

# Monoidal Structures on the Category of Sets

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This chapter contains some material on monoidal structures on Sets.

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## 5.1 The Monoidal Category of Sets and Products

### 5.1.1 Products of Sets

See [Constructions With Sets, Section 4.1.3](#).

### 5.1.2 The Internal Hom of Sets

See [Constructions With Sets, Section 4.3.5](#).

### 5.1.3 The Monoidal Unit

#### DEFINITION 5.1.3.1.1 ► THE MONOIDAL UNIT OF $\times$

The **monoidal unit of the product of sets** is the functor

$$\mathbb{1}^{\text{Sets}} : \text{pt} \rightarrow \text{Sets}$$

defined by

$$\mathbb{1}_{\text{Sets}} \stackrel{\text{def}}{=} \text{pt},$$

where  $\text{pt}$  is the terminal set of [Constructions With Sets, Definition 4.1.1.1](#).

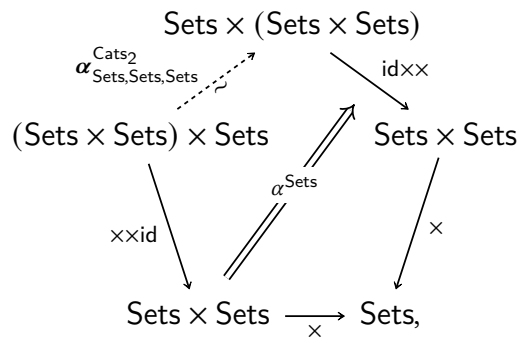
## 5.1.4 The Associator

### DEFINITION 5.1.4.1.1 ► THE ASSOCIATOR OF $\times$

The **associator of the product of sets** is the natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}_2}_{\text{Sets}, \text{Sets}, \text{Sets}},$$

as in the diagram



whose component

$$\alpha^{\text{Sets}}_{X,Y,Z} : (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at  $(X, Y, Z)$  is given by

$$\alpha^{\text{Sets}}_{X,Y,Z}((x, y), z) \stackrel{\text{def}}{=} (x, (y, z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

### PROOF 5.1.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.4.1.1

#### Invertibility

The inverse of  $\alpha^{\text{Sets}}_{X,Y,Z}$  is the morphism

$$\alpha^{\text{Sets}, -1}_{X,Y,Z} : X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \stackrel{\text{def}}{=} ((x, y), z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

· *Invertibility I.* We have

$$\begin{aligned} [\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}}]((x, y), z) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets},-1}(\alpha_{X,Y,Z}^{\text{Sets}}((x, y), z)) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \\ &\stackrel{\text{def}}{=} ((x, y), z) \\ &\stackrel{\text{def}}{=} [\text{id}_{(X \times Y) \times Z}]((x, y), z) \end{aligned}$$

for each  $((x, y), z) \in (X \times Y) \times Z$ , and therefore we have

$$\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} = \text{id}_{(X \times Y) \times Z}.$$

· *Invertibility II.* We have

$$\begin{aligned} [\alpha_{X,Y,Z}^{\text{Sets}} \circ \alpha_{X,Y,Z}^{\text{Sets},-1}](x, (y, z)) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets}}(\alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z))) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets}}((x, y), z) \\ &\stackrel{\text{def}}{=} (x, (y, z)) \\ &\stackrel{\text{def}}{=} [\text{id}_{X \times (Y \times Z)}](x, (y, z)) \end{aligned}$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ , and therefore we have

$$\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} = \text{id}_{X \times (Y \times Z)}.$$

Therefore  $\alpha_{X,Y,Z}^{\text{Sets}}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions

$$f: X \rightarrow X',$$

$$g: Y \rightarrow Y',$$

$$h: Z \rightarrow Z'$$

the diagram


$$\begin{array}{ccc}
 (X \times Y) \times Z & \xrightarrow{(f \times g) \times h} & (X' \times Y') \times Z' \\
 \downarrow \alpha_{X,Y,Z}^{\text{Sets}} & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}} \\
 X \times (Y \times Z) & \xrightarrow{f \times (g \times h)} & X' \times (Y' \times Z')
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 ((x, y), z) & & ((x, y), z) \mapsto ((f(x), g(y)), h(z)) \\
 \downarrow & & \downarrow \\
 (x, (y, z)) \mapsto (f(x), (g(y), h(z))) & & (f(x), (g(y), h(z)))
 \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}}$  to be a natural transformation.

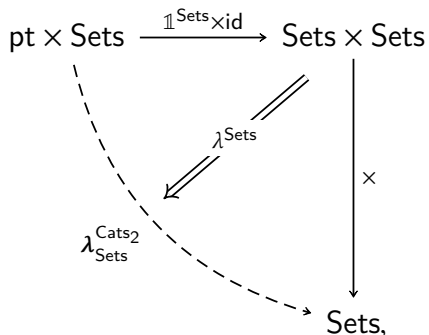
#### Being a Natural Isomorphism

Since  $\alpha^{\text{Sets}}$  is natural and  $\alpha^{\text{Sets}, -1}$  is a componentwise inverse to  $\alpha^{\text{Sets}}$ , it follows from [Categories, Item 2 of Proposition 11.9.7.1.2](#) that  $\alpha^{\text{Sets}, -1}$  is also natural. Thus  $\alpha^{\text{Sets}}$  is a natural isomorphism. 

## 5.1.5 The Left Unitor

### DEFINITION 5.1.5.1.1 ► THE LEFT UNITOR OF $\times$

The **left unitor of the product of sets** is the natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$


whose component

$$\lambda_X^{\text{Sets}}: \text{pt} \times X \dashrightarrow X$$

at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\lambda_X^{\text{Sets}}(\star, x) \stackrel{\text{def}}{=} x$$

for each  $(\star, x) \in \text{pt} \times X$ .

#### PROOF 5.1.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.5.1

##### Invertibility

The inverse of  $\lambda_X^{\text{Sets}}$  is the morphism

$$\lambda_X^{\text{Sets}, -1}: X \dashrightarrow \text{pt} \times X$$

defined by

$$\lambda_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each  $x \in X$ . Indeed:

· *Invertibility I.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}, -1} \circ \lambda_X^{\text{Sets}}](\text{pt}, x) &= \lambda_X^{\text{Sets}, -1}(\lambda_X^{\text{Sets}}(\text{pt}, x)) \\ &= \lambda_X^{\text{Sets}, -1}(x) \\ &= (\star, x) \\ &= [\text{id}_{\text{pt} \times X}](\text{pt}, x) \end{aligned}$$

for each  $(\text{pt}, x) \in \text{pt} \times X$ , and therefore we have

$$\lambda_X^{\text{Sets}, -1} \circ \lambda_X^{\text{Sets}} = \text{id}_{\text{pt} \times X}.$$

· *Invertibility II.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}} \circ \lambda_X^{\text{Sets}, -1}](x) &= \lambda_X^{\text{Sets}}(\lambda_X^{\text{Sets}, -1}(x)) \\ &= \lambda_X^{\text{Sets}}(\star, x) \end{aligned}$$

$$\begin{aligned}
 &= x \\
 &= [\text{id}_X](x)
 \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\text{Sets}} \circ \lambda_X^{\text{Sets}, -1} = \text{id}_X.$$

Therefore  $\lambda_X^{\text{Sets}}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \rightarrow Y$ , the diagram


$$\begin{array}{ccc}
 \text{pt} \times X & \xrightarrow{\text{id}_{\text{pt}} \times f} & \text{pt} \times Y \\
 \lambda_X^{\text{Sets}} \downarrow & & \downarrow \lambda_Y^{\text{Sets}} \\
 X & \xrightarrow{f} & Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (\star, x) & & (\star, x) \mapsto (\star, f(x)) \\
 \downarrow & & \downarrow \\
 x & \mapsto & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\text{Sets}}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\lambda^{\text{Sets}}$  is natural and  $\lambda^{\text{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\text{Sets}}$ , it follows from [Categories, Item 2 of Proposition 11.9.7.1.2](#) that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\lambda^{\text{Sets}}$  is a natural isomorphism. 

## 5.1.6 The Right Unitor

### DEFINITION 5.1.6.1.1 ► THE RIGHT UNITOR OF $\times$

The **right unitor of the product of sets** is the natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}} : X \times \text{pt} \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\rho_X^{\text{Sets}}(x, \star) \stackrel{\text{def}}{=} x$$

for each  $(x, \star) \in X \times \text{pt}$ .

### PROOF 5.1.6.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.6.1.1

#### Invertibility

The inverse of  $\rho_X^{\text{Sets}}$  is the morphism

$$\rho_X^{\text{Sets}, -1} : X \xrightarrow{\sim} X \times \text{pt}$$

defined by

$$\rho_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (x, \star)$$

for each  $x \in X$ . Indeed:

• *Invertibility I.* We have

$$[\rho_X^{\text{Sets}, -1} \circ \rho_X^{\text{Sets}}](x, \star) = \rho_X^{\text{Sets}, -1}(\rho_X^{\text{Sets}}(x, \star))$$



$$\begin{aligned}
&= \rho_X^{\text{Sets}, -1}(x) \\
&= (x, \star) \\
&= [\text{id}_{X \times \text{pt}}](x, \star)
\end{aligned}$$

for each  $(x, \star) \in X \times \text{pt}$ , and therefore we have

$$\rho_X^{\text{Sets}, -1} \circ \rho_X^{\text{Sets}} = \text{id}_{X \times \text{pt}}.$$

• *Invertibility II.* We have

$$\begin{aligned}
[\rho_X^{\text{Sets}} \circ \rho_X^{\text{Sets}, -1}](x) &= \rho_X^{\text{Sets}}(\rho_X^{\text{Sets}, -1}(x)) \\
&= \rho_X^{\text{Sets}, -1}(x, \star) \\
&= x \\
&= [\text{id}_X](x)
\end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\text{Sets}} \circ \rho_X^{\text{Sets}, -1} = \text{id}_X.$$

Therefore  $\rho_X^{\text{Sets}}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \rightarrow Y$ , the diagram


$$\begin{array}{ccc}
X \times \text{pt} & \xrightarrow{f \times \text{id}_{\text{pt}}} & Y \times \text{pt} \\
\rho_X^{\text{Sets}} \downarrow & & \downarrow \rho_Y^{\text{Sets}} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(x, \star) & & (x, \star) \mapsto (f(x), \star) \\
\downarrow & & \downarrow \\
x \mapsto f(x) & & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\rho^{\mathbf{Sets}}$  is a natural transformation.

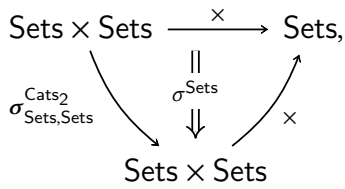
#### Being a Natural Isomorphism

Since  $\rho^{\mathbf{Sets}}$  is natural and  $\rho^{\mathbf{Sets}, -1}$  is a componentwise inverse to  $\rho^{\mathbf{Sets}}$ , it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that  $\rho^{\mathbf{Sets}, -1}$  is also natural. Thus  $\rho^{\mathbf{Sets}}$  is a natural isomorphism. 

## 5.1.7 The Symmetry

### DEFINITION 5.1.7.1.1 ► THE SYMMETRY OF $\times$

The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\mathbf{Sets}} : \times \xrightarrow{\sim} \times \circ \sigma_{\mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}_2}$$


whose component

$$\sigma_{X,Y}^{\mathbf{Sets}} : X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in \text{Obj}(\mathbf{Sets})$  is defined by

$$\sigma_{X,Y}^{\mathbf{Sets}}(x, y) \stackrel{\text{def}}{=} (y, x)$$

for each  $(x, y) \in X \times Y$ .

### PROOF 5.1.7.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.7.1.1

#### Invertibility

The inverse of  $\sigma_{X,Y}^{\mathbf{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\mathbf{Sets}, -1} : Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets},-1}(y, x) \stackrel{\text{def}}{=} (x, y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

· *Invertibility I.* We have

$$\begin{aligned} [\sigma_{X,Y}^{\text{Sets},-1} \circ \sigma_{X,Y}^{\text{Sets}}](x, y) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets},-1}(\sigma_{X,Y}^{\text{Sets}}(x, y)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets},-1}(y, x) \\ &\stackrel{\text{def}}{=} (x, y) \\ &\stackrel{\text{def}}{=} [\text{id}_{X \times Y}](x, y) \end{aligned}$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets},-1} \circ \sigma_{X,Y}^{\text{Sets}} = \text{id}_{X \times Y}.$$

· *Invertibility II.* We have

$$\begin{aligned} [\sigma_{X,Y}^{\text{Sets}} \circ \sigma_{X,Y}^{\text{Sets},-1}](y, x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets}}(\sigma_{X,Y}^{\text{Sets},-1}(y, x)) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets}}(x, y) \\ &\stackrel{\text{def}}{=} (y, x) \\ &\stackrel{\text{def}}{=} [\text{id}_{Y \times X}](y, x) \end{aligned}$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets}} \circ \sigma_{X,Y}^{\text{Sets},-1} = \text{id}_{Y \times X}.$$

Therefore  $\sigma_{X,Y}^{\text{Sets}}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions

$$\begin{aligned} f &: X \rightarrow A, \\ g &: Y \rightarrow B \end{aligned}$$

the diagram


$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & A \times B \\ \sigma_{X,Y}^{\text{Sets}} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}} \\ Y \times X & \xrightarrow{g \times f} & B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x, y) & & (x, y) \mapsto (f(x), g(y)) \\ \downarrow & & \downarrow \\ (y, x) \mapsto (g(y), f(x)) & & (g(y), f(x)) \end{array}$$

and hence indeed commutes, showing  $\sigma^{\text{Sets}}$  to be a natural transformation.

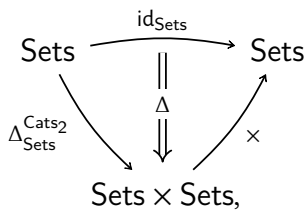
#### Being a Natural Isomorphism

Since  $\sigma^{\text{Sets}}$  is natural and  $\sigma^{\text{Sets}, -1}$  is a componentwise inverse to  $\sigma^{\text{Sets}}$ , it follows from [Categories, Item 2 of Proposition 11.9.7.1.2](#) that  $\sigma^{\text{Sets}, -1}$  is also natural. Thus  $\sigma^{\text{Sets}}$  is a natural isomorphism. 

## 5.1.8 The Diagonal

### DEFINITION 5.1.8.1.1 ► THE DIAGONAL OF $\times$

The **diagonal of the product of sets** is the natural transformation

$$\Delta: \text{id}_{\text{Sets}} \Rightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2},$$


whose component

$$\Delta_X: X \rightarrow X \times X$$

at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .


**PROOF 5.1.8.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.8.1.1**

We need to show that, given a function  $f: X \rightarrow Y$ , the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X \downarrow & & \downarrow \Delta_Y \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & & x \\ \downarrow & \xrightarrow{\quad} & \downarrow \\ (x, x) & \xrightarrow{\quad} & (f(x), f(x)) \end{array}$$

and hence indeed commutes, showing  $\Delta$  to be natural. 

**PROPOSITION 5.1.8.1.3 ► PROPERTIES OF THE DIAGONAL MAP**

Let  $X$  be a set.

1. *Monoidality.* The diagonal map

$$\Delta: \text{id}_{\text{Sets}} \Longrightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2},$$

is a monoidal natural transformation:

- (a) *Compatibility With Strong Monoidality Constraints.* For each  $X, Y \in$

$\text{Obj}(\text{Sets})$ , the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\Delta_X \times \Delta_Y} & (X \times X) \times (Y \times Y) \\ & \searrow \Delta_{X \times Y} & \downarrow \wr \\ & & (X \times Y) \times (X \times Y) \end{array}$$

commutes.

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{array}{ccc} \text{pt} & \xrightarrow{\Delta_{\text{pt}}} & \text{pt} \times \text{pt} \\ & \searrow & \downarrow \lambda_{\text{pt}}^{\text{Sets}} \\ & & \text{pt} \end{array} \quad \begin{array}{ccc} \text{pt} & \xrightarrow{\Delta_{\text{pt}}} & \text{pt} \times \text{pt} \\ & \searrow & \downarrow \rho_{\text{pt}}^{\text{Sets}} \\ & & \text{pt} \end{array}$$

commute, i.e. we have

$$\begin{aligned} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets}, -1} \\ &= \rho_{\text{pt}}^{\text{Sets}, -1}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned} \lambda_{\text{pt}}^{\text{Sets}} &= \rho_{\text{pt}}^{\text{Sets}}, \\ \lambda_{\text{pt}}^{\text{Sets}, -1} &= \rho_{\text{pt}}^{\text{Sets}, -1} \end{aligned}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

2. *The Diagonal of the Unit.* The component

$$\Delta_{\text{pt}} : \text{pt} \xrightarrow{\sim} \text{pt} \times \text{pt}$$

of  $\Delta$  at  $\text{pt}$  is an isomorphism.

## PROOF 5.1.8.1.4 ► PROOF OF PROPOSITION 5.1.8.1.3

## Item 1: Monoidality

We claim that  $\Delta$  is indeed monoidal:

1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\Delta_X \times \Delta_Y} & (X \times X) \times (Y \times Y) \\
 & \searrow \Delta_{X \times Y} & \downarrow \lambda \\
 & & (X \times Y) \times (X \times Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 (x, y) & \longmapsto & ((x, x), (y, y)) & & (x, y) \\
 & & \downarrow & & \searrow \\
 & & ((x, y), (x, y)) & & ((x, y), (x, y))
 \end{array}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of [Definition 5.1.5.1.1](#), the inverse of the left unitor of **Sets** with respect to the product at  $X \in \text{Obj}(\text{Sets})$  is given by

$$\lambda_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each  $x \in X$ , so when  $X = \text{pt}$ , we have

$$\lambda_{\text{pt}}^{\text{Sets}, -1}(\star) \stackrel{\text{def}}{=} (\star, \star),$$


and also

$$\Delta_{\text{pt}}^{\text{Sets}}(\star) \stackrel{\text{def}}{=} (\star, \star),$$

so we have  $\Delta_{\text{pt}} = \lambda_{\text{pt}}^{\text{Sets}, -1}$ .

This finishes the proof.

#### Item 2: The Diagonal of the Unit

This follows from **Item 1** and the invertibility of the left/right unitor of Sets with respect to  $\times$ , proved in the proof of **Definition 5.1.5.1.1** for the left unitor or the proof of **Definition 5.1.6.1.1** for the right unitor. 

## 5.1.9 The Monoidal Category of Sets and Products

### PROPOSITION 5.1.9.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO THE PRODUCT

The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Monoidal Product.* The product functor

$$\times: \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of **Constructions With Sets, Item 1** of **Proposition 4.1.3.1.4**.

- *The Internal Hom.* The internal Hom functor

$$\text{Sets}: \text{Sets}^{\text{op}} \times \text{Sets} \rightarrow \text{Sets}$$

of **Constructions With Sets, Item 1** of **Proposition 4.3.5.1.2**.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\text{Sets}}: \text{pt} \rightarrow \text{Sets}$$

of **Definition 5.1.3.1.1**.

- *The Associators.* The natural isomorphism

$$\alpha^{\text{Sets}}: \times \circ (\times \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of **Definition 5.1.4.1.1**.



- *The Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.5.1.1.**

- *The Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.6.1.1.**

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.7.1.1.**

- *The Diagonals.* The monoidal natural transformation

$$\Delta : \text{id}_{\text{Sets}} \Rightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.8.1.1.**

#### PROOF 5.1.9.1.2 ► PROOF OF PROPOSITION 5.1.9.1.1

The Pentagon Identity

Let  $W, X, Y$  and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \times (X \times Y)) \times Z & \\
 \alpha_{W,X,Y}^{\text{Sets}} \times \text{id}_Z \nearrow & & \searrow \alpha_{W,X \times Y,Z}^{\text{Sets}} \\
 ((W \times X) \times Y) \times Z & & W \times ((X \times Y) \times Z) \\
 \alpha_{W \times X,Y,Z}^{\text{Sets}} \searrow & & \swarrow \text{id}_W \times \alpha_{X,Y,Z}^{\text{Sets}} \\
 (W \times X) \times (Y \times Z) & \xrightarrow{\alpha_{W,X,Y \times Z}^{\text{Sets}}} & W \times (X \times (Y \times Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 & & ((w, (x, y)), z) & & \\
 & \swarrow & & \searrow & \\
 (((w, x), y), z) & & & & ((w, x), y), z) & & (w, ((x, y), z)) \\
 & \searrow & & \swarrow & & \searrow & \\
 ((w, x), (y, z)) \mapsto (w, (x, (y, z))) & & & & & & (w, (x, (y, z))),
 \end{array}$$

and thus the pentagon identity is satisfied.

The Triangle Identity

Let  $X$  and  $Y$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \times \text{pt}) \times Y & \xrightarrow{\alpha_{X,\text{pt},Y}^{\text{Sets}}} & X \times (\text{pt} \times Y) \\
 \searrow \rho_X^{\text{Sets}} \times \text{id}_Y & & \swarrow \text{id}_X \times \lambda_Y^{\text{Sets}} \\
 & X \times Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 ((x, \star), y) & & ((x, \star), y) & \xrightarrow{\quad} & (x, (\star, y)) \\
 \swarrow & & & & \swarrow \\
 & (x, y) & & & (x, y)
 \end{array}$$

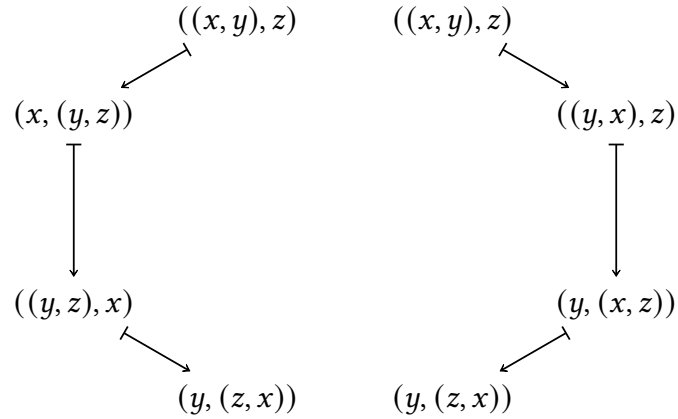
and thus the triangle identity is satisfied.

#### The Left Hexagon Identity

Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & (X \times Y) \times Z & & \\
 & \swarrow \alpha_{X,Y,Z}^{\text{Sets}} & & \searrow \sigma_{X,Y}^{\text{Sets}} \times \text{id}_Z & \\
 X \times (Y \times Z) & & & & (Y \times X) \times Z \\
 \downarrow \sigma_{X,Y \times Z}^{\text{Sets}} & & & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}} \\
 (Y \times Z) \times X & & & & Y \times (X \times Z) \\
 & \swarrow \alpha_{Y,Z,X}^{\text{Sets}} & & \nwarrow \text{id}_Y \times \sigma_{X,Z}^{\text{Sets}} & \\
 & Y \times (Z \times X) & & &
 \end{array}$$

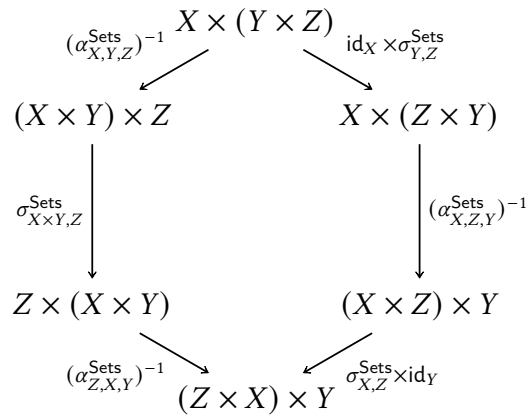
commutes. Indeed, this diagram acts on elements as



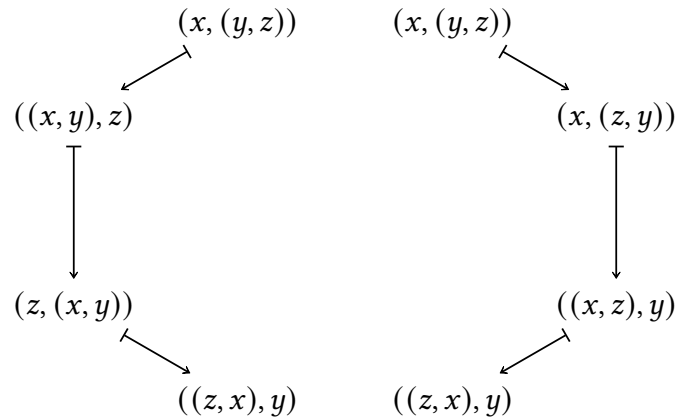
and thus the left hexagon identity is satisfied.

#### The Right Hexagon Identity

Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the right hexagon identity is satisfied.

Monoidal Closedness

This follows from [Constructions With Sets, Item 2 of Proposition 4.3.5.1.2](#)

Existence of Monoidal Diagonals

This follows from [Items 1 and 2 of Proposition 5.1.8.1.3](#).



## 5.1.10 The Universal Property of $(\text{Sets}, \times, \text{pt})$

**THEOREM 5.1.10.1.1 ► THE UNIVERSAL PROPERTY OF  $(\mathbf{Sets}, \times, \text{pt})$** 

The symmetric monoidal structure on the category  $\mathbf{Sets}$  of **Proposition 5.1.9.1.1** is uniquely determined by the following requirements:

1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathbf{Sets}} : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of  $\mathbf{Sets}$  admits an internal Hom  $[-1, -2]_{\mathbf{Sets}}$ .

2. *The Unit Object Is pt.* We have  $\mathbb{1}_{\mathbf{Sets}} \cong \text{pt}$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{B}_{\infty}}^{\text{cl}}(\mathbf{Sets})$  of ?? spanned by the closed symmetric monoidal categories  $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$  satisfying **Items 1** and **2** is contractible (i.e. equivalent to the punctual category).

**PROOF 5.1.10.1.2 ► PROOF OF THEOREM 5.1.10.1.1****Unwinding the Statement**

Let  $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda', \rho', \sigma')$  be a closed symmetric monoidal category satisfying **Items 1** and **2**. We need to show that the identity functor

$$\text{id}_{\mathbf{Sets}} : \mathbf{Sets} \rightarrow \mathbf{Sets}$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \text{id}_{\mathbf{Sets}}^{\otimes} : A \otimes_{\mathbf{Sets}} B &\xrightarrow{\sim} A \times B, \\ \text{id}_{\mathbf{Sets}}^{\text{Hom}} : [A, B]_{\mathbf{Sets}} &\xrightarrow{\sim} \mathbf{Sets}(A, B), \\ \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes} : \mathbb{1}_{\mathbf{Sets}} &\xrightarrow{\sim} \text{pt}, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from  $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda', \rho', \sigma')$  to the closed symmetric monoidal category  $(\mathbf{Sets}, \times, \mathbf{Sets}(-1, -2), \mathbb{1}_{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$  of **Proposition 5.1.9.1.1**.

**Constructing an Isomorphism  $[-1, -2]_{\mathbf{Sets}} \cong \mathbf{Sets}(-1, -2)$** 

By ??, we have a natural isomorphism

$$\mathbf{Sets}(\text{pt}, [-1, -2]_{\mathbf{Sets}}) \cong \mathbf{Sets}(-1, -2).$$

By **Constructions With Sets, Item 3** of **Proposition 4.3.5.1.2**, we also have a natural isomorphism

$$\text{Sets}(\text{pt}, [-1, -2]_{\text{Sets}}) \cong [-1, -2]_{\text{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\text{Sets}(-1, -2) \cong [-1, -2]_{\text{Sets}}.$$

Given  $A, B \in \text{Obj}(\text{Sets})$ , we will write

$$\text{id}_{A,B}^{\text{Hom}} : \text{Sets}(A, B) \xrightarrow{\sim} [A, B]_{\text{Sets}}$$

for the component of this isomorphism at  $(A, B)$ .

**Constructing an Isomorphism  $\otimes_{\text{Sets}} \cong \times$**

Since  $\otimes_{\text{Sets}}$  is adjoint in each variable to  $[-1, -2]_{\text{Sets}}$  by assumption and  $\times$  is adjoint in each variable to  $\text{Sets}(-1, -2)$  by **Constructions With Sets, Item 2** of **Proposition 4.3.5.1.2**, uniqueness of adjoints (??) gives us natural isomorphisms

$$\begin{aligned} A \otimes_{\text{Sets}} - &\cong A \times -, \\ - \otimes_{\text{Sets}} B &\cong B \times -. \end{aligned}$$

By ??, we then have  $\otimes_{\text{Sets}} \cong \times$ . We will write

$$\text{id}_{\text{Sets}|A,B}^{\otimes} : A \otimes_{\text{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at  $(A, B)$ .

**Alternative Construction of an Isomorphism  $\otimes_{\text{Sets}} \cong \times$**

Alternatively, we may construct a natural isomorphism  $\otimes_{\text{Sets}} \cong \times$  as follows:

1. Let  $A \in \text{Obj}(\text{Sets})$ .
2. Since  $\otimes_{\text{Sets}}$  is part of a closed monoidal structure, it preserves colimits in each variable by ??.

3. Since  $A \cong \coprod_{a \in A} \text{pt}$  and  $\otimes_{\text{Sets}}$  preserves colimits in each variable, we have

$$\begin{aligned} A \otimes_{\text{Sets}} B &\cong \left( \coprod_{a \in A} \text{pt} \right) \otimes_{\text{Sets}} B \\ &\cong \coprod_{a \in A} (\text{pt} \otimes_{\text{Sets}} B) \\ &\cong \coprod_{a \in A} B \\ &\cong A \times B, \end{aligned}$$

naturally in  $B \in \text{Obj}(\text{Sets})$ , where we have used that  $\text{pt}$  is the monoidal unit for  $\otimes_{\text{Sets}}$ . Thus  $A \otimes_{\text{Sets}} - \cong A \times -$  for each  $A \in \text{Obj}(\text{Sets})$ .

4. Similarly,  $- \otimes_{\text{Sets}} B \cong - \times B$  for each  $B \in \text{Obj}(\text{Sets})$ .  
 5. By ??, we then have  $\otimes_{\text{Sets}} \cong \times$ .

Below, we'll show that if a natural isomorphism  $\otimes_{\text{Sets}} \cong \times$  exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\text{id}_{\text{Sets}|A,B}^{\otimes}: A \otimes_{\text{Sets}} B \rightarrow A \times B$  from before.

Constructing an isomorphism  $\text{id}_{\mathbb{1}}^{\otimes}: \mathbb{1}_{\text{Sets}} \rightarrow \text{pt}$

We define an isomorphism  $\text{id}_{\mathbb{1}}^{\otimes}: \mathbb{1}_{\text{Sets}} \rightarrow \text{pt}$  as the composition

$$\mathbb{1}_{\text{Sets}} \xrightarrow[\sim]{\rho_{\mathbb{1}_{\text{Sets}}}^{\text{Sets}, -1}} \mathbb{1}_{\text{Sets}} \times \text{pt} \xrightarrow[\sim]{\text{id}_{\text{Sets}|\mathbb{1}_{\text{Sets}}}^{\otimes}} \mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} \text{pt} \xrightarrow[\sim]{\lambda'_{\text{pt}}} \text{pt}$$

in  $\text{Sets}$ .

Monoidal Left Unity of the Isomorphism  $\otimes_{\text{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{ccc} & \text{pt} \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, A}^{\otimes}} \text{pt} \times A \\ \text{id}_{\mathbb{1}_{\text{Sets}}}^{\otimes} \otimes_{\text{Sets}} \text{id}_A \nearrow & & \searrow \lambda_A^{\text{Sets}} \\ \mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} A & \xrightarrow{\lambda_A'} & A \end{array}$$



commutes. First, note that the diagram

$$\begin{array}{ccc}
 & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes} |_{\text{pt}, \text{pt}}} \text{pt} \times \text{pt} \\
 \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_{\text{pt}} \nearrow & & \searrow \lambda_{\text{pt}}^{\mathbf{Sets}} \\
 \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\lambda'_{\text{pt}}} & \text{pt},
 \end{array}$$

corresponding to the case  $A = \text{pt}$ , commutes by the terminality of  $\text{pt}$  (**Constructions With Sets, Construction 4.1.1.2**). Since this diagram commutes, so does the diagram

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes, -1} |_{\text{pt}, \text{pt}}} \text{pt} \otimes_{\mathbf{Sets}} \text{pt} \\
 \lambda_{\text{pt}}^{\mathbf{Sets}, -1} \nearrow & & \searrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_{\text{pt}} \\
 \text{pt} & \xrightarrow{\lambda'^{-1}_{\text{pt}}} & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt}.
 \end{array}
 \quad (\dagger)$$

Now, let  $A \in \text{Obj}(\mathbf{Sets})$ , let  $a \in A$ , and consider the diagram

$$\begin{array}{ccccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes, -1} |_{\text{pt}, \text{pt}}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes, -1} \times \text{id}_{\text{pt}}} & \text{pt} \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}} \\
 \lambda_{\text{pt}}^{\mathbf{Sets}, -1} \nearrow & & & & & \searrow \\
 \text{pt} & \xrightarrow{\lambda'^{-1}_{\text{pt}}} & & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt} & & \\
 \downarrow [a] & \downarrow \text{id}_{\text{pt}} \times [a] & (1) & \downarrow \text{id}_{\text{pt}} \otimes_{\mathbf{Sets}} [a] & (4) & \downarrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}} \times [a] \\
 & \text{pt} \times A & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes, -1} |_{\text{pt}, A}} & \text{pt} \otimes_{\mathbf{Sets}} A & & \\
 \downarrow \lambda_A^{\mathbf{Sets}, -1} & & (2) & & & \downarrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}} \times [a] \\
 A & \xrightarrow{\lambda'^{-1}_A} & & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A & & \\
 & & & & & \downarrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}} \times [a] \\
 & & & & & A.
 \end{array}$$

(3) (5)

Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'^{-1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}_{\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\text{Sets}, -1}$ .

it follows that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times A & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1}} & \text{pt} \otimes_{\text{Sets}} A \\
 & \nearrow \lambda_A^{\text{Sets}, -1} & & & \searrow \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_A \\
 \text{pt} & \xrightarrow{[a]} & A & \xrightarrow{\lambda_A'^{-1}} & \mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} A
 \end{array}$$

Here, a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}
 \lambda_A'^{-1}(a) &= [\lambda_A'^{-1} \circ [a]](\star) \\
 &= [(\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \times \text{id}_A) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\text{Sets}, -1} \circ [a]](\star) \\
 &= [(\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \times \text{id}_A) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\text{Sets}, -1}](a)
 \end{aligned}$$

for each  $a \in A$ , and thus we have

$$\lambda_A'^{-1} = (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \times \text{id}_A) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\text{Sets}, -1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\text{Sets}} \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes} \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \times \text{id}_A),$$

showing that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, A}^{\otimes}} & \text{pt} \times A \\
 & \nearrow \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_A & & & \searrow \lambda_A^{\text{Sets}} \\
 \mathbb{1}_{\text{Sets}} \otimes_{\text{Sets}} A & \xrightarrow{\lambda_A'} & & & A
 \end{array}$$

indeed commutes.

#### Monoidal Right Unity of the Isomorphism $\otimes_{\text{Sets}} \cong \times$

We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\text{Sets}} \cong \times$  above. For completeness, we repeat it below.

We have to show that the diagram

$$\begin{array}{ccc}
 & A \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|A,\text{pt}}^{\otimes}} A \times \text{pt} \\
 \text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \nearrow & & \searrow \rho_A^{\text{Sets}} \\
 A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho'_A} & A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes}} \text{pt} \times \text{pt} \\
 \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \nearrow & & \searrow \rho_{\text{pt}}^{\text{Sets}} \\
 \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho'_{\text{pt}}} & \text{pt},
 \end{array}$$

corresponding to the case  $A = \text{pt}$ , commutes by the terminality of  $\text{pt}$  (**Constructions With Sets, Construction 4.1.1.2**). Since this diagram commutes, so does the diagram

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes,-1}} \text{pt} \otimes_{\text{Sets}} \text{pt} \\
 \rho_{\text{pt}}^{\text{Sets},-1} \nearrow & & \searrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes,-1} \\
 \text{pt} & \xrightarrow{\rho_{\text{pt}}'^{-1}} & \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}.
 \end{array}
 \quad (\dagger)$$

Now, let  $A \in \text{Obj}(\text{Sets})$ , let  $a \in A$ , and consider the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}| \text{pt}, \text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\text{Sets}} \text{pt} \\
 & \nearrow \rho_{\text{pt}}^{\text{Sets}, -1} & \downarrow & \text{(\dagger)} & \downarrow \text{id}_{\text{pt}} \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \\
 \text{pt} & \xrightarrow{\quad} & \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho_{\text{pt}}'^{-1}} & \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} \\
 \downarrow [a] & \downarrow \text{id}_{\text{pt}} \times [a] & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} [a] & \downarrow \text{id}_{\mathbb{1}_{\text{Sets}}} \times [a] & \\
 & \text{(3)} & \text{(1)} & \text{(4)} & \\
 & \downarrow & \downarrow & \downarrow & \\
 & A \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}| A, \text{pt}}^{\otimes, -1}} & A \otimes_{\text{Sets}} \text{pt} & \\
 & \nearrow \rho_A^{\text{Sets}, -1} & \downarrow & \downarrow \text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} & \\
 A & \xrightarrow{\quad} & A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho_A'^{-1}} & A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}
 \end{array}$$

(5)

Since:

- Subdiagram (5) commutes by the naturality of  $\rho'^{-1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}_{\text{Sets}}^{\otimes, -1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\text{Sets}, -1}$ .

it follows that the diagram

$$\begin{array}{ccccc}
 & & A \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}| A, \text{pt}}^{\otimes, -1}} & A \otimes_{\text{Sets}} \text{pt} \\
 & \nearrow \rho_A^{\text{Sets}, -1} & & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \\
 \text{pt} & \xrightarrow{[a]} & A & \xrightarrow{\rho_A'^{-1}} & A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}
 \end{array}$$

Here,s a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}\rho_A'^{-1}(a) &= [\rho_A'^{-1} \circ [a]](\star) \\ &= [(\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1}) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1} \circ [a]](\star) \\ &= [(\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1}) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1}](a)\end{aligned}$$

for each  $a \in A$ , and thus we have

$$\rho_A'^{-1} = (\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1}) \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\text{Sets}} \circ \text{id}_{\text{Sets}|\text{pt}, A}^{\otimes} \circ (\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes}),$$

showing that the diagram

$$\begin{array}{ccc} & A \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|A, \text{pt}}^{\otimes}} A \times \text{pt} \\ \text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \nearrow & & \searrow \rho_A^{\text{Sets}} \\ A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho_A'} & A \end{array}$$

indeed commutes.

Monoidality of the Isomorphism  $\otimes_{\text{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{ccc} & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & \\ \text{id}_{\text{Sets}|A, B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \swarrow & & \searrow \alpha'_{A, B, C} \\ (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\ \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes} & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B, C}^{\otimes} \\ (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\ \alpha_{A, B, C}^{\text{Sets}} \searrow & & \swarrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \\ & A \times (B \times C) & \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & \\
 \text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\text{pt}} \swarrow & & \searrow \alpha'_{\text{pt},\text{pt},\text{pt}} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes} & & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \searrow \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes} \\
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 & \downarrow !_{\text{pt} \times (\text{pt} \times \text{pt})} & \\
 & \text{pt} &
 \end{array}$$

commutes by the terminality of  $\text{pt}$  ([Constructions With Sets, Construction 4.1.1.2](#)). Since the map  $!_{\text{pt} \times (\text{pt} \times \text{pt})} : \text{pt} \times (\text{pt} \times \text{pt}) \rightarrow \text{pt}$  is an isomorphism (e.g. having inverse  $\lambda_{\text{pt}}^{\text{Sets}, -1} \circ \lambda_{\text{pt}}^{\text{Sets}, -1}$ ), it follows that the diagram

$$\begin{array}{ccc}
 & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & \\
 \text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\text{pt}} \swarrow & & \searrow \alpha'_{\text{pt},\text{pt},\text{pt}} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes} & & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \searrow \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes} \\
 & \text{pt} \times (\text{pt} \times \text{pt}) &
 \end{array}$$

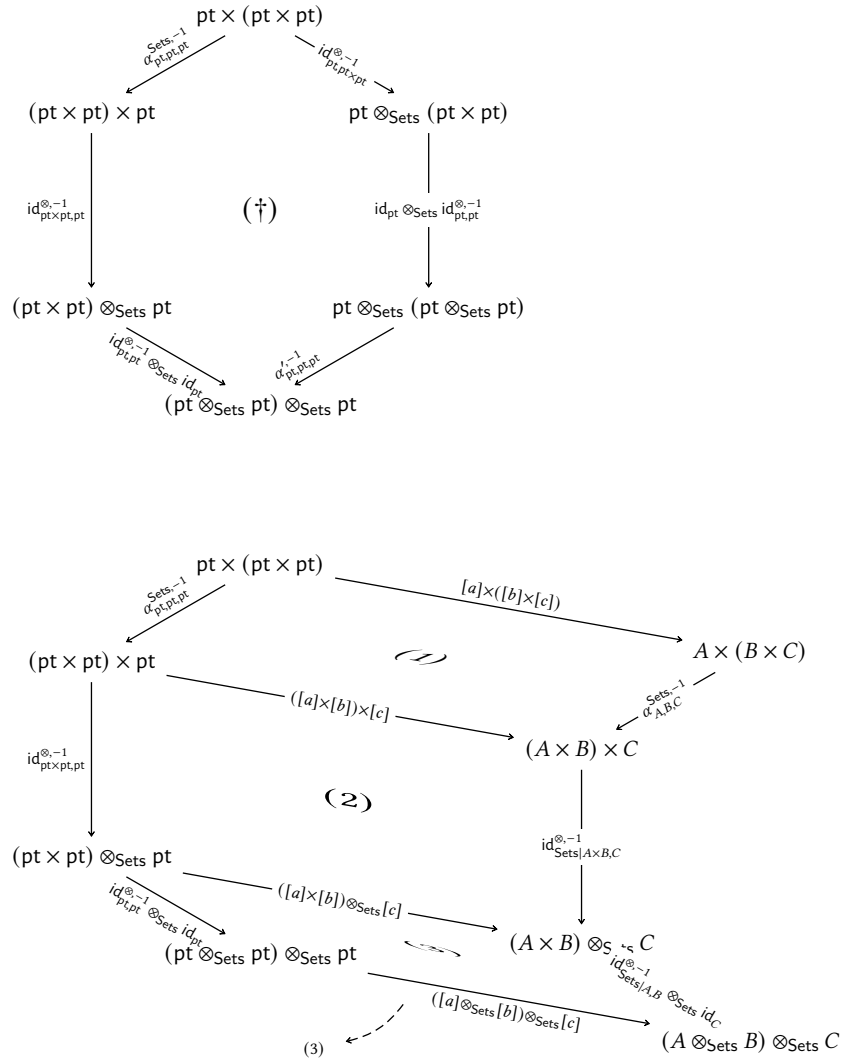
also commutes. Taking inverses, we see that the diagram

$$\begin{array}{ccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\text{Sets}, -1} \swarrow & & \searrow \text{id}_{\text{Sets} | \text{pt}, \text{pt} \times \text{pt}}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \downarrow \text{id}_{\text{Sets} | \text{pt} \times \text{pt}, \text{pt}}^{\otimes, -1} & (+) & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets} | \text{pt}, \text{pt}}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets} | \text{pt}, \text{pt}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_{\text{pt}} & & \downarrow \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\prime, -1} \\
 (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt}
 \end{array}$$

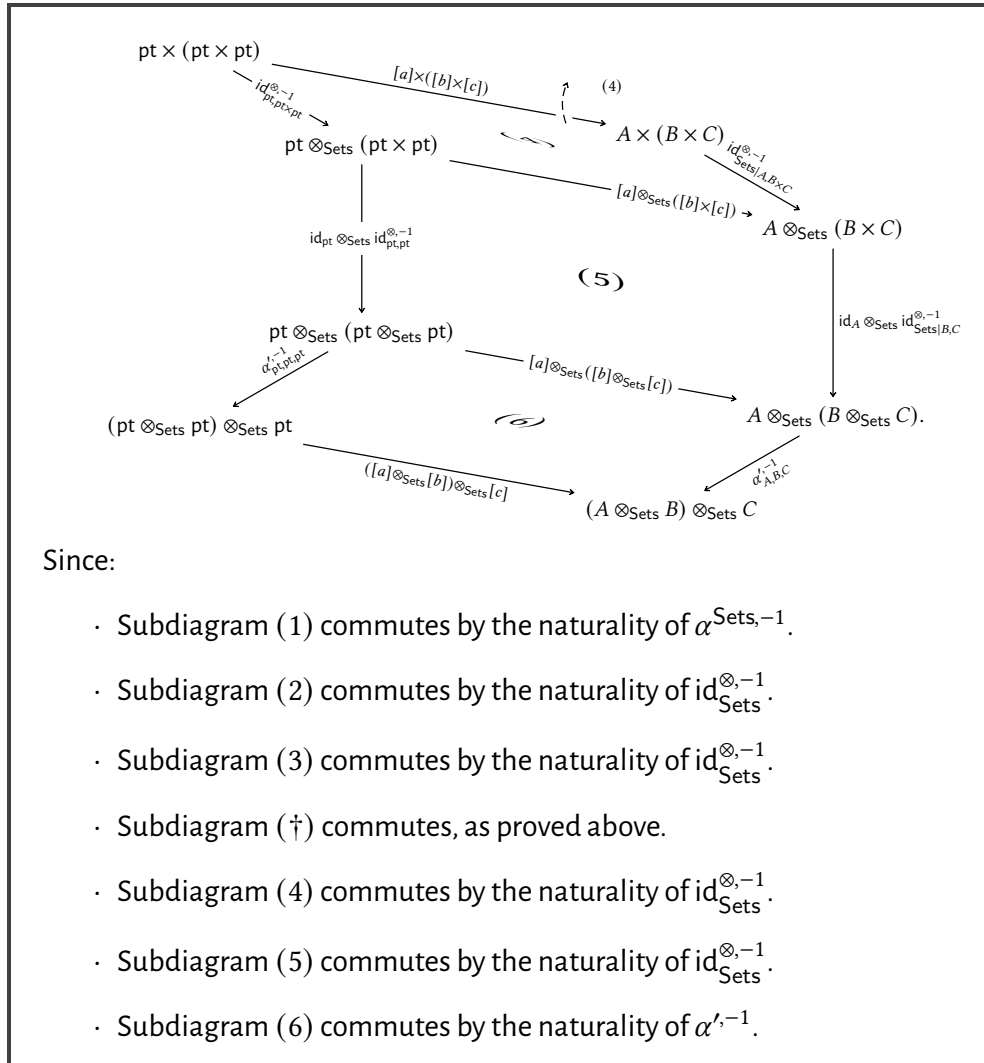
commutes as well. Now, let  $A, B, C \in \text{Obj}(\text{Sets})$ , let  $a \in A$ , let  $b \in B$ , let  $c \in C$ , and consider the diagram

$$\begin{array}{ccccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & & & \\
 \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\text{Sets}, -1} \swarrow & \xrightarrow{[a] \times ([b] \times [c])} & & & \searrow \text{id}_{\text{Sets} | A, B \times C}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) & & A \times (B \times C) \\
 \downarrow \text{id}_{\text{pt} \times \text{pt}, \text{pt}}^{\otimes, -1} & \searrow ([a] \times [b]) \times [c] & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{pt}, \text{pt}}^{\otimes, -1} & \swarrow \alpha_{A, B, C}^{\text{Sets}, -1} & \downarrow \text{id}_{A \otimes_{\text{Sets}} (B \times C)}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) & & A \otimes_{\text{Sets}} (B \times C) \\
 \downarrow \text{id}_{\text{pt} \otimes_{\text{Sets}} \text{pt}}^{\otimes, -1} & \searrow ([a] \times [b]) \otimes_{\text{Sets}} [c] & \downarrow \text{id}_{\text{Sets} | A \times B, C}^{\otimes, -1} & \swarrow \alpha_{A, B, C}^{\text{Sets}, -1} & \downarrow \text{id}_{A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C)}^{\otimes, -1} \\
 (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & & (A \times B) \times C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \downarrow \text{id}_{\text{pt} \otimes_{\text{Sets}} \text{pt}}^{\otimes, -1} & \searrow ([a] \otimes_{\text{Sets}} [b]) \otimes_{\text{Sets}} [c] & \downarrow \text{id}_{\text{Sets} | A \times B, C}^{\otimes, -1} & \swarrow \alpha_{A, B, C}^{\text{Sets}, -1} & \downarrow \text{id}_{A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C)}^{\otimes, -1} \\
 (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & & (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \downarrow \text{id}_{\text{pt} \otimes_{\text{Sets}} \text{pt}}^{\otimes, -1} & \searrow ([a] \otimes_{\text{Sets}} [b]) \otimes_{\text{Sets}} [c] & \downarrow \text{id}_{\text{Sets} | A \times B, C}^{\otimes, -1} & \swarrow \alpha_{A, B, C}^{\text{Sets}, -1} & \downarrow \text{id}_{A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C)}^{\otimes, -1} \\
 (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C)
 \end{array}$$

which we partition into subdiagrams as follows:







it follows that the diagram

$$\begin{array}{ccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 & \downarrow [a] \times ([b] \times [c]) & \\
 & A \times (B \times C) & \\
 \alpha_{A,B,C}^{\text{Sets}, -1} \swarrow & & \searrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} & & \downarrow \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \swarrow \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C & & \nwarrow \alpha'_{A,B,C}{}^{-1} \\
 & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C &
 \end{array}$$

also commutes. We then have

$$\begin{aligned}
 & \left[ (\text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{A,B,C}^{\text{Sets}, -1} \right] (a, (b, c)) = \left[ (\text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{A,B,C}^{\text{Sets}, -1} \circ ([a] \times ([b] \times [c])) \right] (\star, (\star, \star)) \\
 & = \left[ \alpha'_{A,B,C}{}^{-1} \circ (\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1}) \right. \\
 & \quad \left. \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \circ ([a] \times ([b] \times [c])) \right] (\star, (\star, \star)) \\
 & = [\alpha'_{A,B,C}{}^{-1} \circ (\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1}) \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1}] (a, (b, c))
 \end{aligned}$$

for each  $(a, (b, c)) \in A \times (B \times C)$ , and thus we have

$$(\text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \circ \alpha_{A,B,C}^{\text{Sets}, -1} = \alpha'_{A,B,C}{}^{-1} \circ (\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1}) \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\text{Sets}} \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes} \circ (\text{id}_{\text{Sets}|A, B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C) = \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \circ (\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes}) \circ \alpha'_{A,B,C},$$

showing that the diagram

$$\begin{array}{ccc}
 (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & & \\
 \text{id}_{\text{Sets}|A,B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \swarrow & & \searrow \alpha'_{A,B,C} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes} & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B, C}^{\otimes} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \searrow \alpha_{A,B,C}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \\
 & A \times (B \times C) &
 \end{array}$$

indeed commutes.

#### Braidedness of the Isomorphism $\otimes_{\text{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{ccc}
 A \otimes_{\text{Sets}} B & \xrightarrow{\text{id}_{\text{Sets}|A,B}^{\otimes}} & A \times B \\
 \sigma'_{A,B} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}} \\
 B \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|B,A}^{\otimes}} & B \times A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 \sigma'_{\text{pt}, \text{pt}} \downarrow & & \downarrow \sigma_{\text{pt}, \text{pt}}^{\text{Sets}} \\
 \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 & & \searrow !_{\text{pt} \times \text{pt}} \\
 & & \text{pt}
 \end{array}$$

commutes by the terminality of  $\text{pt}$  (**Constructions With Sets, Construction 4.1.1.1.2**). Since the map  $!_{\text{pt} \times \text{pt}}: \text{pt} \times \text{pt} \rightarrow \text{pt}$  is invertible (e.g. with inverse  $\lambda_{\text{pt}}^{\text{Sets}, -1}$ ), the diagram

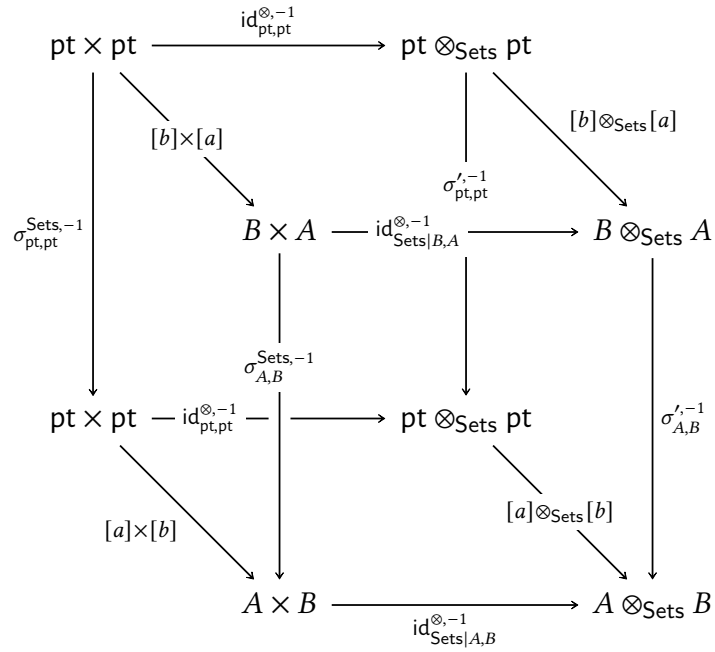
$$\begin{array}{ccc} \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\ \sigma'_{\text{pt}, \text{pt}} \downarrow & & \downarrow \sigma_{\text{pt}, \text{pt}}^{\text{Sets}} \\ \text{pt} \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \end{array}$$

also commutes. Taking inverses, we see that the diagram

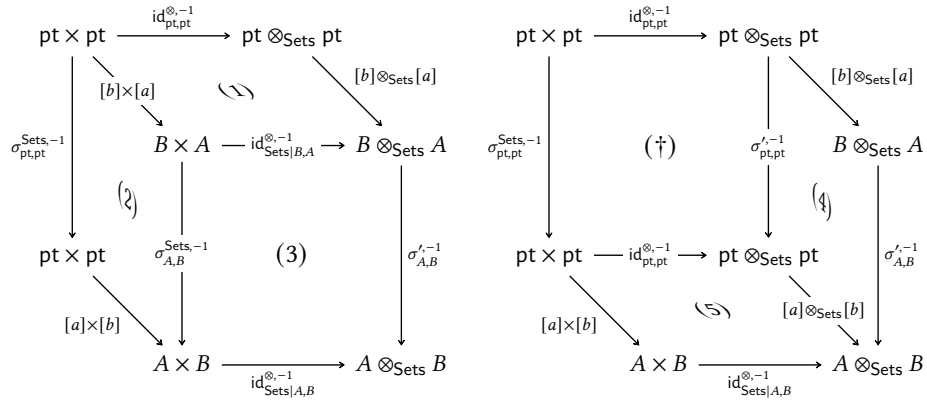
$$\begin{array}{ccc} \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\text{Sets}} \text{pt} \\ \sigma_{\text{pt}, \text{pt}}^{\text{Sets}, -1} \downarrow & (\dagger) & \downarrow \sigma'_{\text{pt}, \text{pt}}{}^{-1} \\ \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt}, \text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\text{Sets}} \text{pt} \end{array}$$

commutes as well. Now, let  $A, B \in \text{Obj}(\text{Sets})$ , let  $a \in A$ , let  $b \in B$ , and

consider the diagram



which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of  $\sigma^{\text{Sets}, -1}$ .

- Subdiagram (5) commutes by the naturality of  $\text{id}^{\otimes, -1}$ .
- Subdiagram  $(\dagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{-1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}^{\otimes, -1}$ .

it follows that the diagram

$$\begin{array}{ccc}
 \text{pt} \times \text{pt} & \xrightarrow{[b] \times [a]} & B \times A \\
 & \searrow & \downarrow \sigma_{A,B}^{\text{Sets}} \\
 & & A \times B \\
 & & \downarrow \sigma'_{A,B} \\
 & & A \otimes_{\text{Sets}} B
 \end{array}
 \quad
 \begin{array}{ccc}
 B \times A & \xrightarrow{\text{id}_{\text{Sets}|B,A}^{\otimes}} & B \otimes_{\text{Sets}} A \\
 \downarrow \sigma_{A,B}^{\text{Sets}} & & \downarrow \sigma'_{A,B} \\
 A \times B & \xrightarrow{\text{id}_{\text{Sets}|A,B}^{\otimes}} & A \otimes_{\text{Sets}} B
 \end{array}$$

commutes. We then have

$$\begin{aligned}
 [\text{id}_{\text{Sets}|A,B}^{\otimes, -1} \circ \sigma_{A,B}^{\text{Sets}, -1}](b, a) &= [\text{id}_{\text{Sets}|A,B}^{\otimes, -1} \circ \sigma_{A,B}^{\text{Sets}, -1} \circ ([b] \times [a])](\star, \star) \\
 &= [\sigma'_{A,B} \circ \text{id}_{\text{Sets}|B,A}^{\otimes, -1} \circ ([b] \times [a])](\star, \star) \\
 &= [\sigma'_{A,B} \circ \text{id}_{\text{Sets}|B,A}^{\otimes, -1}](b, a)
 \end{aligned}$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\text{id}_{\text{Sets}|A,B}^{\otimes, -1} \circ \sigma_{A,B}^{\text{Sets}, -1} = \sigma'_{A,B} \circ \text{id}_{\text{Sets}|B,A}^{\otimes, -1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\text{Sets}} \circ \text{id}_{\text{Sets}|A,B}^{\otimes} = \text{id}_{\text{Sets}|B,A}^{\otimes} \circ \sigma'_{A,B},$$

showing that the diagram

$$\begin{array}{ccc}
 A \otimes_{\text{Sets}} B & \xrightarrow{\text{id}_{\text{Sets}|A,B}^{\otimes}} & A \times B \\
 \downarrow \sigma'_{A,B} & & \downarrow \sigma_{A,B}^{\text{Sets}} \\
 B \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|B,A}^{\otimes}} & B \times A
 \end{array}$$

indeed commutes.

#### Uniqueness of the Isomorphism $\otimes_{\text{Sets}} \cong \times$

Let  $\phi, \psi: -_1 \otimes_{\text{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of  $\text{Sets}$  with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\begin{aligned}\lambda'_B &= \lambda_B^{\text{Sets}} \circ \phi_{\text{pt}, B} \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y), \\ \lambda'_B &= \lambda_B^{\text{Sets}} \circ \psi_{\text{pt}, B} \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y).\end{aligned}$$

Postcomposing both sides with  $\lambda_B^{\text{Sets}, -1}$  gives

$$\begin{aligned}\lambda_B^{\text{Sets}, -1} \circ \lambda'_B \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_Y) &= \phi_{\text{pt}, B}, \\ \lambda_B^{\text{Sets}, -1} \circ \lambda'_B \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y) &= \psi_{\text{pt}, B},\end{aligned}$$

and thus we have

$$\phi_{\text{pt}, B} = \psi_{\text{pt}, B}$$

for each  $B \in \text{Obj}(\text{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams


$$\begin{array}{ccc} \text{pt} \times B & \xrightarrow{[a] \times \text{id}_B} & A \times B \\ \phi_{\text{pt}, B} \downarrow & & \downarrow \phi_{A, B} \\ \text{pt} \otimes_{\text{Sets}} B & \xrightarrow{[a] \otimes_{\text{Sets}} \text{id}_B} & A \otimes_{\text{Sets}} B \end{array} \quad \begin{array}{ccc} \text{pt} \times B & \xrightarrow{[a] \times \text{id}_B} & A \times B \\ \psi_{\text{pt}, B} \downarrow & & \downarrow \psi_{A, B} \\ \text{pt} \otimes_{\text{Sets}} B & \xrightarrow{[a] \otimes_{\text{Sets}} \text{id}_B} & A \otimes_{\text{Sets}} B \end{array}$$

for  $\phi$  and  $\psi$  with respect to the morphisms  $[a]$  and  $\text{id}_B$ . Having shown that  $\phi_{\text{pt}, B} = \psi_{\text{pt}, B}$ , we have

$$\begin{aligned}\phi_{A, B}(a, b) &= [\phi_{A, B} \circ ([a] \times \text{id}_B)](\star, b) \\ &= [([a] \otimes_{\text{Sets}} \text{id}_B) \circ \phi_{\text{pt}, B}](\star, b) \\ &= [([a] \otimes_{\text{Sets}} \text{id}_B) \circ \psi_{\text{pt}, B}](\star, b) \\ &= [\psi_{A, B} \circ ([a] \times \text{id}_B)](\star, b) \\ &= \psi_{A, B}(a, b)\end{aligned}$$

for each  $(a, b) \in A \times B$ . Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each  $A, B \in \text{Obj}(\text{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\text{Sets}} \cong \times$  to be unique. 

#### COROLLARY 5.1.10.1.3 ► A SECOND UNIVERSAL PROPERTY FOR $(\text{Sets}, \times, \text{pt})$

The symmetric monoidal structure on the category  $\text{Sets}$  of [Proposition 5.1.9.1.1](#) is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The tensor product


$$\otimes_{\text{Sets}} : \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of  $\text{Sets}$  preserves colimits separately in each variable.

2. *The Unit Object Is pt.* We have  $\mathbb{1}_{\text{Sets}} \cong \text{pt}$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{B}_\infty}(\text{Sets})$  of ?? spanned by the symmetric monoidal categories  $(\text{Sets}, \otimes_{\text{Sets}}, \mathbb{1}_{\text{Sets}}, \lambda^{\text{Sets}}, \rho^{\text{Sets}}, \sigma^{\text{Sets}})$  satisfying [Items 1](#) and [2](#) is contractible.

#### PROOF 5.1.10.1.4 ► PROOF OF COROLLARY 5.1.10.1.3

Since  $\text{Sets}$  is locally presentable (??), it follows from ?? that [Item 1](#) is equivalent to the existence of an internal Hom as in [Item 1](#) of [Theorem 5.1.10.1.1](#). The result then follows from [Theorem 5.1.10.1.1](#). 

## 5.2 The Monoidal Category of Sets and Coproducts

### 5.2.1 Coproducts of Sets

See [Constructions With Sets, Section 4.2.3](#).



## 5.2.2 The Monoidal Unit

### DEFINITION 5.2.2.1.1 ► THE MONOIDAL UNIT OF $\coprod$

The **monoidal unit of the coproduct of sets** is the functor

$$\mathbb{0}^{\text{Sets}}: \text{pt} \rightarrow \text{Sets}$$

defined by

$$\mathbb{0}_{\text{Sets}} \stackrel{\text{def}}{=} \emptyset,$$

where  $\emptyset$  is the empty set of **Constructions With Sets**, Definition 4.3.1.1.1.

## 5.2.3 The Associator

### DEFINITION 5.2.3.1.1 ► THE ASSOCIATOR OF $\coprod$

The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\text{Sets}, \coprod}: \coprod \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\text{id}_{\text{Sets}} \times \coprod) \circ \alpha^{\text{Cats}_{\text{Sets}, \text{Sets}, \text{Sets}}}$$

as in the diagram

$$\begin{array}{ccccc}
 & & \text{Sets} \times (\text{Sets} \times \text{Sets}) & & \\
 & \nearrow \alpha^{\text{Cats}_{\text{Sets}, \text{Sets}, \text{Sets}}} & & \searrow \text{id} \times \coprod & \\
 (\text{Sets} \times \text{Sets}) \times \text{Sets} & & & & \text{Sets} \times \text{Sets} \\
 \downarrow \coprod \times \text{id} & \nearrow \alpha^{\text{Sets}, \coprod} & & \searrow \coprod & \\
 \text{Sets} \times \text{Sets} & & \xrightarrow{\coprod} & & \text{Sets}
 \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}, \coprod}: (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at  $(X, Y, Z)$  is given by

$$\alpha_{X,Y,Z}^{\text{Sets}, \coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } a = (0, (0, x)), \\ (1, (0, y)) & \text{if } a = (0, (1, y)), \\ (1, (1, a)) & \text{if } a = (1, z) \end{cases}$$

for each  $a \in (X \amalg Y) \amalg Z$ .

**PROOF 5.2.3.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.3.1.1**

**Unwinding the Definitions of  $(X \amalg Y) \amalg Z$  and  $X \amalg (Y \amalg Z)$**

Firstly, we unwind the expressions for  $(X \amalg Y) \amalg Z$  and  $X \amalg (Y \amalg Z)$ . We have

$$\begin{aligned} (X \amalg Y) \amalg Z &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \amalg Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ &= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ &\quad \cup \{(1, z) \in S \mid z \in Z\}, \end{aligned}$$

where  $S = \{0, 1\} \times ((X \amalg Y) \cup Z)$  and

$$\begin{aligned} X \amalg (Y \amalg Z) &\stackrel{\text{def}}{=} \{(0, x) \in S' \mid x \in X\} \cup \{(1, a) \in S' \mid a \in Y \amalg Z\} \\ &= \{(0, x) \in S' \mid x \in X\} \cup \{(1, (0, y)) \in S' \mid y \in Y\} \\ &\quad \cup \{(1, (1, z)) \in S' \mid z \in Z\}, \end{aligned}$$

where  $S' = \{0, 1\} \times (X \cup (Y \amalg Z))$ .

**Invertibility**

The inverse of  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg}$  is the map

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1} : X \amalg (Y \amalg Z) \rightarrow (X \amalg Y) \amalg Z$$

given by

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1}(a) \stackrel{\text{def}}{=} \begin{cases} (0, (0, x)) & \text{if } a = (0, x), \\ (0, (1, y)) & \text{if } a = (1, (0, y)), \\ (1, z) & \text{if } a = (1, (1, z)) \end{cases}$$

for each  $a \in X \amalg Y(\amalg Z)$ . Indeed:

• *Invertibility I.* The map  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1} \circ \alpha_{X,Y,Z}^{\text{Sets}, \amalg}$  acts on elements as

$$(0, (0, x)) \mapsto (0, x) \mapsto (0, (0, x)),$$

$$\begin{aligned} (0, (0, y)) &\mapsto (1, (0, y)) \mapsto (0, (0, y)), \\ (1, z) &\mapsto (1, (1, z)) \mapsto (1, z) \end{aligned}$$

and hence is equal to the identity map of  $(X \amalg Y) \amalg Z$ .

• *Invertibility II.* The map  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg} \circ \alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1}$  acts on elements as

$$\begin{aligned} (0, x) &\mapsto (0, (0, x)) \mapsto (0, x), \\ (1, (0, y)) &\mapsto (0, (0, y)) \mapsto (1, (0, y)), \\ (1, (1, z)) &\mapsto (1, z) \mapsto (1, (1, z)) \end{aligned}$$

and hence is equal to the identity map of  $X \amalg (Y \amalg Z)$ .

Therefore  $\alpha_{X,Y,Z}^{\text{Sets}, \amalg}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions

$$\begin{aligned} f &: X \rightarrow X', \\ g &: Y \rightarrow Y', \\ h &: Z \rightarrow Z' \end{aligned}$$

the diagram


$$\begin{array}{ccc} (X \amalg Y) \amalg Z & \xrightarrow{(f \amalg g) \amalg h} & (X' \amalg Y') \amalg Z' \\ \downarrow \alpha_{X,Y,Z}^{\text{Sets}, \amalg} & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}, \amalg} \\ X \amalg (Y \amalg Z) & \xrightarrow{f \amalg (g \amalg h)} & X' \amalg (Y' \amalg Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0, (0, x)) & & (0, (0, x)) \mapsto (0, (0, f(x))) \\
 \downarrow & & \downarrow \\
 (0, x) \mapsto (0, f(x)) & & (0, f(x)) \\
 \\ 
 (0, (1, y)) & & (0, (1, y)) \mapsto (0, (1, g(y))) \\
 \downarrow & & \downarrow \\
 (1, (0, y)) \mapsto (1, (0, g(y))) & & (1, (0, g(y))) \\
 \\ 
 (1, z) & & (1, z) \mapsto (1, h(z)) \\
 \downarrow & & \downarrow \\
 (1, (1, z)) \mapsto (1, (1, h(z))) & & (1, (1, h(z)))
 \end{array}$$

and hence indeed commutes, showing  $\alpha^{\mathbf{Sets}, \mathbb{I}}$  to be a natural transformation.

#### Being a Natural Isomorphism

Since  $\alpha^{\mathbf{Sets}, \mathbb{I}}$  is natural and  $\alpha^{\mathbf{Sets}, \mathbb{I}, -1}$  is a componentwise inverse to  $\alpha^{\mathbf{Sets}, \mathbb{I}}$ , it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that  $\lambda^{\mathbf{Sets}, -1}$  is also natural. Thus  $\alpha^{\mathbf{Sets}, \mathbb{I}}$  is a natural isomorphism. 

## 5.2.4 The Left Unitor

**DEFINITION 5.2.4.1.1 ► THE LEFT UNITOR OF  $\amalg$** 

The **left unitor of the coproduct of sets** is the natural isomorphism

$$\lambda^{\text{Sets}, \amalg} : \amalg \circ (\emptyset^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}, \amalg} : \emptyset \amalg X \xrightarrow{\sim} X$$

at  $X$  is given by

$$\lambda_X^{\text{Sets}, \amalg}((1, x)) \stackrel{\text{def}}{=} x$$

for each  $(1, x) \in \emptyset \amalg X$ .

**PROOF 5.2.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.4.1.1**
**Unwinding the Definition of  $\emptyset \amalg X$** 

Firstly, we unwind the expressions for  $\emptyset \amalg X$ . We have

$$\begin{aligned} \emptyset \amalg X &\stackrel{\text{def}}{=} \{(0, z) \in S \mid z \in \emptyset\} \cup \{(1, x) \in S \mid x \in X\} \\ &= \emptyset \cup \{(1, x) \in S \mid x \in X\} \\ &= \{(1, x) \in S \mid x \in X\}, \end{aligned}$$

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

**Invertibility**

The inverse of  $\lambda_X^{\text{Sets}, \amalg}$  is the map

$$\lambda_X^{\text{Sets}, \amalg, -1} : X \rightarrow \emptyset \amalg X$$

given by

$$\lambda_X^{\text{Sets}, \amalg, -1}(x) \stackrel{\text{def}}{=} (1, x)$$

for each  $x \in X$ . Indeed:

• *Invertibility I.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}, \amalg, -1} \circ \lambda_X^{\text{Sets}, \amalg}](1, x) &= \lambda_X^{\text{Sets}, \amalg, -1}(\lambda_X^{\text{Sets}, \amalg}(1, x)) \\ &= \lambda_X^{\text{Sets}, \amalg, -1}(x) \\ &= (1, x) \\ &= [\text{id}_{\emptyset \amalg X}](1, x) \end{aligned}$$

for each  $(1, x) \in \emptyset \amalg X$ , and therefore we have

$$\lambda_X^{\text{Sets}, \amalg, -1} \circ \lambda_X^{\text{Sets}, \amalg} = \text{id}_{\emptyset \amalg X}.$$

• *Invertibility II.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}, \amalg} \circ \lambda_X^{\text{Sets}, \amalg, -1}](x) &= \lambda_X^{\text{Sets}, \amalg}(\lambda_X^{\text{Sets}, \amalg, -1}(x)) \\ &= \lambda_X^{\text{Sets}, \amalg, -1}(1, x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\text{Sets}, \amalg} \circ \lambda_X^{\text{Sets}, \amalg, -1} = \text{id}_X.$$

Therefore  $\lambda_X^{\text{Sets}, \amalg}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \rightarrow Y$ , the diagram


$$\begin{array}{ccc} \emptyset \amalg X & \xrightarrow{\text{id}_{\emptyset} \amalg f} & \emptyset \amalg Y \\ \lambda_X^{\text{Sets}, \amalg} \downarrow & & \downarrow \lambda_Y^{\text{Sets}, \amalg} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (1, x) & & (1, x) \mapsto (1, f(x)) \\
 \downarrow & & \downarrow \\
 x & \xmapsto{\quad} & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\text{Sets}, \sqcup}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\lambda^{\text{Sets}, \sqcup}$  is natural and  $\lambda^{\text{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\text{Sets}, \sqcup}$ , it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\lambda^{\text{Sets}, \sqcup}$  is a natural isomorphism. 

## 5.2.5 The Right Unitor

**DEFINITION 5.2.5.1.1 ► THE RIGHT UNITOR OF  $\amalg$** 

The **right unitor of the coproduct of sets** is the natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \emptyset^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}, \amalg} : X \amalg \emptyset \xrightarrow{\sim} X$$

at  $X$  is given by

$$\rho_X^{\text{Sets}, \amalg}((0, x)) \stackrel{\text{def}}{=} x$$

for each  $(0, x) \in X \amalg \emptyset$ .

**PROOF 5.2.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.5.1.1****Unwinding the Definition of  $X \amalg \emptyset$** 

Firstly, we unwind the expression for  $X \amalg \emptyset$ . We have

$$\begin{aligned} X \amalg \emptyset &\stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, z) \in S \mid z \in \emptyset\} \\ &= \{(0, x) \in S \mid x \in X\} \cup \emptyset \\ &= \{(0, x) \in S \mid x \in X\}, \end{aligned}$$

where  $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$ .

**Invertibility**

The inverse of  $\rho_X^{\text{Sets}, \amalg}$  is the map

$$\rho_X^{\text{Sets}, \amalg, -1} : X \rightarrow X \amalg \emptyset$$



given by

$$\rho_X^{\text{Sets}, \amalg, -1}(x) \stackrel{\text{def}}{=} (0, x)$$

for each  $x \in X$ . Indeed:

• *Invertibility I.* We have

$$\begin{aligned} [\rho_X^{\text{Sets}, \amalg, -1} \circ \rho_X^{\text{Sets}, \amalg}](0, x) &= \rho_X^{\text{Sets}, \amalg, -1}(\rho_X^{\text{Sets}, \amalg}(0, x)) \\ &= \rho_X^{\text{Sets}, \amalg, -1}(x) \\ &= (0, x) \\ &= [\text{id}_X \amalg \emptyset](0, x) \end{aligned}$$

for each  $(0, x) \in \emptyset \amalg X$ , and therefore we have

$$\rho_X^{\text{Sets}, \amalg, -1} \circ \rho_X^{\text{Sets}, \amalg} = \text{id}_{\emptyset \amalg X}.$$

• *Invertibility II.* We have

$$\begin{aligned} [\rho_X^{\text{Sets}, \amalg} \circ \rho_X^{\text{Sets}, \amalg, -1}](x) &= \rho_X^{\text{Sets}, \amalg}(\rho_X^{\text{Sets}, \amalg, -1}(x)) \\ &= \rho_X^{\text{Sets}, \amalg, -1}(0, x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\text{Sets}, \amalg} \circ \rho_X^{\text{Sets}, \amalg, -1} = \text{id}_X.$$

Therefore  $\rho_X^{\text{Sets}, \amalg}$  is indeed an isomorphism.

#### Naturality

We need to show that, given a function  $f: X \rightarrow Y$ , the diagram


$$\begin{array}{ccc} X \amalg \emptyset & \xrightarrow{f \amalg \text{id}_{\emptyset}} & Y \amalg \emptyset \\ \rho_X^{\text{Sets}, \amalg} \downarrow & & \downarrow \rho_Y^{\text{Sets}, \amalg} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0, x) & & (0, x) \mapsto (1, f(x)) \\
 \downarrow & & \downarrow \\
 x & \xrightarrow{\quad} & f(x)
 \end{array}$$

and hence indeed commutes. Therefore  $\rho^{\text{Sets}, \amalg}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\rho^{\text{Sets}, \amalg}$  is natural and  $\rho^{\text{Sets}, -1}$  is a componentwise inverse to  $\rho^{\text{Sets}, \amalg}$ , it follows from [Categories, Item 2 of Proposition 11.9.7.1.2](#) that  $\rho^{\text{Sets}, -1}$  is also natural. Thus  $\rho^{\text{Sets}, \amalg}$  is a natural isomorphism. 

## 5.2.6 The Symmetry

### DEFINITION 5.2.6.1.1 ► THE SYMMETRY OF $\amalg$

The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\text{Sets}, \amalg} : \amalg \xrightarrow{\sim} \amalg \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

$$\begin{array}{ccc}
 \text{Sets} \times \text{Sets} & \xrightarrow{\amalg} & \text{Sets} \\
 \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2} \searrow & \amalg \downarrow \sigma^{\text{Sets}, \amalg} & \nearrow \amalg \\
 & \text{Sets} \times \text{Sets} &
 \end{array}$$

whose component

$$\sigma_{X, Y}^{\text{Sets}, \amalg} : X \amalg Y \xrightarrow{\sim} Y \amalg X$$

at  $X, Y \in \text{Obj}(\text{Sets})$  is defined by

$$\sigma_{X, Y}^{\text{Sets}, \amalg}(x, y) \stackrel{\text{def}}{=} (y, x)$$

for each  $(x, y) \in X \times Y$ .

## PROOF 5.2.6.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.6.1.1

Unwinding the Definitions of  $X \amalg Y$  and  $Y \amalg X$ 

Firstly, we unwind the expressions for  $X \amalg Y$  and  $Y \amalg X$ . We have

$$X \amalg Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \amalg X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},$$

where  $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$ .

## Invertibility

The inverse of  $\sigma_{X,Y}^{\text{Sets}, \amalg}$  is the map

$$\sigma_{X,Y}^{\text{Sets}, \amalg, -1} : Y \amalg X \rightarrow X \amalg Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets}, \amalg, -1} \stackrel{\text{def}}{=} \sigma_{Y,X}^{\text{Sets}, \amalg}$$

and hence given by

$$\sigma_{X,Y}^{\text{Sets}, \amalg, -1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = (1, x), \\ (1, y) & \text{if } z = (0, y) \end{cases}$$

for each  $z \in Y \amalg X$ . Indeed:

• *Invertibility I.* We have

$$\begin{aligned} [\sigma_{X,Y}^{\text{Sets}, \amalg, -1} \circ \sigma_{X,Y}^{\text{Sets}, \amalg}](0, x) &= \sigma_X^{\text{Sets}, \amalg, -1}(\sigma_X^{\text{Sets}, \amalg}(0, x)) \\ &= \sigma_X^{\text{Sets}, \amalg, -1}(1, x) \\ &= (0, x) \\ &= [\text{id}_X \amalg \text{id}_Y](0, x) \end{aligned}$$

for each  $(0, x) \in X \amalg Y$  and

$$[\sigma_{X,Y}^{\text{Sets}, \amalg, -1} \circ \sigma_{X,Y}^{\text{Sets}, \amalg}](1, y) = \sigma_X^{\text{Sets}, \amalg, -1}(\sigma_X^{\text{Sets}, \amalg}(1, y))$$

$$\begin{aligned}
&= \sigma_X^{\text{Sets}, \coprod, -1}(0, y) \\
&= (1, y) \\
&= [\text{id}_X \coprod \text{id}_Y](1, y)
\end{aligned}$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} = \text{id}_{X \coprod Y}.$$

· *Invertibility II.* We have

$$\begin{aligned}
[\sigma_{X,Y}^{\text{Sets}, \coprod} \circ \sigma_{X,Y}^{\text{Sets}, \coprod, -1}](0, y) &= \sigma_X^{\text{Sets}, \coprod}(\sigma_X^{\text{Sets}, \coprod, -1}(0, y)) \\
&= \sigma_X^{\text{Sets}, \coprod, -1}(1, y) \\
&= (0, y) \\
&= [\text{id}_Y \coprod \text{id}_X](0, y)
\end{aligned}$$

for each  $(0, y) \in Y \coprod X$  and

$$\begin{aligned}
[\sigma_{X,Y}^{\text{Sets}, \coprod} \circ \sigma_{X,Y}^{\text{Sets}, \coprod, -1}](1, x) &= \sigma_X^{\text{Sets}, \coprod}(\sigma_X^{\text{Sets}, \coprod, -1}(1, x)) \\
&= \sigma_X^{\text{Sets}, \coprod, -1}(0, x) \\
&= (1, x) \\
&= [\text{id}_Y \coprod \text{id}_X](1, x)
\end{aligned}$$

for each  $(1, x) \in Y \coprod X$ , and therefore we have

$$\sigma_X^{\text{Sets}, \coprod} \circ \sigma_X^{\text{Sets}, \coprod, -1} = \text{id}_{Y \coprod X}.$$

Therefore  $\sigma_{X,Y}^{\text{Sets}, \coprod}$  is indeed an isomorphism.

Naturality

We need to show that, given functions  $f: A \rightarrow X$  and  $g: B \rightarrow Y$ , the diagram


$$\begin{array}{ccc} A \amalg B & \xrightarrow{f \amalg g} & X \amalg Y \\ \sigma_{A,B}^{\text{Sets}, \amalg} \downarrow & & \downarrow \sigma_{X,Y}^{\text{Sets}, \amalg} \\ B \amalg A & \xrightarrow{g \amalg f} & Y \amalg X \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (0, a) & \longmapsto & (0, f(a)) \\ \downarrow & & \downarrow \\ (1, a) & \longmapsto & (1, f(a)) \\ \\ (1, b) & \longmapsto & (1, g(b)) \\ \downarrow & & \downarrow \\ (0, b) & \longmapsto & (0, g(b)) \end{array}$$

and hence indeed commutes. Therefore  $\sigma^{\text{Sets}, \amalg}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\sigma^{\text{Sets}, \amalg}$  is natural and  $\sigma^{\text{Sets}, -1}$  is a componentwise inverse to  $\sigma^{\text{Sets}, \amalg}$ , it follows from **Categories, Item 2 of Proposition 11.9.7.1.2** that  $\sigma^{\text{Sets}, -1}$  is also natural. Thus  $\sigma^{\text{Sets}, \amalg}$  is a natural isomorphism. 

## **5.2.7 The Monoidal Category of Sets and Coproducts**

**PROPOSITION 5.2.7.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO  $\amalg$** 

The category **Sets** admits a closed symmetric monoidal category structure consisting of:

- *The Underlying Category.* The category **Sets** of pointed sets.
- *The Monoidal Product.* The coproduct functor

$$\amalg : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Proposition 4.2.3.1.4**.

- *The Monoidal Unit.* The functor

$$\mathbb{0}^{\mathbf{Sets}} : \mathbf{pt} \rightarrow \mathbf{Sets}$$

of **Definition 5.2.2.1.1**.

- *The Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}, \amalg} : \amalg \circ (\amalg \times \mathrm{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \amalg \circ (\mathrm{id}_{\mathbf{Sets}} \times \amalg) \circ \alpha_{\mathbf{Sets}, \mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}}$$

of **Definition 5.2.3.1.1**.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Sets}, \amalg} : \amalg \circ (\mathbb{0}^{\mathbf{Sets}} \times \mathrm{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}}^{\mathbf{Cats}_2}$$

of **Definition 5.2.4.1.1**.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathbf{Sets}, \amalg} : \amalg \circ (\mathrm{id} \times \mathbb{0}^{\mathbf{Sets}}) \xrightarrow{\sim} \rho_{\mathbf{Sets}}^{\mathbf{Cats}_2}$$

of **Definition 5.2.5.1.1**.

- *The Symmetry.* The natural isomorphism

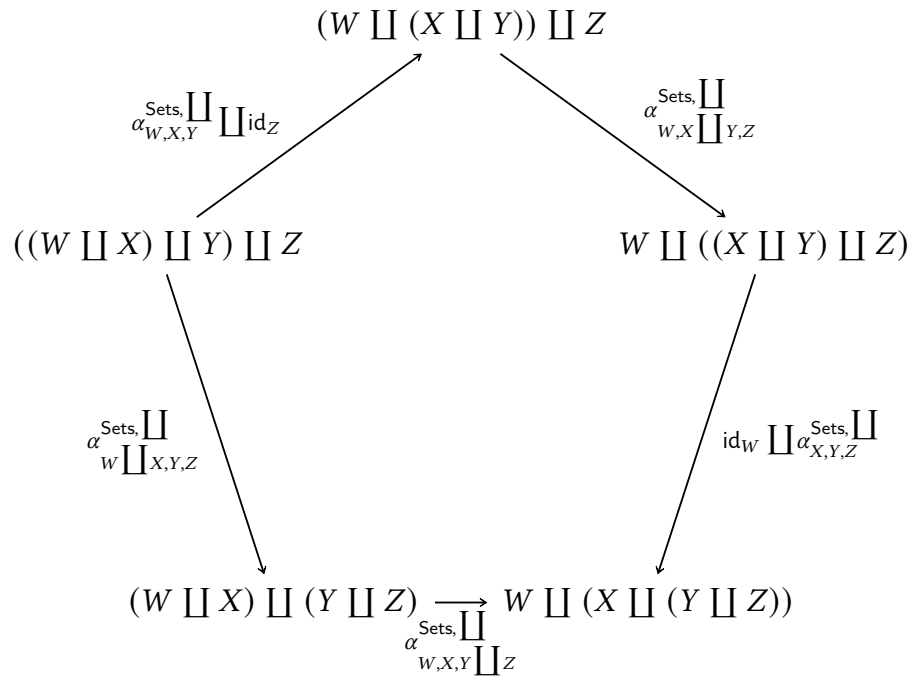
$$\sigma^{\mathbf{Sets}, \amalg} : \times \xrightarrow{\sim} \times \circ \sigma_{\mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}_2}$$

of **Definition 5.2.6.1.1**.

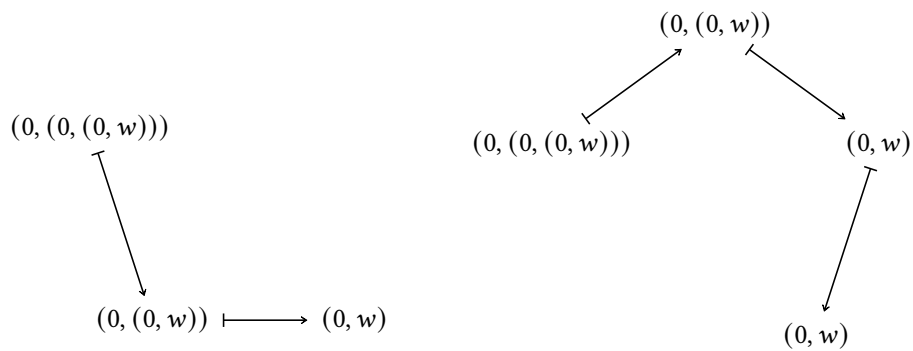
## PROOF 5.2.7.1.2 ► PROOF OF PROPOSITION 5.2.7.1.1

## The Pentagon Identity

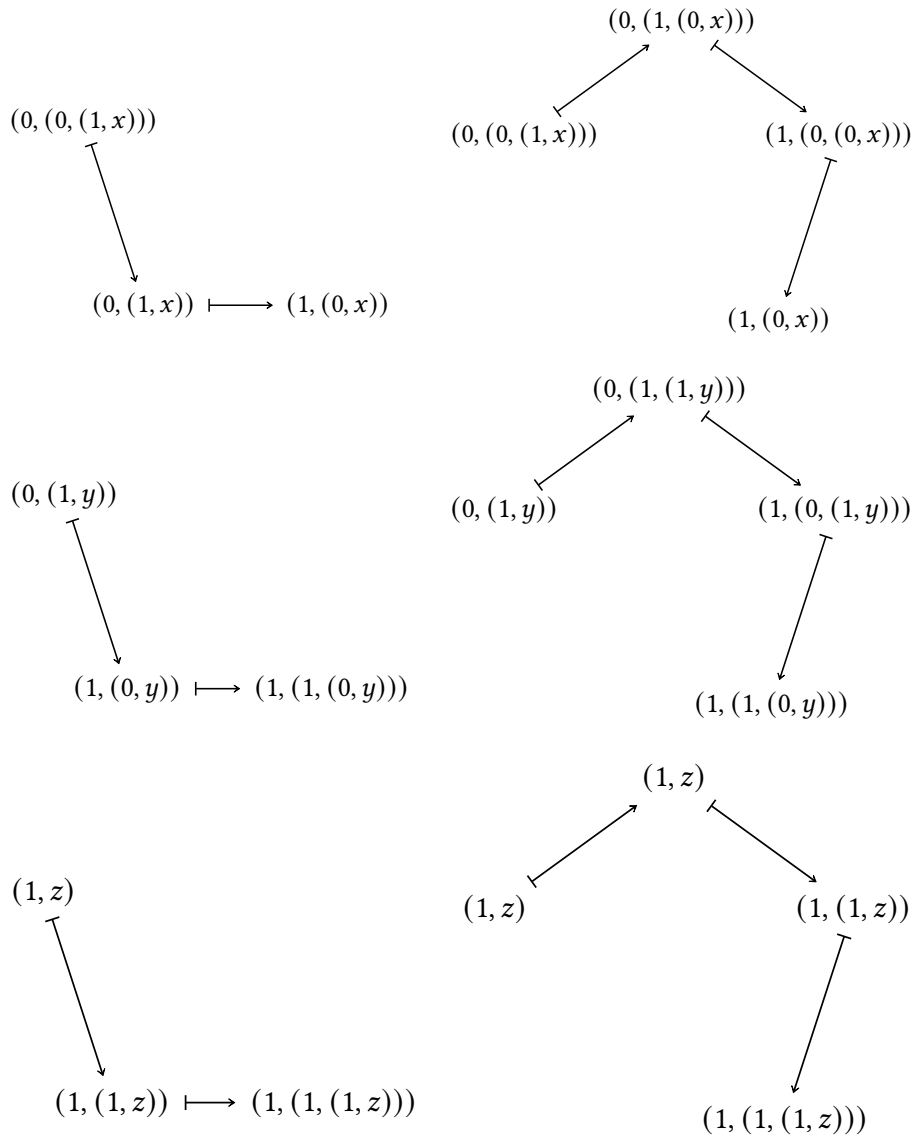
Let  $W, X, Y$  and  $Z$  be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as







and therefore the pentagon identity is satisfied.

The Triangle Identity

Let  $X$  and  $Y$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \amalg \emptyset) \amalg Y & \xrightarrow{\alpha_{X, \emptyset, Y}^{\text{Sets}, \amalg}} & X \amalg (\emptyset \amalg Y) \\
 \rho_X^{\text{Sets}, \amalg} \amalg \text{id}_Y \searrow & & \swarrow \text{id}_X \amalg \lambda_Y^{\text{Sets}, \amalg} \\
 & X \amalg Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0, (1, x)) & \xrightarrow{\quad} & (1, (0, x)) \\
 \searrow & & \swarrow \\
 & (0, x) & \\
 \\ 
 (1, y) & \xrightarrow{\quad} & (1, (1, y)) \\
 \searrow & & \swarrow \\
 & (1, y) &
 \end{array}$$

and therefore the triangle identity is satisfied.

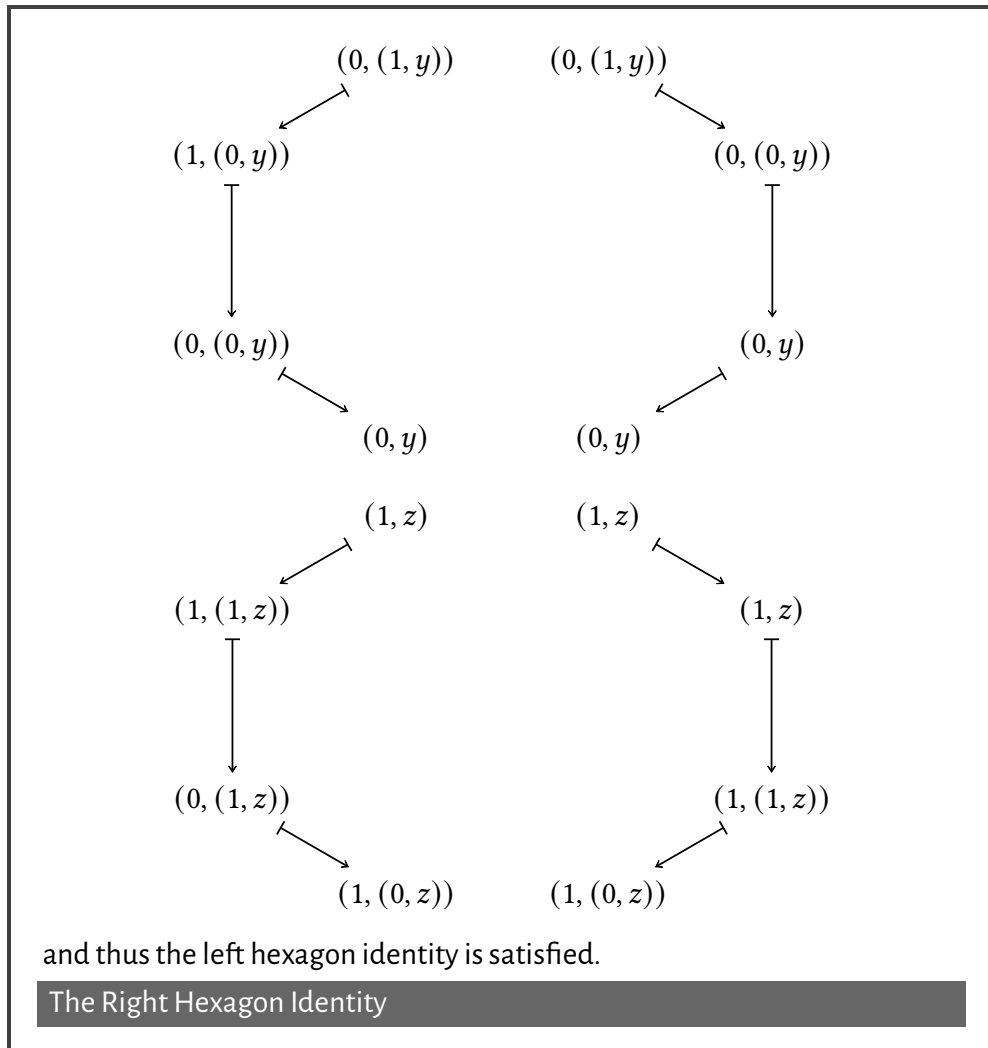
The Left Hexagon Identity

Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \amalg Y) \amalg Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}} \swarrow & & \searrow \sigma_{X,Y}^{\text{Sets}} \amalg \text{id}_Z \\
 X \amalg (Y \amalg Z) & & (Y \amalg X) \amalg Z \\
 \downarrow \sigma_{X,Y}^{\text{Sets}} \amalg \text{id}_Z & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}} \\
 (Y \amalg Z) \amalg X & & Y \amalg (X \amalg Z) \\
 \searrow \alpha_{Y,Z,X}^{\text{Sets}} & & \swarrow \text{id}_Y \amalg \sigma_{X,Z}^{\text{Sets}} \\
 & Y \amalg (Z \amalg X) &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (0, (0, x)) & (0, (0, x)) \\
 \swarrow & & \searrow \\
 (0, x) & & (0, (1, x)) \\
 \downarrow & & \downarrow \\
 (1, x) & & (1, (0, x)) \\
 \searrow & & \swarrow \\
 & (1, (1, x)) & (1, (1, x))
 \end{array}$$

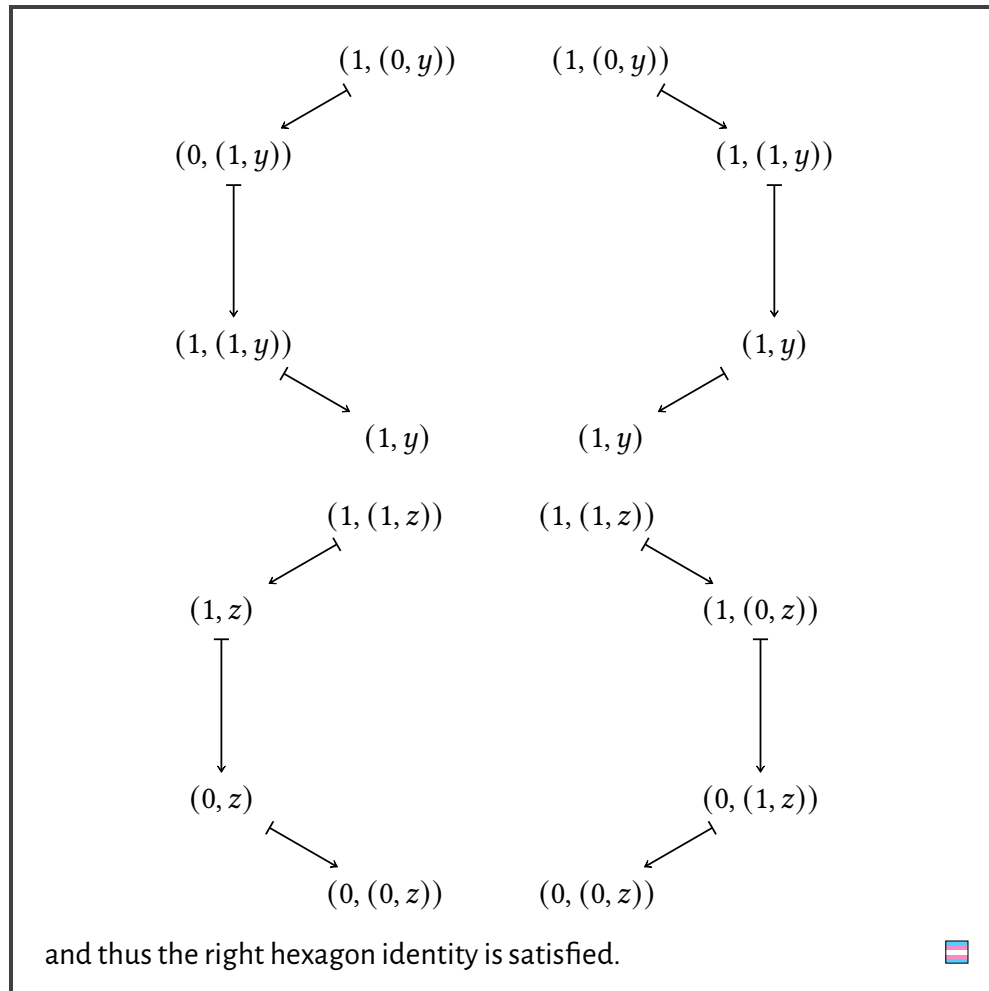


Let  $X$ ,  $Y$ , and  $Z$  be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & X \amalg (Y \amalg Z) & \\
 (\alpha_{X,Y,Z}^{\text{Sets}})^{-1} \swarrow & & \searrow \text{id}_X \amalg \sigma_{Y,Z}^{\text{Sets}} \\
 (X \amalg Y) \amalg Z & & X \amalg (Z \amalg Y) \\
 \downarrow \sigma_{X \amalg Y, Z}^{\text{Sets}} & & \downarrow (\alpha_{X,Z,Y}^{\text{Sets}})^{-1} \\
 Z \amalg (X \amalg Y) & & (X \amalg Z) \amalg Y \\
 & \searrow (\alpha_{Z,X,Y}^{\text{Sets}})^{-1} \quad \swarrow \sigma_{X,Z}^{\text{Sets}} \amalg \text{id}_Y & \\
 & (Z \amalg X) \amalg Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (0, x) & \\
 \swarrow & & \searrow \\
 (0, (0, x)) & & (0, x) \\
 \downarrow & & \downarrow \\
 (1, (0, x)) & & (0, (0, x)) \\
 \searrow & & \swarrow \\
 & (0, (1, x)) &
 \end{array}$$



## **5.3 The Bimonoidal Category of Sets, Products, and Coproducts**

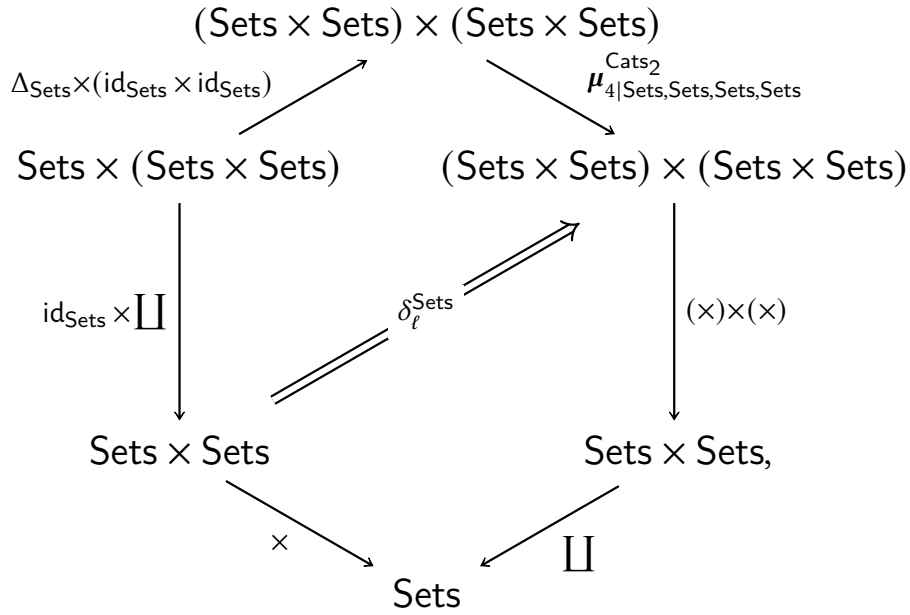
### **5.3.1 The Left Distributor**

**DEFINITION 5.3.1.1.1 ► THE LEFT DISTRIBUTOR OF  $\times$  OVER  $\amalg$** 

The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_{\ell}^{\text{Sets}}: \times \circ (\text{id}_{\text{Sets}} \times \amalg) \xrightarrow{\sim} \amalg \circ ((\times) \times (\times)) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\Delta_{\text{Sets}} \times (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}))$$

as in the diagram



whose component

$$\delta_{\ell|X,Y,Z}^{\text{Sets}}: X \times (Y \amalg Z) \xrightarrow{\sim} (X \times Y) \amalg (X \times Z)$$

at  $(X, Y, Z)$  is defined by

$$\delta_{\ell|X,Y,Z}^{\text{Sets}}(x, a) \stackrel{\text{def}}{=} \begin{cases} (0, (x, y)) & \text{if } a = (0, y), \\ (1, (x, z)) & \text{if } a = (1, z) \end{cases}$$

for each  $(x, a) \in X \times (Y \amalg Z)$ .



## PROOF 5.3.1.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.1.1.1

## Invertibility

The inverse of  $\delta_{\ell|X,Y,Z}^{\text{Sets}}$  is the map

$$\delta_{\ell|X,Y,Z}^{\text{Sets},-1}: (X \times Y) \amalg (X \times Z) \xrightarrow{\sim} X \times (Y \amalg Z)$$

given by

$$\delta_{\ell|X,Y,Z}^{\text{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} (x, (0, y)) & \text{if } a = (0, (x, y)), \\ (x, (1, z)) & \text{if } a = (1, (x, z)) \end{cases}$$

for  $a \in (X \times Y) \amalg (X \times Z)$ . Indeed:

- *Invertibility I.* The map  $\delta_{\ell|X,Y,Z}^{\text{Sets},-1} \circ \delta_{\ell|X,Y,Z}^{\text{Sets}}$  acts on elements as

$$\begin{aligned} (x, (0, y)) &\mapsto (0, (x, y)) \mapsto (x, (0, y)), \\ (x, (1, z)) &\mapsto (1, (x, z)) \mapsto (x, (1, z)), \end{aligned}$$

but these are the two possible cases for elements of  $X \times (Y \amalg Z)$ . Hence the map is equal to the identity.

- *Invertibility II.* The map  $\delta_{\ell|X,Y,Z}^{\text{Sets}} \circ \delta_{\ell|X,Y,Z}^{\text{Sets},-1}$  acts on elements as

$$\begin{aligned} (0, (x, y)) &\mapsto (x, (0, y)) \mapsto (0, (x, y)), \\ (1, (x, z)) &\mapsto (x, (1, z)) \mapsto (1, (x, z)), \end{aligned}$$

but these are the two possible cases for elements of  $(X \times Y) \amalg (X \times Z)$ . Hence the map is equal to the identity.

Thus  $\delta_{\ell|X,Y,Z}^{\text{Sets}}$  is an isomorphism for all  $X, Y, Z$ .

## Naturality

We need to show that, given functions

$$\begin{aligned} f: X &\rightarrow X', \\ g: Y &\rightarrow Y', \end{aligned}$$

$$h: Z \rightarrow Z'$$

the diagram


$$\begin{array}{ccc} X \times (Y \amalg Z) & \xrightarrow{f \times (g \amalg h)} & X' \times (Y' \amalg Z') \\ \downarrow \delta_{\ell|X,Y,Z}^{\text{Sets}} & & \downarrow \delta_{\ell|X',Y',Z'}^{\text{Sets}} \\ (X \times Y) \amalg (X \times Z) & \xrightarrow{(f \times g) \amalg (f \times h)} & (X' \times Y') \amalg (X' \times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x, (0, y)) & \mapsto & (f(x), (0, f(y))) \\ \downarrow & & \downarrow \\ (0, (x, y)) & \mapsto & (0, (f(x), g(y))) \\ \\ (x, (1, z)) & \mapsto & (f(x), (1, h(z))) \\ \downarrow & & \downarrow \\ (1, (x, z)) & \mapsto & (1, (f(x), h(z))), \end{array}$$

so it commutes, showing  $\delta_{\ell}^{\text{Sets}}$  to be a natural transformation.

#### Being a Natural Isomorphism

Since  $\delta_{\ell}^{\text{Sets}}$  is natural and  $\delta_{\ell}^{\text{Sets}, -1}$  is a componentwise inverse to  $\delta_{\ell}^{\text{Sets}}$ , it follows from **Categories, Item 2 of Proposition 11.9.7.1.2** that  $\delta_{\ell}^{\text{Sets}, -1}$  is also natural. Thus  $\delta_{\ell}^{\text{Sets}}$  is a natural isomorphism. 

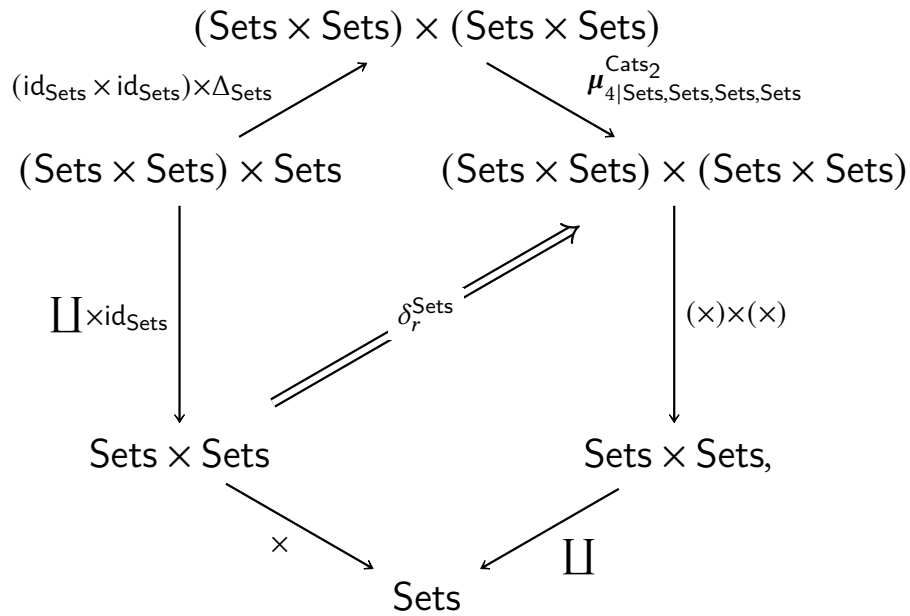
### 5.3.2 The Right Distributor

#### DEFINITION 5.3.2.1.1 ► THE RIGHT DISTRIBUTOR OF $\times$ OVER $\coprod$

The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\text{Sets}}: \times \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ ((\times) \times (\times)) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ ((\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}})$$

as in the diagram



whose component

$$\delta_{r|X,Y,Z}^{\text{Sets}}: (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at  $(X, Y, Z)$  is defined by

$$\delta_{r|X,Y,Z}^{\text{Sets}}(a, z) \stackrel{\text{def}}{=} \begin{cases} (0, (x, z)) & \text{if } a = (0, x), \\ (1, (y, z)) & \text{if } a = (1, y) \end{cases}$$

for each  $(a, z) \in (X \coprod Y) \times Z$ .

## PROOF 5.3.2.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.2.1.1

## Invertibility

The inverse of  $\delta_{r|X,Y,Z}^{\text{Sets}}$  is the map

$$\delta_{r|X,Y,Z}^{\text{Sets},-1}: (X \times Z) \amalg (Y \times Z) \xrightarrow{\sim} (X \amalg Y) \times Z$$

given by

$$\delta_{r|X,Y,Z}^{\text{Sets},-1}(a) \stackrel{\text{def}}{=} \begin{cases} ((0, x), z) & \text{if } a = (0, (x, z)), \\ ((1, y), z) & \text{if } a = (1, (y, z)) \end{cases}$$

for  $a \in (X \times Z) \amalg (Y \times Z)$ . Indeed:

- *Invertibility I.* The map  $\delta_{r|X,Y,Z}^{\text{Sets},-1} \circ \delta_{r|X,Y,Z}^{\text{Sets}}$  acts on elements as

$$\begin{aligned} ((0, x), z) &\mapsto (0, (x, z)) \mapsto (0, (x, z)), \\ ((1, y), z) &\mapsto (1, (y, z)) \mapsto (1, (y, z)), \end{aligned}$$

but these are the two possible cases for elements of  $(X \amalg Y) \times Z$ . Hence the map is equal to the identity.

- *Invertibility II.* The map  $\delta_{r|X,Y,Z}^{\text{Sets}} \circ \delta_{r|X,Y,Z}^{\text{Sets},-1}$  acts on elements as

$$\begin{aligned} (0, (x, z)) &\mapsto ((0, x), z) \mapsto (0, (x, z)), \\ (1, (y, z)) &\mapsto ((1, y), z) \mapsto (1, (y, z)), \end{aligned}$$

but these are the two possible cases for elements of  $(X \times Z) \amalg (Y \times Z)$ . Hence the map is equal to the identity.

So  $\delta_{r|X,Y,Z}^{\text{Sets}}$  is an isomorphism for all  $X, Y, Z$ .

## Naturality

We need to show that, given functions

$$\begin{aligned} f: X &\rightarrow X', \\ g: Y &\rightarrow Y', \end{aligned}$$

$$h: Z \rightarrow Z'$$

the diagram


$$\begin{array}{ccc} (X \amalg Y) \times Z' & \xrightarrow{(f \amalg g) \times h} & (X' \amalg Y') \times Z' \\ \delta_{r|X,Y,Z}^{\text{Sets}} \downarrow & & \downarrow \delta_{r|X',Y',Z'}^{\text{Sets}} \\ (X \times Z) \amalg (Y \times Z) & \xrightarrow{(f \times h) \amalg (g \times h)} & (X' \times Z') \amalg (Y' \times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} ((0, x), z) & \mapsto & ((0, f(x)), h(z)) \\ \downarrow & & \downarrow \\ (0, (x, z)) & \mapsto & (0, (f(x), h(z))) \\ \\ ((1, y), z) & \mapsto & ((1, g(y)), h(z)) \\ \downarrow & & \downarrow \\ (1, (y, z)) & \mapsto & (1, (g(y), h(z))) \end{array}$$

so it commutes and  $\delta_r^{\text{Sets}}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\delta_r^{\text{Sets}}$  is natural and  $\delta_r^{\text{Sets}, -1}$  is a componentwise inverse to  $\delta_r^{\text{Sets}}$ , it follows from **Categories, Item 2 of Proposition 11.9.7.1.2** that  $\delta_r^{\text{Sets}, -1}$  is also natural. Thus  $\delta_r^{\text{Sets}}$  is a natural isomorphism. 

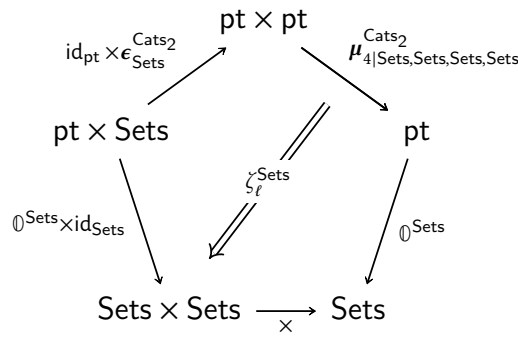
### 5.3.3 The Left Annihilator

#### DEFINITION 5.3.3.1.1 ► THE LEFT ANNIHILATOR OF $\times$

The **left annihilator of the product of sets** is the natural isomorphism

$$\zeta_{\ell}^{\text{Sets}} : \emptyset^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\text{id}_{\text{pt}} \times \epsilon_{\text{Sets}}^{\text{Cats}_2}) \xrightarrow{\sim} \times \circ (\emptyset^{\text{Sets}} \times \text{id}_{\text{Sets}})$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\text{Sets}} : \emptyset \times A \xrightarrow{\sim} \emptyset$$

given by  $\zeta_{\ell|A}^{\text{Sets}} \stackrel{\text{def}}{=} \text{pr}_1$ .

#### PROOF 5.3.3.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.3.1.1

##### Invertibility

The inverse of  $\zeta_{\ell|A}^{\text{Sets}}$  is the map

$$\zeta_{\ell|A}^{\text{Sets}, -1} : \emptyset \xrightarrow{\sim} \emptyset \times A$$

given by

$$\zeta_{\ell|A}^{\text{Sets}, -1} \stackrel{\text{def}}{=} \iota_A,$$

where  $\iota_A$  is as defined in **Constructions With Sets, Construction 4.2.1.1.2:**

- *Invertibility I.* The map  $\zeta_{\ell|A}^{\text{Sets}} \circ \iota_A : \emptyset \rightarrow \emptyset$  is equal to  $\text{id}_{\emptyset}$ , as  $\emptyset$  is the initial object of **Sets**.

- *Invertibility II.* The map  $\iota_A \circ \zeta_{\ell|A}^{\text{Sets}}$  is equal to the identity on every  $(x, a) \in \emptyset \times A$ , of which there are none.

Hence  $\zeta_{\ell|A}^{\text{Sets}}$  is an isomorphism.


#### Naturality

We need to show that given a function  $f: A \rightarrow B$ , the diagram

$$\begin{array}{ccc} \emptyset \times A & \xrightarrow{\text{id}_{\emptyset} \times f} & \emptyset \times B \\ \zeta_{\ell|A}^{\text{Sets}} \downarrow & & \downarrow \zeta_{\ell|B}^{\text{Sets}} \\ \emptyset & \xrightarrow{\text{id}_{\emptyset}} & \emptyset \end{array}$$

commutes. But since  $\emptyset \times A$  has no elements, this is trivially true.

#### Being a Natural Isomorphism

Since  $\zeta_{\ell}^{\text{Sets}}$  is natural and  $\zeta_{\ell}^{\text{Sets}, -1}$  is a componentwise inverse to  $\zeta_{\ell}^{\text{Sets}}$ , it follows from [Categories, Item 2 of Proposition 11.9.7.1.2](#) that  $\zeta_{\ell}^{\text{Sets}, -1}$  is also natural. Thus  $\zeta_{\ell}^{\text{Sets}}$  is a natural isomorphism. 

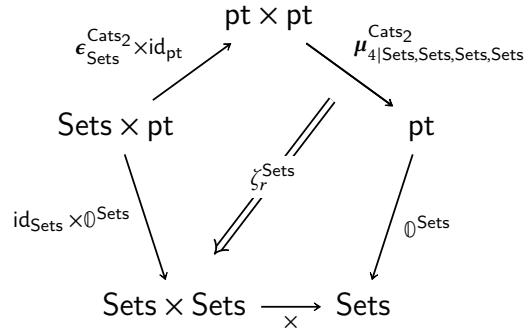
### 5.3.4 The Right Annihilator

#### DEFINITION 5.3.4.1.1 ► THE RIGHT ANNIHILATOR OF $\times$

The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\text{Sets}}: \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\epsilon_{\text{Sets}}^{\text{Cats}_2} \times \text{id}_{\text{pt}}) \dashrightarrow \times \circ (\text{id}_{\text{Sets}} \times \mathbb{0}^{\text{Sets}})$$

as in the diagram



with components

$$\zeta_{r|A}^{\text{Sets}} : A \times \emptyset \dashrightarrow \emptyset$$

given by  $\zeta_{r|A}^{\text{Sets}} \stackrel{\text{def}}{=} \text{pr}_2$ .

#### PROOF 5.3.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.4.1.1

##### Invertibility

The inverse of  $\zeta_{r|A}^{\text{Sets}}$  is the map

$$\zeta_{r|A}^{\text{Sets}, -1} : \emptyset \dashrightarrow A \times \emptyset$$

given by

$$\zeta_{r|A}^{\text{Sets}, -1} \stackrel{\text{def}}{=} \iota_A,$$

where  $\iota_A$  is as defined in [Constructions With Sets, Construction 4.2.1.1.2](#):

- *Invertibility I.* The map  $\zeta_{r|A}^{\text{Sets}} \circ \iota_A : \emptyset \rightarrow \emptyset$  is equal to  $\text{id}_{\emptyset}$ , as  $\emptyset$  is the initial object of **Sets**.
- *Invertibility II.* The map  $\iota_A \circ \zeta_{r|A}^{\text{Sets}}$  is equal to the identity on every  $(a, x) \in A \times \emptyset$ , of which there are none.

Hence  $\zeta_{r|A}^{\text{Sets}}$  is an isomorphism.




**Naturality**

We need to show that given a function  $f: A \rightarrow B$ , the diagram

$$\begin{array}{ccc}
 A \times \emptyset & \xrightarrow{f \times \text{id}_\emptyset} & B \times \emptyset \\
 \zeta_{r|A}^{\text{Sets}} \downarrow & & \downarrow \zeta_{r|B}^{\text{Sets}} \\
 \emptyset & \xrightarrow{\text{id}_\emptyset} & \emptyset
 \end{array}$$

commutes. But since  $A \times \emptyset$  has no elements, this is trivially true.

**Being a Natural Isomorphism**

Since  $\zeta_r^{\text{Sets}}$  is natural and  $\zeta_r^{\text{Sets}, -1}$  is a componentwise inverse to  $\zeta_r^{\text{Sets}}$ , it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that  $\zeta_r^{\text{Sets}, -1}$  is also natural. Thus  $\zeta_r^{\text{Sets}}$  is a natural isomorphism. 

**5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts****PROPOSITION 5.3.5.1.1 ► THE BIMONOIDAL STRUCTURE ON SETS ASSOCIATED TO  $\times$  AND  $\coprod$** 

The category **Sets** admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category.* The category **Sets** of pointed sets.
- *The Additive Monoidal Product.* The coproduct functor

$$\coprod: \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of **Constructions With Sets, Item 1** of **Proposition 4.2.3.1.4**.

- *The Multiplicative Monoidal Product.* The product functor

$$\times: \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of **Constructions With Sets, Item 1** of **Proposition 4.1.3.1.4**.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\text{Sets}}: \text{pt} \rightarrow \text{Sets}$$

of **Definition 5.1.3.1.1.**

- *The Monoidal Zero.* The functor

$$\mathbb{0}^{\text{Sets}}: \text{pt} \rightarrow \text{Sets}$$

of **Definition 5.1.3.1.1.**

- *The Internal Hom.* The internal Hom functor

$$\text{Sets}: \text{Sets}^{\text{op}} \times \text{Sets} \rightarrow \text{Sets}$$

of **Constructions With Sets**, ?? of ??.

- *The Additive Associators.* The natural isomorphism

$$\alpha^{\text{Sets}, \sqcup}: \sqcup \circ (\sqcup \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \sqcup \circ (\text{id}_{\text{Sets}} \times \sqcup) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of **Definition 5.2.3.1.1.**

- *The Additive Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}, \sqcup}: \sqcup \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.2.4.1.1.**

- *The Additive Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}, \sqcup}: \sqcup \circ (\text{id} \times \mathbb{0}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.2.5.1.1.**

- *The Additive Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}, \sqcup}: \sqcup \xrightarrow{\sim} \sqcup \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.2.6.1.1.**

- *The Multiplicative Associators.* The natural isomorphism

$$\alpha^{\text{Sets}}: \times \circ (\times \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of **Definition 5.1.4.1.1.**

- *The Multiplicative Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.5.1.1.**

- *The Multiplicative Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.6.1.1.**

- *The Multiplicative Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.7.1.1.**

- *The Left Distributor.* The natural isomorphism

$$\delta_l^{\text{Sets}} : \times \circ (\text{id}_{\text{Sets}} \times \amalg) \xRightarrow{\sim} \amalg \circ ((\times) \times (\times)) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\Delta_{\text{Sets}} \times (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}))$$

of **Definition 5.3.1.1.1.**

- *The Right Distributor.* The natural isomorphism

$$\delta_r^{\text{Sets}} : \times \circ (\amalg \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \amalg \circ ((\times) \times (\times)) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ ((\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}})$$

of **Definition 5.3.2.1.1.**

- *The Left Annihilator.* The natural isomorphism

$$\zeta_l^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\text{id}_{\text{pt}} \times \epsilon_{\text{Sets}}^{\text{Cats}_2}) \xRightarrow{\sim} \times \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}})$$

of **Definition 5.3.3.1.1.**

- *The Right Annihilator.* The natural isomorphism

$$\zeta_r^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\epsilon_{\text{Sets}}^{\text{Cats}_2} \times \text{id}_{\text{pt}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \mathbb{0}^{\text{Sets}})$$

of **Definition 5.3.4.1.1.**

PROOF 5.3.5.1.2 ► PROOF OF PROPOSITION 5.3.5.1.1

Omitted.



# Appendices

## A Other Chapters

### Preliminaries

1. Introduction
2. A Guide to the Literature

### Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

### Relations

8. Relations
9. Constructions With Relations

### 10. Conditions on Relations

### Categories

11. Categories
12. Presheaves and the Yoneda Lemma

### Monoidal Categories

13. Constructions With Monoidal Categories

### Bicategories

14. Types of Morphisms in Bicategories

### Extra Part

15. Notes