Constructions With Sets

The Clowder Project Authors

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This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 4.2.4.1.1, 4.2.4.1.3, 4.2.5.1.1 and 4.2.5.1.3).

2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:

   (a) A discussion of the internal Hom of a powerset (Section 4.4.7).

   (b) A 0-categorical version of the Yoneda lemma (Presheaves and the Yoneda Lemma, Definition 12.1.5.1.1), which we term the *Yoneda lemma for sets* (Definition 4.5.5.1.1).

   (c) A characterisation of powersets as free cocompletions (Section 4.4.5), mimicking the corresponding statement for categories of presheaves (**??**).

   (d) A characterisation of powersets as free completions (Section 4.4.6), mimicking the corresponding statement for categories of copresheaves (**??**).

   (e) A (-1)-categorical version of un/straightening (Item 2 of Definition 4.5.1.1.4 and Definition 4.5.1.1.5).

   (f) A 0-categorical form of Isbell duality internal to powersets (Section 4.4.8).

3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, including in particular:

   (a) How f^{-1} can be described as a precomposition while $f_!$ and f_* can be described as Kan extensions (Definitions 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).

   (b) An extensive list of the properties of $f_!$, f^{-1}, and f_* (Definitions 4.6.1.1.5, 4.6.1.1.6, 4.6.2.1.3, 4.6.2.1.4, 4.6.3.1.7 and 4.6.3.1.8).

   (c) How the functors $f_!$, f^{-1}, f_*, along with the functors

$$\begin{aligned} -_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) &\to \mathcal{P}(X), \\ [-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) &\to \mathcal{P}(X) \end{aligned}$$

   may be viewed as a six-functor formalism with the empty set $\emptyset$ as the dualising object (Section 4.6.4).

Contents

4.1 Limits of Sets

4.1.1 The Terminal Set

Definition 4.1.1.1.1. The **terminal set** is the terminal object of Sets as in Limits and Colimits, **??**.

Construction 4.1.1.1.2. Concretely, the terminal set is the pair $(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

1. *The Limit.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$.
2. *The Cone.* The collection of maps

$$\left\{!_A : A \longrightarrow \mathrm{pt}\right\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_A(a) \overset{\text{def}}{=} \star$$

for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi\colon A \longrightarrow \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. ◻

4.1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 4.1.2.1.1. The **product**[1] **of** $\{A_i\}_{i\in I}$ is the product of $\{A_i\}_{i\in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.1.2.1.2. Concretely, the product of $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \left\{\mathrm{pr}_i\right\}_{i\in I}\right)$ consisting of:

1. *The Limit.* The set $\prod_{i\in I} A_i$ defined by
$$\prod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \,\middle|\, \begin{array}{l}\text{for each } i \in I, \text{ we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$
2. *The Cone.* The collection
$$\left\{ \mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i \right\}_{i\in I}$$
of maps given by
$$\mathrm{pr}_i(f) \stackrel{\text{def}}{=} f(i)$$
for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. □

Remark 4.1.2.1.3. Less formally, we may think of Cartesian products and projection maps as follows:

1. We think of $\prod_{i \in I} A_i$ as the set whose elements are I-indexed collections $(a_i)_{i \in I}$ with $a_i \in A_i$ for each $i \in I$.

2. We view the projection maps

$$\left\{\mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i\right\}_{i \in I}$$

as being given by

$$\mathrm{pr}_i((a_j)_{j \in I}) \stackrel{\text{def}}{=} a_i$$

for each $(a_j)_{j \in I} \in \prod_{i \in I} A_i$ and each $i \in I$.

Proposition 4.1.2.1.4. Let $\{A_i\}_{i \in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i \in I} \mapsto \prod_{i \in I} A_i$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i \in I}\right]((A_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i \in I}, (B_i)_{i \in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i \in I}\right)_{(A_i)_{i \in I}, (B_i)_{i \in I}} \colon \mathrm{Nat}((A_i)_{i \in I}, (B_i)_{i \in I}) \to \mathsf{Sets}\left(\prod_{i \in I} A_i, \prod_{i \in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i\colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i\colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \overset{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

4.1.3 Binary Products of Sets

Let A and B be sets.

Definition 4.1.3.1.1. The **product of** A **and** B[2] is the product of A and B in Sets as in Limits and Colimits, **??**.

Construction 4.1.3.1.2. Concretely, the product of A and B is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times B$ defined by

$$\begin{aligned}
A \times B &\overset{\text{def}}{=} \prod_{z\in\{A,B\}} z \\
&\overset{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\
&\cong \{\{\{a\}, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\
&\cong \left\{\begin{matrix}\text{ordered pairs } (a,b) \text{ with} \\ a \in A \text{ and } b \in B\end{matrix}\right\}.
\end{aligned}$$

[2] *Further Terminology:* Also called the **Cartesian product of** A **and** B.

2. *The Cone.* The maps

$$\mathrm{pr}_1 : A \times B \to A,$$
$$\mathrm{pr}_2 : A \times B \to B$$

defined by

$$\mathrm{pr}_1(a, b) \stackrel{\text{def}}{=} a,$$
$$\mathrm{pr}_2(a, b) \stackrel{\text{def}}{=} b$$

for each $(a, b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in the category of sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] & \\ A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times B$ making the diagram

$$\begin{tikzcd} & P \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

Proposition 4.1.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -\colon & \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\ - \times B\colon & \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\ -_1 \times -_2\colon & \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A, B) \stackrel{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)}\colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g\colon A \times B \to X \times Y$$

defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness I.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\mathsf{Sets}} \dashv -_1 \times -_2)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\Delta_{\mathsf{Sets}}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \times \mathsf{Sets}, \arrow[l, bend left, "-_1 \times -_2"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}\times\mathsf{Sets}}((A, A), (B, C)) \cong \mathsf{Sets}(A, B \times C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$.

4. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets}}_{A,B,C}\colon (A \times B) \times C \xrightarrow{\sim} A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Unitality.* We have isomorphisms of sets

$$\lambda^{\mathsf{Sets}}_{A}\colon \mathrm{pt} \times A \xrightarrow{\sim} A,$$
$$\rho^{\mathsf{Sets}}_{A}\colon A \times \mathrm{pt} \xrightarrow{\sim} A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Commutativity.* We have an isomorphism of sets

$$\sigma^{\mathsf{Sets}}_{A,B}\colon A \times B \xrightarrow{\sim} B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\delta^{\mathsf{Sets}}_{\ell}\colon A \times (B \textstyle\coprod C) \xrightarrow{\sim} (A \times B) \textstyle\coprod (A \times C),$$
$$\delta^{\mathsf{Sets}}_{r}\colon (A \textstyle\coprod B) \times C \xrightarrow{\sim} (A \times C) \textstyle\coprod (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \emptyset \times A \overset{\sim}{\dashrightarrow} \emptyset,$$
$$\zeta_{r}^{\mathsf{Sets}} \colon A \times \emptyset \overset{\sim}{\dashrightarrow} \emptyset,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Unions.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cup W) = (U \times V) \cup (U \times W),$$
$$(U \cup V) \times W = (U \times W) \cup (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

10. *Distributivity Over Intersections.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \cap W) = (U \times V) \cap (U \times W),$$
$$(U \cap V) \times W = (U \times W) \cap (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

11. *Distributivity Over Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \setminus W) = (U \times V) \setminus (U \times W),$$
$$(U \setminus V) \times W = (U \times W) \setminus (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

12. *Distributivity Over Symmetric Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$U \times (V \triangle W) = (U \times V) \triangle (U \times W),$$
$$(U \triangle V) \times W = (U \times W) \triangle (V \times W)$$

of subsets of $\mathcal{P}(X \times X)$.

13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$\begin{tikzcd}
(\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[r, "\cap\times\cap"] \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[r, "\cap"'] & \mathcal{P}(X \times X)
\end{tikzcd}$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).$$

for each $U, V, W, T \in \mathcal{P}(X)$.

14. *Symmetric Monoidality.* The 8-tuple $(\mathsf{Sets}, \times, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}})$ is a closed symmetric monoidal category.

15. *Symmetric Bimonoidality.* The 18-tuple

$$\left(\begin{aligned}&\mathsf{Sets}, \textstyle\coprod, \times, \emptyset, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}},\\ &\alpha^{\mathsf{Sets},\coprod}, \lambda^{\mathsf{Sets},\coprod}, \rho^{\mathsf{Sets},\coprod}, \sigma^{\mathsf{Sets},\coprod}, \delta^{\mathsf{Sets}}_{\ell}, \delta^{\mathsf{Sets}}_{r}, \zeta^{\mathsf{Sets}}_{\ell}, \zeta^{\mathsf{Sets}}_{r}\end{aligned}\right),$$

is a symmetric closed bimonoidal category, where $\alpha^{\mathsf{Sets},\coprod}$, $\lambda^{\mathsf{Sets},\coprod}$, $\rho^{\mathsf{Sets},\coprod}$, and $\sigma^{\mathsf{Sets},\coprod}$ are the natural transformations from Items 3 to 5 of Definition 4.2.3.1.3.

Proof. *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, *Adjointness*: We prove only that there's an adjunction $- \times B \dashv \mathsf{Sets}(B, -)$, witnessed by a bijection

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathsf{Sets}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

$$\Phi_{B,C} \colon \mathsf{Sets}(A \times B, C) \to \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

by sending a function

$$\xi \colon A \times B \to C$$

to the function

$$\begin{aligned}\xi^{\dagger} \colon A &\longrightarrow \mathsf{Sets}(B, C),\\ a &\longmapsto (\xi^{\dagger}_{a} \colon B \to C),\end{aligned}$$

where we define

$$\xi^{\dagger}_{a}(b) \overset{\text{def}}{=} \xi(a, b)$$

for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$\xi^{\dagger} \overset{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

- *Map II.* We define a map

$$\Psi_{B,C} \colon \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \to \mathsf{Sets}(A \times B, C)$$

given by sending a function

$$\begin{aligned}\xi \colon A &\longrightarrow \mathsf{Sets}(B, C),\\ a &\longmapsto (\xi_{a} \colon B \to C),\end{aligned}$$

to the function

$$\xi^{\dagger} \colon A \times B \to C$$

defined by

$$\begin{aligned}\xi^{\dagger}(a, b) &\overset{\text{def}}{=} \mathrm{ev}_{b}(\mathrm{ev}_{a}(\xi))\\ &\overset{\text{def}}{=} \mathrm{ev}_{b}(\xi_{a})\\ &\overset{\text{def}}{=} \xi_{a}(b)\end{aligned}$$

for each $(a, b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A \times B, C)}.$$

Indeed, given a function $\xi \colon A \times B \to C$, we have

$$\begin{aligned}
[\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi)) \\
&= \Psi_{A,B}(\Phi_{A,B}(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= \Psi_{A,B}(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= \Psi_{A,B}(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b') \rrbracket \rrbracket) \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket b' \mapsto \xi(a',b') \rrbracket \rrbracket)) \rrbracket \\
&= \llbracket (a,b) \mapsto \mathrm{ev}_b(\llbracket b' \mapsto \xi(a,b') \rrbracket) \rrbracket \\
&= \llbracket (a,b) \mapsto \xi(a,b) \rrbracket \\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A, \mathsf{Sets}(B,C))}.$$

Indeed, given a function

$$\begin{aligned}
\xi \colon A &\longrightarrow \mathsf{Sets}(B,C), \\
a &\longmapsto (\xi_a \colon B \to C),
\end{aligned}$$

we have

$$\begin{aligned}
[\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi)) \\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a,b) \mapsto \xi_a(b) \rrbracket) \\
&\overset{\text{def}}{=} \Phi_{A,B}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \mathrm{ev}_{(a,b)}(\llbracket (a',b') \mapsto \xi_{a'}(b') \rrbracket) \rrbracket \rrbracket \\
&\overset{\text{def}}{=} \llbracket a \mapsto \llbracket b \mapsto \xi_a(b) \rrbracket \rrbracket \\
&\overset{\text{def}}{=} \llbracket a \mapsto \xi_a \rrbracket \\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g \colon B \to$

B', the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A \times B', C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A \times g^*"'] & \mathsf{Sets}(A, \mathsf{Sets}(B', C)), \arrow[d, "(g^*)_!"] \\
\mathsf{Sets}(A \times B, C) \arrow[r, "\Phi_{B,C}"'] & \mathsf{Sets}(A, \mathsf{Sets}(B, C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi \colon A \times B' \to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}(\xi(-_1, g(-_2))) \\
&= [\xi(-_1, g(-_2))]^\dagger \\
&= \xi^\dagger_{-_1}(g(-_2)) \\
&= (g^*)_!(\xi^\dagger) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)_! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

Alternatively, using the $[\![a \mapsto f(a)]\!]$ notation of Sets, Definition 3.1.1.1.2, we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}([\mathrm{id}_A \times g^*]([\![(a, b') \mapsto \xi(a, b')]\!])) \\
&= \Phi_{B,C}([\![(a, b) \mapsto \xi(a, g(b))]\!]) \\
&= [\![a \mapsto [\![b \mapsto \xi(a, g(b))]\!]]\!] \\
&= [\![a \mapsto g^*([\![b' \mapsto \xi(a, b')]\!])]\!] \\
&= (g^*)_!([\![a \mapsto [\![b' \mapsto \xi(a, b')]\!]]\!]) \\
&= (g^*)_!(\Phi_{B',C}([\![(a, b') \mapsto \xi(a, b')]\!])) \\
&= (g^*)_!(\Phi_{B',C}(\xi)) \\
&= [(g^*)_! \circ \Phi_{B',C}](\xi).
\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C \to C'$, the diagram

$$
\begin{tikzcd}
\mathsf{Sets}(A \times B, C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_!"'] & \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \arrow[d, "(h_!)_!"] \\
\mathsf{Sets}(A \times B, C') \arrow[r, "\Phi_{B,C'}"'] & \mathsf{Sets}(A, \mathsf{Sets}(B, C'))
\end{tikzcd}
$$

  commutes. Indeed, given a function

$$\xi\colon A \times B \to C,$$

  we have

$$
\begin{aligned}
[\Phi_{B,C} \circ h_!](\xi) &= \Phi_{B,C}(h_!(\xi)) \\
&= \Phi_{B,C}(h_!(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= \Phi_{B,C}(\llbracket (a,b) \mapsto h(\xi(a,b)) \rrbracket) \\
&= \llbracket a \mapsto \llbracket b \mapsto h(\xi(a,b)) \rrbracket \rrbracket \\
&= \llbracket a \mapsto h_!(\llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= (h_!)_!(\llbracket a \mapsto \llbracket b \mapsto \xi(a,b) \rrbracket \rrbracket) \\
&= (h_!)_!(\Phi_{B,C}(\llbracket (a,b) \mapsto \xi(a,b) \rrbracket)) \\
&= (h_!)_!(\Phi_{B,C}(\xi)) \\
&= [(h_!)_! \circ \Phi_{B,C}](\xi).
\end{aligned}
$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.
Item 3, Adjointness II: This follows from the universal property of the product.
Item 4, Associativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.4.1.1.
Item 5, Unitality: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.1.5.1.1 and 5.1.6.1.1.
Item 6, Commutativity: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.1.7.1.1.

Item 7, Distributivity Over Coproducts: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.1.1.1 and 5.3.2.1.1.
Item 8, Annihilation With the Empty Set: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.3.3.1.1 and 5.3.4.1.1.
Item 9, Distributivity Over Unions: See [Pro25c].
Item 10, Distributivity Over Intersections: See [Pro25d, Corollary 1].
Item 11, Distributivity Over Differences: See [Pro25a].
Item 12, Distributivity Over Symmetric Differences: See [Pro25b].
Item 13, Middle-Four Exchange With Respect to Intersections: See [Pro25d, Corollary 1].
Item 14, Symmetric Monoidality: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.1.9.1.1, and is proved there.
Item 15, Symmetric Bimonoidality: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.3.5.1.1, and is proved there. □

Remark 4.1.3.1.4. As shown in Item 1 of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

$$-_1 \times -_2 \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}.$$

This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k,\ell} \colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets})$$

of tensor products of $\mathbb{E}_k$-monoid objects on Sets with $\mathbb{E}_\ell$-monoid objects on Sets; see **??**.

Remark 4.1.3.1.5. We may state the equalities in Items 9 to 12 of Definition 4.1.3.1.3 as the commutativity of the following diagrams:

$$\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[dl, "\cup"] \\
& \mathcal{P}(X \times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)} \times \Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[dl, "\cup"] \\
& \mathcal{P}(X \times X) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cap"] \\
& \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cap"] \\
& \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\setminus"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\setminus"] \\
& \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\setminus\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\setminus"] \\
& \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\triangle"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\triangle"] \\
& \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
& (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\triangle\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\triangle"] \\
& \mathcal{P}(X\times X) &
\end{tikzcd}$$

4.1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 4.1.4.1.1. The **pullback of A and B over C along f and g**[3] is the pullback of A and B over C along f and g in Sets as in Limits and Colimits, **??**.

Construction 4.1.4.1.2. Concretely, the pullback of A and B over C along f and g is the pair $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

1. *The Limit.* The set $A \times_C B$ defined by
$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

[3] *Further Terminology:* Also called the **fibre product of A and B over C along f and g**.

2. *The Cone.* The maps[4]

$$\mathrm{pr}_1 \colon A \times_C B \to A,$$
$$\mathrm{pr}_2 \colon A \times_C B \to B$$

defined by

$$\mathrm{pr}_1(a, b) \overset{\text{def}}{=} a,$$
$$\mathrm{pr}_2(a, b) \overset{\text{def}}{=} b$$

for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b)) \\ &= f(a) \\ &= g(b) \\ &= g(\mathrm{pr}_2(a, b)) \\ &= [g \circ \mathrm{pr}_2](a, b), \end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies

[4] *Further Notation:* Also written $\mathrm{pr}_1^{A \times_C B}$ and $\mathrm{pr}_2^{A \times_C B}$.

the universal property of the pullback. Suppose we have a diagram of the form

$$
\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times_C B$ making the diagram

$$
\begin{tikzcd}
P \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}
$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

Remark 4.1.4.1.3. It is common practice to write $A \times_C B$ for the pullback of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \times_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

Example 4.1.4.1.4. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let X be a set. We have

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook, "\iota_B"] \\ A \arrow[r, hook, "\iota_A"'] & A \cup B \end{tikzcd}$$

   for each $A, B \in \mathcal{P}(X)$.

Proof. Item 1, *Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \left\{(x, y) \in A \times B \,\middle|\, x = y\right\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

Proposition 4.1.4.1.5. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

   where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dd, "g"] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f"] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi\colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & B \arrow[dd, "g"] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f"] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

commute.

2. *Adjointness I.* We have adjunctions

$$\begin{aligned}
\left(A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)\right)\colon &\quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "A\times_X -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X}, \arrow[l, bend left, "\mathbf{Sets}_{/X}(A,-)"] \end{tikzcd} \\
\left(- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)\right)\colon &\quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "-\times_X B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X}, \arrow[l, bend left, "\mathbf{Sets}_{/X}(B,-)"] \end{tikzcd}
\end{aligned}$$

witnessed by bijections

$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}(A, \mathbf{Sets}_{/X}(B, C)),$$
$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}(B, \mathbf{Sets}_{/X}(A, C)),$$

natural in $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$, where $\mathbf{Sets}_{/X}(A, B)$ is the object of $\mathsf{Sets}_{/X}$ consisting of (see Fibred Sets, **??**):

- *The Set.* The set $\mathbf{Sets}_{/X}(A, B)$ defined by

$$\mathbf{Sets}_{/X}(A, B) \overset{\text{def}}{=} \coprod_{x \in X} \mathsf{Sets}(\phi_A^{-1}(x), \phi_Y^{-1}(x))$$

- *The Map to X.* The map

$$\phi_{\mathbf{Sets}_{/X}(A,B)} \colon \mathbf{Sets}_{/X}(A, B) \to X$$

  defined by
$$\phi_{\mathbf{Sets}_{/X}(A,B)}(x, f) \overset{\text{def}}{=} x$$
  for each $(x, f) \in \mathbf{Sets}_{/X}(A, B)$.

3. *Adjointness II.* We have an adjunction

$$\left(\Delta_{\mathsf{Sets}_{/X}} \dashv -_1 \times -_2\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "\Delta_{\mathsf{Sets}_{/X}}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}, \arrow[l, bend left, "-_1 \times -_2"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}}((A, A), (B, C)) \cong \mathsf{Sets}_{/X}(A, B \times_X C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets}_{/X})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X})$.

4. *Associativity.* Given a diagram

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\ & X & & Y & \end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}$$

5. *Interaction With Composition.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "\phi"'] & & & & Y \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$\begin{aligned}
X \times_K^{f \circ \phi, g \circ \psi} Y &\cong (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \times^{p_2, p_1}_{A \times_K^{f,g} B} ((A \times_K^{f,g} B) \times_B^{q_2, \psi} Y) \\
&\cong X \times_A^{\phi, p} ((A \times_K^{f,g} B) \times_B^{q_2, \psi} Y) \\
&\cong (X \times_A^{\phi, q_1} (A \times_K^{f,g} B)) \times_B^{q, \psi} Y
\end{aligned}$$

where

$$\begin{aligned}
q_1 &= \mathrm{pr}_1^{A \times_K^{f,g} B}, & q_2 &= \mathrm{pr}_2^{A \times_K^{f,g} B}, \\
p_1 &= \mathrm{pr}_1^{(A \times_K^{f,g} B) \times_Y^{q_2, \psi}}, & p_2 &= \mathrm{pr}_2^{X \times^{\phi, q_1}_{A \times_K^{f,g} B} (A \times_K^{f,g} B)}, \\
p &= q_1 \circ \mathrm{pr}_1^{(A \times_K^{f,g} B) \times_B^{q_2, \psi} Y}, & q &= q_2 \circ \mathrm{pr}_2^{X \times_A^{\phi, q_1} (A \times_K^{f,g} B)},
\end{aligned}$$

and where these pullbacks are built as in the following diagrams:

$$
\begin{tikzcd}[column sep=tiny]
 & & (X\times_A(A\times_K B))\times_{A\times_K B}((A\times_K B)\times_B Y) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & X\times_A(A\times_K B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & (A\times_K B)\times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
X \arrow[dr, "\phi"'] & & A\times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & X\times_A((A\times_K B)\times_B Y) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & & & (A\times_K B)\times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & \\
X \arrow[dr, "\phi"'] & & A\times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
 & & (X\times_A(A\times_K B))\times_B Y \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & X\times_A(A\times_K B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & & \\
X \arrow[dr, "\phi"'] & & A\times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner"] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & X\times_K Y \arrow[ddll] \arrow[ddrr] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & & & & \\
X \arrow[dr, "\phi"'] & & & & Y. \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

6. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
\lambda^{\mathsf{Sets}_{/X}}_A &\colon X\times_X A \overset{\sim}{\dashrightarrow} A,\\
\rho^{\mathsf{Sets}_{/X}}_A &\colon A\times_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X,
\end{tikzcd}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

7. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \quad \sigma^{\mathsf{Sets}_{/X}}_{A,B} \colon A \times_X B \overset{\sim}{\dashrightarrow} B \times_X A \quad \begin{tikzcd} B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X, \end{tikzcd}$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

8. *Distributivity Over Coproducts.* Let A, B, and C be sets and let $\phi_A \colon A \to X$, $\phi_B \colon B \to X$, and $\phi_C \colon C \to X$ be morphisms of sets. We have isomorphisms of sets

$$\begin{aligned} \delta^{\mathsf{Sets}_{/X}}_{\ell} &\colon A \times_X \left(B \textstyle\coprod C\right) \overset{\sim}{\dashrightarrow} (A \times_X B) \textstyle\coprod (A \times_X C), \\ \delta^{\mathsf{Sets}_{/X}}_{r} &\colon \left(A \textstyle\coprod B\right) \times_X C \overset{\sim}{\dashrightarrow} (A \times_X C) \textstyle\coprod (B \times_X C), \end{aligned}$$

as in the diagrams

$$\begin{tikzcd} (A \times_X B) \coprod (A \times_X C) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B \coprod C \arrow[d, "\phi_B \coprod \phi_C"] \\ A \arrow[r, "\phi_A"'] & X \end{tikzcd} \quad \begin{tikzcd} (A \times_X C) \coprod (B \times_X C) \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\phi_C"] \\ A \coprod B \arrow[r, "\phi_A \coprod \phi_B"'] & X \end{tikzcd}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \quad \begin{aligned} \zeta^{\mathsf{Sets}_{/X}}_{\ell} &\colon A \times_X \emptyset \overset{\sim}{\dashrightarrow} \emptyset, \\ \zeta^{\mathsf{Sets}_{/X}}_{r} &\colon \emptyset \times_X A \overset{\sim}{\dashrightarrow} \emptyset, \end{aligned} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X, \end{tikzcd}$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

10. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

11. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}_{/X}, \times_X, X, \mathbf{Sets}_{/X}, \alpha^{\mathsf{Sets}_{/X}}, \lambda^{\mathsf{Sets}_{/X}}, \rho^{\mathsf{Sets}_{/X}}, \sigma^{\mathsf{Sets}_{/X}}\right)$ is a symmetric closed monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, *Adjointness I*: This is a repetition of Fibred Sets, **??** of **??**, and is proved there.
Item 3, *Adjointness II*: This follows from the universal property of the product (pullbacks are products in $\mathsf{Sets}_{/X}$).
Item 4, *Associativity*: We have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \left\{((a,b),c) \in (A \times B) \times C \,\middle|\, f(a) = g(b) \text{ and } h(b) = k(c)\right\} \\ &\cong \left\{(a,(b,c)) \in A \times (B \times C) \,\middle|\, f(a) = g(b) \text{ and } h(b) = k(c)\right\} \\ &\cong \left\{(a,(b,c)) \in A \times (B \times_Y C) \,\middle|\, f(a) = g(b)\right\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned} (A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\} \\ &\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array}\right\} \\ &\cong \left\{(a,(b,c)) \in A \times (B \times C) \,\middle|\, f(a) = g(b) \text{ and } h(b) = k(c)\right\} \end{aligned}$$

$$\cong A \times_X (B \times_Y C),$$

where we have used Item 6 for the isomorphism $B \times_B B \cong B$.
Item 5, Interaction With Composition: By Item 4, it suffices to construct only the isomorphism

$$X \times_K^{f\circ\phi, g\circ\psi} Y \cong (X \times_A^{\phi,q_1} (A \times_K^{f,g} B)) \times_{A\times_K^{f,g} B}^{p_2,p_1} ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y).$$

We have

$$\begin{aligned}
(X \times_A^{\phi,q_1} (A \times_K^{f,g} B)) &\overset{\text{def}}{=} \left\{(x, (a, b)) \in X \times (A \times_K^{f,g} B) \,\middle|\, \phi(x) = q_1(a, b)\right\} \\
&\overset{\text{def}}{=} \left\{(x, (a, b)) \in X \times (A \times_K^{f,g} B) \,\middle|\, \phi(x) = a\right\} \\
&\cong \left\{(x, (a, b)) \in X \times (A \times B) \,\middle|\, \phi(x) = a \text{ and } f(a) = g(b)\right\}, \\
((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y) &\overset{\text{def}}{=} \left\{((a, b), y) \in (A \times_K^{f,g} B) \times Y \,\middle|\, q_2(a, b) = \psi(y)\right\} \\
&\overset{\text{def}}{=} \left\{((a, b), y) \in (A \times_K^{f,g} B) \times Y \,\middle|\, b = \psi(y)\right\} \\
&\cong \left\{((a, b), y) \in (A \times B) \times Y \,\middle|\, b = \psi(y) \text{ and } f(a) = g(b)\right\},
\end{aligned}$$

so writing

$$\begin{aligned}
S &= (X \times_A^{\phi,q_1} (A \times_K^{f,g} B)) \\
S' &= ((A \times_K^{f,g} B) \times_B^{q_2,\psi} Y),
\end{aligned}$$

we have

$$\begin{aligned}
S \times_{A\times_K^{f,g} B}^{p_2,p_1} S' &\overset{\text{def}}{=} \left\{((x, (a, b)), ((a', b'), y)) \in S \times S' \,\middle|\, p_1(x, (a, b)) = p_2((a', b'), y)\right\} \\
&\overset{\text{def}}{=} \left\{((x, (a, b)), ((a', b'), y)) \in S \times S' \,\middle|\, (a, b) = (a', b')\right\} \\
&\cong \left\{((x, a, b, y)) \in X \times A \times B \times Y \,\middle|\, \phi(x) = a, \psi(y) = b, \text{ and } f(a) = g(b)\right\} \\
&\cong \left\{((x, a, b, y)) \in X \times A \times B \times Y \,\middle|\, f(\phi(x)) = g(\psi(y))\right\} \\
&\overset{\text{def}}{=} X \times_K Y.
\end{aligned}$$

This finishes the proof.
Item 6, Unitality: We have

$$\begin{aligned}
X \times_X A &\cong \left\{(x, a) \in X \times A \,\middle|\, f(a) = x\right\}, \\
A \times_X X &\cong \left\{(a, x) \in X \times A \,\middle|\, f(a) = x\right\},
\end{aligned}$$

which are isomorphic to A via the maps $(x,a)\mapsto a$ and $(a,x)\mapsto a$. The proof of the naturality of $\lambda^{\mathsf{Sets}/X}$ and $\rho^{\mathsf{Sets}/X}$ is omitted.

Item 7, Commutativity: We have

$$\begin{aligned}
A\times_C B &\overset{\text{def}}{=} \left\{(a,b)\in A\times B \,\middle|\, f(a)=g(b)\right\}\\
&= \left\{(a,b)\in A\times B \,\middle|\, g(b)=f(a)\right\}\\
&\cong \left\{(b,a)\in B\times A \,\middle|\, g(b)=f(a)\right\}\\
&\overset{\text{def}}{=} B\times_C A.
\end{aligned}$$

The proof of the naturality of $\sigma^{\mathsf{Sets}/X}$ is omitted.

Item 8, Distributivity Over Coproducts: We have

$$\begin{aligned}
A\times_X \left(B\coprod C\right) &\overset{\text{def}}{=} \left\{(a,z)\in A\times \left(B\coprod C\right) \,\middle|\, \phi_A(a)=\phi_{B\coprod C}(z)\right\}\\
&= \left\{(a,z)\in A\times \left(B\coprod C\right) \,\middle|\, z=(0,b) \text{ and } \phi_A(a)=\phi_{B\coprod C}(z)\right\}\\
&\quad\cup \left\{(a,z)\in A\times \left(B\coprod C\right) \,\middle|\, z=(1,c) \text{ and } \phi_A(a)=\phi_{B\coprod C}(z)\right\}\\
&= \left\{(a,z)\in A\times \left(B\coprod C\right) \,\middle|\, z=(0,b) \text{ and } \phi_A(a)=\phi_B(b)\right\}\\
&\quad\cup \left\{(a,z)\in A\times \left(B\coprod C\right) \,\middle|\, z=(1,c) \text{ and } \phi_A(a)=\phi_C(c)\right\}\\
&\cong \left\{(a,b)\in A\times B \,\middle|\, \phi_A(a)=\phi_B(b)\right\}\\
&\quad\cup \left\{(a,c)\in A\times C \,\middle|\, \phi_A(a)=\phi_C(c)\right\}\\
&\overset{\text{def}}{=} (A\times_X B)\cup(A\times_X C)\\
&\cong (A\times_X B)\coprod(A\times_X C),
\end{aligned}$$

with the construction of the isomorphism

$$\delta_r^{\mathsf{Sets}/X}\colon \left(A\coprod B\right)\times_X C \overset{\sim}{\dashrightarrow} (A\times_X C)\coprod(B\times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\mathsf{Sets}/X}$ and $\delta_r^{\mathsf{Sets}/X}$ is omitted.

Item 9, Annihilation With the Empty Set: We have

$$\begin{aligned}
A\times_X \emptyset &\overset{\text{def}}{=} \left\{(a,b)\in A\times\emptyset \,\middle|\, f(a)=g(b)\right\}\\
&= \left\{k\in\emptyset \,\middle|\, f(a)=g(b)\right\}\\
&= \emptyset,
\end{aligned}$$

and similarly for $\emptyset\times_X A$, where we have used Item 8 of Definition 4.1.3.1.3. The proof of the naturality of $\zeta_\ell^{\mathsf{Sets}/X}$ and $\zeta_r^{\mathsf{Sets}/X}$ is omitted.

Item 10, Interaction With Products: We have

$$\begin{aligned}
A \times_{\mathrm{pt}} B &\overset{\mathrm{def}}{=} \{(a,b) \in A \times B \mid !_A(a) = !_B(b)\} \\
&\overset{\mathrm{def}}{=} \{(a,b) \in A \times B \mid \star = \star\} \\
&= \{(a,b) \in A \times B\} \\
&= A \times B.
\end{aligned}$$

Item 11, Symmetric Monoidality: Omitted. □

4.1.5 Equalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

Definition 4.1.5.1.1. The **equaliser of f and g** is the equaliser of f and g in Sets as in Limits and Colimits, **??**.

Construction 4.1.5.1.2. Concretely, the equaliser of f and g is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

1. *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by
$$\mathrm{Eq}(f,g) \overset{\mathrm{def}}{=} \left\{a \in A \,\middle|\, f(a) = g(a)\right\}.$$
2. *The Cone.* The inclusion map
$$\mathrm{eq}(f,g)\colon\ \mathrm{Eq}(f,g) \to A.$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g)$ making the diagram

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & & \end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. □

Proposition 4.1.5.1.3. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[5]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

[5]That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

   in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\to} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \underset{h}{\overset{f}{\underset{-g\to}{\rightrightarrows}}} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \left\{ a \in A \,\middle|\, f(a) = g(a) = h(a) \right\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

---

and then take the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\to} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))$$

of $\mathrm{Eq}(f, g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\to} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\to} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g, h), h \circ \mathrm{eq}(g, h))$$

of $\mathrm{Eq}(g, h)$.

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\to} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1, Associativity*: We first prove that $\mathrm{Eq}(f,g,h)$ is indeed given by

$$\mathrm{Eq}(f,g,h) \cong \left\{a \in A \,\middle|\, f(a) = g(a) = h(a)\right\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g,h)$.

We now check the equalities

$$\mathrm{Eq}(f\circ\mathrm{eq}(g,h),g\circ\mathrm{eq}(g,h))\cong\mathrm{Eq}(f,g,h)\cong\mathrm{Eq}(f\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}\mathrm{Eq}(f\circ\mathrm{eq}(g,h),g\circ\mathrm{eq}(g,h)) &\cong \left\{x\in\mathrm{Eq}(g,h)\;\middle|\;[f\circ\mathrm{eq}(g,h)](a)=[g\circ\mathrm{eq}(g,h)](a)\right\}\\ &\cong \left\{x\in\mathrm{Eq}(g,h)\;\middle|\;f(a)=g(a)\right\}\\ &\cong \left\{x\in A\;\middle|\;f(a)=g(a)\text{ and }g(a)=h(a)\right\}\\ &\cong \left\{x\in A\;\middle|\;f(a)=g(a)=h(a)\right\}\\ &\cong \mathrm{Eq}(f,g,h).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Eq}(f\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g)) &\cong \left\{x\in\mathrm{Eq}(f,g)\;\middle|\;[f\circ\mathrm{eq}(f,g)](a)=[h\circ\mathrm{eq}(f,g)](a)\right\}\\ &\cong \left\{x\in\mathrm{Eq}(f,g)\;\middle|\;f(a)=h(a)\right\}\\ &\cong \left\{x\in A\;\middle|\;f(a)=h(a)\text{ and }f(a)=g(a)\right\}\\ &\cong \left\{x\in A\;\middle|\;f(a)=g(a)=h(a)\right\}\\ &\cong \mathrm{Eq}(f,g,h).\end{aligned}$$

Item 4, *Unitality*: Indeed, we have

$$\begin{aligned}\mathrm{Eq}(f,f) &\overset{\text{def}}{=} \left\{a\in A\;\middle|\;f(a)=f(a)\right\}\\ &= A.\end{aligned}$$

Item 5, *Commutativity*: Indeed, we have

$$\begin{aligned}\mathrm{Eq}(f,g) &\overset{\text{def}}{=} \left\{a\in A\;\middle|\;f(a)=g(a)\right\}\\ &= \left\{a\in A\;\middle|\;g(a)=f(a)\right\}\\ &\overset{\text{def}}{=} \mathrm{Eq}(g,f).\end{aligned}$$

Item 6, *Interaction With Composition*: Indeed, we have

$$\begin{aligned}\mathrm{Eq}(h\circ f\circ\mathrm{eq}(f,g),k\circ g\circ\mathrm{eq}(f,g)) &\cong \left\{a\in\mathrm{Eq}(f,g)\;\middle|\;h(f(a))=k(g(a))\right\}\\ &\cong \left\{a\in A\;\middle|\;f(a)=g(a)\text{ and }h(f(a))=k(g(a))\right\}.\end{aligned}$$

and

$$\mathrm{Eq}(h\circ f,k\circ g)\cong\left\{a\in A\;\middle|\;h(f(a))=k(g(a))\right\},$$

and thus there's an inclusion from $\mathrm{Eq}(h\circ f\circ\mathrm{eq}(f,g),k\circ g\circ\mathrm{eq}(f,g))$ to $\mathrm{Eq}(h\circ f,k\circ g)$. □

4.1.6 Inverse Limits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I} \colon (I, \preceq) \to \mathsf{Sets}$ be an inverse system of sets.

Definition 4.1.6.1.1. The **inverse limit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ is the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.1.6.1.2. Concretely, the inverse limit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ is the pair $\left(\varprojlim_{\alpha\in I}(X_\alpha), \{\mathrm{pr}_\alpha\}_{\alpha\in I}\right)$ consisting of:

1. *The Limit.* The set $\varprojlim_{\alpha\in I}(X_\alpha)$ defined by

$$\varprojlim_{\alpha\in I}(X_\alpha) \overset{\text{def}}{=} \left\{ (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha \;\middle|\; \begin{array}{l}\text{for each } \alpha,\beta\in I, \text{ if } \alpha\preceq\beta,\\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta)\end{array} \right\}.$$

2. *The Cone.* The collection

$$\left\{ \mathrm{pr}_\gamma \colon \varprojlim_{\alpha\in I}(X_\alpha) \to X_\gamma \right\}_{\gamma\in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{ \mathrm{pr}_\gamma \colon \prod_{\alpha\in I} X_\alpha \to X_\gamma \right\}_{\gamma\in I}$$

of Item 2 of Definition 4.1.2.1.2 to $\varprojlim_{\alpha\in I}(X_\alpha)$ and hence given by

$$\mathrm{pr}_\gamma((x_\alpha)_{\alpha\in I}) \overset{\text{def}}{=} x_\gamma$$

for each $\gamma\in I$ and each $(x_\alpha)_{\alpha\in I} \in \varprojlim_{\alpha\in I}(X_\alpha)$.

Proof. We claim that $\varprojlim_{\alpha\in I}(X_\alpha)$ is the limit of the inverse system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$. First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta}\circ\mathrm{pr}_\alpha = \mathrm{pr}_\beta, \qquad \begin{tikzcd} & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\ X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \end{tikzcd}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma \in I} \in \varprojlim_{\gamma \in I}(X_\gamma)$, we have

$$\begin{aligned}
[f_{\alpha\beta} \circ \mathrm{pr}_\alpha]((x_\gamma)_{\gamma\in I}) &\overset{\text{def}}{=} f_{\alpha\beta}(\mathrm{pr}_\alpha((x_\gamma)_{\gamma\in I}))\\
&\overset{\text{def}}{=} f_{\alpha\beta}(x_\alpha)\\
&= x_\beta\\
&\overset{\text{def}}{=} \mathrm{pr}_\beta((x_\gamma)_{\gamma\in I}),
\end{aligned}$$

where the third equality comes from the definition of $\varprojlim_{\alpha\in I}(X_\alpha)$. Next, we prove that $\varprojlim_{\alpha\in I}(X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{tikzcd}
 & L \arrow[ddl, bend right, "p_\alpha"'] \arrow[ddr, bend left, "p_\beta"] & \\
 & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\
X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta
\end{tikzcd}$$

in Sets. Then there indeed exists a unique map $\phi \colon L \xrightarrow{\exists!} \varprojlim_{\alpha\in I}(X_\alpha)$ making the diagram

$$\begin{tikzcd}
 & L \arrow[ddl, bend right, "p_\alpha"'] \arrow[ddr, bend left, "p_\beta"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
 & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\
X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta
\end{tikzcd}$$

commute, being uniquely determined by the family of conditions

$$\left\{p_\alpha = \mathrm{pr}_\alpha \circ \phi\right\}_{\alpha\in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha\in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha$ indeed lies in $\varprojlim_{\alpha\in I}(X_\alpha)$, as we have

$$\begin{aligned} f_{\alpha\beta}(p_\alpha(\ell)) &\stackrel{\text{def}}{=} [f_{\alpha\beta}\circ p_\alpha](\ell) \\ &\stackrel{\text{def}}{=} p_\beta(\ell) \end{aligned}$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $(L, \{p_\alpha\}_{\alpha\in I})$. □

Example 4.1.6.1.3. Here are some examples of inverse limits of sets.

1. *The p-Adic Integers.* The ring of p-adic integers $\mathbb{Z}_p$ of **??** is the inverse limit

$$\mathbb{Z}_p \cong \varprojlim_{n\in\mathbb{N}}(\mathbb{Z}_{/p^n});$$

see **??**.

2. *Rings of Formal Power Series.* The ring $R[\![t]\!]$ of formal power series in a variable t is the inverse limit

$$R[\![t]\!] \cong \varprojlim_{n\in\mathbb{N}}(R[t]/t^nR[t]);$$

see **??**.

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see **??**.

4.2 Colimits of Sets

4.2.1 The Initial Set

Definition 4.2.1.1.1. The **initial set** is the initial object of Sets as in Limits and Colimits, **??**.

Construction 4.2.1.1.2. Concretely, the initial set is the pair $(\emptyset, \{\iota_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})})$ consisting of:

1. *The Colimit.* The empty set $\emptyset$ of Definition 4.3.1.1.1.

2. *The Cocone.* The collection of maps

$$\{\iota_A \colon \emptyset \to A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

   given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi \colon \emptyset \to A$ making the diagram

$$\emptyset \overset{\phi}{\underset{\exists!}{\dashrightarrow}} A$$

commute, namely the inclusion map ι_A. □

4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 4.2.2.1.1. The **coproduct of** $\{A_i\}_{i\in I}$[6] is the coproduct of $\{A_i\}_{i\in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.2.2.1.2. Concretely, the disjoint union of $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of:

1. *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by

$$\coprod_{i\in I} A_i \overset{\text{def}}{=} \left\{ (i, x) \in I \times \left(\bigcup_{i\in I} A_i \right) \;\middle|\; x \in A_i \right\}.$$

2. *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i \right\}_{i\in I}$$

   of maps given by

$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

   for each $x \in A_i$ and each $i \in I$.

[6] *Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i\in I}$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod\limits_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i, x)) = \iota_i(x)$$

for each $(i, x) \in \coprod_{i\in I} A_i$. $\square$

Proposition 4.2.2.1.3. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \overset{\text{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}((A_i)_{i\in I},(B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I},(B_i)_{i\in I})$ is defined by sending a map

$$\left\{f_i\colon A_i \to B_i\right\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I},(B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i\colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i,a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i,a) \in \coprod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

4.2.3 Binary Coproducts

Let A and B be sets.

Definition 4.2.3.1.1. The **coproduct of A and B**[7] is the coproduct of A and B in Sets as in Limits and Colimits, **??**.

Construction 4.2.3.1.2. Concretely, the coproduct of A and B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

1. *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\stackrel{\text{def}}{=} \{(0,a) \in S \mid a \in A\} \cup \{(1,b) \in S \mid b \in B\}, \end{aligned}$$

   where $S = \{0,1\} \times (A \cup B)$.

[7] *Further Terminology:* Also called the **disjoint union of A and B**.

2. *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 &\colon A \to A \coprod B,\\ \mathrm{inj}_2 &\colon B \to A \coprod B,\end{aligned}$$

given by

$$\begin{aligned}\mathrm{inj}_1(a) &\overset{\mathrm{def}}{=} (0, a),\\ \mathrm{inj}_2(b) &\overset{\mathrm{def}}{=} (1, b),\end{aligned}$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[r, hook, "\mathrm{inj}_1"'] \arrow[ur, bend left, "\iota_1"] & A \coprod B & B \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"'] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[r, hook, "\mathrm{inj}_1"'] \arrow[ur, bend left, "\iota_1"] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"'] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_A &= \iota_A,\\ \phi \circ \mathrm{inj}_B &= \iota_B\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a),\\ \iota_B(b) & \text{if } x = (1, b)\end{cases}$$

for each $x \in A \coprod B$. $\square$

Proposition 4.2.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned} A \coprod - \colon \quad & \mathsf{Sets} \to \mathsf{Sets}, \\ - \coprod B \colon \quad & \mathsf{Sets} \to \mathsf{Sets}, \\ -_1 \coprod -_2 \colon \quad & \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}, \end{aligned}$$

where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$

for each $x \in A \coprod B$.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have an adjunction

$$(-_1 \coprod -_2 \dashv \Delta_{\mathsf{Sets}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \times \mathsf{Sets} \arrow[r, bend left, "-_1 \coprod -_2"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\Delta_{\mathsf{Sets}}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}(A \coprod B, C), \cong \mathrm{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A, B), (C, C))$$

natural in $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$ and in $C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z} \colon (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z),$$

natural in $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned}\lambda^{\mathsf{Sets},\coprod}_{X} &\colon \emptyset \coprod X \xrightarrow{\sim} X,\\ \rho^{\mathsf{Sets},\coprod}_{X} &\colon X \coprod \emptyset \xrightarrow{\sim} X,\end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$\sigma^{\mathsf{Sets},\coprod}_{X,Y} \colon X \coprod Y \xrightarrow{\sim} Y \coprod X,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Symmetric Monoidality.* The 7-tuple $\left(\mathsf{Sets}, \coprod, \emptyset, \alpha^{\mathsf{Sets}}_{\coprod}, \lambda^{\mathsf{Sets}}_{\coprod}, \rho^{\mathsf{Sets}}_{\coprod}, \sigma^{\mathsf{Sets}}\right)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, *Adjointness*: This follows from the universal property of the coproduct.
Item 3, *Associativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.3.1.1.
Item 4, *Unitality*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definitions 5.2.4.1.1 and 5.2.5.1.1.
Item 5, *Commutativity*: This is proved in the proof of Monoidal Structures on the Category of Sets, Definition 5.2.6.1.1.
Item 6, *Symmetric Monoidality*: This is a repetition of Monoidal Structures on the Category of Sets, Definition 5.2.7.1.1, and is proved there. □

4.2.4 Pushouts

Let A, B, and C be sets and let $f \colon C \to A$ and $g \colon C \to B$ be functions.

Definition 4.2.4.1.1. The **pushout of** A **and** B **over** C **along** f **and** g[8] is the pushout of A and B over C along f and g in Sets as in Limits and Colimits, **??**.

Construction 4.2.4.1.2. Concretely, the pushout of A and B over C along f and g is the pair $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

1. *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \stackrel{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

2. *The Cocone.* The maps

$$\mathrm{inj}_1 \colon A \to A \coprod_C B,$$
$$\mathrm{inj}_2 \colon B \to A \coprod_C B$$

given by

$$\mathrm{inj}_1(a) \stackrel{\text{def}}{=} [(0, a)]$$
$$\mathrm{inj}_2(b) \stackrel{\text{def}}{=} [(1, b)]$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$[\mathrm{inj}_1 \circ f](c) = \mathrm{inj}_1(f(c))$$

---

[8] *Further Terminology:* Also called the **fibre coproduct of** A **and** B **over** C **along** f **and** g.

$$
\begin{aligned}
&= [(0, f(c))] \\
&= [(1, g(c))] \\
&= \mathrm{inj}_2(g(c)) \\
&= [\mathrm{inj}_2 \circ g](c),
\end{aligned}
$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$
\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, phantom] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, very near end, "\ulcorner"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

in Sets. Then there exists a unique map $\phi \colon A \coprod_C B \to P$ making the diagram

$$
\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\exists!", "\phi"'] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, very near end, "\ulcorner"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}
$$

via

$$
\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}
$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by Definition 4.2.4.1.3, we have a sequence
$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$
2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by Definition 4.2.4.1.3, we have a sequence
$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$
3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by Definition 4.2.4.1.3, we have a sequence
$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned}
\phi([x]) &= \phi([(0, f(c))]) \\
&\stackrel{\text{def}}{=} \iota_1(f(c)) \\
&= \iota_2(g(c)) \\
&\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\
&= \phi([y]),
\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}
(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\
(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\
(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)
\end{aligned}$$

gives

$$\phi([(0, a)]) = \phi([(0, a')]),$$

$$\phi([(1, b)]) = \phi([(1, b')]),$$
$$\phi([(0, a)]) = \phi([(1, b)]),$$

showing ϕ to be well-defined. □

Remark 4.2.4.1.3. In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a, b \in A$ and $a = b$.

2. We have $a, b \in B$ and $a = b$.

3. There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:

   (a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
   (b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

   In other words, there exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

   (c) There exists $c_0 \in C$ satisfying one of the following conditions:
      i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.
      ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.
   (d) For each $1 \leq i \leq n - 1$, there exists $c_i \in C$ satisfying one of the following conditions:
      i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
      ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
   (e) There exists $c_n \in C$ satisfying one of the following conditions:
      i. We have $x_n = f(c_n)$ and $b = g(c_n)$.
      ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

Remark 4.2.4.1.4. It is common practice to write $A \coprod_C B$ for the pushout of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \coprod_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \coprod_{f,C,g} B$ or $A \coprod_C^{f,g} B$ for $A \coprod_C B$.

Example 4.2.4.1.5. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Pointed Sets, Definition 6.3.3.1.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let X be a set. We have

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u] & A \cap B \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

   for each $A, B \in \mathcal{P}(X)$.

Proof. *Item 1, Wedge Sums of Pointed Sets*: This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.
Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. □

Proposition 4.2.4.1.6. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

   where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Adjointness.* We have an adjunction

$$\left(-_1 \coprod X -_2 \dashv \Delta_{\mathsf{Sets}_{X/}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/} \arrow[r, bend left, "-_1 \coprod_X -_2"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{X/}, \arrow[l, bend left, "\Delta_{\mathsf{Sets}_{X/}}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_{X/}(A \textstyle\coprod_X B, C), \cong \mathrm{Hom}_{\mathsf{Sets}_{X/}\times\mathsf{Sets}_{X/}}((A,B),(C,C))$$

natural in $(A,B) \in \mathrm{Obj}(\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/})$ and in $C \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

3. *Associativity.* Given a diagram

$$\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \textstyle\coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C)$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small]
& & (A \coprod_X B) \coprod_Y C & & \\
& A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\begin{tikzcd}[column sep=small]
& & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
& A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\begin{tikzcd}[column sep=small]
& & A \coprod_X (B \coprod_Y C) & & \\
& & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

4. *Interaction With Composition.* Given a diagram

$$\begin{tikzcd}
X & & & & Y \\
& A \arrow[ul, "\phi"] & & B \arrow[ur, "\psi"'] & \\
& & K \arrow[ul, "f"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$\begin{aligned}
X \textstyle\coprod_K^{\phi\circ f,\psi\circ g} Y &\cong (X \textstyle\coprod_A^{\phi,j_1} (A \coprod_K^{f,g} B)) \coprod_{A \coprod_K^{f,g} B}^{i_2,i_1} ((A \coprod_K^{f,g} B) \coprod_B^{j_2,\psi} Y) \\
&\cong X \textstyle\coprod_A^{\phi,i} ((A \coprod_K^{f,g} B) \coprod_B^{j_2,\psi} Y)
\end{aligned}$$

$$\cong (X \coprod\nolimits_A^{\phi,i_1} (A \coprod\nolimits_K^{f,g} B)) \coprod\nolimits_B^{j,\psi} Y$$

where

$$\begin{aligned}
j_1 &= \mathrm{inj}_1^{A\times_K^{f,g} B}, & j_2 &= \mathrm{inj}_2^{A\times_K^{f,g} B}, \\
i_1 &= \mathrm{inj}_1^{(A\times_K^{f,g} B)\times_Y^{q_2,\psi}}, & i_2 &= \mathrm{inj}_2^{X\times^{\phi,q_1}_{A\times_K^{f,g} B}(A\times_K^{f,g} B)}, \\
i &= j_1 \circ \mathrm{inj}_1^{(A\times_K^{f,g} B)\times_B^{q_2,\psi} Y}, & j &= j_2 \circ \mathrm{inj}_2^{X\times_A^{\phi,q_1}(A\times_K^{f,g} B)},
\end{aligned}$$

and where these pullbacks are built as in the diagrams

$$\begin{tikzcd}
 & & (X \coprod_A (A \coprod_K B)) \coprod_{A \coprod_K B} ((A \coprod_K B) \coprod_B Y) & & \\
 & X \coprod_A (A \coprod_K B) \arrow[ur, hook] & & (A \coprod_K B) \coprod_B Y \arrow[ul, hook'] & \\
X \arrow[ur, hook] & & A \coprod_K B \arrow[ul, hook'] \arrow[ur, hook] \arrow[uu, phantom, "\wedge" near end] & & Y, \arrow[ul, hook'] \\
 & A \arrow[ul, "\phi"] \arrow[ur, hook] \arrow[uu, phantom, "\wedge" near end] & & B \arrow[ul, hook'] \arrow[ur, "\psi"'] \arrow[uu, phantom, "\wedge" near end] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\wedge" near end] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X \coprod_A ((A \coprod_K B) \coprod_B Y) & & \\
 & & & (A \coprod_K B) \coprod_B Y \arrow[ul, hook'] & \\
X \arrow[uurr, hook] & & A \coprod_K B \arrow[ur, hook] \arrow[uu, phantom, "\wedge" near end] & & Y, \arrow[ul, hook'] \\
 & A \arrow[ul, "\phi"] \arrow[ur, hook] & & B \arrow[ul, hook'] \arrow[ur, "\psi"'] \arrow[uu, phantom, "\wedge" near end] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\wedge" near end] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & (X \coprod_A (A \coprod_K B)) \coprod_B Y & & \\
 & X \coprod_A (A \coprod_K B) \arrow[ur, hook] & & & \\
X \arrow[ur, hook] & & A \coprod_K B \arrow[ul, hook'] \arrow[uu, phantom, "\wedge" near end] & & Y, \arrow[uull, hook'] \\
 & A \arrow[ul, "\phi"] \arrow[ur, hook] \arrow[uu, phantom, "\wedge" near end] & & B \arrow[ul, hook'] \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, phantom, "\wedge" near end] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X \coprod_K Y & & \\
 & & & & \\
X \arrow[uurr, hook] & & & & Y. \arrow[uull, hook'] \\
 & A \arrow[ul, "\phi"] & & B \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uuuu, phantom, "\wedge" very near end] & &
\end{tikzcd}$$

5. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
\lambda^{\mathsf{Sets}_{X/}}_{A} &\colon X \textstyle\coprod_X A \overset{\sim}{\dashrightarrow} A,\\
\rho^{\mathsf{Sets}_{X/}}_{A} &\colon A \textstyle\coprod_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X, \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

6. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
A \coprod_X B & B \arrow[l] \\
A \arrow[u] & X, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
\sigma^{\mathsf{Sets}_{X/}}_{A} \colon A \textstyle\coprod_X B \overset{\sim}{\dashrightarrow} B \textstyle\coprod_X A
\qquad
\begin{tikzcd}
B \coprod_X A & A \arrow[l] \\
B \arrow[u] & X. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}
$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

7. *Interaction With Coproducts.* We have

$$
A \textstyle\coprod_{\emptyset} B \cong A \textstyle\coprod B,
\qquad
\begin{tikzcd}
A \coprod B & B \arrow[l] \\
A \arrow[u] & \emptyset. \arrow[l, hook, "\iota_A"] \arrow[u, hook, "\iota_B"']
\end{tikzcd}
$$

8. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_{X/}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *: Adjointness*: This follows from the universal property of the coproduct (pushouts are coproducts in $\mathsf{Sets}_{X/}$).
Item 3, *Associativity*: Omitted.
Item 4, *Interaction With Composition*: Omitted.

Item 5, *Unitality*: Omitted.
Item 6, *Commutativity*: Omitted.
Item 7, *Interaction With Coproducts*: Omitted.
Item 8, *Symmetric Monoidality*: Omitted. □

4.2.5 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 4.2.5.1.1. The **coequaliser of f and g** is the coequaliser of f and g in Sets as in Limits and Colimits, **??**.

Construction 4.2.5.1.2. Concretely, the coequaliser of f and g is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

1. *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

$$\mathrm{CoEq}(f,g) \stackrel{\mathrm{def}}{=} B/\sim,$$

   where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

2. *The Cocone.* The map

$$\mathrm{coeq}(f,g) \colon B \twoheadrightarrow \mathrm{CoEq}(f,g)$$

   given by the quotient map $\pi \colon B \twoheadrightarrow B/\sim$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\stackrel{\mathrm{def}}{=} [f(a)] \\
&= [g(a)] \\
&\stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \\ & & C \end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

commute. □

Remark 4.2.5.1.3. In detail, by Conditions on Relations, Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a = b$;
2. There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
   (a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
   (b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

   In other words, there exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

   (a) There exists $z_0 \in A$ satisfying one of the following conditions:
      i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.
      ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.

(b) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:

   i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.

   ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.

(c) There exists $z_n \in A$ satisfying one of the following conditions:

   i. We have $x_n = f(z_n)$ and $b = g(z_n)$.

   ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 4.2.5.1.4. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}(R \to X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X).$$

Proof. *Item 1, Quotients by Equivalence Relations*: See [Pro25z]. □

Proposition 4.2.5.1.5. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[9]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

[9]That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

of $\mathrm{CoEq}(g,h)$.

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$A \overset{f}{\underset{h}{\overset{\xrightarrow{g}}{\rightrightarrows}}} B$$

in Sets.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \overset{h}{\underset{k}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Omitted.
Item 5, *Commutativity*: Omitted.
Item 6, *Interaction With Composition*: Omitted. □

4.2.6 Direct Colimits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \to \mathsf{T}$ be a direct system of sets.

Definition 4.2.6.1.1. The **direct colimit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in Limits and Colimits, **??**.

Construction 4.2.6.1.2. Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ is the pair $\left(\underset{\substack{\longrightarrow \\ \alpha\in I}}{\operatorname{colim}}(X_\alpha), \{\mathrm{inj}_\alpha\}_{\alpha\in I}\right)$ consisting of:

1. *The Colimit.* The set $\underset{\substack{\longrightarrow \\ \alpha\in I}}{\operatorname{colim}}(X_\alpha)$ defined by

$$\underset{\substack{\longrightarrow \\ \alpha\in I}}{\operatorname{colim}}(X_\alpha) \overset{\text{def}}{=} \left(\coprod_{\alpha\in I} X_\alpha\right)\Big/\sim,$$

   where $\sim$ is the equivalence relation on $\coprod_{\alpha\in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

   (a) We have $\alpha \preceq \gamma$.

   (b) We have $\beta \preceq \gamma$.

   (c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

2. *The Cocone.* The collection

$$\left\{\mathrm{inj}_\gamma \colon X_\gamma \to \underset{\substack{\longrightarrow \\ \alpha\in I}}{\operatorname{colim}}(X_\alpha)\right\}_{\gamma\in I}$$

   of maps of sets defined by

$$\mathrm{inj}_\gamma(x) \overset{\text{def}}{=} [(\gamma, x)]$$

   for each $\gamma \in I$ and each $x \in X_\gamma$.

Proof. We will prove Definition 4.2.6.1.2 below in a bit, but first we need a lemma (which is interesting in its own right). □

Lemma 4.2.6.1.3. For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have

$$(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$$

in $\underset{\substack{\longrightarrow \\ \alpha\in I}}{\operatorname{colim}}(X_\alpha)$.

Proof. Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \stackrel{\text{def}}{=} \mathrm{id}_{X_\beta}$, and we have

$$\begin{aligned} f_{\alpha\beta}(x) &= f_{\beta\beta}(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} \mathrm{id}_{X_\beta}(f_{\alpha\beta}(x)), \\ &= f_{\alpha\beta}(x). \end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, Items 1a to 1c of Definition 4.2.6.1.2 are met. Thus we have $(\alpha, x) \sim (\beta, f_{\alpha\beta}(x))$. □

We can now prove Definition 4.2.6.1.2:

Proof. We claim that $\underset{\overrightarrow{\alpha\in I}}{\mathrm{colim}}(X_\alpha)$ is the colimit of the direct system of sets $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$.

Commutativity of the Colimit Diagram: First, we need to check that the colimit diagram defined by $\underset{\overrightarrow{\alpha\in I}}{\mathrm{colim}}(X_\alpha)$ commutes, i.e. that we have

$$\mathrm{inj}_\alpha = \mathrm{inj}_\beta \circ f_{\alpha\beta}, \qquad \begin{tikzcd} & \underset{\overrightarrow{\alpha\in I}}{\mathrm{colim}}(X_\alpha) & \\ X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[ul, "\mathrm{inj}_\beta"'] \end{tikzcd}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$\begin{aligned} [\mathrm{inj}_\beta \circ f_{\alpha\beta}](x) &\stackrel{\text{def}}{=} \mathrm{inj}_\beta(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\ &= [(\alpha, x)] \\ &\stackrel{\text{def}}{=} \mathrm{inj}_\alpha(x), \end{aligned}$$

where we have used Definition 4.2.6.1.3 for the third equality.

Proof of the Universal Property of the Colimit: Next, we prove that $\underset{\overrightarrow{\alpha\in I}}{\mathrm{colim}}(X_\alpha)$ as constructed in Definition 4.2.6.1.2 satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{tikzcd}
& C & \\
& \varinjlim_{\alpha\in I}(X_\alpha) & \\
X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[uur, bend left, "i_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[ul, "\mathrm{inj}_\beta"'] \arrow[uul, bend right, "i_\beta"']
\end{tikzcd}$$

in Sets. We claim that there exists a unique map $\phi\colon \varinjlim_{\alpha\in I}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
& C & \\
& \varinjlim_{\alpha\in I}(X_\alpha) \arrow[u, dashed, "\phi", "\exists!"'] & \\
X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[uur, bend left, "i_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[ul, "\mathrm{inj}_\beta"'] \arrow[uul, bend right, "i_\beta"']
\end{tikzcd}$$

commute. To this end, first consider the diagram

$$\begin{tikzcd}
\coprod_{\alpha\in I} X_\alpha \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, "\coprod_{\alpha\in I} i_\alpha"'] & \varinjlim_{\alpha\in I}(X_\alpha) \\
& C.
\end{tikzcd}$$

Lemma. If $(\alpha, x) \sim (\beta, y)$, then we have

$$\left[\coprod_{\alpha\in I} i_\alpha\right](x) = \left[\coprod_{\alpha\in I} i_\alpha\right](y).$$

Proof. Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

1. We have $\alpha \preceq \gamma$.
2. We have $\beta \preceq \gamma$.
3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$\begin{aligned}\left[\coprod_{\alpha\in I} i_\alpha\right](x) &\overset{\text{def}}{=} i_\alpha(x)\\ &\overset{\text{def}}{=} [i_\gamma \circ f_{\alpha\gamma}](x)\\ &\overset{\text{def}}{=} i_\gamma(f_{\alpha\gamma}(x))\\ &= i_\gamma(f_{\beta\gamma}(x))\\ &\overset{\text{def}}{=} [i_\gamma \circ f_{\beta\gamma}](x)\\ &= i_\beta(y)\\ &\overset{\text{def}}{=} \left[\coprod_{\alpha\in I} i_\alpha\right](y).\end{aligned}$$

This finishes the proof of the lemma. Continuing, by Conditions on Relations, **??** of Definition 10.6.2.1.3, there then exists a map $\phi\colon \underset{\alpha\in I}{\underrightarrow{\mathrm{colim}}}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
\coprod_{\alpha\in I} X_\alpha \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, "\coprod_{\alpha\in I} i_\alpha"'] & \underset{\alpha\in I}{\underrightarrow{\mathrm{colim}}}(X_\alpha) \arrow[d, "\phi"] \\
& C
\end{tikzcd}$$

commute. In particular, this implies that the diagram

$$\begin{tikzcd}
X_\alpha \arrow[r, two heads, "\mathrm{inj}_\alpha"] \arrow[dr, "i_\alpha"'] & \underset{\alpha\in I}{\underrightarrow{\mathrm{colim}}}(X_\alpha) \arrow[d, "\phi"] \\
& C
\end{tikzcd}$$

also commutes, and thus so does the diagram

$$
\begin{tikzcd}
& C & \\
& \underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha) \arrow[u, dashed, "\phi", "\exists!"'] & \\
X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[uur, bend left, "i_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta . \arrow[ul, "\mathrm{inj}_\beta"'] \arrow[uul, bend right, "i_\beta"']
\end{tikzcd}
$$

This finishes the proof.[10] □

Example 4.2.6.1.4. Here are some examples of direct colimits of sets.

1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit

$$\mathbb{Z}(p^\infty) \overset{\text{def}}{=} \underset{\overrightarrow{n\in\mathbb{N}}}{\operatorname{colim}}(\mathbb{Z}_{/p^n});$$

   see **??**.

4.3 Operations With Sets

4.3.1 The Empty Set

Definition 4.3.1.1.1. The **empty set** is the set $\varnothing$ defined by

$$\varnothing \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where X is the set in the set existence axiom, **??** of **??**.

[10] Incidentally, the conditions

$$\left\{i_\alpha = \phi \circ \mathrm{inj}_\alpha\right\}_{\alpha\in I}$$

show that ϕ must be given by

$$\phi([(\alpha, x)]) = (i_\alpha(x))_{\alpha\in I}$$

for each $[(\alpha, x)] \in \underset{\overrightarrow{\alpha\in I}}{\operatorname{colim}}(X_\alpha)$, although we would need to show that this assignment is well-defined were we to prove Definition 4.2.6.1.2 in this way. Instead, invoking Conditions on

4.3.2 Singleton Sets

Let X be a set.

Definition 4.3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself of Definition 4.3.3.1.1.

4.3.3 Pairings of Sets

Let X and Y be sets.

Definition 4.3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

4.3.4 Ordered Pairs

Let A and B be sets.

Definition 4.3.4.1.1. The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 4.3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof. Item 1, Uniqueness: See [Cie97, Theorem 1.2.3]. □

4.3.5 Sets of Maps

Let A and B be sets.

Definition 4.3.5.1.1. The **set of maps from A to B**[11] is the set $\mathsf{Sets}(A, B)$[12] whose elements are the functions from A to B.

Proposition 4.3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathsf{Sets}(X, -)\colon &\quad \mathsf{Sets} \to \mathsf{Sets}, \\ \mathsf{Sets}(-, Y)\colon &\quad \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathsf{Sets}(-_1, -_2)\colon &\quad \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd},$$

$$(- \times B \dashv \mathsf{Sets}(B, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd},$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Maps From the Punctual Set.* We have a bijection

$$\mathsf{Sets}(\mathrm{pt}, A) \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

Relations, **??** of Definition 10.6.2.1.3 gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

[11] *Further Terminology:* Also called the **Hom set from A to B**.

[12] *Further Notation:* Also written $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$.

4. *Maps to the Punctual Set.* We have a bijection

$$\mathsf{Sets}(A, \mathrm{pt}) \cong \mathrm{pt},$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: This follows from Categories, Items 2 and 5 of Definition 11.1.4.1.2.
Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.1.3.1.3 and is proved there.
Item 3, Maps From the Punctual Set: The bijection

$$\Phi_A \colon \mathsf{Sets}(\mathrm{pt}, A) \xrightarrow{\sim} A$$

is given by

$$\Phi_A(f) \stackrel{\text{def}}{=} f(\star)$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$, admitting an inverse

$$\Phi_A^{-1} \colon A \xrightarrow{\sim} \mathsf{Sets}(\mathrm{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \stackrel{\text{def}}{=} [\![\star \mapsto a]\!]$$

for each $a \in A$. Indeed, we have

$$\begin{aligned}
[\Phi_A^{-1} \circ \Phi_A](f) &\stackrel{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\
&\stackrel{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\
&\stackrel{\text{def}}{=} [\![\star \mapsto f(\star)]\!] \\
&\stackrel{\text{def}}{=} f \\
&\stackrel{\text{def}}{=} [\mathrm{id}_{\mathsf{Sets}(\mathrm{pt}, A)}](f)
\end{aligned}$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$ and

$$\begin{aligned}
[\Phi_A \circ \Phi_A^{-1}](a) &\stackrel{\text{def}}{=} \Phi_A(\Phi_A^{-1}(a)) \\
&\stackrel{\text{def}}{=} \Phi_A([\![\star \mapsto a]\!]) \\
&\stackrel{\text{def}}{=} \mathrm{ev}_\star([\![\star \mapsto a]\!]) \\
&\stackrel{\text{def}}{=} a \\
&\stackrel{\text{def}}{=} [\mathrm{id}_A](a)
\end{aligned}$$

for each $a \in A$, and thus we have

$$\begin{aligned}\Phi_A^{-1} \circ \Phi_A &= \mathrm{id}_{\mathsf{Sets}(\mathrm{pt},A)}\\ \Phi_A \circ \Phi_A^{-1} &= \mathrm{id}_A.\end{aligned}$$

To prove naturality, we need to show that the diagram

$$\begin{tikzcd} \mathsf{Sets}(\mathrm{pt},A) \arrow[r, "f_!"] \arrow[d, dashed, "\Phi_A"', "\wr"] & \mathsf{Sets}(\mathrm{pt},B) \arrow[d, dashed, "\wr"', "\Phi_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}[f \circ \Phi_A](\phi) &\stackrel{\text{def}}{=} f(\Phi_A(\phi))\\ &\stackrel{\text{def}}{=} f(\phi(\star))\\ &\stackrel{\text{def}}{=} [f \circ \phi](\star)\\ &\stackrel{\text{def}}{=} \Phi_B(f \circ \phi)\\ &\stackrel{\text{def}}{=} \Phi_B(f_!(\phi))\\ &\stackrel{\text{def}}{=} [\Phi_B \circ f_!](\phi)\end{aligned}$$

for each $\phi \in \mathsf{Sets}(\mathrm{pt},A)$. This finishes the proof.

Item 4, Maps to the Punctual Set: This follows from the universal property of pt as the terminal set, Definition 4.1.1.1.1. □

4.3.6 Unions of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.6.1.1. The **union of** $\mathcal{U}$ is the set $\bigcup_{U \in \mathcal{U}} U$ defined by

$$\bigcup_{U \in \mathcal{U}} U \stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } U \in \mathcal{U}\\ \text{such that we have } x \in U\end{array} \right\}.$$

Proposition 4.3.6.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U \in \mathcal{U}} U$ defines a functor

   $$\bigcup \colon (\mathcal{P}(\mathcal{P}(X)), \subset) \to (\mathcal{P}(X), \subset).$$

   In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

   (★) If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V$.

2. *Associativity.* The diagram

   $$\begin{tikzcd} \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\ \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X) \end{tikzcd}$$

   commutes, i.e. we have

   $$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcup_{A \in \mathcal{A}} \Big(\bigcup_{U \in A} U \Big)$$

   for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

   $$\begin{tikzcd} \mathcal{P}(X) \arrow[d, "\chi_{\mathcal{P}(X)}"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\ \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X) \end{tikzcd}$$

   commutes, i.e. we have

   $$\bigcup_{V \in \{U\}} V = U$$

   for each $U \in \mathcal{P}(X)$.

4. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\mathcal{P}(\chi_X)"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{\{u\}\in\chi_X(U)} \{u\} = U$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Unions I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cup"] \arrow[d, "\bigcup\times\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{W\in\mathcal{U}\cup\mathcal{V}} W = \left(\bigcup_{U\in\mathcal{U}} U\right) \cup \left(\bigcup_{V\in\mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "U\cup-"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\} \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "-\cup V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U \cup \left(\bigcup_{V\in\mathcal{V}} V\right) = \bigcup_{V\in\mathcal{V}} (U \cup V),$$

$$\left(\bigcup_{U\in\mathcal{U}} U\right)\cup V = \bigcup_{U\in\mathcal{U}}(U\cup V)$$

for each nonempty $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U,V\in\mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))\times\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cap"] \arrow[d, "\bigcup\times\bigcup"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X),
\end{tikzcd}$$

with components

$$\bigcup_{W\in\mathcal{U}\cap\mathcal{V}} W \subset \left(\bigcup_{U\in\mathcal{U}} U\right)\cap\left(\bigcup_{V\in\mathcal{V}} V\right)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cap-)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "U\cap-"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cap V)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "-\cap V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U\cap\left(\bigcup_{V\in\mathcal{V}} V\right) = \bigcup_{V\in\mathcal{V}}(U\cap V),$$

$$\left(\bigcup_{U\in\mathcal{U}} U\right)\cap V = \bigcup_{U\in\mathcal{U}}(U\cap V)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U,V\in\mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\setminus"] \arrow[d, "\bigcup \times \bigcup"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\setminus"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U\right) \setminus \left(\bigcup_{V \in \mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[d, "\bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{U \in \mathcal{U}^{\mathsf{c}}} U \neq \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[drr, dashed, "\sim"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[dr, "\bigcup"'] & & & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dl, "\bigcap^{\mathsf{op}}"] \\
 & \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, &
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{\textsf{c}} = \bigcap_{U\in\mathcal{U}} U^{\textsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[dr, dashed, "\sim"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\textsf{c}}"] \arrow[d, "\bigcap"'] & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[d, "\bigcup^{\mathsf{op}}"] \\
\mathcal{P}(X) \arrow[rr, "(-)^{\textsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}},
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcap_{U\in\mathcal{U}} U\right)^{\textsf{c}} = \bigcup_{U\in\mathcal{U}} U^{\textsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\triangle"] \arrow[d, "\bigcup\times\bigcup"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcup^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X}"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V\right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f : X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f : X \to Y$ be a map of sets. The

diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Intersections of Families I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(x)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "\bigcap"'] & X
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Intersections of Families II.* Let X be a set and consider

the compositions

$$
\begin{tikzcd}
& & & \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[dlll, "\bigcap\star\mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dl, "\bigcup\star\mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}\star\bigcap"'] \arrow[drrr, "\mathrm{id}_{\mathcal{P}(X)}\star\bigcup"] & & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[drrr, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dr, "\bigcap"] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dl, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dlll, "\bigcap"] \\
& & & \mathcal{P}(X) & & &
\end{tikzcd}
$$

given by

$$
\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,
$$

$$
\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)
$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\begin{tikzcd}
& \displaystyle\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U \arrow[dr, hook] & \\
\displaystyle\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) & & \displaystyle\bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \\
& \displaystyle\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \arrow[ul, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

All other possible inclusions fail to hold in general.

Proof. *Item 1*, *Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition $(\star)$. So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

Item 2, *Associativity*: We have

$$\begin{aligned}
\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } U \in \displaystyle\bigcup_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U\end{array}\right\} \\
&\overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U\right).
\end{aligned}
$$

This finishes the proof.

Item 3, *Left Unitality*: We have

$$
\begin{aligned}
\bigcup_{V \in \{U\}} V &\overset{\text{def}}{=} \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } V \in \{U\} \\ \text{such that we have } x \in U\end{array}\right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.

Item 4, *Right Unitality*: We have

$$
\begin{aligned}
\bigcup_{\{u\} \in \chi_X(U)} \{u\} &\overset{\text{def}}{=} \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\}\end{array}\right\} \\
&= \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u\end{array}\right\} \\
&= \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } u \in U \\ \text{such that we have } x = u\end{array}\right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.

Item 5, *Interaction With Unions I*: We have

$$
\begin{aligned}
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W &\overset{\text{def}}{=} \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W\end{array}\right\} \\
&= \left\{x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \\ \text{such that we have } x \in W \end{array} \right\} \\
&\quad \cup \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
&\overset{\text{def}}{=} \left(\bigcup_{W \in \mathcal{U}} W \right) \cup \left(\bigcup_{W \in \mathcal{V}} W \right) \\
&= \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}
$$

This finishes the proof.

Item 6, Interaction With Unions II: Assume $\mathcal{V}$ is nonempty. We have

$$
\begin{aligned}
U \cup \bigcup_{V \in \mathcal{V}} V &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; x \in U \text{ or } x \in \bigcup_{V \in \mathcal{V}} V \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} x \in U \text{ or there exists some} \\ V \in \mathcal{V} \text{ such that } x \in V \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \text{ or } x \in V \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } V \in \mathcal{V} \\ \text{such that } x \in U \cup V \end{array} \right\} \\
&\overset{\text{def}}{=} \bigcup_{V \in \mathcal{V}} U \cup V.
\end{aligned}
$$

This concludes the proof of the first statement. For the second statement, use Item 4 of Definition 4.3.8.1.2 to rewrite

$$
\begin{aligned}
\left(\bigcup_{U \in \mathcal{U}} U \right) \cup V &= V \cup \left(\bigcup_{U \in \mathcal{U}} U \right), \\
\bigcup_{U \in \mathcal{U}} (U \cup V) &= \bigcup_{U \in \mathcal{U}} (V \cup U).
\end{aligned}
$$

But these two sets are equal by the first statement.

Item 7, Interaction With Intersections I: We have

$$
\begin{aligned}
\bigcup_{W\in\mathcal{U}\cap\mathcal{V}} W &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{array}{l}\text{there exists some } W\in\mathcal{U}\cap\mathcal{V}\\ \text{such that we have } x\in W\end{array}\right\}\\
&\subset \left\{x\in X \,\middle|\, \begin{array}{l}\text{there exists some } U\in\mathcal{U} \text{ and some } V\in\mathcal{V}\\ \text{such that we have } x\in U \text{ and } x\in V\end{array}\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}\text{there exists some } U\in\mathcal{U}\\ \text{such that we have } x\in U\end{array}\right\}\\
&\quad\cup \left\{x\in X \,\middle|\, \begin{array}{l}\text{there exists some } V\in\mathcal{V}\\ \text{such that we have } x\in V\end{array}\right\}\\
&\overset{\text{def}}{=} \left(\bigcup_{U\in\mathcal{U}} U\right)\cap\left(\bigcup_{V\in\mathcal{V}} V\right).
\end{aligned}
$$

This finishes the proof.
Item 8, Interaction With Intersections II: We have

$$
\begin{aligned}
U\cap\bigcup_{V\in\mathcal{V}} V &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, x\in U \text{ and } x\in \bigcup_{V\in\mathcal{V}} V\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}x\in U \text{ and there exists some}\\ V\in\mathcal{V} \text{ such that } x\in V\end{array}\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}\text{there exists some } V\in\mathcal{V}\\ \text{such that } x\in U \text{ and } x\in V\end{array}\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}\text{there exists some } V\in\mathcal{V}\\ \text{such that } x\in U\cap V\end{array}\right\}\\
&\overset{\text{def}}{=} \bigcup_{V\in\mathcal{V}} U\cap V.
\end{aligned}
$$

This concludes the proof of the first statement. For the second statement, use Item 5 of Definition 4.3.9.1.2 to rewrite

$$
\left(\bigcup_{U\in\mathcal{U}} U\right)\cap V = V\cap\left(\bigcup_{U\in\mathcal{U}} U\right),
$$

$$\bigcup_{U \in \mathcal{U}} (U \cap V) = \bigcup_{U \in \mathcal{U}} (V \cap U).$$

But these two sets are equal by the first statement.
Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned}\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} U &= \bigcup_{W \in \{\{0,1\}\}} W \\ &= \{0, 1\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcup_{U \in \mathcal{U}} U\right) \setminus \left(\bigcup_{V \in \mathcal{V}} V\right) &= \{0, 1\} \setminus \{0\} \\ &= \{1\}.\end{aligned}$$

Thus we have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W = \{0, 1\} \neq \{1\} = \left(\bigcup_{U \in \mathcal{U}} U\right) \setminus \left(\bigcup_{V \in \mathcal{V}} V\right).$$

This finishes the proof.
Item 10, Interaction With Complements I: Let $X = \{0, 1\}$ and let $\mathcal{U} = \{0\}$. We have

$$\begin{aligned}\bigcup_{U \in \mathcal{U}^{\mathsf{c}}} U &= \bigcup_{U \in \{\emptyset, \{1\}, \{0,1\}\}} U \\ &= \{0, 1\},\end{aligned}$$

whereas

$$\begin{aligned}\bigcup_{U \in \mathcal{U}} U^{\mathsf{c}} &= \{0\}^{\mathsf{c}} \\ &= \{1\}.\end{aligned}$$

Thus we have

$$\bigcup_{U \in \mathcal{U}^{\mathsf{c}}} U = \{0, 1\} \neq \{1\} = \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.
Item 11, Interaction With Complements II: We have

$$\begin{aligned}
\left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathsf{c}} &\overset{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{there exists no } U\in\mathcal{U}\\ \text{such that we have } x\in U\end{array}\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}\text{for all } U\in\mathcal{U}\\ \text{we have } x\notin U\end{array}\right\}\\
&\overset{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for all } U\in\mathcal{U}\\ \text{we have } x\in U^{\mathsf{c}}\end{array}\right\}\\
&\overset{\text{def}}{=} \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}.
\end{aligned}$$

Item 12, Interaction With Complements III: By Item 11 Item 3 of Definition 4.3.11.1.2, we have

$$\begin{aligned}
\left(\bigcap_{U\in\mathcal{U}} U\right)^{\mathsf{c}} &= \left(\bigcap_{U\in\mathcal{U}} (U^{\mathsf{c}})^{\mathsf{c}}\right)^{\mathsf{c}}\\
&= \left(\left(\bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}\right)^{\mathsf{c}}\right)^{\mathsf{c}}\\
&= \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}.
\end{aligned}$$

Item 13, Interaction With Symmetric Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$\begin{aligned}
\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W &= \bigcup_{W\in\{\{0\}\}} W\\
&= \{0\},
\end{aligned}$$

whereas

$$\begin{aligned}
\left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right) &= \{0,1\} \triangle \{0,1\}\\
&= \emptyset,
\end{aligned}$$

Thus we have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right).$$

This finishes the proof.
Item 14, Interaction With Internal Homs I: This is a repetition of Item 7 of Definition 4.4.7.1.3 and is proved there.
Item 15, Interaction With Internal Homs II: This is a repetition of Item 8 of Definition 4.4.7.1.3 and is proved there.
Item 16, Interaction With Internal Homs III: This is a repetition of Item 9 of Definition 4.4.7.1.3 and is proved there.
Item 17, Interaction With Direct Images: This is a repetition of Item 3 of Definition 4.6.1.1.5 and is proved there.
Item 18, Interaction With Inverse Images: This is a repetition of Item 3 of Definition 4.6.2.1.3 and is proved there.
Item 19, Interaction With Codirect Images: This is a repetition of Item 3 of Definition 4.6.3.1.7 and is proved there.
Item 20, Interaction With Intersections of Families I: We have

$$\begin{aligned}
\bigcap_{U\in\bigcup_{A\in\mathcal{A}} A} U &\overset{\text{def}}{=} \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } U\in\bigcup_{A\in\mathcal{A}} A,\\ \text{we have } x\in U\end{array}\right\}\\
&= \left\{x\in X \,\middle|\, \begin{array}{l}\text{for each } A\in\mathcal{A} \text{ and each}\\ U\in A, \text{ we have } x\in U\end{array}\right\}\\
&\overset{\text{def}}{=} \bigcap_{A\in\mathcal{A}}\left(\bigcap_{U\in A} U\right).
\end{aligned}$$

This finishes the proof.
Item 21, Interaction With Intersections of Families II: Omitted. □

4.3.7 Intersections of Families of Subsets

Let X be a set and let $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.7.1.1. The **intersection of** $\mathcal{U}$ is the set $\bigcap_{U\in\mathcal{U}} U$ defined by

$$\bigcap_{U\in\mathcal{U}} U \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{for each } U \in \mathcal{U},\\ \text{we have } x \in U\end{array} \right\}.$$

Proposition 4.3.7.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcap_{U\in\mathcal{U}} U$ defines a functor

$$\bigcap\colon (\mathcal{P}(\mathcal{P}(X)), \supset) \to (\mathcal{P}(X), \subset).$$

   In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

   ($\star$) If $\mathcal{U} \subset \mathcal{V}$, then $\bigcap_{V\in\mathcal{V}} V \subset \bigcap_{U\in\mathcal{U}} U$.

2. *Oplax Associativity.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

   with components

$$\bigcap_{A\in\mathcal{A}} \left(\bigcap_{U\in A} U \right) \subset \bigcap_{U\in\bigcap_{A\in\mathcal{A}} A} U$$

   for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\chi_{\mathcal{P}(X)}"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V\in\{U\}} V = U.$$

for each $U \in \mathcal{P}(X)$.

4. *Oplax Right Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\mathcal{P}(\chi_X)"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] \arrow[ur, phantom, "\times"] & \mathcal{P}(X)
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{\{x\}\in\chi_X(U)} \{x\} \neq U$$

in general, where $U \in \mathcal{P}(X)$. However, when U is nonempty, we have

$$\bigcap_{\{x\}\in\chi_X(U)} \{x\} \subset U.$$

5. *Interaction With Unions I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cup"] \arrow[d, "\bigcap\times\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{W\in\mathcal{U}\cup\mathcal{V}} W = \left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagram

$$
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "U\cup-"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "-\cup V"'] & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have

$$
\begin{aligned}
U \cup \left(\bigcap_{V\in\mathcal{V}} V\right) &= \bigcap_{V\in\mathcal{V}} (U \cup V),\\
\left(\bigcap_{U\in\mathcal{U}} U\right) \cup V &= \bigcap_{U\in\mathcal{U}} (U \cup V)
\end{aligned}
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cap"] \arrow[d, "\bigcap\times\bigcap"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X),
\end{tikzcd}
$$

with components

$$
\left(\bigcap_{U\in\mathcal{U}} U\right) \cap \left(\bigcap_{V\in\mathcal{V}} V\right) \subset \bigcap_{W\in\mathcal{U}\cap\mathcal{V}} W
$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cap-)"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "U\cap-"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cap V)"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "-\cap V"'] & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V \in \mathcal{V}} V \right) = \bigcap_{V \in \mathcal{V}} (U \cup V),$$

$$\left(\bigcap_{U \in \mathcal{U}} U \right) \cup V = \bigcap_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\setminus"] \arrow[d, "\cap \times \cap"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\setminus"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U \right) \setminus \left(\bigcap_{V \in \mathcal{V}} V \right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[d, "\cap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U}^{\mathsf{c}}} W \neq \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$
\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[drr, dashed, "\sim"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[dr, "\bigcap"'] & & & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dl, "\bigcup^{\mathsf{op}}"] \\
 & \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, &
\end{tikzcd}
$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U\right)^{\mathsf{c}} = \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$
\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[drr, dashed, "\sim"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[dr, "\bigcup"'] & & & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dl, "\bigcap^{\mathsf{op}}"] \\
 & \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, &
\end{tikzcd}
$$

commutes, i.e. we have

$$\left(\bigcup_{U \in \mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\triangle"] \arrow[d, "\bigcap \times \bigcap"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \triangle \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U\right) \triangle \left(\bigcap_{V \in \mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\begin{tikzcd}
& & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[drr, "\bigcap^{\mathsf{op}}"] & & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[urr, dashed, "\sim"] \arrow[dr, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & & & \mathcal{P}(X)^{\mathsf{op}} \arrow[dl, "{[-,V]_X}"] \\
& \mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcup"'] & & \mathcal{P}(X) &
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcap_{U\in\mathcal{U}} U, V\right]_X = \bigcup_{U\in\mathcal{U}} [U,V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\star[U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V\in\mathcal{V}} V\right]_X = \bigcap_{V\in\mathcal{V}} [U,V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U\in\mathcal{U}} f_!(U) = \bigcap_{V\in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a map of sets. The

diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Unions of Families I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(x)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "\bigcap"'] & X
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Unions of Families II.* Let X be a set and consider the compositions

$$\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[dll, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dl, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"'] \arrow[drr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[drr, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[dr, "\bigcap"] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dl, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[dll, "\bigcap"] \\
 & & \mathcal{P}(X) & &
\end{tikzcd}$$

given by

$$\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$
\begin{tikzcd}
 & \displaystyle\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U \arrow[dr, hook] & \\
\displaystyle\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) & & \displaystyle\bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \\
 & \displaystyle\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \arrow[ul, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

All other possible inclusions fail to hold in general.

Proof. Item 1, Functoriality: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition ($\star$). So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion.

Item 2, Oplax Associativity: We have

$$
\begin{aligned}
\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \displaystyle\bigcap_{U \in A} U \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \displaystyle\bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
&\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.
\end{aligned}
$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 3, Left Unitality: We have

$$
\begin{aligned}
\bigcap_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in U \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.

Item 4, Oplax Right Unitality: If $U = \emptyset$, then we have

$$
\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \bigcap_{\{u\} \in \emptyset} \{u\} \\
&= X,
\end{aligned}
$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$. When U is nonempty, we have two cases:

1. If U is a singleton, say $U = \{u\}$, we have

$$
\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \{u\} \\
&\stackrel{\text{def}}{=} U.
\end{aligned}
$$

2. If U contains at least two elements, we have

$$
\begin{aligned}
\bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \emptyset \\
&\subset U.
\end{aligned}
$$

This finishes the proof.
Item 5, Interaction With Unions I: We have

$$\begin{aligned}
\bigcap_{W\in\mathcal{U}\cup\mathcal{V}} W &\overset{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } W\in\mathcal{U}\cup\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } W\in\mathcal{U} \text{ and each}\\ W\in\mathcal{V}, \text{ we have } x\in W\end{array}\right\}\\
&\overset{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } W\in\mathcal{U},\\ \text{we have } x\in W\end{array}\right\}\\
&\qquad\cap \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } W\in\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&\overset{\text{def}}{=} \left(\bigcap_{W\in\mathcal{U}} W\right)\cap\left(\bigcap_{W\in\mathcal{V}} W\right)\\
&= \left(\bigcap_{U\in\mathcal{U}} U\right)\cap\left(\bigcap_{V\in\mathcal{V}} V\right).
\end{aligned}$$

This finishes the proof.
Item 6, Interaction With Unions II: Omitted.
Item 7, Interaction With Intersections I: We have

$$\begin{aligned}
\left(\bigcap_{U\in\mathcal{U}} U\right)\cap\left(\bigcap_{V\in\mathcal{V}} V\right) &\overset{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } U\in\mathcal{U},\\ \text{we have } x\in U\end{array}\right\}\\
&\qquad\cup \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } V\in\mathcal{V},\\ \text{we have } x\in V\end{array}\right\}\\
&= \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } W\in\mathcal{U}\cap\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&\subset \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } W\in\mathcal{U}\cup\mathcal{V},\\ \text{we have } x\in W\end{array}\right\}\\
&\overset{\text{def}}{=} \bigcap_{W\in\mathcal{U}\cap\mathcal{V}} W.
\end{aligned}$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0\},\{0,1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$\begin{aligned}\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} U &= \bigcap_{W\in\{\{0,1\}\}} W\\ &= \{0,1\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcap_{U\in\mathcal{U}} U\right)\setminus\left(\bigcap_{V\in\mathcal{V}} V\right) &= \{0\}\setminus\{0\}\\ &= \emptyset.\end{aligned}$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} W = \{0,1\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right)\setminus\left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0,1\}$ and let $\mathcal{U} = \{\{0\}\}$. We have

$$\begin{aligned}\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U &= \bigcap_{W\in\{\emptyset,\{1\},\{0,1\}\}} W\\ &= \emptyset,\end{aligned}$$

whereas

$$\begin{aligned}\bigcap_{U\in\mathcal{U}} U^{\mathsf{c}} &= \{0\}^{\mathsf{c}}\\ &= \{1\}.\end{aligned}$$

Thus we have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U = \emptyset \neq \{1\} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.

Item 11, Interaction With Complements II: This is a repetition of Item 12 of Definition 4.3.6.1.2 and is proved there.

Item 12, Interaction With Complements III: This is a repetition of Item 11 of Definition 4.3.6.1.2 and is proved there.

Item 13, Interaction With Symmetric Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$\begin{aligned}\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W &= \bigcap_{W\in\{\{0\}\}} W\\ &= \{0\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcap_{U\in\mathcal{U}} U\right)\triangle\left(\bigcap_{V\in\mathcal{V}} V\right) &= \{0,1\}\triangle\{0\}\\ &= \emptyset,\end{aligned}$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right)\triangle\left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of Item 10 of Definition 4.4.7.1.3 and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of Item 11 of Definition 4.4.7.1.3 and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of Item 12 of Definition 4.4.7.1.3 and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of Item 4 of Definition 4.6.1.1.5 and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of Item 4 of Definition 4.6.2.1.3 and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of Item 4 of Definition 4.6.3.1.7 and is proved there.

Item 20, Interaction With Unions of Families I: This is a repetition of Item 20 of Definition 4.3.6.1.2 and is proved there.

Item 21, Interaction With Unions of Families II: This is a repetition of Item 21 of Definition 4.3.6.1.2 and is proved there. □

4.3.8 Binary Unions

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.8.1.1. The **union of** U **and** V is the set $U \cup V$ defined by

$$
\begin{aligned}
U \cup V &\overset{\text{def}}{=} \bigcup_{z \in \{U, V\}} z \\
&\overset{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}.
\end{aligned}
$$

Proposition 4.3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$
\begin{aligned}
U \cup - \colon &\quad (\mathcal{P}(X), \subset) &&\to (\mathcal{P}(X), \subset), \\
- \cup V \colon &\quad (\mathcal{P}(X), \subset) &&\to (\mathcal{P}(X), \subset), \\
-_1 \cup -_2 \colon &\quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &&\to (\mathcal{P}(X), \subset).
\end{aligned}
$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \cup V \subset A \cup V$.

   (b) If $V \subset B$, then $U \cup V \subset U \cup B$.

   (c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

2. *Associativity.* The diagram

$$
\begin{tikzcd}
& \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X), \mathcal{P}(X), \mathcal{P}(X)}", "\sim"'] \arrow[d, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X),
\end{tikzcd}
$$

   commutes, i.e. we have an equality of sets

$$
(U \cup V) \cup W = U \cup (V \cup W)
$$

   for each $U, V, W \in \mathcal{P}(X)$.

3. *Unitality.* The diagrams

$$
\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[\emptyset]\times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\sim", "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
& \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [\emptyset]}"] \arrow[dr, dashed, "\sim", "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
& \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
\emptyset \cup U &= U, \\
U \cup \emptyset &= U
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

4. *Commutativity.* The diagram

$$
\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
& \mathcal{P}(X)
\end{tikzcd}
$$

commutes, i.e. we have an equality of sets

$$
U \cup V = V \cup U
$$

for each $U, V \in \mathcal{P}(X)$.

5. *Annihilation With X.* The diagrams

$$
\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathrm{pt} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \varepsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}"] \arrow[d, "{[X]\times \mathrm{id}_{\mathcal{P}(X)}}"'] & & \mathrm{pt} \arrow[d, "{[X]}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times \mathrm{pt} \arrow[ur, "\varepsilon^{\mathsf{Sets}}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \times [X]}"'] & & \mathrm{pt} \arrow[d, "{[X]}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
U \cup X = X,
$$

$$X \cup V = X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Distributivity of Unions Over Intersections.* The diagrams

$$\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cap"] \\
& \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cap"] \\
& \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\
(U \cap V) \cup W &= (U \cup W) \cap (V \cup W)
\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Distributivity of Intersections Over Unions.* The diagrams

$$\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cup"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cup"] \\
& \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cup"] \\
& \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\
(U \cup V) \cap W &= (U \cap W) \cup (V \cap W)
\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Idempotency.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\Delta_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each $U \in \mathcal{P}(X)$.

9. *Via Intersections and Symmetric Differences.* The diagram

$$\begin{tikzcd} (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[r, "\triangle \times \cap"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\triangle"] & \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[u, "\Delta_{\mathcal{P}(X) \times \mathcal{P}(X)}"] \arrow[rr, "\cup"'] & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f : X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cup"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f : X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f : X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cup"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: See [Pro25an].
Item 2, *Associativity*: See [Pro25ba].
Item 3, *Unitality*: This follows from [Pro25bd] and Item 4.
Item 4, *Commutativity*: See [Pro25bb].
Item 5, *Annihilation With* X: We have

$$\begin{aligned} U \cup X &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\ &= \{x \in X \mid x \in X\}, \\ &= X \end{aligned}$$

and

$$\begin{aligned} X \cup V &\stackrel{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\ &= \{x \in X \mid x \in X\} \\ &= X. \end{aligned}$$

This finishes the proof.
Item 6, *Distributivity of Unions Over Intersections*: See [Pro25az].
Item 7, *Distributivity of Intersections Over Unions*: See [Pro25aj].
Item 8, *Idempotency*: See [Pro25am].
Item 9, *Via Intersections and Symmetric Differences*: See [Pro25ay].
Item 10, *Interaction With Characteristic Functions I*: See [Pro25h].
Item 11, *Interaction With Characteristic Functions II*: See [Pro25h].
Item 12, *Interaction With Direct Images*: See [Pro25p].
Item 13, *Interaction With Inverse Images*: See [Pro25y].
Item 14, *Interaction With Codirect Images*: This is a repetition of Item 5 of Definition 4.6.3.1.7 and is proved there.
Item 15, *Interaction With Powersets and Semirings*: This follows from Items 2 to 4 and 8 of this propostition and Items 3 to 6 and 8 of Definition 4.3.9.1.2. □

4.3.9 Binary Intersections

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.9.1.1. The **intersection of** U **and** V is the set $U \cap V$ defined by

$$\begin{aligned} U \cap V &\stackrel{\text{def}}{=} \bigcap_{z \in \{U, V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

Proposition 4.3.9.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap - \colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ - \cap V \colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2 \colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \cap V \subset A \cap V$.

   (b) If $V \subset B$, then $U \cap V \subset U \cap B$.

   (c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U \cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[U,-]_X}"] \end{tikzcd}$$

$$(- \cap V \dashv [V, -]_X) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "- \cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[V,-]_X}"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X), \end{aligned}$$

   natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

   is the bifunctor of Section 4.4.7. In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

i. We have $U \cap V \subset W$.

ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

i. We have $U \cap V \subset W$.

ii. We have $V \subset [U, W]_X$.

3. *Associativity.* The diagram

$$
\begin{tikzcd}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \cap"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}", "\sim"'] \arrow[d, "\cap \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X),
\end{tikzcd}
$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* The diagrams

$$
\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[X]} \times \mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, dashed, "\sim", "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \times {[X]}"] \arrow[dr, dashed, "\sim", "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
X \cap U &= U, \\
U \cap X &= U
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

5. *Commutativity.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
& \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each $U, V \in \mathcal{P}(X)$.

6. *Annihilation With the Empty Set.* The diagrams

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathrm{pt} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \varepsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}"] \arrow[d, "{[\varnothing] \times \mathrm{id}_{\mathcal{P}(X)}}"'] & & \mathrm{pt} \arrow[d, "{[\varnothing]}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times \mathrm{pt} \arrow[ur, "\varepsilon^{\mathsf{Sets}}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \times [\varnothing]}"'] & & \mathrm{pt} \arrow[d, "{[\varnothing]}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
\varnothing \cap X &= \varnothing, \\
X \cap \varnothing &= \varnothing
\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

7. *Distributivity of Unions Over Intersections.* The diagrams

$$\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)} \times (\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \times \cap"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cup \times \cup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cap"] \\
& \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)}) \times \Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap \times \mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cup \times \cup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cap"] \\
& \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}U \cup (V \cap W) &= (U \cup V) \cap (U \cup W),\\ (U \cap V) \cup W &= (U \cup W) \cap (V \cup W)\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Distributivity of Intersections Over Unions.* The diagrams

$$\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)} \times (\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cap \times \cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cup"] \\
& \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)} \times \mathrm{id}_{\mathcal{P}(X)}) \times \Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[d, "\cap \times \cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X) \times \mathcal{P}(X), \arrow[dl, "\cup"] \\
& \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}U \cap (V \cup W) &= (U \cap V) \cup (U \cap W),\\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W)\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Idempotency.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\Delta_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
& \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f : X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cap"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f : X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cap"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro25al].
Item 2, Adjointness: See [MSE 267469].
Item 3, Associativity: See [Pro25r].
Item 4, Unitality: This follows from [Pro25v] and Item 5.
Item 5, Commutativity: See [Pro25s].
Item 6, Annihilation With the Empty Set: This follows from [Pro25t] and Item 5.
Item 7, Distributivity of Unions Over Intersections: See [Pro25az].
Item 8, Distributivity of Intersections Over Unions: See [Pro25aj].
Item 9, Idempotency: See [Pro25ak].
Item 10, Interaction With Characteristic Functions I: See [Pro25e].
Item 11, Interaction With Characteristic Functions II: See [Pro25e].
Item 12, Interaction With Direct Images: See [Pro25n].
Item 13, Interaction With Inverse Images: See [Pro25w].
Item 14, Interaction With Codirect Images: This is a repetition of Item 6 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 6.
Item 16, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 and Items 3 to 6 and 8 of Definition 4.3.9.1.2. □

4.3.10 Differences

Let X and Y be sets.

Definition 4.3.10.1.1. The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 4.3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon & \quad (\mathcal{P}(X), \supset) && \to (\mathcal{P}(X), \subset),\\ - \setminus V\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset),\\ -_1 \setminus -_2\colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.

   (b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.

   (c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V),\\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

   for each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

   for each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

   for each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \bigtriangleup (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

13. *Right Annihilation.* We have

$$U \setminus X = \emptyset$$

for each $U \in \mathcal{P}(X)$.

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

15. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Direct Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

18. *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\setminus"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\setminus"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

19. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: See [Pro25ad] and [Pro25ah].
Item 2, De Morgan's Laws: See [Pro25k].
Item 3, Interaction With Unions I: See [Pro25l].
Item 4, Interaction With Unions II: We have

$$\begin{aligned}
(U \setminus V) \cup W &\overset{\text{def}}{=} \{x \in X \mid (x \in U \text{ and } x \notin V) \text{ or } x \in W\} \\
&= \{x \in X \mid (x \in U \text{ or } x \in W) \text{ and } (x \notin V \text{ or } x \in W)\} \\
&= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \text{ and } x \notin W)\} \\
&= \{x \in X \mid (x \in U \cup W) \text{ and not } (x \in V \setminus W)\}
\end{aligned}$$

$$
\begin{aligned}
&= \{x \in X \mid (x \in (U \cup W) \setminus (V \setminus W))\} \\
&= (U \cup W) \setminus (V \setminus W).
\end{aligned}
$$

Item 5, Interaction With Unions III: See [Pro25ai].
Item 6, Interaction With Unions IV: See [Pro25ac].
Item 7, Interaction With Intersections: See [Pro25u].
Item 8, Interaction With Complements: See [Pro25aa].
Item 9, Interaction With Symmetric Differences: See [Pro25ab].
Item 10, Triple Differences: See [Pro25ag].
Item 11, Left Annihilation: The direction $\emptyset \subset \emptyset \setminus U$ always holds. Now assume $x \in \emptyset \setminus U$. Then, $x \in \emptyset$ and $x \notin U$. Hence $\emptyset \setminus U \subset \emptyset$ must hold and the sets are equal.
Item 12, Right Unitality: See [Pro25ae].
Item 13, Right Annihilation: It suffices to show that no $x \in X$ can be an element of $U \setminus X$. Assume $x \in U \setminus X$. Then $x \notin X$, contradicting $x \in X$. This completes the proof.
Item 14, Invertibility: See [Pro25af].
Item 15, Interaction With Containment: The conditions are symmetric in U, W, hence it suffices to show that $V \setminus U \subset W$ implies $V \setminus W \subset U$. So assume $V \setminus U \subset W$, $x \in V \setminus W$. Then $x \in V$, $x \notin W$. So by contraposition, $x \notin V \setminus U$. But $x \in V$, so we must have $x \in U$, completing the proof.
Item 16, Interaction With Characteristic Functions: See [Pro25f].
Item 17, Interaction With Direct Images: See [Pro25o].
Item 18, Interaction With Inverse Images: See [Pro25x]. □

4.3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.3.11.1.1. The **complement of** U is the set U^{c} defined by

$$
\begin{aligned}
U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\
&\overset{\text{def}}{=} \{a \in X \mid a \notin U\}.
\end{aligned}
$$

Proposition 4.3.11.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X).$$

   In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

(★) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* The diagrams

$$
\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cup^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cup"'] & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
(U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}}, \\
(U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}}
\end{aligned}
$$

for each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* The diagram

$$
\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}"'] & \mathcal{P}(X) \arrow[d, "(-)^{\mathsf{c},\mathsf{op}}"] \\
& \mathcal{P}(X)^{\mathsf{op}}
\end{tikzcd}
$$

commutes, i.e. we have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. The diagram

$$
\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}
$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

6. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: This follows from Item 1 of Definition 4.3.10.1.2.
Item 2, *De Morgan's Laws*: See [Pro25k].
Item 3, *Involutority*: See [Pro25i].
Item 4, *Interaction With Characteristic Functions*: We consider the two cases $x \in U, x \notin U$.

1. If $x \in U$, then $x \notin U^{\mathsf{c}}$. So $\chi_U(x) = 1$ and

$$\begin{aligned}\chi_{U^{\mathsf{c}}}(x) &= 0\\ &= 1 - \chi_U(x).\end{aligned}$$

2. If $x \notin U$, then $x \in U^{\mathsf{c}}$. So $\chi_U(x) = 0$ and

$$\begin{aligned}\chi_{U^{\mathsf{c}}}(x) &= 1\\ &= 1 - \chi_U(x).\end{aligned}$$

Hence, the equation holds for all $x \in X$.
Item 5, Interaction With Direct Images: This is a repetition of Item 8 of Definition 4.6.1.1.5 and is proved there.
Item 6, Interaction With Inverse Images: This is a repetition of Item 8 of Definition 4.6.2.1.3 and is proved there.
Item 7, Interaction With Codirect Images: This is a repetition of Item 7 of Definition 4.6.3.1.7 and is proved there. □

4.3.12 Symmetric Differences

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.12.1.1. The **symmetric difference of** U **and** V is the set $U \triangle V$ defined by[13]

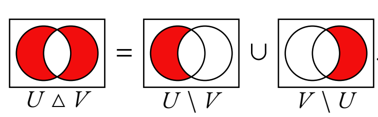
$$U \triangle V \overset{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

Proposition 4.3.12.1.2. Let X be a set.

1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \triangle V$ ***does not*** in general define functors

$$\begin{aligned}U \triangle -\colon & (\mathcal{P}(X), \subset) &&\to (\mathcal{P}(X), \subset),\\ - \triangle V\colon & (\mathcal{P}(X), \subset) &&\to (\mathcal{P}(X), \subset),\\ -_1 \triangle -_2\colon & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &&\to (\mathcal{P}(X), \subset).\end{aligned}$$

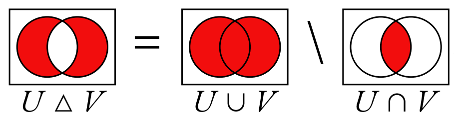
[13] *Illustration:*



2. *Via Unions and Intersections.* We have

$$U \mathbin{\triangle} V = (U \cup V) \setminus (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram



3. *Symmetric Differences of Disjoint Sets.* If U and V are disjoint, then we have

$$U \mathbin{\triangle} V = U \cup V.$$

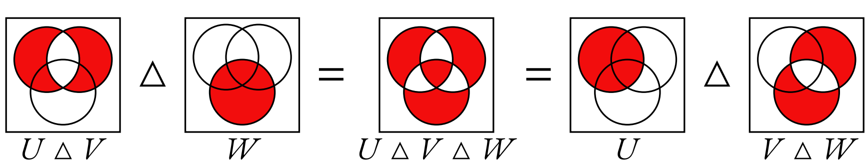
4. *Associativity.* The diagram

$$\begin{tikzcd}
& \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \triangle"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}", "\sim"'] \arrow[d, "\triangle \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\triangle"'] & & \mathcal{P}(X),
\end{tikzcd}$$

commutes, i.e. we have

$$(U \mathbin{\triangle} V) \mathbin{\triangle} W = U \mathbin{\triangle} (V \mathbin{\triangle} W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram



5. *Unitality.* The diagrams

$$\begin{tikzcd} \mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[\emptyset]\times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ & \mathcal{P}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [\emptyset]}"] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

6. *Commutativity.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\triangle"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$U \triangle V = V \triangle U$$

for each $U, V \in \mathcal{P}(X)$.

7. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Unions.* We have

$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Interaction With Complements I.* We have

$$U \mathbin{\triangle} U^{\mathsf{c}} = X$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements II.* We have

$$\begin{aligned} U \mathbin{\triangle} X &= U^{\mathsf{c}}, \\ X \mathbin{\triangle} U &= U^{\mathsf{c}} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

11. *Interaction With Complements III.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X) \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$U^{\mathsf{c}} \mathbin{\triangle} V^{\mathsf{c}} = U \mathbin{\triangle} V$$

for each $U, V \in \mathcal{P}(X)$.

12. *"Transitivity".* We have

$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$

for each $U, V, W \in \mathcal{P}(X)$.

13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \mathbin{\triangle} W \subset U \mathbin{\triangle} V \cup V \mathbin{\triangle} W$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \mathbin{\triangle} W) &= (U \cap V) \mathbin{\triangle} (U \cap W), \\ (U \mathbin{\triangle} V) \cap W &= (U \cap W) \mathbin{\triangle} (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

16. *Bijectivity.* Given $U, V \in \mathcal{P}(X)$, the maps

$$\begin{aligned} U \triangle - &: \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle V &: \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are self-inverse bijections. Moreover, the map

$$\begin{aligned} \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ C &\longmapsto C \triangle (U \triangle V) \end{aligned}$$

is a bijection of $\mathcal{P}(X)$ onto itself sending U to V and V to U.

17. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)})$ is an abelian group.[14]

    (b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

---

[14] Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:

$$(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}) \cong \mathrm{pt}.$$

2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:

$$(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}) \cong \mathbb{Z}_{/2}.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:

$$(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

4. *Interaction With Powersets and Vector Spaces I.* The pair $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ consisting of

    - The group $\mathcal{P}(X)$ of Item 17;
    - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

    $$\begin{aligned} 0 \cdot U &\stackrel{\text{def}}{=} \emptyset, \\ 1 \cdot U &\stackrel{\text{def}}{=} U; \end{aligned}$$

    is an $\mathbb{F}_2$-vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

    (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $(\mathcal{P}(X), \alpha_{\mathcal{P}(X)})$ of Item 4.

    (b) We have

    $$\dim(\mathcal{P}(X)) = \#X.$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[15]

7. *Interaction With Direct Images.* We have a natural transformation

    $$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

    with components

    $$f_!(U) \triangle f_!(V) \subset f_!(U \triangle V)$$

    indexed by $U, V \in \mathcal{P}(X)$.

---

[15] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25aw] for a proof.

8. *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd} \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\triangle"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

i.e. we have

$$f^{-1}(U) \triangle f^{-1}(V) = f^{-1}(U \triangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

9. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_*"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_*(U \triangle V) \subset f_*(U) \triangle f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Lack of Functoriality*: Let $X = \{0, 1\}$, $U = \{0\}$. Then $\emptyset \subset U$, but $U \triangle \emptyset = U \not\subset \emptyset = U \triangle U$ from Item 5 and Item 7. This gives a counterexample to the first statement. By using Item 6, we can adapt it to the second and third statement.
Item 2, Via Unions and Intersections: See [Pro25m].
Item 3, Symmetric Differences of Disjoint Sets: Since U and V are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$\begin{aligned} U \triangle V &= (U \cup V) \setminus (U \cap V) \\ &= (U \cup V) \setminus \emptyset \end{aligned}$$

$$= U \cup V,$$

where we've used Item 2 and Item 12 of Definition 4.3.10.1.2.
Item 4, Associativity: See [Pro25ao].
Item 5, Unitality: This follows from Item 6 and [Pro25at].
Item 6, Commutativity: See [Pro25ap].
Item 7, Invertibility: See [Pro25av].
Item 8, Interaction With Unions: See [Pro25bc].
Item 9, Interaction With Complements I: See [Pro25as].
Item 10, Interaction With Complements II: This follows from Item 6 and [Pro25ax].
Item 11, Interaction With Complements III: See [Pro25aq].
Item 12, "Transitivity": We have

$$\begin{aligned}
(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 4)}\\
&= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 4)}\\
&= U \triangle (\emptyset \triangle W) && \text{(by Item 7)}\\
&= U \triangle W. && \text{(by Item 5)}
\end{aligned}$$

This finishes the proof.
Item 13, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 12.
Item 14, Distributivity Over Intersections: See [Pro25q].
Item 15, Interaction With Characteristic Functions: See [Pro25g].
Item 16, Bijectivity:

- We show that
$$(U \triangle -)\colon \mathcal{P}(X) \to \mathcal{P}(X)$$
is self-inverse.

  Let $W \in \mathcal{P}(X)$. Then,
$$\begin{aligned}
U \triangle (U \triangle W) &= (U \triangle U) \triangle W && \text{(by Item 4)}\\
&= \emptyset \triangle W && \text{(by Item 7)}\\
&= W. && \text{(by Item 5)}
\end{aligned}$$

- By Item 6, $(- \triangle V) = (V \triangle -)$, hence the former is also self-inverse by the first point.

- The map $- \triangle (U \triangle V)$ is a bijection as a special case of the second point.

From the first two points and Item 6, we get

$$U \triangle (U \triangle V) = V, \; V \triangle (U \triangle V) = V \triangle (V \triangle U) = U.$$

Hence the function maps U to V and V to U.

Item 17, Interaction With Powersets and Groups: Item 17a follows from Items 4 to 7, while Item 3b follows from Item 7.[16]
Item 4, Interaction With Powersets and Vector Spaces I: See [`MSE 2719059`].
Item 5, Interaction With Powersets and Vector Spaces II: See [`MSE 2719059`].
Item 6, Interaction With Powersets and Rings: This follows from Items 6 and 15 of Definition 4.3.9.1.2 and Items 14 and 17.[17]
Item 7, Interaction With Direct Images: This is a repetition of Item 9 of Definition 4.6.1.1.5 and is proved there.
Item 8, Interaction With Inverse Images: This is a repetition of Item 9 of Definition 4.6.2.1.3 and is proved there.
Item 9, Interaction With Codirect Images: This is a repetition of Item 8 of Definition 4.6.3.1.7 and is proved there. □

4.4 Powersets

4.4.1 Foundations

Let X be a set.

Definition 4.4.1.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.4.1.1.2. Under the analogy that $\{\mathsf{t}, \mathsf{f}\}$ should be the (-1)-categorical analogue of Sets, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set X is equivalently (Item 2 of Definition 4.5.1.1.4) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

[16] *Reference:* [Pro25ar].
[17] *Reference:* [Pro25au].

- The category of presheaves on a category C is the category

$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$

of functors from C^{op} to the category Sets of sets.

Notation 4.4.1.1.3. Let X be a set.

1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of X.

2. We write $\mathcal{P}_{\mathrm{fin}}(X)$ for the set of finite subsets of X.

Proposition 4.4.1.1.4. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

   (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

   (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.

3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X),\\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}),\end{aligned}$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Interaction With Products I.* The map

$$\begin{aligned}\mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \textstyle\coprod Y)\\ (U, V) &\longmapsto U \cup V\end{aligned}$$

   is an isomorphism of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

5. *Interaction With Products II.* The map

$$\begin{aligned} \mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \textstyle\coprod Y) \\ (U, V) &\longmapsto U \boxtimes_{X\times Y} V, \end{aligned}$$

   where[18]

$$U \boxtimes_{X\times Y} V \stackrel{\text{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

   is an inclusion of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

6. *Interaction With Products III.* We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

   natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1, where $\otimes$ denotes the tensor product of suplattices of **??**. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

Proof. *Item 1, Co/Completeness*: Omitted.
Item 2, Cartesian Closedness: See Section 4.4.7.
Item 3, Powersets as Sets of Relations: Indeed, we have

$$\begin{aligned} \mathrm{Rel}(\mathrm{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X) \\ &\cong \mathcal{P}(X) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Rel}(X, \mathrm{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt}) \\ &\cong \mathcal{P}(X), \end{aligned}$$

where we have used Item 5 of Definition 4.1.3.1.3.

[18]The set $U \boxtimes_{X\times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see

Item 4, Interaction With Products I: The inverse of the map in the statement is the map

$$\Phi\colon \mathcal{P}(X \coprod Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \overset{\text{def}}{=} (S_X, S_Y)$$

for each $S \in \mathcal{P}(X \coprod Y)$, where

$$S_X \overset{\text{def}}{=} \{x \in X \mid (0, x) \in S\}$$
$$S_Y \overset{\text{def}}{=} \left\{y \in Y \,\middle|\, (1, y) \in S\right\}.$$

The rest of the proof is omitted.
Item 5, Interaction With Products II: Omitted.
Item 6, Interaction With Products III: Omitted. □

4.4.2 Functoriality of Powersets

Proposition 4.4.2.1.1. Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_!\colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f\colon A \to B$ to the map

$$\mathcal{P}_!(f)\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \overset{\text{def}}{=} f_!,$$

as in Definition 4.6.1.1.1.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

  of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

  defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

  as in Definition 4.6.2.1.1.

3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

also Section 4.6.4).

of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f : A \to B$ to the map

$$\mathcal{P}_*(f) : \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in Definition 4.6.3.1.1.

Proof. *Item 1, Functoriality I*: This follows from Items 3 and 4 of Definition 4.6.1.1.6.
Item 2, Functoriality II: This follows from Items 3 and 4 of Definition 4.6.2.1.4.
Item 3, Functoriality III: This follows from Items 3 and 4 of Definition 4.6.3.1.8. □

4.4.3 Adjointness of Powersets I

Proposition 4.4.3.1.1. We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right) : \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

Proof. We have

$$\begin{aligned} \mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\stackrel{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\ &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)} \end{aligned}$$

where all bijections are natural in A and B.[19] □

[19] Here we are using Item 3 of Definition 4.5.1.1.4.

4.4.4 Adjointness of Powersets II

Proposition 4.4.4.1.1. We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_!)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_!"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(X), Y) \cong \mathsf{Sets}(X, \mathcal{P}(Y))$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Relations, Item 1 of Definition 8.2.2.1.2 and $\mathcal{P}_!$ is the functor of Relations, Definition 8.7.5.1.1.

Proof. We have

$$\begin{aligned} \mathrm{Rel}(\mathrm{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)} \end{aligned}$$

where all bijections are natural in A, (where we are using Item 3 of Definition 4.5.1.1.4). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \nrightarrow B$ to the map $R^\dagger\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Relations, Definition 8.1.1.1.1.

Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$, the diagram

$$\begin{tikzcd} \mathrm{Rel}(\mathrm{Gr}(A), B) \arrow[r, "R\diamond -"] \arrow[d, dashed, "\wr"'] & \mathrm{Rel}(\mathrm{Gr}(A), B') \arrow[d, dashed, "\wr"] \\ \mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_!"'] & \mathsf{Sets}(A, \mathcal{P}(B')) \end{tikzcd}$$

commutes, which follows from Relations, Definition 8.7.1.1.3. $\square$

4.4.5 Powersets as Free Cocompletions

Let X be a set.

Proposition 4.4.5.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of X of Definition 4.4.1.1.1;
- The characteristic embedding $\chi_{(-)} \colon X \to \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

- A suplattice $(Y, \preceq)$;
- A function $f \colon X \to Y$;

there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

Proof. This is a rephrasing of Definition 4.4.5.1.2, which we prove below.[20] □

Proposition 4.4.5.1.2. We have an adjunction

$$(\mathcal{P} \dashv 忘) \colon \quad \mathsf{Sets} \underset{忘}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{SupLat},$$

[20]Here we only remark that the unique morphism of suplattices in the statement is given by

witnessed by a bijection

$$\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{SupLat})$, where:

- The category SupLat is the category of suplattices of **??**.
- The map

  $$\chi_X^* \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  witnessing the above bijection is defined by

  $$\chi_X^*(f) \overset{\mathrm{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of suplattices $f \colon \mathcal{P}(X) \to Y$ to the composition

  $$X \xrightarrow{\chi_X} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  witnessing the above bijection is given by sending a function $f \colon X \to Y$ to its left Kan extension along χ_X,

  $$\mathrm{Lan}_{\chi_X}(f) \colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & Y. \end{tikzcd}$$

  Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

  $$[\mathrm{Lan}_{\chi_X}(f)](U) = \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x)$$

the left Kan extension $\mathrm{Lan}_{\chi_X}(f)$ of f along χ_X.

$$
\begin{aligned}
&= \int^{x\in X} \chi_U(x) \odot f(x) \\
&= \bigvee_{x\in X} (\chi_U(x) \odot f(x)) \\
&= \left(\bigvee_{x\in U} (\chi_U(x) \odot f(x))\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} (\chi_U(x) \odot f(x))\right) \\
&= \left(\bigvee_{x\in U} f(x)\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} \varnothing_Y\right) \\
&= \bigvee_{x\in U} f(x)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigvee$ denotes the join in $(Y, \preceq)$.
- The symbol $\odot$ denotes the tensor of an element of Y by a truth value as in **??**. In particular, we have

  $$
  \begin{aligned}
  \mathsf{true} \odot f(x) &\overset{\text{def}}{=} f(x), \\
  \mathsf{false} \odot f(x) &\overset{\text{def}}{=} \varnothing_Y,
  \end{aligned}
  $$

  where $\varnothing_Y$ is the bottom element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathrm{Lan}_{\chi_X}(f)$ is given by

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f)](U) &= \bigvee_{x\in U} f(x) \\
&= \bigcup_{x\in U} f(x)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y} \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.
Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift left=4, "\Uparrow"] & Y, \end{tikzcd}$$

for each $f \in \mathsf{Sets}(X, Y)$.
Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))}.$$

We have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. Indeed, we have

$$\left[\mathrm{Lan}_{\chi_X}(f \circ \chi_X)\right](U) = \bigvee_{x \in U} f(\chi_X(x))$$

$$
\begin{aligned}
&= f\left(\bigvee_{x\in U} \chi_X(x)\right)\\
&= f\left(\bigcup_{x\in U} \{x\}\right)\\
&= f(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))}$ of $\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.
Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

We have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))\\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathrm{Lan}_{\chi_X}(f))\\
&\stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f) \circ \chi_X
\end{aligned}
$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$
\begin{aligned}
[\mathrm{Lan}_{\chi_X}(f) \circ \chi_X](x) &= \bigvee_{y\in\{x\}} f(y)\\
&= f(x)
\end{aligned}
$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, *Part I*: We need to show that, given a function $f\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_!(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!) \\
&\stackrel{\text{def}}{=} (\xi \circ f_!) \circ \chi_X \\
&= \xi \circ (f_! \circ \chi_X) \\
&\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X'), \subset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, *Part II*: We need to show that, given a morphism of suplattices

$$g\colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_!"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_!"] \\
\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, we have

$$[\Phi_{X,Y'} \circ g_!](\xi) \stackrel{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi))$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\
&\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X\\
&= g \circ (\xi \circ \chi_X)\\
&\stackrel{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))\\
&\stackrel{\text{def}}{=} g_!(\Phi_{X,Y}(\xi))\\
&\stackrel{\text{def}}{=} [g_! \circ \Phi_{X,Y}](\xi).
\end{aligned}
$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.
Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. □

Warning 4.4.5.1.3. Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion of* X, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.

4.4.6 Powersets as Free Completions

Let X be a set.

Proposition 4.4.6.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset of X together with reverse inclusion $\mathcal{P}(X)^{\mathsf{op}} = (\mathcal{P}(X), \supset)$ of Definition 4.4.1.1.1;
- The characteristic embedding $\chi_{(-)} \colon X \longrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

  - An inflattice $(Y, \preceq)$;
  - A function $f \colon X \to Y$;

  there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

making the diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

Proof. This is a rephrasing of Definition 4.4.6.1.2, which we prove below.[21] □

Proposition 4.4.6.1.2. We have an adjunction

$$(\mathcal{P} \dashv 忘): \quad \mathsf{Sets} \underset{忘}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{InfLat},$$

witnessed by a bijection

$$\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{InfLat})$, where:

- The category InfLat is the category of inflattices of **??**.
- The map

  $$\chi_X^* \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  witnessing the above bijection is defined by

  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of inflattices $f \colon \mathcal{P}(X)^{\mathsf{op}} \to Y$ to the composition

  $$X \xrightarrow{\chi_X} \mathcal{P}(X)^{\mathsf{op}} \xrightarrow{f} Y.$$

[21] Here we only remark that the unique morphism of inflattices in the statement is given by the right Kan extension $\mathrm{Ran}_{\chi_X}(f)$ of f along χ_X.

- The map

$$\mathrm{Ran}_{\chi_X}\colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

witnessing the above bijection is given by sending a function $f\colon X \to Y$ to its right Kan extension along χ_X,

$$\mathrm{Ran}_{\chi_X}(f)\colon \mathcal{P}(X)^{\mathsf{op}} \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\mathrm{Ran}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y. \end{tikzcd}$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned}
[\mathrm{Ran}_{\chi_X}(f)](U) &= \int_{x\in X} \chi_{\mathcal{P}(X)^{\mathsf{op}}}(\chi_x, U) \pitchfork f(x) \\
&= \int_{x\in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \pitchfork f(x) \\
&= \int_{x\in X} \chi_U(x) \pitchfork f(x) \\
&= \bigwedge_{x\in X} \chi_U(x) \pitchfork f(x) \\
&= \left(\bigwedge_{x\in U} \chi_U(x) \pitchfork f(x)\right) \wedge \left(\bigwedge_{x\in U^{\mathsf{c}}} \chi_U(x) \pitchfork f(x)\right) \\
&= \left(\bigwedge_{x\in U} f(x)\right) \wedge \left(\bigwedge_{x\in U^{\mathsf{c}}} \infty_Y\right) \\
&= \left(\bigwedge_{x\in U} f(x)\right) \wedge \infty_Y \\
&= \bigwedge_{x\in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

  – We have used **??** for the first equality.

- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigwedge$ denotes the meet in $(Y, \preceq)$.
- The symbol $\pitchfork$ denotes the cotensor of an element of Y by a truth value as in **??**. In particular, we have

$$\begin{aligned} \mathsf{true} \pitchfork f(x) &\stackrel{\text{def}}{=} f(x), \\ \mathsf{false} \pitchfork f(x) &\stackrel{\text{def}}{=} \infty_Y, \end{aligned}$$

  where ∞_Y is the top element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathrm{Ran}_{\chi_X}(f)$ is given by

$$\begin{aligned} [\mathrm{Ran}_{\chi_X}(f)](U) &= \bigwedge_{x \in U} f(x) \\ &= \bigcap_{x \in U} f(x) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. Map I: We define a map

$$\Phi_{X,Y} \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\mathrm{Ran}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Leftarrow"] & Y, \end{tikzcd}$$

for each $f \in \mathsf{Sets}(X, Y)$.
Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))}.$$

We have

$$\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](f) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\
&\overset{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\
&\overset{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f \circ \chi_X)
\end{aligned}$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned}
\left[\mathrm{Ran}_{\chi_X}(f \circ \chi_X)\right](U) &= \bigwedge_{x \in U} f(\chi_X(x)) \\
&= f\left(\bigwedge_{x \in U} \chi_X(x)\right) \\
&= f\left(\bigcup_{x \in U} \{x\}\right) \\
&= f(U)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](f) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))}$ of $\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

We have

$$\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Ran}_{\chi_X}(f))\\
&\overset{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f) \circ \chi_X
\end{aligned}$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned}
[\mathrm{Ran}_{\chi_X}(f) \circ \chi_X](x) &= \bigwedge_{y \in \{x\}} f(y)\\
&= f(x)
\end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$[\Phi_{X,Y} \circ \Psi_{X,Y}](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, *Part I*: We need to show that, given a function $f : X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_!(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_!(f)^*(\xi))\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!)\\
&\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X\\
&= \xi \circ (f_! \circ \chi_X)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f)\\
&= (\xi \circ \chi_{X'}) \circ f\\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f\\
&\overset{\text{def}}{=} f^{*}(\Phi_{X',Y}(\xi))\\
&\overset{\text{def}}{=} [f^{*} \circ \Phi_{X',Y}](\xi),
\end{aligned}
$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ*, Part II*: We need to show that, given a cocontinuous morphism of posets

$$
g\colon (Y, \preceq_{Y}) \to (Y', \preceq_{Y'}),
$$

the diagram

$$
\begin{tikzcd}
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_{!}"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_{!}"] \\
\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}
$$

commutes. Indeed, we have

$$
\begin{aligned}
[\Phi_{X,Y'} \circ g_{!}](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_{!}(\xi))\\
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_{X}\\
&= g \circ (\xi \circ \chi_{X})\\
&\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))\\
&\overset{\text{def}}{=} g_{!}(\Phi_{X,Y}(\xi))\\
&\overset{\text{def}}{=} [g_{!} \circ \Phi_{X,Y}](\xi).
\end{aligned}
$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.

Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. □

Warning 4.4.6.1.3. Although the assignment $X \mapsto \mathcal{P}(X)^{\mathsf{op}}$ is called the *free completion of* X, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}})^{\mathsf{op}} \neq \mathcal{P}(X)^{\mathsf{op}}$.

4.4.7 The Internal Hom of a Powerset

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Proposition 4.4.7.1.1. The **internal Hom of** $\mathcal{P}(X)$ **from** U **to** V is the subset $[U, V]_X$[22] of X given by

$$\begin{aligned}[U, V]_X &= U^{\mathsf{c}} \cup V \\ &= (U \setminus V)^{\mathsf{c}}\end{aligned}$$

where U^{c} is the complement of U of Definition 4.3.11.1.1.

Proof. Proof of the Equality $U^{\mathsf{c}} \cup V = (U \setminus V)^{\mathsf{c}}$: We have

$$\begin{aligned}(U \setminus V)^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus (U \setminus V) \\ &= (X \cap V) \cup (X \setminus U) \\ &= V \cup (X \setminus U) \\ &\overset{\text{def}}{=} V \cup U^{\mathsf{c}} \\ &= U^{\mathsf{c}} \cup V,\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the second equality.

2. Item 4 of Definition 4.3.9.1.2 for the third equality.

3. Item 4 of Definition 4.3.8.1.2 for the last equality.

This finishes the proof.
Proof that $U^{\mathsf{c}} \cup V$ *Is Indeed the Internal Hom*: This follows from Item 2 of Definition 4.3.9.1.2. □

Remark 4.4.7.1.2. Henning Makholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from U to V ([`MSE 267365`]):

1. Since products in $\mathcal{P}(X)$ are given by binary intersections (Item 1 of Definition 4.4.1.1.4), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.

[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$.

2. Under the Curry–Howard correspondence (**??**), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.

3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.

4. The expression $\neg U \vee V$ then corresponds to the set $U^{\mathsf{c}} \cup V$ in $\mathcal{P}(X)$.

5. The set $U^{\mathsf{c}} \vee V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

Proposition 4.4.7.1.3. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$\begin{aligned} [U, -]_X &\colon && (\mathcal{P}(X), \supset) && \to (\mathcal{P}(X), \subset), \\ [-, V]_X &\colon && (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ [-_1, -_2]_X &\colon && (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

(a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.

(b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.

(c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[U,-]_X}"] \end{tikzcd}$$

$$(- \cap V \dashv [V, -]_X)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[V,-]_X}"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X). \end{aligned}$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

i. We have $U \cap V \subset W$.
ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

i. We have $U \cap V \subset W$.
ii. We have $V \subset [U, W]_X$.

3. *Interaction With the Empty Set I.* We have

$$\begin{aligned}[U, \emptyset]_X &= U^{\mathsf{c}}, \\ [\emptyset, V]_X &= X,\end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Interaction With X.* We have

$$\begin{aligned}[U, X]_X &= X, \\ [X, V]_X &= V,\end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Interaction With the Empty Set II.* The functor

$$D_X \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

defined by

$$\begin{aligned}D_X &\overset{\text{def}}{=} [-, \emptyset]_X \\ &= (-)^{\mathsf{c}}\end{aligned}$$

is an involutory isomorphism of categories, making $\emptyset$ into a dualising object for $(\mathcal{P}(X), \cap, X, [-, -]_X)$ in the sense of **??**. In particular:

(a) The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "D_X"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}"'] & \mathcal{P}(X) \arrow[d, "D_X"] \\ & \mathcal{P}(X)^{\mathsf{op}} \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\overset{\text{def}}{=}[[U,\emptyset]_X,\emptyset]_X} = U$$

for each $U \in \mathcal{P}(X)$.

(b) The diagram

$$\begin{tikzcd}
& \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] & \mathcal{P}(X)^{\mathsf{op}} \arrow[dr, "D_X"] & \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X"] \arrow[rrr, "{[-_1,-_2]_X}"'] & & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\text{def}}{=}[U\cap[V,\emptyset]_X,\emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With the Empty Set III.* Let $f : X \to Y$ be a function.

   (a) *Interaction With Direct Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(b) *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "D_Y"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "D_X"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

(c) *Interaction With Codirect Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcup^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcup_{U\in\mathcal{U}} U, V\right]_X = \bigcap_{U\in\mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V\in\mathcal{V}} V\right]_X = \bigcup_{V\in\mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcap^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[dl, "{[-,V]_X}"] \\
 & \mathcal{P}(\mathcal{P}(X)) \xrightarrow[\bigcup]{} \mathcal{P}(X) &
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V\right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{V}} V\right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Binary Unions.* We have equalities of sets

$$\begin{aligned}[U \cap V, W]_X &= [U, W]_X \cup [V, W]_X, \\ [U, V \cap W]_X &= [U, V]_X \cap [U, W]_X\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Binary Intersections.* We have equalities of sets

$$\begin{aligned}[U \cup V, W]_X &= [U, W]_X \cap [V, W]_X, \\ [U, V \cup W]_X &= [U, V]_X \cup [U, W]_X\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Differences.* We have equalities of sets

$$\begin{aligned}[U \setminus V, W]_X &= [U, W]_X \cup [V^{\mathsf{c}}, W]_X \\ &= [U, W]_X \cup [U, V]_X, \\ [U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

16. *Interaction With Complements.* We have equalities of sets

$$\begin{aligned}[U^{\mathsf{c}}, V]_X &= U \cup V, \\ [U, V^{\mathsf{c}}]_X &= U \cap V, \\ [U, V]^{\mathsf{c}}_X &= U \setminus V\end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

18. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_!"] \arrow[d, "{[-_1, -_2]_X}"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1, -_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f_!([U,V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

19. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

20. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_*"] \arrow[d, "{[-_1,-_2]_X}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U,V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove Items 1a to 1c.

1. *Proof of Item 1a:* We have

$$\begin{aligned}[A, V]_X &\stackrel{\text{def}}{=} A^{\mathsf{c}} \cup V \\ &\subset U^{\mathsf{c}} \cup V \\ &\stackrel{\text{def}}{=} [U, V]_X,\end{aligned}$$

   where we have used:

   (a) Item 1 of Definition 4.3.11.1.2, which states that if $U \subset A$, then $A^{\mathsf{c}} \subset U^{\mathsf{c}}$.

   (b) Item 1a of Item 1 of Definition 4.3.11.1.2, which states that if $A^{\mathsf{c}} \subset U^{\mathsf{c}}$, then $A^{\mathsf{c}} \cup K \subset U^{\mathsf{c}} \cup K$ for any $K \in \mathcal{P}(X)$.

2. *Proof of Item 1b:* We have

$$\begin{aligned}[U, V]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup V \\ &\subset U^{\mathsf{c}} \cup B \\ &\stackrel{\text{def}}{=} [U, B]_X,\end{aligned}$$

   where we have used Item 1b of Item 1 of Definition 4.3.11.1.2, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

3. *Proof of Item 1c:* We have

$$\begin{aligned}[A, V]_X &\subset [U, V]_X \\ &\subset [U, B]_X,\end{aligned}$$

   where we have used Items 1a and 1b.

This finishes the proof.

Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.3.9.1.2 and is proved there.

Item 3, Interaction With the Empty Set I: We have

$$\begin{aligned}[U, \emptyset]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup \emptyset \\ &= U^{\mathsf{c}},\end{aligned}$$

where we have used Item 3 of Definition 4.3.8.1.2, and we have

$$[\emptyset, V]_X \stackrel{\text{def}}{=} \emptyset^{\mathsf{c}} \cup V$$

$$\begin{aligned}
&\overset{\text{def}}{=} (X \setminus \varnothing) \cup V \\
&= X \cup V \\
&= X,
\end{aligned}$$

where we have used:

1. Item 12 of Definition 4.3.10.1.2 for the first equality.

2. Item 5 of Definition 4.3.8.1.2 for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

Item 4, Interaction With X: We have

$$\begin{aligned}
[U, X]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup X \\
&= X,
\end{aligned}$$

where we have used Item 5 of Definition 4.3.8.1.2, and we have

$$\begin{aligned}
[X, V]_X &\overset{\text{def}}{=} X^{\mathsf{c}} \cup V \\
&\overset{\text{def}}{=} (X \setminus X) \cup V \\
&= \varnothing \cup V \\
&= V,
\end{aligned}$$

where we have used Item 3 of Definition 4.3.8.1.2 for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

Item 5, Interaction With the Empty Set II: We have

$$\begin{aligned}
D_X(D_X(U)) &\overset{\text{def}}{=} [[U, \varnothing]_X, \varnothing]_X \\
&= [U^{\mathsf{c}}, \varnothing]_X \\
&= (U^{\mathsf{c}})^{\mathsf{c}} \\
&= U,
\end{aligned}$$

where we have used:

1. Item 3 for the second and third equalities.

2. Item 3 of Definition 4.3.11.1.2 for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and thus we have

$$[[-, \emptyset]_X, \emptyset]_X \cong \mathrm{id}_{\mathcal{P}(X)}$$

This finishes the proof.
Item 6, Interaction With the Empty Set III: Since $D_X = (-)^{\mathsf{c}}$, this is essentially a repetition of the corresponding results for $(-)^{\mathsf{c}}$, namely Items 5 to 7 of Definition 4.3.11.1.2.
Item 7, Interaction With Unions of Families of Subsets I: By Item 3 of Definition 4.4.7.1.3, we have

$$\begin{aligned}[\mathcal{U}, \emptyset]_{\mathcal{P}(X)} &= \mathcal{U}^{\mathsf{c}},\\ [U, \emptyset]_X &= U^{\mathsf{c}}.\end{aligned}$$

With this, the counterexample given in the proof of Item 10 of Definition 4.3.6.1.2 then applies.
Item 8, Interaction With Unions of Families of Subsets II: We have

$$\begin{aligned}\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X &\stackrel{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U\right)^{\mathsf{c}} \cup V\\ &= \left(\bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}\right) \cup V\\ &= \bigcap_{U \in \mathcal{U}} (U^{\mathsf{c}} \cup V)\\ &\stackrel{\text{def}}{=} \bigcap_{U \in \mathcal{U}} [U, V]_X,\end{aligned}$$

where we have used:

1. Item 11 of Definition 4.3.6.1.2 for the second equality.

2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.
Item 9, Interaction With Unions of Families of Subsets III: We have

$$\bigcup_{V \in \mathcal{V}} [U, V]_X \stackrel{\text{def}}{=} \bigcup_{V \in \mathcal{V}} (U^{\mathsf{c}} \cup V)$$

$$
\begin{aligned}
&= U^{\mathsf{c}} \cup \left(\bigcup_{V \in \mathcal{V}} V\right) \\
&\overset{\text{def}}{=} \left[U, \bigcup_{V \in \mathcal{V}} V\right]_X.
\end{aligned}
$$

where we have used Item 6. This finishes the proof.

Item 10, Interaction With Intersections of Families of Subsets I: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$
\begin{aligned}
\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W &= \bigcap_{W \in \mathcal{P}(X)} W \\
&= \{0, 1\},
\end{aligned}
$$

whereas

$$
\begin{aligned}
\left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X &= [\{0, 1\}, \{0\}] \\
&= \{0\},
\end{aligned}
$$

Thus we have

$$
\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X.
$$

This finishes the proof.

Item 11, Interaction With Intersections of Families of Subsets II: We have

$$
\begin{aligned}
\left[\bigcap_{U \in \mathcal{U}} U, V\right]_X &\overset{\text{def}}{=} \left(\bigcap_{U \in \mathcal{U}} U\right)^{\mathsf{c}} \cup V \\
&= \left(\bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}\right) \cup V \\
&= \bigcup_{U \in \mathcal{U}} (U^{\mathsf{c}} \cup V) \\
&\overset{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X,
\end{aligned}
$$

where we have used:

1. Item 12 of Definition 4.3.6.1.2 for the second equality.
2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.
Item 12, Interaction With Intersections of Families of Subsets III: We have

$$\begin{aligned}\bigcap_{V\in\mathcal{U}}[U,V]_X &\stackrel{\text{def}}{=} \bigcap_{V\in\mathcal{U}}(U^{\mathsf{c}}\cup V)\\ &= U^{\mathsf{c}}\cup\left(\bigcap_{V\in\mathcal{U}}V\right)\\ &\stackrel{\text{def}}{=} \left[U,\bigcap_{V\in\mathcal{U}}V\right]_X.\end{aligned}$$

where we have used Item 6. This finishes the proof.
Item 13, Interaction With Binary Unions: We have

$$\begin{aligned}[U\cap V,W]_X &\stackrel{\text{def}}{=} (U\cap V)^{\mathsf{c}}\cup W\\ &= (U^{\mathsf{c}}\cup V^{\mathsf{c}})\cup W\\ &= (U^{\mathsf{c}}\cup V^{\mathsf{c}})\cup(W\cup W)\\ &= (U^{\mathsf{c}}\cup W)\cup(V^{\mathsf{c}}\cup W)\\ &\stackrel{\text{def}}{=} [U,W]_X\cup[V,W]_X,\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.
2. Item 8 of Definition 4.3.8.1.2 for the third equality.
3. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the fourth equality.

For the second equality in the statement, we have

$$\begin{aligned}[U,V\cap W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}}\cup(V\cap W)\\ &= (U^{\mathsf{c}}\cup V)\cap(U^{\mathsf{c}}\cap W)\\ &\stackrel{\text{def}}{=} [U,V]_X\cap[U,W]_X,\end{aligned}$$

where we have used Item 6 of Definition 4.3.8.1.2 for the second equality.

Item 14, Interaction With Binary Intersections: We have

$$\begin{aligned}[U \cup V, W]_X &\overset{\text{def}}{=} (U \cup V)^{\mathsf{c}} \cup W \\ &= (U^{\mathsf{c}} \cap V^{\mathsf{c}}) \cup W \\ &= (U^{\mathsf{c}} \cup W) \cap (V^{\mathsf{c}} \cup W) \\ &\overset{\text{def}}{=} [U, W]_X \cap [V, W]_X,\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.
2. Item 6 of Definition 4.3.8.1.2 for the third equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}[U, V \cup W]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup (V \cup W) \\ &= (U^{\mathsf{c}} \cup U^{\mathsf{c}}) \cup (V \cup W) \\ &= (U^{\mathsf{c}} \cup V) \cup (U^{\mathsf{c}} \cup W) \\ &\overset{\text{def}}{=} [U, V]_X \cup [U, W]_X,\end{aligned}$$

where we have used:

1. Item 8 of Definition 4.3.8.1.2 for the second equality.
2. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the third equality.

This finishes the proof.

Item 15, Interaction With Differences: We have

$$\begin{aligned}[U \setminus V, W]_X &\overset{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\ &\overset{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\ &= ((X \cap V) \cup (X \setminus U)) \cup W \\ &= (V \cup (X \setminus U)) \cup W \\ &\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\ &= (V \cup (U^{\mathsf{c}} \cup U^{\mathsf{c}})) \cup W \\ &= (U^{\mathsf{c}} \cup W) \cup (U^{\mathsf{c}} \cup V) \\ &\overset{\text{def}}{=} [U, W]_X \cup [U, V]_X,\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.
2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.
3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.
4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

We also have

$$\begin{aligned}
[U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W \\
&\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\
&= (V \cup U^{\mathsf{c}}) \cup (W \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup (V \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup ((V^{\mathsf{c}})^{\mathsf{c}} \cup W) \\
&\stackrel{\text{def}}{=} [U, W]_X \cup [V^{\mathsf{c}}, W]_X,
\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.
2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.
3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.
4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.
5. Item 3 of Definition 4.3.11.1.2 for the eighth equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
[U, V \setminus W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup (V \setminus W) \\
&= (V \setminus W) \cup U^{\mathsf{c}} \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \setminus U^{\mathsf{c}})
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{def}}{=} (V \cup U^{\mathsf{c}}) \setminus (W \setminus (X \setminus U)) \\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup (W \setminus X)) \\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup \emptyset) \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \cap U) \\
&= (V \cup U^{\mathsf{c}}) \setminus (U \cap W) \\
&\overset{\text{def}}{=} [U, V]_X \setminus (U \cap W)
\end{aligned}
$$

where we have used:

1. Item 4 of Definition 4.3.8.1.2 for the second equality.
2. Item 4 of Definition 4.3.10.1.2 for the third equality.
3. Item 10 of Definition 4.3.10.1.2 for the fifth equality.
4. Item 13 of Definition 4.3.10.1.2 for the sixth equality.
5. Item 3 of Definition 4.3.8.1.2 for the seventh equality.
6. Item 5 of Definition 4.3.9.1.2 for the eighth equality.

This finishes the proof.
Item 16, Interaction With Complements: We have

$$
\begin{aligned}
[U^{\mathsf{c}}, V]_X &\overset{\text{def}}{=} (U^{\mathsf{c}})^{\mathsf{c}} \cup V, \\
&= U \cup V,
\end{aligned}
$$

where we have used Item 3 of Definition 4.3.11.1.2. We also have

$$
\begin{aligned}
[U, V^{\mathsf{c}}]_X &\overset{\text{def}}{=} U^{\mathsf{c}} \cup V^{\mathsf{c}} \\
&= U \cap V
\end{aligned}
$$

where we have used Item 2 of Definition 4.3.11.1.2. Finally, we have

$$
\begin{aligned}
[U, V]_X^{\mathsf{c}} &= ((U \setminus V)^{\mathsf{c}})^{\mathsf{c}} \\
&= U \setminus V,
\end{aligned}
$$

where we have used Item 2 of Definition 4.3.11.1.2.

Item 17, Interaction With Characteristic Functions: We have

$$\begin{aligned}\chi_{[U,V]_{\mathcal{P}(X)}}(x) &\stackrel{\text{def}}{=} \chi_{U^{\mathsf{c}}\cup V}(x)\\ &= \max(\chi_{U^{\mathsf{c}}}, \chi_V)\\ &= \max(1-\chi_U \pmod 2, \chi_V),\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.8.1.2 for the second equality.
2. Item 4 of Definition 4.3.11.1.2 for the third equality.

This finishes the proof.
Item 18, Interaction With Direct Images: This is a repetition of Item 10 of Definition 4.6.1.1.5 and is proved there.
Item 19, Interaction With Inverse Images: This is a repetition of Item 10 of Definition 4.6.2.1.3 and is proved there.
Item 20, Interaction With Codirect Images: This is a repetition of Item 9 of Definition 4.6.3.1.7 and is proved there. □

4.4.8 Isbell Duality for Sets

Let X be a set.

Definition 4.4.8.1.1. The **Isbell function** of X is the map

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

defined by

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [\![x \mapsto [U, \{x\}]_X]\!]$$

for each $U \in \mathcal{P}(X)$.

Remark 4.4.8.1.2. Recall from Definition 4.4.1.1.2 that we may view the powerset $\mathcal{P}(X)$ of a set X as the decategorification of the category of presheaves $\mathsf{PSh}(\mathcal{C})$ of a category $\mathcal{C}$. Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\mathsf{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \mathrm{Nat}(\mathcal{F}, h_{(-)})$$

for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$. To this end, we could define

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [U, \chi_{(-)}]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of C into $\mathsf{PSh}(C)$ with the characteristic embedding $x \mapsto \chi_x$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1.
- The internal Hom Nat of $\mathsf{PSh}(C)$ with the internal Hom $[-,-]_X$ of $\mathcal{P}(X)$ of Definition 4.4.7.1.1.

However, since $[U, \chi_x]_X$ is a subset of U instead of a truth value, we get a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathcal{P}(X).$$

This makes some of the properties involving I a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $\mathsf{I}_! \circ \mathsf{I}$ evaluates to $\mathrm{id}_{\mathcal{P}(X)}$ in the sense of Definition 4.4.8.1.3.

Proposition 4.4.8.1.3. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\mathsf{I}"] \arrow[dr, "\Delta_{\Delta_{\mathrm{id}_{\mathcal{P}(X)}}}"'] & \mathsf{Sets}(X, \mathcal{P}(X)) \arrow[d, "\mathsf{I}_!"] \\
& \mathsf{Sets}(X, \mathsf{Sets}(X, \mathcal{P}(X)))
\end{tikzcd}$$

commutes, i.e. we have

$$\mathsf{I}_!(\mathsf{I}(U)) = [\![x \mapsto [\![y \mapsto U]\!]]\!]$$

for each $U \in \mathcal{P}(X)$.

Proof. We have

$$\begin{aligned}
\mathsf{I}_!(\mathsf{I}(U)) &\overset{\text{def}}{=} \mathsf{I}_!([\![x \mapsto U^{\mathsf{c}} \cup \{x\}]\!]) \\
&\overset{\text{def}}{=} [\![x \mapsto \mathsf{I}(U^{\mathsf{c}} \cup \{x\})]\!] \\
&\overset{\text{def}}{=} [\![x \mapsto [\![y \mapsto (U^{\mathsf{c}} \cup \{x\})^{\mathsf{c}} \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto (U \setminus \{x\}) \cup \{x\}]\!]]\!] \\
&= [\![x \mapsto [\![y \mapsto U]\!]]\!],
\end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2 for the fourth equality above. □

4.5 Characteristic Functions

4.5.1 The Characteristic Function of a Subset

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.5.1.1.1. The **characteristic function of** U[23] is the function $\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$[24] defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

Remark 4.5.1.1.2. Under the analogy that $\{\mathsf{t}, \mathsf{f}\}$ should be the (-1)-categorical analogue of Sets, we may view a function

$$f \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

as a decategorification of presheaves and copresheaves

$$\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
$$F \colon \mathcal{C} \to \mathsf{Sets}.$$

The characteristic functions χ_U of the subsets of X are then the primordial examples of such functions (and, in fact, all of them).

Notation 4.5.1.1.3. We will often employ the bijection $\{\mathsf{t}, \mathsf{f}\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when disucssing characteristic functions.

Examples of this include Items 4 to 11 of Definition 4.5.1.1.4 below.

Proposition 4.5.1.1.4. Let X be a set.

1. *Functionality.* The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)} \colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}).$$

2. *Bijectivity.* The function $\chi_{(-)}$ from Item 1 is bijective.

[23] *Further Terminology:* Also called the **indicator function of** U.
[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

3. *Naturality.* The collection

$$\left\{\chi_{(-)} \colon \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})\right\}_{X \in \mathrm{Obj}(\mathsf{Sets})}$$

defines a natural isomorphism between $\mathcal{P}^{-1}$ and $\mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\})$. In particular, given a function $f \colon X \to Y$, the diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\
\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^*"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})
\end{tikzcd}$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \vartriangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \vartriangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Internal Homs.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functionality*: There is nothing to prove.
Item 2, *Bijectivity*: We proceed in three steps:

1. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi \colon \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \overset{\sim}{\dashrightarrow} \mathcal{P}(X),$$

defined by

$$\begin{aligned}\Phi(f) &\overset{\text{def}}{=} U_f \\ &\overset{\text{def}}{=} f^{-1}(\mathsf{true}) \\ &\overset{\text{def}}{=} \left\{x \in X \,\middle|\, f(x) = \mathsf{true}\right\}\end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$.

2. *Invertibility I.* We have

$$\begin{aligned}[\Phi\circ\chi_{(-)}](U) &\overset{\text{def}}{=} \Phi(\chi_U)\\ &\overset{\text{def}}{=} \chi_U^{-1}(\mathsf{true})\\ &\overset{\text{def}}{=} \left\{x\in X\,\middle|\,\chi_U(x)=\mathsf{true}\right\}\\ &\overset{\text{def}}{=} \{x\in X\mid x\in U\}\\ &= U\\ &\overset{\text{def}}{=} [\mathrm{id}_{\mathcal{P}(X)}](U)\end{aligned}$$

for each $U\in\mathcal{P}(X)$. Thus, we have

$$\Phi\circ\chi_{(-)}=\mathrm{id}_{\mathcal{P}(X)}.$$

3. *Invertibility II.* We have

$$\begin{aligned}[\chi_{(-)}\circ\Phi](U) &\overset{\text{def}}{=} \chi_{\Phi(f)}\\ &\overset{\text{def}}{=} \chi_{f^{-1}(\mathsf{true})}\\ &\overset{\text{def}}{=} \left[\!\left[x\mapsto\begin{cases}\mathsf{true} & \text{if } x\in f^{-1}(\mathsf{true})\\ \mathsf{false} & \text{otherwise}\end{cases}\right]\!\right]\\ &= [\![x\mapsto f(x)]\!]\\ &= f\\ &\overset{\text{def}}{=} [\mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})}](f)\end{aligned}$$

for each $f\in\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})$. Thus, we have

$$\chi_{(-)}\circ\Phi=\mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})}.$$

This finishes the proof.
Item 3, *Naturality*: We proceed in two steps:

1. *Naturality of $\chi_{(-)}$.* We have

$$\begin{aligned}[\chi_V\circ f](v) &\overset{\text{def}}{=} \chi_V(f(v))\\ &= \begin{cases}\mathsf{true} & \text{if } f(v)\in V,\\ \mathsf{false} & \text{otherwise}\end{cases}\end{aligned}$$

$$\begin{aligned} &= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v) \end{aligned}$$

for each $v \in V$.

2. *Naturality of* Φ. Since $\chi_{(-)}$ is natural and a componentwise inverse to Φ, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Φ is also natural in each argument.

This finishes the proof.

Item 4, Interaction With Unions I: This is a repetition of Item 10 of Definition 4.3.8.1.2 and is proved there.

Item 5, Interaction With Unions II: This is a repetition of Item 11 of Definition 4.3.8.1.2 and is proved there.

Item 6, Interaction With Intersections I: This is a repetition of Item 10 of Definition 4.3.9.1.2 and is proved there.

Item 7, Interaction With Intersections II: This is a repetition of Item 11 of Definition 4.3.9.1.2 and is proved there.

Item 8, Interaction With Differences: This is a repetition of Item 16 of Definition 4.3.10.1.2 and is proved there.

Item 9, Interaction With Complements: This is a repetition of Item 4 of Definition 4.3.11.1.2 and is proved there.

Item 10, Interaction With Symmetric Differences: This is a repetition of Item 15 of Definition 4.3.12.1.2 and is proved there.

Item 11, Interaction With Internal Homs: This is a repetition of Item 17 of Definition 4.4.7.1.3 and is proved there. □

Remark 4.5.1.1.5. The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 2 of Definition 4.5.1.1.4, which

- Takes a subset $U \to X$ of X and *straightens* it to a function $\chi_U : X \to \{\mathsf{true}, \mathsf{false}\}$;
- Takes a function $f : X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \to X$ of X;

may be viewed as the (-1)-categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\mathsf{FibSets}_X}_{\stackrel{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}_X}_{\stackrel{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}},\mathsf{Sets})}$$

of Un/Straightening for Indexed and Fibred Sets, **??**. Here we view:

- Subsets $U \hookrightarrow X$ as being analogous to X-fibred sets $\phi_X \colon A \to X$.
- Functions $f \colon X \to \{\mathsf{t},\mathsf{f}\}$ as being analogous to X-indexed sets $A \colon X_{\mathsf{disc}} \to \mathsf{Sets}$.

4.5.2 The Characteristic Function of a Point

Let X be a set and let $x \in X$.

Definition 4.5.2.1.1. The **characteristic function of** x is the function[25]

$$\chi_x \colon X \to \{\mathsf{t},\mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

Remark 4.5.2.1.2. Expanding upon Definition 4.5.1.1.2, we may think of the characteristic function

$$\chi_x \colon X \to \{\mathsf{t},\mathsf{f}\}$$

of an *element* x of X as a decategorification of the representable presheaf and of the representable copresheaf

$$h_X \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
$$h^X \colon \mathcal{C} \to \mathsf{Sets}$$

associated of an *object* X of a category $\mathcal{C}$.

[25] *Further Notation:* Also written χ^x, $\chi_X(x,-)$, or $\chi_X(-,x)$.

4.5.3 The Characteristic Relation of a Set

Let X be a set.

Definition 4.5.3.1.1. The **characteristic relation on** X[26] is the relation[27]

$$\chi_X(-_1, -_2)\colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

Remark 4.5.3.1.2. Expanding upon Definitions 4.5.1.1.2 and 4.5.2.1.2, we may view the characteristic relation

$$\chi_X(-_1, -_2)\colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

of X as a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

Proposition 4.5.3.1.3. Let $f\colon X \to Y$ be a function.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f\colon A \to B$ be a function. We have an inclusion[29]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f \times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t}, \mathsf{f}\}. & \end{tikzcd}$$

Proof. *Item 1, The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. $\square$

[26] *Further Terminology:* Also called the **identity relation on** X.

[27] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\mathrm{id}}$ in the context of relations.

[28] Under the bijection $\mathsf{Sets}(X \times X, \{\mathsf{t}, \mathsf{f}\}) \cong \mathcal{P}(X \times X)$ of Item 2 of Definition 4.5.1.1.4, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

[29] *Note:* This is the 0-categorical version of Categories, Definition 11.5.4.1.1.

4.5.4 The Characteristic Embedding of a Set

Let X be a set.

Definition 4.5.4.1.1. The **characteristic embedding**[30] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)}\colon X \to \mathcal{P}(X)$$

defined by[31]

$$\begin{aligned}\chi_{(-)}(x) &\overset{\text{def}}{=} \chi_x\\ &= \{x\}\end{aligned}$$

for each $x \in X$.

Remark 4.5.4.1.2. Expanding upon Definitions 4.5.1.1.2, 4.5.2.1.2 and 4.5.3.1.2, we may view the characteristic embedding

$$\chi_{(-)}\colon X \to \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$よ\colon C^{\mathsf{op}} \to \mathsf{PSh}(C)$$

of a category C into $\mathsf{PSh}(C)$.

Proposition 4.5.4.1.3. Let $f\colon X \to Y$ be a map of sets.

1. *Interaction With Functions.* We have

$$f_! \circ \chi_X = \chi_Y \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & Y \arrow[d, "\chi_Y"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y). \end{tikzcd}$$

[30]The name "characteristic *embedding*" is justified by Definition 4.5.5.1.2, which gives an analogue of fully faithfulness for $\chi_{(-)}$.

[31]Here we are identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ as per Item 2 of Definition 4.5.1.1.4.

Proof. *Item 1*, *Interaction With Functions*: Indeed, we have

$$\begin{aligned}[f_! \circ \chi_X](x) &\stackrel{\text{def}}{=} f_!(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f_!(\{x\}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x),\end{aligned}$$

for each $x \in X$, showing the desired equality. □

4.5.5 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 4.5.5.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}(\chi_{(-)}, \chi_U) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } U \subset V, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned}\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) &\stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } \{x\} \subset U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } x \in U \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_U(x).\end{aligned}$$

This finishes the proof. □

Corollary 4.5.5.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) \cong \chi_X(x, y)$$

for each $x, y \in X$.

Proof. We have

$$
\begin{aligned}
\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) &= \chi_y(x) \\
&\overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in \{y\} \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if } x = y \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&\overset{\text{def}}{=} \chi_X(x, y).
\end{aligned}
$$

where we have used Definition 4.5.5.1.1 for the first equality. □

4.6 The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

4.6.1 Direct Images

Let $f : X \to Y$ be a function.

Definition 4.6.1.1.1. The **direct image function associated to** f is the function[32]

$$f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[33]

$$
\begin{aligned}
f_!(U) &\overset{\text{def}}{=} \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\} \\
&= \{ f(x) \in Y \mid x \in U \}
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.1.1.2. Sometimes one finds the notation

$$\exists_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

[32] *Further Notation:* Also written simply $f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$.

[33] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

- We have $y \in \exists_f(U)$.
- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.1.1.3. Notation for direct images between powersets is tricky:

1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:
   (a) *Powersets.* Given a function $f : X \to Y$, we have an inverse image functor
   $$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X).$$
   The *left* adjoint of this functor is the usual direct image, defined above in Definition 4.6.1.1.1.
   (b) *Presheaves.* Given a morphism of topological spaces $f : X \to Y$, we have an inverse image functor
   $$f^{-1} : \mathsf{PSh}(Y) \to \mathsf{PSh}(X).$$
   The *right* adjoint of this functor is the direct image functor of presheaves, defined in **??**.
2. The presheaf direct image functor is denoted f_*, but the direct image functor for powersets is denoted $f_!$ (as it's a left adjoint).
3. Adding to the confusion, it's somewhat common for $f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$ to be denoted f_*.

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

Situation	Adjoint String
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*) \colon \mathcal{P}(X) \overset{\rightleftarrows}{\to} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*) \colon \mathsf{PSh}(X) \overset{\rightleftarrows}{\to} \mathsf{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*) \colon \mathcal{C}_{/X} \overset{\rightleftarrows}{\to} \mathcal{C}_{/Y}$
Kan Extensions	$(F_! \dashv F^* \dashv F_*) \colon \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \overset{\rightleftarrows}{\to} \mathsf{Fun}(\mathcal{D}, \mathcal{E})$

Remark 4.6.1.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the direct image function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\overset{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \operatorname{colim}\left(\left(\underline{f \overset{\rightarrow}{\times} (-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\ &= \operatorname*{colim}_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)) \\ &= \bigvee_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)), \end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned} [f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $y \in Y$.

Proposition 4.6.1.1.5. Let $f : X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

   ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{P}(Y) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

   witnessed by:

   (a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\to f^{-1} \circ f_!, & \mathrm{id}_{\mathcal{P}(Y)} &\to f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\to \mathrm{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\to \mathrm{id}_{\mathcal{P}(X)}, \end{aligned}$$

   having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

   (b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

   natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

   i. The following conditions are equivalent:
      A. We have $f_!(U) \subset V$.

B. We have $U \subset f^{-1}(V)$.

ii. The following conditions are equivalent:

A. We have $f^{-1}(U) \subset V$.

B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cup"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cap"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

7. *Interaction With Differences.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \mathbin{\triangle} f_!(V) \subset f_!(U \mathbin{\triangle} V)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_!"] \arrow[d, "{[-_1,-_2]_X}"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_!(U) \cup f_!(V) &= f_!(U \cup V), \\ f_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U \cap V) &\subset f_!(U) \cap f_!(V), \\ f_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f_{!|U,V}^{\otimes} &\colon f_!(U) \cup f_!(V) \xrightarrow{=} f_!(U \cup V), \\ f_{!|\mathbb{1}}^{\otimes} &\colon \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} f_{!|U,V}^{\otimes} &\colon f_!(U \cap V) \to f_!(U) \cap f_!(V), \\ f_{!|\mathbb{1}}^{\otimes} &\colon f_!(X) \to Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

15. *Interaction With Coproducts.* Let $f: X \to X'$ and $g: Y \to Y'$ be maps of sets. We have

$$(f \coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Interaction With Products.* Let $f: X \to X'$ and $g: Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

17. *Relation to Codirect Images.* We have

$$\begin{aligned} f_!(U) &= f_*(U^{\mathsf{c}})^{\mathsf{c}} \\ &\overset{\text{def}}{=} Y \setminus f_*(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
Item 3, *Interaction With Unions of Families of Subsets*: We have

$$\begin{aligned} \bigcup_{V \in f_!(\mathcal{U})} V &= \bigcup_{V \in \left\{ f_!(U) \in \mathcal{P}(X) \,\middle|\, U \in \mathcal{U} \right\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.
Item 4, *Interaction With Intersections of Families of Subsets*: We have

$$\begin{aligned} \bigcap_{V \in f_!(\mathcal{U})} V &= \bigcap_{V \in \left\{ f_!(U) \in \mathcal{P}(X) \,\middle|\, U \in \mathcal{U} \right\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_!(U). \end{aligned}$$

This finishes the proof.
Item 5, Interaction With Binary Unions: See [Pro25p].
Item 6, Interaction With Binary Intersections: See [Pro25n].
Item 7, Interaction With Differences: See [Pro25o].
Item 8, Interaction With Complements: Applying Item 17 to $X \setminus U$, we have

$$\begin{aligned} f_!(U^{\mathsf{c}}) &= f_!(X \setminus U) \\ &= Y \setminus f_*(X \setminus (X \setminus U)) \\ &= Y \setminus f_*(U) \\ &= f_*(U)^{\mathsf{c}}. \end{aligned}$$

This finishes the proof.
Item 9, Interaction With Symmetric Differences: We have

$$\begin{aligned} f_!(U) \triangle f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\ &\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\ &= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\ &\subset f_!((U \cup V) \setminus (U \cap V)) \\ &= f_!(U \triangle V), \end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.
2. Item 6 of this proposition together with Item 1 of Definition 4.3.10.1.2 for the first inclusion.
3. Item 5 for the second equality.
4. Item 7 for the second inclusion.
5. Item 2 of Definition 4.3.12.1.2 for the tchird equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
Item 10, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned} f_!([U, V]_X) &\overset{\text{def}}{=} f_!(U^{\mathsf{c}} \cup V) \\ &= f_!(U^{\mathsf{c}}) \cup f_!(V) \end{aligned}$$

$$
\begin{aligned}
&= f_*(U)^{\mathsf{c}} \cup f_!(V) \\
&\overset{\text{def}}{=} [f_*(U), f_!(V)]_Y,
\end{aligned}
$$

where we have used:

1. Item 5 for the second equality.

2. Item 17 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.

Item 11, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.[34]

Item 12, Oplax Preservation of Limits: The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25n] for the other inclusions.

Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.

Item 14, Symmetric Oplax Monoidality With Respect to Intersections: The inclusions in the statement follow from Item 12. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (Categories, Item 4 of Definition 11.2.7.1.2).

Item 15, Interaction With Coproducts: Omitted.

Item 16, Interaction With Products: Omitted.

Item 17, Relation to Codirect Images: Applying Item 16 of Definition 4.6.3.1.7 to $X \setminus U$, we have

$$
\begin{aligned}
f_*(X \setminus U) &= B \setminus f_!(X \setminus (X \setminus U)) \\
&= B \setminus f_!(U).
\end{aligned}
$$

Taking complements, we then obtain

$$
\begin{aligned}
f_!(U) &= B \setminus (B \setminus f_!(U)), \\
&= B \setminus f_*(X \setminus U),
\end{aligned}
$$

which finishes the proof. □

Proposition 4.6.1.1.6. Let $f\colon X \to Y$ be a function.

[34] *Reference:* [Pro25p].

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon X \to Y$ and $g \colon Y \to Z$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(Y) \arrow[d, "g_!"] \\ & \mathcal{P}(Z). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.1.1.5.
Item 3, Interaction With Identities: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**.
Item 4, Interaction With Composition: This follows from Definition 4.6.1.1.4 and Kan Extensions, **??** of **??**. □

4.6.2 Inverse Images

Let $f \colon X \to Y$ be a function.

Definition 4.6.2.1.1. The **inverse image function associated to** f is the function[35]

$$f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

[35] *Further Notation:* Also written $f^* \colon \mathcal{P}(Y) \to \mathcal{P}(X)$.

defined by[36]

$$f^{-1}(V) \overset{\text{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 4.6.2.1.2. Identifying $\mathcal{P}(Y)$ with $\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the inverse image function associated to f is equivalently the function

$$f^{*} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by

$$f^{*}(\chi_V) \overset{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 4.6.2.1.3. Let $f \colon X \to Y$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} \colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

   (★) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[rr, bend left=45, "f_!"] \arrow[rr, bend right=45, "f_*"'] & & \mathcal{P}(Y) \arrow[ll, "f^{-1}" description] \end{tikzcd},$$

   witnessed by:

[36] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of V by f**.

(a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\to f^{-1} \circ f_!, & \mathrm{id}_{\mathcal{P}(Y)} &\to f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\to \mathrm{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\to \mathrm{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*(f^{-1}(V)), \\ f_!(f^{-1}(V)) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f^{-1}(V)), \\ \mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

i. The following conditions are equivalent:
   A. We have $f_!(U) \subset V$.
   B. We have $U \subset f^{-1}(V)$.

ii. The following conditions are equivalent:
   A. We have $f^{-1}(U) \subset V$.
   B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \stackrel{\mathrm{def}}{=} (f^{-1})^{-1}(\mathcal{V})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{U}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{U} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{U}) \stackrel{\text{def}}{=} (f^{-1})^{-1}(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

6. *Interaction With Binary Intersections.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Differences.* The diagram

$$\begin{tikzcd} \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\setminus"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\setminus"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$\begin{tikzcd} \mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd} \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\triangle"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\triangle"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

i.e. we have

$$f^{-1}(U) \triangle f^{-1}(V) = f^{-1}(U \triangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V), \\ f^{-1}(\emptyset) = \emptyset, \end{gathered}$$

natural in $U, V \in \mathcal{P}(Y)$.

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V), \\ f^{-1}(Y) = X, \end{gathered}$$

natural in $U, V \in \mathcal{P}(Y)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}) \colon (\mathcal{P}(Y), \cup, \varnothing) \to (\mathcal{P}(X), \cup, \varnothing),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon \varnothing \xrightarrow{=} f^{-1}(\varnothing),$$

natural in $U, V \in \mathcal{P}(Y)$.

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}) \colon (\mathcal{P}(Y), \cap, Y) \to (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$f_{U,V}^{-1,\otimes} \colon f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V),$$
$$f_{\mathbb{1}}^{-1,\otimes} \colon X \xrightarrow{=} f^{-1}(Y),$$

natural in $U, V \in \mathcal{P}(Y)$.

15. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \textstyle\coprod g)^{-1}(U' \textstyle\coprod V') = f^{-1}(U') \textstyle\coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

16. *Interaction With Products.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

Proof. *Item 1*, *Functoriality*: Omitted.

Item 2, Triple Adjointness: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
Item 3, Interaction With Unions of Families of Subsets: We have

$$\begin{aligned}\bigcup_{U\in f^{-1}(\mathcal{V})} U &= \bigcup_{U\in\{f^{-1}(V)\in\mathcal{P}(X)\mid V\in\mathcal{V}\}} U\\ &= \bigcup_{V\in\mathcal{V}} f^{-1}(V).\end{aligned}$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned}\bigcap_{U\in f^{-1}(\mathcal{V})} U &= \bigcap_{U\in\{f^{-1}(V)\in\mathcal{P}(X)\mid V\in\mathcal{V}\}} U\\ &= \bigcap_{V\in\mathcal{V}} f^{-1}(V).\end{aligned}$$

This finishes the proof.
Item 5, Interaction With Binary Unions: See [Pro25y].
Item 6, Interaction With Binary Intersections: See [Pro25w].
Item 7, Interaction With Differences: See [Pro25x].
Item 8, Interaction With Complements: See [Pro25j].
Item 9, Interaction With Symmetric Differences: We have

$$\begin{aligned}f^{-1}(U \bigtriangleup V) &= f^{-1}((U\cup V)\setminus(U\cap V))\\ &= f^{-1}(U\cup V)\setminus f^{-1}(U\cap V)\\ &= f^{-1}(U)\cup f^{-1}(V)\setminus f^{-1}(U\cap V)\\ &= f^{-1}(U)\cup f^{-1}(V)\setminus f^{-1}(U)\cap f^{-1}(V)\\ &= f^{-1}(U)\bigtriangleup f^{-1}(V),\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.
2. Item 7 for the second equality.
3. Item 5 for the third equality.

4. Item 6 for the fourth equality.

5. Item 2 of Definition 4.3.12.1.2 for the fifth equality.

This finishes the proof.
Item 10, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned} f^{-1}([U,V]_Y) &\stackrel{\text{def}}{=} f^{-1}(U^{\mathsf{c}} \cup V) \\ &= f^{-1}(U^{\mathsf{c}}) \cup f^{-1}(V) \\ &= f^{-1}(U)^{\mathsf{c}} \cup f^{-1}(V) \\ &\stackrel{\text{def}}{=} [f^{-1}(U), f^{-1}(V)]_X, \end{aligned}$$

where we have used:

1. Item 8 for the second equality.

2. Item 5 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
Item 11, Preservation of Colimits: This follows from Item 2 and **??**, **??** of **??**.[37]
Item 12, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.[38]
Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.
Item 14, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 12.
Item 15, Interaction With Coproducts: Omitted.
Item 16, Interaction With Products: Omitted. □

Proposition 4.6.2.1.4. Let $f : X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{X,Y} : \mathsf{Sets}(X,Y) \to \mathsf{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{X,Y} : \mathsf{Sets}(X,Y) \to \mathsf{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

[37] *Reference:* [Pro25y].
[38] *Reference:* [Pro25w].

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_X^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f\colon X \to Y$ and $g\colon Y \to Z$, we have
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(Z) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(Y) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.2.1.3.
Item 3, Interaction With Identities: This follows from Definition 4.6.2.1.2 and Categories, Item 5 of Definition 11.1.4.1.2.
Item 4, Interaction With Composition: This follows from Definition 4.6.2.1.2 and Categories, Item 2 of Definition 11.1.4.1.2. □

4.6.3 Codirect Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.3.1.1. The **codirect image function associated to f** is the function
$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$
defined by[39,40]
$$f_*(U) \overset{\text{def}}{=} \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{for each } x \in X, \text{ if we have} \\ f(x) = y, \text{ then } x \in U \end{array} \right\}$$

[39] *Further Terminology:* The set $f_*(U)$ is called the **codirect image of U by f**.
[40] We also have
$$\begin{aligned} f_*(U) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \\ &\overset{\text{def}}{=} Y \setminus f_!(X \setminus U); \end{aligned}$$
see Item 16 of Definition 4.6.3.1.7.

$$= \left\{ y \in Y \,\middle|\, \text{we have } f^{-1}(y) \subset U \right\}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.3.1.2. Sometimes one finds the notation

$$\forall_f \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.
- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.3.1.3. See Definition 4.6.1.1.3.

Remark 4.6.3.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the codirect image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_*(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\ &= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} X \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\ &= \lim_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)) \\ &= \bigwedge_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)). \end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$[f_*(\chi_U)](y) = \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x))$$

$$
\begin{aligned}
&= \begin{cases} \mathsf{true} & \text{if, for each } x \in X \text{ such that} \\ & f(x) = y, \text{ we have } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if } f^{-1}(y) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $y \in Y$.

Definition 4.6.3.1.5. Let U be a subset of X.[41,42]

1. The **image part of the codirect image $f_*(U)$ of U** is the set $f_{*,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{*,\mathrm{im}}(U) &\overset{\mathrm{def}}{=} f_*(U) \cap \mathrm{Im}(f) \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{array} \right\}.
\end{aligned}
$$

2. The **complement part of the codirect image $f_*(U)$ of U** is the set $f_{*,\mathrm{cp}}(U)$ defined by

$$
f_{*,\mathrm{cp}}(U) \overset{\mathrm{def}}{=} f_*(U) \cap (Y \setminus \mathrm{Im}(f))
$$

[41] Note that we have

$$
f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U),
$$

as

$$
\begin{aligned}
f_*(U) &= f_*(U) \cap Y \\
&= f_*(U) \cap (\mathrm{Im}(f) \cup (Y \setminus \mathrm{Im}(f))) \\
&= (f_*(U) \cap \mathrm{Im}(f)) \cup (f_*(U) \cap (Y \setminus \mathrm{Im}(f))) \\
&\overset{\mathrm{def}}{=} f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U).
\end{aligned}
$$

[42] In terms of the meet computation of $f_*(U)$ of Definition 4.6.3.1.4, namely

$$
f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -1}} (\chi_U(x)),
$$

we see that $f_{*,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

$$
\begin{aligned}
&= Y \setminus \mathrm{Im}(f) \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; f^{-1}(y) = \emptyset \right\}.
\end{aligned}
$$

Example 4.6.3.1.6. Here are some examples of codirect images.

1. *Multiplication by Two.* Consider the function $f \colon \mathbb{N} \to \mathbb{N}$ given by

   $$f(n) \stackrel{\text{def}}{=} 2n$$

   for each $n \in \mathbb{N}$. Since f is injective, we have

   $$
   \begin{aligned}
   f_{*,\mathrm{im}}(U) &= f_!(U) \\
   f_{*,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\}
   \end{aligned}
   $$

   for any $U \subset \mathbb{N}$. In particular, we have

   $$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

2. *Parabolas.* Consider the function $f \colon \mathbb{R} \to \mathbb{R}$ given by

   $$f(x) \stackrel{\text{def}}{=} x^2$$

   for each $x \in \mathbb{R}$. We have

   $$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

   for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

   $$
   \begin{aligned}
   f_{*,\mathrm{im}}([0,1]) &= \{0\}, \\
   f_{*,\mathrm{im}}([-1,1]) &= [0,1], \\
   f_{*,\mathrm{im}}([1,2]) &= \emptyset, \\
   f_{*,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4].
   \end{aligned}
   $$

3. *Circles.* Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(x,y) \stackrel{\text{def}}{=} x^2 + y^2$$

for each $(x, y) \in \mathbb{R}^2$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$f_{*,\mathrm{im}}([-1,1] \times [-1,1]) = [0,1],$$
$$f_{*,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) = \emptyset.$$

Proposition 4.6.3.1.7. Let $f : X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_* \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

   (★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left=50, "f_!"] \arrow[r, bend left=25, phantom, "\perp"] \arrow[r, bend right=25, phantom, "\perp"] \arrow[r, bend right=50, "f_*"'] & \mathcal{P}(Y) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

   witnessed by:

   (a) Units and counits of the form

$$\mathrm{id}_{\mathcal{P}(X)} \to f^{-1} \circ f_!, \qquad \mathrm{id}_{\mathcal{P}(Y)} \to f_* \circ f^{-1},$$
$$f_! \circ f^{-1} \to \mathrm{id}_{\mathcal{P}(Y)}, \qquad f^{-1} \circ f_* \to \mathrm{id}_{\mathcal{P}(X)},$$

   having components of the form

$$U \subset f^{-1}(f_!(U)), \qquad V \subset f_*(f^{-1}(V)),$$
$$f_!(f^{-1}(V)) \subset V, \qquad f^{-1}(f_*(U)) \subset U$$

   indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U),V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U,f^{-1}(V)),\\ \mathrm{Hom}_{\mathcal{P}(X)}(f^{-1}(U),V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U,f_*(V)),\end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

i. The following conditions are equivalent:
   A. We have $f_!(U) \subset V$.
   B. We have $U \subset f^{-1}(V)$.

ii. The following conditions are equivalent:
   A. We have $f^{-1}(U) \subset V$.
   B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd}\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U\in\mathcal{U}} f_*(U) = \bigcup_{V\in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd}\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

5. *Interaction With Binary Unions.* Let $f : X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cup"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cap"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_*(U^\mathsf{c}) = f_!(U)^\mathsf{c}$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_*"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U \triangle V) \subset f_*(U) \triangle f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_*"] \arrow[d, "{[-_1,-_2]_X}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} f_*(U) \cup f_*(V) \to f_*(U \cup V), \\ \emptyset \to f_*(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Limits.* We have an equality of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{gathered} f^{-1}(U \cap V) = f_*(U) \cap f^{-1}(V), \\ f_*(X) = Y, \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cup, \emptyset) \longrightarrow (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{gathered} f_{*|U,V}^{\otimes} \colon f_*(U) \cup f_*(V) \to f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} \colon \emptyset \to f_*(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cap, X) \longrightarrow (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{gathered} f_{*|U,V}^{\otimes} \colon f_*(U \cap V) \xrightarrow{=} f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} \colon f_*(X) \xrightarrow{=} Y, \end{gathered}$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Interaction With Coproducts.* Let $f: X \to X'$ and $g: Y \to Y'$ be maps of sets. We have

$$(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

15. *Interaction With Products.* Let $f: X \to X'$ and $g: Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_*(U \boxtimes_{X \times Y} V) = f_*(U) \boxtimes_{X' \times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Relation to Direct Images.* We have

$$\begin{aligned} f_*(U) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \\ &= Y \setminus f_!(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

17. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &= f_!(U), \\ f_{*,\mathrm{cp}}(U) &= Y \setminus \mathrm{Im}(f), \end{aligned}$$

and so

$$\begin{aligned} f_*(U) &= f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U) \\ &= f_!(U) \cup (Y \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

18. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &\subset f_!(U), \\ f_{*,\mathrm{cp}}(U) &= \emptyset, \end{aligned}$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Triple Adjointness: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and Kan Extensions, **??** of **??**.
Item 3, Interaction With Unions of Families of Subsets: We have

$$\begin{aligned}\bigcup_{V\in f_*(\mathcal{U})} V &= \bigcup_{V\in\{f_*(U)\in\mathcal{P}(X)\mid U\in\mathcal{U}\}} V\\ &= \bigcup_{U\in\mathcal{U}} f_*(U).\end{aligned}$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned}\bigcap_{V\in f_*(\mathcal{U})} V &= \bigcap_{V\in\{f_*(U)\in\mathcal{P}(X)\mid U\in\mathcal{U}\}} V\\ &= \bigcap_{U\in\mathcal{U}} f_*(U).\end{aligned}$$

This finishes the proof.
Item 5, Interaction With Binary Unions: We have

$$\begin{aligned}f_*(U)\cup f_*(V) &= f_!(U^{\mathsf{c}})^{\mathsf{c}}\cup f_!(V^{\mathsf{c}})^{\mathsf{c}}\\ &= (f_!(U^{\mathsf{c}})\cap f_!(V^{\mathsf{c}}))^{\mathsf{c}}\\ &\subset (f_!(U^{\mathsf{c}}\cap V^{\mathsf{c}}))^{\mathsf{c}}\\ &= f_!((U\cup V)^{\mathsf{c}})^{\mathsf{c}}\\ &= f_*(U\cup V),\end{aligned}$$

where:

1. We have used Item 16 for the first equality.
2. We have used Item 2 of Definition 4.3.11.1.2 for the second equality.
3. We have used Item 6 of Definition 4.6.1.1.5 for the third equality.
4. We have used Item 2 of Definition 4.3.11.1.2 for the fourth equality.
5. We have used Item 16 for the last equality.

This finishes the proof.
Item 6, Interaction With Binary Intersections: This follows from Item 11.
Item 7, Interaction With Complements: Omitted.
Item 8, Interaction With Symmetric Differences: Omitted.
Item 9, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned}\left[f_!(U), f^!(V)\right]_X &\stackrel{\text{def}}{=} f_!(U)^{\mathsf{c}} \cup f_*(V) \\ &= f_*(U^{\mathsf{c}}) \cup f_*(V) \\ &\subset f_*(U^{\mathsf{c}} \cup V) \\ &\stackrel{\text{def}}{=} f_*([U, V]_X),\end{aligned}$$

where we have used:

1. Item 7 of Definition 4.6.3.1.7 for the second equality.
2. Item 5 of Definition 4.6.3.1.7 for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2). This finishes the proof.
Item 10, Lax Preservation of Colimits: Omitted.
Item 11, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.
Item 12, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 10.
Item 13, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 11.
Item 14, Interaction With Coproducts: Omitted.
Item 15, Interaction With Products: Omitted.
Item 16, Relation to Direct Images: We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication.* We claim that

  $$f_*(U) \subset Y \setminus f_!(X \setminus U).$$

  Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.

  This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).

  Thus $y \in Y \setminus f_!(X \setminus U)$.

- *The Second Implication.* We claim that

  $$Y \setminus f_!(X \setminus U) \subset f_*(U).$$

  Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.

  Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.

  Thus $y \in f_*(U)$.

This finishes the proof of Item 16.
Item 17, Interaction With Injections: Omitted.
Item 18, Interaction With Surjections: Omitted. □

Proposition 4.6.3.1.8. Let $f \colon X \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

   $$(-)_{!|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

   $$(-)_{!|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon X \to Y$ and $g \colon Y \to Z$, we have

   $$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(Y) \arrow[d, "g_*"] \\ & \mathcal{P}(Z). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.3.1.7.
Item 3, Interaction With Identities: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**.
Item 4, Interaction With Composition: This follows from Definition 4.6.3.1.4 and Kan Extensions, **??** of **??**. □

4.6.4 A Six-Functor Formalism for Sets

Remark 4.6.4.1.1. The assignment $X \mapsto \mathcal{P}(X)$ together with the functors f_*, f^{-1}, and $f_!$ of Item 1 of Definition 4.6.1.1.5, Item 1 of Definition 4.6.2.1.3, and Item 1 of Definition 4.6.3.1.7, and the functors

$$\begin{aligned} -_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) &\to \mathcal{P}(X), \\ [-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) &\to \mathcal{P}(X) \end{aligned}$$

of Item 1 of Definition 4.3.9.1.2 and Item 1 of Definition 4.4.7.1.3 satisfy several properties reminiscent of a six functor formalism in the sense of **??**.

We collect these properties in Definition 4.6.4.1.2 below.[43]

Proposition 4.6.4.1.2. Let X be a set.

1. *The Beck–Chevalley Condition.* Let

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a pullback diagram in Sets. We have

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\mathrm{pr}_1^{-1}"] \arrow[d, "f_!"'] & \mathcal{P}(X \times_Z Y) \arrow[d, "(\mathrm{pr}_2)_!"] \\ \mathcal{P}(Z) \arrow[r, "g^{-1}"'] & \mathcal{P}(Y), \end{tikzcd} \qquad \begin{aligned} g^{-1} \circ f_! &= (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}, \\ f^{-1} \circ g_! &= (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}, \end{aligned} \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "\mathrm{pr}_2^{-1}"] \arrow[d, "g_!"'] & \mathcal{P}(X \times_Z Y) \arrow[d, "(\mathrm{pr}_1)_!"] \\ \mathcal{P}(Z) \arrow[r, "f^{-1}"'] & \mathcal{P}(Y). \end{tikzcd}$$

[43] See also [nLa25].

2. *The Projection Formula I.* The diagram

$$
\begin{tikzcd}
& \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"] & \\
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}"] \arrow[d, "f_! \times \mathrm{id}_{\mathcal{P}(Y)}"'] & & \mathcal{P}(X) \arrow[d, "f_!"] \\
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[rr, "\cap"'] & & \mathcal{P}(Y),
\end{tikzcd}
$$

commutes, i.e. we have

$$f_!(U \cap f^{-1}(V)) = f_!(U) \cap V$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

3. *The Projection Formula II.* We have a natural transformation

$$
\begin{tikzcd}
& \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\cap"] & \\
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}"] \arrow[d, "f_* \times \mathrm{id}_{\mathcal{P}(Y)}"'] \arrow[rr, phantom, "\cup"] & & \mathcal{P}(X) \arrow[d, "f_*"] \\
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[rr, "\cap"'] & & \mathcal{P}(Y),
\end{tikzcd}
$$

with components

$$f_*(U) \cap V \subset f_*(U \cap f^{-1}(V))$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

4. *Strong Closed Monoidality.* The diagram

$$
\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}
$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = [f^{-1}(U), f^{-1}(V)]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *The External Tensor Product.* We have an external tensor product

$$-_1 \boxtimes_{X\times Y} -_2 \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$$

given by

$$\begin{aligned} U \boxtimes_{X\times Y} V &\overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\ &= \{(u,v) \in X \times Y \mid u \in U \text{ and } v \in V\}. \end{aligned}$$

This is the same map as the one in Item 5 of Definition 4.4.1.1.4. Moreover, the following conditions are satisfied:

(a) *Interaction With Direct Images.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_! \times g_!"] \arrow[d, "\boxtimes_{X\times Y}"'] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, "\boxtimes_{X'\times Y'}"] \\ \mathcal{P}(X \times Y) \arrow[r, "f_! \times g_!"'] & \mathcal{P}(X' \times Y') \end{tikzcd}$$

commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X\times Y} V) = f_!(U) \boxtimes_{X'\times Y'} g_!(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(b) *Interaction With Inverse Images.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd} \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[r, "f^{-1} \times g^{-1}"] \arrow[d, "\boxtimes_{X'\times Y'}"'] & \mathcal{P}(X) \times \mathcal{P}(Y) \arrow[d, "\boxtimes_{X\times Y}"] \\ \mathcal{P}(X' \times Y') \arrow[r, "f^{-1} \times g^{-1}"'] & \mathcal{P}(X \times Y) \end{tikzcd}$$

commutes, i.e. we have

$$[f^{-1}\times g^{-1}](U\boxtimes_{X'\times Y'}V)=f^{-1}(U)\boxtimes_{X\times Y}g^{-1}(V)$$

for each $(U,V)\in\mathcal{P}(X')\times\mathcal{P}(Y')$.

(c) *Interaction With Codirect Images.* Let $f\colon X\to X'$ and $g\colon Y\to Y'$ be functions. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(Y) \arrow[r, "f_*\times g_*"] \arrow[d, "\boxtimes_{X\times Y}"'] & \mathcal{P}(X')\times\mathcal{P}(Y') \arrow[d, "\boxtimes_{X'\times Y'}"] \\
\mathcal{P}(X\times Y) \arrow[r, "f_*\times g_*"'] & \mathcal{P}(X'\times Y')
\end{tikzcd}$$

commutes, i.e. we have

$$[f_*\times g_*](U\boxtimes_{X\times Y}V)=f_*(U)\boxtimes_{X'\times Y'}g_*(V)$$

for each $(U,V)\in\mathcal{P}(X)\times\mathcal{P}(Y)$.

(d) *Interaction With Diagonals.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "\boxtimes_{X\times X}"] \arrow[dr, "\cap"'] & \mathcal{P}(X\times X) \arrow[d, "\Delta_X^{-1}"] \\
& \mathcal{P}(X),
\end{tikzcd}$$

i.e. we have

$$U\cap V=\Delta_X^{-1}(U\boxtimes_{X\times X}V)$$

for each $U,V\in\mathcal{P}(X)$.

6. *The Dualisation Functor.* We have a functor

$$D_X\colon\mathcal{P}(X)^{\mathsf{op}}\to\mathcal{P}(X)$$

given by

$$\begin{aligned}D_X(U)&\overset{\text{def}}{=}[U,\emptyset]_X\\&\overset{\text{def}}{=}U^{\mathsf{c}}\end{aligned}$$

for each $U\in\mathcal{P}(X)$, as in Item 5 of Definition 4.4.7.1.3, satisfying the following conditions:

(a) *Duality.* We have

$$D_X(D_X(U)) = U, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "D_X"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \arrow[d, "D_X"] \\ & \mathcal{P}(X). \end{tikzcd}$$

(b) *Duality.* The diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] & \mathcal{P}(X)^{\mathsf{op}} \arrow[dr, "D_X"] & \\ \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X"] \arrow[rrr, "{[-_1,-_2]_X}"'] & & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\stackrel{\text{def}}{=}[U\cap[V,\emptyset]_X,\emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

(c) *Interaction With Direct Images.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(d) *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "D_Y"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "D_X"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X(f^{-1}(U))$$

for each $U \in \mathcal{P}(X)$.

(e) *Interaction With Codirect Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, The Beck–Chevalley Condition*: We have

$$\begin{aligned}
[g^{-1} \circ f_!](U) &\overset{\text{def}}{=} g^{-1}(f_!(U)) \\
&\overset{\text{def}}{=} \left\{ y \in Y \,\middle|\, g(y) \in f_!(U) \right\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \,\middle|\, x \in U \right\} \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; x \in U \right\} \\ \text{such that } y = y \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; x \in U \right\} \\ \text{such that } \mathrm{pr}_2(x, y) = y \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; x \in U \right\}) \\
&= (\mathrm{pr}_2)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; \mathrm{pr}_1(x, y) \in U \right\}) \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\mathrm{pr}_1^{-1}(U)) \\
&\overset{\text{def}}{=} [(\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$
g^{-1} \circ f_! = (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}.
$$

For the second equality, we have

$$
\begin{aligned}
[f^{-1} \circ g_!](U) &\overset{\text{def}}{=} f^{-1}(g_!(U)) \\
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; f(x) \in g_!(V) \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\} \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\} \\ \text{such that } x = x \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x, y) \in \left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\} \\ \text{such that } \mathrm{pr}_1(x, y) = x \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; y \in V \right\}) \\
&= (\mathrm{pr}_1)_!(\left\{ (x, y) \in X \times_Z Y \;\middle|\; \mathrm{pr}_2(x, y) \in V \right\}) \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\mathrm{pr}_2^{-1}(V))
\end{aligned}
$$

$$\stackrel{\text{def}}{=} [(\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}](V)$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$f^{-1} \circ g_! = (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}.$$

This finishes the proof.
Item 2, The Projection Formula I: We claim that

$$f_!(U) \cap V \subset f_!(U \cap f^{-1}(V)).$$

Indeed, we have

$$\begin{aligned} f_!(U) \cap V &\subset f_!(U) \cap f_!(f^{-1}(V)) \\ &= f_!(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

1. Item 2 of Definition 4.6.1.1.5 for the inclusion.
2. Item 6 of Definition 4.6.1.1.5 for the equality.

Conversely, we claim that

$$f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V.$$

Indeed:

1. Let $y \in f_!(U \cap f^{-1}(V))$.
2. Since $y \in f_!(U \cap f^{-1}(V))$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.
3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.
4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.
5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.
6. But $y = f(x)$, so $y \in f_!(U) \cap V$.
7. Thus $f_!(U \cap f^{-1}(V)) \subset f_!(U) \cap V$.

This finishes the proof.
Item 3, The Projection Formula II: We have

$$\begin{aligned} f_*(U) \cap V &\subset f_*(U) \cap f_*(f^{-1}(V)) \\ &= f_*(U \cap f^{-1}(V)), \end{aligned}$$

where we have used:

1. Item 2 of Definition 4.6.3.1.7 for the inclusion.

2. Item 6 of Definition 4.6.3.1.7 for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (Categories, Item 4 of Definition 11.2.7.1.2).
Item 4, Strong Closed Monoidality: This is a repetition of Item 19 of Definition 4.4.7.1.3 and is proved there.
Item 5, The External Tensor Product: We have

$$\begin{aligned} U \boxtimes_{X\times Y} V &\stackrel{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\ &\stackrel{\text{def}}{=} \left\{(x,y) \in X \times Y \,\middle|\, \mathrm{pr}_1(x,y) \in U\right\} \\ &\quad \cup \left\{(x,y) \in X \times Y \,\middle|\, \mathrm{pr}_2(x,y) \in V\right\} \\ &= \left\{(x,y) \in X \times Y \,\middle|\, x \in U\right\} \\ &\quad \cup \left\{(x,y) \in X \times Y \,\middle|\, y \in V\right\} \\ &= \left\{(x,y) \in X \times Y \,\middle|\, x \in U \text{ and } y \in V\right\} \\ &\stackrel{\text{def}}{=} U \times V. \end{aligned}$$

Next, we claim that Items 5a to 5d are indeed true:

1. *Proof of Item 5a:* This is a repetition of Item 16 of Definition 4.6.1.1.5 and is proved there.

2. *Proof of Item 5b:* This is a repetition of Item 16 of Definition 4.6.2.1.3 and is proved there.

3. *Proof of Item 5c:* This is a repetition of Item 15 of Definition 4.6.3.1.7 and is proved there.

4. *Proof of Item 5d:* We have

$$\begin{aligned}\Delta_X^{-1}(U \boxtimes_{X\times X} V) &\stackrel{\text{def}}{=} \{x \in X \mid (x,x) \in U \boxtimes_{X\times X} V\}\\ &= \{x \in X \mid (x,x) \in \{(u,v) \in X \times X \mid u \in U \text{ and } v \in V\}\}\\ &= U \cap V.\end{aligned}$$

This finishes the proof.
Item 6, The Dualisation Functor: This is a repetition of Items 5 and 6 of Definition 4.4.7.1.3 and is proved there. □

4.7 Miscellany

4.7.1 Injective Functions

Let A and B be sets.

Definition 4.7.1.1.1. A function $f \colon A \to B$ is **injective** if it satisfies the following condition:

(★) For each $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.

Proposition 4.7.1.1.2. Let $f \colon A \to B$ be a function.

1. *Characterisations.* The following conditions are equivalent:[44]

   (a) The function f is injective.

   (b) The function f is a monomorphism in Sets.

   (c) The direct image function

   $$f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

   associated to f is injective.

[44] Items 1c to 1f unwind respectively to the following statements:

- For each $U, V \in \mathcal{P}(A)$, if $f_!(U) = f_!(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_*(U) = f_*(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_!(U) \subset f_!(V)$, then $U \subset V$.
- For each $U, V \in \mathcal{P}(A)$, if $f_*(U) \subset f_*(V)$, then $U \subset V$.

(d) The codirect image function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to f is injective.

(e) The direct image functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

associated to f is full.

(f) The codirect image function

$$f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to f is full.

(g) The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, hook, "\chi_A"'] & B \arrow[d, "f^{-1}"] \\
 & \mathcal{P}(A)
\end{tikzcd}$$

commutes. That is, we have

$$f^{-1}(f(a)) = \{a\}$$

for each $a \in A$.

(h) We have

$$f^{-1} \circ f_! = \mathrm{id}_{\mathcal{P}(A)} \qquad \begin{tikzcd}
\mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, equals] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\
 & \mathcal{P}(A).
\end{tikzcd}$$

In other words, we have

$$\left\{a \in A \,\middle|\, f(a) \in f(U)\right\} = U$$

for each $U \in \mathcal{P}(A)$.

(i) We have

$$f^{-1} \circ f_* = \mathrm{id}_{\mathcal{P}(A)} \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

In other words, we have

$$\left\{a \in A \,\middle|\, f^{-1}(f(a)) \subset U\right\} = U$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Characterisations*: We will proceed by showing:

- Step 1: Item 1a $\iff$ Item 1b.
- Step 2: Item 1a $\iff$ Item 1c.
- Step 3: Item 1a $\iff$ Item 1d.
- Step 4: Item 1c $\iff$ Item 1e.
- Step 5: Item 1e $\iff$ Item 1f.
- Step 6: Item 1a $\iff$ Item 1g.
- Step 7: Item 1g $\iff$ Item 1h.
- Step 8: Item 1a $\iff$ Item 1i.

Step 1: Item 1a $\iff$ Item 1b. We claim that Items 1a and 1b are equivalent:

- *Item 1a $\implies$ Item 1b:* We proceed in a few steps:
  - Proceeding by contrapositive, we claim that given a pair of maps $g, h\colon C \rightrightarrows A$ such that $g \neq h$, we have $f \circ g \neq f \circ h$.
  - Indeed, as g and h are different maps, there must exist at least one element $x \in C$ such that $g(x) \neq h(x)$.
  - But then we have $f(g(x)) \neq f(h(x))$, as f is injective.

   - Thus $f \circ g \neq f \circ h$, and we are done.

- *Item 1b* $\implies$ *Item 1a:* We proceed in a few steps:

   - Consider the diagram

$$\mathrm{pt} \underset{[y]}{\overset{[x]}{\rightrightarrows}} A \xrightarrow{f} B,$$

     where $[x]$ and $[y]$ are the morphisms picking the elements x and y of A.

   - Note that we have $f(x) = f(y)$ iff $f \circ [x] = f \circ [y]$.
   - Since f is assumed to be a monomorphism, if $f(x) = f(y)$, then $f \circ [x] = f \circ [y]$ and therefore $[x] = [y]$.
   - This shows that if $f(x) = f(y)$, then $x = y$, so f is injective.

Step 2: Item 1a $\iff$ Item 1c. We claim that Items 1a and 1c are indeed equivalent:

- *Item 1a* $\implies$ *Item 1c:* We proceed in a few steps:

   - Assume that f is injective and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$. We wish to show that $U = V$.
   - To show that $U \subset V$, let $u \in U$.
   - By the definition of the direct image, we have $f(u) \in f_!(U)$.
   - Since $f_!(U) = f_!(V)$, it follows that $f(u) \in f_!(V)$.
   - Thus, there exists some $v \in V$ such that $f(v) = f(u)$.
   - Since f is injective, the equality $f(v) = f(u)$ implies that $v = u$.
   - Thus $u \in V$ and $U \subset V$.
   - A symmetric argument shows that $V \subset U$.
   - Therefore $U = V$, showing $f_!$ to be injective.

- *Item 1c* $\implies$ *Item 1a:* We proceed in a few steps:

   - Assume that the direct image function $f_!$ is injective and let $a, a' \in A$ such that $f(a) = f(a')$. We wish to show that $a = a'$.

- Since

$$\begin{aligned} f_!(\{a\}) &= \{f(a)\} \\ &= \{f(a')\} \\ &= f_!(\{a'\}), \end{aligned}$$

we must have $\{a\} = \{a'\}$, as $f_!$ is injective, so $a = a'$, showing f to be injective.

Step 3: Item 1c $\iff$ Item 1d. This follows from Item 17 of Definition 4.6.1.1.5.

Step 4: Item 1c $\iff$ Item 1e. We claim that Items 1c and 1e are equivalent:

- *Item 1c $\implies$ Item 1e:* We proceed in a few steps:
  - Let $U, V \in \mathcal{P}(A)$ such that $f_!(U) \subset f_!(V)$, assume $f_!$ to be injective, and consider the set $U \cup V$.
  - Since $f_!(U) \subset f_!(V)$, we have

    $$\begin{aligned} f_!(U \cup V) &= f_!(U) \cup f_!(V) \\ &= f_!(V), \end{aligned}$$

    where we have used Item 5 of Definition 4.6.1.1.5 for the first equality.
  - Since $f_!$ is injective, this implies $U \cup V = V$.
  - Thus $U \subset V$, as we wished to show.
- *Item 1c $\implies$ Item 1e:* We proceed in a few steps:
  - Suppose Item 1e holds, and let $U, V \in \mathcal{P}(A)$ such that $f_!(U) = f_!(V)$.
  - Since $f_!(U) = f_!(V)$, we have $f_!(U) \subset f_!(V)$ and $f_!(V) \subset f_!(U)$.
  - By assumption, this implies $U \subset V$ and $V \subset U$.
  - Thus $U = V$, showing $f_!$ to be injective.

Step 5: Item 1e $\iff$ Item 1f. This follows from Item 17 of Definition 4.6.1.1.5.

Step 6: Item 1a $\iff$ Item 1g. We have

$$f^{-1}(f(a)) = \{a' \in A \,|\, f(a') = f(a)\}$$

so the condition $f^{-1}(f(a)) = \{a\}$ states precisely that if $f(a') = f(a)$, then $a' = a$.

Step 7: Item 1g $\iff$ Item 1h. We claim that Items 1g and 1h are indeed equivalent:

- *Item 1g $\implies$ Item 1h:* We have

$$\begin{aligned}
[f^{-1} \circ f_!](U) &\overset{\text{def}}{=} f^{-1}(f_!(U)) \\
&= f^{-1}\left(f_!\left(\bigcup_{u \in U} \{u\}\right)\right) \\
&= f^{-1}\left(\bigcup_{u \in U} f_!(\{u\})\right) \\
&= \bigcup_{u \in U} f^{-1}(f_!(\{u\})) \\
&= \bigcup_{u \in U} f^{-1}(f_!(u)) \\
&= \bigcup_{u \in U} \{u\} \\
&= U
\end{aligned}$$

  for each $U \in \mathcal{P}(A)$, where we have used Item 5 of Definition 4.6.1.1.5 for the third equality and Item 5 of Definition 4.6.2.1.3 for the fourth equality.

- *Item 1h $\implies$ Item 1g:* Applying the condition $f^{-1} \circ f_! = \mathrm{id}_{\mathcal{P}(A)}$ to $U = \{a\}$ gives

$$f^{-1}(f_!(\{a\})) = \{a\}.$$

Step 8: Item 1a $\iff$ Item 1i. We claim that Items 1a and 1i are equivalent:

- *Item 1a $\implies$ Item 1i:* If f is injective, then $f^{-1}(f(a)) = \{a\}$, so we have

$$\begin{aligned}
f^{-1}(f_*(a)) &= \{a \in A \mid \{a\} \subset U\} \\
&= U.
\end{aligned}$$

- *Item 1i $\implies$ Item 1a:* For $U = \{a\}$, the condition $f^{-1}(f_*(U)) = U$ becomes

$$\left\{a' \in A \,\middle|\, f^{-1}(f(a')) \subset \{a\}\right\} = \{a\}.$$

Since the set $f^{-1}(f(a'))$ is given by

$$\left\{a \in A \,\middle|\, f(a) = f(a')\right\},$$

it follows that f is injective.

This finishes the proof. □

4.7.2 Surjective Functions

Let A and B be sets.

Definition 4.7.2.1.1. A function $f\colon A \to B$ is **surjective** if it satisfies the following condition:

($\star$) For each $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Proposition 4.7.2.1.2. Let $f\colon A \to B$ be a function.

1. *Characterisations.* The following conditions are equivalent:

   (a) The function f is surjective.

   (b) The function f is an epimorphism in Sets.

   (c) The inverse image function

   $$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

   associated to f is injective.

   (d) The inverse image functor

   $$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   associated to f is full.

   (e) The diagram

   $$\begin{tikzcd} B \arrow[r, "f^{-1}"] \arrow[dr, "\chi_B"'] & \mathcal{P}(A) \arrow[d, "f_!"] \\ & \mathcal{P}(B) \end{tikzcd}$$

commutes. That is, we have

$$f_!(f^{-1}(b)) = \{b\}$$

for each $b \in B$.

(f) We have

$$f_! \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "f^{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "f_!"] \\ & \mathcal{P}(B). \end{tikzcd}$$

In other words, we have

$$\left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in f^{-1}(U) \\ \text{such that } f(a) = b \end{array} \right\} = U$$

for each $U \in \mathcal{P}(A)$.

(g) We have

$$f_* \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "f^{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "f_*"] \\ & \mathcal{P}(B). \end{tikzcd}$$

In other words, we have

$$\left\{ b \in B \,\middle|\, f^{-1}(b) \subset f^{-1}(U) \right\} = U$$

for each $U \in \mathcal{P}(B)$.

Proof. Item 1, Characterisations: We will proceed by showing:

- Step 1: Item 1a $\iff$ Item 1b.
- Step 2: Item 1a $\iff$ Item 1c.

- Step 3: Item 1c $\iff$ Item 1d.
- Step 4: Item 1a $\iff$ Item 1e.
- Step 5: Item 1e $\iff$ Item 1f.
- Step 6: Item 1a $\iff$ Item 1g.

Step 1: Item 1a $\iff$ Item 1b. We claim Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* We proceed in a few steps:
  - Let $g, h\colon B \rightrightarrows C$ be morphisms such that $g \circ f = h \circ f$.
  - For each $a \in A$, we have
  $$g(f(a)) = h(f(a)).$$
  - However, this implies that
  $$g(b) = h(b)$$
  for each $b \in B$, as f is surjective.
  - Thus $g = h$ and f is an epimorphism.
- *Item 1b $\implies$ Item 1a:* We proceed by contrapositive. Consider the diagram
$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C,$$
where h is the map defined by $h(b) = 0$ for each $b \in B$ and g is the map defined by
$$g(b) = \begin{cases} 1 & \text{if } b \in \mathrm{Im}(f), \\ 0 & \text{otherwise}. \end{cases}$$
Then $h \circ f = g \circ f$, as $h(f(a)) = 1 = g(f(a))$ for each $a \in A$. However, for any $b \in B \setminus \mathrm{Im}(f)$, we have
$$g(b) = 0 \neq 1 = h(b).$$
Therefore $g \neq h$ and f is not an epimorphism.

Step 2: Item 1a $\iff$ Item 1c. We claim Items 1a and 1c are indeed equivalent:

- *Item 1a $\implies$ Item 1c:* We proceed in a few steps:
  - Assume that f is surjective. Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$. We wish to show that $U = V$.
  - To show that $U \subset V$, let $b \in U$.
  - Since f is surjective, there must exist some $a \in A$ such that $f(a) = b$.
  - By the definition of the inverse image, since $f(a) = b$ and $b \in U$, we have $a \in f^{-1}(U)$.
  - By our initial assumption, $f^{-1}(U) = f^{-1}(V)$, so it follows that $a \in f^{-1}(V)$.
  - Again, by the definition of the inverse image, $a \in f^{-1}(V)$ means that $f(a) \in V$.
  - Since $f(a) = b$, we have shown that $b \in V$.
  - This establishes that $U \subset V$. A symmetric argument shows that $V \subset U$.
  - Thus $U = V$, proving that f^{-1} is injective.
- *Item 1c $\implies$ Item 1a:* We proceed in a few steps:
  - Assume that the inverse image function f^{-1} is injective. Suppose, for the sake of contradiction, that f is not surjective.
  - The assumption that f is not surjective means there exists some $b_0 \in B$ such that for all $a \in A$, we have $f(a) \neq b_0$.
  - By the definition of the inverse image, this is equivalent to stating that $f^{-1}(\{b_0\}) = \emptyset$.
  - Since $f^{-1}(\emptyset) = \emptyset$, we have $f^{-1}(\{b_0\}) = f^{-1}(\emptyset)$.
  - Since f^{-1} is injective, this implies that $\{b_0\} = \emptyset$.
  - This is a contradiction, as the singleton set $\{b_0\}$ is non-empty.
  - Therefore, f is surjective.

Step 3: Item 1c $\iff$ Item 1d. We claim that Items 1c and 1d are equivalent:

- *Item 1c $\implies$ Item 1d:* We proceed in a few steps:

- Let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) \subset f^{-1}(V)$, assume f^{-1} to be injective, and consider the set $U \cup V$.
- Since $f^{-1}(U) \subset f^{-1}(V)$, we have

$$\begin{aligned} f^{-1}(U \cup V) &= f^{-1}(U) \cup f^{-1}(V) \\ &= f^{-1}(V), \end{aligned}$$

  where we have used Item 5 of Definition 4.6.2.1.3 for the first equality.
- Since f^{-1} is injective, this implies $U \cup V = V$.
- Thus $U \subset V$, as we wished to show.

- *Item 1d $\Longrightarrow$ Item 1c:* We proceed in a few steps:
  - Suppose Item 1d holds, and let $U, V \in \mathcal{P}(B)$ such that $f^{-1}(U) = f^{-1}(V)$.
  - Since $f^{-1}(U) = f^{-1}(V)$, we have $f^{-1}(U) \subset f^{-1}(V)$ and $f^{-1}(V) \subset f^{-1}(U)$.
  - By assumption, this implies $U \subset V$ and $V \subset U$.
  - Thus $U = V$, showing f^{-1} to be injective.

Step 4: Item 1a $\Longleftrightarrow$ Item 1e. We have

$$f_!(f^{-1}(b)) = \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in f^{-1}(b) \\ \text{such that } f(a) = b \end{array} \right\},$$

so the condition $f_!(f^{-1}(b)) = \{b\}$ holds iff f is surjective.
Step 5: Item 1e $\Longleftrightarrow$ Item 1f. We claim that Items 1e and 1f are indeed equivalent:

- *Item 1e $\Longrightarrow$ Item 1f:* We have

$$\begin{aligned} [f_! \circ f^{-1}](U) &\overset{\text{def}}{=} f_!(f^{-1}(U)) \\ &= f_!\left(f^{-1}\left(\bigcup_{u \in U} \{u\}\right)\right) \\ &= f_!\left(\bigcup_{u \in U} f^{-1}(\{u\})\right) \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{u \in U} f_!(f^{-1}(\{u\})) \\
&= \bigcup_{u \in U} f_!(f^{-1}(u)) \\
&= \bigcup_{u \in U} \{u\} \\
&= U
\end{aligned}
$$

for each $U \in \mathcal{P}(B)$, where we have used Item 5 of Definition 4.6.1.1.5 for the third equality and Item 5 of Definition 4.6.2.1.3 for the fourth equality.

- *Item 1f* $\Longrightarrow$ *Item 1e:* Applying the condition $f_! \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)}$ to $U = \{b\}$ gives
$$f_!(f^{-1}(\{b\})) = \{b\}.$$

Step 6: Item 1a $\Longleftrightarrow$ Item 1g. First, note that for the condition $f^{-1}(b) \subset f^{-1}(U)$ to hold, we must have $b \in U$ or $f^{-1}(b) = \emptyset$. Thus

$$f_*(f^{-1}(U)) = (U \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)).$$

We now claim that Items 1a and 1g are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1g:* If f is surjective, we have
$$
\begin{aligned}
(U \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)) &= U \cup \emptyset \\
&= U,
\end{aligned}
$$
so $f_* \circ f^{-1} = \mathrm{id}_{\mathcal{P}(B)}$.

- *Item 1g* $\Longrightarrow$ *Item 1a:* Taking $U = \emptyset$ gives
$$
\begin{aligned}
f_*(f^{-1}(\emptyset)) &= (\emptyset \cap \mathrm{Im}(f)) \cup (B \setminus \mathrm{Im}(f)) \\
&= B \setminus \mathrm{Im}(f),
\end{aligned}
$$
so the condition $f_*(f^{-1}(\emptyset)) = \emptyset$ implies $B \setminus \mathrm{Im}(f) = \emptyset$. Thus $\mathrm{Im}(f) = B$ and f is surjective.

This finishes the proof. □

Appendices

A Other Chapters

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