# Monoidal Structures on the Category of Sets

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**O1NK** This chapter contains some material on monoidal structures on Sets.

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# **01NQ DEFINITION 5.1.3.1.1** ► THE MONOIDAL UNIT OF ×

The monoidal unit of the product of sets is the functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{1}_{\mathsf{Sets}} \stackrel{\scriptscriptstyle \mathrm{def}}{=} \mathrm{pt},$$

where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.1.

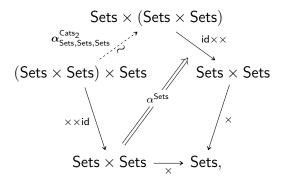
## 01NR 5.1.4 The Associator

# **01NS DEFINITION 5.1.4.1.1** ► THE ASSOCIATOR OF ×

The associator of the product of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}},$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z) \stackrel{\text{def}}{=} (x,(y,z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

## PROOF 5.1.4.1.2 ▶ Proof of the Claims Made in Definition 5.1.4.1.1

## Invertibility

The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is the morphism

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \stackrel{\scriptscriptstyle \mathsf{def}}{=} ((x,y),z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}}\right] &((x,y),z) \stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1} \Big(\alpha_{X,Y,Z}^{\mathsf{Sets}}((x,y),z)\Big) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z)) \\ &\stackrel{\text{def}}{=} \big((x,y),z\big) \\ &\stackrel{\text{def}}{=} \Big[\mathrm{id}_{(X\times Y)\times Z}\Big] &((x,y),z) \end{split}$$

for each  $((x,y),z) \in (X \times Y) \times Z$ , and therefore we have  $\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{(X \times Y) \times Z}$ .

• Invertibility II. We have

$$\begin{split} \left[\alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1}\right] &(x,(y,z)) \stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} \left(\alpha_{X,Y,Z}^{\mathsf{Sets},-1}(x,(y,z))\right) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} ((x,y),z) \\ &\stackrel{\text{def}}{=} (x,(y,z)) \\ &\stackrel{\text{def}}{=} \left[\mathrm{id}_{(X\times Y)\times Z}\right] &(x,(y,z)) \end{split}$$

for each  $(x,(y,z)) \in X \times (Y \times Z)$ , and therefore we have  $\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{X \times (Y \times Z)} \,.$ 

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

commutes. Indeed, this diagram acts on elements as

$$((x,y),z) \qquad \qquad ((x,y),z) \longmapsto ((f(x),g(y)),h(z))$$
 
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$(x,(y,z)) \longmapsto (f(x),(g(y),h(z))) \qquad \qquad (f(x),(g(y),h(z)))$$

and hence indeed commutes, showing  $\alpha^{\mathsf{Sets}}$  to be a natural transformation.

## Being a Natural Isomorphism

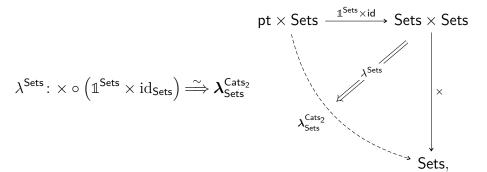
Since  $\alpha^{\mathsf{Sets}}$  is natural and  $\alpha^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\alpha^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\alpha^{\mathsf{Sets},-1}$  is also natural. Thus  $\alpha^{\mathsf{Sets}}$  is a natural isomorphism.

## 01NT 5.1.5 The Left Unitor

#### **01NU**

#### **DEFINITION 5.1.5.1.1** ► THE LEFT UNITOR OF ×

The left unitor of the product of sets is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets}} \colon \operatorname{pt} \times X \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\scriptscriptstyle \mathrm{def}}{=} x$$

for each  $(\star, x) \in \text{pt} \times X$ .

## PROOF 5.1.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.5.1.1

#### Invertibility

The inverse of  $\lambda_X^{\mathsf{Sets}}$  is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathrm{pt} \times X$$

defined by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\text{def}}{=} (\star,x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}\right] &(\mathsf{pt},x) = \lambda_X^{\mathsf{Sets},-1} \Big(\lambda_X^{\mathsf{Sets}} (\mathsf{pt},x)\Big) \\ &= \lambda_X^{\mathsf{Sets},-1} (x) \end{split}$$

$$= (pt, x)$$
$$= [id_{pt \times X}](pt, x)$$

for each  $(pt, x) \in pt \times X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}} = \mathrm{id}_{\mathrm{pt} \times X}$$
 .

• Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}\right] (x) &= \lambda_X^{\mathsf{Sets}} \Big(\lambda_X^{\mathsf{Sets},-1}(x)\Big) \\ &= \lambda_X^{\mathsf{Sets},-1} \big(\mathrm{pt},x\big) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X$$
 .

Therefore  $\lambda_X^{\mathsf{Sets}}$  is indeed an isomorphism.

## Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc} \operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\ \lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\ X & & & f \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(\star, x) & (\star, x) & \longrightarrow (\star, f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets}}$  is a natural transformation.

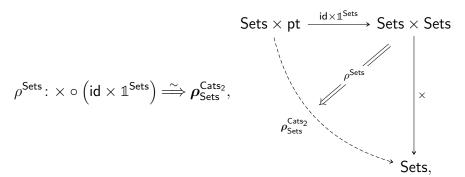
# Being a Natural Isomorphism

Since  $\lambda^{\mathsf{Sets}}$  is natural and  $\lambda^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\lambda^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets},-1}$  is also natural. Thus  $\lambda^{\mathsf{Sets}}$  is a natural isomorphism.

# 01NV 5.1.6 The Right Unitor

#### **01NW DEFINITION 5.1.6.1.1** ► THE RIGHT UNITOR OF ×

The right unitor of the product of sets is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \operatorname{pt} \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\rho_X^{\mathsf{Sets}}(x,\star) \stackrel{\scriptscriptstyle \mathrm{def}}{=} x$$

for each  $(x, \star) \in X \times \text{pt.}$ 

## PROOF 5.1.6.1.2 ▶ Proof of the Claims Made in Definition 5.1.6.1.1

## Invertibility

The inverse of  $\rho_X^{\sf Sets}$  is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (x,\star)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} \right] &(x,\star) = \rho_X^{\mathsf{Sets},-1} \Big( \rho_X^{\mathsf{Sets}} (x,\star) \Big) \\ &= \rho_X^{\mathsf{Sets},-1} (x) \\ &= (x,\star) \\ &= [\mathrm{id}_{X \times \mathrm{pt}}] (x,\star) \end{split}$$

for each  $(x,\star) \in X \times pt$ , and therefore we have

$$\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} = \mathrm{id}_{X \times \mathrm{pt}} \,.$$

• Invertibility II. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} \right] (x) &= \rho_X^{\mathsf{Sets}} \Big( \rho_X^{\mathsf{Sets},-1} (x) \Big) \\ &= \rho_X^{\mathsf{Sets},-1} (x, \star) \\ &= x \\ &= [\mathrm{id}_X] (x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X \,.$$

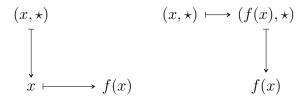
Therefore  $\rho_X^{\sf Sets}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc} X \times \operatorname{pt} & \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} Y \times \operatorname{pt} \\ & & & \downarrow \rho_{X}^{\operatorname{Sets}} \\ \downarrow & & & \downarrow \rho_{Y}^{\operatorname{Sets}} \\ X & & & & f \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\rho^{\mathsf{Sets}}$  is a natural transformation.

## Being a Natural Isomorphism

Since  $\rho^{\mathsf{Sets}}$  is natural and  $\rho^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\rho^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\rho^{\mathsf{Sets},-1}$  is also natural. Thus  $\rho^{\mathsf{Sets}}$  is a natural isomorphism.

## 01NX 5.1.7 The Symmetry

**01NY** 

#### **DEFINITION 5.1.7.1.1** $\blacktriangleright$ The Symmetry of imes

The symmetry of the product of sets is the natural isomorphism



whose component

$$\sigma_{X,Y}^{\mathsf{Sets}} \colon X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in \text{Obj}(\mathsf{Sets})$  is defined by

$$\sigma_{X,Y}^{\mathsf{Sets}}(x,y) \stackrel{\scriptscriptstyle \mathrm{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

#### PROOF 5.1.7.1.2 ▶ Proof of the Claims Made in Definition 5.1.7.1.1

## Invertibility

The inverse of  $\sigma_{X,Y}^{\mathsf{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \colon Y \times X \stackrel{\sim}{\dashrightarrow} X \times Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},-1}(y,x) \stackrel{\text{def}}{=} (x,y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}}\right] &(x,y) \stackrel{\text{\tiny def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \Big(\sigma_{X,Y}^{\mathsf{Sets}}(x,y)\Big) \\ \stackrel{\text{\tiny def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(y,x) \\ \stackrel{\text{\tiny def}}{=} (x,y) \\ \stackrel{\text{\tiny def}}{=} [\mathrm{id}_{X\times Y}](x,y) \end{split}$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}} = \mathrm{id}_{X \times Y} \,.$$

• Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1}\right] &(y,x) \stackrel{\scriptscriptstyle \mathsf{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \Big(\sigma_{X,Y}^{\mathsf{Sets}}(y,x)\Big) \\ &\stackrel{\scriptscriptstyle \mathsf{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(x,y) \\ &\stackrel{\scriptscriptstyle \mathsf{def}}{=} (y,x) \\ &\stackrel{\scriptscriptstyle \mathsf{def}}{=} [\mathrm{id}_{Y\times X}] (y,x) \end{split}$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \operatorname{id}_{Y \times X}.$$

Therefore  $\sigma_{X,Y}^{\mathsf{Sets}}$  is indeed an isomorphism.

## Naturality

We need to show that, given functions

$$f: X \to A,$$
  
 $g: Y \to B$ 

the diagram

$$\begin{array}{c|c} X \times Y & \xrightarrow{f \times g} & A \times B \\ \sigma_{X,Y}^{\mathsf{Sets}} & & & & & \downarrow \sigma_{A,B}^{\mathsf{Sets}} \\ Y \times X & \xrightarrow{g \times f} & B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes, showing  $\sigma^{\mathsf{Sets}}$  to be a natural transformation.

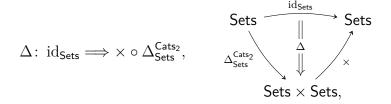
## Being a Natural Isomorphism

Since  $\sigma^{\mathsf{Sets}}$  is natural and  $\sigma^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\sigma^{\mathsf{Sets}}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\sigma^{\mathsf{Sets},-1}$  is also natural. Thus  $\sigma^{\mathsf{Sets}}$  is a natural isomorphism.

# 01NZ 5.1.8 The Diagonal

01P0 DEFINITION 5.1.8.1.1 ► THE DIAGONAL OF ×

The diagonal of the product of sets is the natural transformation



whose component

$$\Delta_X \colon X \to X \times X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .

#### PROOF 5.1.8.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.8.1.1

We need to show that, given a function  $f: X \to Y$ , the diagram

$$X \xrightarrow{f} Y$$

$$\downarrow^{\Delta_X} \qquad \downarrow^{\Delta_Y}$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
x & & x & & \\
\downarrow & & & \downarrow \\
(x,x) & \longmapsto (f(x),f(x)) & & (f(x),f(x))
\end{array}$$

and hence indeed commutes, showing  $\Delta$  to be natural.

#### 01P1 PROPOSITION 5.1.8.1.3 ▶ PROPERTIES OF THE DIAGONAL MAP

Let X be a set.

01P2

01P3

01P4

1. Monoidality. The diagonal map

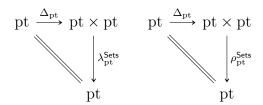
$$\Delta : \operatorname{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

is a monoidal natural transformation:

(a) Compatibility With Strong Monoidality Constraints. For each  $X, Y \in \text{Obj}(\mathsf{Sets})$ , the diagram

commutes.

(b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\mathrm{pt}} &= \lambda_{\mathrm{pt}}^{\mathsf{Sets},-1} \\ &= \rho_{\mathrm{pt}}^{\mathsf{Sets},-1}, \end{split}$$

where we recall that the equalities

$$\begin{split} \lambda_{\mathrm{pt}}^{\mathsf{Sets}} &= \rho_{\mathrm{pt}}^{\mathsf{Sets}}, \\ \lambda_{\mathrm{pt}}^{\mathsf{Sets}, -1} &= \rho_{\mathrm{pt}}^{\mathsf{Sets}, -1} \end{split}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

01P5

2. The Diagonal of the Unit. The component

$$\Delta_{\rm pt}$$
: pt  $\stackrel{\sim}{--}$  pt  $\times$  pt

of  $\Delta$  at pt is an isomorphism.

#### PROOF 5.1.8.1.4 ▶ PROOF OF PROPOSITION 5.1.8.1.3

## Item 1: Monoidality

We claim that  $\Delta$  is indeed monoidal:

024S

1. Item 1a: Compatibility With Strong Monoidality Constraints: We need to show that the diagram

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} (X \times X) \times (Y \times Y)$$

$$\downarrow \\ (X \times Y) \times (X \times Y)$$

commutes. Indeed, this diagram acts on elements as

and hence indeed commutes.

024T

2. Item 1b: Compatibility With Strong Unitality Constraints: As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each  $x \in X$ , so when X = pt, we have

$$\lambda_{\mathrm{pt}}^{\mathsf{Sets},-1}(\star) \stackrel{\scriptscriptstyle\mathrm{def}}{=} (\star,\star),$$

and also

$$\Delta_{\mathrm{pt}}^{\mathsf{Sets}}(\star) \stackrel{\scriptscriptstyle\mathrm{def}}{=} (\star, \star),$$

so we have  $\Delta_{\mathrm{pt}} = \lambda_{\mathrm{pt}}^{\mathsf{Sets},-1}$ .

This finishes the proof.

## Item 2: The Diagonal of the Unit

This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to  $\times$ , proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor.

# 01P6 5.1.9 The Monoidal Category of Sets and Products

#### Ø1P7 PROPOSITION 5.1.9.1.1 ➤ THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO THE PRODUCT

The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- The Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.3.5.1.2.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

ullet The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Right Unitors. The natural isomorphism

$$ho^{\mathsf{Sets}} \colon imes \circ \left(\mathsf{id} imes \mathbb{1}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} oldsymbol{
ho}_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} \colon \times \stackrel{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Diagonals. The monoidal natural transformation

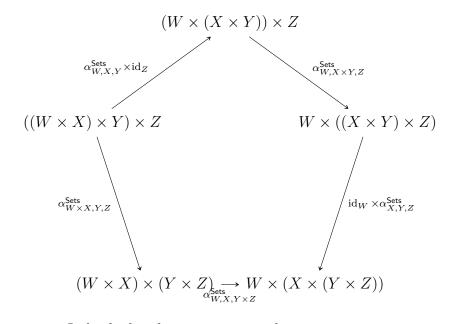
$$\Delta \colon \operatorname{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

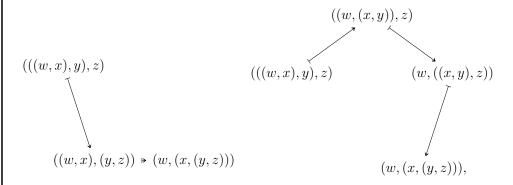
## PROOF 5.1.9.1.2 ▶ PROOF OF PROPOSITION 5.1.9.1.1

The Pentagon Identity

Let W, X, Y and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the pentagon identity is satisfied.

#### The Triangle Identity

Let X and Y be sets. We have to show that the diagram

$$(X \times \mathrm{pt}) \times Y \xrightarrow{\alpha_{X,\mathrm{pt},Y}^{\mathsf{Sets}}} X \times (\mathrm{pt} \times Y)$$

$$\rho_X^{\mathsf{Sets}} \times \mathrm{id}_Y \xrightarrow{\mathrm{id}_X \times \lambda_Y^{\mathsf{Sets}}} X \times Y$$

commutes. Indeed, this diagram acts on elements as

$$((x,\star),y) \qquad ((x,\star),y) \longmapsto (x,(\star,y))$$

$$(x,y) \qquad (x,y)$$

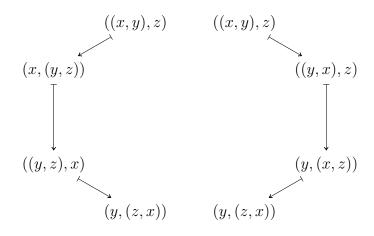
and thus the triangle identity is satisfied.

## The Left Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram

$$\begin{array}{c|c} \alpha_{X,Y,Z}^{\mathsf{Sets}} & (X \times Y) \times Z \\ \hline \alpha_{X,Y,Z}^{\mathsf{Sets}} & (Y \times X) \times Z \\ \hline X \times (Y \times Z) & (Y \times X) \times Z \\ \hline \\ \sigma_{X,Y \times Z}^{\mathsf{Sets}} & & & & & \\ \alpha_{Y,X,Z}^{\mathsf{Sets}} & & & & \\ (Y \times Z) \times X & & & & & \\ Y \times (X \times Z) & & & & \\ \hline \alpha_{Y,X,Z}^{\mathsf{Sets}} & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \end{array}$$

commutes. Indeed, this diagram acts on elements as

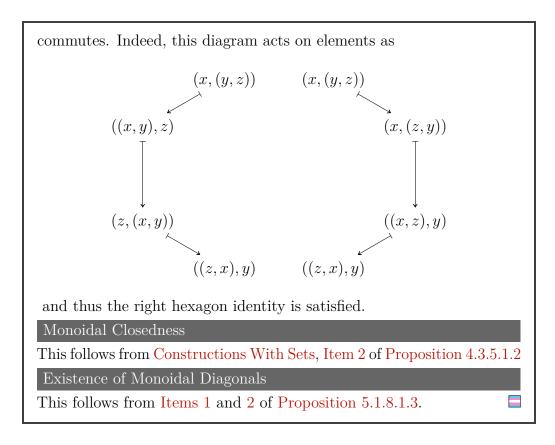


and thus the left hexagon identity is satisfied.

#### The Right Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram

$$(\alpha_{X,Y,Z}^{\mathsf{Sets}})^{-1}X \times (Y \times Z) \\ \mathrm{id}_{X} \times \sigma_{Y,Z}^{\mathsf{Sets}} \\ (X \times Y) \times Z \qquad \qquad X \times (Z \times Y) \\ \sigma_{X \times Y,Z}^{\mathsf{Sets}} \bigvee_{\left(\alpha_{X,Z,Y}^{\mathsf{Sets}}\right)^{-1}} \\ Z \times (X \times Y) \qquad \qquad (X \times Z) \times Y \\ (\alpha_{Z,X,Y}^{\mathsf{Sets}})^{-1} \bigvee_{\left(Z \times X\right) \times Y} \sigma_{X,Z}^{\mathsf{Sets}} \times \mathrm{id}_{Y}$$



01P8 5.1.10 The Universal Property of  $(Sets, \times, pt)$ 

## 01P9 THEOREM 5.1.10.1.1 $\blacktriangleright$ The Universal Property of (Sets, $\times$ , $\operatorname{pt}$ )

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Existence of an Internal Hom. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom  $[-1, -2]_{Sets}$ 

2. The Unit Object Is pt. We have  $\mathbb{1}_{\mathsf{Sets}} \cong \mathsf{pt}$ .

More precisely, the full subcategory of the category  $\mathcal{M}^{\mathrm{cld}}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of ?? spanned by the closed symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,

 $[-1, -2]_{\text{Sets}}$ ,  $\mathbb{1}_{\text{Sets}}$ ,  $\lambda^{\text{Sets}}$ ,  $\rho^{\text{Sets}}$ ,  $\sigma^{\text{Sets}}$ ) satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

#### PROOF 5.1.10.1.2 ▶ PROOF OF THEOREM 5.1.10.1.1

## Unwinding the Statement

01PA

01PB

Let  $(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, [-_1, -_2]_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda', \rho', \sigma')$  be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$\mathrm{id}_{\mathsf{Sets}} \colon \mathsf{Sets} \to \mathsf{Sets}$$

admits a unique closed symmetric monoidal functor structure

$$\begin{array}{cccc} \operatorname{id}_{\mathsf{Sets}}^{\otimes} \colon & A \otimes_{\mathsf{Sets}} B \stackrel{\sim}{--} & A \times B, \\ \operatorname{id}_{\mathsf{Sets}}^{\mathsf{Hom}} \colon & [A,B]_{\mathsf{Sets}} \stackrel{\sim}{--} & \mathsf{Sets}(A,B), \\ \operatorname{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \colon & \mathbb{1}_{\mathsf{Sets}} & \stackrel{\sim}{--} & \mathrm{pt}, \end{array}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) to the closed symmetric monoidal category (Sets,  $\times$ , Sets( $-_1$ ,  $-_2$ ),  $\mathbb{1}_{Sets}$ ,  $\lambda^{Sets}$ ,  $\rho^{Sets}$ ,  $\sigma^{Sets}$ ) of Proposition 5.1.9.1.1.

Constructing an Isomorphism  $[-1, -2]_{Sets} \cong Sets(-1, -2)$ 

By ??, we have a natural isomorphism

$$\mathsf{Sets}(\mathrm{pt},[-_1,-_2]_{\mathsf{Sets}}) \cong \mathsf{Sets}(-_1,-_2).$$

By Constructions With Sets, Item 3 of Proposition 4.3.5.1.2, we also have a natural isomorphism

$$\mathsf{Sets}(\mathsf{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}(-1, -2) \cong [-1, -2]_{\mathsf{Sets}}$$
.

Given  $A, B \in \text{Obj}(\mathsf{Sets})$ , we will write

$$\operatorname{id}_{A,B}^{\operatorname{Hom}} : \operatorname{\mathsf{Sets}}(A,B) \stackrel{\sim}{\dashrightarrow} [A,B]_{\operatorname{\mathsf{Sets}}}$$

for the component of this isomorphism at (A, B).

## Constructing an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Since  $\otimes_{\mathsf{Sets}}$  is adjoint in each variable to  $[-_1, -_2]_{\mathsf{Sets}}$  by assumption and  $\times$  is adjoint in each variable to  $\mathsf{Sets}(-_1, -_2)$  by Constructions With Sets, Item 2 of Proposition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -,$$
$$- \otimes_{\mathsf{Sets}} B \cong B \times -.$$

By  $\ref{eq:solution}$ , we then have  $\otimes_{\mathsf{Sets}} \cong \times$ . We will write

$$\operatorname{id}_{\operatorname{Sets}|A}^{\otimes} : A \otimes_{\operatorname{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

## Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Alternatively, we may construct a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  as follows:

1. Let  $A \in \text{Obj}(\mathsf{Sets})$ .

01PC

01PD

2. Since  $\otimes_{\mathsf{Sets}}$  is part of a closed monoidal structure, it preserves colimits in each variable by ??.

01PE

3. Since  $A \cong \coprod_{a \in A} \text{pt}$  and  $\otimes_{\mathsf{Sets}}$  preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong \left( \coprod_{a \in A} \mathsf{pt} \right) \otimes_{\mathsf{Sets}} B$$

$$\cong \coprod_{a \in A} (\mathsf{pt} \otimes_{\mathsf{Sets}} B)$$

$$\cong \coprod_{a \in A} B$$

$$\cong A \times B,$$

naturally in  $B \in \text{Obj}(\mathsf{Sets})$ , where we have used that pt is the monoidal unit for  $\otimes_{\mathsf{Sets}}$ . Thus  $A \otimes_{\mathsf{Sets}} - \cong A \times -$  for each  $A \in \text{Obj}(\mathsf{Sets})$ .

01PF

4. Similarly,  $-\otimes_{\mathsf{Sets}} B \cong -\times B$  for each  $B \in \mathsf{Obj}(\mathsf{Sets})$ .

01PG

5. By ??, we then have  $\otimes_{\mathsf{Sets}} \cong \times$ .

Below, we'll show that if a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\mathrm{id}_{\mathsf{Sets}|A,B}^\otimes\colon A\otimes_{\mathsf{Sets}} B\to A\times B$  from before.

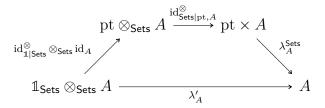
# Constructing an Isomorphism $id_1^{\otimes} : \mathbb{1}_{\mathsf{Sets}} \to \mathsf{pt}$

We define an isomorphism  $\mathrm{id}_{1}^{\otimes} \colon \mathbb{1}_{\mathsf{Sets}} \to \mathrm{pt}$  as the composition

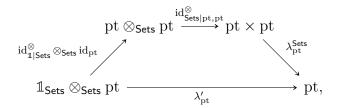
in Sets.

Monoidal Left Unity of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ 

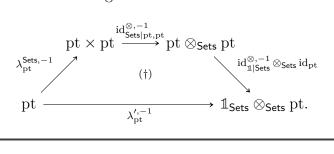
We have to show that the diagram

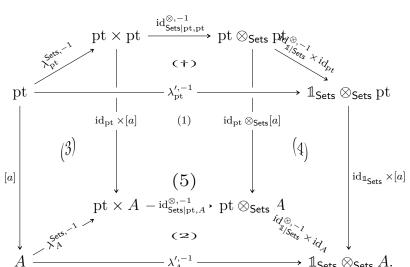


commutes. First, note that the diagram



corresponding to the case  $A={\rm pt}$ , commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so does the diagram





Now, let  $A \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , and consider the diagram

Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'^{,-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{1|Sets}^{\otimes,-1}$
- Subdiagram (1) commutes by the naturality of  $id_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\mathsf{Sets},-1}$ .

it follows that the diagram

$$\operatorname{pt} \times A \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\operatorname{pt},A}^{\otimes,-1}} \operatorname{pt} \otimes_{\mathsf{Sets}} A$$

$$\uparrow^{\mathsf{Sets},-1} \nearrow \qquad \qquad \downarrow^{\operatorname{id}_{\mathsf{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_{A}}$$

$$\operatorname{pt} \xrightarrow{[a]} A \xrightarrow{\lambda_{A}^{\prime,-1}} \qquad \qquad \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\lambda_A^{\prime,-1}(a) = \left[\lambda_A^{\prime,-1} \circ [a]\right](\star)$$

$$\begin{split} &= \left[ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_{A} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_{A}^{\mathsf{Sets},-1} \circ [a] \right] (\star) \\ &= \left[ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_{A} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_{A}^{\mathsf{Sets},-1} \right] (a) \end{split}$$

for each  $a \in A$ , and thus we have

$$\lambda_A^{\prime,-1} = \left(\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ (\mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \times \mathrm{id}_A),$$

showing that the diagram

$$\operatorname{pt} \otimes_{\mathsf{Sets}} A \xrightarrow{\operatorname{id}_{\mathsf{Sets}}^{\otimes} | \operatorname{pt} \times A} \operatorname{pt} \times A$$

$$\operatorname{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \operatorname{id}_{A} \xrightarrow{\lambda_{A}'} A$$

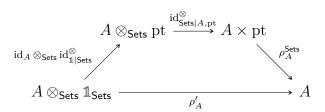
$$\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \xrightarrow{\lambda_{A}'} A$$

indeed commutes.

## Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  above. For completeness, we repeat it below.

We have to show that the diagram

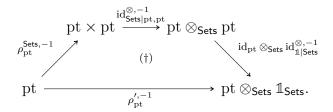


commutes. First, note that the diagram

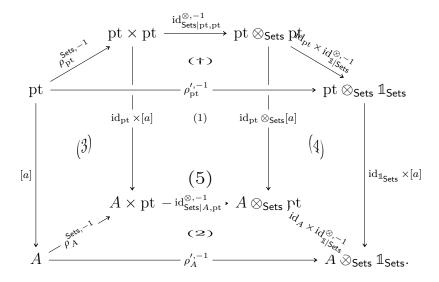
$$\begin{array}{c} \operatorname{pt} \otimes_{\mathsf{Sets}} \operatorname{pt} \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\operatorname{pt},\operatorname{pt}}^{\mathsf{d}}} \operatorname{pt} \times \operatorname{pt} \\ \operatorname{id}_{\operatorname{pt}} \otimes_{\mathsf{Sets}} \operatorname{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} & & \rho_{\operatorname{pt}}^{\mathsf{Sets}} \end{array}$$

$$\operatorname{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \xrightarrow{\rho_{\operatorname{pt}}'} \operatorname{pt}$$

corresponding to the case  $A={\rm pt}$ , commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so does the diagram



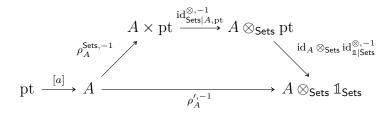
Now, let  $A \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , and consider the diagram



#### Since:

- Subdiagram (5) commutes by the naturality of  $\rho'^{,-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of id<sub>1|Sets</sub>.
- Subdiagram (1) commutes by the naturality of id<sub>Sets</sub><sup>⊗,-1</sup>.
- Subdiagram (3) commutes by the naturality of  $\rho^{\mathsf{Sets},-1}.$

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{aligned} \rho_A^{\prime,-1}(a) &= \left[ \rho_A^{\prime,-1} \circ [a] \right] (\star) \\ &= \left[ \left( \mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1} \circ [a] \right] (\star) \\ &= \left[ \left( \mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1} \right] (a) \end{aligned}$$

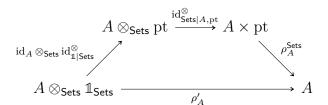
for each  $a \in A$ , and thus we have

$$\rho_A^{\prime,-1} = \left(\mathrm{id}_A \times \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ \left(\mathrm{id}_A \times \mathrm{id}_{\mathsf{1}|\mathsf{Sets}}^{\otimes}\right).$$

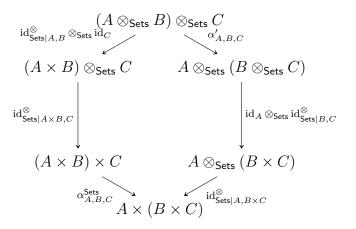
showing that the diagram



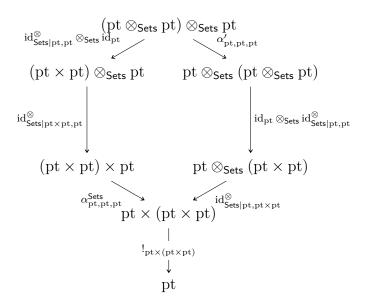
indeed commutes.

Monoidality of the Isomorphism  $\otimes_{\mathsf{Sets}} = \times$ 

We have to show that the diagram

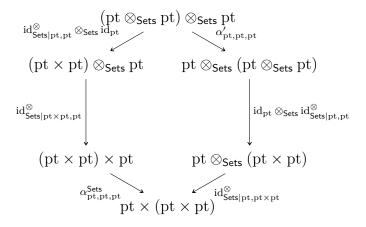


commutes. First, note that the diagram

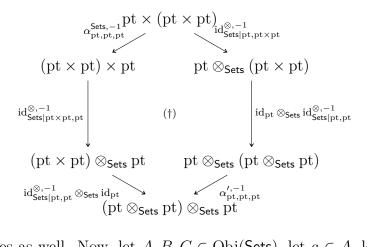


commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.2). Since the map  $!_{pt \times (pt \times pt)} : pt \times (pt \times pt) \to pt$  is an isomorphism (e.g. having inverse  $\lambda_{pt}^{\mathsf{Sets},-1} \circ \lambda_{pt}^{\mathsf{Sets},-1}$ ), it follows that the

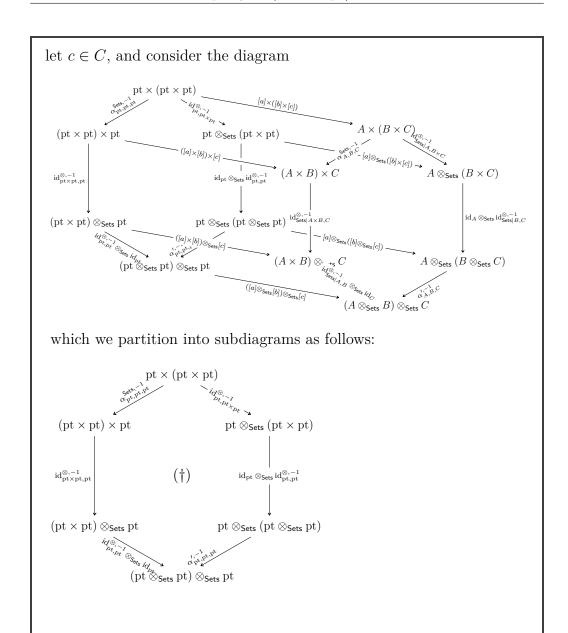
## diagram

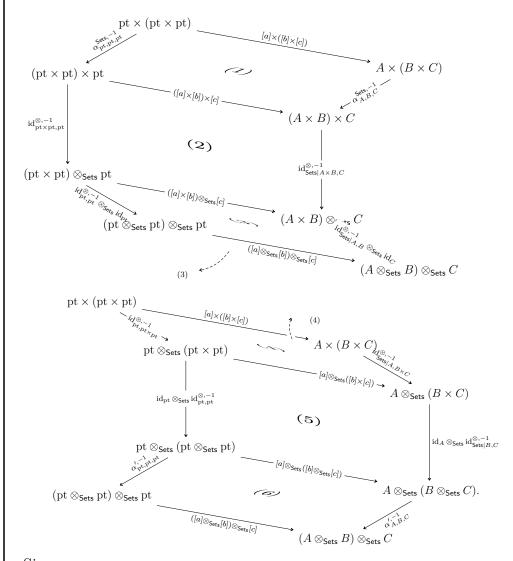


also commutes. Taking inverses, we see that the diagram



commutes as well. Now, let  $A, B, C \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ ,





## Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\mathsf{Sets},-1}$ .
- Subdiagram (2) commutes by the naturality of  $\mathrm{id}_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $id_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.

- Subdiagram (4) commutes by the naturality of  $id_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (5) commutes by the naturality of  $id_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .

it follows that the diagram

$$\begin{array}{c|c} \operatorname{pt} \times (\operatorname{pt} \times \operatorname{pt}) \\ & | \\ & | \\ (a) \times ([b] \times [c]) \\ \downarrow \\ & \downarrow \\ (A \times B) \times C \\ & A \otimes_{\mathsf{Sets}} (B \times C) \\ \operatorname{id}_{\mathsf{Sets}|A \times B, C} \\ \downarrow \\ (A \times B) \otimes_{\mathsf{Sets}} C \\ & A \otimes_{\mathsf{Sets}} (B \times C) \\ & \downarrow \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B, C}^{\otimes, -1} \\ & \downarrow \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B, C}^{\otimes, -1} \\ & \downarrow \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B, C}^{\otimes, -1} \\ & (A \times B) \otimes_{\mathsf{Sets}} C \\ & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \\ & \operatorname{id}_{\mathsf{Sets}|A, B}^{\otimes, -1} \otimes_{\mathsf{Sets}} \operatorname{id}_{C} \\ & (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C \end{array}$$

also commutes. We then have

$$\begin{split} \left[ \left( \operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_C \right) \circ \operatorname{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ &\circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) = \left[ \left( \operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_C \right) \circ \operatorname{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ &\circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ \left( [a] \times \left( [b] \times [c] \right) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \operatorname{id}_A \times \operatorname{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \\ &\circ \operatorname{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ \left( [a] \times \left( [b] \times [c] \right) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \operatorname{id}_A \times \operatorname{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \circ \operatorname{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

for each  $(a,(b,c)) \in A \times (B \times C)$ , and thus we have

$$\left(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_{C}\right) \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} = \alpha_{A,B,C}^{\prime,-1} \circ \left(\mathrm{id}_{A} \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1}$$

Taking inverses then gives

 $\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ \left( \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_{C} \right) = \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ \left( \mathrm{id}_{A} \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes} \right) \circ \alpha_{A,B,C}',$  showing that the diagram

$$(A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C$$

$$(A \times B) \otimes_{\mathsf{Sets}} C$$

$$(A \times B) \otimes_{\mathsf{Sets}} C$$

$$A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C)$$

$$\mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \qquad \mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes}$$

$$(A \times B) \times C$$

$$A \otimes_{\mathsf{Sets}} (B \times C)$$

$$\mathsf{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \qquad \mathsf{id}_{\mathsf{Sets}|A,B \times C}^{\otimes}$$

$$A \times (B \times C)$$

indeed commutes.

## Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{c|c} A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B \\ \\ \sigma'_{A,B} \bigg| & & \bigg| \sigma^{\mathsf{Sets}}_{A,B} \\ B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{c|c} \operatorname{pt} \otimes_{\mathsf{Sets}} \operatorname{pt} \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\operatorname{pt},\operatorname{pt}}^{\otimes}} \operatorname{pt} \times \operatorname{pt} \\ \\ \sigma'_{\operatorname{pt},\operatorname{pt}} \bigg| & & & & & \\ \sigma'_{\operatorname{pt},\operatorname{pt}} \bigg| & & & & \\ \operatorname{pt} \otimes_{\mathsf{Sets}} \operatorname{pt} \xrightarrow{\operatorname{id}_{\overset{\otimes}{\mathsf{Sets}|\operatorname{pt},\operatorname{pt}}}} \operatorname{pt} \times \operatorname{pt} \\ \\ \operatorname{pt} \otimes_{\mathsf{Sets}} \operatorname{pt} \xrightarrow{\operatorname{id}_{\overset{\otimes}{\mathsf{Sets}|\operatorname{pt},\operatorname{pt}}}} \operatorname{pt} \times \operatorname{pt} \\ \end{array}$$

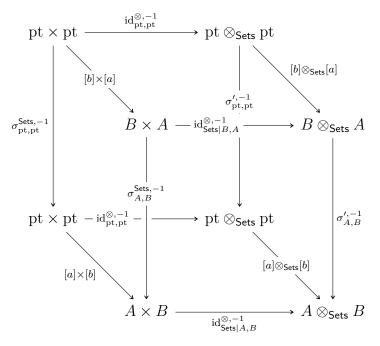
commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since the map  $!_{pt \times pt} : pt \times pt \rightarrow pt$  is invertible (e.g. with inverse  $\lambda_{pt}^{\mathsf{Sets},-1}$ ), the diagram

also commutes. Taking inverses, we see that the diagram

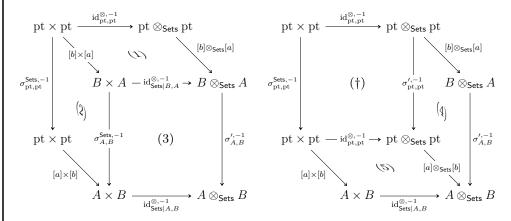
$$\begin{array}{c|c} pt \times pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} pt \otimes_{\mathsf{Sets}} pt \\ \\ \sigma_{\mathrm{pt,pt}}^{\mathsf{Sets},-1} & & (\dagger) & & \sigma_{\mathrm{pt,pt}}^{\prime,-1} \\ \\ pt \times pt & \xrightarrow{\mathrm{id}_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} pt \otimes_{\mathsf{Sets}} pt \end{array}$$

commutes as well. Now, let  $A, B \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , and





which we partition into subdiagrams as follows:

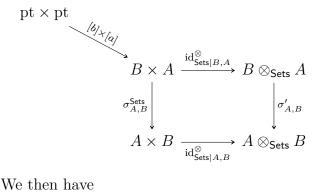


#### Since:

- Subdiagram (2) commutes by the naturality of  $\sigma^{\mathsf{Sets},-1}$ .
- Subdiagram (5) commutes by the naturality of  $id^{\otimes,-1}$ .

- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{,-1}$ .
- Subdiagram (1) commutes by the naturality of  $id^{\otimes,-1}$ .

it follows that the diagram



commutes. We then have

$$\begin{split} \left[ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \right] (b,a) &= \left[ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a]) \right] (\star, \star) \\ &= \left[ \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a]) \right] (\star, \star) \\ &= \left[ \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \right] (b,a) \end{split}$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}^{\prime,-1} \circ \operatorname{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}',$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\sigma'_{A,B} \downarrow \qquad \qquad \qquad \downarrow \sigma^{\mathsf{Sets}}_{A,B}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

indeed commutes.

#### Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Let  $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of  $\mathsf{Sets}$  with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right),$$
$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right).$$

Postcomposing both sides with  $\lambda_B^{\mathsf{Sets},-1}$  gives

$$\lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right) = \phi_{\mathrm{pt},B},$$
$$\lambda_B^{\mathsf{Sets},-1} \circ \lambda_B' \circ \left( \mathrm{id}_{\mathbb{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right) = \psi_{\mathrm{pt},B},$$

and thus we have

$$\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$$

for each  $B \in \text{Obj}(\mathsf{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams

for  $\phi$  and  $\psi$  with respect to the morphisms [a] and  $\mathrm{id}_B$ . Having shown that  $\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$ , we have

$$\begin{split} \phi_{A,B}(a,b) &= [\phi_{A,B} \circ ([a] \times \mathrm{id}_B)](\star,b) \\ &= [([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \phi_{\mathsf{pt},B}](\star,b) \\ &= [([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \psi_{\mathsf{pt},B}](\star,b) \\ &= [\psi_{A,B} \circ ([a] \times \mathrm{id}_B)](\star,b) \\ &= \psi_{A,B}(a,b) \end{split}$$

for each  $(a, b) \in A \times B$ . Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each  $A, B \in \text{Obj}(\mathsf{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  to be unique.

#### **O1PH** COROLLARY 5.1.10.1.3 ► A SECOND UNIVERSAL PROPERTY FOR (Sets, $\times$ , pt)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

2. The Unit Object Is pt. We have  $\mathbb{1}_{\mathsf{Sets}} \cong \mathsf{pt}$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:subcategory}$  spanned by the symmetric monoidal categories  $\left(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\right)$  satisfying Items 1 and 2 is contractible.

#### PROOF 5.1.10.1.4 ▶ PROOF OF COROLLARY 5.1.10.1.3

Since Sets is locally presentable (??), it follows from ?? that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Theorem 5.1.10.1.1. The result then follows from Theorem 5.1.10.1.1.

# oppl 5.2 The Monoidal Category of Sets and Coproducts

#### 01PM 5.2.1 Coproducts of Sets

01PJ

01PK

See Constructions With Sets, Section 4.2.3.

#### 01PN 5.2.2 The Monoidal Unit

#### 01PP DEFINITION 5.2.2.1.1 ► THE MONOIDAL UNIT OF

The monoidal unit of the coproduct of sets is the functor

$$\mathbb{0}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{O}_{\mathsf{Sets}} \stackrel{\text{def}}{=} \emptyset$$

where Ø is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

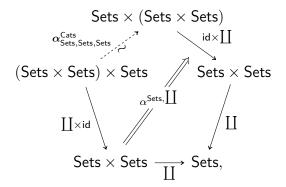
#### 01PQ 5.2.3 The Associator

#### 01PR DEFINITION 5.2.3.1.1 ► THE ASSOCIATOR OF [

The associator of the coproduct of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} \colon \coprod \circ (\coprod \times \operatorname{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\operatorname{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}},$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \colon (X \coprod Y) \coprod Z \stackrel{\sim}{\dashrightarrow} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each  $a \in (X \coprod Y) \coprod Z$ .

#### PROOF 5.2.3.1.2 ▶ Proof of the Claims Made in Definition 5.2.3.1.1

#### Unwinding the Definitions of $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$

Firstly, we unwind the expressions for  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ . We have

$$\begin{split} (X \coprod Y) \coprod Z & \stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \coprod Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ & = \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ & \cup \{(1, z) \in S \mid z \in Z\}, \end{split}$$

where  $S = \{0, 1\} \times ((X \coprod Y) \cup Z)$  and

$$\begin{split} X \coprod (Y \coprod Z) &\stackrel{\text{def}}{=} \{(0,x) \in S' \mid x \in X\} \cup \{(1,a) \in S' \mid a \in Y \coprod Z\} \\ &= \{(0,x) \in S' \mid x \in X\} \cup \{(1,(0,y)) \in S' \mid y \in Y\} \\ & \cup \{(1,(1,z)) \in S' \mid z \in Z\}, \end{split}$$

where  $S' = \{0, 1\} \times (X \cup (Y \coprod Z)).$ 

#### Invertibility

The inverse of  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\text{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each  $a \in X \coprod Y(\coprod Z)$ . Indeed:

• Invertibility I. The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  acts on elements as

$$(0,(0,x)) \mapsto (0,x) \mapsto (0,(0,x)),$$

$$(0, (0, y)) \mapsto (1, (0, y)) \mapsto (0, (0, y)),$$
  
$$(1, z) \mapsto (1, (1, z)) \mapsto (1, z)$$

and hence is equal to the identity map of  $(X \coprod Y) \coprod Z$ .

• Invertibility II. The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$  acts on elements as

$$\begin{array}{cccc} (0,x) & \mapsto & (0,(0,x)) & \mapsto & (0,x), \\ (1,(0,y)) & \mapsto & (0,(0,y)) & \mapsto & (1,(0,y)), \\ (1,(1,z)) & \mapsto & (1,z) & \mapsto & (1,(1,z)) \end{array}$$

and hence is equal to the identity map of  $X \coprod (Y \coprod Z)$ .

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

#### Naturality

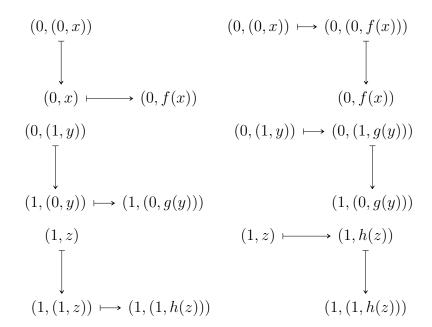
We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$\begin{array}{c|c} (X \coprod Y) \coprod Z & \xrightarrow{\left(f \coprod g\right) \coprod h} & (X' \coprod Y') \coprod Z' \\ & & \downarrow \\ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} & & \downarrow \\ X \coprod (Y \coprod Z) & \xrightarrow{f \coprod (g \coprod h)} & X' \coprod (Y' \coprod Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing  $\alpha^{\mathsf{Sets},\coprod}$  to be a natural transformation.

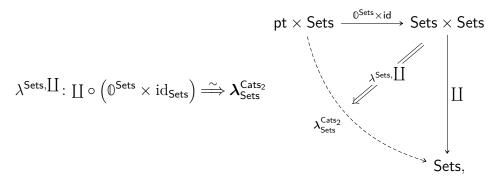
#### Being a Natural Isomorphism

Since  $\alpha^{\mathsf{Sets}, \coprod}$  is natural and  $\alpha^{\mathsf{Sets}, \coprod, -1}$  is a componentwise inverse to  $\alpha^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets}, -1}$  is also natural. Thus  $\alpha^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

#### 01PS 5.2.4 The Left Unitor

01PT DEFINITION 5.2.4.1.1 ► THE LEFT UNITOR OF

The left unitor of the coproduct of sets is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} \colon \varnothing \coprod X \xrightarrow{\sim} X$$

at X is given by

$$\lambda_X^{\mathsf{Sets},\coprod}((1,x))\stackrel{\scriptscriptstyle\mathrm{def}}{=} x$$

for each  $(1, x) \in \emptyset \coprod X$ .

#### PROOF 5.2.4.1.2 ▶ Proof of the Claims Made in Definition 5.2.4.1.1

#### Unwinding the Definition of $\emptyset \coprod X$

Firstly, we unwind the expressions for  $\emptyset \coprod X$ . We have

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

#### Invertibility

The inverse of  $\lambda_X^{\mathsf{Sets},\coprod}$  is the map

$$\lambda_X^{\mathsf{Sets},\coprod,-1} \colon X \to \emptyset \coprod X$$

given by

$$\lambda_X^{\mathsf{Sets},\coprod,-1}(x)\stackrel{\scriptscriptstyle\rm def}{=} (1,x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets},\coprod,-1} \circ \lambda_X^{\mathsf{Sets},\coprod}\right] (1,x) &= \lambda_X^{\mathsf{Sets},\coprod,-1} \bigg(\lambda_X^{\mathsf{Sets},\coprod} (1,x)\bigg) \\ &= \lambda_X^{\mathsf{Sets},\coprod,-1} (x) \\ &= (1,x) \\ &= \left[\mathrm{id}_{\varnothing \coprod X}\right] (1,x) \end{split}$$

for each  $(1, x) \in \emptyset \coprod X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod,-1} \circ \lambda_X^{\mathsf{Sets},\coprod} = \mathrm{id}_{\varnothing \coprod X}$$
 .

• Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1}\right] (x) &= \lambda_X^{\mathsf{Sets},\coprod} \left(\lambda_X^{\mathsf{Sets},\coprod,-1}(x)\right) \\ &= \lambda_X^{\mathsf{Sets},\coprod,-1}(1,x) \\ &= x \\ &= [\mathrm{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X$$
 .

Therefore  $\lambda_X^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(1,x) & (1,x) & \longrightarrow (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets},\coprod}$  is a natural transformation.

#### Being a Natural Isomorphism

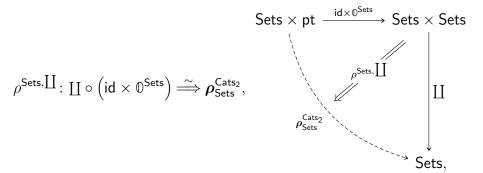
Since  $\lambda^{\mathsf{Sets}, \coprod}$  is natural and  $\lambda^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\lambda^{\mathsf{Sets}, -1}$  is also natural. Thus  $\lambda^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

#### 01PU 5.2.5 The Right Unitor

#### 01PV

#### **DEFINITION 5.2.5.1.1** ► THE RIGHT UNITOR OF

The right unitor of the coproduct of sets is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets},\coprod} \colon X \coprod \emptyset \xrightarrow{\sim} X$$

at X is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x))\stackrel{\scriptscriptstyle\mathrm{def}}{=} x$$

for each  $(0, x) \in X \coprod \emptyset$ .

#### PROOF 5.2.5.1.2 ▶ Proof of the Claims Made in Definition 5.2.5.1.1

#### Unwinding the Definition of $X \coprod \emptyset$

Firstly, we unwind the expression for  $X \coprod \emptyset$ . We have

$$\begin{split} X \coprod \varnothing &\stackrel{\text{def}}{=} \{(0,x) \in S \mid x \in X\} \cup \{(1,z) \in S \mid z \in \varnothing\} \\ &= \{(0,x) \in S \mid x \in X\} \cup \varnothing \\ &= \{(0,x) \in S \mid x \in X\}, \end{split}$$

where  $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$ .

#### Invertibility

The inverse of  $\rho_X^{\mathsf{Sets},\coprod}$  is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \emptyset$$

given by

$$\rho_X^{\mathsf{Sets},\coprod,-1}(x) \stackrel{\text{def}}{=} (0,x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} \right] (0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} \left( \rho_X^{\mathsf{Sets}, \coprod} (0, x) \right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (0, x) \\ &= \left[ \mathrm{id}_{X \coprod \varnothing} \right] (0, x) \end{split}$$

for each  $(0, x) \in \emptyset \coprod X$ , and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod,-1} \circ \rho_X^{\mathsf{Sets},\coprod} = \mathrm{id}_{\emptyset \coprod X} \,.$$

• Invertibility II. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1} \right] (x) &= \rho_X^{\mathsf{Sets}, \coprod} \left( \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (0, x) \\ &= x \\ &= [\mathrm{id}_X] (x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X$$
 .

Therefore  $\rho_X^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc} X \coprod \varnothing & \xrightarrow{f \coprod \operatorname{id}_{\varnothing}} Y \coprod \varnothing \\ \downarrow^{\operatorname{Sets}, \coprod} & & & \downarrow^{\rho_Y^{\operatorname{Sets}, \coprod}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(0,x) & (0,x) & \longmapsto (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\rho^{\mathsf{Sets},\coprod}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\rho^{\mathsf{Sets}, \coprod}$  is natural and  $\rho^{\mathsf{Sets}, -1}$  is a componentwise inverse to  $\rho^{\mathsf{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\rho^{\mathsf{Sets}, -1}$  is also natural. Thus  $\rho^{\mathsf{Sets}, \coprod}$  is a natural isomorphism.

#### 01PW 5.2.6 The Symmetry

#### **01PX** DEFINITION 5.2.6.1.1 ► THE SYMMETRY OF ∐

The symmetry of the coproduct of sets is the natural isomorphism

$$\sigma^{\mathsf{Sets}, \coprod} \colon \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}, \qquad \begin{array}{c} \mathsf{Sets} \times \mathsf{Sets} & \overset{\coprod}{\longrightarrow} \mathsf{Sets}, \\ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}} & \downarrow & \downarrow \\ \mathsf{Sets} \times \mathsf{Sets} & \mathsf{Sets} \end{array}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod} \colon X \coprod Y \stackrel{\sim}{\dashrightarrow} Y \coprod X$$

at  $X, Y \in \text{Obj}(\mathsf{Sets})$  is defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}(x,y) \stackrel{\scriptscriptstyle \mathsf{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

#### PROOF 5.2.6.1.2 ▶ Proof of the Claims Made in Definition 5.2.6.1.1

#### Unwinding the Definitions of $X \coprod Y$ and $Y \coprod X$

Firstly, we unwind the expressions for  $X \coprod Y$  and  $Y \coprod X$ . We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},\$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \coprod X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},\$$

where 
$$S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$$
.

#### Invertibility

The inverse of  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is the map

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\scriptscriptstyle \mathrm{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = (1,x), \\ (1,y) & \text{if } z = (0,y) \end{cases}$$

for each  $z \in Y \coprod X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod}\right] (0,x) &= \sigma_X^{\mathsf{Sets},\coprod,-1} \bigg(\sigma_X^{\mathsf{Sets},\coprod}(0,x)\bigg) \\ &= \sigma_X^{\mathsf{Sets},\coprod,-1} (1,x) \\ &= (0,x) \\ &= \left[\mathrm{id}_{X\coprod Y}\right] (0,x) \end{split}$$

for each  $(0, x) \in X \coprod Y$  and

$$\left[\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod}\right]\!(1,y) = \sigma_X^{\mathsf{Sets},\coprod,-1}\!\left(\sigma_X^{\mathsf{Sets},\coprod}(1,y)\right)$$

$$= \sigma_X^{\mathsf{Sets}, \coprod, -1}(0, y)$$

$$= (1, y)$$

$$= \left[ \mathrm{id}_{X \coprod Y} \right] (1, y)$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod} = \mathrm{id}_{X\coprod Y} \,.$$

• Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}\right] (0,y) &= \sigma_X^{\mathsf{Sets},\coprod} \Big(\sigma_X^{\mathsf{Sets},\coprod,-1}(0,y)\Big) \\ &= \sigma_X^{\mathsf{Sets},\coprod,-1}(1,y) \\ &= (0,y) \\ &= \left[\mathrm{id}_{Y\coprod X}\right] (0,y) \end{split}$$

for each  $(0, y) \in Y \coprod X$  and

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets},\coprod} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}\right] &(1,x) = \sigma_X^{\mathsf{Sets},\coprod} \left(\sigma_X^{\mathsf{Sets},\coprod,-1}(1,x)\right) \\ &= \sigma_X^{\mathsf{Sets},\coprod,-1}(0,x) \\ &= (1,x) \\ &= \left[\mathrm{id}_{Y\coprod X}\right] &(1,x) \end{split}$$

for each  $(1, x) \in Y \coprod X$ , and therefore we have

$$\sigma_X^{\mathsf{Sets},\coprod} \circ \sigma_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_{Y\coprod X} \,.$$

Therefore  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions  $f: A \to X$  and  $g: B \to Y$ , the

diagram

commutes. Indeed, this diagram acts on elements as

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes. Therefore  $\sigma^{\mathsf{Sets},\coprod}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\sigma^{\mathsf{Sets},\coprod}$  is natural and  $\sigma^{\mathsf{Sets},-1}$  is a componentwise inverse to  $\sigma^{\mathsf{Sets},\coprod}$  it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that  $\sigma^{\mathsf{Sets},-1}$  is also natural. Thus  $\sigma^{\mathsf{Sets},\coprod}$  is a natural isomorphism.

#### **101PY** 5.2.7 The Monoidal Category of Sets and Coproducts

#### 01PZ PROPOSITION 5.2.7.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO []

The category **Sets** admits a closed symmetric monoidal category structure consisting of:

• The Underlying Category. The category Sets of pointed sets.

• The Monoidal Product. The coproduct functor

$$\coprod$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

• The Monoidal Unit. The functor

$$\mathbb{0}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

of Definition 5.2.3.1.1.

• The Associators. The natural isomorphism  $\alpha^{\mathsf{Sets}, \coprod} \colon \coprod \circ (\coprod \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$ 

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets},\coprod} \colon \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}\right) \stackrel{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

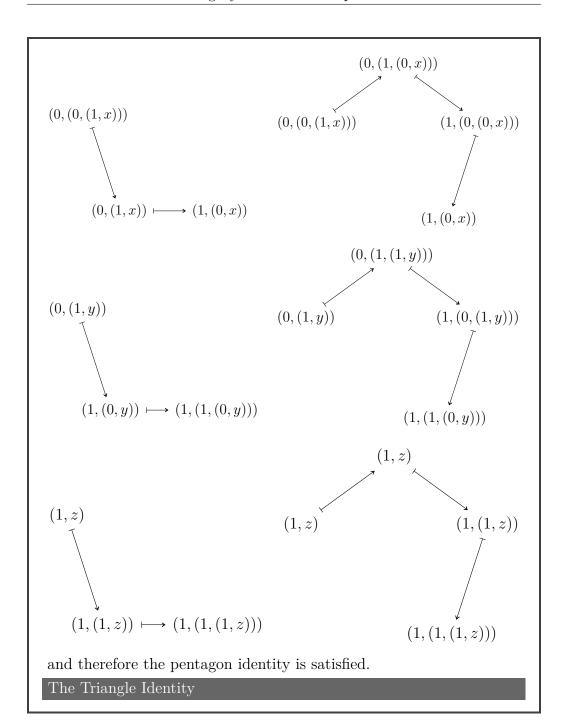
of Definition 5.2.5.1.1.

• The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} : imes \stackrel{\sim}{\Longrightarrow} imes \circ {m{\sigma}}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

## PROOF 5.2.7.1.2 ▶ PROOF OF PROPOSITION 5.2.7.1.1 The Pentagon Identity Let W, X, Y and Z be sets. We have to show that the diagram $(W \coprod (X \coprod Y)) \coprod Z$ $((W \coprod X) \coprod Y) \coprod Z$ $W \coprod ((X \coprod Y) \coprod Z)$ $(W \coprod X) \coprod (Y \coprod Z) \underset{\alpha}{\underset{K,X,Y \coprod Z}{\longrightarrow}} W \coprod (X \coprod (Y \coprod Z))$ commutes. Indeed, this diagram acts on elements as (0,(0,w))(0,(0,(0,w)))(0,(0,(0,w)))(0, w) $(0,(0,w)) \longmapsto (0,w)$ (0, w)



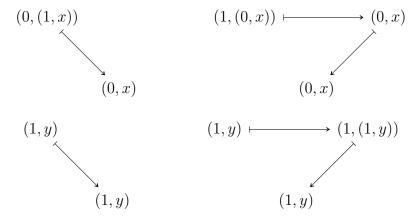
Let X and Y be sets. We have to show that the diagram

$$(X \coprod \varnothing) \coprod Y \xrightarrow{\alpha_{X,\varnothing,Y}^{\mathsf{Sets}, \coprod}} X \coprod (\varnothing \coprod Y)$$

$$\rho_X^{\mathsf{Sets}, \coprod} \coprod_{\mathrm{id}_X} \lambda_Y^{\mathsf{Sets}, \coprod}$$

$$X \coprod Y$$

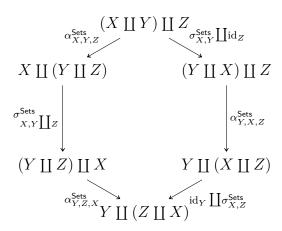
commutes. Indeed, this diagram acts on elements as

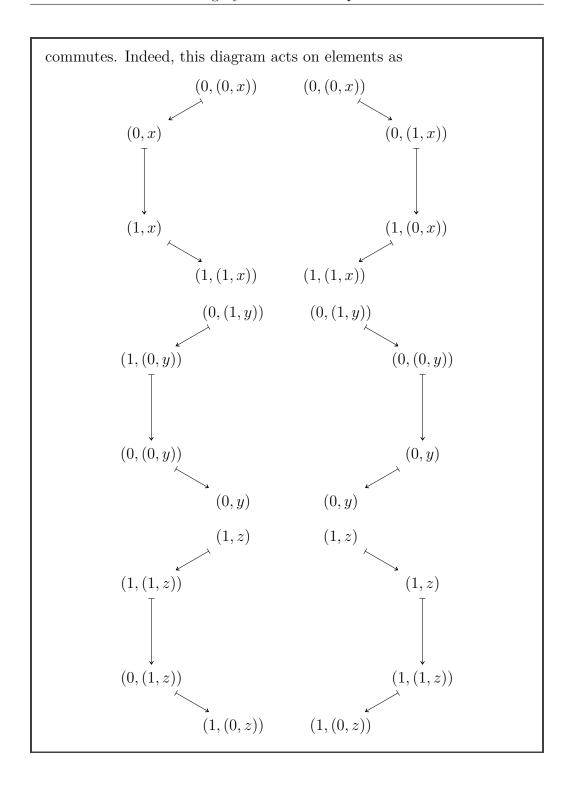


and therefore the triangle identity is satisfied.

#### The Left Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram

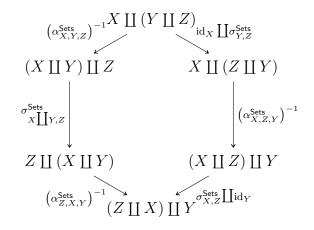




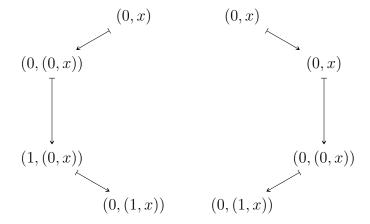
and thus the left hexagon identity is satisfied.

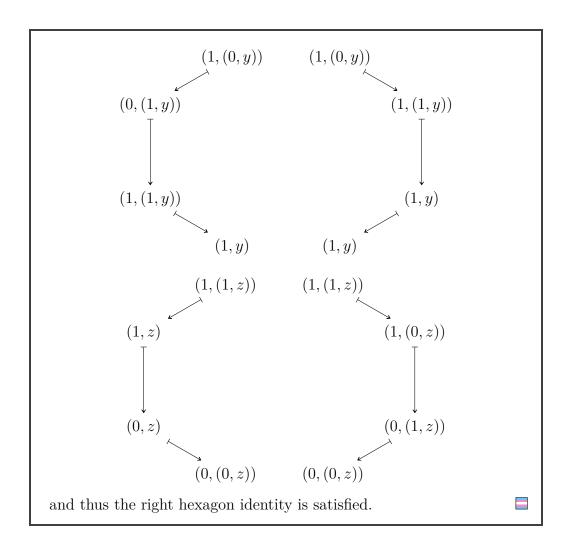
#### The Right Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as





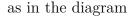
# oloo 5.3 The Bimonoidal Category of Sets, Products, and Coproducts

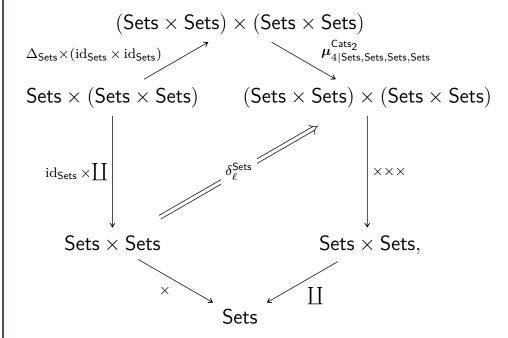
#### 0101 5.3.1 The Left Distributor

#### 01Q2 DEFINITION 5.3.1.1.1 ► THE LEFT DISTRIBUTOR OF × OVER ]

The left distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{4 \mid \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$





whose component

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z} \colon X \times (Y \coprod Z) \stackrel{\sim}{\dashrightarrow} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z}(x,a) \stackrel{\text{\tiny def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each  $(x, a) \in X \times (Y \coprod Z)$ .

#### PROOF 5.3.1.1.2 ▶ Proof of the Claims Made in Definition 5.3.1.1.1

Omitted.



#### 0103 5.3.2 The Right Distributor

#### 01Q4 DEFINITION 5.3.2.1.1 ► THE RIGHT DISTRIBUTOR OF × OVER []

The right distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ (\coprod \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
 as in the diagram

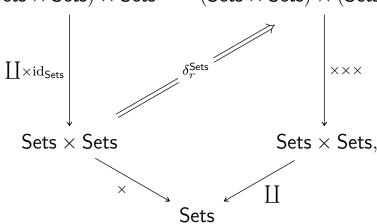
$$(\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets})$$

$$(\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}}$$

$$(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets}$$

$$(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets}$$

$$(\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets})$$



whose component

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each  $(a, z) \in (X \coprod Y) \times Z$ .

#### PROOF 5.3.2.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.2.1.1

Omitted.

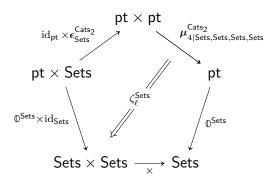
#### 0105 5.3.3 The Left Annihilator

#### 01Q6 DEFINITION 5.3.3.1.1 $\blacktriangleright$ The Left Annihilator of $\times$

The left annihilator of the product of sets is the natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \emptyset \times A \stackrel{\sim}{\dashrightarrow} \emptyset.$$

#### PROOF 5.3.3.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.3.1.1

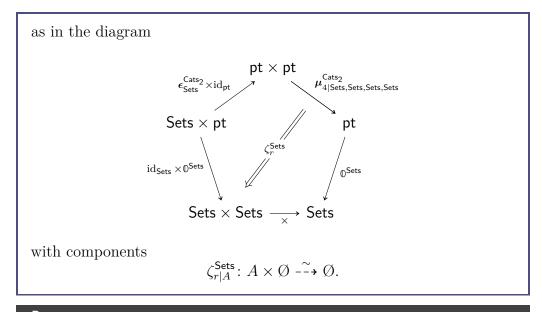
Omitted. For a partial proof, see [Pro25].

### 01Q7 5.3.4 The Right Annihilator

#### 0108 DEFINITION 5.3.4.1.1 $\triangleright$ The Right Annihilator of $\times$

The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}\right) \stackrel{\sim}{\dashrightarrow} \times \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$



#### PROOF 5.3.4.1.2 ▶ Proof of the Claims Made in Definition 5.3.4.1.1

Omitted. For a partial proof, see [Pro25].

### 0109 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

#### 01QA PROPOSITION 5.3.5.1.1 ► THE BIMONOIDAL STRUCTURE ON SETS ASSOCIATED TO × AND \[ \]

The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- The Additive Monoidal Product. The coproduct functor

II: Sets 
$$\times$$
 Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

• The Multiplicative Monoidal Product. The product functor

$$\times : \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

• The Monoidal Unit. The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Monoidal Zero. The functor

$$\mathbb{0}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Internal Hom. The internal Hom functor

$$\mathsf{Sets} \colon \mathsf{Sets}^\mathsf{op} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Constructions With Sets, ?? of ??.

- The Additive Associators. The natural isomorphism  $\alpha^{\mathsf{Sets}, \coprod} \colon \coprod \circ (\coprod \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$  of Definition 5.2.3.1.1.
- The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} \colon \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Additive Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathsf{id} \times \mathbb{O}^{\mathsf{Sets}}\right) \stackrel{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.5.1.1.

• The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} : \coprod \stackrel{\sim}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

- The Multiplicative Associators. The natural isomorphism  $\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$  of Definition 5.1.4.1.1.
- The Multiplicative Left Unitors. The natural isomorphism  $\lambda^{\mathsf{Sets}} \colon \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$  of Definition 5.1.5.1.1.
- The Multiplicative Right Unitors. The natural isomorphism  $\rho^{\mathsf{Sets}} \colon \times \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}}\right) \stackrel{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}}$  of Definition 5.1.6.1.1.
- The Multiplicative Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$
 of Definition 5.3.1.1.1.

• The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} \colon \times \circ (\coprod \times \mathrm{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.1.1.

• The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left( \mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \right) \stackrel{\sim}{\Longrightarrow} \times \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}} \right)$$
of Definition 5.3.3.1.1.

• The Right Annihilator. The natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\boldsymbol{\epsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}}\right) \stackrel{\sim}{\dashrightarrow} \times \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$
of Definition 5.3.4.1.1.

# PROOF 5.3.5.1.2 ► PROOF OF PROPOSITION 5.3.5.1.1 Omitted.

### Appendices

### A Other Chapters

#### **Preliminaries**

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

#### Relations

- 8. Relations
- 9. Constructions With Relations

#### 10. Conditions on Relations

#### Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

13. Constructions With Monoidal Categories

#### **Bicategories**

14. Types of Morphisms in Bicategories

#### Extra Part

15. Notes

#### References

[Pro25] Proof Wiki Contributors. Cartesian Product Is Empty Iff Factor Is Empty — Proof Wiki. 2025. URL: https://proofwiki.org/wiki/ Cartesian\_Product\_is\_Empty\_iff\_Factor\_is\_Empty (cit. on pp. 63, 64).