## Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

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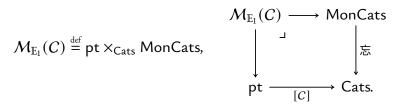
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### 13.1 Moduli Categories of Monoidal Structures

# 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let *C* be a category.

**Definition 13.1.1.1.1.** The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by



Remark 13.1.1.2. In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- Objects. The objects of  $\mathcal{M}_{\mathbb{E}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is C.
- *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{1|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C: C \to C$  of C.

• *Identities*. For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$  at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\stackrel{\operatorname{def}}{=} (\operatorname{id}_{C}^{\otimes},\operatorname{id}_{1|C}^{\otimes}),$$

where  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  is the identity monoidal functor of C of  $\ref{C}$ ?

• Composition. For each  $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the composition map

$$\circ_{\mathcal{M}, N, P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(N, P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M, N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M, P)$$
 of  $\mathcal{M}_{\mathbb{E}_{1}}(C)$  at  $(M, N, P)$  is defined by 
$$\left( \operatorname{id}_{C}^{\otimes, \prime}, \operatorname{id}_{1|C}^{\otimes, \prime} \right) \circ_{\mathcal{M}, N, P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \left( \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes} \right) \stackrel{\text{def}}{=} \left( \operatorname{id}_{C}^{\otimes, \prime} \circ \operatorname{id}_{C}^{\otimes}, \operatorname{id}_{1|C}^{\otimes, \prime} \circ \operatorname{id}_{1|C}^{\otimes} \right).$$

**Remark 13.1.1.13.** In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

I. *Naturality.* For each pair  $f:A\to X$  and  $g:B\to Y$  of morphisms of C, the diagram

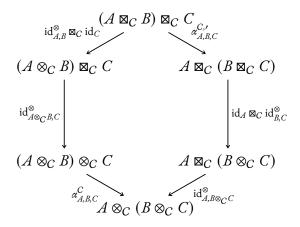
$$A \boxtimes_{C} B \xrightarrow{f \boxtimes_{C} g} X \boxtimes_{C} Y$$

$$\downarrow_{\mathrm{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow_{\mathrm{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{C} B \xrightarrow{f \otimes_{C} g} X \otimes_{C} Y$$

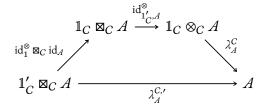
commutes.

2. *Monoidality*. For each  $A, B, C \in Obj(C)$ , the diagram



commutes.

3. Left Monoidal Unity. For each  $A \in Obj(C)$ , the diagram



commutes.

4. Right Monoidal Unity. For each  $A \in Obj(C)$ , the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbf{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbf{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A$$

commutes.

#### **Proposition 13.1.1.1.4.** Let C be a category.

- 1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B\boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{E_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
  - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

commutes.

3. Mixed Associators. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^C)$  be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} : -_1 \boxtimes_C -_2 \longrightarrow -_1 \otimes_C -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes}: (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ \downarrow^{\operatorname{id}_{A \otimes_C B,C}} & & \downarrow^{\operatorname{id}_A \otimes_C \operatorname{id}_{B,C}^{\otimes}} \\ (A \otimes_C B) \otimes_C C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

commute, then the natural transformation id<sup>®</sup> satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} \colon (A \boxtimes_C B) \otimes_C C \longrightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

and

$$\begin{array}{c|c} (A\boxtimes_{C}B)\boxtimes_{C}C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A\boxtimes_{C}(B\boxtimes_{C}C) \\ \operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes} & \operatorname{id}_{A\boxtimes_{C}}\operatorname{id}_{B,C}^{\otimes} \\ (A\boxtimes_{C}B)\otimes_{C}C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A\boxtimes_{C}(B\otimes_{C}C) \end{array}$$

commute, then the natural transformation id<sup>®</sup> satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

and

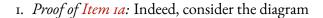
commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

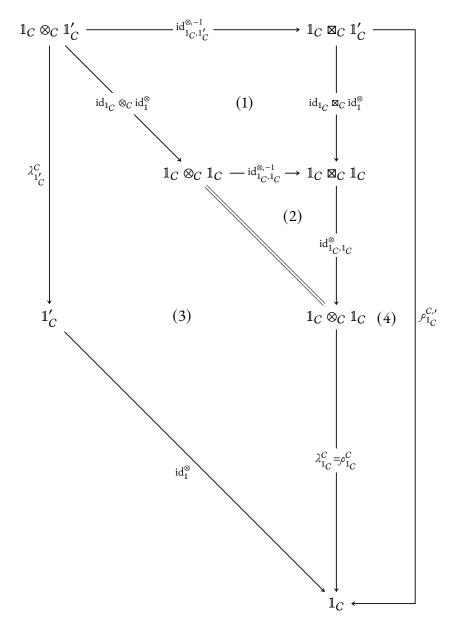
*Proof. Item 1*, *Extra Monoidality Conditions*: We claim that Items 1a and 1b are indeed true:

- I. *Proof of Item 1a:* This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A,B}^{\otimes}$  and  $id_{C}$ .
- 2. *Proof of Item 1b*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

*Item 2, Extra Monoidal Unity Constraints*: We claim that Items 2a and 2b are indeed true:



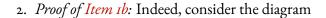


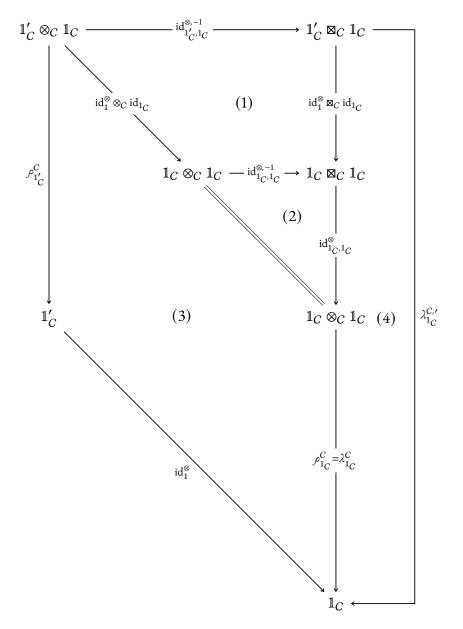
whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{1_C}^C=\lambda_{1_C}^C$  comes from  $\ref{eq:compare}$ ;
- Subdiagram (4) commutes by the right monoidal unity of (id $_C$ , id $_C^{\otimes}$ , id $_{C|1}^{\otimes}$ );

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

• Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{1_C}^C = \lambda_{1_C}^C$  comes from  $\ref{eq:compare}$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.

3. Proof of Item 2c: Indeed, consider the diagram

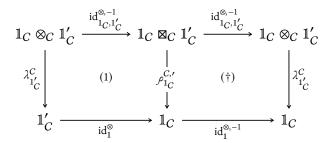
Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $\mathrm{id}_{1_C,1_C'}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram



Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$1_{C} \otimes_{C} 1_{C}' \xrightarrow{\operatorname{id}_{1_{C},1_{C}'}^{\otimes,-1}} 1_{C} \boxtimes_{C} 1_{C}' \xrightarrow{\operatorname{id}_{1_{C},1_{C}'}^{\otimes,-1}} 1_{C} \otimes_{C} 1_{C}'$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{1_{C}'}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{1_{C}'}^{C}$$

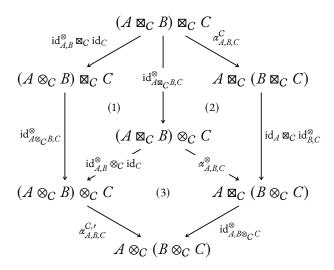
commutes. But since  $id_1^{\otimes,-1}$  is an isomorphism, it follows that the diagram  $(\dagger)$  also commutes, and we are done.

This finishes the proof.

*Item 3, Mixed Associators*: We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for  $id^{\otimes}$  of





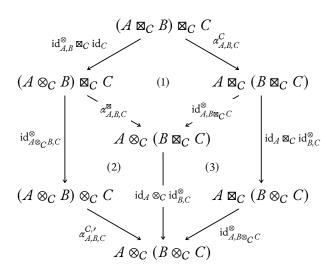
#### Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. *Proof of Item 3b*: We may partition the monoidality diagram for  $id^{\otimes}$  of

Item 2 of Definition 13.1.1.1.3 as follows:

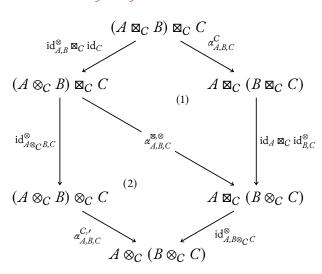


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item 3c:* We may partition the monoidality diagram for  $id^{\otimes}$  of



Item 2 of Definition 13.1.1.1.3 as follows:

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id<sup>®</sup> satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
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