## Constructions With Monoidal Categories

# The Clowder Project Authors July 21, 2025

This chapter contains some material on constructions with monoidal categories.

## Contents

13.1	Moduli Categories of Monoidal Structures	<b>2</b>
	13.1.1 The Moduli Category of Monoidal Structures on a Cate-	
gory.		2
	13.1.2 The Moduli Category of Braided Monoidal Structures	
on a	Category	14
	13.1.3 The Moduli Category of Symmetric Monoidal Structures	
on a	Category	14
13.2	Moduli Categories of Closed Monoidal Structures	14
13.3	Moduli Categories of Refinements of Monoidal Struc-	
ture	S	14
ture	s	14

## 13.1 Moduli Categories of Monoidal Structures

## 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let C be a category.

Definition 13.1.1.1. The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  defined by

Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{E}_1}(C)$  where:

- Objects. The objects of  $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$  are monoidal categories  $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$  whose underlying category is  $\mathcal{C}$ .
- Morphisms. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$  of  $\mathcal{C}$ .

• *Identities.* For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C)),$  the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$  at M is defined by

$$\mathrm{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(\mathcal{C})} \stackrel{\mathrm{def}}{=} \left(\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right),$$

where  $\left(\mathrm{id}_{\mathcal{C}}^{\otimes},\mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)$  is the identity monoidal functor of  $\mathcal{C}$  of ??.

• Composition. For each  $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_{1}}(C))$ , the composition map  $\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}$ :  $\mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(N,P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M,N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_{1}}(C)}(M,P)$  of  $\mathcal{M}_{\mathbb{E}_{1}}(C)$  at (M,N,P) is defined by  $\left(\mathrm{id}_{C}^{\otimes,\prime},\mathrm{id}_{\mathbb{I}|C}^{\otimes,\prime}\right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_{1}}(C)}\left(\mathrm{id}_{C}^{\otimes},\mathrm{id}_{\mathbb{I}|C}^{\otimes}\right) \stackrel{\mathrm{def}}{=} \left(\mathrm{id}_{C}^{\otimes,\prime}\circ\mathrm{id}_{C}^{\otimes},\mathrm{id}_{\mathbb{I}|C}^{\otimes,\prime}\circ\mathrm{id}_{\mathbb{I}|C}^{\otimes}\right).$ 

**Remark 13.1.1.1.3.** In particular, a morphism in  $\mathcal{M}_{\mathbb{E}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:

1. Naturality. For each pair  $f: A \to X$  and  $g: B \to Y$  of morphisms of  $\mathcal{C}$ , the diagram

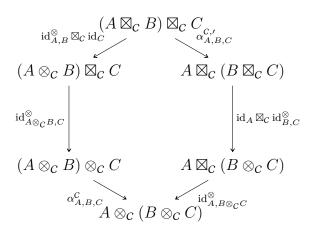
$$A \boxtimes_{\mathcal{C}} B \xrightarrow{f \boxtimes_{\mathcal{C}} g} X \boxtimes_{\mathcal{C}} Y$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{X,Y}^{\otimes}}$$

$$A \otimes_{\mathcal{C}} B \xrightarrow{f \otimes_{\mathcal{C}} g} X \otimes_{\mathcal{C}} Y$$

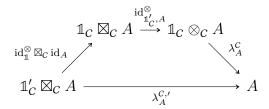
commutes.

2. Monoidality. For each  $A, B, C \in \text{Obj}(\mathcal{C})$ , the diagram



commutes.

3. Left Monoidal Unity. For each  $A \in \text{Obj}(\mathcal{C})$ , the diagram



commutes.

4. Right Monoidal Unity. For each  $A \in \text{Obj}(\mathcal{C})$ , the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}'_{C}}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \longrightarrow A$$

$$A \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\rho_{A'}^{C,\prime}} A$$

commutes.

## **Proposition 13.1.1.1.4.** Let C be a category.

- 1. Extra Monoidality Conditions. Let  $\left(\mathrm{id}_{\mathcal{C}}^{\otimes},\mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_{1}}(\mathcal{C})$  from  $\left(\mathcal{C},\otimes_{\mathcal{C}},\mathbb{1}_{\mathcal{C}},\alpha^{\mathcal{C}},\lambda^{\mathcal{C}},\rho^{\mathcal{C}}\right)$  to  $\left(\mathcal{C},\boxtimes_{\mathcal{C}},\mathbb{1}_{\mathcal{C}}',\alpha^{\mathcal{C},\prime},\lambda^{\mathcal{C},\prime},\rho^{\mathcal{C},\prime}\right)$ .
  - (a) The diagram

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\operatorname{id}_{A,B}^{\otimes} \boxtimes_{C} \operatorname{id}_{C}} (A \otimes_{C} B) \boxtimes_{C} C$$

$$\operatorname{id}_{A\boxtimes_{C} B,C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A\otimes_{C} B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\operatorname{id}_{A}\otimes_{C} \operatorname{id}_{C}} (A \otimes_{C} B) \otimes_{C} C$$

commutes.

(b) The diagram

$$A \boxtimes_{C} (B \boxtimes_{C} C) \xrightarrow{\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\operatorname{id}_{A,B\boxtimes_{C} C}^{\otimes} \downarrow \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C} C}^{\otimes}$$

$$A \otimes_{C} (B \boxtimes_{C} C) \xrightarrow[\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}]{} A \otimes_{C} (B \otimes_{C} C)$$

commutes.

- 2. Extra Monoidal Unity Constraints. Let  $\left(\operatorname{id}_{\mathcal{C}}^{\otimes}, \operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)$  be a morphism of  $\mathcal{M}_{\mathbb{E}_{1}}(\mathcal{C})$  from  $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$  to  $\left(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}', \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime}\right)$ .
  - (a) The diagram

commutes.

(b) The diagram

commutes.

(c) The diagram

commutes.

(d) The diagram

$$\mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow^{\lambda_{\mathbb{1}'_{C}}^{C,\prime}} \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes.

3. Mixed Associators. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\operatorname{id}_{-1,-2}^{\otimes} : -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes}: (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A \otimes_{C} B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \boxtimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B \boxtimes_{C} C}}$$

$$(A \otimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} C}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{C}B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A}\boxtimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes,\otimes} : (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A,B}^{\otimes} \otimes_{C} \operatorname{id}_{C}} \qquad \qquad \downarrow^{\operatorname{id}_{A} \otimes_{C} \operatorname{id}_{B,C}^{\otimes}}$$

$$(A \otimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_{C} (B \otimes_{C} C)$$

and

$$(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \xrightarrow{\alpha_{A,B,C}^{\mathcal{C},\prime}} A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{\mathcal{C}}B,C}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B\boxtimes_{\mathcal{C}}C}^{\otimes}}$$

$$(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

*Proof.* Item 1, Extra Monoidality Conditions: We claim that Items 1a and 1b are indeed true:

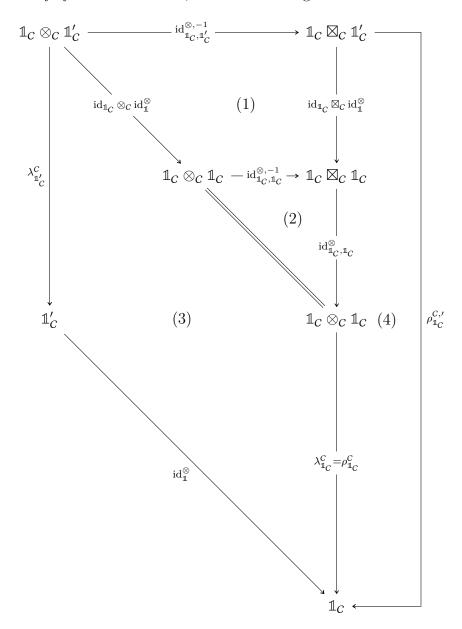
1. Proof of Item 1a: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A,B}^{\otimes}$  and  $id_{C}$ .

2. Proof of Item 1b: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{B,C}^{\otimes}$ .

This finishes the proof.

*Item 2, Extra Monoidal Unity Constraints*: We claim that *Items 2a* and *2b* are indeed true:

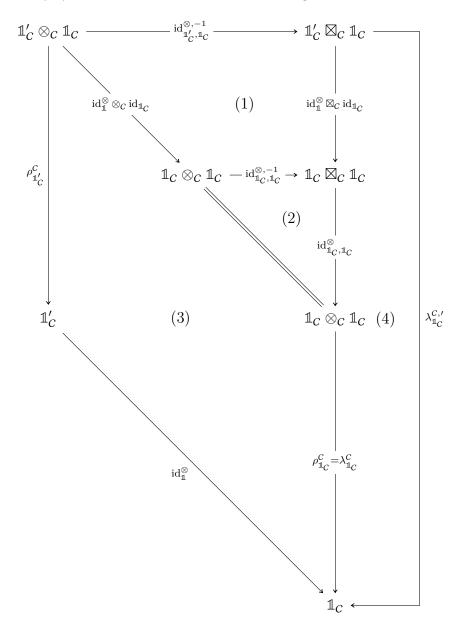
1. Proof of Item 1a: Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $id_{\mathcal{C}}^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^{\mathcal{C}}$ , where the equality  $\rho_{\mathbb{1}_{\mathcal{C}}}^{\mathcal{C}} = \lambda_{\mathbb{1}_{\mathcal{C}}}^{\mathcal{C}}$  comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|\mathbb{1}}^{\otimes})$ ; so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_{\mathcal{C}}^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of  $\rho^{\mathcal{C}}$ , where the equality  $\rho_{\mathbb{1}_{\mathcal{C}}}^{\mathcal{C}} = \lambda_{\mathbb{1}_{\mathcal{C}}}^{\mathcal{C}}$  comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|\mathbb{1}}^{\otimes})$ ; so does the boundary diagram, and we are done.
- 3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

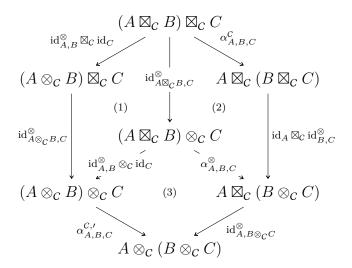
$$\downarrow \qquad \qquad \downarrow^{\lambda_{\mathbb{1}'_{C}}^{C}}$$

commutes. But since  $id_{1}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

1. Proof of Item 3a: We may partition the monoidality diagram for  $id^{\otimes}$  of Item 2 of Definition 13.1.1.1.3 as follows:



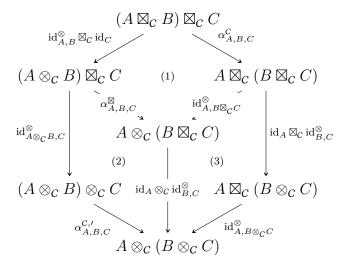
Since:

• Subdiagram (1) commutes by Item 1a of Item 1.

- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for  $id^{\otimes}$  of Item 2 of Definition 13.1.1.1.3 as follows:



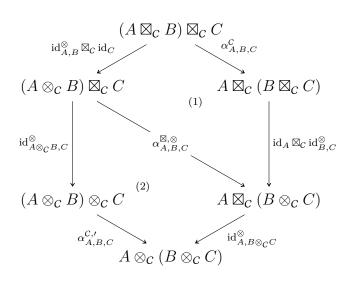
Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. Proof of Item 3c: We may partition the monoidality diagram for id $^{\otimes}$ 

#### of Item 2 of Definition 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 13.2 Moduli Categories of Closed Monoidal Structures
- 13.3 Moduli Categories of Refinements of Monoidal Structures
- 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

## Appendices

## A Other Chapters

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Pre	lın	nını	aries

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

## Relations

- 8. Relations
- 9. Constructions With Relations

#### 10. Conditions on Relations

## Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

## **Monoidal Categories**

13. Constructions With Monoidal Categories

## **Bicategories**

14. Types of Morphisms in Bicategories

#### Extra Part

15. Notes