Types of Morphisms in Bicategories

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019H   In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2)*. There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

Contents

019J # 14.1 Monomorphisms in Bicategories

019K ## 14.1.1 Representably Faithful Morphisms

Let C be a bicategory.

019L **Definition 14.1.1.1.1.** A 1-morphism $f: A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*: \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

019M **Remark 14.1.1.1.2.** In detail, f is representably faithful if, for all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow\Downarrow \beta}{\rightrightarrows}}} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

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[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Definition 14.1.1.1.3.

019N **Example 14.1.1.1.3.** Here are some examples of representably faithful morphisms.

019P   1. *Representably Faithful Morphisms in* Cats$_2$. The representably faithful morphisms in Cats$_2$ are precisely the faithful functors; see Categories, Item 2 of Definition 11.6.1.1.2.

019Q   2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, Item 1 of Definition 8.5.11.1.1.

019R **14.1.2   Representably Full Morphisms**

Let C be a bicategory.

019S **Definition 14.1.2.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably full**[2] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is full.

019T **Remark 14.1.2.1.2.** In detail, f is representably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f\circ\psi}{\overset{f\circ\phi}{\Longrightarrow\beta}} B$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow\alpha}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Rightarrow\alpha}} A \xrightarrow{f} B \quad = \quad X \underset{f\circ\psi}{\overset{f\circ\phi}{\Rightarrow\beta}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

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[2] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Definition 14.1.2.1.3.

019U   **Example 14.1.2.1.3.** Here are some examples of representably full morphisms.

019V   1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in
                Cats_2 are precisely the full functors; see Categories, **??** of Definition 11.6.2.1.2.

019W   2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel**
                are characterised in Relations, Item 2 of Definition 8.5.11.1.1.

019X   **14.1.3   Representably Fully Faithful Morphisms**

Let C be a bicategory.

019Y   **Definition 14.1.3.1.1.** A 1-morphism $f : A \rightarrow B$ of C is **representably fully faithful**[3]
if the following equivalent conditions are satisfied:

019Z   1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1) and repre-
                sentably full (Definition 14.1.2.1.1).

01A0   2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathrm{Hom}_C(X, A) \rightarrow \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

01A1   **Remark 14.1.3.1.2.** In detail, f is representably fully faithful if the conditions in
Definition 14.1.1.1.2 and Definition 14.1.2.1.2 hold:

        1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A \xrightarrow{f} B,$$

        if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

        then $\alpha = \beta$.

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[3]*Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of
Definition 14.1.3.1.3.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \quad X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A \xrightarrow{f} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01A2 **Example 14.1.3.1.3.** Here are some examples of representably fully faithful morphisms.

01A3 1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 6 of Definition 11.6.3.1.2.

01A4 2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Definition 8.5.11.1.1) with the representably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Definition 8.5.11.1.1.

01A5 ## 14.1.4   Morphisms Representably Faithful on Cores

Let C be a bicategory.

01A6   **Definition 14.1.4.1.1.** A 1-morphism $f : A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

01A7   **Remark 14.1.4.1.2.** In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \Downarrow \Downarrow \beta}}} A \xrightarrow{\ f\ } B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01A8   14.1.5   Morphisms Representably Full on Cores

Let C be a bicategory.

01A9   **Definition 14.1.5.1.1.** A 1-morphism $f : A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is full.

01AA   **Remark 14.1.5.1.2.** In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta : f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\underset{\Longrightarrow}{\beta \Downarrow}}} B$$

of C, there exists a 2-isomorphism

$$\alpha : \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \Downarrow}}} A$$

of C such that we have an equality

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow\alpha\Downarrow}} A \xrightarrow{f} B \quad = \quad X \overset{f\circ\phi}{\underset{f\circ\psi}{\Longrightarrow\beta\Downarrow}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AB   14.1.6   Morphisms Representably Fully Faithful on Cores

Let C be a bicategory.

01AC   **Definition 14.1.6.1.1.** A 1-morphism $f \colon A \to B$ of C is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

01AD   1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).

01AE   2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Core}(\mathrm{Hom}_C(X, A)) \to \mathrm{Core}(\mathrm{Hom}_C(X, B))$$

given by postcomposition by f is fully faithful.

01AF   **Remark 14.1.6.1.2.** In detail, f is representably fully faithful on cores if the conditions in Definition 14.1.4.1.2 and Definition 14.1.5.1.2 hold:

1. For all diagrams in C of the form

$$X \overset{\phi}{\underset{\psi}{\Longrightarrow\alpha\Downarrow\Downarrow\beta}} A \xrightarrow{f} B,$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}}_{\beta \Downarrow} B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}}_{\beta \Downarrow} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AG 14.1.7   Representably Essentially Injective Morphisms

Let C be a bicategory.

01AH **Definition 14.1.7.1.1.** A 1-morphism $f \colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

01AJ **Remark 14.1.7.1.2.** In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi \colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

01AK   **14.1.8   Representably Conservative Morphisms**

Let C be a bicategory.

01AL   **Definition 14.1.8.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

01AM   **Remark 14.1.8.1.2.** In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad X \underset{\psi}{\overset{\phi}{\Longrightarrow}} A$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\Longrightarrow}} B$$

is a 2-isomorphism, then so is α.

01AN   **14.1.9   Strict Monomorphisms**

Let C be a bicategory.

01AP   **Definition 14.1.9.1.1.** A 1-morphism $f\colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$

is injective.

01AQ **Remark 14.1.9.1.2.** In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

01AR **Example 14.1.9.1.3.** Here are some examples of strict monomorphisms.

01AS   1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Definition 11.7.2.1.2.

01AT   2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Relations, Definition 8.5.10.1.1.

01AU **14.1.10   Pseudomonic Morphisms**

Let C be a bicategory.

01AV **Definition 14.1.10.1.1.** A 1-morphism $f \colon A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

01AW **Remark 14.1.10.1.2.** In detail, a 1-morphism $f \colon A \to B$ of C is pseudomonic if it satisfies the following conditions:

01AX   1. For all diagrams in C of the form

$$X \underset{\psi}{\overset{\phi}{\underset{\Downarrow\beta}{\overset{\alpha\Downarrow}{\rightrightarrows}}}} A \xrightarrow{f} B,$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01AY   2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad X \underset{f \circ \psi}{\overset{f \circ \phi}{\underset{\beta \Downarrow}{\rightrightarrows}}} B$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow}{\rightrightarrows}}} A$$

of C such that we have an equality

$$X \underset{\psi}{\overset{\phi}{\underset{\alpha \Downarrow}{\rightrightarrows}}} A \overset{f}{\longrightarrow} B \quad = \quad X \underset{f \circ \psi}{\overset{f \circ \phi}{\underset{\beta \Downarrow}{\rightrightarrows}}} B$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AZ   **Proposition 14.1.10.1.3.** Let $f \colon A \to B$ be a 1-morphism of C.

01B0   1. *Characterisations.* The following conditions are equivalent:

01B1   (a) The morphism f is pseudomonic.

01B2   (b) The morphism f is representably full on cores and representably faithful.

01B3   (c) We have an isocomma square of the form

$$A \overset{\mathrm{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad
\begin{array}{ccc}
A & \overset{\mathrm{id}_A}{\longrightarrow} & A \\
{\scriptstyle \mathrm{id}_A} \downarrow & \nearrow & \downarrow {\scriptstyle F} \\
A & \underset{F}{\longrightarrow} & B
\end{array}$$

in C up to equivalence.

01B4   2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

     (a) The morphism f is pseudomonic.

     (b) We have an isocomma square of the form

$$A \stackrel{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B,$$

in C up to equivalence.

Proof. Item 1, *Characterisations*: Omitted.
Item 2, *Interaction With Cotensors*: Omitted. $\square$

01B5   14.2   Epimorphisms in Bicategories

01B6   14.2.1   Corepresentably Faithful Morphisms

Let C be a bicategory.

01B7   **Definition 14.2.1.1.1.** A 1-morphism $f : A \to B$ of C is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is faithful.

01B8   **Remark 14.2.1.1.2.** In detail, f is corepresentably faithful if, for all diagrams in C of the form

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01B9   **Example 14.2.1.1.3.** Here are some examples of corepresentably faithful morphisms.

01BA   1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 5 of Definition 11.6.1.1.2.

01BB   2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Relations, Item 1 of Definition 8.5.13.1.1.

01BC   ## 14.2.2   Corepresentably Full Morphisms

Let C be a bicategory.

01BD   **Definition 14.2.2.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is full.

01BE   **Remark 14.2.2.1.2.** In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Rightarrow}} X \;\; \beta \Downarrow$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad B \underset{\psi}{\overset{\phi}{\Rightarrow}} X \;\; \alpha \Downarrow$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Rightarrow}} X \;\; \alpha\Downarrow \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Rightarrow}} X \;\; \beta\Downarrow$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f\,.$$

01BF   **Example 14.2.2.1.3.** Here are some examples of corepresentably full morphisms.

01BG   1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 7 of Definition 11.6.2.1.2.

01BH   2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in Relations, Item 2 of Definition 8.5.13.1.1.

01BJ   ## 14.2.3   Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

01BK   **Definition 14.2.3.1.1.** A 1-morphism $f \colon A \to B$ of C is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

01BL   1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).

01BM   2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is fully faithful.

01BN   **Remark 14.2.3.1.2.** In detail, f is corepresentably fully faithful if the conditions in Definition 14.2.1.1.2 and Definition 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}}{\alpha \Downarrow\Downarrow \beta}\, X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

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[4]*Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \quad B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X$$

of C such that we have an equality

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\Longrightarrow}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01BP   **Example 14.2.3.1.3.** Here are some examples of corepresentably fully faithful morphisms.

01BQ   1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 10 of Definition 11.6.3.1.2.

01BR   2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, Item 3 of Definition 8.5.13.1.1) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, Item 2 of Definition 8.5.13.1.1.

01BS   ## 14.2.4   Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

01BT   **Definition 14.2.4.1.1.** A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B, X)) \to \mathrm{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is faithful.

01BU   **Remark 14.2.4.1.2.** In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \| \| \beta}}} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01BV   ## 14.2.5   Morphisms Corepresentably Full on Cores

Let C be a bicategory.

01BW   **Definition 14.2.5.1.1.** A 1-morphism $f \colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Core}(\mathrm{Hom}_C(B, X)) \to \mathsf{Core}(\mathrm{Hom}_C(A, X))$$

given by precomposition by f is full.

01BX   **Remark 14.2.5.1.2.** In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\underset{\Longrightarrow}{\beta \|}}} X$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \|}}} X$$

of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha \|}}} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\underset{\Longrightarrow}{\beta \|}}} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f \,.$$

01BY   ## 14.2.6   Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

01BZ   **Definition 14.2.6.1.1.** A 1-morphism $f\colon A \rightarrow B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

01C0   1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

01C1   2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Core}(\mathrm{Hom}_C(B,X)) \rightarrow \mathrm{Core}(\mathrm{Hom}_C(A,X))$$

given by precomposition by f is fully faithful.

01C2   **Remark 14.2.6.1.2.** In detail, f is corepresentably fully faithful on cores if the conditions in Definition 14.2.4.1.2 and Definition 14.2.5.1.2 hold:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha\Downarrow\Downarrow\beta}}} X,$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi\circ f}{\overset{\phi\circ f}{\underset{\Longrightarrow}{\beta\Downarrow}}} X$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\underset{\Longrightarrow}{\alpha\Downarrow}}} X$$

of C such that we have an equality

$$
A \xrightarrow{f} B \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha} X \quad = \quad A \overset{\phi \circ f}{\underset{\psi \circ f}{\Longrightarrow}}_{\beta} X
$$

of pasting diagrams in C, i.e. such that we have

$$
\beta = \alpha \star \mathrm{id}_f \,.
$$

01C3   14.2.7   Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

01C4   **Definition 14.2.7.1.1.** A 1-morphism $f \colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$
f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)
$$

given by precomposition by f is essentially injective.

01C5   **Remark 14.2.7.1.2.** In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ of C, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

01C6   14.2.8   Corepresentably Conservative Morphisms

Let C be a bicategory.

01C7   **Definition 14.2.8.1.1.** A 1-morphism $f \colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$
f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)
$$

given by precomposition by f is conservative.

01C8   **Remark 14.2.8.1.2.** In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ and each 2-morphism

$$
\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \quad B \overset{\phi}{\underset{\psi}{\Longrightarrow}}_{\alpha} X
$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f : \phi \circ f \Longrightarrow \psi \circ f, \qquad A \quad \overset{\overset{\phi \circ f}{\overset{\parallel}{\underset{\alpha \star \mathrm{id}_f}{\Downarrow}}}}{\underset{\psi \circ f}{\longrightarrow}} \quad X$$

is a 2-isomorphism, then so is α.

01C9 ## 14.2.9   Strict Epimorphisms

Let C be a bicategory.

01CA **Definition 14.2.9.1.1.** A 1-morphism $f : A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* : \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* : \ \mathrm{Obj}(\mathrm{Hom}_C(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_C(A, X))$$

is injective.

01CB **Remark 14.2.9.1.2.** In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

01CC **Example 14.2.9.1.3.** Here are some examples of strict epimorphisms.

01CD   1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Definition 11.7.3.1.2.

01CE   2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, Definition 8.5.12.1.1.

14.2.10   Pseudoepic Morphisms

Let C be a bicategory.

Definition 14.2.10.1.1. A 1-morphism $f\colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is pseudomonic.

Remark 14.2.10.1.2. In detail, a 1-morphism $f\colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow \Downarrow \beta} X,$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}}_{\beta \Downarrow} X$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad B \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow} X$$

of C such that we have an equality

$$A \xrightarrow{\ f\ } B \underset{\psi}{\overset{\phi}{\Longrightarrow}}_{\alpha \Downarrow} X \quad = \quad A \underset{\psi \circ f}{\overset{\phi \circ f}{\Longrightarrow}}_{\beta \Downarrow} X$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f\,.$$

01CL **Proposition 14.2.10.1.3.** Let $f\colon A \to B$ be a 1-morphism of C.

01CM 1. *Characterisations.* The following conditions are equivalent:

01CN   (a) The morphism f is pseudoepic.

01CP   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

01CQ   (c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad
\begin{array}{ccc}
B & \xleftarrow{\ \ \text{id}_B\ \ } & B \\
{\scriptstyle \text{id}_B}\big\uparrow & \nearrow \kern-1.5em \swarrow & \big\uparrow{\scriptstyle F} \\
B & \xleftarrow[\ \ F\ \ ]{} & A
\end{array}$$

    in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted. □

Appendices

A   Other Chapters

Preliminaries

References

[Adá+01]   Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 14).