Notes

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This chapter contains some notes.

Contents

15.1 TikZ Code for Commutative Diagrams

In this section we gather some useful examples of `tikzcd` code for commutative diagrams.

15.1.1 Product Diagram With Circular Arrows

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

in the preamble, as well as

```
\tikzcdset{
    productArrows/.style args={#1#2#3}{
    execute at end picture={
        % FIRST ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=0,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-1-a] (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatrixnar
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
        \path[name path=curve-1-b] (\tikzcdmatrixname-1-2) arc[start angle=90,end ar
        \fill [name intersections={of=curve-1-a and curve-1-b}] (intersection-
2);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.east);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-2.center);
        \draw let
            \p1 = ($(intersection-2) - (arc-center)$), % \p1 is the vector from the
2 for the 2nd intersection)
```

```
          \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degrees
          \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees from the
        in [->] (intersection-2) -- ++(\n2:0.1pt);
        % SECOND ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=180,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-2-a] (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatrixnar
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
        \path[name path=curve-2-b] (\tikzcdmatrixname-1-2) arc[start angle=90,end ar
        \fill [name intersections={of=curve-2-a and curve-2-b}] (intersection-
2);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.west);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-2.center);
        \draw let
            \p1 = ($(intersection-2) - (arc-center)$), % \p1 is the vector from the
2 for the 2nd intersection)
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degrees
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees from the
          in [<-] (intersection-2) -- ++(\n2:0.1pt);
          % Labels
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=180,radius=#1]
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=0,radius=#1] no
    }
  }
}
```

The code

```
\begin{tikzcd}[row sep={4.5*\the\DL,between origins}, column sep={4.5*\the\DL,betwee
    {}% Don't remove this line, it's important!
```

```
    \&
    P
    \arrow[d,"\phi"'{pos=0.475},"\exists!"{pos=0.475}, dashed]
    \&
    {}% Don't remove this line, it's important!
    \\
    A
    \&
    A\times B
    \arrow[l,"\pr_{1}"{pos=0.425},two heads]
    \arrow[r,"\pr_{2}"'{pos=0.425},two heads]
    \&
    B
\end{tikzcd}
```

will then produce the following diagram:

$$
\begin{tikzcd}
 & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"'] \arrow[r, two heads, "\mathrm{pr}_2"'] & B
\end{tikzcd}
$$

15.1.2 Coproduct Diagram With Circular Arrows

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

in the preamble, as well as

```
\tikzcdset{
    coproductArrows/.style args={#1#2#3}{
    execute at end picture={
        % FIRST ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
```

```
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=0,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-1-a] (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatrixnam
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
        \path[name path=curve-1-b] (\tikzcdmatrixname-1-2) arc[start angle=90,end an
        \fill [name intersections={of=curve-1-a and curve-1-b}] (intersection-
1);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.east);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-2.center);
        \draw let
            \p1 = ($(intersection-1) - (arc-center)$), % \p1 is the vector from the
2 for the 2nd intersection)
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degrees
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees from the
          in [<-] (intersection-1) -- ++(\n2:0.1pt);
        % SECOND ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=180,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-2-a] (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatrixnam
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
        \path[name path=curve-2-b] (\tikzcdmatrixname-1-2) arc[start angle=90,end an
        \fill [name intersections={of=curve-2-a and curve-2-b}] (intersection-
1);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.west);
```

```
        \coordinate (arc-center) at (\tikzcdmatrixname-2-2.center);
        \draw let
            \p1 = ($(intersection-1) - (arc-center)$), % \p1 is the vector from the
2 for the 2nd intersection)
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degrees
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees from the
          in [->] (intersection-1) -- ++(\n2:0.1pt);
          % Labels
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=180,radius=#1]
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=0,radius=#1] no
    }
  }
}
```

The code

```
\begin{tikzcd}[row sep={4.5*\the\DL,between origins}, column sep={4.5*\the\DL,betwee
    {}% Don't remove this line, it's important!
    \&
    C
    \arrow[from=d,"\phi","\exists!"', dashed]
    \&
    {}% Don't remove this line, it's important!
    \\
    A
    \&
    A\icoprod B
    \arrow[from=l,"\inj_{1}"',hook]
    \arrow[from=r,"\inj_{2}",hook']
    \&
    B
\end{tikzcd}
```

will then produce the following diagram:

$$
\begin{tikzcd}
 & C & \\
A \arrow[r, hook, "\mathrm{inj}_1"'] \arrow[ur, bend left, "\iota_1"] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"']
\end{tikzcd}
$$

15.1.3 Cube Diagram

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

The code

```
\begin{tikzcd}[row sep={4.0*\the\DL,between origins}, column sep={4.0*\the\DL,betwee
    1
    \&
    \&
    2
    \&
    \\
    \&
    1'
    \&
    \&
    2'
    \\
    3
    \&
    \&
    4
    \&
    \\
    \&
    3'
    \&
    \&
    4'
    % 1-Arrows
    % First Square
    \arrow[from=1-1,to=3-1,"f"']%
    \arrow[from=3-1,to=3-3,"h"{description,pos=0.25}]%
    \arrow[from=1-1,to=1-3,"g"]%
    \arrow[from=1-3,to=3-3,"k"{description,pos=0.25}]%
    % Second Square
    \arrow[from=2-2,to=4-2,"f'"{description,pos=0.3},crossing over]%
```

```
    \arrow[from=4-2,to=4-4,"h'"']%
    \arrow[from=2-2,to=2-4,"g'"{description,pos=0.3},crossing over]%
    \arrow[from=2-4,to=4-4,"k'"]%
    % Connecting Arrows
    \arrow[from=1-1,to=2-2,"c_{1}"description]%
    \arrow[from=1-3,to=2-4,"c_{2}"]%
    \arrow[from=3-1,to=4-2,"c_{3}"']%
    \arrow[from=3-3,to=4-4,"c_{4}"description]%
\end{tikzcd}
```

will produce the following diagram:

$$\begin{tikzcd}
1 \arrow[rr, "g"] \arrow[dd, "f"'] \arrow[dr, "c_1" description] & & 2 \arrow[dd, "k" description] \arrow[dr, "c_2"] & \\
& 1' \arrow[rr, "g'" {description, pos=0.3}, crossing over] \arrow[dd, "f'" description] & & 2' \arrow[dd, "k'"] \\
3 \arrow[rr, "h" {description, pos=0.3}] \arrow[dr, "c_3"'] & & 4 \arrow[dr, "c_4" description] & \\
& 3' \arrow[rr, "h'"'] & & 4'
\end{tikzcd}$$

15.1.4 Cube Diagram With Labelled Faces

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

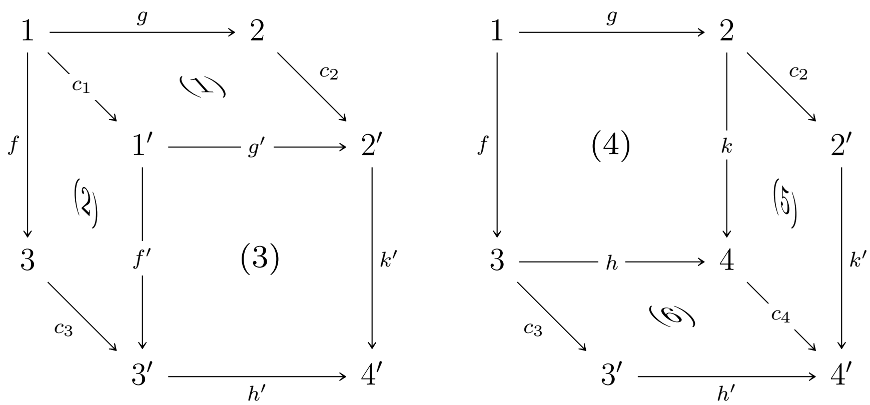
The code

```
\begin{tikzcd}[row sep={4.0*\the\DL,between origins}, column sep={4.0*\the\DL,betwee
    1
    \&
    \&
    2
    \&
    \\
    \&
    1'
```

```
    \&
    \&
    2'
    \\
    3
    \&
    \&
    \&
    \\
    \&
    3'
    \&
    \&
    4'
    % 1-Arrows
    % First Square
    \arrow[from=1-1,to=3-1,"f"']%
    \arrow[from=1-1,to=1-3,"g"]%
    % Second Square
    \arrow[from=2-2,to=4-2,"f'"{description},crossing over]%
    \arrow[from=4-2,to=4-4,"h'"']%
    \arrow[from=2-2,to=2-4,"g'"{description},crossing over]%
    \arrow[from=2-4,to=4-4,"k'"]%
    % Connecting Arrows
    \arrow[from=1-1,to=2-2,"c_{1}"description]%
    \arrow[from=1-3,to=2-4,"c_{2}"]%
    \arrow[from=3-1,to=4-2,"c_{3}"']%
    % Subdiagrams
    \arrow[from=2-2,to=1-3,"\scriptstyle(1)"{rotate=-0.3,xslant=-
0.903569337,yslant=0,xscale=7.0341,yscale=4.4454,xscale=0.225,yscale=0.225},phantom]
    \arrow[from=3-1,to=2-2,"\scriptstyle(2)"{rotate=-44.6,xslant=-
0.965688775,yslant=0,xscale=8.6931,yscale=8.2852,xscale=0.15,yscale=0.15},phantom]%
    \arrow[from=4-2,to=2-4,"\scriptstyle(3)"{rotate=0,xslant=0,yslant=0,xscale=1.5,y
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep={4.0*\the\DL,between origins}, column sep={4.0*\the\DL,betwee
    1
    \&
    \&
```

```
    2
    \&
    \\
    \&
    \&
    \&
    2'
    \\
    3
    \&
    \&
    4
    \&
    \\
    \&
    3'
    \&
    \&
    4'
    % 1-Arrows
    % First Square
    \arrow[from=1-1,to=3-1,"f"']%
    \arrow[from=3-1,to=3-3,"h"{description}]%
    \arrow[from=1-1,to=1-3,"g"]%
    \arrow[from=1-3,to=3-3,"k"{description}]%
    % Second Square
    \arrow[from=4-2,to=4-4,"h'"']%
    \arrow[from=2-4,to=4-4,"k'"]%
    % Connecting Arrows
    \arrow[from=1-3,to=2-4,"c_{2}"]%
    \arrow[from=3-1,to=4-2,"c_{3}"']%
    \arrow[from=3-3,to=4-4,"c_{4}"description]%
    % Subdiagrams
    \arrow[from=1-1,to=3-3,"\scriptstyle(4)"{rotate=0,xslant=0,yslant=0,xscale=1.5,y
    \arrow[from=3-3,to=2-4,"\scriptstyle(5)"{rotate=-44.6,xslant=-
0.965688775,yslant=0,xscale=8.6931,yscale=8.2852,xscale=0.15,yscale=0.15},phantom]%
    \arrow[from=4-2,to=3-3,"\scriptstyle(6)"{rotate=-0.3,xslant=-
0.903569337,yslant=0,xscale=7.0341,yscale=4.4454,xscale=0.225,yscale=0.225},phantom]
\end{tikzcd}
```

will produce the following diagram:



15.1.5 Pentagon Diagram

Define

```
\newlength{\ThreeCm}
\setlength{\ThreeCm}{3.0cm}
```

The code

```
\begin{tikzcd}[row sep={0*\the\DL,between origins}, column sep={0*\the\DL,between or
    \&[0.30901699437\ThreeCm]
    \&[0.5\ThreeCm]
    A\otimes_{R}(A\otimes_{R}A)
    \&[0.5\ThreeCm]
    \&[0.30901699437\ThreeCm]
    \\[0.58778525229\ThreeCm]
    (A\otimes_{R}A)\otimes_{R}A
    \&[0.30901699437\ThreeCm]
    \&[0.5\ThreeCm]
    \&[0.5\ThreeCm]
    \&[0.30901699437\ThreeCm]
    A\otimes_{R}A
    \\[0.95105651629\ThreeCm]
    \&[0.30901699437\ThreeCm]
    A\otimes_{R}A
    \&[0.5\ThreeCm]
    \&[0.5\ThreeCm]
```

```
    A
    \&[0.30901699437\ThreeCm]
    % 1-Arrows
    % Left Boundary
    \arrow[from=2-1,to=1-3,"\alpha^{\Mod_{R}}_{A,A,A}"{pos=0.4125}]%
    \arrow[from=1-3,to=2-5,"\id_{A}\otimes_{R}\mu^{\times}_{A}"{pos=0.6}]%
    \arrow[from=2-5,to=3-4,"\mu^{\times}_{A}"{pos=0.425}]%
    % Right Boundary
    \arrow[from=2-1,to=3-2,"\mu^{\times}_{A}\otimes_{R}\id_{A}"'{pos=0.425}]%
    \arrow[from=3-2,to=3-4,"\mu^{\times}_{A}"']%
\end{tikzcd}
```

will produce the following pentagon diagram:

$$\begin{tikzcd}
 & A\otimes_R(A\otimes_R A) \arrow[dr, "\mathrm{id}_A\otimes_R\mu^{\times}_A"] & \\
(A\otimes_R A)\otimes_R A \arrow[ur, "\alpha^{\mathsf{Mod}_R}_{A,A,A}"] \arrow[d, "\mu^{\times}_A\otimes_R\mathrm{id}_A"'] & & A\otimes_R A \arrow[d, "\mu^{\times}_A"] \\
A\otimes_R A \arrow[rr, "\mu^{\times}_A"'] & & A
\end{tikzcd}$$

To make the diagram larger, one could use e.g.

```
\newlength{\FourCm}
\setlength{\FourCm}{2.0cm}
```

and replace all instances of `\ThreeCm` with `\FourCm` in the code above.

15.1.6 Hexagon Diagram

Define

```
\newlength{\OneCmPlusHalf}
\setlength{\OneCmPlusHalf}{1.5cm}
```

The code

```
\begin{tikzcd}[row sep={0.0*\the\DL,between origins}, column sep={0.0*\the\DL,betwee
    \&[0.86602540378\OneCmPlusHalf]
    1
    \&[0.86602540378\OneCmPlusHalf]
    \\[0.5\OneCmPlusHalf]
    2
    \&[0.86602540378\OneCmPlusHalf]
    \&[0.86602540378\OneCmPlusHalf]
    3
    \\[\OneCmPlusHalf]
    4
    \&[0.86602540378\OneCmPlusHalf]
    \&[0.86602540378\OneCmPlusHalf]
    5
    \\[0.5\OneCmPlusHalf]
    \&[0.86602540378\OneCmPlusHalf]
    6
    \&[0.86602540378\OneCmPlusHalf]
    % 1-Arrows
    % Left Boundary
    \arrow[from=1-2,to=2-1,"L_{1}"']%
    \arrow[from=2-1,to=3-1,"L_{2}"']%
    \arrow[from=3-1,to=4-2,"L_{3}"']%
    % Right Boundary
    \arrow[from=1-2,to=2-3,"R_{1}"]%
    \arrow[from=2-3,to=3-3,"R_{2}"]%
    \arrow[from=3-3,to=4-2,"R_{3}"]%
\end{tikzcd}
```

will produce the following hexagon diagram:

$$\begin{tikzcd}
 & 1 \arrow[dl, "L_1"'] \arrow[dr, "R_1"] & \\
2 \arrow[d, "L_2"'] & & 3 \arrow[d, "R_2"] \\
4 \arrow[dr, "L_3"'] & & 5 \arrow[dl, "R_3"] \\
 & 6 &
\end{tikzcd}$$

To make the diagram larger, one could use e.g.

```
\newlength{\TwoCm}
\setlength{\TwoCm}{2.0cm}
```

and replace all instances of `\OneCmPlusHalf` with `\TwoCm` in the code above.

15.1.7 Double Square Diagram

Define

```
\newlength{\DL}
\setlength{\DL}{0.9cm}
```

The code

```
\begin{tikzcd}[row sep={10.0*\the\DL,between origins}, column sep={10.0*\the\DL,betw
    \bullet
    \&
    \&
    \&
    \bullet
    \\
    \&
    \bullet
    \&
    \bullet
    \&
    \\
    \&
    \bullet
    \&
    \bullet
    \&
    \\
    \bullet
    \&
    \&
    \&
    \bullet
    % Arrows
    % Outer Square
    \arrow[from=1-1,to=1-4]%
```

```
    \arrow[from=1-4,to=4-4]%
    %
    \arrow[from=1-1,to=4-1]%
    \arrow[from=4-1,to=4-4]%
    % Inner Square
    \arrow[from=2-2,to=2-3]%
    \arrow[from=2-3,to=3-3]%
    %
    \arrow[from=2-2,to=3-2]%
    \arrow[from=3-2,to=3-3]%
    % Connecting Arrows
    \arrow[from=1-1,to=2-2]%
    \arrow[from=1-4,to=2-3]%
    \arrow[from=3-2,to=4-1]%
    \arrow[from=3-3,to=4-4]%
    % Subdiagrams
    \arrow[from=2-2,to=3-3,"\scriptstyle(1)",phantom,yshift=10.0*\the\DL]%
    \arrow[from=2-2,to=3-2,"\scriptstyle(2)",phantom,xshift=-5.0*\the\DL]%
    \arrow[from=2-2,to=3-3,"\scriptstyle(3)",phantom]%
    \arrow[from=2-3,to=3-3,"\scriptstyle(4)",phantom,xshift=5.0*\the\DL]%
    \arrow[from=2-2,to=3-3,"\scriptstyle(5)",phantom,yshift=-10.0*\the\DL]%
\end{tikzcd}
```

will produce the following double square diagram:

$$\begin{tikzcd}
\bullet \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \bullet \arrow[dl] \arrow[ddd] \\
& \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\scriptstyle(1)", yshift=50pt] \arrow[d, phantom, "\scriptstyle(2)", xshift=-25pt] \arrow[dr, phantom, "\scriptstyle(3)"] \arrow[dr, phantom, "\scriptstyle(5)", yshift=-50pt] & \bullet \arrow[d] \arrow[d, phantom, "\scriptstyle(4)", xshift=25pt] & \\
& \bullet \arrow[r] \arrow[dl] & \bullet \arrow[dr] & \\
\bullet \arrow[rrr] & & & \bullet
\end{tikzcd}$$

15.1.8 Double Hexagon Diagram

Define

```
\newlength{\OneCm}
\setlength{\OneCm}{1.0cm}
```

The code

```
\begin{tikzcd}[row sep={0.0*\the\DL,between origins}, column sep={0.0*\the\DL,betwee
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{1-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \\[2.0*\OneCm]
    \text{2-1}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{2-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{2-5}
    \\[1.0*\OneCm]
    \&[1.73205081*\OneCm]
    \text{3-2}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{3-4}
    \&[1.73205081*\OneCm]
    \\[2.0*\OneCm]
    \&[1.73205081*\OneCm]
    \text{4-2}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{4-4}
    \&[1.73205081*\OneCm]
    \\[1.0*\OneCm]
    \text{5-1}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
```

```
    \text{5-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{5-5}
    \\[2.0*\OneCm]
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{6-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    % Arrows
    \arrow[from=1-3,to=2-1,”1”']%
    \arrow[from=2-1,to=5-1,”2”']%
    \arrow[from=5-1,to=6-3,”3”']%
    %
    \arrow[from=1-3,to=2-5,”4”]%
    \arrow[from=2-5,to=5-5,”5”]%
    \arrow[from=5-5,to=6-3,”6”]%
    %
    \arrow[from=2-3,to=3-2,”1'”description]%
    \arrow[from=3-2,to=4-2,”2'”description]%
    \arrow[from=4-2,to=5-3,”3'”description]%
    %
    \arrow[from=2-3,to=3-4,”4'”description]%
    \arrow[from=3-4,to=4-4,”5'”description]%
    \arrow[from=4-4,to=5-3,”6'”description]%
    %
    \arrow[from=1-3,to=2-3,”a”description]%
    \arrow[from=2-1,to=3-2,”b”description]%
    \arrow[from=2-5,to=3-4,”c”description]%
    \arrow[from=5-1,to=4-2,”d”description]%
    \arrow[from=5-5,to=4-4,”e”description]%
    \arrow[from=5-3,to=6-3,”f”description]%
\end{tikzcd}
```

will produce the following double hexagon diagram:

$$\begin{tikzcd}
 & & \text{1-3} \arrow[dll, "1" description] \arrow[d, "a" description] \arrow[drr, "4" description] & & \\
\text{2-1} \arrow[dr, "b" description] \arrow[ddd, "2" description] & & \text{2-3} \arrow[dl, "1'" description] \arrow[dr, "4'" description] & & \text{2-5} \arrow[dl, "c" description] \arrow[ddd, "5" description] \\
 & \text{3-2} \arrow[d, "2'" description] & & \text{3-4} \arrow[d, "5'" description] & \\
 & \text{4-2} \arrow[dr, "3'" description] & & \text{4-4} \arrow[dl, "6'" description] & \\
\text{5-1} \arrow[ur, "d" description] \arrow[drr, "3" description] & & \text{5-3} \arrow[d, "f" description] & & \text{5-5} \arrow[ul, "e" description] \arrow[dll, "6" description] \\
 & & \text{6-3} & &
\end{tikzcd}$$

To make the diagram larger, one could use e.g.

```
\newlength{\TwoCm}
\setlength{\TwoCm}{2.0cm}
```

and replace all instances of `\OneCm` with `\TwoCm` in the code above.

15.2 Retired Tags

15.2.1 Relations

Old Tag 15.2.1.1.1 ▶ Equivalent Definitions of Relations

The content of this tag has been moved to Relations, Definition 8.1.1.1.1.

Old Tag 15.2.1.1.2 ▶ Interaction Between Composition and Characteristic Relations

The original statement of this tag was false.

Old Tag 15.2.1.1.3 ▶ Interaction Between Composition and Characteristic Relations

The original statement of this tag was false.

Old Tag 15.2.1.1.4 ▶ Explicit Description of Internal Left Kan Extensions Along Functions

This was a question. Now an explicit description is available as Relations, ??.

Old Tag 15.2.1.1.5 ▶ Explicit Description of Internal Left Kan Lifts Along Functions

This was a question. Now an explicit description is available as Relations, ??.

Old Tag 15.2.1.1.6 ▶ Internal Kan Extensions and Lifts

This tag is obsolete; see Relations, Sections 8.5.13 to 8.5.16 instead.

Old Tag 15.2.1.1.7 ▶ Internal Kan Extensions and Lifts

This tag is obsolete; see Relations, Sections 8.5.13 to 8.5.16 instead.

Old Tag 15.2.1.1.8 ▶ Internal Kan Extensions and Lifts

This tag is obsolete; see Relations, Sections 8.5.13 to 8.5.16 instead.

Old Tag 15.2.1.1.9 ▶ Better Characterisations of Representably Full Morphisms in Rel

This was originally a question. It has been answered in Relations, ??.

Old Tag 15.2.1.1.10 ▶ Better Characterisations of Corepresentably Full Morphisms in Rel

This was originally a question. It has been answered in Relations, Section 8.5.11.

Old Tag 15.2.1.1.11 ▶ Characterisation of Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.12 ▶ Characterisation of 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.13 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.14 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.15 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.16 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.17 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.18 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.19 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.20 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

Old Tag 15.2.1.1.21 ▶ 2-Categorical Monomorphisms in Rel

Superseded by Relations, ??.

OLD TAG 15.2.1.1.22 ▶ 2-CATEGORICAL MONOMORPHISMS IN Rel

Superseded by Relations, ??.

OLD TAG 15.2.1.1.23 ▶ CHARACTERISATION OF EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.24 ▶ CHARACTERISATION OF EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.25 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.26 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.27 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.28 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.29 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.30 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

OLD TAG 15.2.1.1.31 ▶ 2-CATEGORICAL EPIMORPHISMS IN Rel

Superseded by Relations, Section 8.5.11.

Old Tag 15.2.1.1.32 ▶ 2-Categorical Epimorphisms in Rel

Superseded by Relations, Section 8.5.11.

Old Tag 15.2.1.1.33 ▶ 2-Categorical Epimorphisms in Rel

Superseded by Relations, Section 8.5.11.

Old Tag 15.2.1.1.34 ▶ 2-Categorical Epimorphisms in Rel

Superseded by Relations, Section 8.5.11.

Old Tag 15.2.1.1.35 ▶ 2-Categorical Epimorphisms in Rel

Superseded by Relations, Section 8.5.11.

Old Tag 15.2.1.1.36 ▶ Epimorphisms in Rel

Superseded by Relations, Section 8.5.11.

Old Tag 15.2.1.1.37 ▶ Epimorphisms in Rel

Superseded by Relations, Section 8.5.11.

15.2.2 Pointed Sets

Old Tag 15.2.2.1.1 ▶ The Underlying Pointed Set of a Semimodule

The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Old Tag 15.2.2.1.2 ▶ The Underlying Pointed Set of a Module

The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

15.2.3 Tensor Products of Pointed Sets

Old Tag 15.2.3.1.1 ▶ Section on Universal Properties of the Smash Product of Pointed Sets I

Absorbed into Tensor Products of Pointed Sets, Section 7.5.10.

Old Tag 15.2.3.1.2 ▶ Section on Universal Properties of the Smash Product of Pointed Sets II

Absorbed into Tensor Products of Pointed Sets, Section 7.5.10.

Old Tag 15.2.3.1.3 ▶ Universal Properties of the Smash Product of Pointed Sets I

Absorbed into Tensor Products of Pointed Sets, Section 7.5.10.

Old Tag 15.2.3.1.4 ▶ Universal Properties of the Smash Product of Pointed Sets II

Absorbed into Tensor Products of Pointed Sets, Section 7.5.10.

15.2.4 Categories

Old Tag 15.2.4.1.1 ▶ Picturing Natural Transformations in Diagrams

We denote natural transformations in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D}. \end{tikzcd}$$

(This tag has been removed and is now part of Categories, Remark 11.9.2.1.2.)

Old Tag 15.2.4.1.2 ▶ Interaction Between Fullness and Postcomposition Functors

(This Tag was an item of Categories, Proposition 11.6.2.1.2, but has since been removed because its statement is incorrect. Naïm Camille Favier provided a counterexample, and the corrected statements now appear as Categories, Items 2 and 3 of Proposition 11.6.2.1.2.)

1. *Interaction With Postcomposition.* The following conditions are

equivalent:

(a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.

(b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is full.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in Cats_2 in the sense of Types of Morphisms in Bicategories, Definition 14.1.2.1.1.

15.3 Miscellany

15.3.1 List of Things To Explore/Add

Here we list things to be explored in or added to this work in the future. This is a very quick and dirty list; some items may not be fully intelligible.

Remark 15.3.1.1.1 ▶ Things To Explore/Add

Set Theory:

1. https://math.stackexchange.com/questions/200389/show-that-the-set-of-all-finite-subsets-of-mathbbn-is-countable
2. https://mathoverflow.net/a/479528
3. https://www.maths.ed.ac.uk/~tl/ast/ast.pdf

Type Theory:

1. https://mathoverflow.net/questions/497570/universes-dont-need-to-be-indexed-by-natural-numbers

Pointed sets:

1. Universal properties (plural!) of the left tensor product of pointed sets

2. Universal properties (plural!) of the right tensor product of pointed sets

Relations:

1. Internal fibrations in **Rel**, like discrete fibrations and Street fibrations
2. Return to Eilenberg–Moore and Kleisli objects in **Rel** once the general theory has been set up for internal monads

Spans:

1. https://arxiv.org/abs/2505.22832
2. Spans: study certain compositions of spans like composing $B \xleftarrow{f} A = A$ and $A = A \xleftarrow{g} B$ into a span $B \xleftarrow{f} A \xleftarrow{g} B$
3. Comparison *double functor* from Span to Rel and vice versa
4. Apartness composition for spans and alternate compositions for spans in general
5. non-Cartesian analogue of spans
   (a) View spans as morphisms $S \to A \times B$ and consider instead morphisms $S \to A \otimes_{C} B$
6. Record the universal property of the bicategory of spans of https://ncatlab.org/nlab/show/span
7. https://ncatlab.org/nlab/show/span+trace
8. Cospans.
9. Multispans.

Un/Straightening for Indexed and Fibred Sets:

1. Analogue of adjoints for Grothendieck construction for indexed and fibred sets
2. Write proper sections on straightening for lax functors from Sets to Rel or Span (displayed sets)

3. co/units for un/straightening adjunction

Categories:

1. https://www.numdam.org/actas/SE/, https://www.numdam.org/journals/CTGDC/
2. https://www.numdam.org/item/CTGDC_1966__8__A5_0.pdf
3. https://mathoverflow.net/questions/493931/is-the-category-of-posets-locally-cartesian-closed
4. From Keith: Presheaves on a topological space X valued in $\{\mathsf{t}, \mathsf{f}\}$
   (a) They are the same as collections of open subsets of X
   (b) They are sheaves iff that collection is closed under union
   (c) Their sheafification is the closure of that collection under unions
5. https://arxiv.org/abs/2504.20949
6. Notion of equality that is weaker than equivalence but stronger than adjunction
7. Tangent categories, Beck modules, categorical derivations
8. Flat functors
9. Is the classifying space of a category isomorphic to Ex^{∞} of the nerve of the category? If so, an intuition for having an initial/terminal object implying being homotopically contractible is that taking the free ∞-groupoid generated by that identifies every object with the terminal one.
10. https://en.wikipedia.org/wiki/Category_algebra
11. simple objects
12. https://mathoverflow.net/questions/442212/properties-of-categorical-zeta-function

13. Polynomial functors, https://ncatlab.org/nlab/show/polynomial+functor, https://arxiv.org/abs/2312.00990
14. https://ncatlab.org/nlab/show/simple+object
15. https://mathoverflow.net/questions/442212/properties-of-categorical-zeta-function
16. https://arxiv.org/abs/2409.17489
17. https://mathoverflow.net/a/478644
18. Posetal category associated to a poset as a right adjoint
19. "Presetal category" associated to a preordered set
20. Vopenka's principle simplifies stuff in the theory of locally presentable categories. If we build categories using type theory or HoTT, what stuff from vopenka holds?
21. Are pseudoepic functors those functors whose restricted Yoneda embedding is pseudomonic and Yoneda preserves absolute colimits?
22. Absolutely dense functors enriched over $\mathbb{R}^+$ apparently reduce to topological density
23. Is there a reasonable notion of category homology? It is very common for the geometric realisation of a category to be contractible (e.g. having an initial or terminal object), but maybe some notion of directed homology could work here
24. Nerves of categories:
    (a) Dihedral and symmetric nerves of categories via groupoids (define them first for groupoids and then Kan extend along $\mathsf{Grpd} \hookrightarrow \mathsf{Cats}$)
        i. Same applies to twisted nerves
    (b) Cyclic nerve of a category
    (c) Crossed Simplicial Group Categorical Nerves, https://arxiv.org/abs/1603.08768

25. Define contractible categories and add a discussion of universal properties as stating that certain categories are contractible. (Example of non-unique isomorphisms as e.g. being a group of order 5 corresponds to all objects being isomorphic but the category not being contractible)

26. Expand ?? and add a proof to it.

27. Sections and retractions; retracts, https://ncatlab.org/nlab/show/retract.

28. Groupoid cardinality

    (a) https://mathoverflow.net/questions/376175/category-theory-and-arithmetical-identities/376223#376223
    (b) https://mathoverflow.net/questions/420088/groupoid-cardinality-of-the-class-of-abelian-p-groups?rq=1
    (c) https://mathoverflow.net/questions/363292/what-is-the-groupoid-cardinality-of-the-category-of-vector-spaces-over-a-finite
    (d) The groupoid cardinality of the core of the category of finite sets is e. What is the groupoid cardinality of the core of $\mathsf{FinSets}_G$?
    (e) groupoid cardinality of the core of the category of finite G-sets, https://www.arxiv.org/pdf/2502.03585
    (f) https://ncatlab.org/nlab/show/groupoid+cardinality
    (g) https://arxiv.org/abs/2104.11399
    (h) https://terrytao.wordpress.com/2017/04/13/counting-objects-up-to-isomorphism-groupoid-cardinality/
    (i) https://arxiv.org/abs/0809.2130
    (j) https://qchu.wordpress.com/2012/11/08/groupoid-cardinality/
    (k) https://mathoverflow.net/questions/363292/what-is-the-groupoid-cardinality-of-the-category-of-vector-spaces-over-a-finite

29. combinatorial species
    (a) https://ncatlab.org/nlab/show/Schur+functor
        i. Equivalence between twisted commutative algebras and algebras on categories of polynomial functors, https://mathweb.ucsd.edu/~ssam/talks/2014/ihp-tca.pdf
    (b) https://mathoverflow.net/questions/22462/what-are-some-examples-of-interesting-uses-of-the-theory-of-combinatorial-specie
    (c) https://en.wikipedia.org/wiki/Combinatorial_species
30. Leinster's the eventual image, https://arxiv.org/abs/2210.00302
    (a) Telescope notation $\mathrm{tel}_\phi(X) \stackrel{\mathrm{def}}{=} \mathrm{colim}\Big(X \xrightarrow{\phi} X \xrightarrow{\phi}\xrightarrow{\phi} \cdots\Big)$ introduced in https://arxiv.org/abs/2505.06979
31. https://ncatlab.org/nlab/show/separable+functor
32. Dagger categories:
    (a) https://en.wikipedia.org/wiki/Dagger_category
    (b) https://ncatlab.org/nlab/show/dagger+category
    (c) Dagger compact categories, https://en.wikipedia.org/wiki/Dagger_compact_category
    (d) https://mathoverflow.net/questions/220032/are-dagger-categories-truly-evil
    (e) generalisation of dagger categories to categories with duality, i.e. categories $\mathcal{C}$ together with a functor $\dagger\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$
        i. Perhaps with the additional condition that $\dagger \circ \dagger = \mathrm{id}$
        ii. categories with involutions in general

Regular Categories:

1. https://arxiv.org/pdf/2004.08964.pdf.
2. Internal relations

Types of Morphisms in Categories:

1. https://mathoverflow.net/questions/490476/duality-of-injectivity-surjectivity-of-precomposition-map for motivation of monomorphisms/epimorphisms

2. Characterisation of epimorphisms in the category of fields, https://math.stackexchange.com/q/4941660

3. Strong epimorphisms

4. Behaviour in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, e.g. pointwise sections vs. sections in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

5. Faithful functors from balanced categories are conservative

6. Natural cotransformations:

   (a) If there is a natural transformation between functors between categories, taking nerves gives a homotopy equivalence (or something like that). What happens for natural cotransformations?

   (b) Natural transformations come with a vertical composition map

   $$\circ\colon \coprod_{G \in \mathsf{Fun}(\mathcal{C},\mathcal{D})} \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$

   As Morgan Rogers shows here, there's no vertical cocomposition map of the form

   $$\mathrm{CoNat}(F, H) \to \prod_{G \in \mathsf{Fun}(\mathcal{C},\mathcal{D})} \mathrm{CoNat}(G, H) \times \mathrm{CoNat}(F, G)$$

   or of the form

   $$\mathrm{CoNat}(F, H) \to \prod_{G \in \mathsf{Fun}(\mathcal{C},\mathcal{D})} \mathrm{CoNat}(G, H) \coprod \mathrm{CoNat}(F, G)$$

   for natural cotransformations.

   (c) Cap product for CoNat and Nat

        i. recovers map $\mathrm{Z}(G) \times \mathrm{Cl}(G) \to \mathrm{Cl}(G)$.

   (d) What is the geometric realisation of $\mathrm{CoTrans}(F, G)$?

        i. Related: https://mathoverflow.net/questions/89753/geometric-realization-of-hochschild-complex

   (e) What is the totalisation of $\mathrm{Trans}(F, G)$?

        i. If we view sets as discrete topological spaces, what are the homotopy/homology groups of it? The nLab says this (https://ncatlab.org/nlab/show/totalization):

> The homotopy groups of the totalization of a cosimplicial space are computed by a Bousfield-Kan spectral sequence.
> The homology groups by an Eilenberg-Moore spectral sequence.

   (f) Abstract

Adjunctions:

1. Relative adjunctions: message Alyssa asking for her notes
2. Adjunctions, units, counits, and fully faithfulness as in https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints.
3. Morphisms between adjunctions and bicategory $\mathsf{Adj}(\mathcal{C})$.
4. https://ncatlab.org/nlab/show/transformation+of+adjoints

Presheaves and the Yoneda Lemma:

1. https://mathoverflow.net/questions/498069/products-and-coproducts-in-the-category-of-elements-of-a-presheaf
2. Yoneda extension along $よ_{\mathcal{D}} \circ F \colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$, giving a functor left adjoint to the precomposition functor $F^* \colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$.

3. Consider the diagram

$$\begin{tikzcd}
& \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed] \arrow[dr, dashed] & \\
\mathcal{C} \arrow[ur, hook] \arrow[r] & \mathcal{D} \arrow[r, hook] & \mathsf{PSh}(\mathcal{D})
\end{tikzcd}$$

4. Does the functor tensor product admit a right adjoint ("Hom") in some sense?
5. Yoneda embedding preserves limits
6. universal objects and universal elements
7. adjoints to the Yoneda embedding and total categories
8. The co-Yoneda lemma: co/presheaves are colimits of co/representables
9. Properties of categories of copresheaves
10. Contravariant restricted Yoneda embedding
11. Contravariant Yoneda extensions
12. Make table of $\mathrm{Lift}_{よ}(よ)$, $\mathrm{Ran}_{よ}(よ)$, $\mathrm{Ran}_{よ}(\rotatebox[origin=c]{180}{よ})$, etc.
13. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: https://qchu.wordpress.com/2015/05/17/generators/
14. Tensor product of functors and relation to profunctors
15. rifts and rans and lifts and lans involving yoneda in Cats and Prof
16. Tensor product of functors and relation to rifts and rans of profunctors

Isbell Duality:

1. enriched Isbell over walking chain complex

2. Isbell self-dual presheaves for Lawvere metric spaces; when

$$f(x) = \sup_{x\in X}\left(\left|f(x) - \sup_{y\in X}(|f(y) - \mathrm{d}_X(y,x)|)\right|\right)$$

holds.

3. https://ncatlab.org/nlab/show/Fr%C3%B6licher+spaces+and+Isbell+envelopes

4. https://ncatlab.org/nlab/show/envelope+of+an+adjunction

5. https://ncatlab.org/nlab/show/nucleus+of+a+profunctor

6. https://ncatlab.org/nlab/show/nuclear+adjunction

7. https://ncatlab.org/nlab/show/fixed+point+of+an+adjunction

8. **Important:** I should reconsider going with the notation O and Spec. Although a bit common in the (somewhat scarce) literature on Isbell duality, I have doubts regarding how useful/nice of a choice O and Spec are, and whether there are better choices of notation for them.

9. Interaction with $\times$, Hom, $F_!$, F^*, and F_*

10. Interactions between presheaves and copresheaves:

    (a) Natural transformations from a presheaf to a copresheaf and vice versa

    (b) Mixed Day convolution?

11. Isbell duality for monoids:

    (a) Set up a dictionary between properties of $\mathsf{Sets}_A^{\mathrm{L}}$ or $\mathsf{Sets}_A^{\mathrm{R}}$ and properties of A

    (b) Do the same for O given by $A \mapsto \mathsf{Sets}_A^{\mathrm{L}}(X, A)$

    (c) Do the same for Spec given by $A \mapsto \mathsf{Sets}_A^{\mathrm{R}}(X, A)$

    (d) Do the same for $\mathsf{O} \circ \mathsf{Spec}$

(e) Do the same for $\mathsf{Spec} \circ \mathsf{O}$

(f) Algebras for $\mathsf{Spec} \circ \mathsf{O}$

(g) Coalgebras for $\mathsf{O} \circ \mathsf{Spec}$

12. Properties of Spec (e.g. fully faithfulness) vs. properties of $\mathcal{C}$

13. Properties of O (e.g. fully faithfulness) vs. properties of $\mathcal{C}$

14. co/unit being monomorphism/epimorphism

15. reflexive completion

16. Isbell duality for simplicial sets; what's the reflexive completion?

17. Isbell envelope

18. What does Isbell duality look like, when Cat(Aop,Set) is identified with the category of discrete opfibrations over A, using A.5.14?

19. Generalizations of Isbell duality:

(a) Monoidal Isbell duality: monoidality for Isbell adjunction with day convolution (6.3 of coend cofriend)

(b) Isbell duality with sheaves

(c) Isbell duality with Lawvere theories, product preserving functors or whatever

(d) Isbell duality for profunctors

i. In view of **??** of **??**, can we just use right Kan lifts/extensions?

ii. Right Kan lift/extension of Hom functors (there's probably a version of the Yoneda lemma here)

A. What is $\mathrm{Rift}_F(\mathrm{Hom}_{\mathcal{C}})$

B. What is $\mathrm{Ran}_F(\mathrm{Hom}_{\mathcal{C}})$

C. What is $\mathrm{Rift}_{\mathrm{Hom}_{\mathcal{C}}}(F)$

D. What is $\mathrm{Ran}_{\mathrm{Hom}_{\mathcal{C}}}(F)$

E. What is $\mathrm{Lift}_F(\mathrm{Hom}_{\mathcal{C}})$

F. What is $\mathrm{Lan}_F(\mathrm{Hom}_{\mathcal{C}})$

G. What is $\mathrm{Lift}_{\mathrm{Hom}_{\mathcal{C}}}(F)$
H. What is $\mathrm{Lan}_{\mathrm{Hom}_{\mathcal{C}}}(F)$

20. Tensor product of functors and Isbell duality
    (a) What is $\mathcal{F} \boxtimes_{\mathcal{C}} \mathsf{O}(\mathcal{F})$?
    (b) What is $\mathsf{Spec}(F) \boxtimes_{\mathcal{C}} F$?
    (c) I think there is a canonical morphism
    $$\mathcal{F} \boxtimes_{\mathcal{C}} \mathsf{O}(\mathcal{F}) \to \mathrm{Tr}(\mathcal{C}).$$
    By the way, what is $\mathrm{Tr}(\mathbb{\Delta})$? What is $\mathrm{Tr}(\mathsf{B}A)$? What about $\mathrm{Nat}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$ for $\mathcal{C} = \mathsf{B}A$ or $\mathcal{C} = \mathbb{\Delta}$
21. Isbell with coends:
    (a) $\mathrm{Hom}(F(A), h_A)$ but it's a coend
    (b) Conatural transformations and all that
22. Co/limit preservation for O/Spec
23. Isbell duality for N vs. N + N
24. What do we get if we replace $\mathsf{O} \overset{\mathrm{def}}{=} \mathrm{Nat}(-, h_X)$ by $\mathrm{Nat}^{[W]}(-, h_X)$, and in particular by $\mathrm{DiNat}(-, h_X)$?

Species:

1. Joyal–Street's q-species; via promonoidal structures https://arxiv.org/pdf/1201.2991#page=22
2. associators, braidings, unitors; $\mathbb{F}_q^n \to \mathbb{F}_q^n$ centre of $\mathrm{GL}_n(\mathbb{F}_q)$ trick
3. group completion of $\mathcal{GL}(\mathbb{F}_q)$ as algebraic k-theory

Constructions With Categories:

1. https://arxiv.org/abs/2504.21764
2. Comparison between pseudopullbacks and isocomma categories: the "evident" functor $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$ is essentially surjective and full, but not faithful in general.

3. Quotients of categories by actions of monoidal categories
   (a) Quotients of categories by actions of monoids $\mathsf{B}A$
   (b) Quotients of categories by actions of monoids A_{disc}
   (c) Lax, oplax, pseudo, strict, etc. quotients of categories
   (d) lax Kan extensions along $\mathsf{B}\mathcal{C} \to \mathsf{B}\mathcal{D}$ for $\mathcal{C} \to \mathcal{D}$ a monoidal functor
4. Quotient of $\mathsf{Fun}(\mathsf{B}A, \mathcal{C})$ by the A-action.
   (a) This is used to build the cycle and p-cycle categories from the paracycle category.
   (b) The quotient of $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C})$ by the $\mathbb{N}$-action should act as a kind of cyclic directed loop space of $\mathcal{C}$
5. $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C})$ as a homotopy pullback in Cats_2
   (a) $\mathsf{Fun}(\mathsf{B}\mathbb{Z}, \mathcal{C})$ as a homotopy pullback in Grpd_2
   (b) Free loop space objects

Limits and colimits:

1. adjunction between co/product and diagonal; abstract version of **??** and **??**
2. Examples of kan extensions along functors of the form $\mathsf{FinSets} \hookrightarrow \mathsf{Sets}$
3. Initial/terminal objects as left/right adjoints to $!_{\mathcal{C}} \colon \mathcal{C} \to \mathsf{pt}$.
4. A small cocomplete category is a poset, https://mathoverflow.net/questions/108737/small-categories-and-completeness
5. Co/limits in $\mathsf{B}A$, including e.g. co/equalisers in $\mathsf{B}A$
6. Add the characterisations of absolutely dense functors given in **??** to **??**.
7. Absolutely dense functors, https://ncatlab.org/nlab/show/absolutely+dense+functor. Also theorem 1.1 here: http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf.

8. Dense functors, codense functors, and absolutely codense functors.
9. van kampen colimits

Completions and cocompletions:

1. https://mathoverflow.net/questions/429003/manifolds-as-cauchy-completed-objects
2. what is the conservative cocompletion of smooth manifolds? Is it related to diffeological spaces?
3. what is the conservative completion of smooth manifolds? Is it related to diffeological spaces?
4. what is the conservative bicompletion of smooth manifolds? Is it related to diffeological spaces?
5. completion of a category under exponentials
6. https://mathoverflow.net/questions/468897/cocompletion-without-cocontinuous-functors
7. The free cocompletion of a category;
8. The free completion of a category;
9. The free completion under finite products;
10. The free cocompletion under finite coproducts;
11. The free bicompletion of a category;
12. The free bicompletion of a category under nonempty products and nonempty coproducts (https://ncatlab.org/nlab/show/free+bicompletion);
13. Cauchy completions
14. Dedekind–MacNeille completions
15. Isbell completion (https://ncatlab.org/nlab/show/reflexive+completion)

16. Isbell envelope

Ends and Coends:

1. motivate co/ends as co/limits of profunctors
2. Ask Fosco about whether composition of dinatural transformations into higher dinaturals could be useful for https://arxiv.org/abs/2409.10237
3. Cyclic co/ends
   (a) Try to mimic the construction given in Haugseng for the cycle, paracycle, cube, etc. categories
   (b) cyclotomic stuff for cyclic co/ends
      i. Check out Ayala–Mazel-Gee–Rozenblyum's *Symmetries of the cyclic nerve*
      ii. isogenetic $\mathbb{N}^{\times}$-action (what the fuck does this mean?)
4. After stating the co/ends

$$\int^{A\in C} h_A \odot \mathcal{F}^A, \qquad \int_{A\in C} \mathsf{Sets}\big(h_A, \mathcal{F}^A\big),$$
$$\int^{A\in C} h^A \odot F_A, \qquad \int_{A\in C} \mathsf{Sets}\big(h^A, F_A\big)$$

in the co/end version of the Yoneda lemma, add a remark explaining what the co/ends

$$\int_{A\in C} h_A \odot \mathcal{F}^A, \qquad \int^{A\in C} \mathsf{Sets}\big(h_A, \mathcal{F}^A\big),$$
$$\int_{A\in C} h^A \odot F_A, \qquad \int^{A\in C} \mathsf{Sets}\big(h^A, F_A\big)$$

and the co/ends

$$\int^{A\in C} \mathcal{F}^A \odot h_A, \qquad \int_{A\in C} \mathsf{Sets}\big(\mathcal{F}^A, h_A\big),$$
$$\int^{A\in C} F_A \odot h^A, \qquad \int_{A\in C} \mathsf{Sets}\big(F_A, h^A\big),$$
$$\int_{A\in C} \mathcal{F}^A \odot h_A, \qquad \int^{A\in C} \mathsf{Sets}\big(\mathcal{F}^A, h_A\big),$$
$$\int_{A\in C} F_A \odot h^A, \qquad \int^{A\in C} \mathsf{Sets}\big(F_A, h^A\big)$$

are.

5. ends $\mathcal{C} \to \mathcal{D}$ with $\odot$ is a special case of ends for a certain enrichment over $\mathcal{D}$

6. try to figure out what the end/coend

$$\int^{X \in \mathcal{C}} h^A_X \times h^X_B, \qquad \int_{X \in \mathcal{C}} h^A_X \times h^X_B$$

are for $\mathcal{C} = \mathsf{B}A$. (I think the coend is like tensor product of A as a left A-set with it as a right A-set)

7. Cyclic ends

8. Dihedral ends

9. Does Haugseng's constructions give a way to define cyclic co/homology with coefficients in a bimodule?

10. Category of elements of dinatural transformation classifier

11. Examples of co/ends: https://mathoverflow.net/a/461814

12. Cofinality for co/ends, https://mathoverflow.net/questions/353876

13. "Fourier transforms" as in https://arxiv.org/pdf/1501.02503#page=168 or https://tetrapharmakon.github.io/stuff/itaca.pdf

Weighted/diagonal category theory:

1. co/ends as centre/trace-infused co/limits: compare the co/end of $\mathrm{Hom}_{\mathcal{C}}$ with the co/limit of $\mathrm{Hom}_{\mathcal{C}}$

2. Codensity W-weighted monads, $\mathrm{Ran}^{[W]}_F(F)$;

3. Codensity diagonal monads, $\mathrm{DiRan}_F(F)$;

Profunctors:

1. Apartness defines a composition for relations, but its analogue

$$\mathfrak{q} \mathbin{\square} \mathfrak{p} \stackrel{\text{def}}{=} \int_{A \in \mathcal{C}} \mathfrak{p}^{-1}_{A} \coprod \mathfrak{q}^{A}_{-_2}$$

   fails to be unital for profunctors with the unit h^{A}_{-}. Is it unital for some other unit? Is there a less obvious analogue of apartness composition for profunctors? Or maybe does Prof equipped with $\square$ and units h^{A}_{-} form a skew bicategory?

   Is $\Delta_{\emptyset}$ a unit?

2. Figure what monoidal category structures on Sets induce associative and unital compositions on Prof.

3. https://mathoverflow.net/questions/470213/a-distributor-between-categories-induces-a-distributor-between-their-categories

4. Different compositions for profunctors from monoidal structures on the category of sets (e.g. https://mathoverflow.net/questions/155939/what-other-monoidal-structures-exist-on-the-category-of-sets)

5. Nucleus of a profunctor;

6. Isbell duality for profunctors:

   (a) https://mathoverflow.net/questions/259525/isbell-duality-for-profunctors

   (b) https://mathoverflow.net/questions/260322/the-mathfrak-l-functor-on-textsfprof

   (c) https://mathoverflow.net/questions/262462/again-on-the-mathfrak-l-functor-on-mathsfprof

Centres and Traces of Categories:

1. $\mathrm{K}_0(\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C}))$ vs. $\pi_0(\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C}))$ vs. $\mathrm{Tr}(\mathcal{C})$, and how these are generalisations of conjugacy classes for monoids

2. Explicitly work out the trace and $\pi_0\mathsf{Fun}(\mathrm{B}\mathbb{N}, -)$ for monoids with few elements.

3. $[1_A]$ can contain more than one element. An example is $\mathsf{Sets}(\mathbb{N}, \mathbb{N})$ and the maps given by

$$\{0, 1, 2, 3, \dots\} \mapsto \{0, 0, 1, 2, \dots\},$$
$$\{0, 1, 2, 3, \dots\} \mapsto \{2, 3, 4, 5, \dots\}.$$

   Show also that if $c \in [1_A]$, then c is idempotent.

4. Drinfeld centre

5. trace of the symmetric simplex category; it's probably different from that of $\mathsf{FinSets}$

6. Trace of Rep_G and interaction with induction, restriction, etc.

7. $\pi_0(\mathsf{B}\mathbb{N}, \mathsf{B}A)$, $K(\mathsf{B}\mathbb{N}, \mathsf{B}A)$, and $\mathrm{Tr}(\mathsf{B}\mathbb{N}, \mathsf{B}A)$ as concepts of conjugacy for monoids, their equivalents for categories, and comparison with traces

8. Comparison between $\pi_0(\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}))$ and $K(\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}))$

9. Lax, oplax, pseudo, and strict trace of simplex 2-category

10. duality over Γ might give a map from product of a monoid with a set to $\mathrm{Tr}(\Gamma)$

11. Studying the set $\mathrm{Nat}(\mathrm{id}_{\mathcal{C}}, F)$ as a notion of categorical trace:

    (a) Ganter–Kapranov define the trace of a 1-endomorphism $f\colon A \to A$ in a 2-category $\mathcal{C}$ to be the set $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_A, f)$;

        i. https://arxiv.org/abs/math/0602510

        ii. https://golem.ph.utexas.edu/string/archives/000757.html

        iii. https://ncatlab.org/nlab/show/categorical+trace

    We should study this notion in detail, and also study $\mathrm{Nat}(F, \mathrm{id}_{\mathcal{C}})$ as well as $\mathrm{CoNat}(\mathrm{id}_{\mathcal{C}}, F)$ and $\mathrm{CoNat}(F, \mathrm{id}_{\mathcal{C}})$.

12. Centre of bicategories

13. Lax centres and lax traces

14. Examples of traces:
    (a) Discrete categories
    (b) Posets
        i. $\mathsf{Open}(X)$
    (c) Trace of small but non-finite categories:
        i. Sets
        ii. $\mathsf{Rep}(G)$
        iii. category of finite groups
        iv. category of finite abelian groups
        v. category of finite p-groups for fixed p
        vi. category of finite p-groups for all p
        vii. category of finite fields
        viii. category of finite topological spaces
        ix. category of finite [insert a mathematical object here]
15. When is the trace of a groupoid just the disjoint sum of sets of conjugacy classes?
16. Set-theoretical issues when defining traces
    (a) Sets is a large category, and yet we can speak of its centre

$$\begin{aligned} \mathrm{Z}(\mathsf{Sets}) &\stackrel{\text{def}}{=} \int_{A \in \mathsf{Sets}} \mathsf{Sets}(X, X) \\ &\cong \mathrm{Nat}(\mathrm{id}_{\mathsf{Sets}}, \mathrm{id}_{\mathsf{Sets}}) \\ &\cong \mathrm{pt}. \end{aligned}$$

    Is there a way to do the same for the trace of sets, or otherwise work with traces of large categories?
17. Understand how traces are defined via universal properties in Xinwen Zhu's Geometric Satake, categorical traces, and arithmetic of Shimura varieties.
18. trace as an $\mathrm{Obj}(\mathcal{C})$-indexed set
    (a) properties, functoriality, etc.

19. Maybe actually call $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C})$ the categorical directed loop space of $\mathcal{C}$?

20. Cyclic version of $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C})$

21. Traces of categories, nerves of categories, and the cycle category

Categorical Hochschild Homology:

1. To any functor we have an associated natural transformation (**??**). Do we have sharp transformations associated to natural transformation?

2. build Hochschild co/simplicial set and study its homotopy groups

3. $\mathsf{Fun}(\mathsf{B}\mathbb{N}, X_\bullet)$ vs. $\mathsf{Fun}(\Delta^1/\partial\Delta^1, X_\bullet)$

   (a) Their π_0's vs. the π_0's of $\mathrm{Hom}_{X_\bullet}(x, x)$, of $\mathrm{Hom}^{\mathrm{L}}_{X_\bullet}(x, x)$, and $\mathrm{Hom}^{\mathrm{R}}_{X_\bullet}(x, x)$.

Monoidal Categories:

1. https://mathoverflow.net/questions/380302

2. Analogue of Picard rings for dualisable objects

3. Moduli of associators, braidings, etc. for species, q-species

4. When is the left Kan extension along a fully faithful functor of monoidal categories a strong monoidal functor?

5. Interaction between Day convolution and Isbell duality

6. general theory for lifting pseudomonads from Cat to Prof along the equipment embedding

7. definition of prostrength on a functor between promonoidal categories, differential 2-rigs fosco

8. Promonoidal structure in https://arxiv.org/pdf/1201.2991#page=22

9. Day convolution as a colimit over category of factorizations $F(A) \otimes_{\mathcal{C}} G(B) \to V$

10. Day convolution with respect to Cartesian monoidal structure is Cartesian monoidal. There's an easy proof of this with coend Yoneda
11. https://mathoverflow.net/questions/491234
12. https://mathoverflow.net/questions/488426/adjunction-of-monoidal-closed-categories
13. https://arxiv.org/abs/2502.02532
14. Does the forgetful functor $忘\colon \mathsf{IdemMon}(\mathcal{C}) \to \mathsf{Mon}(\mathcal{C})$ admit a left adjoint? What about $忘\colon \mathsf{IdemMon}(\mathcal{C}) \to \mathcal{C}$?
15. Clifford algebras in monoidal categories
16. Exterior algebras in monoidal categories
    (a) https://mathoverflow.net/questions/70607/exterior-powers-in-tensor-categories
    (b) https://mathoverflow.net/questions/127476/analogy-between-the-exterior-power-and-the-power-set
    (c) https://mathoverflow.net/questions/182476/delignes-exterior-power
    (d) martin brandenburg's phd thesis
17. Different monoidal products in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$ and their distributivity
    (a) Composition
    (b) Pointwise product
    (c) Day convolution
    (d) Relative monad version of Day convolution
18. Classification of monoidal structures on $\mathbb{\Delta}$
19. Classification of monoidal structures on Λ
20. Tensor Categories, 8.5.4

21. https://ncatlab.org/nlab/show/monoidal+action+of+a+monoidal+category
22. https://arxiv.org/abs/2203.16351
23. Para construction
24. Drinfeld center; Symmetric center; JY's books on bimonoidal categories
25. Picard and Brauer 2-groups
    (a) Differential Picard and Brauer Groups via $\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathsf{Mod}_R)$.
    (b) Brauer and Picard groups of $(\mathsf{Fun}(\mathcal{C},\mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$
    (c) Brauer and Picard groups of $\mathsf{Rep}(G)$
    (d) Brauer and Picard groups of Sets
    (e) Brauer and Picard groups of $\mathsf{Ch}_{\mathbb{Z}}(R)$
    (f) Brauer and Picard groups of $\mathsf{Shv}(X)$
    (g) Brauer and Picard groups of dgMod_R
26. Explore examples in which Day convolution gives weird things, like $\mathsf{Fun}\Big(\mathrm{B}\mathbb{Z}_{/n}, \mathsf{Sets}\Big)$.
27. Day convolution is a left Kan extension; explore the right Kan extension
28. Further develop the theory of moduli categories of monoidal structures
29. Picard group
    (a) Picard group for Day convolution. A special case is one of Kaplansky's conjectures, https://en.wikipedia.org/wiki/Kaplansky%27s_conjectures, about units of group rings
30. Day convolution between representable and an arbitrary presheaf $\mathcal{F}$ — can we prove something nice using the colimit formula for $\mathcal{F}$ in terms of representables?

31. Notion of braided monoidal categories in which the braiding is not an isomorphism. Relation to https://arxiv.org/abs/1307.5969

32. Proving a certain diagram between free monoidal categories commutes involves Fermat's little theorem. Can we reverse this and prove Fermat's little theorem from the commutativty of that diagram?

33. https://nilesjohnson.net/notes/grPic-P2S.pdf

34. Proof that monoidal equivalences F of monoidal categories automatically admit monoidal natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong F^{-1} \circ F$ and $\mathrm{id}_{\mathcal{D}} \cong F \circ F^{-1}$.

35. Proof that category with products is monoidal under the Cartesian monoidal structure, [MO 382264].

36. Explore 2-categorical algebra:

    (a) Find a construction of the free 2-group on a monoidal category. Apply it to the multiplicative structure on the category of finite sets and permutations, as well as to the multiplicative structure on the 1-truncation of the sphere spectrum, and try to figure out whether this looks like a categorification of $\mathbb{Q}$.

    (b) What is the free 2-group on $(\mathbb{\Delta}, \oplus, [0])$?

37. Categorify the preorder $\leq$ on $\mathbb{N}$ to a promonad $\mathfrak{p}$ on the groupoid of finite sets and permutations $\mathbb{F}$:

    (a) A preorder is a monad in Rel

    (b) A promonad is a monad in Prof.

    (c) There's a promonad $\mathfrak{p}$ in $\mathbb{F}$ defined by

    $$\mathfrak{p}(m, n) \overset{\text{def}}{=} \{\text{surjections from } \{1, \ldots, m\} \text{ to } \{1, \ldots, n\}\}$$

    This promonad categorifies $\leq$ in that its values are the witnesses to the fact that m is bigger than n (i.e. surjections).

    (d) Figure out whether this promonad extends to the 1-truncation of the sphere spectrum, and perhaps to other categorified analogues of monoids/groups/rings.

38. https://arxiv.org/abs/1307.5969
39. https://arxiv.org/abs/1306.3215
40. https://mathoverflow.net/questions/477219/reference-for-the-monoidal-category-structure-x-otimes-y-x-y-x-times-y
41. Include an explicit proof of ??
42. Include an explicit proof of ??
43. ??
44. obstruction theory for braided enhancements of monoidal categories, using the "moduli category of braided enhancements"
45. Define symmetric and exterior algebras internal to braided monoidal categories
    (a) https://mathoverflow.net/questions/471372/is-there-an-alternating-power-functor-on-braided-monoidal-categories
    (b) https://arxiv.org/abs/math/0504155
46. https://mathoverflow.net/q/382364
47. https://mathoverflow.net/q/471490
48. Concepts of bicategories applied to monoidal categories (e.g. internal adjunctions lead to dualisable objects)
49. Involutive Category Theory
50. https://mathoverflow.net/questions/474662/the-analogy-between-dualizable-categories-and-compact-hausdorff-spaces

Bimonoidal Categories:

1. Bimonoidal structures on the category of species
2. Include an explicit proof of ??

Six Functor Formalisms:

1. Michael Shulman:

   > A lot of the ”six functor formalism” makes sense in the context of an arbitrary indexed monoidal category (= monoidal fibration), particularly with cartesian base. In particular, I studied the external tensor product in this generality in my paper on Framed bicategories and monoidal fibrations.
   >
   > The internal-hom of powersets in particular, with Ø as a dualizing object, is well-known in constructive mathematics and topos theory, where powersets are in general a Heyting algebra rather than a Boolean algebra.

   Morgan Rogers:

   > I second this: you’re discovering (and making pleasingly explicit, I might add) a special case of ”thin category theory”: a lot of what you’ve discovered will work for posets, with the powerset replaced with the frame of downsets :D

2. A six functor formalism for monoids

3. https://mathoverflow.net/questions/258159/yoga-of-six-functors-for-group-representations

4. Is the 1-categorical analogue of six functor formalisms given by Mann interesting?

   (a) Mann defines:

   > A six functor formalism is an ∞-functor $f\colon \mathsf{Corr}(C, E) \to \mathsf{Cats}_\infty$ such that $-\otimes A$, f^*, and $f_!$ admit right adjoints

   (b) Is the notion

   > A 1-categorical six functor formalism is a (lax?) 2-functor $f\colon \mathsf{Corr}(C, E) \to \mathsf{Cats}_2$ (or should Cats be the target?) such that $-\otimes A$, f^*, and $f_!$ admit right adjoints

interesting?

5. Interaction of the six functors with Kan extensions (e.g. how the left Kan extension of $-\otimes A$ may interact with the other functors)

6. Contexts like Wirthmuller Grothendieck etc

7. formalisation by cisinski and deglise

8. How do the following examples fit?

   (a) base change between $\mathsf{C}_{/X}$ and $\mathsf{C}_{/Y}$
   (b) $f_! \dashv f_* \dashv f^*$ adjunction between powersets
   (c) $f_! \dashv f_* \dashv f^*$ adjunction between $\mathsf{Span}(\mathrm{pt}, A)$ and $\mathsf{Span}(\mathrm{pt}, B)$
   (d) quadruple adjunction between powersets induced by a relation
   (e) adjunctions between categories of presheaves induced by a functor or a profunctor
   (f) Adjunction between left A-sets and left B-sets

   Do they have exceptional $f^!$? Is there a notion of Fourier–Mukai transform for them? What kind of compatibility conditions (proper base change, etc.) do we have?

Skew Monoidal Categories:

1. https://arxiv.org/abs/2506.06847

2. Try to come up with examples of skew monoidal categories by twisting a tensor product $A \otimes B$ into $T(A) \otimes B$. Related idea: product of G-sets but twisted on the left by an automorphism of G, so that $(ag, b) \sim (a, gb)$ becomes $(a\phi(g), b) \sim (a, gb)$.

3. Skew monoidal category induced from G-sets in analogy to Rel

4. Free monoidal category on a skew monoidal category

5. Skew monoidal structures associated to a locally Cartesian closed category

6. Does the $\mathbb{E}_1$ tensor product of monoids admit a skew monoidal category structure?

7. Is there a (right?) skew monoidal category structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ using right Kan extensions instead of left Kan extensions?
8. Similarly, are there skew monoidal category structures on the subcategory of $\mathbf{Rel}(A, B)$ spanned by the functions using left Kan extensions and left Kan lifts?
9. Add example: $\mathcal{C}$ with coproducts, take $\mathcal{C}_{X/}$ and define

$$\left(X \xrightarrow{f} A\right) \oplus \left(X \xrightarrow{g} B\right) \stackrel{\text{def}}{=} \left[X \to X \coprod X \xrightarrow{f \coprod g} A \coprod B\right]$$

10. Duals:
    (a) Dualisable objects in monoidal categories and traces of endomorphisms of them, including also examples for monoidal categories which are not autonomous/rigid, such as $(\mathsf{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$.
    (b) compact closed categories
    (c) star autonomous categories
    (d) Chu construction
    (e) Balanced monoidal categories, https://ncatlab.org/nlab/show/balanced+monoidal+category
    (f) Traced monoidal categories, https://ncatlab.org/nlab/show/traced+monoidal+category
11. Invertible objects and Picard groupoids
12. https://mathoverflow.net/questions/155939/what-other-monoidal-structures-exist-on-the-category-of-sets
13. Free braided monoidal category with a braided monoid: https://ncatlab.org/nlab/show/vine
14. https://golem.ph.utexas.edu/category/2024/08/skew_monoidal_categories_throu.html

Fibred Category Theory:

1. https://arxiv.org/abs/2402.11644
2. https://categorytheory.zulipchat.com/#narrow/channel/229136-theory.3A-category-theory/topic/A.20.22change.20of.20variables.22.20for.20the.20Grothendieck.20construction/near/495776958
3. Internal **Hom** in categories of co/Cartesian fibrations.
4. *Tensor structures on fibered categories* by Luca Terenzi: https://arxiv.org/abs/2401.13491. Check also the other papers by Luca Terenzi.
5. https://ncatlab.org/nlab/show/cartesian+natural+transformation (this is a cartesian morphism in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ apparently)
6. CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

Operads and Multicategories:

1. Simplicial lists in operad theory I

Monads:

1. Relative monads: message Alyssa asking for her notes
2. https://ncatlab.org/nlab/show/adjoint+monad
3. Kantorovich monad (https://ncatlab.org/nlab/show/Kantorovich+monad) and probability monads in general, https://ncatlab.org/nlab/show/monads+of+probability%2C+measures%2C+and+valuations.

Enriched Categories:

1. $\mathcal{V}$-matrices

Bicategories:

1. Bicategories of Lax Fractions, https://arxiv.org/abs/2507.12044

2. Linear bicategories, https://ncatlab.org/nlab/show/linear+bicategory
   (a) Linearly distributive category, https://ncatlab.org/nlab/show/linearly+distributive+category
   (b) Diagrammatic Algebra of First Order Logic
   (c) Constructing linear bicategories
   (d) Introduction to linear bicategories
3. Allegories, https://ncatlab.org/nlab/show/allegory
4. Skew bicategories
5. Bigroupoid cardinality
6. Bicategory where objects are groups and a morphism $G \nrightarrow H$ is a representation of $G^{\mathsf{op}} \times H$. (I.e. functors $\mathsf{B}G^{\mathsf{op}} \times \mathsf{B}H \to \mathsf{Vect}_k$).
7. Relative monads internal to a bicategory
8. Bicategory of monoid actions
9. https://arxiv.org/abs/0809.1760
10. $\mathrm{Rel}_G \overset{\mathrm{def}}{=} \mathsf{Fun}(\mathsf{B}G, \mathrm{Rel})$
11. Rel but for Ab, where morphisms are pairings of the form $A \otimes_{\mathbb{Z}} B \to \mathbb{Z}$.
12. 2-dimensional co/limits in 2-category of categories and adjoint functors
13. Category of equivalence classes
   (a) Given a category $\mathcal{C}$, we have a set $\mathrm{K}_0(\mathcal{C})$ of isomorphism classes of objects
   (b) Given a bicategory $\mathcal{C}$, there should be a category $\mathsf{K}_0(\mathcal{C})$ with $\mathrm{Hom}_{\mathsf{K}_0(\mathcal{C})}(A, B) \overset{\mathrm{def}}{=} \mathrm{K}_0(\mathsf{Hom}_{\mathcal{C}}(A, B))$
   (c) The set $\mathrm{K}_0^{\mathrm{eq}}(\mathcal{C})$ of equivalence classes of objects of $\mathcal{C}$ should then satisfy

$$\mathrm{K}_0^{\mathrm{eq}}(\mathcal{C}) \cong \mathrm{K}_0(\mathsf{K}_0(\mathcal{C})).$$

14. bicategory of chain complexes, section "Second Example: Differential Complexes of an Abelian Category" on Gabriel–Zisman's calculus of fractions

15. 2-vector spaces

16. Morita equivalence is equivalence internal to bimod

17. `https://mathoverflow.net/questions/478867/2-category-structure-on-modr`

18. Bicategories of matrices, as in Street's Variation through enrichment, also `https://arxiv.org/abs/2410.18877`

19. `https://mathoverflow.net/a/86933`

20. What are the internal 2-adjunctions in the fundamental 2-groupoid of a space?

21. 2-category structure on Mod_R, where a 2-morphism is a commutative square. Characterisation of adjuntions therein

22. Cook up a very large list of examples of bicategories, like the ones I made for the AI problems. In particular, find an interesting bicategory of representations qualitatively different from the one I described in the Epoch AI problem

23. 2-category structure on category of R-algebras as enriched Mod_R-categories

24. Let $\mathcal{C}$ be a bicategory, let $A, B \in \mathrm{Obj}(\mathcal{C})$, and let $F, G \in \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(A, B))$.

    (a) Does precomposition with $\lambda^{\mathcal{C}}_{A|F} \colon \mathrm{id}_A \circ F \Rightarrow F$ induce an isomorphism of sets

    $$\mathrm{Hom}_{\mathsf{Hom}_{\mathcal{C}}(A,B)}(F, G) \cong \mathrm{Hom}_{\mathsf{Hom}_{\mathcal{C}}(A,B)}(F \circ \mathrm{id}_A, G)$$

    for each $F, G \in \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(A, B))$?

    (b) Similarly, do we have an induced isomorphism of the form

    $$\mathrm{Hom}_{\mathsf{Hom}_{\mathcal{C}}(A,B)}(F, G) \cong \mathrm{Hom}_{\mathsf{Hom}_{\mathcal{C}}(A,B)}(F, \mathrm{id}_B \circ G)$$

    and so on?

25. Are there two Duskin nerve functors? (lax/oplax/etc.?)

26. Interaction with cotransformations:

    (a) Can we abstract the structure provided to Cats_2 by natural cotransformations?

    (b) Are there analogues of cotransformations for **Rel**, Span, BiMod, MonAct, etc.?

    (c) Perhaps this might also make sense as a 1-categorical definition, e.g. comorphisms of groups from A to B as $\mathsf{Sets}(A, B)$ quotiented by $f(ab) \sim f(a)f(b)$.

27. Consider developing the analogue of traces for endomorphisms of dualisable objects in monoidal categories to the setting of bicategories, including e.g. the trace of a category as a trace internal to Prof.

28. Centres of bicategories (lax, strict, etc.)

29. Concepts of monoidal categories applied to bicategories (e.g. traces)

30. Internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].

31. Comonads in the bicategory of profunctors.

32. 2-limit of $\mathrm{id}, \mathrm{id} \colon \mathsf{Sets} \rightrightarrows \mathsf{Sets}$ is $\mathrm{B}\mathbb{Z}$, https://mathoverflow.net/questions/209904/van-kampen-colimits?rq=1#comment520288_209904

33. https://mathoverflow.net/questions/473527/universal-property-of-2-presheaves-and-pseudo-lax-colax-natural-transformations

34. https://mathoverflow.net/questions/473526/free-cocompletion-of-a-2-category-under-pseudo-colimits-lax-colimits-and-colax

Types of Morphisms in Bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.
2. Statements like "coequifiers are lax epimorphisms", Item 2 of Examples 2.4 of https://arxiv.org/abs/2109.09836, along with most of the other statements/examples there.
3. Dense, absolutely dense, etc. morphisms in bicategories

Internal adjunctions:

1. https://www.google.com/search?q=mate+of+an+adjunction
2. Moreover, by uniqueness of adjoints (Internal Adjunctions, **??** of **??**), this implies also that $S = f^{-1}$.
3. define bicategory $\mathsf{Adj}(\mathcal{C})$
4. walking monad
5. proposition: 2-functors preserve unitors and associators
6. https://ncatlab.org/nlab/show/2-category+of+adjunctions. Is there a 3-category too?
7. https://ncatlab.org/nlab/show/free+monad
8. https://ncatlab.org/nlab/show/CatAdj
9. https://ncatlab.org/nlab/show/Adj
10. $\mathsf{Adj}(\mathsf{Adj}(\mathcal{C}))$
11. Examples of internal adjunctions
    (a) Internal adjunctions in Mod.
    (b) Internal adjunctions in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$.
    (c) Internal adjunctions in $\mathsf{LaxFun}(\mathcal{C}, \mathcal{D})$.
    (d) Internal adjunctions in 2-categories related to fibrations.

2-Categorical Limits:

1. https://sorilee.github.io/posts/strict-bilimit-and-its-proper-examples

Double Categories:

1. Ehresmann
2. https://arxiv.org/abs/2505.08766
3. https://arxiv.org/abs/2504.18065
4. https://arxiv.org/abs/2504.11099
5. Pinwheel/Yojouhan diagrams and compositionality, section on nLab at https://ncatlab.org/nlab/show/double+category

Homological Algebra:

1. https://arxiv.org/abs/2505.08321
2. https://mathoverflow.net/questions/418676/derived-functor-of-functor-tensor-product
3. https://math.stackexchange.com/questions/3665036/higher-chain-homotopies

Topos theory:

1. https://arxiv.org/abs/2505.08766
2. https://arxiv.org/abs/2304.05338
3. https://arxiv.org/abs/2503.20664
4. https://arxiv.org/abs/2204.08351
5. https://arxiv.org/abs/2404.12313
6. https://www.teses.usp.br/teses/disponiveis/45/45131/tde-31082023-163143/en.php
7. https://teses.usp.br/teses/disponiveis/45/45131/tde-24042019-195658/pt-br.php

8. https://mathoverflow.net/q/479496
9. Grothendieck topologies on $\mathsf{B}A$
10. Enriched Grothendieck topologies
    (a) Borceux–Quintero, https://www.numdam.org/item/CTGDC_1996__37_2_145_0/
    (b) https://arxiv.org/abs/2405.19529
11. Cotopos theory:
    (a) Copresheaves and copresheaf cotopoi
    (b) Elementary cotopoi
        i. https://mathoverflow.net/questions/474287/intuition-for-the-internal-logic-of-a-cotopos
        ii. https://mathoverflow.net/questions/394098/what-is-a-cotopos

            In case you haven't seen it yet, Grothendieck studies (pseudo) cotopos in pursuing stacks

Formal category theory:

1. Yosegi boxes https://arxiv.org/abs/1901.01594

Homotopical Algebra:

1. https://arxiv.org/abs/2109.07803

Simplicial stuff:

1. https://arxiv.org/abs/2507.15341
2. https://arxiv.org/abs/2503.13663
3. https://www.math.univ-paris13.fr/~harpaz/quasi_unital.pdf
    (a) slogan: geometric definition of ∞-categories should be geometric for identities too

(b) In an ∞-category, define a **quasi-unit** to be a 1-morphism f such that

$$
\begin{aligned}
[f]_* &\colon \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Spaces})}(\mathrm{Hom}_{\mathcal{C}}(X, A)\, \mathrm{Hom}_{\mathcal{C}}(X, B)),\\
[f]^* &\colon \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Spaces})}(\mathrm{Hom}_{\mathcal{C}}(B, X)\, \mathrm{Hom}_{\mathcal{C}}(A, X))
\end{aligned}
$$

are the identity in $\mathsf{Ho}(\mathsf{Spaces})$. Explore equivalent conditions,

(c) https://arxiv.org/abs/1606.05669

(d) https://arxiv.org/abs/1702.08696

4. https://arxiv.org/abs/math/0507116, https://arxiv.org/abs/2503.11338

5. https://arxiv.org/abs/2302.02484 and https://arxiv.org/abs/2411.19751

6. Internal adjunctions in $\mathbb{\Delta}$ are the same as Galois connections between $[n]$ and $[m]$.

7. https://mathoverflow.net/q/478461

8. draw coherence for lax functors using the diagram for Δ^2

9. characterisation of simplicial sets such that left, right, and two-sided homotopies agree

10. every continuous simplicial set arises as the nerve of a poset.

11. Functor sd is convolution of $\text{よ}_{\mathbb{\Delta}}$ with itself; see https://arxiv.org/pdf/1501.02503.pdf#page=109

12. Extra degeneracies

    (a) https://www.google.com/search?client=firefox-b-d&q=augmented+simplicial+objects+with+extra+degeneracies

    (b) https://leanprover-community.github.io/mathlib_docs/algebraic_topology/extra_degeneracy.html

13. Comparison between $\Delta^1/\partial\Delta^1$ and $\mathsf{B}\mathbb{N}$

∞-Categories:

1. https://arxiv.org/abs/2505.22640
2. https://arxiv.org/abs/2410.17102
3. https://arxiv.org/abs/2410.02578, https://scholar.colorado.edu/concern/graduate_thesis_or_dissertations/st74cr650, https://arxiv.org/abs/2206.00849
4. https://mathoverflow.net/questions/479716/non-strictly-unital-functors-of-infinity-categories
5. https://mathoverflow.net/questions/472253/whats-the-localization-of-the-infty-category-of-categories-under-inverting-f

Condensed Mathematics:

1. https://golem.ph.utexas.edu/category/2020/03/pyknoticity_versus_cohesivenes.html#c057724
2. https://golem.ph.utexas.edu/category/2020/03/pyknoticity_versus_cohesivenes.html#c057810
3. https://maths.anu.edu.au/news-events/events/universal-property-category-condensed-sets
4. https://grossack.site/2024/07/03/life-in-johnstones-topological-topos
5. https://grossack.site/2024/07/03/topological-topos-2-algebras
6. https://grossack.site/2024/07/03/topological-topos-3-bonus-axioms
7. https://terrytao.wordpress.com/2025/04/23/stonean-spaces-projective-objects-the-riesz-representation-theorem-and-possibly-condensed-mathematics/

Monoids:

1. https://mathoverflow.net/questions/278429/
2. Homological algebra of A-sets, https://arxiv.org/abs/1503.02309
3. Catalan monoids, https://arxiv.org/abs/1309.6120
4. https://mathoverflow.net/questions/438305/grothendieck-group-of-the-fibonacci-monoid
5. https://math.stackexchange.com/questions/2662005/how-much-of-a-group-g-is-determined-by-the-category-of-g-sets
6. https://math.stackexchange.com/a/4996051/603207, https://arxiv.org/abs/1006.5687
7. Six functor formalism for monoids, following Constructions With Sets, Section 4.6.4, but in which $\cap$ and $[-,-]$ are replaced with Day convolution.
8. Monoid $(\{1,\ldots,n\}\cup\infty,\gcd)$. The element ∞ can be replaced by $p_1^{\min\left(e_1^1,\ldots,e_1^m\right)}\cdots p_k^{\min\left(e_k^1,\ldots,e_k^m\right)}$.
9. Universal property of localisation of monoids as a left adjoint to the forgetful functor $\mathcal{C}\to\mathcal{D}$, where:
   - $\mathcal{C}$ is the category whose objects are pairs (A,S) with A a monoid and S a submonoid of A.
   - $\mathcal{D}$ is the category whose objects are pairs (A,S) with A a monoid and S a submonoid of A which is also a group.

   Explore this also for localisations of rings

   Explore if we can define field spectra with an approach like this
10. Adjunction between monoids and monoids with zero corresponding to $(-)^{-}\dashv(-)^{+}$
11. Rock paper scissors as an example of a non-associative operation

12. https://mathoverflow.net/questions/438305/grothendieck-group-of-the-fibonacci-monoid

13. Witt monoid, https://www.google.com/search?q=Witt+monoid

14. semi-direct product of monoids, https://ncatlab.org/nlab/show/semidirect+product+group

15. morphisms of monoids as natural transformation between left A-sets over A and B_A.

16. Figure out if 2-morphisms of monoids coming from $\mathsf{Fun}^{\otimes}(A_{\mathsf{disc}}, B_{\mathsf{disc}})$, $\mathsf{PseudoFun}(\mathsf{B}A, \mathsf{B}B)$, etc. are interesting

17. Write sections on the quotient and set of fixed points of a set by a monoid action

18. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:

    (a) A morphism $f\colon A \to B$ of semigroups is an epimorphism.

    (b) For each $b \in B$, one of the following conditions is satisfied:

    - We have $f(a) = b$.
    - There exist some $m \in \mathbb{N}_{\geq 1}$ and two factorisations

      $$\begin{aligned} b &= a_0 y_1, \\ b &= x_m a_{2m} \end{aligned}$$

      connected by relations

      $$\begin{aligned} a_0 &= x_1 a_1, \\ a_1 y_1 &= a_2 y_2, \\ x_1 a_2 &= x_2 a_3, \\ a_{2m-1} y_m &= a_{2m} \end{aligned}$$

      such that, for each $1 \leq i \leq m$, we have $a_i \in \mathrm{Im}(f)$.

    Wikipedia says in https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem:

For monoids, this theorem can be written more concisely:

19. Representation theory of monoids
    (a) https://mathoverflow.net/questions/37115/why-arent-representations-of-monoids-studied-so-much
    (b) Representation theory of groups associated to monoids (groups of units, group completions, etc.)

Monoid Actions:

1. https://link.springer.com/book/10.1007/978-3-642-11297-3
2. https://ncatlab.org/schreiber/files/EquivariantInfinityBundles_220809.pdf has some interesting things, like a fully faithful embedding of $\mathsf{Mon}\left(\mathsf{Sets}_A^{\mathrm{L}}\right)$ into $\mathsf{Mon}_{/A}$ whose essential image is given by those monoids of the form $X \rtimes_\alpha A$.
3. $f_! \dashv f^* \dashv f_*$ adjunction
    (a) Is it related to the Kan extensions adjunction for $f \colon \mathsf{B}A \to \mathsf{B}B$ and the categories $\mathsf{Sets}_A^{\mathrm{L}} \cong \mathsf{PSh}(\mathsf{B}A^{\mathsf{op}}, \mathsf{Sets})$ and $\mathsf{Sets}_B^{\mathrm{L}} \cong \mathsf{PSh}(\mathsf{B}B^{\mathsf{op}}, \mathsf{Sets})$?
    (b) Is it related to the cobase change adjunction of https://ncatlab.org/nlab/show/base+change? Maybe we can take a morphism of monoids $f \colon A \to B$ and consider B_A^{L} as a left A-set, and then $\left(\mathsf{Sets}_A^{\mathrm{L}}\right)_{A/}$ and $\left(\mathsf{Sets}_A^{\mathrm{L}}\right)_{B_A^{\mathrm{L}}/}$
4. https://arxiv.org/abs/2112.10198
5. double category of monoid actions
6. Analogue of Brauer groups for A-sets
7. Hochschild homology for A-sets

Group Theory:

1. https://mathoverflow.net/questions/45651/is-there-a-q-analog-to-the-braid-group

2. https://johncarlosbaez.wordpress.com/2025/03/27/the-mcgee-group/
3. https://bookstore.ams.org/memo-1-2/
4. https://link.springer.com/book/10.1007/978-3-662-59144-4
5. https://en.wikipedia.org/wiki/Tits_group
6. https://en.wikipedia.org/wiki/Group_of_Lie_type
7. https://mathoverflow.net/questions/251769/what-meanings-does-chevalley-group-have
8. https://encyclopediaofmath.org/wiki/Chevalley_group
9. https://en.wikipedia.org/wiki/Group_of_Lie_type
10. MO: cardinality of $\mathrm{Cl}(\mathrm{Aut}(\mathrm{GL}_n(\mathbb{F}_q)))$
11. https://math.stackexchange.com/questions/4419869/do-the-groups-operatornamesl-operatornamepgl-and-operatornamepsl
12. https://groupprops.subwiki.org/wiki/Order_formulas_for_linear_groups
13. https://groupprops.subwiki.org/wiki/Order_of_semidirect_product_is_product_of_orders
14. https://groupprops.subwiki.org/wiki/Central_automorphism_group_of_general_linear_group
15. https://groupprops.subwiki.org/wiki/Automorphism_group_of_general_linear_group_over_a_field
16. https://groupprops.subwiki.org/wiki/Inner-centralizing_automorphism
17. https://math.stackexchange.com/questions/2519372/number-of-conjugacy-classes-for-the-modular-group

18. $\mathrm{GL}_n(K)$ for K a skew field
19. https://arxiv.org/abs/1212.6157, https://arxiv.org/abs/0708.1608, https://en.wikipedia.org/wiki/Wild_problem, https://www.google.com/search?q=matrix+pair+problem, https://arxiv.org/abs/2007.09242, https://mathoverflow.net/questions/291815/rational-canonical-form-over-mathbbz-pk-mathbbz, https://mathoverflow.net/questions/291815/rational-canonical-form-over-mathbbz-pk-mathbbz
20. https://link.springer.com/book/10.1007/978-981-13-2895-4
21. https://ysharifi.wordpress.com/2022/09/14/automorphisms-of-dihedral-groups/
22. https://en.wikipedia.org/wiki/PSL(2,7)
23. https://arxiv.org/abs/2304.08617
24. https://johncarlosbaez.wordpress.com/2016/03/22/the-involute-of-a-cubical-parabola/#comment-78884
25. https://arxiv.org/abs/0904.1876
26. finite subgroups of $\mathrm{SU}(2)$, and viewing them as groups of rotations and such
27. https://arxiv.org/abs/1201.2363
28. https://ncatlab.org/nlab/show/group+extension#SchreierTheory, https://ncatlab.org/nlab/show/nonabelian+cohomology, https://ncatlab.org/nlab/show/nonabelian+group+cohomology
29. https://en.wikipedia.org/wiki/Fibonacci_group
30. Study the functoriality properties of $G \mapsto \mathrm{Aut}(G)$ via functoriality of ends
31. Is $\sum_{[g]\in\mathrm{Cl}(G)} \frac{1}{|g|}$ an interesting invariant of G?

32. Idempotent endomorphism $f\colon A \to A$ is the same as a decomposition $A \cong B \oplus C$ via $B \cong \mathrm{Im}(f)$ and $C \cong \mathrm{Ker}(f)$.
    (a) https://mathstrek.blog/2015/03/02/idempotents-and-decomposition/
33. https://math.stackexchange.com/questions/34271/order-of-general-and-special-linear-groups-over-finite-fields

Linear Algebra:

1. Size of conjugacy class $[A]$ of $A \in \mathrm{GL}_n(\mathbb{F}_q)$ is given by $\#\mathrm{GL}_n(\mathbb{F}_q)$ divided by the centralizer $\mathrm{Z}_{\mathrm{GL}_n(\mathbb{F}_q)}(A)$ of A in $\mathrm{GL}_n(\mathbb{F}_q)$, whose order is given by
$$\begin{aligned}\#\mathrm{Z}_{\mathrm{GL}_n(\mathbb{F}_q)}(A) &= \prod_{i=1}^{k} \#\mathrm{GL}_{r_i}(\mathbb{F}_q)\\ &= q^{\sum_{i=1}^{k}\binom{r_i}{2}} \prod_{i=1}^{k}\prod_{j=0}^{r_i-1}\left(q^{r_i-j}-1\right)\end{aligned}$$
if A is diagonalisable with eigenvalues $\lambda_1, \ldots, \lambda_k$ having multiplicities $r_1, \ldots, r_k$. More generally, see https://groupprops.subwiki.org/wiki/Conjugacy_class_size_formula_in_general_linear_group_over_a_finite_field
2. https://en.wikipedia.org/wiki/Semilinear_map
3. conjugacy for $\mathrm{GL}_n(\mathbb{F}_q)$, https://mathoverflow.net/a/104457
4. https://en.wikipedia.org/wiki/Dieudonn%C3%A9_determinant, https://ncatlab.org/nlab/show/Dieudonn%C3%A9+determinant#Dieudonne
5. https://ncatlab.org/nlab/show/Pfaffian
6. https://math.stackexchange.com/questions/1715249/the-number-of-subspaces-over-a-finite-field
7. https://math.stackexchange.com/questions/70801/how-many-k-dimensional-subspaces-there-are-in-n-dimensional-vector-space-over

8. https://en.wikipedia.org/wiki/Gaussian_binomial_coefficient
9. https://en.wikipedia.org/wiki/List_of_q-analogs

Noncommutative Algebra:

1. https://arxiv.org/abs/1608.08140
2. https://arxiv.org/abs/2401.12884
3. https://ncatlab.org/nlab/show/dihedral+homology
4. https://www.sciencedirect.com/science/article/pii/0022404995000836
5. https://arxiv.org/abs/2008.11569, https://www.lakeheadu.ca/sites/default/files/uploads/77/docs/Cox%20Daniel.pdf

Commutative Algebra:

1. If $M \in \mathrm{Pic}(R)$, then $\mathrm{Aut}(M) \cong R^{\times}$.
2. https://math.stackexchange.com/questions/637918/
3. https://categorytheory.zulipchat.com/#narrow/stream/411257-theory.3A-mathematics/topic/Big.20Witt.20ring
4. https://math.stackexchange.com/questions/535623/how-many-irreducible-factors-does-xn-1-have-over-finite-field
5. Derivations between morphisms of R-algebras, after https://mathoverflow.net/questions/434488
   (a) Namely, a derivation from a morphism $f\colon A \to B$ of R-algebras to a morphism $g\colon A \to B$ of R-algebras is a map $D\colon B \to B$ such that we have
   $$D(ab) = g(a)D(b) + D(a)f(b)$$
   for each $a, b \in B$.

Hyper Algebra:

1. https://arxiv.org/abs/2205.02362
2. http://www.numdam.org/item/SD_1959-1960__13_1_A9_0/
3. https://www.worldscientific.com/worldscibooks/10.1142/13652#t=aboutBook

Coalgebra:

1. https://mathoverflow.net/questions/483668/textrepd-4-and-its-three-fiber-functors

Topological Algebra:

1. https://golem.ph.utexas.edu/category/2014/08/holy_crap_do_you_know_what_a_c.html
2. https://categorytheory.zulipchat.com/#narrow/channel/411257-theory.3A-mathematics/topic/topological.20rings.20and.20fields
3. https://mathoverflow.net/q/477757
4. https://math.stackexchange.com/questions/2593556/galois-theory-for-topological-fields

Differential Graded Algebras:

1. https://mathoverflow.net/questions/476150/constructing-an-adjunction-between-algebras-and-differential-graded-algebras

Topology:

1. https://arxiv.org/abs/2507.18418
2. Topologies on $\mathcal{P}(\mathcal{P}(X))$, https://mathoverflow.net/questions/496630/topological-analogues-of-gromov-hausdorff-convergence
3. https://mathoverflow.net/questions/255912/what-is-the-structure-associated-to-almost-everywhere-convergence

4. https://arxiv.org/abs/2504.12965
5. https://mathoverflow.net/questions/485669/exponential-law-for-topological-spaces-for-the-topology-of-pointwise-convergence and comments therein
6. This paper has some cool references on convergence spaces: https://arxiv.org/abs/2410.18245
7. https://arxiv.org/abs/2402.12316
8. Write about the 6-functor formalism for sheaves on topological spaces and for topological stacks, with lots of examples.
   (a) MO question titled *6-functor formalism for topological stacks*: https://mathoverflow.net/q/471758

Measure Theory:

1. https://mathoverflow.net/questions/126994/beck-chevalley-for-measures
2. https://mathoverflow.net/questions/483726
3. https://en.wikipedia.org/wiki/Valuation_%28measure_theory%29
4. There's a theorem saying that there does not exist an infinite-dimensional "Lebesgue" measure, i.e. (from https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure):

   > Let X be an infinite-dimensional, separable Banach space. Then, the only locally finite and translation invariant Borel measure μ on X is a trivial measure. Equivalently, there is no locally finite, strictly positive, and translation invariant measure on X.

   What kind of measures exist/not exist that satisfy all conditions above except being locally finite?
5. https://ncatlab.org/nlab/show/categories+of+measure+theory

6. Functions $f_!$, f^*, and f_* between spaces of (probability) measures on probability/measurable spaces, mimicking how a map of sets $f\colon X \to Y$ induces morphisms of sets $f_!$, f^*, and f_* between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

7. Analogies between representable presheaves and the Yoneda lemma on the one hand and Dirac probability measures on the other hand

   (a) Universal property of the embedding of a space X into the space of probability measures on X

   (b) Same question but for distributions

   (c) non-symmetric metric on space of probability measures where we define $\mathrm{d}(\mu, \nu)$ to be the measure given by

   $$U \mapsto \int_U \rho_\mu \,\mathrm{d}\nu,$$

   where ρ_μ is the probability density of μ. Can we make this idea work?

8. https://arxiv.org/abs/0801.2250

9. https://mathoverflow.net/questions/325861

   In particular, I came across a PhD thesis by Martial Agueh. I thought it was interesting because it explicitly investigated the geodesics of Wasserstein space to produce solutions to a type of parabolic PDE.

Probability Theory:

1. https://en.wikipedia.org/wiki/Wiener_sausage

2. https://link.springer.com/book/10.1007/978-3-319-20828-2

3. https://arxiv.org/abs/2406.10676

4. Lévy's forgery theorem

5. https://www.epatters.org/wiki/stats-ml/categorical-probability-theory

6. https://ncatlab.org/nlab/show/category-theoretic+approaches+to+probability+theory
7. Categorical probability theory
8. https://golem.ph.utexas.edu/category/2024/08/introduction_to_categorical_pr.html
9. https://arxiv.org/abs/1109.1880
10. Connection between fractional differential operators and stochastic processes with jumps

Statistics:

1. https://towardsdatascience.com/t-test-from-application-to-theory-5e5051b0f9dc

Metric Spaces:

1. Lawvere metric spaces: object of $\mathcal{V}$-natural transformations corresponds to $\inf(\mathrm{d}(f(x), g(x)))$.
2. Does the assignment $d(x, y) \mapsto d(x, y)/(1 + d(x, y))$ constructing a bounded metric from a metric be given a universal property?
3. Explore Lawvere metric spaces in a comprehensive manner
4. metric $\operatorname{lcm}(x, y)/\gcd(x, y)$ on $\mathbb{N}$, https://mathoverflow.net/questions/461588/. What shape do balls on $\mathbb{N} \times \mathbb{N}$ have with respect to this metric?
5. https://golem.ph.utexas.edu/category/2023/05/metric_spaces_as_enriched_categories_ii.html
6. Simon Willerton's work on the Legendre–Fenchel transform:
   (a) https://golem.ph.utexas.edu/category/2014/04/enrichment_and_the_legendrefen.html
   (b) https://golem.ph.utexas.edu/category/2014/05/enrichment_and_the_legendrefen_1.html
   (c) https://arxiv.org/abs/1501.03791

Special Functions:

1. https://en.wikipedia.org/wiki/Dickson_polynomial

p-Adic Analysis:

1. https://arxiv.org/abs/2503.08909
2. Analysis of functions $\mathbb{Z}_p \to \mathbb{Q}_q$, $\mathbb{Q}_p \to \mathbb{Q}_q$, $\mathbb{Z}_p \to \mathbb{C}_q$, etc.
   (a) https://siegelmaxwellc.wordpress.com/publications-pre-prints/

Partial Differential Equations:

1. Moduli of PDEs
   (a) https://arxiv.org/abs/2312.05226, https://arxiv.org/abs/2406.16825
   (b) https://arxiv.org/abs/2304.08671, https://arxiv.org/abs/2404.07931
   (c) https://arxiv.org/abs/2507.07937
2. https://en.wikipedia.org/wiki/Homotopy_principle
3. https://mathoverflow.net/questions/125166/wild-solutions-of-the-heat-equation-how-to-graph-them
4. https://math.stackexchange.com/questions/2112841/difference-between-linear-semilinear-and-quasilinear-pdes/5036699#5036699
5. Proof of the smoothing property of the heat equation via:
   (a) Feynman–Kac formula
   (b) Radon–Nikodym + Wiener process has Gaussian as PDF
   (c) Convolution of locally integrable with smooth is smooth
6. Geometry of PDEs:
   (a) https://mathoverflow.net/questions/457268/pdes-and-algebraic-varieties

(b) Can we build a kind of algebraic geometry of PDEs starting with the notion of the zero locus of a differential operator?

i. https://ncatlab.org/nlab/show/diffiety

Functional Analysis:

1. https://www.numdam.org/item/SE_1957-1958__1__A3_0/
2. https://thenumb.at/Functions-are-Vectors/
3. Tate vector spaces
4. Analytic sheaves, https://mathoverflow.net/questions/484408/literature-on-fr%c3%a9chet-quasi-coherent-sheaves
5. https://mathscinet.ams.org/mathscinet/article?mr=1257171
6. Vidav–Palmer theorem
7. In the Hilbert space $\ell^2(\mathbb{N};\mathbb{C})$, the operator $(x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$ admits $(x_n)_{n\in\mathbb{N}} \mapsto (0, x_0, x_1, \ldots)$ as its adjoint.
8. https://arxiv.org/abs/2110.06300

Lie algebras:

1. Pre-Lie algebras
2. Post-Lie algebras
3. https://arxiv.org/abs/2504.05929

Modular Representation Theory:

1. https://en.wikipedia.org/wiki/Deligne%E2%80%93Lusztig_theory
2. https://math.stackexchange.com/questions/167979/representation-of-cyclic-group-over-finite-field
3. https://math.stackexchange.com/questions/153429/irreducible-representations-of-a-cyclic-group-over-a-field-of-prime-order

Homotopy theory:

1. https://mathoverflow.net/questions/495229
2. https://ncatlab.org/nlab/show/Moore+path+category, https://mathoverflow.net/questions/486905/has-the-path-category-of-a-topological-space-been-studied/487212#487212
3. https://ncatlab.org/nlab/show/group+actions+on+spheres, https://www.maths.ed.ac.uk/~v1ranick/papers/wall7.pdf, https://math.stackexchange.com/questions/1575798/which-groups-act-freely-on-sn, https://arxiv.org/abs/math/0212280.
4. Pascal's triangle via homology of n-tori, https://topospaces.subwiki.org/wiki/Homology_of_torus
5. Conditions on morphisms of spaces $f\colon X \to Y$ such that $f^*\colon [Y,K] \to [X,K]$ or $f_*\colon [K,X] \to [K,Y]$ are injective/surjective (so, epi/monomorphisms in $\mathsf{Ho}(\mathsf{Top})$) or other conditions.

Algebraic Geometry:

1. Galois points, https://bdtd.ibict.br/vufind/Record/USP_c5e6638812a74657c40fcd402a894514
2. https://arxiv.org/abs/2407.09256

Differential Geometry:

1. https://en.wikipedia.org/wiki/Spherical_3-manifold
2. functor of points approach to differential geometry

Number Theory:

1. https://math.stackexchange.com/questions/10233/uses-of-quadratic-reciprocity-theorem/10719#10719
2. https://mathoverflow.net/questions/120067/what-do-theta-functions-have-to-do-with-quadratic-reciprocity

Classical Mechanics:

1. Koopman–von Neumann formalism
2. Relativistic Lagrangian and Hamiltonian mechanics

Quantum Mechanics:

1. https://ncatlab.org/nlab/show/geometrical+formulation+of+quantum+mechanics

Quantum Field Theory:

1. https://arxiv.org/abs/2309.15913 and https://arxiv.org/abs/2311.09284
2. The current ongoing work on higher gauge theory, specially Christian Saemann's
3. The recent work about determining the value of the strong coupling constant in the long-distance range, some pointers and keywords for this are available at this scientific american article.

Combinatorics:

1. Catalan numbers, https://mathstrek.blog/2012/02/19/power-series-and-generating-functions-ii-formal-power-series/

Other:

1. https://arxiv.org/abs/2202.00084
2. Are sedenions and higher useful for anything?
3. https://mathstodon.xyz/@pschwahn/113388126188923908
4. Tambara functors, https://arxiv.org/abs/2410.23052
5. 2-vector spaces
6. 2-term chain complexes. They form a 2-category and middle-four exchange holds, the proof using the fact that we have

$$h_1 \circ \alpha + \beta \circ g_2 = k_1 \circ \alpha + \beta \circ f_2,$$

which uses the chain homotopy identities

$$d_V \circ \alpha = g_2 - f_2,$$
$$-\beta \circ d_V = h_1 - k_1.$$

Can we modify this to work for usual chain complexes, seeking an answer to https://mathoverflow.net/questions/424268? What seems to make things go wrong in that case is that the chain homotopy identities are replaced with

$$\alpha_{n+1} \circ d_n^V + d_{n-1}^W \circ \alpha_n = g_n - f_n,$$
$$\beta_{n+1} \circ d_n^V + d_{n-1}^W \circ \beta_n = k_n - h_n.$$

7. https://arxiv.org/abs/1402.2600
8. https://grossack.site/blog
9. Classifying space of $\mathbb{Q}_p$
10. https://www.valth.eu/proc.htm
11. Construction of $\mathbb{R}$ via slopes:
    (a) http://maths.mq.edu.au/~street/EffR.pdf
    (b) https://arxiv.org/abs/math/0301015
    (c) Pierre Colmez's comment "Et si on remplace $\mathbb{Z}$ par $\mathbb{Q}$, on obtient les adèles."
    (d) I wonder if one could apply an analogue of this construction to the sphere spectrum and obtain a kind of spectral version of the real numbers, as in e.g. following the spirit of https://mathoverflow.net/questions/443018.
12. https://arxiv.org/abs/2406.04936
13. https://mathoverflow.net/a/471510
14. https://mathoverflow.net/questions/279478/the-category-theory-of-span-enriched-categories-2-segal-spaces/448523#448523

15. The works of David Kern, https://dskern.github.io/writings
16. https://qchu.wordpress.com/
17. https://aroundtoposes.com/
18. https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor
19. https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration
20. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
21. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html (Isbell conjugacy and the reflexive completion)
22. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
23. The works of Philip Saville, https://philipsaville.co.uk/
24. https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html
25. https://mathoverflow.net/q/463855 (One-object lax transformations)
26. https://ncatlab.org/nlab/show/analytic+completion+of+a+ring
27. https://en.wikipedia.org/wiki/Quaternionic_analysis
28. https://arxiv.org/abs/2401.15051 (The Norm Functor over Schemes)
29. https://mathoverflow.net/questions/407291/ (Adjunctions with respect to profunctors)
30. https://mathoverflow.net/a/462726 (Prof is free completion of Cats under right extensions)

31. there's some cool stuff in https://arxiv.org/abs/2312.00990 (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.

32. https://ncatlab.org/nlab/show/adjoint+lifting+theorem

33. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

General TODO:

1. https://arxiv.org/abs/2108.11952

2. https://mathoverflow.net/questions/483243/is-there-a-theory-of-completions-of-semirings-similar-to-i-adic-completions-of

3. https://mathoverflow.net/questions/9218/probabilistic-proofs-of-analytic-facts

4. https://x.com/cihanpoststhms

5. Special graded rings, https://mathoverflow.net/questions/403448/in-search-of-lost-graded-rings

    (a) https://arxiv.org/abs/1209.5122

6. Counterexamples in category theory

7. https://math.stackexchange.com/questions/279347/counterexample-math-books

8. Browse MO questions/answers for interesting ideas/topics

9. Change Longrightarrow to Rightarrow where appropriate

10. Try to minimize the amount of footnotes throughout the project. There should be no long footnotes.

Appendices

A Other Chapters

References

[MO 382264] Neil Strickland. *Proof that a cartesian category is monoidal.* MathOverflow. URL: https://mathoverflow.net/q/382264 (cit. on p. 46).

[JY21] Niles Johnson and Donald Yau. *2-Dimensional Categories.* Oxford University Press, Oxford, 2021, pp. xix+615. ISBN: 978-0-19-887138-5; 978-0-19-887137-8. DOI: 10.1093/oso/9780198871378.001.0001. URL: https://doi.org/10.1093/oso/9780198871378.001.0001 (cit. on p. 54).