  $\text{id}^\otimes$  of **Item 2** of **Definition 13.1.1.1.3** as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $\text{id}^\otimes$  satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

This finishes the proof.

□

- 01V2 **13.1.2 The Moduli Category of Braided Monoidal Structures on a Category**
- 01V3 **13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category**
- 01V4 **13.2 Moduli Categories of Closed Monoidal Structures**
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