Relations

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This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 8.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 8.1.2).
3. The various kinds of categories that relations form, namely:
   (a) A category (Section 8.3.2).
   (b) A monoidal category (Section 8.3.3).
   (c) A 2-category (Section 8.3.4).
   (d) A double category (Section 8.3.5).
4. The various categorical properties of the 2-category of relations, including:
   (a) The self-duality of Rel and $\mathbf{Rel}$ (Proposition 8.5.1.1.1).
   (b) Identifications of equivalences and isomorphisms in $\mathbf{Rel}$ with bijections (Proposition 8.5.2.1.3).
   (c) Identifications of adjunctions in $\mathbf{Rel}$ with functions (Proposition 8.5.3.1.1).
   (d) Identifications of monads in $\mathbf{Rel}$ with preorders (**??**).
   (e) Identifications of comonads in $\mathbf{Rel}$ with subsets (**??**).
   (f) A description of the monoids and comonoids in $\mathbf{Rel}$ with respect to the Cartesian product (Remark 8.5.9.1.1).

(g) Characterisations of monomorphisms in Rel (Proposition 8.5.10.1.1).

(h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (Proposition 8.5.11.1.1).

(i) Characterisations of epimorphisms in Rel (Proposition 8.5.12.1.1).

(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Proposition 8.5.13.1.1).

(k) The partial co/completeness of Rel (Proposition 8.5.14.1.1).

(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (**??**).

(m) The closedness of **Rel** (Proposition 8.5.19.1.1).

(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Proposition 8.5.20.1.1).

5. The adjoint pairs

$$R_! \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_* \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R \colon A \nrightarrow B$, as well as the properties of $R_!$, R_{-1}, R^{-1}, and R_* (Section 8.7).

Of particular note are the following points:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_! \dashv f^{-1} \dashv f_*$ induced by a function $f \colon A \to B$ studied in Constructions With Sets, Section 4.6.

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 8.7.2.1.4).

(c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

$$R_! \dashv R_{-1} = R^{-1} \dashv R_*$$

reduces to the triple adjunction

$$f_! \dashv f^{-1} \dashv f_*$$

from Constructions With Sets, Section 4.6.

(d) The pairs $R_! \dashv R_{-1}$ and $R^{-1} \dashv R_*$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (Topological Spaces, **??**).

6. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ appearing in the definition of a relative monad (Sections 8.8 and 8.9).

This chapter is under revision. TODO:

1. Replicate Section 8.5 for apartness composition
2. Revise Section 8.7
3. Add subsection "A Six Functor Formalism for Sets, Part 2", now with relations, building upon Section 8.7.
4. Replicate Section 8.7 for apartness composition
5. Revise sections on skew monoidal structures on $\mathbf{Rel}(A, B)$
6. Replicate the sections on skew monoidal structures on $\mathbf{Rel}(A, B)$ for apartness composition.
7. Explore relative co/monads in Rel, defined to be co/monoids in $\mathbf{Rel}(A, B)$ with its left/right skew monoidal structures of Relations, Sections 8.8 and 8.9
8. functional total relations defined with "satisfying the following equivalent conditions:"

Contents

8.1 Relations

8.1.1 Foundations

Let A and B be sets.

Definition 8.1.1.1.1 ▶ Relations

A **relation** $R\colon A \mathrel{\mapstochar\mkern-4mu\rightarrow} B$ **from** A **to** B[1,2] is equivalently:

1. A subset R of $A \times B$.
2. A function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$.
3. A function from A to $\mathcal{P}(B)$.
4. A function from B to $\mathcal{P}(A)$.
5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.
6. A continuous morphism of posets from $(\mathcal{P}(B), \supset)$ to $(\mathcal{P}(A), \supset)$.

[1] *Further Terminology:* Also called a **multivalued function from A to B**.
[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on A**.

Proof 8.1.1.1.2 ▶ Proof of the Equivalences in Definition 8.1.1.1.1

(We will prove that Items 1 to 6 are indeed equivalent in a bit.)

Remark 8.1.1.1.3 ▶ Unwinding Item 1, I

We may think of a relation $R\colon A \nrightarrow B$ as a function from A to B that is *multivalued*, assigning to each element a in A a set $R(a)$ of elements of B, thought of as the *set of values of R at a*.
Note that this includes also the possibility of R having no value at all on a given $a \in A$ when $R(a) = \emptyset$.

Remark 8.1.1.1.4 ▶ Unwinding Item 2, II

Another way of stating the equivalence between Items 1 to 5 of Definition 8.1.1.1.1 is by saying that we have bijections of sets

$$\begin{aligned}\{\text{relations from } A \text{ to } B\} &\cong \mathcal{P}(A \times B)\\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\})\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B))\\ &\cong \mathsf{Sets}(B, \mathcal{P}(A))\\ &\cong \mathsf{Pos}^{\mathcal{D}}(\mathcal{P}(A), \mathcal{P}(B))\\ &\cong \mathsf{Pos}^{\mathcal{C}}(\mathcal{P}(B), \mathcal{P}(A))\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are endowed with the poset structure given by inclusion.

Proof 8.1.1.1.5 ▶ Proof of the Equivalences in Definition 8.1.1.1.1

We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This is a special case of Constructions With Sets, Items 2 and 3 of Proposition 4.5.1.1.4.
- *Item 2* $\iff$ *Item 3:* This follows from the bijections

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from Constructions With Sets, Items 2 and 3 of Proposition 4.5.1.1.4.

- *Item 2* $\iff$ *Item 4:* This follows from the bijections

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\})) \\ &\cong \mathsf{Sets}(B, \mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from Constructions With Sets, Items 2 and 3 of Proposition 4.5.1.1.4.

- *Item 2* $\iff$ *Item 5:* This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X \colon X \hookrightarrow \mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$, as in Constructions With Sets, Proposition 4.4.5.1.1. In particular, the bijection

$$\mathsf{Sets}(A, \mathcal{P}(B)) \cong \mathsf{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B))$$

  is given by extending each $f \colon A \to \mathcal{P}(B)$ in $\mathsf{Sets}(A, \mathcal{P}(B))$ from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X, recovering the direct image function $f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$ of f of Constructions With Sets, Definition 4.6.1.1.1.

- *Item 5* $\iff$ *Item 6:* Omitted.

This finishes the proof. ∎

NOTATION 8.1.1.1.6 ▶ FURTHER NOTATION FOR RELATIONS

Let A and B be sets and let $R \colon \nrightarrow B$ be a relation from A to B.

1. We write $\mathrm{Rel}(A, B)$ for the set of relations from A to B.
2. We write $\mathbf{Rel}(A, B)$ for the sub-poset of $(\mathcal{P}(A \times B), \subset)$ spanned by the relations from A to B.
3. Given $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.
4. When viewing R as a function

$$R \colon A \times B \to \{\mathsf{t}, \mathsf{f}\},$$

we write R^b_a for the value of R at (a,b).[1]

[1]The choice to write R^b_a in place of R^a_b is to keep the notation consistent with the notation we will later employ for profunctors in ??.

Proposition 8.1.1.1.7 ▶ Properties of Relations

Let A and B be sets and let $R, S\colon A \nrightarrow B$ be relations.

1. *End Formula for the Set of Inclusions of Relations.* We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right).$$

Proof 8.1.1.1.8 ▶ Proof of Proposition 8.1.1.1.7

Item 1: End Formula for the Set of Inclusions of Relations

Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a\in A \text{ and each } b \in B, \\ & \text{we have } \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) \cong \mathrm{pt} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since we have $\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) = \{\mathsf{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathsf{false}$ or $R^b_a = S^b_a = \mathsf{true}$, we get

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a\in A \text{ and each } b \in B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $(a \sim_R b \implies a \sim_S b)$ iff $R \subset S$, the two sets above are isomorphic. This finishes the proof. ∎

8.1.2 Relations as Decategorifications of Profunctors

Remark 8.1.2.1.1 ▶ Relations as Decategorifications of Profunctors I

The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor
$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}.$$
2. A relation on sets A and B is a function
$$R\colon A \times B \to \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on Sets.
- The values that profunctors and relations take are analogous:
  - A category is enriched over the category
  $$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$
  of sets, with profunctors taking values on it.
  - A set is enriched over the set
  $$\{\mathsf{true}, \mathsf{false}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$
  of classical truth values, with relations taking values on it.

Remark 8.1.2.1.2 ▶ Relations as Decategorifications of Profunctors II

Extending Remark 8.1.2.1.1, the equivalent definitions of relations in Definition 8.1.1.1.1 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$.

3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{CoPSh}(\mathcal{C})$.
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.
5. A limit-preserving functor $\mathfrak{p}\colon \mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}} \to \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\end{aligned}$$

  and

$$\begin{aligned}\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 2 and 4 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$, as stated and proved in Constructions With Sets, Proposition 4.4.5.1.1.
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in Presheaves and the Yoneda Lemma, **??** of Proposition 12.1.4.1.3.
- The equivalence between Items 3 and 5 follows from the universal properties of:

- The powerset $\mathcal{P}(X)$ of a set X as the free completion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$, as stated and proved in Constructions With Sets, Proposition 4.4.6.1.1.

- The category $\mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$ of copresheaves on a category $\mathcal{D}$ as the free completion of $\mathcal{D}$ via the dual Yoneda embedding

$$よ\colon \mathcal{D} \hookrightarrow \mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$$

of $\mathcal{D}$ into $\mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$, as stated and proved in Presheaves and the Yoneda Lemma, ?? of Proposition 12.1.4.1.3.

8.1.3 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 8.1.3.1.1 ▶ Composition of Relations

The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

1. Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

2. Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned}(S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{b \in B} S^{-1}_{b} \times R^{b}_{-2}\\ &= \bigvee_{b \in B} S^{-1}_{b} \times R^{b}_{-2},\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

3. Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{b \in B} \chi_{\mathcal{P}(B)}(\chi_b, V) \odot S(b) \\ &\cong \int^{b \in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b \in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b \in V} S(b) \end{aligned}$$

for each $V \in \mathcal{P}(B)$, so we have[1]

$$\begin{aligned} [S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{b \in R(a)} S(b). \end{aligned}$$

for each $a \in A$.

[1]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{ \{c_{j_i}\}_{j_i \in J_i} \right\}_{i \in I}$ in C.

Remark 8.1.3.1.2 ▶ Composing Relations With Right Kan Extensions

You might wonder what happens if we instead define an alternative composition of relations $\diamond'$ via right Kan extensions. In this case, we would take the right Kan extension of S along the dual characteristic embedding $B \hookrightarrow \mathcal{P}(B)^{\mathsf{op}}$:

$$S \diamond' R \overset{\text{def}}{=} \mathrm{Ran}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, "\chi_B"'] \arrow[dr, phantom, "\Uparrow"] & \mathcal{P}(C). \\ A \arrow[r, "R"'] & \mathcal{P}(B)^{\mathsf{op}} \arrow[ur, bend right, "\mathrm{Ran}_{\chi'_B}(S)"'] & \end{tikzcd}$$

In this case, we would have[1]

$$[S \diamond' R](a) \overset{\text{def}}{=} \bigcap_{b \in R(a)} S(b).$$

This alternative composition turns out to actually be a different kind of structure: it's an internal right Kan extension in **Rel**, namely $\mathrm{Ran}_{R^{\dagger}}(S)$ — see Section 8.5.17.

[1]If we replace $R(a)$ with $B \setminus R(a)$, defining

$$S \,\square\, R \overset{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b),$$

we instead obtain the apartness composition of relations; see Section 8.1.4.

Example 8.1.3.1.3 ▶ Examples of Composition of Relations

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq. \end{aligned}$$

Proposition 8.1.3.1.4 ▶ Properties of Composition of Relations

Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Functoriality.* The assignments $R, S, (R, S) \mapsto S \diamond R$ define functors

$$\begin{aligned} S \diamond -\colon &\qquad \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C), \\ - \diamond R\colon &\qquad \mathbf{Rel}(B, C) \to \mathbf{Rel}(A, C), \\ -_1 \diamond -_2\colon &\ \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C). \end{aligned}$$

In particular, given relations

$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \underset{S_2}{\overset{S_1}{\rightrightarrows}} C,$$

if $R_1 \subset R_2$ and $S_1 \subset S_2$, then $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

That is, we have

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)$$

for each $a \in A$.

3. *Unitality.* We have

$$\begin{aligned} \Delta_B \diamond R &= R, \\ R \diamond \Delta_A &= R. \end{aligned}$$

That is, we have

$$\begin{aligned} \bigcup_{b \in R(a)} \{b\} &= R(a), \\ \bigcup_{a \in \{a\}} R(a) &= R(a) \end{aligned}$$

for each $a \in A$.

4. *Relation to Apartness Composition of Relations.* We have

$$(S \diamond R)^{\mathsf{c}} = S^{\mathsf{c}} \mathbin{\Box} R^{\mathsf{c}},$$
$$(S \mathbin{\Box} R)^{\mathsf{c}} = S^{\mathsf{c}} \diamond R^{\mathsf{c}},$$

where $(-)^{\mathsf{c}}$ is the complement functor of Constructions With Sets, Section 4.3.11. In particular, $\diamond$ is a special case of apartness composition of relations, as we have

$$S \diamond R = (S^{\mathsf{c}} \mathbin{\Box} R^{\mathsf{c}})^{\mathsf{c}}.$$

This is also compatible with units, as we have $\Delta_A^{\mathsf{c}} = \nabla_A$.

5. *Linear Distributivity.* We have inclusions of relations

$$T \diamond (S \mathbin{\Box} R) \subset (T \diamond S) \mathbin{\Box} R,$$
$$(T \mathbin{\Box} S) \diamond R \subset T \mathbin{\Box} (S \diamond R).$$

That is, we have

$$T\left(\bigcap_{b \in B \setminus R(a)} S(b)\right) \subset \bigcap_{b \in B \setminus R(a)} T(S(b))$$
$$\bigcup_{b \in R(a)} \bigcap_{c \in C \setminus S(b)} T(c) \subset \bigcap_{c \in C \setminus S(R(a))} T(c)$$

or, unwinding the expression for $S(R(a))$, we have

$$\bigcup_{c \in \bigcap_{b \in B \setminus R(a)} S(b)} T(c) \subset \bigcap_{b \in B \setminus R(a)} \bigcup_{c \in S(b)} T(c)$$
$$\bigcup_{b \in R(a)} \bigcap_{c \in C \setminus S(b)} T(c) \subset \bigcap_{c \in C \setminus \bigcup_{b \in R(a)} S(b)} T(c)$$

for each $a \in A$.

6. *Interaction With Converses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

7. *Interaction With Ranges and Domains.* We have

$$\mathrm{dom}(S \diamond R) \subset \mathrm{dom}(R),$$
$$\mathrm{range}(S \diamond R) \subset \mathrm{range}(S).$$

Proof 8.1.3.1.5 ▶ Proof of Proposition 8.1.3.1.4

Item 1: Functoriality

We have

$$\begin{aligned}
S_1 \diamond R_1 &\overset{\text{def}}{=} \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B\text{, such}\\ \text{that } a \sim_{R_1} b \text{ or } b \sim_{S_1} c\end{array} \right\}\\
&\subset \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B\text{, such}\\ \text{that } a \sim_{R_2} b \text{ or } b \sim_{S_2} c\end{array} \right\}\\
&\overset{\text{def}}{=} S_2 \diamond R_2.
\end{aligned}$$

This finishes the proof.

Item 2: Associativity, Proof I

Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\overset{\text{def}}{=} \left(\int^{c \in C} T^{-1}_{c} \times S^{c}_{-_2} \right) \diamond R\\
&\overset{\text{def}}{=} \int^{b \in B} \left(\int^{c \in C} T^{-1}_{c} \times S^{c}_{b} \right) \times R^{b}_{-_2}\\
&= \int^{b \in B} \int^{c \in C} \left(T^{-1}_{c} \times S^{c}_{b} \right) \times R^{b}_{-_2}\\
&= \int^{c \in C} \int^{b \in B} \left(T^{-1}_{c} \times S^{c}_{b} \right) \times R^{b}_{-_2}\\
&= \int^{c \in C} \int^{b \in B} T^{-1}_{c} \times \left(S^{c}_{b} \times R^{b}_{-_2} \right)\\
&= \int^{c \in C} T^{-1}_{c} \times \left(\int^{b \in B} S^{c}_{b} \times R^{b}_{-_2} \right)\\
&\overset{\text{def}}{=} \int^{c \in C} T^{-1}_{c} \times (S \diamond R)^{c}_{-_2}\\
&\overset{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:

- We have $a \sim_R b$;
- We have $b \sim_{T\diamond S} d$, i.e. there exists some $c \in C$ such that:
  - We have $b \sim_S c$;
  - We have $c \sim_T d$;

2. We have $a \sim_{T\diamond(S\diamond R)} d$, i.e. there exists some $c \in C$ such that:
   - We have $a \sim_{S\diamond R} c$, i.e. there exists some $b \in B$ such that:
     - We have $a \sim_R b$;
     - We have $b \sim_S c$;
   - We have $c \sim_T d$;

both of which are equivalent to the statement

($\star$) There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 2: Associativity, Proof II

Using Item 3 of Definition 8.1.3.1.1, we have

$$\begin{aligned}[(T \diamond S) \diamond R](a) &\overset{\text{def}}{=} \bigcup_{b \in R(a)} (T \diamond S)(b)\\ &\overset{\text{def}}{=} \bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c)\end{aligned}$$

on the one hand and

$$\begin{aligned}[T \diamond (S \diamond R)](a) &\overset{\text{def}}{=} \bigcup_{c \in [S\diamond R](a)} T(c)\\ &\overset{\text{def}}{=} \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)\end{aligned}$$

on the other, so we want to prove an equality of the form

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c).$$

This then follows from an application of Constructions With Sets, Item 2 of Proposition 4.3.6.1.2 in which we consider $X = D$, consider $\mathcal{P}(\mathcal{P}(\mathcal{P}(D)))$, take $U = U_c = T(c)$, take A to be

$$A_b \overset{\text{def}}{=} \{T(c) \in \mathcal{P}(D) \mid c \in S(b)\},$$

and then finally take

$$\begin{aligned}\mathcal{A} &\overset{\text{def}}{=} \{A_b \in \mathcal{P}(\mathcal{P}(D)) \mid b \in R(a)\} \\ &\overset{\text{def}}{=} \{\{T(c) \in \mathcal{P}(D) \mid c \in S(b)\} \mid b \in R(a)\}.\end{aligned}$$

Indeed, we have

$$\begin{aligned}\bigcup_{A\in\mathcal{A}}\left(\bigcup_{U\in A} U\right) &= \bigcup_{A_b\in\mathcal{A}}\left(\bigcup_{c\in S(b)} T(c)\right) \\ &= \bigcup_{b\in R(a)}\left(\bigcup_{c\in S(b)} T(c)\right)\end{aligned}$$

and

$$\begin{aligned}\bigcup_{U\in\bigcup_{A\in\mathcal{A}} A} U &= \bigcup_{U_c\in\bigcup_{b\in R(a)} A_b} U_c \\ &= \bigcup_{T(c)\in\bigcup_{b\in R(a)} A_b} T(c) \\ &= \bigcup_{c\in\bigcup_{b\in R(a)} S(b)} T(c).\end{aligned}$$

This finishes the proof.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}\Delta_B \diamond R &\overset{\text{def}}{=} \int^{b\in B} (\Delta_B)^{-1}_{b} \times R^{b}_{-_2} \\ &= \bigvee_{b\in B} (\Delta_B)^{-1}_{b} \times R^{b}_{-_2}\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\substack{b\in B\\ b=-_1}} R^b_{-_2}\\
&= R^{-_1}_{-_2},
\end{aligned}
$$

and

$$
\begin{aligned}
R \diamond \Delta_A &\overset{\text{def}}{=} \int^{a\in A} R^{-_1}_a \times (\Delta_A)^a_{-_2}\\
&= \bigvee_{a\in B} R^{-_1}_a \times (\Delta_A)^a_{-_2}\\
&= \bigvee_{\substack{a\in B\\ a=-_2}} R^{-_1}_a\\
&= R^{-_1}_{-_2}.
\end{aligned}
$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\Delta_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  - We have $a \sim_b B$.
  - There exists some $b' \in B$ such that:
    * We have $a \sim_R b'$
    * We have $b' \sim_{\Delta_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \Delta_A = R$$
witnesses the equivalence of the following two statements:
  - There exists some $a' \in A$ such that:
    * We have $a \sim_{\Delta_B} a'$, i.e. $a = a'$.
    * We have $a' \sim_R b$
  - We have $a \sim_b B$.

Item 4: Relation to Apartness Composition of Relations

This is a repetition of Item 4 of Proposition 8.1.4.1.3 and is proved there.

Item 5: Linear Distributivity

This is a repetition of Item 5 of Proposition 8.1.4.1.3 and is proved there.

Item 6: Interaction With Converses

This is a repetition of Item 3 of Proposition 8.1.5.1.3 and is proved there.

Item 7: Interaction With Ranges and Domains

We have

$$\begin{aligned} \mathrm{dom}(S \diamond R) &\overset{\text{def}}{=} \{a \in A \mid a \sim_{S\diamond R} c \text{ for some } c \in C\}, \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ and } c \in C \\ \text{such that } a \sim_R b \text{ and } b \sim_R c \end{array} \right\}, \\ &\subset \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}, \\ &\overset{\text{def}}{=} \mathrm{dom}(R) \end{aligned}$$

and

$$\begin{aligned} \mathrm{range}(S \diamond R) &\overset{\text{def}}{=} \{c \in C \mid a \sim_{S\diamond R} c \text{ for some } a \in A\}, \\ &= \left\{ c \in C \;\middle|\; \begin{array}{l} \text{there exists some } a \in A \text{ and } b \in B \\ \text{such that } a \sim_R b \text{ and } b \sim_R c \end{array} \right\}, \\ &\subset \left\{ c \in C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } b \sim_S c \end{array} \right\}, \\ &\overset{\text{def}}{=} \mathrm{range}(S). \end{aligned}$$

This finishes the proof.

8.1.4 Apartness Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 8.1.4.1.1 ▶ Apartness Composition of Relations

The **apartness composition of** R **and** S is the relation $S \mathbin{\square} R$ defined as follows:

- Viewing relations as subsets of $A \times C$, we define

$$S \mathbin{\square} R \stackrel{\text{def}}{=} \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times C \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned}(S \mathbin{\square} R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int_{b \in B} S^{-1}_b \amalg R^b_{-2} \\ &= \bigwedge_{b \in B} S^{-1}_b \amalg R^b_{-2},\end{aligned}$$

  where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

- Viewing relations as functions $A \to \mathcal{P}(C)$, we define

$$[S \mathbin{\square} R](a) \stackrel{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b)$$

  for each $a \in A$.

Example 8.1.4.1.2 ▶ Examples of Apartness Composition of Relations

Here are some examples of apartness composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} {\leq} \mathbin{\square} {\geq} &= \emptyset, \\ {\geq} \mathbin{\square} {\leq} &= \emptyset. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} {\leq} \mathbin{\square} {\leq} &= \emptyset, \\ {\geq} \mathbin{\square} {\geq} &= \emptyset. \end{aligned}$$

3. *Equality and Inequality.* Let $A = B = C = \mathbb{Z}$. We have

$$\begin{aligned} = \mathbin{\square} \neq &= =, \\ \neq \mathbin{\square} = &= =. \end{aligned}$$

4. *Subset Inclusion.* Let X be a set with at least three elements and consider the relations $\subset$ and $\supset$ in $\mathcal{P}(X)$. We have

$$\supset \mathbin{\square} \subset = \{(U, V) \in \mathcal{P}(X) \mid U = \emptyset \text{ or } V = \emptyset\}.$$

Proposition 8.1.4.1.3 ▶ Properties of Apartness Composition of Relations

Let $R\colon A \nvrightarrow B$, $S\colon B \nvrightarrow C$, and $T\colon C \nvrightarrow D$ be relations.

1. *Functoriality.* The assignments $R, S, (R, S) \mapsto S \mathbin{\square} R$ define functors

$$\begin{aligned} S \mathbin{\square} -\colon \quad & \mathbf{Rel}(A, B) && \to \mathbf{Rel}(A, C), \\ - \mathbin{\square} R\colon \quad & \mathbf{Rel}(B, C) && \to \mathbf{Rel}(A, C), \\ -_1 \mathbin{\square} -_2\colon \mathbf{Rel}(B, C) \times{} & \mathbf{Rel}(A, B) && \to \mathbf{Rel}(A, C). \end{aligned}$$

In particular, given relations

$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \underset{S_2}{\overset{S_1}{\rightrightarrows}} C,$$

if $R_1 \subset R_2$ and $S_1 \subset S_2$, then $S_1 \mathbin{\square} R_1 \subset S_2 \mathbin{\square} R_2$.

2. *Associativity.* We have

$$(T \mathbin{\square} S) \mathbin{\square} R = T \mathbin{\square} (S \mathbin{\square} R).$$

3. *Unitality.* We have

$$\begin{aligned} \nabla_B \mathbin{\square} R &= R, \\ R \mathbin{\square} \nabla_A &= R. \end{aligned}$$

4. *Relation to Composition of Relations.* We have

$$(S \mathbin{\square} R)^{\mathsf{c}} = S^{\mathsf{c}} \diamond R^{\mathsf{c}},$$

$$(S \diamond R)^{\mathsf{c}} = S^{\mathsf{c}} \mathbin{\Box} R^{\mathsf{c}},$$

where $(-)^{\mathsf{c}}$ is the complement functor of Constructions With Sets, Section 4.3.11. In particular, $\Box$ is a special case of composition of relations, as we have

$$S \mathbin{\Box} R = (S^{\mathsf{c}} \diamond R^{\mathsf{c}})^{\mathsf{c}}.$$

This is also compatible with units, as we have $\nabla_A^{\mathsf{c}} = \Delta_A$.

5. *Linear Distributivity.* We have inclusions of relations

$$\begin{aligned} T \diamond (S \mathbin{\Box} R) &\subset (T \diamond S) \mathbin{\Box} R, \\ (T \mathbin{\Box} S) \diamond R &\subset T \mathbin{\Box} (S \diamond R). \end{aligned}$$

6. *Interaction With Converses.* We have

$$(S \mathbin{\Box} R)^{\dagger} = R^{\dagger} \mathbin{\Box} S^{\dagger}.$$

Proof 8.1.4.1.4 ▶ Proof of Proposition 8.1.4.1.3

Item 1: Functoriality

We have

$$\begin{aligned} S_1 \mathbin{\Box} R_1 &\stackrel{\text{def}}{=} \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_{R_1} b \text{ or } b \sim_{S_1} c \end{array} \right\} \\ &\subset \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_{R_2} b \text{ or } b \sim_{S_2} c \end{array} \right\} \\ &\stackrel{\text{def}}{=} S_2 \mathbin{\Box} R_2. \end{aligned}$$

This finishes the proof.

Item 2: Associativity

Indeed, we have

$$(T \mathbin{\Box} S) \mathbin{\Box} R \stackrel{\text{def}}{=} \left(\int_{c \in C} T_c^{-1} \coprod S_{-2}^{c} \right) \mathbin{\Box} R$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \int_{b\in B}\left(\int_{c\in C} T^{-1}_{c} \amalg S^{c}_{b}\right) \amalg R^{b}_{-_2}\\
&= \int_{b\in B}\int_{c\in C}\left(T^{-1}_{c} \amalg S^{c}_{b}\right) \amalg R^{b}_{-_2}\\
&= \int_{c\in C}\int_{b\in B}\left(T^{-1}_{c} \amalg S^{c}_{b}\right) \amalg R^{b}_{-_2}\\
&= \int_{c\in C}\int_{b\in B} T^{-1}_{c} \amalg \left(S^{c}_{b} \amalg R^{b}_{-_2}\right)\\
&= \int_{c\in C} T^{-1}_{c} \amalg \left(\int_{b\in B} S^{c}_{b} \amalg R^{b}_{-_2}\right)\\
&\overset{\text{def}}{=} \int_{c\in C} T^{-1}_{c} \amalg (S \,\square\, R)^{c}_{-_2}\\
&\overset{\text{def}}{=} T \,\square\, (S \,\square\, R).
\end{aligned}
$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

- We have $a \sim_{(T\square S)\square R} d$, i.e. there exists some $b \in B$ such that:
  - We have $a \sim_R b$;
  - We have $b \sim_{T\square S} d$, i.e. there exists some $c \in C$ such that:
    - We have $b \sim_S c$;
    - We have $c \sim_T d$;
- We have $a \sim_{T\square(S\square R)} d$, i.e. there exists some $c \in C$ such that:
  - We have $a \sim_{S\square R} c$, i.e. there exists some $b \in B$ such that:
    - We have $a \sim_R b$;
    - We have $b \sim_S c$;
  - We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$
\begin{aligned}
\nabla_B \mathbin{\square} R &\overset{\text{def}}{=} \int_{b\in B} (\nabla_B)^{-1}_{b} \amalg R^{b}_{-_2} \\
&= \bigwedge_{b\in B} (\nabla_B)^{-1}_{b} \amalg R^{b}_{-_2} \\
&= \left(\bigwedge_{\substack{b\in B \\ b=-_1}} (\nabla_B)^{-1}_{b} \amalg R^{b}_{-_2} \right) \wedge \left(\bigwedge_{\substack{b\in B \\ b\neq -_1}} (\nabla_B)^{-1}_{b} \amalg R^{b}_{-_2} \right) \\
&= \left((\nabla_B)^{-1}_{-_1} \amalg R^{-_1}_{-_2} \right) \wedge \left(\bigwedge_{\substack{b\in B \\ b\neq -_1}} \mathsf{t} \amalg R^{b}_{-_2} \right) \\
&= \left(\mathsf{f} \amalg R^{-_1}_{-_2} \right) \wedge \left(\bigwedge_{\substack{b\in B \\ b\neq -_1}} \mathsf{t} \right) \\
&= R^{-_1}_{-_2} \wedge \mathsf{t} \\
&= R^{-_1}_{-_2},
\end{aligned}
$$

and

$$
\begin{aligned}
R \mathbin{\square} \nabla_A &\overset{\text{def}}{=} \int_{a\in A} R^{-_1}_{a} \amalg (\nabla_A)^{a}_{-_2} \\
&= \bigwedge_{a\in A} R^{-_1}_{a} \amalg (\nabla_A)^{a}_{-_2} \\
&= \left(\bigwedge_{\substack{a\in A \\ a=-_2}} R^{-_1}_{a} \amalg (\nabla_A)^{a}_{-_2} \right) \wedge \left(\bigwedge_{\substack{a\in A \\ a\neq -_2}} R^{-_1}_{a} \amalg (\nabla_A)^{a}_{-_2} \right) \\
&= \left(R^{-_1}_{-_2} \amalg (\nabla_A)^{-_2}_{-_2} \right) \wedge \left(\bigwedge_{\substack{a\in A \\ a\neq -_2}} R^{-_1}_{a} \amalg \mathsf{t} \right) \\
&= \left(R^{-_1}_{-_2} \amalg \mathsf{f} \right) \wedge \left(\bigwedge_{\substack{a\in A \\ a\neq -_2}} \mathsf{t} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= R^{-1}_{-2} \wedge \mathsf{t} \\
&= R^{-1}_{-2},
\end{aligned}
$$

This finishes the proof.

Item 4: Relation to Composition of Relations

We proceed in a few steps.

- We have $a \sim_{(S \boxdot R)^{\mathsf{c}}} b$ iff $a \nsim_{S \boxdot R} b$.
- We have $a \nsim_{S \boxdot R} b$ iff the assertion "for each $b \in B$, we have $a \sim_R b$ or $b \sim_S c$" is false.
- That happens iff there exists some $b \in B$ such that $a \nsim_R b$ and $b \nsim_S c$.
- That happens iff there exists some $b \in B$ such that $a \sim_{R^{\mathsf{c}}} b$ and $b \sim_{S^{\mathsf{c}}} c$.

The second equality then follows from the first one by Constructions With Sets, Item 3 of Proposition 4.3.11.1.2.

Item 5: Linear Distributivity

We have

$$
\begin{aligned}
T \diamond (S \boxdot R) &\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } c \in C \text{ such} \\ \text{that } a \sim_{S \boxdot R} c \text{ and } c \sim_T d \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } c \in C \text{ such that} \\ c \sim_T d \text{ and, for each } b \in B, \\ \text{we have } a \sim_R b \text{ or } b \sim_S c \end{array} \right\} \\
&= \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{the following conditions are satisfied:} \\ \quad 1.\ \text{For each } b \in B \text{, we have } a \sim_R b \text{ or } b \sim_S c. \\ \quad 2.\ \text{There exists } c \in C \text{ such that } c \sim_T d. \end{array} \right\} \\
&\subset \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B \text{, at least one of the} \\ \text{following conditions is satisfied:} \\ \quad 1.\ \text{We have } a \sim_R b. \\ \quad 2.\ \text{There exists } c \in C \text{ such that } b \sim_S c \\ \qquad \text{and } c \sim_T d. \end{array} \right\}
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_R b \text{ or there exists some } c \in C \\ \text{such that } b \sim_S c \text{ and } c \sim_T d \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_R b \text{ or } b \sim_{T \diamond S} d \end{array} \right\} \\
&\overset{\text{def}}{=} (T \diamond S) \mathbin{\square} R
\end{aligned}$$

and

$$\begin{aligned}
(T \mathbin{\square} S) \diamond R &\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_{T \square S} d \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and, for each } c \in C, \\ \text{we have } b \sim_S c \text{ or } c \sim_T d \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ satisfying} \\ \text{the following conditions:} \\ \quad 1.\ \text{We have } a \sim_R b. \\ \quad 2.\ \text{For each } c \in C \text{, we have } b \sim_S c \\ \quad\quad \text{or } c \sim_T d. \end{array} \right\} \\
&\subset \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C \text{, at least one of the} \\ \text{following conditions is satisfied:} \\ \quad 1.\ \text{We have } c \sim_T d. \\ \quad 2.\ \text{There exists some } b \in B \text{ such that} \\ \quad\quad \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C \text{, we have } c \sim_T d \\ \text{or there exists some } b \in B \text{, such that} \\ \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C \text{, we have} \\ a \sim_{S \diamond R} c \text{ or } c \sim_T d \end{array} \right\} \\
&\subset T \mathbin{\square} (S \diamond R).
\end{aligned}$$

This finishes the proof.

Item 6: Interaction With Converses

This is a repetition of Item 4 of Proposition 8.1.5.1.3 and is proved there. ∎

8.1.5 The Converse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 8.1.5.1.1 ▶ The Converse of a Relation

The **converse of** R[1] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \overset{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } a \sim_R b\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\left[R^\dagger\right]^a_b \overset{\text{def}}{=} R^b_a$$

  for each $(b,a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define[2]

$$R^\dagger(b) \overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}$$

  for each $b \in B$.

[1] *Further Terminology:* Also called the **opposite of** R or the **transpose of** R.
[2] Note that $R^\dagger(b) = R^{-1}(\{b\})$.

Example 8.1.5.1.2 ▶ Examples of Converses of Relations

Here are some examples of converses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^\dagger = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^\dagger = \leq$.
3. *Functions.* Let $f \colon A \to B$ be a function. We have

$$\begin{aligned} \mathrm{Gr}(f)^\dagger &= f^{-1}, \\ \left(f^{-1}\right)^\dagger &= \mathrm{Gr}(f), \end{aligned}$$

where $\mathrm{Gr}(f)$ and f^{-1} are the relations of Sections 8.2.2 and 8.2.3.

Proposition 8.1.5.1.3 ▶ Properties of Converses of Relations

Let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^{\dagger}$ defines a functor (i.e. morphism of posets)

$$(-)^{\dagger}\colon \mathbf{Rel}(A,B) \to \mathbf{Rel}(B,A).$$

   In other words, given relations $R, S\colon A \rightrightarrows B$, we have:

   (★) If $R \subset S$, then $R^{\dagger} \subset S^{\dagger}$.

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}\left(R^{\dagger}\right) &= \mathrm{range}(R),\\ \mathrm{range}\left(R^{\dagger}\right) &= \mathrm{dom}(R).\end{aligned}$$

3. *Interaction With Composition.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

4. *Interaction With Apartness Composition.* We have

$$(S \mathbin{\Box} R)^{\dagger} = R^{\dagger} \mathbin{\Box} S^{\dagger}.$$

5. *Invertibility.* We have

$$\left(R^{\dagger}\right)^{\dagger} = R.$$

6. *Identity I.* We have

$$\Delta_A^{\dagger} = \Delta_A.$$

7. *Identity II.* We have

$$\nabla_A^{\dagger} = \nabla_A.$$

Proof 8.1.5.1.4 ▶ Proof of Proposition 8.1.5.1.3

Item 1: Functoriality

We have

$$\begin{aligned} R^{\dagger} &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\} \\ &\subset \{a \in A \mid b \in S(a)\} \\ &\overset{\text{def}}{=} S^{\dagger}. \end{aligned}$$

This finishes the proof.

Item 2: Interaction With Ranges and Domains

We have

$$\begin{aligned} \mathrm{dom}\Big(R^{\dagger}\Big) &\overset{\text{def}}{=} \{b \in B \mid b \sim_{R^{\dagger}} a \text{ for some } a \in A\} \\ &= \{b \in B \mid a \sim_{R} b \text{ for some } a \in A\} \\ &\overset{\text{def}}{=} \mathrm{range}(R) \end{aligned}$$

and

$$\begin{aligned} \mathrm{range}\Big(R^{\dagger}\Big) &\overset{\text{def}}{=} \{a \in A \mid b \sim_{R^{\dagger}} a \text{ for some } b \in B\} \\ &= \{a \in A \mid a \sim_{R} b \text{ for some } b \in B\} \\ &\overset{\text{def}}{=} \mathrm{dom}(R). \end{aligned}$$

This finishes the proof.

Item 3: Interaction With Composition

We have

$$\begin{aligned} (S \diamond R)^{\dagger} &\overset{\text{def}}{=} \left\{(c,a) \in C \times A \;\middle|\; c \sim_{(S\diamond R)^{\dagger}} a\right\} \\ &= \{(c,a) \in C \times A \mid a \sim_{S\diamond R} c\} \\ &= \left\{(c,a) \in C \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_{R} b \text{ and } b \sim_{S} c \end{array}\right\} \\ &= \left\{(c,a) \in C \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } b \sim_{R^{\dagger}} a \text{ and } c \sim_{S^{\dagger}} b \end{array}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left\{ (c,a) \in C \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } c \sim_{S^\dagger} b \text{ and } b \sim_{R^\dagger} a \end{array} \right\} \\
&\stackrel{\text{def}}{=} R^\dagger \diamond S^\dagger.
\end{aligned}
$$

This finishes the proof.

Item 4: Interaction With Apartness Composition

We have

$$
\begin{aligned}
(S \mathbin{\Box} R)^\dagger &\stackrel{\text{def}}{=} \left\{ (c,a) \in C \times A \;\middle|\; c \sim_{(S \Box R)^\dagger} a \right\} \\
&= \{ (c,a) \in C \times A \mid a \sim_{S \Box R} c \} \\
&= \left\{ (c,a) \in C \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_S c \end{array} \right\} \\
&= \left\{ (c,a) \in C \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ b \sim_{R^\dagger} a \text{ or } c \sim_{S^\dagger} b \end{array} \right\} \\
&= \left\{ (c,a) \in C \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B, \text{ we have} \\ c \sim_{S^\dagger} b \text{ or } b \sim_{R^\dagger} a \end{array} \right\} \\
&\stackrel{\text{def}}{=} R^\dagger \mathbin{\Box} S^\dagger.
\end{aligned}
$$

This finishes the proof.

Item 5: Invertibility

We have

$$
\begin{aligned}
\left(R^\dagger\right)^\dagger &\stackrel{\text{def}}{=} \{ (a,b) \in A \times B \mid b \sim_{R^\dagger} a \} \\
&= \{ (a,b) \in A \times B \mid a \sim_R b \} \\
&\stackrel{\text{def}}{=} R.
\end{aligned}
$$

This finishes the proof.

Item 6: Identity I

We have

$$
\begin{aligned}
\Delta_A^\dagger &\stackrel{\text{def}}{=} \{ (a,b) \in A \times A \mid a \sim_{\Delta_A} b \} \\
&= \{ (a,b) \in A \times A \mid a = b \}
\end{aligned}
$$

$$= \Delta_A.$$

This finishes the proof.

Item 7: Identity II

We have

$$\begin{aligned}\nabla_A^\dagger &\stackrel{\text{def}}{=} \{(a,b) \in A \times A \mid a \sim_{\nabla_A} b\}\\ &= \{(a,b) \in A \times A \mid a \neq b\}\\ &= \nabla_A.\end{aligned}$$

This finishes the proof.

8.2 Examples of Relations

8.2.1 Elementary Examples of Relations

Example 8.2.1.1.1 ▶ The Trivial Relation

The **trivial relation on** A **and** B is the relation $\sim_{\text{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have
$$\sim_{\text{triv}} \overset{\text{def}}{=} A \times B.$$
2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\text{triv}}$ is the constant function
$$\Delta_{\mathsf{true}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$
from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value true.
3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{triv}}$ is the function
$$\Delta_{\mathsf{true}} \colon A \to \mathcal{P}(B)$$
defined by
$$\Delta_{\mathsf{true}}(a) \overset{\text{def}}{=} B$$
for each $a \in A$.
4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

Example 8.2.1.1.2 ▶ The Cotrivial Relation

The **cotrivial relation on** A **and** B is the relation $\sim_{\text{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have
$$\sim_{\text{cotriv}} \overset{\text{def}}{=} \emptyset.$$
2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\text{cotriv}}$ is the constant function
$$\Delta_{\mathsf{false}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$
from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value false.
3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{cotriv}}$ is the function
$$\Delta_{\mathsf{false}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{false}}(a) \overset{\text{def}}{=} \emptyset$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \nsim_R b$ for each $a \in A$ and each $b \in B$.

Example 8.2.1.1.3 ▶ The Characteristic Relation of a Set

The characteristic relation χ_X on X of Constructions With Sets, Definition 4.5.3.1.1:

1. As a subset of $X \times X$, we have

$$\begin{aligned} \sim_{\chi_X} &\overset{\text{def}}{=} \Delta_X \\ &\overset{\text{def}}{=} \{(x, x) \in X \times X\}. \end{aligned}$$

2. As a function from $X \times X$ to $\{\mathsf{true}, \mathsf{false}\}$, we have

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

3. As a function from X to $\mathcal{P}(X)$, we have

$$\chi_X(x) \overset{\text{def}}{=} \{x\}$$

for each $x \in X$.

Example 8.2.1.1.4 ▶ The Antidiagonal Relation on a Set

The **antidiagonal relation on** X is the relation ∇_X defined equivalently as follows:

1. As a subset of $X \times X$, we have

$$\begin{aligned} \sim_{\nabla_X} &\overset{\text{def}}{=} \nabla_X \\ &\overset{\text{def}}{=} X \setminus \Delta_X \\ &= \{(x, y) \in X \times X \mid x \neq y\}. \end{aligned}$$

2. As a function from $X \times X$ to $\{\mathsf{true}, \mathsf{false}\}$, we have

$$\nabla_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a \neq b, \\ \mathsf{false} & \text{if } a = b \end{cases}$$

for each $x, y \in X$.

3. As a function from X to $\mathcal{P}(X)$, we have

$$\nabla_X(x) \stackrel{\text{def}}{=} X \setminus \{x\}$$

for each $x \in X$.

Example 8.2.1.1.5 ▶ Partial Functions

Partial functions may be viewed (or defined) as being exactly those relations which are functional; see Conditions on Relations, Section 10.1.1.

Example 8.2.1.1.6 ▶ Square Roots

Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in* $\mathbb{Q}$. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-} \colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 8.2.1.1.7 ▶ Complex Logarithms

The complex logarithm defines a relation

$$\log \colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \overset{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z}\right\}$$

for each $a + bi \in \mathbb{C}$.

Example 8.2.1.1.8 ▶ More Examples of Relations

See [Wik25] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

8.2.2 The Graph of a Function

Let $f\colon A \to B$ be a function.

Definition 8.2.2.1.1 ▶ The Graph of a Function

The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define
$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$
- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define
$$\mathrm{Gr}(f)^b_a \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$
for each $(a, b) \in A \times B$.
- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define
$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$
for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition
$$A \xrightarrow{f} B \overset{\chi_B}{\hookrightarrow} \mathcal{P}(B).$$

[1] *Further Terminology and Notation:* When $f = \mathrm{id}_A$, we write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, calling it the **graph of** A.

Proposition 8.2.2.1.2 ▶ Properties of Graphs of Functions

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor
$$\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$$
where
   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
   $$\mathrm{Gr}(A) \stackrel{\text{def}}{=} A.$$
   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
   $$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\text{def}}{=}\mathrm{Rel}(A,B)}$$
   of Gr at (A, B) is defined by
   $$\mathrm{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \mathrm{Gr}(f),$$
   where $\mathrm{Gr}(f)$ is the graph of f as in Definition 8.2.2.1.1.

   In particular, the following statements are true:
   - *Preservation of Identities.* We have
   $$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$
   for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
   - *Preservation of Composition.* We have
   $$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$
   for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness.* We have an adjunction
$$(\mathrm{Gr} \dashv \mathcal{P}_!)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_!"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

3. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

4. *Adjointness Inside* **Rel**. We have an internal adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid" marking] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid" marking] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 8.2.3.1.1.

5. *Interaction With Converses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof 8.2.2.1.3 ▶ Proof of Proposition 8.2.2.1.2

Item 1: Functoriality

Omitted.

Item 2: Adjointness

This is a repetition of Constructions With Sets, Proposition 4.4.4.1.1, and is proved there.

Item 3: Cocontinuity

This follows from Item 2 and ??.

Item 4: Adjointness Inside Rel

We need to check that there are inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 5: Interaction With Converses

Omitted.

Item 6: Characterisations

We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of Proposition 8.5.2.1.3.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton. Since the conditions

  - $R(a) \cap V \neq \emptyset$;
  - $R(a) \subset V$;

  are equivalent when $R(a)$ is a singleton, it follows that the sets

  $$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\},$$
  $$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

  are equal for all $V \in \mathcal{P}(B)$.

- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality*. We proceed in a few steps:
    * If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$.
    * But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction.
    * Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality*. If $R^{-1} = R_{-1}$, then we have

    $$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, so R is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Relations, Proposition 8.5.3.1.1.

This finishes the proof. □

8.2.3 The Inverse of a Function

Let $f\colon A \to B$ be a function.

Definition 8.2.3.1.1 ▶ The Inverse of a Function

The **inverse of** f is the relation $f^{-1}\colon B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

  $$f^{-1} \overset{\text{def}}{=} \left\{ \left(b, f^{-1}(b)\right) \in B \times A \;\middle|\; a \in A \right\},$$

  where

  $$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.
- Viewing relations from B to A as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$,

we define

$$\left[f^{-1}\right]_a^b \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$

for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

Proposition 8.2.3.1.2 ▶ Properties of Inverses of Functions

Let $f: A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A$, $f \mapsto f^{-1}$ defines a functor

$$(-)^{-1}: \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \overset{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}: \mathsf{Sets}(A, B) \to \mathrm{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \overset{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 8.2.3.1.1.

In particular, the following statements are true:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Converses of Relations.* We have

$$\begin{aligned} \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f), \\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}. \end{aligned}$$

Proof 8.2.3.1.3 ▶ Proof of Proposition 8.2.3.1.2

Item 1: Functoriality

Omitted.

Item 2: Adjointness Inside Rel

This is a repetition of Item 4 of Proposition 8.2.2.1.2 and is proved there.

Item 3: Interaction With Converses of Relations

This is a repetition of Item 5 of Proposition 8.2.2.1.2 and is proved there.

□

8.2.4 Representable Relations

Let A and B be sets.

Let $f\colon A\to B$ and $g\colon B\to A$ be functions.[1]

1. The **representable relation associated to** f is the relation $\chi_f\colon A\nrightarrow B$ defined as the composition

$$A\times B\xrightarrow{f\times\mathrm{id}_B}B\times B\xrightarrow{\chi_B}\{\mathsf{true},\mathsf{false}\},$$

i.e. given by declaring $a\sim_{\chi_f}b$ iff $f(a)=b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B\nrightarrow A$ defined as the composition

$$B\times A\xrightarrow{g\times\mathrm{id}_A}A\times A\xrightarrow{\chi_A}\{\mathsf{true},\mathsf{false}\},$$

i.e. given by declaring $b\sim_{\chi^g}a$ iff $g(b)=a$.

[1]More generally, given functions

$$\begin{aligned}f&\colon A\to C,\\ g&\colon B\to D\end{aligned}$$

and a relation $B\nrightarrow D$, we may consider the composite relation

$$A\times B\xrightarrow{f\times g}C\times D\xrightarrow{R}\{\mathsf{true},\mathsf{false}\},$$

for which we have $a\sim_{R\circ(f\times g)}b$ iff $f(a)\sim_R g(b)$.

8.3 Categories of Relations

8.3.1 The Category of Relations Between Two Sets

Definition 8.3.1.1.1 ▶ The Category of Relations Between Two Sets

The **category of relations from** A **to** B is the category $\mathbf{Rel}(A,B)$ defined by[1]

$$\mathbf{Rel}(A,B)\overset{\text{def}}{=}\mathbf{Rel}(A,B)_{\mathsf{pos}},$$

where $\mathbf{Rel}(A,B)_{\mathsf{pos}}$ is the posetal category associated to the poset $\mathbf{Rel}(A,B)$ of Item 2 of Notation 8.1.1.1.6 and Categories, Definition 11.2.7.1.1.

[1]Here we choose to abuse notation by writing $\mathbf{Rel}(A, B)$ instead of $\mathbf{Rel}(A, B)_{\mathsf{pos}}$ for the posetal category of relations from A to B, even though the same notation is used for the poset of relations from A to B.

8.3.2 The Category of Relations

DEFINITION 8.3.2.1.1 ▶ THE CATEGORY OF RELATIONS

The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}(A, B) \stackrel{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \to \mathrm{Rel}(A, A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Example 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

8.3.3 The Closed Symmetric Monoidal Category of Relations

8.3.3.1 The Monoidal Product

Definition 8.3.3.1.1 ▶ The Monoidal Product of Rel

The **monoidal product of** Rel is the functor

$$\times\colon \mathsf{Rel} \times \mathsf{Rel} \to \mathsf{Rel}$$

where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have

  $$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

  where $A \times B$ is the Cartesian product of sets of Constructions With Sets, Definition 4.1.3.1.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

  $$\times_{(A,C),(B,D)}\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \to \mathrm{Rel}(A \times C, B \times D)$$

  of $\times$ is given by sending a pair of morphisms (R, S) of the form

  $$\begin{aligned} R&\colon A \nrightarrow B, \\ S&\colon C \nrightarrow D \end{aligned}$$

  to the relation

  $$R \times S\colon A \times C \nrightarrow B \times D$$

  of Constructions With Relations, Definition 9.2.6.1.1.

8.3.3.2 The Monoidal Unit

Definition 8.3.3.2.1 ▶ The Monoidal Unit of Rel

The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \to \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \overset{\mathrm{def}}{=} \mathrm{pt}$$

of Rel.

8.3.3.3 The Associator

Definition 8.3.3.3.1 ▶ The Associator of Rel

The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathsf{id}) \overset{\sim}{\Longrightarrow} \times \circ (\mathsf{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Rel} \times (\mathsf{Rel} \times \mathsf{Rel}) \arrow[dr, "\mathsf{id}\times(\times)", "|" marking] & \\
(\mathsf{Rel} \times \mathsf{Rel}) \times \mathsf{Rel} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\wr"'] \arrow[d, "(\times)\times\mathsf{id}"', "|" marking] \arrow[rr, phantom, "\Uparrow \alpha^{\mathsf{Rel}}"] & & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times", "|" marking] \\
\mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"', "|" marking] & & \mathsf{Rel},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} \colon (A \times B) \times C \nrightarrow A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

8.3.3.4 The Left Unitor

Definition 8.3.3.4.1 ▶ The Left Unitor of Rel

The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_A \colon \mathbb{1}_{\mathsf{Rel}} \times A \rightarrow\!\!\!\!\!\!\mapstochar\;\; A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a = b$.

8.3.3.5 The Right Unitor

Definition 8.3.3.5.1 ▶ The Right Unitor of Rel

The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \rightarrow\!\!\!\!\!\!\mapstochar\;\; A$$

at A is defined by declaring

$$(a,\star) \sim_{\rho_A^{\mathsf{Rel}}} b$$

iff $a = b$.

8.3.3.6 The Symmetry

Definition 8.3.3.6.1 ▶ The Symmetry of Rel

The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\overset{\|}{\Downarrow} \sigma^{\mathsf{Rel}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at (A, B) is defined by declaring

$$(a,b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

8.3.3.7 The Internal Hom

Definition 8.3.3.7.1 ▶ The Internal Hom of Rel

The **internal Hom of** Rel is the functor

$$\mathrm{Rel}\colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

defined

- On objects by sending $A, B \in \mathrm{Obj}(\mathsf{Rel})$ to the set $\mathrm{Rel}(A, B)$ of **??** of **??**.
- On morphisms by pre/post-composition defined as in Definition 8.1.3.1.1.

Proposition 8.3.3.7.2 ▶ Properties of the Internal Hom of Rel

Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathrm{Rel}(A, -))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathrm{Rel}(B, -))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathrm{Rel}(B, C)), \\ \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathrm{Rel}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

Proof 8.3.3.7.3 ▶ Proof of Proposition 8.3.3.7.2

Item 1: Adjointness

Indeed, we have

$$\begin{aligned}\mathrm{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A \times B \times C, \{\mathsf{true}, \mathsf{false}\}) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, \mathrm{Rel}(B, C)),\end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathrm{Rel}(A, C))$.

8.3.3.8 The Closed Symmetric Monoidal Category of Relations

Proposition 8.3.3.8.1 ▶ The Closed Symmetric Monoidal Category of Relations

The category Rel admits a closed symmetric monoidal category structure consisting of[1]

- *The Underlying Category.* The category Rel of sets and relations of Definition 8.3.2.1.1.
- *The Monoidal Product.* The functor
$$\times \colon \mathrm{Rel} \times \mathrm{Rel} \to \mathrm{Rel}$$
of Definition 8.3.3.1.1.
- *The Internal Hom.* The internal Hom functor
$$\mathbf{Rel} \colon \mathrm{Rel}^{\mathsf{op}} \times \mathrm{Rel} \to \mathrm{Rel}$$
of Definition 8.3.3.7.1.
- *The Monoidal Unit.* The functor
$$\mathbb{1}^{\mathrm{Rel}} \colon \mathsf{pt} \to \mathrm{Rel}$$
of Definition 8.3.3.2.1.
- *The Associators.* The natural isomorphism
$$\alpha^{\mathrm{Rel}} \colon \times \circ (\times \times \mathrm{id}_{\mathrm{Rel}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathrm{Rel}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$
of Definition 8.3.3.3.1.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathrm{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathrm{Rel}} \times \mathrm{id}_{\mathrm{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 8.3.3.4.1.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathrm{Rel}} \colon \times \circ \left(\mathsf{id} \times \mathbb{1}^{\mathrm{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 8.3.3.5.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathrm{Rel}} \colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 8.3.3.6.1.

[1] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in fact given by the disjoint union of sets; see Constructions With Relations, ??.

Proof 8.3.3.8.2 ▶ Proof of Proposition 8.3.3.8.1

Omitted. ∎

8.3.4 The 2-Category of Relations

Definition 8.3.4.1.1 ▶ The 2-Category of Relations

The **2-category of relations** is the locally posetal 2-category Rel where

- *Objects.* The objects of Rel are sets.

- **Hom**-*Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Rel}}(A, B) &\overset{\text{def}}{=} \mathbf{Rel}(A, B)\\ &\overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_{A} \colon \mathrm{pt} \to \mathbf{Rel}(A, A)$$

of **Rel** at A is defined by

$$\mathrm{id}_A^{\mathsf{Rel}} \overset{\text{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Example 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map[1]

  $$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

  of **Rel** at (A, B, C) is defined by

  $$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

[1] That this is indeed a morphism of posets is proven in **??** of Proposition 8.1.3.1.4.

8.3.5 The Double Category of Relations

8.3.5.1 The Double Category of Relations

Definition 8.3.5.1.1 ▶ The Double Category of Relations

The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets.
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f \colon A \to B$.
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R \colon A \nrightarrow X$.

- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "|" marking] & Y \end{tikzcd}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 8.3.5.2.1.

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \stackrel{\text{def}}{=} \mathrm{id}_A\,.$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "|" marking] & B \end{tikzcd}$$

of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 8.3.5.3.1.
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have
$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$
- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 8.3.5.4.1.
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.5.1.
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.6.1.
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.7.1.

8.3.5.2 Horizontal Identities

Definition 8.3.5.2.1 ▶ The Horizontal Identities of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor
$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$
of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have
$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2).$$
- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism
$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of Constructions With Sets, Item 1 of Proposition 4.5.3.1.3.

8.3.5.3 Horizontal Composition

Definition 8.3.5.3.1 ▶ The Horizontal Composition of $\mathsf{Rel}^{\mathsf{dbl}}$

The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}_1^{\mathsf{dbl}} \underset{\mathsf{Rel}_0^{\mathsf{dbl}}}{\times} \mathsf{Rel}_1^{\mathsf{dbl}} \to \mathsf{Rel}_1^{\mathsf{dbl}}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\text{def}}{=} S \diamond R,$$

  where $S \diamond R$ is the composition of R and S of Definition 8.1.3.1.1.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "|"] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "|"] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$
\begin{tikzcd}
A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\}
\end{tikzcd}
$$

of inclusions of relations, the horizontal composition

$$
\begin{tikzcd}
A \arrow[r, "S \odot R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \beta \odot \alpha"] & C \arrow[d, "h"] \\
X \arrow[r, "U \odot T"', "\shortmid"{marking}] & Z
\end{tikzcd}
$$

of α and β is the inclusion of relations

$$
(U \diamond T) \circ (f \times h) \subset (S \diamond R)
\qquad
\begin{tikzcd}
A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}.
\end{tikzcd}
$$

Proof 8.3.5.3.2 ▶ Proof of the Inclusion in Definition 8.3.5.3.1

The inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R)$$

follows from the fact that the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  - We have $f(a) \sim_T y$.
  - We have $y \sim_U h(c)$.

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  - We have $a \sim_R b$.
  - We have $b \sim_S c$.

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

This finishes the proof. □

8.3.5.4 Vertical Composition of 2-Morphisms

Definition 8.3.5.4.1 ▶ The Vertical Composition of 2-Morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$

The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "|" marking] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h\circ f"'] \arrow[dr, phantom, "\overset{\beta\circ\alpha}{\Downarrow}"] & X \arrow[d, "k\circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, dashed, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof 8.3.5.4.2 ▶ Proof of the Inclusion in Definition 8.3.5.4.1

The inclusion

$$T \circ [(h \circ f) \times (k \circ g)] \subset R$$

follows from the fact that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

This finishes the proof. ∎

8.3.5.5 The Associators

Definition 8.3.5.5.1 ▶ The Associators of $\mathsf{Rel}^{\mathsf{dbl}}$

For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "|"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "|"] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "|"] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "|"] & B \arrow[r, "S"', "|"] & C \arrow[r, "T"', "|"] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[1]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T\diamond S)\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "/\!\!/"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T\diamond(S\diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1] As proved in Item 2 of Proposition 8.1.3.1.4.

8.3.5.6 The Left Unitors

Definition 8.3.5.6.1 ▶ The Left Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$
\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R} \colon \mathbb{1}_B \odot R \xRightarrow{\sim} R,
\qquad
\begin{tikzcd}
A \arrow[r, "R", "|" marking] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "|" marking] \arrow[d, phantom, "\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[rr, "R"', "|" marking] & & B
\end{tikzcd}
$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$
R = \chi_B \diamond R,
\qquad
\begin{tikzcd}
A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\sslash"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\
A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}.
\end{tikzcd}
$$

[1] As proved in Item 3 of Proposition 8.1.3.1.4.

8.3.5.7 The Right Unitors

Definition 8.3.5.7.1 ▶ The Right Unitors of $\mathsf{Rel}^{\mathsf{dbl}}$

For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$
\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_{R} \colon R \odot \mathbb{1}_A \xRightarrow{\sim} R,
\qquad
\begin{tikzcd}
A \arrow[r, "\mathbb{1}_A", "|" marking] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|" marking] \arrow[d, phantom, "\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[rr, "R"', "|" marking] & & B
\end{tikzcd}
$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[1]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "/\!\!/"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[1]As proved in Item 3 of Proposition 8.1.3.1.4.

8.4 Categories of Relations With Apartness Composition

8.4.1 The Category of Relations With Apartness Composition

Definition 8.4.1.1.1 ▶ The Category of Relations With Apartness Composition

The **category of relations with apartness composition** is the category $\mathsf{Rel}^{\square}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\square}$ are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}^{\square}(A, B) \overset{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}\big(\mathsf{Rel}^{\square}\big)$, the unit map

$$\mathbb{1}^{\mathsf{Rel}^{\square}}_A : \mathrm{pt} \to \mathrm{Rel}(A, A)$$

of $\mathsf{Rel}^{\square}$ at A is defined by

$$\mathrm{id}^{\mathsf{Rel}^{\square}}_A \overset{\text{def}}{=} \nabla_A(-_1, -_2),$$

where $\nabla_A(-_1, -_2)$ is the antidiagonal relation of A of Example 8.2.1.1.4.

- *Composition.* For each $A, B, C \in \mathrm{Obj}\big(\mathsf{Rel}^{\square}\big)$, the composition map

$$\circ^{\mathsf{Rel}^{\square}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, C)$$

of $\mathsf{Rel}^{\square}$ at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}^{\square}}_{A,B,C} R \stackrel{\text{def}}{=} S \,\square\, R$$

for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.4.1.1.

Proposition 8.4.1.1.2 ▶ Isomorphism Between Rel and $\mathsf{Rel}^{\square}$

The functor

$$(-)^{\mathsf{c}} \colon \mathsf{Rel} \to \mathsf{Rel}^{\square}$$

given by the identity on objects and by $R \mapsto R^{\mathsf{c}}$ on morphisms is an isomorphism of categories.

Proof 8.4.1.1.3 ▶ Proof of Proposition 8.4.1.1.2

By Item 4 of Proposition 8.1.4.1.3, we see that $(-)^{\mathsf{c}}$ is indeed a functor. By Categories, Item 1 of Proposition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\mathsf{c}} \colon \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, B)$$

defined by the assignment $R \mapsto R^{\mathsf{c}}$ is a bijection by Constructions With Sets, Item 3 of Proposition 4.3.11.1.2. Thus $(-)^{\mathsf{c}}$ is an isomorphism of categories. ∎

8.4.2 The 2-Category of Relations With Apartness Composition

Definition 8.4.2.1.1 ▶ The 2-Category of Relations With Apartness Composition

The **2-category of relations with apartness composition** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets.
- **Hom**-*Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A,B) &\overset{\text{def}}{=} \mathbf{Rel}(A,B)\\ &\overset{\text{def}}{=} (\mathrm{Rel}(A,B),\subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_{A} \colon \mathrm{pt} \to \mathbf{Rel}(A,A)$$

  of **Rel** at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_{A} \overset{\text{def}}{=} \chi_A(-_1,-_2),$$

  where $\chi_A(-_1,-_2)$ is the characteristic relation of A of Example 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[1]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B,C) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,C)$$

  of **Rel** at (A,B,C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \overset{\text{def}}{=} S \diamond R$$

  for each $(S,R) \in \mathbf{Rel}(B,C) \times \mathbf{Rel}(A,B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

[1] That this is indeed a morphism of posets is proven in **??** of Proposition 8.1.4.1.3.

Proposition 8.4.2.1.2 ▶ 2-Isomorphism Between Rel and $\mathsf{Rel}^{\Box,\mathrm{co}}$

The functor

$$(-)^{\mathsf{c}} \colon \mathsf{Rel} \to \mathsf{Rel}^{\Box,\mathrm{co}}$$

given by the identity on objects and by $R \mapsto R^{\mathsf{c}}$ on 1-morphisms is a 2-isomorphism of 2-categories.

Proof 8.4.2.1.3 ▶ Proof of Proposition 8.4.2.1.2

By Item 4 of Proposition 8.1.4.1.3, we see that $(-)^{\mathsf{c}}$ is indeed a functor. By Constructions With Sets, Item 1 of Proposition 4.3.11.1.2, it is also a 2-functor.
By **??**, it suffices to show that $(-)^{\mathsf{c}}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Definition 8.4.1.1.1.
- Bijective on 2-morphisms, which follows from Constructions With Sets, Item 1 of Proposition 4.3.11.1.2.

Thus $(-)^{\mathsf{c}}$ is indeed a 2-isomorphism of categories. ∎

8.4.3 The Linear Bicategory of Relations

Definition 8.4.3.1.1 ▶ The Linear Bicategory of Relations

The **linear bicategory of relations** is the linear bicategory consisting of:

- *The Underlying Bicategory I.* The bicategory Rel of Definition 8.3.4.1.1.
- *The Underlying Bicategory II.* The bicategory Rel of Definition 8.4.2.1.1.
- *Linear Distributors.* The inclusions

$$\delta^{\ell}_{R,S,T} \colon T \diamond (S \mathbin{\Box} R) \hookrightarrow (T \diamond S) \mathbin{\Box} R,$$
$$\delta^{r}_{R,S,T} \colon (T \mathbin{\Box} S) \diamond R \hookrightarrow T \mathbin{\Box} (S \diamond R)$$

of Item 5 of Proposition 8.1.4.1.3.

Proof 8.4.3.1.2 ▶ Proof of the Claims in Definition 8.4.3.1.1

Since Rel and $\mathsf{Rel}^{\Box}$ are locally posetal, the commutativity of the coherence conditions for linear bicategories follows automatically (Categories, Item 4 of Proposition 11.2.7.1.2). ∎

8.4.4 Other Categorical Structures With Apartness Composition

Remark 8.4.4.1.1 ▶ Other Categorical Structures With Apartness Composition

It seems apartness composition fails to form the following categorical structures:

- *Monoidal Category With Products.* Products don't seem to endow $\mathsf{Rel}^{\square}$ with a monoidal structure.
- *Monoidal Category With Coproducts.* Coproducts also don't seem to endow $\mathsf{Rel}^{\square}$ with a monoidal structure.
- *Double Categorical Structure.* It seems the apartness composition of relations doesn't form a double category in a natural[1] way.

[1] I.e. such that the composition of vertical morphisms is the usual composition of functions, as in Sets.

8.5 Properties of the 2-Category of Relations

8.5.1 Self-Duality

Proposition 8.5.1.1.1 ▶ Self-Duality for the (2-)Category of Relations

The 2-/category of relations is self-dual:

1. *Self-Duality I.* We have an isomorphism

$$\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$$

   of categories.

2. *Self-Duality II.* We have a 2-isomorphism

$$\boldsymbol{\mathsf{Rel}}^{\mathsf{op}} \cong \boldsymbol{\mathsf{Rel}}$$

   of 2-categories.

Proof 8.5.1.1.2 ▶ Proof of Proposition 8.5.1.1.1

Item 1: Self-Duality I

We claim that the functor

$$(-)^{\dagger} \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories. Note that this is indeed a functor by Items 3 and 6 of Proposition 8.1.5.1.3.

By Categories, Item 1 of Proposition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\dagger} \colon \mathrm{Rel}(A, B) \to \mathrm{Rel}(B, A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Proposition 8.1.5.1.3, showing $(-)^{\dagger}$ to be fully faithful.

Item 2: Self-Duality II

We claim that the 2-functor

$$(-)^{\dagger} \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Proposition 8.1.5.1.3, is an isomorphism of categories.

By **??**, it suffices to show that $(-)^{\dagger}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Proposition 8.1.5.1.3.

Thus $(-)^{\dagger}$ is indeed a 2-isomorphism of categories. □

8.5.2 Isomorphisms and Equivalences

Let $R \colon A \nrightarrow B$ be a relation from A to B.

The conditions below are row-wise equivalent:

Condition	Inclusion
R is functional	$R \diamond R^\dagger \subset \Delta_B$
R is total	$\Delta_A \subset R^\dagger \diamond R$
R is injective	$R^\dagger \diamond R \subset \Delta_A$
R is surjective	$\Delta_B \subset R \diamond R^\dagger$

Proof 8.5.2.1.2 ▶ Proof of Lemma 8.5.2.1.1

Functionality Is Equivalent to $R \diamond R^\dagger \subset \Delta_B$

The condition $R \diamond R^\dagger \subset \Delta_B$ unwinds to

($\star$) For each $b, b' \in B$, if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$.

Since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, the condition says that $a \sim_R b$ and $a \sim_R b'$ imply $b = b'$. This is precisely the condition for R to be functional.

Totality Is Equivalent to $\Delta_A \subset R^\dagger \diamond R$

The condition $\Delta_A \subset R^\dagger \diamond R$ unwinds to

($\star$) For each $a, a' \in A$, if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$.

Since $b \sim_{R^\dagger} a'$ is the same as $a' \sim_R b$, the condition says that for each $a \in A$, there is some $b \in B$ with $b \in R(a)$, so $R(a) \neq \emptyset$. This is precisely the condition for R to be total.

Injectivity Is Equivalent to $R^\dagger \diamond R \subset \Delta_A$

The condition $R^\dagger \diamond R \subset \Delta_A$ unwinds to

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$, then $a = a'$.

Since $b \sim_{R^\dagger} a'$ is the same as $a' \sim_R b$, the condition says that for each $b \in B$, if $a \sim_R b$ and $a' \sim_R b$, then $a = a'$. This is precisely the condition for R to be injective.

Surjectivity Is Equivalent to $\Delta_B \subset R \diamond R^\dagger$

The condition $\Delta_B \subset R \diamond R^\dagger$ unwinds to

($\star$) For each $b, b' \in B$, if $b = b'$, then there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$.

Since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, the condition says that for each $b \in B$, there is some $a \in A$ with $b \in R(a)$, so $R^{-1}(b) \neq \emptyset$. This is precisely the condition for R to be surjective.

Proposition 8.5.2.1.3 ▶ Isomorphisms and Equivalences in Rel

The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

$$
\begin{aligned}
R^{-1} \diamond R &\cong \Delta_A, \\
R \diamond R^{-1} &\cong \Delta_B.
\end{aligned}
$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$
\begin{aligned}
R^{-1} \diamond R &= \Delta_A, \\
R \diamond R^{-1} &= \Delta_B.
\end{aligned}
$$

3. There exists a bijection $f\colon A \xrightarrow{\sim} B$ with $R = \mathrm{Gr}(f)$.

Proof 8.5.2.1.4 ▶ Proof of Proposition 8.5.2.1.3

We claim that Items 1 to 3 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.

- *Item 2* $\implies$ *Item 3:* We proceed in a few steps:
  - First, note that the equalities in Item 2 imply $R \dashv R^{-1}$ and thus, by Proposition 8.5.3.1.1, there exists a function $f_R \colon A \to B$ associated to R.
  - By Lemma 8.5.2.1.1, f_R is a bijection.
- *Item 3* $\implies$ *Item 2:* By Item 4 of Proposition 8.2.2.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\begin{aligned}\Delta_A &\subset f^{-1} \diamond \mathrm{Gr}(f),\\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \Delta_B.\end{aligned}$$

  If f is bijective, then the reverse inclusions are also true by Lemma 8.5.2.1.1.

This finishes the proof. ∎

8.5.3 Internal Adjunctions

Let A and B be sets.

Proposition 8.5.3.1.1 ▶ Adjunctions in Rel

We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof 8.5.3.1.2 ▶ Proof of Proposition 8.5.3.1.1

We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations
$$R\colon A \nrightarrow B,$$
$$S\colon B \nrightarrow A,$$
together with inclusions
$$\Delta_A \subset S \diamond R,$$
$$R \diamond S \subset \Delta_B.$$
By Lemma 8.5.2.1.1, R is total and functional. In particular, $R(a)$ is a singleton for all $a \in A$. Defining f_R such that $f_R(a)$ is the unique element of $R(a)$ then gives us our desired function, forming a map
$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\}.$$
Moreover, by uniqueness of adjoints (**??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 4 of Proposition 8.2.2.1.2, every function $f\colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map
$$\left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This follows form the fact that we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S} \colon A \to B$, we have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &= R, \\ f_{R,S}^{-1} &= S. \end{aligned}$$

We check these explicitly:

- $\mathrm{Gr}(f_{R,S}) = R$. We have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &\stackrel{\text{def}}{=} \{(a, f_{R,S}(a)) \in A \times B \mid a \in A\} \\ &\stackrel{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$

- $f_{R,S}^{-1} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* We proceed in a few steps.
    * Since $\Delta_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
    * Since $a \sim_R b$ and R is functional, we have $k = b$.
    * Thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* We proceed in a few steps.
    * First note that, since R is total, we have $a \sim_R b'$ for some $b' \in B$.
    * Since $R \diamond S \subset \Delta_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$.
    * Thus $a \sim_R b$.

  Having show this, we now have

$$f_{R,S}^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f_{R,S}(a) = b\}$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\}\\
&= \{a \in A \mid b \sim_S a\}\\
&\overset{\text{def}}{=} S(b).
\end{aligned}
$$

for each $b \in B$, and thus $f^{-1}_{R,S} = S$.

This finishes the proof.

8.5.4 Internal Monads

Let X be a set.

Proposition 8.5.4.1.1 ▶ Internal Monads in Rel

We have a natural identification[1]

$$\left\{\begin{matrix}\text{Monads in}\\ \textbf{Rel} \text{ on } X\end{matrix}\right\} \cong \{\text{Preorders on } X\}.$$

[1]See also **??** for an extension of this correspondence to "relative monads in **Rel**".

Proof 8.5.4.1.2 ▶ Proof of Proposition 8.5.4.1.1

A monad in **Rel** on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$
\begin{aligned}
\mu_R &\colon R \diamond R \subset R,\\
\eta_R &\colon \Delta_X \subset R
\end{aligned}
$$

making the diagrams

$$
\begin{tikzcd}
\Delta_X \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R"] \arrow[dr, equal, "\lambda^{\textbf{Rel}}_R"'] & R \diamond R \arrow[d, "\mu_R"]\\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \diamond (R \diamond R) \arrow[dr, "\mathrm{id}_R \diamond \mu_R"] & \\
(R \diamond R) \diamond R \arrow[ur, equal, "\alpha^{\textbf{Rel}}_{R,R,R}"] \arrow[d, "\mu_R \diamond \mathrm{id}_R"'] & & R \diamond R \arrow[d, "\mu_R"]\\
R \diamond R \arrow[rr, "\mu_R"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \Delta_X \arrow[r, "\mathrm{id}_R \diamond \eta_R"] \arrow[dr, equal, "\rho^{\textbf{Rel}}_R"'] & R \diamond R \arrow[d, "\mu_R"]\\
 & R
\end{tikzcd}
$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $x, z \in X$, if there exists some $y \in Y$ such that $x \sim_R y$ and $y \sim_R z$, then $x \sim_R z$.
2. For each $x \in X$, we have $x \sim_R x$.

These are exactly the requirements for R to be a preorder (**??**). Conversely, any preorder $\preceq$ gives rise to a pair of maps $\mu_{\preceq}$ and $\eta_{\preceq}$, forming a monad on X.

Example 8.5.4.1.3 ▶ Codensity Monads in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. The codensity monad $\mathrm{Ran}_R(R)\colon B \nrightarrow B$ is given by

$$[\mathrm{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b)$$

$$\begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(R)", "\shortmid" marking] \\ A \arrow[ur, "R", "\shortmid" marking] \arrow[r, "R"', "\shortmid" marking] \arrow[ur, phantom, "\Swarrow", shift left=3] & B \end{tikzcd}$$

   for each $b \in B$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Ran}_R(R)}\colon B \times B \to \{\mathsf{t}, \mathsf{f}\}$$

   on B obtained by declaring $b \preceq_{\mathrm{Ran}_R(R)} b'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a \sim_R b'$.

   (b) We have $R^{-1}(b) \subset R^{-1}(b')$.

2. The dual codensity monad $\mathrm{Rift}_R(R)\colon A \nrightarrow A$ is given by

$$[\mathrm{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\}$$

$$\begin{tikzcd} & A \arrow[d, "R", "\shortmid"{marking}] \arrow[dl, phantom, "\Swarrow"] \\ A \arrow[ur, dashed, "\mathrm{Rift}_R(R)", "\shortmid"{marking}] \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

for each $a \in A$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Rift}_R(R)}\colon A \times A \to \{\mathsf{t}, \mathsf{f}\}$$

on A obtained by declaring $a \preceq_{\mathrm{Rift}_R(R)} a'$ iff the following equivalent conditions are satisfied:

(a) For each $a \in A$, if $a \sim_R b$, then $a' \sim_R b$.

(b) We have $R(a') \subset R(a)$.

8.5.5 Internal Comonads

Let X be a set.

Proposition 8.5.5.1.1 ▶ Internal Comonads in Rel

We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in} \\ \textsf{Rel} \text{ on } X\end{matrix}\right\} \cong \{\text{Subsets of } X\}.$$

Proof 8.5.5.1.2 ▶ Proof of Proposition 8.5.5.1.1

A comonad in **Rel** on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\Delta_R\colon R \subset R \diamond R,$$
$$\epsilon_R\colon R \subset \Delta_X$$

making the diagrams

$$
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\mathsf{Rel},-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\
& \Delta_X \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & \\
R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \diamond (R \diamond R) \arrow[d, equal, "\alpha^{\mathsf{Rel},-1}_{R,R,R}"'] \\
R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\mathsf{Rel},-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\
& R \diamond \Delta_X
\end{tikzcd}
$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $x, y \in X$, if $x \sim_R y$, then there exists some $k \in X$ such that $x \sim_R k$ and $k \sim_R y$.

2. For each $x, y \in X$, if $x \sim_R y$, then $x = y$.

The second condition implies that $R \subset \Delta_X$, so R must be a subset of X. Taking $k = y$ in the first condition above then shows it to be trivially satisfied. Conversely, any subset U of X satisfies $U \subset \Delta_X$, defining a comonad as above.

Example 8.5.5.1.3 ▶ Density Comonads in Rel

Let $f\colon A \to B$ be a function.

1. The density comonad $\mathrm{Lan}_f(f)\colon B \nrightarrow B$ is given by

$$
[\mathrm{Lan}_f(f)](b) = \bigcup_{a \in f^{-1}(b)} f(a)
\qquad
\begin{tikzcd}
& B \arrow[d, dashed, "\mathrm{Lan}_f(f)", "\shortmid"{marking}] \\
A \arrow[ur, "f", "\shortmid"{marking}] \arrow[r, "f"', "\shortmid"{marking}] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & B
\end{tikzcd}
$$

for each $b \in B$. Thus, it corresponds to the image $\mathrm{Im}(f)$ of f as a subset of B.

2. The dual density comonad $\mathrm{Lift}_{f^{\dagger}}\left(f^{\dagger}\right): A \nrightarrow A$ is given by

$$\left[\mathrm{Lift}_{f^{\dagger}}\left(f^{\dagger}\right)\right](b) = \bigcup_{a \in f^{-1}(b)} f(a)$$

$$\begin{tikzcd} & B \arrow[d, "f^{\dagger}", "\shortmid"{marking}] \\ B \arrow[ur, dashed, "\mathrm{Lift}_{f^{\dagger}}\left(f^{\dagger}\right)", "\times"{marking}] \arrow[r, "f^{\dagger}"', "\shortmid"{marking}] & A \arrow[u, phantom, shift right=3, "\Uparrow"] \end{tikzcd}$$

for each $b \in B$. Thus, it also corresponds to the image $\mathrm{Im}(f)$ of f as a subset of B.

8.5.6 Modules Over Internal Monads

Let A be a set.

Proposition 8.5.6.1.1 ▶ Modules Over Internal Monads in Rel

Let $\preceq_A$ be a preorder on A, viewed also as an internal monad on A via Proposition 8.5.4.1.1.

1. *Left Modules.* We have a natural identification

$$\{\text{Left modules over } \preceq_A\} \cong \left\{\begin{array}{l}\text{Relations } R\colon B \nrightarrow A \text{ such that,}\\ \text{for each } b \in B\text{, the set } R(b) \text{ is}\\ \text{upward-closed in } A\end{array}\right\}.$$

2. *Right Modules.* We have a natural identification

$$\{\text{Right modules over } \preceq_A\} \cong \left\{\begin{array}{l}\text{Relations } R\colon A \nrightarrow B \text{ such that,}\\ \text{for each } b \in B\text{, the set } R^{-1}(b) \text{ is}\\ \text{downward-closed in } A\end{array}\right\}.$$

3. *Bimodules.* We have a natural identification

$$\{\text{Bimodules over } \preceq_A\} \cong \left\{\begin{array}{l}\text{Quadruples } (B, C, R, S) \text{ such that:}\\ 1.\ \text{For each } b \in B\text{, the set } R(b) \text{ is}\\ \quad\ \text{upward-closed in } A.\\ 2.\ \text{For each } c \in C\text{, the set } S^{-1}(c) \text{ is}\\ \quad\ \text{downward-closed in } A.\end{array}\right\}.$$

Proof 8.5.6.1.2 ▶ Proof of Proposition 8.5.6.1.1

Item 1: Left Modules

A left module over $\preceq_A$ in **Rel** consists of a relation $R \colon B \nrightarrow A$ together with an inclusion

$$\alpha_B \colon \preceq_A \diamond R \subset R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and hence all that is left is the data of the inclusion α_B. This corresponds to the following condition:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $b \sim_R a$ and $a \preceq_a a'$, then $b \sim_R a'$.

This condition is equivalent to $R(b)$ being downward-closed for all $b \in B$.

Item 2: Right Modules

The proof is dual to Item 1, and is therefore omitted.

Item 3: Bimodules

Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (Categories, Item 4 of Proposition 11.2.7.1.2), and hence a bimodule is just a left module along with a right module. □

8.5.7 Comodules Over Internal Comonads

Let A be a set.

Proposition 8.5.7.1.1 ▶ Comodules Over Internal Comonads in Rel

Let U be a subset of A, viewed also as an internal comonad on A via Proposition 8.5.5.1.1.

1. *Left Comodules.* We have a natural identification

$$\{\text{Left comodules over } U\} \cong \left\{\begin{matrix}\text{Relations } R \colon B \nrightarrow A \text{ such that,} \\ \text{for each } b \in B \text{, we have } R(b) \subset U\end{matrix}\right\}.$$

2. *Right Comodules.* We have a natural identification

$$\{\text{Right comodules over } U\} \cong \left\{\begin{matrix}\text{Relations } R\colon A \nrightarrow B \text{ such that,}\\ \text{for each } b \in B, \text{ we have } R^{-1}(b) \subset U\end{matrix}\right\}.$$

3. *Bicomodules.* We have a natural identification

$$\{\text{Bicomodules over } U\} \cong \left\{\begin{aligned}&\quad\text{Quadruples } (B, C, R, S) \text{ such that:}\\ &\text{1. For each } b \in B, \text{ we have } R(b) \subset U\\ &\text{2. For each } c \in C, \text{ we have } S^{-1}(c) \subset U\end{aligned}\right\}.$$

Proof 8.5.7.1.2 ▶ Proof of Proposition 8.5.7.1.1

Item 1: Left Comodules

A left comodule over U in **Rel** consists of a relation $R\colon B \nrightarrow A$ together with an inclusion

$$R \subset U \diamond R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

(⋆) For each $b \in B$, if $b \sim_R a$, then there exists some $a' \in A$ such that $b \sim_R a'$ and $a' \sim_U a$.

Since $a' \sim_U a$ is true if $a = a'$ and $a \in U$, this condition ends up being equivalent to $R(b) \subset U$.

Item 2: Right Comodules

A right comodule over U in **Rel** consists of a relation $R\colon A \nrightarrow B$ together with an inclusion

$$R \subset R \diamond U$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

(⋆) For each $a \in A$, if $a \sim_R b$, then there exists some $x \in A$ such that $a \sim_U x$ and $x \sim_R b$.

Since $a \sim_U x$ is true if $a = x$ and $a \in U$, this condition ends up being equivalent to $R^{-1}(b) \subset U$.

Item 3: Bicomodules

Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (Categories, Item 4 of Proposition 11.2.7.1.2), and hence a bicomodule is just a left comodule along with a right comodule.

8.5.8 Eilenberg–Moore and Kleisli Objects

Let X be a set.

Proposition 8.5.8.1.1 ▶ Eilenberg–Moore and Kleisli Objects in Rel

Let R be a preorder on X, viewed as an internal monad on X via Proposition 8.5.4.1.1.

1. *Eilenberg–Moore Objects in* **Rel**. The Eilenberg–Moore object for R exists iff it is an equivalence relation, in which case it is the quotient $X/{\sim_R}$ of X by R.

2. *Kleisli Objects in* **Rel**. [...]

Proof 8.5.8.1.2 ▶ Proof of Proposition 8.5.8.1.1

Omitted.

8.5.9 Co/Monoids

Remark 8.5.9.1.1 ▶ Co/Monoids in Rel

The monoids in **Rel** with respect to the Cartesian monoidal structure of Proposition 8.3.3.8.1 are called *hypermonoids*, and their theory is explored in ??. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in ??.

8.5.10 Monomorphisms

In this section we characterise the epimorphisms in the category Rel, following ??, ??.

Proposition 8.5.10.1.1 ▶ Characterisations of Monomorphisms in Rel

Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is a monomorphism in Rel.
2. The direct image function
$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
3. The codirect image function
$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that
$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$
then $a = a'$.

Proof 8.5.10.1.2 ▶ Proof of Proposition 8.5.10.1.1

First Proof of the Equivalence of Items 1 to 3

Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 8.7.1.1.4. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

  By Remark 8.7.1.1.3, we have

$$\begin{aligned} R_!(U) &= R \diamond U, \\ R_!(V) &= R \diamond V. \end{aligned}$$

  Now, if $R \diamond U = R \diamond V$, i.e. $R_!(U) = R_!(V)$, then $U = V$ since R is assumed to be a monomorphism, showing $R_!$ to be injective.

- *Item 2* $\implies$ *Item 1:* Conversely, suppose that $R_!$ is injective, consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

  and suppose that $R \diamond S = R \diamond T$. Note that, since $R_!$ is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

  if $R_!(U) = R \diamond U = R \diamond V = R_!(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

  where we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

  for each $x \in X$. Thus $S = T$ and R is a monomorphism.

Second Proof of the Equivalence of Items 1 to 3

A more abstract proof can also be given, following [`MSE 350788`]:

- *Item 1* $\implies$ *Item 2:* Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to $R_!$ by Remark 8.7.1.1.3.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by Limits and Colimits, **??** of **??**, it follows by **??**, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to $R_!$, it follows that $R_!$ is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that $R_!$ is injective.
- *Item 2* $\implies$ *Item 1:* Assume that $R_!$ is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to $R_!$ by Remark 8.7.1.1.3.
  - Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that $R_!$ is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??**, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since $R_!$ is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to $R_!$, it follows that R is also a monomorphism.

Proof of the Second Half of Proposition 8.5.10.1.1

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[a]}", shift left, "\shortmid"{marking}] \arrow[r, "{[a']}"', shift right, "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B. \end{tikzcd}$$

Then:

- Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$.
- Similarly, $\star \sim_{R \diamond [a']} b$.
- Thus $R \diamond [a] = R \diamond [a']$.
- Since R is a monomorphism, we have $[a] = [a']$, so $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$

then $a = a'$.

consider the diagram

$$X \overset{S}{\underset{T}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $x \in X$ and $a \in A$ such that $x \sim_S a$.

- Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$.
- In this case, we have $x \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $x \sim_{R \diamond T} b$.
- Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$.
- However, since $a \sim_R b$ and $a' \sim_R b$, we must have $a = a'$.
- Thus $x \sim_T a$ as well.
- A similar argument shows that if $x \sim_T a$, then $x \sim_S a$.
- Thus $S = T$ and it follows that R is a monomorphism.

This finishes the proof. ∎

8.5.11 2-Categorical Monomorphisms

In this section we characterise (for now, some of) the 2-categorical monomorphisms in **Rel**, following Types of Morphisms in Bicategories, Section 14.1.

Proposition 8.5.11.1.1 ▶ 2-Categorical Monomorphisms in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a representably faithful morphism.

2. *Representably Full Morphisms in* **Rel**. The following conditions are equivalent:

   (a) The morphism $R\colon A \nrightarrow B$ is a representably full morphism.

   (b) For each pair of relations $S, T\colon X \rightrightarrows A$, the following condition is satisfied:

   (★) If $R \diamond S \subset R \diamond T$, then $S \subset T$.

   (c) The direct image functor
   $$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
   of Item 2 of Proposition 8.7.1.1.6 is full.

   (d) The codirect image functor
   $$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
   of Item 2 of Proposition 8.7.4.1.6 is full.

   (e) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

   (f) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

3. *Representably Fully Faithful Morphisms in* **Rel**. Every representably full morphism in **Rel** is a representably fully faithful morphism.

Proof 8.5.11.1.2 ▶ Proof of Proposition 8.5.11.1.1

Item 1: Representably Faithful Morphisms in Rel

The relation R is a representably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R_! \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

is faithful, i.e. iff the morphism

$$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$. However, $\mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T)$ is either empty or a singleton, in either case of which the map $R_{*|S,T}$ is necessarily injective.

Item 2: Representably Full Morphisms in Rel

We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a representably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

  $$R_! \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

  is full, i.e. iff the morphism

  $$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$, i.e. iff, whenever $R \diamond S \subset R \diamond T$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2e:* This is also simply a matter of unwinding definitions: The functor

  $$R_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

  $$R_{*|U,V} \colon \mathrm{Hom}_{\mathcal{P}(A)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(B)}(R_!(U), R_!(V))$$

  is surjective, i.e. iff whenever $R_!(U) \subset R_!(V)$, we also necessarily have $U \subset V$.

- *Item 2d* $\iff$ *Item 2f:* This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2e given above.

- *Item 2e* $\implies$ *Item 2f:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  – Suppose we have $U, V \in \mathcal{P}(A)$ with $R_*(U) \subset R_*(V)$.
  – By Constructions With Relations, **??** of **??**, we have

  $$\begin{aligned} R_*(U) &= B \setminus R_!(A \setminus U), \\ R_*(V) &= B \setminus R_!(A \setminus V). \end{aligned}$$

  – By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 we have $R_!(A \setminus V) \subset R_!(A \setminus U)$.
  – By assumption, we then have $A \setminus V \subset A \setminus U$.
  – By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2e:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  – Suppose we have $U, V \in \mathcal{P}(A)$ with $R_!(U) \subset R_!(V)$.

- By Constructions With Relations, **??** of **??**, we have

$$\begin{aligned}R_!(U) &= B \setminus R_*(A \setminus U),\\ R_!(V) &= B \setminus R_*(A \setminus V).\end{aligned}$$

- By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 we have $R_*(A \setminus V) \subset R_*(A \setminus U)$.
- By assumption, we then have $A \setminus V \subset A \setminus U$.
- By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\Longrightarrow$ *Item 2e:* Consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

if $R_!(U) = R \diamond U \subset R \diamond V = R_!(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2e* $\Longrightarrow$ *Item 2b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

By Remark 8.7.1.1.3, we have

$$R_!(U) = R \diamond U,$$
$$R_!(V) = R \diamond V.$$

Now, if $R_!(U) \subset R_!(V)$, i.e. $R \diamond U \subset R \diamond V$, then $U \subset V$ by assumption.

Item 3: Representably Fully Faithful Morphisms in Rel

This follows from Items 1 and 2.

Question 8.5.11.1.3 ▶ Better Characterisations of Representably Full Morphisms in Rel

Item 2 of Proposition 8.5.11.1.1 gives a characterisation of the representably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

8.5.12 Epimorphisms

In this section we characterise the epimorphisms in the category Rel, following ??, ??.

Proposition 8.5.12.1.1 ▶ Characterisations of Epimorphisms in Rel

Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is an epimorphism in Rel.
2. The weak inverse image function
$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.
3. The strong inverse image function
$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.

4. The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":

   ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is total and an epimorphism, then it satisfies the following equivalent conditions:

1. For each $b \in B$, there exists some $a \in A$ such that $a \sim_R b$.
2. We have $\mathrm{Im}(R) = B$.

Proof 8.5.12.1.2 ▶ Proof of Proposition 8.5.12.1.1

First Proof of the Equivalence of Items 1 to 3

Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 8.7.2.1.4. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

  $$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Constructions With Relations, **??**, we have

  $$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\implies$ *Item 1:* Conversely, suppose that R^{-1} is injective, consider the diagram

  $$A \overset{R}{\nrightarrow} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

  and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

  $$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

for each $x \in X$. Thus $S = T$ and R is an epimorphism.

Second Proof of the Equivalence of Items 1 to 3

A more abstract proof can also be given, following [`MSE 350788`]:

- *Item 1* $\Longrightarrow$ *Item 2:* Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt})\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Constructions With Relations, **??**.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by Limits and Colimits, **??** of **??**, it follows by **??**, **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathsf{op}}$ to monomorphisms in Sets.
  - The monomorphisms $\mathrm{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??**, **??** of **??**.
  - Since R is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R^{-1} is injective.
- *Item 2* $\Longrightarrow$ *Item 1:* Assume that R^{-1} is injective.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt})\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Constructions With Relations, **??**.

- Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R^{-1} is a monomorphism.
- Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??**, **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects monomorphisms.
- That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathsf{op}}$.
- The monomorphisms $\mathrm{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??**, **??** of **??**.
- Since R^{-1} is a monomorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

Proof of the Equivalence of Items 2 and 4

We claim that Items 2 and 4 are equivalent, following [MO 350788]:

- *Item 2* $\Longrightarrow$ *Item 4:* We proceed in two steps.
  - Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$.
  - Taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$, we obtain an element of A such that $R(a) = \{b\}$.
- *Item 4* $\Longrightarrow$ *Item 2:* We proceed in a few steps.
  - Let $U, V \subset B$ with $U \neq V$.
  - Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V.
  - Let then $b \in U \setminus V$.
  - By assumption, there exists an $a \in A$ with $R(a) = \{b\}$.
  - Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, so $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Proof of the Second Half of Proposition 8.5.12.1.1

Finally, we prove the second part of the statement. Assume R is a total epimorphism in Rel and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} \{0,1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$.

- Since R is total, we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ for all $a \in A$, and no element of A is related to 1 by $S \diamond R$ or $T \diamond R$.
- Thus $S \diamond R = T \diamond R$.
- Since R is an epimorphism, we have $S = T$.
- But by the definition of T, this implies $\mathrm{Im}(R) = B$.

This finishes the proof. □

8.5.13 2-Categorical Epimorphisms

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following Types of Morphisms in Bicategories, Section 14.2.

Proposition 8.5.13.1.1 ▶ 2-Categorical Epimorphisms in Rel

Let $R \colon A \nrightarrow B$ be a relation.

1. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a corepresentably faithful morphism.
2. *Corepresentably Full Morphisms in* **Rel**. The following conditions are equivalent:
   (a) The morphism $R \colon A \nrightarrow B$ is a corepresentably full morphism.

(b) For each pair of relations $S, T \colon X \nrightarrow\!\!\!\!\nrightarrow A$, the following condition is satisfied:

($\star$) If $S \diamond R \subset T \diamond R$, then $S \subset T$.

(c) The functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full.

(d) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

(e) The functor

$$R_{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full.

(f) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

3. *Corepresentably Fully Faithful Morphisms in* **Rel**. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

Proof 8.5.13.1.2 ▶ Proof of Proposition 8.5.13.1.1

Item 1: Corepresentably Faithful Morphisms in Rel

The relation R is a corepresentably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^{*} \colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R^{*}_{S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$. However, $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, in either case of which the map $R^{*}_{S,T}$ is necessarily injective.

Item 2: Corepresentably Full Morphisms in Rel

We claim Items 2a to 2f are indeed equivalent:

- *Item 2a $\iff$ Item 2b:* This is simply a matter of unwinding definitions: The relation R is a corepresentably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathsf{Rel})$, the functor

$$R^* \colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is full, i.e. iff the morphism

$$R^*_{S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$, i.e. iff, whenever $S \diamond R \subset T \diamond R$, we also have $S \subset T$.

- *Item 2c $\iff$ Item 2d:* This is also simply a matter of unwinding definitions: The functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

$$R^{-1}_{U,V} \colon \mathrm{Hom}_{\mathcal{P}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(A)}\Big(R^{-1}(U), R^{-1}(V)\Big)$$

is surjective, i.e. iff whenever $R^{-1}(U) \subset R^{-1}(V)$, we also necessarily have $U \subset V$.

- *Item 2e $\iff$ Item 2f:* This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d $\implies$ Item 2f:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

- Suppose we have $U, V \in \mathcal{P}(B)$ with $R_{-1}(U) \subset R_{-1}(V)$.
- By Constructions With Relations, ?? of ??, we have

$$R_{-1}(U) = B \setminus R^{-1}(A \setminus U),$$
$$R_{-1}(V) = B \setminus R^{-1}(A \setminus V).$$

- By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 we have $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$.
- By assumption, we then have $A \setminus V \subset A \setminus U$.
- By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\Longrightarrow$ *Item 2d:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  - Suppose we have $U, V \in \mathcal{P}(B)$ with $R^{-1}(U) \subset R^{-1}(V)$.
  - By Constructions With Relations, ?? of ??, we have

$$R^{-1}(U) = B \setminus R_{-1}(A \setminus U),$$
$$R^{-1}(V) = B \setminus R_{-1}(A \setminus V).$$

  - By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 we have $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$.
  - By assumption, we then have $A \setminus V \subset A \setminus U$.
  - By Constructions With Sets, Item 1 of Proposition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\implies$ *Item 2d:* Consider the diagram

$$A \overset{R}{\nrightarrow} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form

$$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\nrightarrow} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\nrightarrow} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2e* $\implies$ *Item 2b:* Let $U, V \in \mathcal{P}(B)$ and consider the diagram

$$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

By Remark 8.7.1.1.3, we have

$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$

Now, if $R^{-1}(U) \subset R^{-1}(V)$, i.e. $U \diamond R \subset V \diamond R$, then $U \subset V$ by assumption.

Item 3: Corepresentably Fully Faithful Morphisms in **Rel**

This follows from Items 1 and 2. ∎

Question 8.5.13.1.3 ▶ Better Characterisations of Corepresentably Full Morphisms in Rel

Item 2 of Proposition 8.5.13.1.1 gives a characterisation of the corepresentably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

8.5.14 Co/Limits

Proposition 8.5.14.1.1 ▶ Co/Limits in Rel

This will be properly written later on.

Proof 8.5.14.1.2 ▶ Proof of Proposition 8.5.14.1.1

Omitted.

8.5.15 Internal Left Kan Extensions

Proposition 8.5.15.1.1 ▶ Internal Left Kan Extensions in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Internal Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Internal Left Kan Extensions Along Them.* The following conditions are equivalent:

   (a) The left Kan extension

   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 8.2.2.1.1) for some function f.

Proof 8.5.15.1.2 ▶ Proof of Proposition 8.5.15.1.1

Item 1: Non-Existence of All Internal Left Kan Extensions in Rel

By Item 2, it suffices to take a relation that doesn't have a left adjoint.

Item 2: Characterisation of Relations Admitting Left Kan Extensions

This proof is mostly due to Tim Campion, via [MO 460693].

- We may view precomposition

$$- \diamond R \colon \mathrm{Rel}(B, C) \to \mathrm{Rel}(A, C)$$

  with $R \colon A \nrightarrow B$ as a cocontinuous functor from $\mathcal{P}(B \times C)$ to $\mathcal{P}(A \times C)$ (via Item 5 of Definition 8.1.1.1.1).

- By the adjoint functor theorem (**??**), this map has a left adjoint iff it preserves limits.

- If $C = \emptyset$, this holds trivially.

- Otherwise, C admits pt as a retract, and we reduce to the case $C = \mathrm{pt}$ via **??**.

- For the case $C = \mathrm{pt}$, a relation $T \colon B \nrightarrow \mathrm{pt}$ is the same as a subset of B, and $- \diamond R$ becomes the weak inverse image functor R^{-1} of Section 8.7.3.

- Now, again by the adjoint functor theorem, R^{-1} preserves limits exactly when it has a left adjoint.

- Finally R^{-1} has a left adjoint precisely when $R = \mathrm{Gr}(f)$ for f a function (Item 8 of Proposition 8.7.3.1.4).

This finishes the proof. □

Example 8.5.15.1.3 ▶ Internal Left Kan Extensions Along Functions

Given a function $f \colon A \to B$, the left Kan extension

$$\mathrm{Lan}_f \colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

along f exists by Item 2 of Proposition 8.5.15.1.1. Explicitly, given a relation $R\colon A \nrightarrow X$, the left Kan extension

$$\mathrm{Lan}_f(R)\colon B \nrightarrow X, \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Lan}_f(R)"] \\ A \arrow[ur, "f"] \arrow[r, "R"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & X \end{tikzcd}$$

may be described as follows:

1. We declare $b \sim_{\mathrm{Lan}_f(R)} x$ iff there exists some $a \in R$ such that $b = f(a)$ and $a \sim_R x$.

2. We have[1]
$$[\mathrm{Lan}_f(R)](b) = \bigcup_{a \in f^{-1}(b)} R(a)$$
for each $b \in B$.

[1]Cf. Item 3 of Proposition 8.5.17.1.2.

Remark 8.5.15.1.4 ▶ Illustrating the Failure of Internal Left Kan Extensions in Rel to Exist

Following Example 8.5.15.1.3, given a relation $R\colon A \nrightarrow B$ and a relation $F\colon A \nrightarrow X$, we could perhaps try to define an "honorary" left Kan extension
$$\mathrm{Lan}'_R(F)\colon B \nrightarrow X$$
by
$$\left[\mathrm{Lan}'_F(F)\right](b) \overset{\mathrm{def}}{=} \bigcup_{a \in R^{-1}(b)} F(a)$$
for each $b \in B$.
The failure of $\mathrm{Lan}'_R(F)$ to be a Kan extension can then be seen as follows. Let $G\colon B \nrightarrow X$ be a relation. If $\mathrm{Lan}'_R(F)$ were a left Kan extension, then the following conditions ***would be*** equivalent:

1. For each $b \in B$, we have $\bigcup_{a \in R^{-1}(b)} F(a) \subset G(b)$.

2. For each $a \in A$, we have $F(a) \subset \bigcup_{b \in R(a)} G(b)$.

The issue is two-fold:

- *Totality.* If R isn't total, then the implication Item 1 $\Rightarrow$ Item 2 fails.
- *Functionality.* If R isn't functional, then the implication Item 2 $\Rightarrow$ Item 1 fails.

Question 8.5.15.1.5 ▶ Existence of Specific Internal Left Kan Extensions of Relations

Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension[1]

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

[1] Specifically for R and S, not Lan_S the functor.

8.5.16 Internal Left Kan Lifts

Proposition 8.5.16.1.1 ▶ Internal Left Kan Lifts in Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Internal Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Internal Left Kan Lifts Along Them.* The following conditions are equivalent:
   (a) The left Kan lift
   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X, B) \to \mathbf{Rel}(X, A)$$
   along R exists.
   (b) The relation R admits a right adjoint in **Rel**.
   (c) The relation R is of the form f^{-1} (as in Definition 8.2.3.1.1) for some function f.

Proof 8.5.16.1.2 ▶ Proof of Proposition 8.5.16.1.1

Item 1: Non-Existence of All Internal Left Kan Lifts in Rel

By Item 2, it suffices to take a relation that doesn't have a right adjoint.

Item 2: Characterisation of Relations Admitting Left Kan Lifts Along

This proof is dual to that of Item 2 of Proposition 8.5.15.1.1, and is therefore omitted.

Example 8.5.16.1.3 ▶ Internal Left Kan Lifts Along Functions

Given a function $f\colon A \to B$, the left Kan lift

$$\mathrm{Lift}_{f^{\dagger}}\colon \mathbf{Rel}(X,A) \to \mathbf{Rel}(X,B)$$

along $f^{\dagger}$ exists by Item 2 of Proposition 8.5.16.1.1. Explicitly, given a relation $R\colon X \nrightarrow A$, the left Kan lift

$$\mathrm{Lift}_{f^{\dagger}}(R)\colon X \nrightarrow B, \qquad \begin{tikzcd} & B \arrow[d, "f^{\dagger}", "|" marking] \\ X \arrow[ur, dashed, "\mathrm{Lift}_{f^{\dagger}}(R)", "|" marking] \arrow[r, "R"', "|" marking] & A. \arrow[ul, phantom, "\Uparrow"] \end{tikzcd}$$

is given by

$$\begin{aligned} [\mathrm{Lift}_{f}(R)](x) &= [\mathrm{Gr}(f) \diamond R](a) \\ &= \bigcup_{a \in R(x)} f(a) \end{aligned}$$

for each $x \in X$.

Question 8.5.16.1.4 ▶ Existence of Specific Internal Left Kan Lifts of Relations

Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan lift[1]

$$\mathrm{Lift}_{S}(R)\colon X \nrightarrow A$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

[1]Specifically for R and S, not Lift_S the functor.

8.5.17 Internal Right Kan Extensions

Let A, B, and X be sets and let $R\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} B$ and $F\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} X$ be relations.

Motivation 8.5.17.1.1 ▶ Setting for Internal Right Kan Extensions in Rel

We want to understand internal right Kan extensions in **Rel**, which look like this:

$$\begin{tikzcd}
 & B \arrow[d, dashed, "\mathrm{Ran}_R(F)", "\mapstochar" marking] \arrow[dl, phantom, shift left=2ex, "\Swarrow"] \\
A \arrow[ur, "R", "\times" marking] \arrow[r, "F"', "\mapstochar" marking] & X.
\end{tikzcd}$$

Note in particular here that $F\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} X$ is a relation from A to X. These will form a functor

$$\mathrm{Ran}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

that is right adjoint to the precomposition by R functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X).$$

Proposition 8.5.17.1.2 ▶ Internal Right Kan Extensions in Rel

The internal right Kan extension of F along R is the relation $\mathrm{Ran}_R(F)$ described as follows:

1. Viewing relations from B to X as subsets of $B \times X$, we have

$$\mathrm{Ran}_R(F) = \left\{ (b, x) \in B \times X \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, if } a \sim_R b, \\ \text{then we have } a \sim_F x \end{array} \right\}.$$

2. Viewing relations as functions $B \times X \to \{\mathsf{true}, \mathsf{false}\}$, we have

$$\begin{aligned} (\mathrm{Ran}_R(F))^{-1}_{-2} &= \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, F^{-1}_a\right) \\ &= \bigwedge_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, F^{-1}_a\right), \end{aligned}$$

where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

3. Viewing relations as functions $B \to \mathcal{P}(X)$, we have

$$\mathrm{Ran}_R(F) = \mathrm{Ran}_{\chi'_A}(F) \circ R^{-1}, \qquad \begin{tikzcd} & A \arrow[r, "F"] \arrow[d, hook, "\chi_A"'] & \mathcal{P}(X), \\ B \arrow[r, "R^{-1}"'] & \mathcal{P}(A)^{\mathsf{op}} \arrow[ur, bend right, "\mathrm{Ran}_{\chi_A}(F)"'] \arrow[ur, phantom, "\Leftarrow"] & \end{tikzcd}$$

where $\mathrm{Ran}_{\chi'_B}(F)$ is computed by the formula

$$\begin{aligned}\left[\mathrm{Ran}_{\chi'_A}(F)\right](V) &\cong \int_{a\in A} \chi_{\mathcal{P}(A)^{\mathsf{op}}}(V, \chi_a) \pitchfork F(a)\\ &\cong \int_{a\in A} \chi_{\mathcal{P}(A)}(\chi_a, V) \pitchfork F(a)\\ &\cong \int_{a\in A} \chi_V(a) \pitchfork F(a)\\ &\cong \bigcap_{a\in A} \chi_V(a) \pitchfork F(a)\\ &\cong \bigcap_{a\in V} F(a)\end{aligned}$$

for each $V \in \mathcal{P}(B)$, so we have

$$[\mathrm{Ran}_R(F)](b) = \bigcap_{a\in R^{-1}(b)} F(a)$$

for each $b \in B$.

Proof 8.5.17.1.3 ▶ Proof of Proposition 8.5.17.1.2

We have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}(A,X)}(F \diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((F\diamond R)^x_a, T^x_a)\\ &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} F^x_b \times R^b_a\right), T^x_a\right)\end{aligned}$$

$$
\begin{aligned}
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(F^x_b\times R^b_a, T^x_a\Big)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(F^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, T^x_a\Big)\Big)\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(F^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, T^x_a\Big)\Big)\\
&\cong \int_{b\in B}\int_{x\in X}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^x_b, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, T^x_a\Big)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(F, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(R^{-2}_a, T^{-1}_a\big)\right)
\end{aligned}
$$

naturally in each $F\in\mathbf{Rel}(B,X)$ and each $T\in\mathbf{Rel}(A,X)$, showing that

$$\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^{-2}_a, T^{-1}_a\Big)$$

is right adjoint to the precomposition functor $-\diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Proposition 8.1.1.1.7.
2. Definition 8.1.3.1.1.
3. Ends and Coends, **??** of **??**.
4. Sets, Proposition 3.2.2.1.5.
5. Ends and Coends, **??** of **??**.
6. Ends and Coends, **??** of **??**.
7. Relations, Item 1 of Proposition 8.1.1.1.7.

This finishes the proof. □

Example 8.5.17.1.4 ▶ Examples of Internal Right Kan Extensions of Relations

Here are some examples of internal right Kan extensions of relations.

1. *Orthogonal Complements.* Let $A=B=X=\mathcal{V}$ be an inner product space, and let $R=F=\perp$ be the orthogonality relation,

so that we have

$$R(v) = v^{\perp}$$
$$F(u) = u^{\perp},$$

for each $u, v \in \mathcal{V}$, where

$$v^{\perp} \stackrel{\text{def}}{=} \{u \in V \mid v \perp u\}$$

is the orthogonal complement of v. The right Kan extension $\mathrm{Ran}_R(F)$ is then given by

$$\begin{aligned}[\mathrm{Ran}_R(F)](v) &= \bigcap_{u \in R^{-1}(v)} F(u)\\ &= \bigcap_{\substack{u \in V\\ u \perp v}} u^{\perp}\\ &= \left(v^{\perp}\right)^{\perp},\end{aligned}$$

the double orthogonal complement. In particular:

- If $\mathcal{V}$ is finite-dimensional, then $[\mathrm{Ran}_R(F)](v) = \mathrm{Span}(v)$.
- If $\mathcal{V}$ is a Hilbert space, then $[\mathrm{Ran}_R(F)](v) = \overline{\mathrm{Span(v)}}$.

2. *Galois Connections and Closure Operators.* Let:

   - $B = X = (P, \preceq_P)$ and $A = (Q, \preceq_Q)$ be posets;
   - (f, g) be a Galois connection (adjunction) between P and Q;
   - $R, F \colon Q \nrightarrow P$ be the relations defined by

     $$R(q) \stackrel{\text{def}}{=} \{p \in P \mid q \preceq_Q f(p)\},$$
     $$F(q) \stackrel{\text{def}}{=} \{p \in P \mid p \preceq_P g(q)\}$$

     for each $q \in Q$.

   We have

   $$[\mathrm{Ran}_R(F)](p) = \bigcap_{q \in R^{-1}(p)} F(q)$$

$$
\begin{aligned}
&= \bigcap_{\substack{q \in Q \\ q \preceq_Q f(p)}} \{p \in P \mid p \preceq_P g(q)\} \\
&= \{p \in P \mid p \preceq_P g(f(q))\} \\
&= \downarrow g(f(p)),
\end{aligned}
$$

the down set of $g(f(p))$. In other words, $\mathrm{Ran}_R(F)$ is the closure operator on P associated with the Galois connection (f, g).

Proposition 8.5.17.1.5 ▶ Properties of Internal Right Kan Extensions in Rel

Let A, B, C and X be sets and let $R \colon A \nrightarrow B$, $S \colon B \nrightarrow C$, and $F \colon A \nrightarrow X$ be relations.

1. *Functoriality.* The assignments $R, F, (R, F) \mapsto \mathrm{Ran}_R(F)$ define functors

$$
\begin{aligned}
\mathrm{Ran}_{(-)}(F) \colon &\quad \mathbf{Rel}(A, B)^{\mathsf{op}} \to \mathbf{Rel}(B, X), \\
\mathrm{Ran}_R \colon &\quad \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X), \\
\mathrm{Ran}_{(-_1)}(-_2) \colon &\quad \mathbf{Rel}(A, X) \times \mathbf{Rel}(A, B)^{\mathsf{op}} \to \mathbf{Rel}(B, X).
\end{aligned}
$$

   In other words, given relations

$$
A \overset{R_1}{\underset{R_2}{\rightrightarrows}} B \qquad A \overset{F_1}{\underset{F_2}{\rightrightarrows}} X,
$$

   if $R_1 \subset R_2$ and $F_1 \subset F_2$, then $\mathrm{Ran}_{R_2}(F_1) \subset \mathrm{Ran}_{R_1}(F_2)$.

2. *Interaction With Composition.* We have

$$
\mathrm{Ran}_{S \diamond R}(F) = \mathrm{Ran}_S(\mathrm{Ran}_R(F))
$$

   and an equality

$$
\begin{tikzcd}
 & & C \arrow[dd, dashed, "\mathrm{Ran}_S(\mathrm{Ran}_R(F))"] \arrow[dl, phantom, "\Downarrow"] \\
 & B \arrow[ur, "S"] \arrow[dr, dashed, "\mathrm{Ran}_R(F)"'] \arrow[d, phantom, "\Downarrow"] & \\
A \arrow[ur, "R"] \arrow[rr, "F"'] & {} & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & C \arrow[dd, dashed, "\mathrm{Ran}_{S \diamond R}(F)"] \arrow[ddl, phantom, "\Downarrow"] \\
 & B \arrow[ur, "S"] & \\
A \arrow[ur, "R"] \arrow[rr, "F"'] & {} & X
\end{tikzcd}
$$

   of pasting diagrams in **Rel**.

3. *Interaction With Converses.* We have

$$\mathrm{Ran}_R(F)^\dagger = \mathrm{Rift}_{R^\dagger}\left(F^\dagger\right).$$

4. *Interaction With Weak Inverse Images.* We have

$$[\mathrm{Ran}_R(F)]^{-1}(x) = \left\{b \in B \;\middle|\; R^{-1}(b) \subset F^{-1}(x)\right\}$$

for each $x \in X$.

Proof 8.5.17.1.6 ▶ Proof of Proposition 8.5.17.1.5

Item 1: Functoriality

We have

$$\begin{aligned}[\mathrm{Ran}_{R_2}(F_1)](b) &= \bigcap_{a \in R_2^{-1}(b)} F_1(a) \\ &\subset \bigcap_{a \in R_1^{-1}(b)} F_1(a) \\ &\subset \bigcap_{a \in R_1^{-1}(b)} F_2(a) \\ &= [\mathrm{Ran}_{R_1}(F_2)](b)\end{aligned}$$

for each $b \in B$, so we therefore have $\mathrm{Ran}_{R_2}(F_1) \subset \mathrm{Ran}_{R_1}(F_2)$.

Item 2: Interaction With Composition

This holds in a general bicategory with the necessary right Kan extensions, being therefore a special case of **??**.

Item 3: Interaction With Converses

We have

$$\begin{aligned}\left[\mathrm{Rift}_{R^\dagger}\left(F^\dagger\right)\right](x) &= \left\{b \in B \;\middle|\; R^\dagger(b) \subset F^\dagger(x)\right\} \\ &= \left\{b \in B \;\middle|\; R^{-1}(b) \subset F^{-1}(x)\right\} \\ &= \mathrm{Ran}_R(F)^{-1}(x) \\ &= \mathrm{Ran}_R(F)^\dagger(x)\end{aligned}$$

where we have used Proposition 8.5.18.1.2 and Item 4.

Item 4: Interaction With Weak Inverse Images

We proceed in a few steps.

- We have $b \in [\mathrm{Ran}_R(F)]^{-1}(x)$ iff, for each $a \in R^{-1}(b)$, we have $b \in F(a)$.
- This holds iff, for each $a \in R^{-1}(b)$, we have $a \in F^{-1}(b)$.
- This holds iff $R^{-1}(b) \subset F^{-1}(b)$.

This finishes the proof.

8.5.18 Internal Right Kan Lifts

Let A, B, and X be sets and let $R\colon A \nrightarrow B$ and $F\colon X \nrightarrow B$ be relations.

Motivation 8.5.18.1.1 ▶ Setting for Internal Right Kan Lifts in Rel

We want to understand internal right Kan lifts in **Rel**, which look like this:

$$\begin{tikzcd} & A \arrow[d, "R", "\mid"'] \\ X \arrow[ur, dashed, "\mathrm{Rift}_R(F)"] \arrow[r, "F"', "\mid"] & B. \arrow[ul, phantom, "\Swarrow"] \end{tikzcd}$$

Note in particular here that $F\colon B \nrightarrow X$ is a relation from B to X. These will form a functor

$$\mathrm{Rift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$

that is right adjoint to the postcomposition by R functor

$$R_*\colon \mathbf{Rel}(X,A) \to \mathbf{Rel}(X,B).$$

Proposition 8.5.18.1.2 ▶ Internal Right Kan Lifts in Rel

The internal right Kan lift of F along R is the relation $\mathrm{Rift}_R(F)$ described as follows:

1. Viewing relations from X to A as subsets of $X \times A$, we have

$$\mathrm{Rift}_R(F) = \left\{ (x,a) \in X \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B, \text{ if } a \sim_R b, \\ \text{then we have } x \sim_F b \end{array} \right\}.$$

2. Viewing relations as functions $X \times A \to \{\mathsf{true}, \mathsf{false}\}$, we have

$$\begin{aligned}(\mathrm{Rift}_R(F))^{-1}_{-2} &= \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_{-1}, F^b_{-2}\Big) \\ &= \bigwedge_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_{-1}, F^b_{-2}\Big),\end{aligned}$$

   where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

3. Viewing relations as functions $X \to \mathcal{P}(A)$, we have

$$[\mathrm{Rift}_R(F)](x) = \{a \in A \mid R(a) \subset F(x)\}$$

   for each $a \in A$.

Proof 8.5.18.1.3 ▶ Proof of Proposition 8.5.18.1.2

We have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond F, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big((R \diamond F)^b_x, T^b_x\Big) \\ &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a \in A} R^b_a \times F^a_x\right), T^b_x\right) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a \times F^a_x, T^b_x\Big) \\ &\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(F^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, T^b_x\Big)\Big) \\ &\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(F^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, T^b_x\Big)\Big) \\ &\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^a_x, \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, T^b_x\Big)\right)\end{aligned}$$

$$\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(F, \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{b}_{-_1}, T^{b}_{-_2}\right)\right)$$

naturally in each $F \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{b}_{-_1}, F^{b}_{-_2}\right)$$

is right adjoint to the postcomposition functor $R\diamond-$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Proposition 8.1.1.1.7.
2. Definition 8.1.3.1.1.
3. Ends and Coends, **??** of **??**.
4. Sets, Proposition 3.2.2.1.5.
5. Ends and Coends, **??** of **??**.
6. Ends and Coends, **??** of **??**.
7. Relations, Item 1 of Proposition 8.1.1.1.7.

This finishes the proof.

Example 8.5.18.1.4 ▶ Examples of Internal Right Kan Extensions of Relations

Here are some examples of internal right Kan lifts of relations.

1. *Pullbacks.* Let $p\colon A \to B$ and $f\colon X \to B$ be functions. We have

$$\begin{aligned}\left[\mathrm{Rift}_{\mathrm{Gr}(p)}(\mathrm{Gr}(f))\right](x) &= \{a \in A \mid [\mathrm{Gr}(p)](a) \subset [\mathrm{Gr}(f)](x)\}\\ &= \{a \in A \mid p(a) = f(x)\}.\end{aligned}$$

   Thus, as a subset of $X \times A$, the right Kan lift $\mathrm{Rift}_{\mathrm{Gr}(p)}(\mathrm{Gr}(f))$ corresponds precisely to the pullback $X \times_B A$ of X and A along p and f of Constructions With Sets, Section 4.1.4.

PROPOSITION 8.5.18.1.5 ▶ PROPERTIES OF INTERNAL RIGHT KAN LIFTS IN Rel

Let A, B, C and X be sets and let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $F\colon X \nrightarrow B$ be relations.

1. *Functoriality.* The assignments $R, F, (R, F) \mapsto \mathrm{Rift}_R(F)$ define functors

$$\begin{aligned} \mathrm{Rift}_{(-)}(F)\colon & \qquad \mathbf{Rel}(A,B)^{\mathsf{op}} && \to \mathbf{Rel}(B,X), \\ \mathrm{Rift}_R\colon & \qquad \mathbf{Rel}(A,X) && \to \mathbf{Rel}(B,X), \\ \mathrm{Rift}_{(-_1)}(-_2)\colon & \mathbf{Rel}(A,X) \times \mathbf{Rel}(A,B)^{\mathsf{op}} && \to \mathbf{Rel}(B,X). \end{aligned}$$

   In other words, given relations

$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \qquad\qquad A \underset{F_2}{\overset{F_1}{\rightrightarrows}} X,$$

   if $R_1 \subset R_2$ and $F_1 \subset F_2$, then $\mathrm{Rift}_{R_2}(F_1) \subset \mathrm{Rift}_{R_1}(F_2)$.

2. *Interaction With Composition.* We have

$$\mathrm{Rift}_{S \diamond R}(F) = \mathrm{Rift}_R(\mathrm{Ran}_S(F))$$

   and an equality

$$\begin{tikzcd} & A \arrow[d, "R"] \\ & B \arrow[d, "S"] \arrow[ul, phantom] \\ X \arrow[r, "F"'] \arrow[uur, dashed, "\mathrm{Rift}_R(\mathrm{Rift}_S(F))"] \arrow[ur, dashed, "\mathrm{Rift}_S(F)"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} & A \arrow[d, "R"] \\ & B \arrow[d, "S"] \\ X \arrow[r, "F"'] \arrow[uur, dashed, "\mathrm{Rift}_{S \diamond R}(F)"] & C \end{tikzcd}$$

   of pasting diagrams in **Rel**.

3. *Interaction With Converses.* We have

$$\mathrm{Rift}_R(F)^{\dagger} = \mathrm{Ran}_{R^{\dagger}}\left(F^{\dagger}\right).$$

4. *Interaction With Weak Inverse Images.* We have

$$\mathrm{Rift}_R(F)^\dagger = \mathrm{Ran}_{\chi'_B}\left(F^\dagger\right) \circ R,$$

$$\begin{tikzcd} & B \arrow[r, "F^\dagger"] \arrow[d, hook, "\chi_B"'] \arrow[d, phantom, shift left=3, "\Swarrow"] & \mathcal{P}(X), \\ A \arrow[r, "R"'] & \mathcal{P}(B)^{\mathsf{op}} \arrow[ur, bend right, "\mathrm{Ran}_{\chi_A}\left(F^{-1}\right)"'] & \end{tikzcd}$$

where $\mathrm{Ran}_{\chi_A}\left(F^\dagger\right)$ is computed by the formula

$$\begin{aligned}\left[\mathrm{Ran}_{\chi_A}\left(F^\dagger\right)\right](U) &\cong \int_{a\in A} \chi_{\mathcal{P}(B)^{\mathsf{op}}}(U, \chi_a) \pitchfork F^\dagger(a) \\ &\cong \int_{a\in A} \chi_{\mathcal{P}(B)}(\chi_a, U) \pitchfork F^{-1}(a) \\ &\cong \int_{a\in A} \chi_U(a) \pitchfork F(a) \\ &\cong \bigcap_{a\in A} \chi_U(a) \pitchfork F(a) \\ &\cong \bigcap_{a\in U} F(a)\end{aligned}$$

for each $U \in \mathcal{P}(A)$, so we have

$$[\mathrm{Rift}_R(F)]^{-1}(a) = \bigcap_{b\in R(a)} F^{-1}(b)$$

for each $a \in A$.

PROOF 8.5.18.1.6 ▶ PROOF OF PROPOSITION 8.5.18.1.5

Item 1: Functoriality

We have

$$\begin{aligned}[\mathrm{Rift}_{R_2}(F_1)](x) &= \{a \in A \mid R_2(a) \subset F_1(x)\} \\ &\subset \{a \in A \mid R_1(a) \subset F_1(x)\} \\ &\subset \{a \in A \mid R_1(a) \subset F_2(x)\} \\ &= \mathrm{Rift}_{R_1}(F_2)\end{aligned}$$

for each $x \in X$, so we therefore have $\mathrm{Rift}_{R_2}(F_1) \subset \mathrm{Rift}_{R_1}(F_2)$.

Item 2: Interaction With Composition

This holds in a general bicategory with the necessary right Kan lifts, being therefore a special case of **??**.

Item 3: Interaction With Converses

This follows from Item 3 of Proposition 8.5.17.1.5 by duality.

Item 4: Interaction With Weak Inverse Images

We proceed in a few steps.

- We have $x \in \mathrm{Rift}_R(F)^\dagger(a)$ iff $a \in \mathrm{Rift}_R(F)(x)$.
- This holds iff $R(a) \subset F(x)$.
- This holds iff, for each $b \in R(a)$, we have $b \in F(x)$.
- This holds iff, for each $b \in R(a)$, we have $x \in F^{-1}(b)$.
- This holds iff $x \in \bigcap_{b \in R(a)} F^{-1}(b)$.

This finishes the proof.

8.5.19 Closedness

Proposition 8.5.19.1.1 ▶ Closedness of Rel

The 2-category **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_! \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

Proof 8.5.19.1.2 ▶ Proof of Proposition 8.5.19.1.1

This follows from Constructions With Relations, ????. ∎

8.5.20 Rel as a Category of Free Algebras

Proposition 8.5.20.1.1 ▶ Rel as a Category of Free Algebras

We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}_!}(\mathsf{Sets}),$$

where $\mathcal{P}_!$ is the powerset monad of ??, ??.

Proof 8.5.20.1.2 ▶ Proof of Proposition 8.5.20.1.1

Omitted. ∎

8.6 Properties of the 2-Category of Relations With Apartness Composition

8.6.1 Self-Duality

Proposition 8.6.1.1.1 ▶ Self-Duality for the (2-)Category of Relations With Apartness Composition

The 2-/category of relations with apartness-composition-is self-dual:

1. *Self-Duality I.* We have an isomorphism

$$\left(\mathsf{Rel}^{\square}\right)^{\mathsf{op}} \cong \mathsf{Rel}^{\square}$$

of categories.

2. *Self-Duality II.* We have a 2-isomorphism

$$\left(\mathbf{Rel}^{\square}\right)^{\mathsf{op}} \cong \mathbf{Rel}^{\square}$$

of 2-categories.

Proof 8.6.1.1.2 ▶ Proof of Proposition 8.6.1.1.1

Item 1: Self-Duality I

We claim that the functor

$$(-)^{\dagger}\colon \left(\mathsf{Rel}^{\square}\right)^{\mathsf{op}} \to \mathsf{Rel}^{\square}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories. Note that this is indeed a functor by Items 4 and 7 of Proposition 8.1.5.1.3.

By Categories, Item 1 of Proposition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\dagger}\colon \mathrm{Rel}(A, B) \to \mathrm{Rel}(B, A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Proposition 8.1.5.1.3, showing $(-)^{\dagger}$ to be fully faithful.

Item 2: Self-Duality II

We claim that the 2-functor

$$(-)^{\dagger}\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Proposition 8.1.5.1.3, is an isomorphism of categories.

By **??**, it suffices to show that $(-)^{\dagger}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Proposition 8.1.5.1.3.

Thus $(-)^{\dagger}$ is indeed a 2-isomorphism of categories. ∎

8.6.2 Isomorphisms and Equivalences

Let $R\colon A \nrightarrow B$ be a relation from A to B, and recall that $R^{\mathsf{c}} \overset{\text{def}}{=} B \times A \setminus R$.

Lemma 8.6.2.1.1 ▶ Conditions Involving a Relation and Its Converse II

The conditions below are row-wise equivalent:

Condition	Inclusion
R^{c} is functional	$\nabla_B \subset R \mathbin{\square} R^{\dagger}$
R^{c} is total	$R \mathbin{\square} R^{\dagger} \subset \nabla_A$
R^{c} is injective	$\nabla_A \subset R^{\dagger} \mathbin{\square} R$
R^{c} is surjective	$R^{\dagger} \mathbin{\square} R \subset \nabla_B$

Proof 8.6.2.1.2 ▶ Proof of Lemma 8.6.2.1.1

This follows from Lemma 8.5.2.1.1 and Item 4 of Proposition 8.1.4.1.3. For instance:

- Suppose we have $R \mathbin{\square} R^{\dagger} \subset \nabla_B$.
- Taking complements, we obtain $\nabla_B^{\mathsf{c}} \subset \left(R \mathbin{\square} R^{\dagger}\right)^{\mathsf{c}}$.
- Applying Item 4 of Proposition 8.1.4.1.3, this becomes $\Delta_B \subset R^{\mathsf{c}} \diamond \left(R^{\dagger}\right)^{\mathsf{c}}$.
- Then, by Lemma 8.5.2.1.1, this is equivalent to R^{c} being total.

The proof of the other equivalences is similar, and thus omitted. ∎

Remark 8.6.2.1.3 ▶ Unwinding Lemma 8.6.2.1.1

The statements in Lemma 8.6.2.1.1 unwind to the following:

Inclusion	Quantifier	Condition
$\nabla_B \subset R \mathbin{\Box} R^{\dagger}$	For each $b_1, b_2 \in B$	If $b_1 \neq b_2$, then, for each $a \in A$, we have $a \sim_R b_1$ or $a \sim_R b_2$.
$R \mathbin{\Box} R^{\dagger} \subset \nabla_B$	For each $b_1, b_2 \in B$	If, for each $a \in A$, $a \sim_R b_1$ or $a \sim_R b_2$, then $b_1 \neq b_2$.
$\nabla_A \subset R^{\dagger} \mathbin{\Box} R$	For each $a_1, a_2 \in A$	If $a_1 \neq a_2$, then, for each $b \in B$, we have $a_1 \sim_R b$ or $a_2 \sim_R b$.
$R^{\dagger} \mathbin{\Box} R \subset \nabla_A$	For each $a_1, a_2 \in A$	If, for each $b \in B$, $a_1 \sim_R b$ or $a_2 \sim_R b$, then $a_1 \neq a_2$.

Equivalently:

Inclusion	Quantifier	If	Then
$\nabla_B \subset R \mathbin{\Box} R^{\dagger}$	For each $b_1, b_2 \in B$	$b_1 \neq b_2$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$
$R \mathbin{\Box} R^{\dagger} \subset \nabla_B$	For each $b_1, b_2 \in B$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$	$b_1 \neq b_2$
$\nabla_A \subset R^{\dagger} \mathbin{\Box} R$	For each $a_1, a_2 \in A$	$a_1 \neq a_2$	$R(a_1) \cup R(a_2) = B$
$R^{\dagger} \mathbin{\Box} R \subset \nabla_A$	For each $a_1, a_2 \in A$	$R(a_1) \cup R(a_2) = B$	$a_1 \neq a_2$

Proposition 8.6.2.1.4 ▶ Isomorphisms and Equivalences in $\mathbf{Rel}^{\Box}$

The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in $\mathbf{Rel}^{\Box}$, i.e.:

   ($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

$$R^{-1} \mathbin{\Box} R \cong \nabla_A,$$
$$R \mathbin{\Box} R^{-1} \cong \nabla_B.$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

($\star$) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$
\begin{aligned}
R^{-1} \mathbin{\Box} R &= \nabla_A,\\
R \mathbin{\Box} R^{-1} &= \nabla_B.
\end{aligned}
$$

3. There exists a bijection $f\colon B \overset{\sim}{\dashrightarrow} A$ with $R^{\mathsf{c}} = f^{-1}$.

Proof 8.6.2.1.5 ▶ Proof of Proposition 8.6.2.1.4

This follows from Proposition 8.5.2.1.3 and Item 4 of Proposition 8.1.4.1.3. ∎

8.6.3 Internal Adjunctions

Let A and B be sets.

Proposition 8.6.3.1.1 ▶ Adjunctions in $\mathsf{Rel}^{\Box}$

We have a natural bijection

$$
\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel}^{\Box}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions}\\ \text{from } B \text{ to } A\end{matrix}\right\},
$$

with every adjunction in $\mathsf{Rel}^{\Box}$ being of the form $(f^{-1})^{\mathsf{c}} \dashv \mathrm{Gr}(f)^{\mathsf{c}}$ for some function $f\colon B \to A$.

Proof 8.6.3.1.2 ▶ Proof of Proposition 8.6.3.1.1

This follows from Proposition 8.5.3.1.1 and Item 4 of Proposition 8.1.4.1.3. ∎

8.6.4 Internal Monads

Let X be a set.

Proposition 8.6.4.1.1 ▶ Internal Monads in $\mathsf{Rel}^{\square}$

We have a natural identification

$$\left\{\begin{matrix}\text{Monads in}\\ \mathsf{Rel}^{\square}\text{ on }X\end{matrix}\right\} \cong \{\text{Subsets of }X\}.$$

Proof 8.6.4.1.2 ▶ Proof of Proposition 8.6.4.1.1

This follows from Proposition 8.6.4.1.1 and Item 4 of Proposition 8.1.4.1.3.

8.6.5 Internal Comonads

Let X be a set.

Proposition 8.6.5.1.1 ▶ Internal Comonads in $\mathsf{Rel}^{\square}$

We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \mathsf{Rel}^{\square}\text{ on }X\end{matrix}\right\} \cong \{\text{Strict total orders on }X\}.$$

Proof 8.6.5.1.2 ▶ Proof of Proposition 8.6.5.1.1

A comonad in $\mathsf{Rel}^{\square}$ on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\Delta_R\colon R &\subset R \mathbin{\square} R,\\ \epsilon_R\colon R &\subset \nabla_X\end{aligned}$$

making the diagrams

$$
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\mathsf{Rel}^{\square},-1}_{R}"'] & R \mathbin{\square} R \arrow[d, "\epsilon_R \square \mathrm{id}_R", "\shortmid"{marking}] \\
& \nabla_X \mathbin{\square} R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \mathbin{\square} R \arrow[dr, "\mathrm{id}_R \square \Delta_R", "\shortmid"{marking}] & \\
R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \mathbin{\square} (R \mathbin{\square} R) \arrow[d, equal, "\alpha^{\mathsf{Rel}^{\square},-1}_{R,R,R}"] \\
R \mathbin{\square} R \arrow[rr, "\Delta_R \square \mathrm{id}_R"', "\shortmid"{marking}] & & (R \mathbin{\square} R) \mathbin{\square} R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\mathsf{Rel}^{\square},-1}_{R}"'] & R \mathbin{\square} R \arrow[d, "\mathrm{id}_R \square \epsilon_R", "\shortmid"{marking}] \\
& R \mathbin{\square} \nabla_X
\end{tikzcd}
$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $x, z \in X$, if $x \sim_R z$, then, for each $y \in X$, we have $x \sim_R y$ or $y \sim_R z$.

2. For each $x, y \in X$, if $x \sim_R y$, then $x \neq y$.

Replacing $\sim_R$ with $<_R$ and taking the contrapositive of each condition, we obtain:

1. For each $x, z \in X$, if there exists some $y \in X$ such that $x <_R y$ and $y <_R z$, then $x <_R z$.

2. For each $x \in X$, we have $x \nless_R x$.

These are exactly the requirements for R to be a strict linear order (**??**). Conversely, any strict linear order $<_R$ gives rise to a pair of maps $\Delta_{<_R}$ and $\epsilon_{<_R}$, forming a comonad on X.

EXAMPLE 8.6.5.1.3 ▶ DENSITY COMONADS IN Rel

Let $R\colon A \nrightarrow B$ be a relation.

1. The codensity monad $\mathrm{Ran}_R(R)\colon B \nrightarrow B$ is given by

$$[\mathrm{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b) \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(R)"] \\ A \arrow[ur, "R"] \arrow[r, "R"'] & B \end{tikzcd}$$

   for each $b \in B$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Ran}_R(R)}\colon B \times B \to \{\mathsf{t}, \mathsf{f}\}$$

   on B obtained by declaring $b \preceq_{\mathrm{Ran}_R(R)} b'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a \sim_R b'$.

   (b) We have $R^{-1}(b) \subset R^{-1}(b')$.

2. The dual codensity monad $\mathrm{Rift}_R(R)\colon A \nrightarrow A$ is given by

$$[\mathrm{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\} \qquad \begin{tikzcd} & A \arrow[d, "R"] \\ A \arrow[ur, dashed, "\mathrm{Rift}_R(R)"] \arrow[r, "R"'] & B \end{tikzcd}$$

   for each $a \in A$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Rift}_R(R)}\colon A \times A \to \{\mathsf{t}, \mathsf{f}\}$$

   on A obtained by declaring $a \preceq_{\mathrm{Rift}_R(R)} a'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a' \sim_R b$.

   (b) We have $R(a') \subset R(a)$.

8.6.6 Modules Over Internal Monads

8.6.7 Comodules Over Internal Comonads

8.6.8 Eilenberg–Moore and Kleisli Objects

8.6.9 Monomorphisms

8.6.10 2-Categorical Monomorphisms

8.6.11 Epimorphisms

8.6.12 2-Categorical Epimorphisms

8.6.13 Co/Limits

This will be expanded later on.

8.6.14 Internal Left Kan Extensions

8.6.15 Internal Left Kan Lifts

8.6.16 Internal Right Kan Extensions

8.6.17 Internal Right Kan Lifts

8.6.18 Coclosedness

8.7 The Adjoint Pairs $R_! \dashv R_{-1}$ and $R^{-1} \dashv R_*$

8.7.1 Direct Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.1.1.1 ▶ Direct Images

The **direct image function associated to** R is the function[1]

$$R_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[2]

$$R_!(U) \overset{\text{def}}{=} \bigcup_{a \in U} R(a)$$

$$= \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\}$$

for each $U \in \mathcal{P}(X)$.

[1] *Further Notation:* Also written simply $R\colon \mathcal{P}(X) \to \mathcal{P}(Y)$.
[2] *Further Terminology:* The set $R(U)$ is called the **direct image of U by R**.

Warning 8.7.1.1.2 ▶ Notation for Direct Images Is Confusing

Notation for direct images between powersets is tricky; see Constructions With Sets, Warning 4.6.1.1.3. Here we'll try to align our notation for relations with that for functions.

Remark 8.7.1.1.3 ▶ Unwinding Definition 8.7.1.1.1

Identifying subsets of X with relations from pt to X via Constructions With Sets, Item 3 of Proposition 4.4.1.1.4, we see that the direct image function associated to R is equivalently the function

$$R_!\colon \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(\mathrm{pt},X)} \to \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(\mathrm{pt},Y)}$$

defined by

$$R_!(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(X)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\mathrel{+\!\!\!\!\to}} X \overset{R}{\mathrel{+\!\!\!\!\to}} Y.$$

Proposition 8.7.1.1.4 ▶ Properties of Direct Image Functions

Let $R\colon X \mathrel{+\!\!\!\!\to} Y$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(X)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1})\colon\quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by:

(a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow R_{-1} \circ R_!, \\ R_! \circ R_{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset R_{-1}(R_!(U)), \\ R_!(R_{-1}(V)) &\subset V \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$

(b) A bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$. In particular:

($\star$) The following conditions are equivalent:
  - We have $R_!(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_!(U) \cup R_!(V) &= R_!(U \cup V), \\ R_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_!\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_!(U \cap V) &\subset R_!(U) \cap R_!(V), \\ R_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_!(U) \cup R_!(V) &\xrightarrow{=} R_!(U \cup V), \\ R_{*|\mathbb{1}}^{\otimes} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_!(U \cap V) &\subset R_!(U) \cap R_!(V), \\ R_{*|\mathbb{1}}^{\otimes} \colon R_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

7. *Relation to Codirect Images.* We have

$$R_!(U) = Y \setminus R_*(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof 8.7.1.1.5 ▶ Proof of Proposition 8.7.1.1.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Preservation of Colimits

This follows from Item 2 and **??**, **??** of **??**.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Codirect Images

The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 17 of Proposition 4.6.1.1.5): applying Item 7 of Proposition 8.7.4.1.4 to $A \setminus U$, we have

$$\begin{aligned} R_*(X \setminus U) &= Y \setminus R_!(X \setminus (X \setminus U)) \\ &= Y \setminus R_!(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_!(U) &= Y \setminus (Y \setminus R_!(U)), \\ &= Y \setminus R_*(X \setminus U), \end{aligned}$$

which finishes the proof. ∎

Proposition 8.7.1.1.6 ▶ Properties of the Direct Image Function Operation

Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathrm{Rel}(X,Y) \to \mathsf{Sets}(\mathcal{P}(X),\mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathrm{Rel}(X,Y) \to \mathsf{Pos}((\mathcal{P}(X),\subset),(\mathcal{P}(Y),\subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]

$$(\chi_X)_! = \mathrm{id}_{\mathcal{P}(X)}\,.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have[2]

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(Y) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

---

[1]That is, the postcomposition function

$$(\chi_X)_!\colon \mathrm{Rel}(\mathrm{pt},X) \to \mathrm{Rel}(\mathrm{pt},X)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},X)}$.

[2]That is, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt},X) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathrm{Rel}(\mathrm{pt},Y) \arrow[d, "S_!"] \\ & \mathrm{Rel}(\mathrm{pt},C). \end{tikzcd}$$

Proof 8.7.1.1.7 ▶ Proof of Proposition 8.7.1.1.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_X)_!(U) &\overset{\text{def}}{=} \bigcup_{a\in U}\chi_X(a)\\ &\overset{\text{def}}{=} \bigcup_{a\in U}\{a\}\\ &= U\\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(X)}(U)\end{aligned}$$

for each $U\in\mathcal{P}(X)$. Thus $(\chi_X)_! = \mathrm{id}_{\mathcal{P}(X)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S\diamond R)_!(U) &\overset{\text{def}}{=} \bigcup_{a\in U}[S\diamond R](a)\\ &\overset{\text{def}}{=} \bigcup_{a\in U}S(R(a))\\ &\overset{\text{def}}{=} \bigcup_{a\in U}S_!(R(a))\\ &= S_!\left(\bigcup_{a\in U}R(a)\right)\\ &\overset{\text{def}}{=} S_!(R_!(U))\\ &\overset{\text{def}}{=} [S_!\circ R_!](U)\end{aligned}$$

for each $U\in\mathcal{P}(X)$, where we used Item 3 of Proposition 8.7.1.1.4. Thus $(S\diamond R)_! = S_!\circ R_!$. ∎

8.7.2 Strong Inverse Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[1]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in X \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(Y)$.

[1] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 8.7.2.1.2 ▶ Unwinding Definition 8.7.2.1.1

Identifying subsets of Y with relations from pt to Y via Constructions With Sets, Item 3 of Proposition 4.4.1.1.4, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(\mathrm{pt},Y)} \to \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(\mathrm{pt},X)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & X \arrow[d, "R", "\shortmid"{marking}] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\shortmid"{marking}] \arrow[r, "V"', "\shortmid"{marking}] & Y, \arrow[ul, phantom, "\Leftarrow"] \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b\in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{b}_{-1}, V^{b}_{-2}\right), \end{aligned}$$

where we have used **??**.

Proof 8.7.2.1.3 ▶ Proof of Remark 8.7.2.1.2

We have

$$\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, V^b_{-2}\right)\\
&= \left\{a \in X \;\middle|\; \int_{b\in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, V^b_\star\right) = \mathsf{true}\right\}\\
&= \left\{a \in X \;\middle|\; \begin{array}{l}\text{for each } b \in Y\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^b_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^b_a = \mathsf{true}\\ \qquad \text{(b) We have } V^b_\star = \mathsf{true}\end{array}\right\}\\
&= \left\{a \in X \;\middle|\; \begin{array}{l}\text{for each } b \in Y\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } b \notin R(a)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } b \in R(a)\\ \qquad \text{(b) We have } b \in V\end{array}\right\}\\
&= \{a \in X \mid \text{for each } b \in R(a)\text{, we have } b \in V\}\\
&= \{a \in X \mid R(a) \subset V\}\\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}$$

This finishes the proof.

Proposition 8.7.2.1.4 ▶ Properties of Strong Inverse Images

Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(Y)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(Y)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_!(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have inclusions

$$R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$\emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(Y)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(Y) = Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$R_{-1|U,V}^{\otimes}: R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R_{-1|\mathbb{1}}^{\otimes}: \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(Y)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right): (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$R_{-1|U,V}^{\otimes}: R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1|\mathbb{1}}^{\otimes}: R_{-1}(X) \xrightarrow{=} Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = X \setminus R^{-1}(Y \setminus V)$$

for each $V \in \mathcal{P}(Y)$.

8. *Interaction With Weak Inverse Images II.* Let $R\colon X \nrightarrow Y$ be a relation from X to Y.

(a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(Y)$.

(b) If R is total and functional, then the above inclusion is in fact an equality.

(c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

PROOF 8.7.2.1.5 ▶ PROOF OF PROPOSITION 8.7.2.1.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??**, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Weak Inverse Images I

We claim we have an equality

$$R_{-1}(Y \setminus V) = X \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(Y \setminus V) &= \{a \in X \mid R(a) \subset Y \setminus V\}, \\ X \setminus R^{-1}(V) &= \{a \in X \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = Y \setminus V$ then implies the original statement.

Item 8: Interaction With Weak Inverse Images II

Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 8.2.2.1.2.

Proposition 8.7.2.1.6 ▶ Properties of the Strong Inverse Image Function Operation

Let $R\colon X \rightarrow\!\!\!\!\!\mapsto Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}(\mathcal{P}(X),\mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}((\mathcal{P}(X),\subset),(\mathcal{P}(Y),\subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_{-1} = \mathrm{id}_{\mathcal{P}(X)}\,.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \rightarrow\!\!\!\!\!\mapsto Y$ and $S\colon Y \rightarrow\!\!\!\!\!\mapsto C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(Y) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

Proof 8.7.2.1.7 ▶ Proof of Proposition 8.7.2.1.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_X)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in X \mid \chi_X(a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_{-1} = \mathrm{id}_{\mathcal{P}(X)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in X \mid [S \diamond R](a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid S(R(a)) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid S_!(R(a)) \subset U\}\\ &= \{a \in X \mid R(a) \subset S_{-1}(U)\}\\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U))\\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 8.7.2.1.4, which implies that the conditions

- We have $S_!(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

8.7.3 Weak Inverse Images

Let X and Y be sets and let $R \colon X \nrightarrow Y$ be a relation.

Definition 8.7.3.1.1 ▶ Weak Inverse Images

The **weak inverse image function associated to** R[1] is the function

$$R^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[2]

$$R^{-1}(V) \overset{\text{def}}{=} \{a \in X \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(Y)$.

[1] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[2] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Remark 8.7.3.1.2 ▶ Unwinding Definition 8.7.3.1.1

Identifying subsets of Y with relations from Y to pt via Constructions With Sets, Item 3 of Proposition 4.4.1.1.4, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1} \colon \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(Y,\mathrm{pt})} \to \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(X,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \overset{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(X)$, where $R \diamond V$ is the composition

$$X \overset{R}{\nrightarrow} Y \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} V \diamond R \\ &\overset{\text{def}}{=} \int^{b \in Y} V^{-1}_{b} \times R^{b}_{-_2}. \end{aligned}$$

Proof 8.7.3.1.3 ▶ Proof of Remark 8.7.3.1.2

We have

$$V \diamond R \overset{\text{def}}{=} \int^{b \in Y} V^{-1}_{b} \times R^{b}_{-_2}$$

$$
\begin{aligned}
&= \left\{ a \in X \;\middle|\; \int^{b \in Y} V^\star_b \times R^b_a = \mathsf{true} \right\} \\
&= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V^\star_b = \mathsf{true} \\ \quad 2.\ \text{We have } R^b_a = \mathsf{true} \end{array} \right\} \\
&= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \in V \\ \quad 2.\ \text{We have } b \in R(a) \end{array} \right\} \\
&= \{ a \in X \mid \text{there exists } b \in V \text{ such that } b \in R(a) \} \\
&= \{ a \in X \mid R(a) \cap V \neq \emptyset \} \\
&\overset{\text{def}}{=} R^{-1}(V)
\end{aligned}
$$

This finishes the proof. ∎

Proposition 8.7.3.1.4 ▶ Properties of Weak Inverse Image Functions

Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}\colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(Y)$, we have

$$\left[R^{-1}\right](V) \overset{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(Y)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\Big(R^{-1} \dashv R_*\Big)\colon \quad \begin{tikzcd} \mathcal{P}(Y) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X) \arrow[l, bend left, "R_*"] \end{tikzcd},$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}\Big(R^{-1}(U), V\Big) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

($\star$) The following conditions are equivalent:
- We have $R^{-1}(U) \subset V$.
- We have $U \subset R_*(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(Y)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(X) \subset Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = X \setminus R_{-1}(Y \setminus V)$$

for each $V \in \mathcal{P}(Y)$.

8. *Interaction With Strong Inverse Images II.* Let $R \colon X \nrightarrow Y$ be a relation from X to Y.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(Y)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 8.7.3.1.5 ▶ Proof of Proposition 8.7.3.1.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, ?? of ??.

Item 3: Preservation of Colimits

This follows from Item 2 and ??, ?? of ??.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Strong Inverse Images I

This follows from Item 7 of Proposition 8.7.2.1.4.

Item 8: Interaction With Strong Inverse Images II

This was proved in Item 8 of Proposition 8.7.2.1.4.

Proposition 8.7.3.1.6 ▶ Properties of the Weak Inverse Image Function Operation

Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(X,Y) \to \mathsf{Sets}(\mathcal{P}(X),\mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(X,Y) \to \mathsf{Pos}((\mathcal{P}(X),\subset),(\mathcal{P}(Y),\subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]

$$(\chi_X)^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have[2]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(Y) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

[1]That is, the postcomposition

$$(\chi_X)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, X) \to \mathrm{Rel}(\mathrm{pt}, X)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},X)}$.

[2]That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, Y) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, X). \end{tikzcd}$$

Proof 8.7.3.1.7 ▶ Proof of Proposition 8.7.3.1.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, Item 5 of Proposition 11.1.4.1.2.

Item 4: Interaction With Composition

This follows from Categories, Item 2 of Proposition 11.1.4.1.2.

8.7.4 Codirect Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.4.1.1 ▶ Codirect Images

The **codirect image function associated to** R is the function

$$R_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[1,2]

$$\begin{aligned}R_*(U) &\overset{\text{def}}{=} \left\{ b \in Y \;\middle|\; \begin{gathered}\text{for each } a \in X\text{, if we have}\\ b \in R(a)\text{, then } a \in U\end{gathered} \right\}\\ &= \left\{ b \in Y \;\middle|\; R^{-1}(b) \subset U \right\}\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

---

[1] *Further Terminology:* The set $R_*(U)$ is called the **codirect image of** U **by** R.

[2] We also have

$$R_*(U) = Y \setminus R_!(X \setminus U);$$

see Item 7 of Proposition 8.7.4.1.4.

Remark 8.7.4.1.2 ▶ Unwinding Definition 8.7.4.1.1

Identifying subsets of Y with relations from pt to Y via Constructions With Sets, Item 3 of Proposition 4.4.1.1.4, we see that the codirect image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(X,\mathrm{pt})} \to \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(Y,\mathrm{pt})}$$

defined by

$$R_*(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & Y \arrow[d, dashed, "\mathrm{Ran}_R(U)", "+"'] \\ X \arrow[ur, "R", "\times"'] \arrow[r, "U"', "+"] & \mathrm{pt}, \arrow[from=1-2, to=2-1, phantom, "\Downarrow"] \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned}R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U)\\ &\cong \int_{a\in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, U_a^{-1}\right),\end{aligned}$$

where we have used ??.

We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, U_a^{-1}) \\
&= \left\{ b \in Y \;\middle|\; \int_{a\in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^b, U_a^\star\right) = \mathsf{true} \right\} \\
&= \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{for each } a \in X\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R_a^b = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R_a^b = \mathsf{true} \\ \qquad \text{(b) We have } U_a^\star = \mathsf{true} \end{array} \right\} \\
&= \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{for each } a \in X\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(X) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\
&= \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{for each } a \in X\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\
&= \left\{ b \in Y \;\middle|\; R^{-1}(b) \subset U \right\} \\
&\stackrel{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof.

Proposition 8.7.4.1.4 ▶ Properties of Codirect Images

Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(X)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_*\right)\colon \quad \begin{tikzcd} \mathcal{P}(Y) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "R_*"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_*(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_*(U_i) \subset R_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$R_*(U) \cup R_*(V) \subset R_*(U \cup V),$$
$$\emptyset \subset R_*(\emptyset),$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_*(U \cap V) &= R_*(U) \cap R_*(V), \\ R_*(X) &= Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} R_{!|U,V}^{\otimes} &: R_*(U) \cup R_*(V) \subset R_*(U \cup V), \\ R_{!|\mathbb{1}}^{\otimes} &: \emptyset \subset R_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} R_{!|U,V}^{\otimes} &: R_*(U \cap V) \xrightarrow{=} R_*(U) \cap R_*(V), \\ R_{!|\mathbb{1}}^{\otimes} &: R_*(X) \xrightarrow{=} Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

7. *Relation to Direct Images.* We have

$$R_*(U) = Y \setminus R_!(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof 8.7.4.1.5 ▶ Proof of Proposition 8.7.4.1.4

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, **??** of **??**.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and **??**, **??** of **??**.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

This follows from Item 7 of Proposition 8.7.1.1.4. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Constructions With Sets, Item 16 of Proposition 4.6.3.1.7).

We claim that $R_*(U) = Y \setminus R_!(X \setminus U)$:

- *The First Implication.* We claim that

  $$R_*(U) \subset Y \setminus R_!(X \setminus U).$$

  Let $b \in R_*(U)$. We need to show that $b \notin R_!(X \setminus U)$, i.e. that there is no $a \in X \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_*(U)$).

  Thus $b \in Y \setminus R_!(X \setminus U)$.

- *The Second Implication.* We claim that

  $$Y \setminus R_!(X \setminus U) \subset R_*(U).$$

Let $b \in Y \setminus R_!(X \setminus U)$. We need to show that $b \in R_*(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_!(X \setminus U)$, there exists no $a \in X \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_*(U)$.

This finishes the proof. ∎

Proposition 8.7.4.1.6 ▶ Properties of the Codirect Image Function Operation

Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Sets}(X, Y) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(Y) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof 8.7.4.1.7 ▶ Proof of Proposition 8.7.4.1.6

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_X)_*(U) &\stackrel{\text{def}}{=} \left\{a \in X \;\middle|\; \chi_X^{-1}(a) \subset U\right\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_* = \mathrm{id}_{\mathcal{P}(X)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_*(U) &\stackrel{\text{def}}{=} \left\{c \in C \;\middle|\; [S \diamond R]^{-1}(c) \subset U\right\}\\ &\stackrel{\text{def}}{=} \left\{c \in C \;\middle|\; S^{-1}\Big(R^{-1}(c)\Big) \subset U\right\}\\ &= \left\{c \in C \;\middle|\; R^{-1}(c) \subset S_*(U)\right\}\\ &\stackrel{\text{def}}{=} R_*(S_*(U))\\ &\stackrel{\text{def}}{=} [R_* \circ S_*](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 8.7.4.1.4, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_*(U)$.

are equivalent. Thus $(S \diamond R)_* = S_* \circ R_*$. □

8.7.5 Functoriality of Powersets

Proposition 8.7.5.1.1 ▶ Functoriality of Powersets I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors[1]

$$\begin{aligned} \mathcal{P}_! &: \mathrm{Rel} \to \mathsf{Sets}, \\ \mathcal{P}_{-1} &: \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathcal{P}^{-1} &: \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathcal{P}_* &: \mathrm{Rel} \to \mathsf{Sets} \end{aligned}$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathrm{Rel})$, we have

$$\begin{aligned} \mathcal{P}_!(X) &\stackrel{\text{def}}{=} \mathcal{P}(X), \\ \mathcal{P}_{-1}(X) &\stackrel{\text{def}}{=} \mathcal{P}(X), \\ \mathcal{P}^{-1}(X) &\stackrel{\text{def}}{=} \mathcal{P}(X), \\ \mathcal{P}_*(X) &\stackrel{\text{def}}{=} \mathcal{P}(X). \end{aligned}$$

- *Action on Morphisms.* For each morphism $R \colon X \nrightarrow Y$ of Rel, the images

$$\begin{aligned} \mathcal{P}_!(R) &: \mathcal{P}(X) \to \mathcal{P}(Y), \\ \mathcal{P}_{-1}(R) &: \mathcal{P}(Y) \to \mathcal{P}(X), \\ \mathcal{P}^{-1}(R) &: \mathcal{P}(Y) \to \mathcal{P}(X), \\ \mathcal{P}_*(R) &: \mathcal{P}(X) \to \mathcal{P}(Y) \end{aligned}$$

  of R by $\mathcal{P}_!$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ are defined by

$$\begin{aligned} \mathcal{P}_!(R) &\stackrel{\text{def}}{=} R_!, \\ \mathcal{P}_{-1}(R) &\stackrel{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\stackrel{\text{def}}{=} R^{-1}, \\ \mathcal{P}_*(R) &\stackrel{\text{def}}{=} R_*, \end{aligned}$$

  as in Definitions 8.7.1.1.1, 8.7.2.1.1, 8.7.3.1.1 and 8.7.4.1.1.

[1] The functor $\mathcal{P}_! \colon \mathrm{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 2 of Proposition 8.2.2.1.2.

This follows from Items 3 and 4 of Proposition 8.7.1.1.6, Items 3 and 4 of Proposition 8.7.2.1.6, Items 3 and 4 of Proposition 8.7.3.1.6, and Items 3 and 4 of Proposition 8.7.4.1.6.

8.7.6 Functoriality of Powersets: Relations on Powersets

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.6.1.1 ▶ The Relation on Powersets Associated to a Relation

The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$$

defined by[1]

$$\mathcal{P}(R)^{V}_{U} \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\mathrm{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

[1]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"', "|" marking] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}", "|" marking] & X \arrow[r, "R"', "|" marking] & Y \arrow[r, "V"', "|" marking] & \mathrm{pt}. \end{tikzcd}$$

Remark 8.7.6.1.2 ▶ Unwinding Definition 8.7.6.1.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathsf{true}$, i.e. we have

$$\int^{a\in X}\int^{b\in Y} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathsf{true}.$$

- There exists some $a \in X$ and some $b \in Y$ such that:
  - We have $U^{a}_{\star} = \mathsf{true}$.
  - We have $R^{b}_{a} = \mathsf{true}$.
  - We have $V^{\star}_{b} = \mathsf{true}$.

- There exists some $a \in X$ and some $b \in Y$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

Proposition 8.7.6.1.3 ▶ Functoriality of Powersets II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathrm{Rel} \to \mathrm{Rel}.$$

Proof 8.7.6.1.4 ▶ Proof of Proposition 8.7.6.1.3

Omitted. □

8.8 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

8.8.1 The Left Skew Monoidal Product

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 8.8.1.1.1 ▶ The Left J-Skew Monoidal Product of $\mathbf{Rel}(A, B)$

The **left J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\lhd_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\text{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] & B. \\ A \arrow[r, "R"', "\shortmid"{marking}] \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] & B \arrow[u, phantom, shift right=3, "\Swarrow"] \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S,S'\big) \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(R,R'\big) \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S \lhd_J R, S' \lhd_J R'\big)$$

of $\lhd_J$ at $((R,S),(R',S'))$ is defined by[1]

$$\beta \lhd_J \alpha \overset{\mathrm{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\beta \Downarrow"] \arrow[d, "J"] & B \\ A \arrow[ur, dashed, bend left=40, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[ur, phantom, bend left=20, "\mathrm{Rift}_J(\alpha)"] \arrow[r, "R"] \arrow[r, bend right=50, "R'"'] \arrow[r, phantom, bend right=25, "\alpha \Downarrow"] & B & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R')$.

[1] Since $\mathbf{Rel}(A,B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset S' \lhd_J R'$.

8.8.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 8.8.2.1.1 ▶ The Left J-Skew Monoidal Unit of $\mathbf{Rel}(A,B)$

The **left J-skew monoidal unit of $\mathbf{Rel}(A,B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \colon \mathsf{pt} \to \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\lhd_J}_{\mathbf{Rel}(A,B)} \overset{\mathrm{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

8.8.3 The Left Skew Associators

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.3.1.1 ▶ The Left J-Skew Associator of $\mathbf{Rel}(A,B)$

The **left J-skew associator of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J\circ(\triangleleft_J\times\mathsf{id})\Longrightarrow\triangleleft_J\circ(\mathsf{id}\times\triangleleft_J)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & \mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[dr, "\mathsf{id}\times\triangleleft_J"] & \\
(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\wr"'] \arrow[d, "\triangleleft_J\times\mathsf{id}"'] & & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \\
\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[rr, "\triangleleft_J"] \arrow[urr, phantom, "\Nearrow\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\colon\underbrace{(T\triangleleft_J S)\triangleleft_J R}_{\overset{\text{def}}{=}T\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)}\hookrightarrow\underbrace{T\triangleleft_J(S\triangleleft_J R)}_{\overset{\text{def}}{=}T\diamond\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\overset{\text{def}}{=}\mathrm{id}_T\diamond\gamma,$$

where

$$\gamma\colon\ \mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)\hookrightarrow\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S\star\mathrm{id}_{\mathrm{Rift}_J(R)}\colon\ \underbrace{J\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)}_{\overset{\text{def}}{=}J_!(\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R))}\hookrightarrow S\diamond\mathrm{Rift}_J(R)$$

under the adjunction $J_!\dashv\mathrm{Rift}_J$, where $\epsilon\colon J\diamond\mathrm{Rift}_J\Longrightarrow\mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_!\dashv\mathrm{Rift}_J$.

8.8.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.4.1.1 ▶ The Left J-Skew Left Unitor of $\mathbf{Rel}(A,B)$

The **left J-skew left unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathsf{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
 & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \colon \underbrace{J \triangleleft_J R}_{\overset{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J_! \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.5 The Left Skew Right Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.5.1.1 ▶ The Left J-Skew Right Unitor of $\mathbf{Rel}(A,B)$

The **left J-skew right unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\right)$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B)\times\mathsf{pt} \arrow[r, "\mathsf{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\overset{\rho^{\mathbf{Rel}(A,B),\lhd_J}}{\Nearrow}"] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B), \arrow[d, "\lhd_J"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho_R^{\mathbf{Rel}(A,B),\lhd_J}\colon R \hookrightarrow \underbrace{R \lhd_J J}_{\overset{\text{def}}{=} R\diamond\mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} R\diamond\chi_A \\
&\overset{\mathrm{id}_R\diamond\eta_{\chi_A}}{\Longrightarrow} R\diamond\mathrm{Rift}_J(J_!(\chi_A)) \\
&\overset{\text{def}}{=} R\diamond\mathrm{Rift}_J(J\diamond\chi_A) \\
&\overset{\sim}{\Longrightarrow} R\diamond\mathrm{Rift}_J(J) \\
&\overset{\text{def}}{=} R\lhd_J J,
\end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(A,A)} \Longrightarrow \mathrm{Rift}_J\circ J_!$ is the unit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Proposition 8.8.6.1.1 ▶ The Left J-Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

The category $\mathbf{Rel}(A, B)$ admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of **??** of **??**.
- *The Left Skew Monoidal Product.* The left J-skew monoidal

product

$$\lhd_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

of Definition 8.8.1.1.1.

- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J} : \mathsf{pt} \to \mathbf{Rel}(A, B)$$

of Definition 8.8.2.1.1.

- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\lhd_J} : \lhd_J \circ (\lhd_J \times \mathsf{id}) \Longrightarrow \lhd_J \circ (\mathsf{id} \times \lhd_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

of Definition 8.8.3.1.1.

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\lhd_J} : \lhd_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \times \mathsf{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

of Definition 8.8.4.1.1.

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\lhd_J} : \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \lhd_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}\right)$$

of Definition 8.8.5.1.1.

Proof 8.8.6.1.2 ▶ Proof of Proposition 8.8.6.1.1

Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a left skew monoidal category. ∎

8.9 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

8.9.1 The Right Skew Monoidal Product

Definition 8.9.1.1.1 ▶ The Right J-Skew Monoidal Product of $\mathbf{Rel}(A, B)$

The **right J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\triangleright_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mapstochar" marking] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\mapstochar" marking] & B. \\ & A \arrow[u, "J", "\mapstochar" marking] \arrow[ur, "S"', "\mapstochar" marking] \arrow[u, phantom, shift left=4, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S, S'\big) \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(R, R'\big) \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}\big(S \triangleright_J R, S' \triangleright_J R'\big)$$

  of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[1]

$$\beta \triangleright_J \alpha \overset{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'", "\mapstochar" marking] \arrow[r, bend right, "R"', "\mapstochar" marking] \arrow[r, phantom, "\Uparrow\alpha"] & B \arrow[r, dashed, bend left, "\mathrm{Ran}_J(S')", "\mapstochar" marking] \arrow[r, dashed, "\mathrm{Ran}_J(S)"', "\mapstochar" marking] \arrow[r, phantom, shift left=3, "\mathrm{Ran}_J(\beta)\Uparrow"] & B \\ & A \arrow[u, "J", "\mapstochar" marking] \arrow[ur, "S'", "\mapstochar" marking] \arrow[ur, bend right, "S"', "\mapstochar" marking] \arrow[ur, phantom, shift right=3, "\Leftarrow\beta"] & \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[1]Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

8.9.2 The Right Skew Monoidal Unit

Definition 8.9.2.1.1 ▶ The Right J-Skew Monoidal Unit of $\mathbf{Rel}(A, B)$

The **right J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

8.9.3 The Right Skew Associators

Definition 8.9.3.1.1 ▶ The Right J-Skew Associator of $\mathbf{Rel}(A, B)$

The **right J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) \times \mathbf{Rel}(A, B) \arrow[dr, "\triangleright_J \times \mathsf{id}"] & \\
\mathbf{Rel}(A, B) \times (\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B)) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\mathsf{id} \times \triangleright_J"'] & & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleright_J"] \\
\mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[rr, "\triangleright_J"'] \arrow[from=3-1, to=1-2, phantom, "\Uparrow \alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & & \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\overset{\text{def}}{=} \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond R} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where
$$\gamma\colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S)$$
is the inclusion adjunct to the inclusion
$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$
under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon\colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.4 The Right Skew Left Unitors

Definition 8.9.4.1.1 ▶ The Right J-Skew Left Unitor of $\mathbf{Rel}(A, B)$

The **right J-skew left unitor of $\mathbf{Rel}(A, B)$** is the natural transformation
$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathsf{id}\right),$$
as in the diagram
$$\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\Nearrow \lambda^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
& \mathbf{Rel}(A,B),
\end{tikzcd}$$
whose component
$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$
at R is given by the composition
$$\begin{aligned} R &\overset{\sim}{\Longrightarrow} \chi_B \diamond R \\ &\xRightarrow{\eta_{\chi_B} \diamond \mathrm{id}_R} \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{def}}{=} && \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R\\
&\overset{\sim}{\Longrightarrow} && \mathrm{Ran}_J(J) \diamond R\\
&\overset{\text{def}}{=} && R \rhd_J J,
\end{aligned}
$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.5 The Right Skew Right Unitors

Definition 8.9.5.1.1 ▶ The Right J-Skew Right Unitor of $\mathbf{Rel}(A,B)$

The **right J-skew right unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\rhd_J}\colon \rhd_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

as in the diagram

$$
\begin{tikzcd}
\mathbf{Rel}(A,B) \times \mathsf{pt} \arrow[r, "\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[r, phantom, shift right=6ex, "\Swarrow\rho^{\mathbf{Rel}(A,B),\rhd_J}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\rhd_J"]\\
& \mathbf{Rel}(A,B)
\end{tikzcd}
$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\rhd_J}_S\colon \underbrace{S \rhd_J J}_{\overset{\text{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\rhd_J}_S \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J^* \circ \mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A,B)$

Proposition 8.9.6.1.1 ▶ The Right J-Skew Monoidal Structure on $\mathbf{Rel}(A,B)$

The category $\mathbf{Rel}(A,B)$ admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A,B)$ of relations from A to B of **??** of **??**.

- *The Right Skew Monoidal Product.* The right J-skew monoidal product

$$\lhd_J \colon \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,B)$$

  of Definition 8.9.1.1.1.

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J} \colon \mathsf{pt} \to \mathbf{Rel}(A,B)$$

  of Definition 8.9.2.1.1.

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\rhd_J} \colon \rhd_J \circ (\mathsf{id} \times \rhd_J) \Longrightarrow \rhd_J \circ (\rhd_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 8.9.3.1.1.

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\rhd_J} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \rhd_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd} \times \mathsf{id}\right)$$

  of Definition 8.9.4.1.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\rhd_J} \colon \rhd_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 8.9.5.1.1.

PROOF 8.9.6.1.2 ▶ PROOF OF PROPOSITION 8.9.6.1.1

Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic (Categories, Item 4 of Proposition 11.2.7.1.2), and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a right skew monoidal category. ∎

Appendices

A Other Chapters

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