Pointed Sets

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This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

Definition 6.1.1.1.1. A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 6.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 6.1.1.1.3. The 0**-sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

Example 6.1.1.1.4. The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

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[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$**-modules**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 6.1.1.1.5. The **standard pointed set with** $n+1$ **elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \overset{\text{def}}{=} \{*\} \cup \{1, \dots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n \rangle$.

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1. A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathrm{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathrm{Sets}, \mathrm{pt})$.

Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f\colon (X, x_0) \longrightarrow (Y, y_0)$ is a morphism of sets $f\colon X \longrightarrow Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1. The **category of pointed sets** is the category Sets_* defined equivalently as:

[4] *Further Terminology:* Also called a **pointed function**.

[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**.
- The category Sets_* of Constructions With Categories, **??**.

Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[6]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\text{def}}{=} \mathrm{id}_X .$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\text{def}}{=} g \circ f.$$

6.1.4 Elementary Properties of Pointed Sets

Proposition 6.1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

[6] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[7] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned} \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

(a) Products, described as in Definition 6.2.3.1.1.

(b) Pullbacks, described as in Definition 6.2.4.1.1.

(c) Equalisers, described as in Definition 6.2.5.1.1.

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

   (a) Coproducts, described as in Definition 6.3.3.1.1.

   (b) Pushouts, described as in Definition 6.3.4.1.1;

   (c) Coequalisers, described as in Definition 6.3.5.1.1.

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

[8] The category Sets_* does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

[9] In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi\colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

i. A pointed set (X, x_0) to X.
ii. A pointed function

$$f\colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f\colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\mathsf{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.
ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. *Item 1, Completeness*: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.
Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.
Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].
Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}: S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.
Item 5, Relation to Partial Functions: See [MSE 884460]. □

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1. Let $f: (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2. We write $\mathsf{Sets}_*^{\mathsf{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

1. The map $\mu: \langle 2 \rangle \to \langle 1 \rangle$ given by

$$\begin{aligned} 1 &\longmapsto 1 \\ 2 &\longmapsto 1 \\ * &\longmapsto * \end{aligned}$$

is active but not inert.

2. The map $f\colon \langle 2\rangle \to \langle 2\rangle$ given by

$$
\begin{tikzcd}[row sep=small]
1 \arrow[r, mapsto] & 1 \\
2 \arrow[dr, mapsto] & 2 \\
* \arrow[r, mapsto] & *
\end{tikzcd}
$$

is inert but not active.

3. The map $f\colon \langle 3\rangle \to \langle 1\rangle$ given by

$$
\begin{tikzcd}[row sep=small]
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & \\
3 \arrow[dr, mapsto] & \\
* \arrow[r, mapsto] & *
\end{tikzcd}
$$

is neither inert nor active. However, it factors as $f = a \circ i$, where

$$
\begin{aligned}
i &\colon \langle 3\rangle \to \langle 2\rangle,\\
a &\colon \langle 2\rangle \to \langle 1\rangle
\end{aligned}
$$

are the morphisms of pointed sets given by

$$
\begin{tikzcd}[row sep=small]
1 \arrow[r, mapsto] & 1 & 1 \arrow[r, mapsto] & 1 \\
2 \arrow[r, mapsto] & 2 & 2 \arrow[ur, mapsto] & \\
3 \arrow[dr, mapsto] & & * \arrow[r, mapsto] & *, \\
* \arrow[r, mapsto] & * & &
\end{tikzcd}
$$

with i being inert and a being active.

Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

(a) The map $i\colon (X, x_0) \longrightarrow (K, k_0)$ is an inert morphism of pointed sets

(b) The map $a\colon (K, k_0) \longrightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_*.

Proof. *Item 1, Active-Inert Factorisation*: Let $f\colon X \longrightarrow Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= \big(X \setminus f^{-1}(y_0)\big) \cup \{x_0\}; \end{aligned}$$

- $i\colon X \longrightarrow K$ is the inert morphism of pointed sets given by

$$i(x) \overset{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

  for each $x \in X$;

- $a\colon K \longrightarrow Y$ is the active morphism of pointed sets given by

$$a(x) \overset{\text{def}}{=} f(x)$$

  for each $x \in K$.

Next, let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y \longrightarrow A$ given by

$$\phi(y) = f\big(i^{-1}(y)\big)$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\exists!"', "\phi"] \\ A \arrow[r, "a"'] & B. \end{tikzcd}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\overset{\text{def}}{=} \phi(i(x)) \\ &\overset{\text{def}}{=} f\big(i^{-1}(i(x))\big) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [a \circ \phi](y) &\overset{\text{def}}{=} a(\phi(y)) \\ &\overset{\text{def}}{=} a\big(f\big(i^{-1}(y)\big)\big) \\ &\overset{\text{def}}{=} [a \circ f]\big(i^{-1}(y)\big) \\ &= [g \circ i]\big(i^{-1}(y)\big) \\ &\overset{\text{def}}{=} g\big(i\big(i^{-1}(y)\big)\big) \\ &\overset{\text{def}}{=} g(y) \end{aligned}$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned} \psi(y) &= \psi(i(x)) \\ &= f(x) \\ &= f\big(i^{-1}(y)\big) \\ &\overset{\text{def}}{=} \phi(y). \end{aligned}$$

This finishes the proof. □

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1. The **terminal pointed set** is the terminal object of Sets_* as in Limits and Colimits, **??**.

Construction 6.2.1.1.2. Concretely, the **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{!_X : (X, x_0) \longrightarrow (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$!_X(x) \stackrel{\text{def}}{=} \star$$

  for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad\qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \longrightarrow (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\mathrm{pt}, \star)$$

commute, namely $!_X$. □

6.2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.2.2.1.1. The **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the product of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.2.1.2. Concretely, the **product of** $\{(X_i, x_0^i)\}_{i\in I}$ is the pair $((\prod_{i\in I} X_i, (x_0^i)_{i\in I}), \{\mathrm{pr}_i\}_{i\in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i\colon \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \to (X_i, x_0^i)\right\}_{i\in I}$$

  of maps given by

$$\mathrm{pr}_i\left((x_j)_{j\in I}\right) \overset{\text{def}}{=} x_i$$

  for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
(P, *) \arrow[dr, "p_i"] & \\
(\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \to \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd}
(P, *) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\
(\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i)
\end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_i(*))_{i\in I}\\ &= \left(x_0^i\right)_{i\in I},\end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

Proposition 6.2.2.1.3. Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ defines a functor
$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \longrightarrow \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.3.1.2. Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets
$$\begin{aligned}\mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \longrightarrow (X, x_0),\\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \longrightarrow (Y, y_0)\end{aligned}$$
defined by
$$\begin{aligned}\mathrm{pr}_1(x, y) &\overset{\mathrm{def}}{=} x,\\ \mathrm{pr}_2(x, y) &\overset{\mathrm{def}}{=} y\end{aligned}$$
for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and

(Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & (P,*) \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] & \\
(X,x_0) & (X\times Y,(x_0,y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y,y_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to (X\times Y,(x_0,y_0))$$

making the diagram

$$\begin{tikzcd}
 & (P,*) \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
(X,x_0) & (X\times Y,(x_0,y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y,y_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1\circ\phi &= p_1,\\ \mathrm{pr}_2\circ\phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.3.1.3. Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned} A \times - \colon & \quad \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\ - \times B \colon & \quad \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\ -_1 \times -_2 \colon & \mathsf{Sets}_* \times \mathsf{Sets}_* & \to \mathsf{Sets}_*, \end{aligned}$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.
Item 2, *Lack of Adjointness*: See [MSE 2855868].
Item 3, *Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.
Item 4, *Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.
Item 5, *Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.
Item 6, *Symmetric Monoidality*: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3. □

6.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.4.1.2. Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned} \mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

  defined by

  $$\begin{aligned} \mathrm{pr}_1(x, y) &\overset{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\overset{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \end{aligned}$$

$$
\begin{aligned}
&= g(\mathrm{pr}_2(x,y)) \\
&= [g \circ \mathrm{pr}_2](x,y),
\end{aligned}
$$

where $f(x) = g(y)$ since $(x,y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$
\begin{tikzcd}
(P,*) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0,y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to (X \times_Z Y, (x_0,y_0))$$

making the diagram

$$
\begin{tikzcd}
(P,*) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] \arrow[rd, dashed, "\phi", "\exists!"'] & & \\
& (X \times_Z Y, (x_0,y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}
$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.4.1.3. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

   where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

   In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd} X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\ & X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\ X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\ & X' \arrow[rr, "f'"'] & & Z' \end{tikzcd}$$

   in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \left(X' \times_{Z'} Y', \left(x'_0, y'_0\right)\right)$$

   given by

$$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\vee"] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & & \\
X \arrow[dr, "f"'] & {} & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & {} & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
X \arrow[dr, "f"'] & {} & Y \arrow[dl, "g"] \arrow[dr, "h"'] & {} & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\vee"] & & \\
& & {} & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & {} & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad A \times_X B \cong B \times_X A \qquad \begin{tikzcd} B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X. \end{tikzcd}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad \begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\ X \arrow[r, "!_X"'] & \mathrm{pt}. \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, *Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.
Item 3, *Unitality*: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.
Item 4, *Commutativity*: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.
Item 5, *Interaction With Products*: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.
Item 6, *Symmetric Monoidality*: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5. □

6.2.5 Equalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1. The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

  $$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= e(*)\\ &= x_0,\end{aligned}$$

where we have used that e is a morphism of pointed sets. □

Proposition 6.2.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \overset{f}{\underset{h}{\overset{-g\to}{\rightrightarrows}}} (Y, y_0)$$

   in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.
Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.
Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3. □

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1. The **initial pointed set** is the initial object of Sets_* as in Limits and Colimits, **??**.

Construction 6.3.1.1.2. Concretely, the **initial pointed set** is the pair $\left((\mathrm{pt},\star),\{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt},\star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X\colon(\mathrm{pt},\star)\to(X,x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$\iota_X(\star)\overset{\mathrm{def}}{=}x_0.$$

Proof. We claim that $(\mathrm{pt},\star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt},\star)\qquad\qquad(X,x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon(\mathrm{pt},\star)\to(X,x_0)$$

making the diagram

$$(\mathrm{pt},\star)\ \overset{\phi}{\underset{\exists!}{\dashrightarrow}}\ (X,x_0)$$

commute, namely ι_X. □

6.3.2 Coproducts of Families of Pointed Sets

Let $\left\{(X_i,x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.3.2.1.1. The **coproduct of the family** $\left\{(X_i,x_0^i)\right\}_{i\in I}$[11] is the coproduct of $\left\{(X_i,x_0^i)\right\}_{i\in I}$ in Sets_* as in Limits and Colimits, **??**.

[11] *Further Terminology:* Also called the **wedge sum of the family** $\left\{(X_i,x_0^i)\right\}_{i\in I}$.

Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i\in I}$ is the pair $((\bigvee_{i\in I} X_i, p_0), \{\mathrm{inj}_i\}_{i\in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by
  $$\bigvee_{i\in I} X_i \overset{\text{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim,$$
  where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring
  $$(i, x_0^i) \sim \left(j, x_0^j\right)$$
  for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by
  $$\begin{aligned} p_0 &\overset{\text{def}}{=} \left[(i, x_0^i)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$
  for any $i, j \in I$.
- *The Cocone.* The collection
$$\left\{\mathrm{inj}_i \colon (X_i, x_0^i) \to \left(\bigvee_{i\in I} X_i, p_0\right)\right\}_{i\in I}$$
of morphism of pointed sets given by
$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$
for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C, *) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon \left(\bigvee_{i\in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & & (C,*) \\ (X_i, x_0^i) \arrow[urr, "\iota_i"] \arrow[r, "\mathsf{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \arrow[ur, dashed, "\phi", "\exists!"'] & \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathsf{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i\left(\left[\left(i, x_0^i\right)\right]\right) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 6.3.2.1.3. Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I} \mapsto \left(\bigvee_{i\in I} X_i, p_0\right)$ defines a functor
$$\bigvee_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0)[12] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned}(X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0)\\ &\cong \left(X \coprod\nolimits_{\mathrm{pt}} Y, p_0\right)\\ &\cong (X \coprod Y/\sim, p_0),\end{aligned} \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner" very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}, \arrow[l, hook, "{[x_0]}"'] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned}p_0 &\stackrel{\text{def}}{=} [(0, x_0)]\\ &= [(1, y_0)].\end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathsf{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\ \mathsf{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),\end{aligned}$$

  given by

$$\begin{aligned}\mathsf{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)],\\ \mathsf{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0)

[12] *Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & (X \vee Y, p_0) & (Y,y_0) \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"]
\end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \vee Y, p_0) \to (C,*)$$

making the diagram

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y,y_0) \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"]
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_X &= \iota_X, \\
\phi \circ \mathrm{inj}_Y &= \iota_Y
\end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0,x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1,y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \iota_X([(0,x_0)]) \\
&= \iota_Y([(1,y_0)]) \\
&= *,
\end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

Proposition 6.3.3.1.3. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments
$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$
define functors
$$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ - \vee Y\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets
$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$
natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets
$$\begin{aligned} (\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0), \end{aligned}$$
natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets
$$X \vee Y \cong Y \vee X,$$
natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation
$$\nabla\colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow \nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & {} & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\text{def}}{=} \begin{cases} x & \text{if } p = [(0,x)], \\ x & \text{if } p = [(1,x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, Associativity: Omitted.
Item 3, Unitality: Omitted.
Item 4, Commutativity: Omitted.
Item 5, Symmetric Monoidality: Omitted.
Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f \colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{aligned}$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. □

6.3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f \colon (Z, z_0) \to (X, x_0)$ and $g \colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1. The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

Construction 6.3.4.1.2. Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)]\\ \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)]\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}x_0 &= f(z_0),\\ y_0 &= g(z_0)\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}[\mathrm{inj}_1 \circ f](z) &= \mathrm{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \mathrm{inj}_2(g(z)) \\ &= [\mathrm{inj}_2 \circ g](z),\end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} (P, *) & & \\ & (X \coprod_Z Y, p_0) \arrow[dr, phantom, "\ulcorner"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd} (P, *) & & \\ & (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!"', "\phi"] \arrow[dr, phantom, "\ulcorner"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_1 = \iota_1,$$

$$\phi \circ \mathrm{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0,x)], \\ \iota_2(y) & \text{if } x = [(1,y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0,x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1,y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. □

Proposition 6.3.4.1.3. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \coprod_Z Y & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in Fun($\mathcal{P}$, Sets$_*$) to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} \left(X' \coprod_{Z'} Y', p_0'\right)$$

given by

$$\xi(p) \overset{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \textstyle\coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small]
 & & (X \coprod_W Y) \coprod_V Z & & \\
 & X \coprod_W Y \arrow[ur] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
 & W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
 & X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
 & W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & & X \coprod_W (Y \coprod_V Z) & & \\
 & & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
 & W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \textstyle\coprod_X A &\cong A, \\
A \textstyle\coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
X \coprod_Z Y & Y \arrow[l] \\
X \arrow[u] & Z, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
X \textstyle\coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{tikzcd}
Y \coprod_Z X & X \arrow[l] \\
Y \arrow[u] & Z. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$X \textstyle\coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad
\begin{tikzcd}
X \vee Y & Y \arrow[l] \\
X \arrow[u] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"']
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *Associativity*: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.
Item 3, *Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.
Item 4, *Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.
Item 5, *Interaction With Coproducts*: Omitted.
Item 6, *Symmetric Monoidality*: Omitted. □

6.3.5 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.

- *The Cocone.* The map

$$\mathrm{coeq}(f, g)\colon Y \twoheadrightarrow (\mathrm{CoEq}(f, g), [y_0])$$

  given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](f(x)) \\
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)]
\end{aligned}$$

$$\begin{aligned}&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)](g(x))\\&\stackrel{\text{def}}{=} [\mathrm{coeq}(f,g)\circ g](x)\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}(X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \\ & & (C,*)\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\phi\colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}(X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\ & & (C,*)\end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}\phi([y_0]) &= [\phi \circ \mathrm{coeq}(f,g)]([y_0])\\&= c([y_0])\\&= *,\end{aligned}$$

where we have used that c is a morphism of pointed sets. □

Proposition 6.3.5.1.3. Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd}(X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g"] \arrow[r, shift right=2, "h"'] & (Y,y_0)\end{tikzcd}$$

in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathsf{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathsf{CoEq}(f, g) \cong \mathsf{CoEq}(g, f).$$

Proof. *Item 1*, *Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.
Item 2, *Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.
Item 3, *Commutativity*: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5. □

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let X be a set.

Definition 6.4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[13]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 6.4.1.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \longrightarrow \mathsf{Sets}_*,$$

where:

[13] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{+}\right](X) \overset{\text{def}}{=} X^{+},$$

where X^{+} is the pointed set of Definition 6.4.1.1.1.

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^{+} \colon X^{+} \to Y^{+}$$

of f by $(-)^{+}$ is the map of pointed sets defined by

$$f^{+}(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^{+} \dashv \text{忘}\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*}, \arrow[l, bend left, "\text{忘}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_{*}\left((X^{+}, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^{+} \vee Y^{+} \overset{\sim}{\dashrightarrow} (X \textstyle\coprod Y)^{+},$$

$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+},(-)^{+},(-)^{+}_{\mathbb{1}}\right)\colon (\mathsf{Sets},\times,\mathrm{pt}) \longrightarrow \left(\mathsf{Sets}_{*},\wedge,S^{0}\right),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned}(-)^{+}_{X,Y}\colon X^{+}\wedge Y^{+} &\overset{\sim}{\dashrightarrow} (X\times Y)^{+},\\ (-)^{+}_{\mathbb{1}}\colon S^{0} &\overset{\sim}{\dashrightarrow} \mathrm{pt}^{+},\end{aligned}$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: We claim that $(-)^{+}$ is indeed a functor:

- *Preservation of Identities.* Let $X\in\mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}^{+}_{X}(x)\overset{\mathrm{def}}{=}\begin{cases}x & \text{if } x\in X,\\ \star_X & \text{if } x=\star_X,\end{cases}$$

for each $x\in X^{+}$, so $\mathrm{id}^{+}_{X}=\mathrm{id}_{X^{+}}$.

- *Preservation of Composition.* Given morphisms of sets

$$\begin{aligned}f\colon X&\longrightarrow Y,\\ g\colon Y&\longrightarrow Z,\end{aligned}$$

we have

$$\begin{aligned}\left[g^{+}\circ f^{+}\right](x)&\overset{\mathrm{def}}{=}g^{+}\left(f^{+}(x)\right)\\ &\overset{\mathrm{def}}{=}g^{+}(f(x))\\ &\overset{\mathrm{def}}{=}g(f(x))\\ &\overset{\mathrm{def}}{=}[g\circ f]^{+}(x)\end{aligned}$$

for each $x\in X$ and

$$\begin{aligned}\left[g^{+}\circ f^{+}\right](\star_X)&\overset{\mathrm{def}}{=}g^{+}\left(f^{+}(\star_X)\right)\\ &\overset{\mathrm{def}}{=}g^{+}(\star_Y)\\ &\overset{\mathrm{def}}{=}\star_Z\\ &\overset{\mathrm{def}}{=}[g\circ f]^{+}(\star_X),\end{aligned}$$

so $(g\circ f)^{+}=g^{+}\circ f^{+}$.

This finishes the proof.
Item 2, *Adjointness*: We proceed in a few steps:

- *Map I.* We define a map
$$\Phi_{X,Y}\colon \mathsf{Sets}_*\left(X^+, Y\right) \to \mathsf{Sets}(X, Y)$$
by sending a morphism of pointed sets
$$\xi\colon \left(X^+, \star_X\right) \to (Y, y_0)$$
to the function
$$\xi^\dagger : X \to Y$$
given by
$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$
for each $x \in X$.
- *Map II.* We define a map
$$\Psi_{X,Y}\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*\left(X^+, Y\right)$$
given by sending a function $\xi\colon X \to Y$ to the morphism of pointed sets
$$\xi^\dagger\colon \left(X^+, \star_X\right) \to (Y, y_0)$$
defined by
$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$
for each $x \in X^+$.
- *Invertibility I.* Given a morphism of pointed sets
$$\xi\colon \left(X^+, \star_X\right) \to (Y, y_0),$$
we have
$$\begin{aligned} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}\left(\Phi_{X,Y}(\xi)\right) \\ &= \Psi_{X,Y}\left(\xi^\dagger\right) \end{aligned}$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} \left[\!\left[x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right] \\
&= \left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right] \\
&= \xi \\
&\stackrel{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}\right](\xi).
\end{aligned}
$$

Therefore we have

$$
\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}.
$$

- *Invertibility II.* Given a map of sets $\xi\colon X \to Y$, we have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right) \\
&= \Phi_{X,Y}\left(\xi^{\dagger}\right) \\
&= \Phi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right]\right) \\
&= \left[\!\left[x \mapsto \xi(x) \right]\!\right] \\
&= \xi \\
&\stackrel{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,Y)}\right](\xi).
\end{aligned}
$$

Therefore we have

$$
\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.
$$

- *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$
f\colon (X, x_0) \to \left(X', x_0'\right),
$$

the diagram

$$
\begin{tikzcd}
\mathsf{Sets}_*(X'^{,+}, Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}
$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+} \to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*\big(\Phi_{X',Y}(\xi)\big) \\
&= f^*\big(\Phi_{X',Y}(\xi)\big) \\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

  $$g\colon (Y, y_0) \to \big(Y', y'_0\big),$$

  the diagram

  $$\begin{tikzcd}
  \mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
  \mathsf{Sets}_*(X^+, Y'), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
  \end{tikzcd}$$

  commutes. Indeed, given a morphism of pointed sets

  $$\xi^\dagger\colon X^+ \to Y,$$

  we have

  $$\begin{aligned}
  \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
  &= \Phi_{X,Y'}(g \circ \xi) \\
  &= g \circ \xi \\
  &= g \circ \Phi_{X,Y'}(\xi)
  \end{aligned}$$

$$\begin{aligned} &= g_*\big(\Phi_{X,Y'}(\xi)\big) \\ &= \big[g_* \circ \Phi_{X,Y'}\big](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.
Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \coprod Y)^+$$

is given by

$$(-)^{+,\coprod}_{X,Y}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)^{+,\coprod,-1}_{X,Y} \colon (X \coprod Y)^+ \overset{\sim}{\dashrightarrow} X^+ \vee Y^+$$

given by

$$(-)^{+,\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\coprod,\mathbb{1}}_{X,Y}\colon \mathrm{pt} \dashrightarrow{\sim} \emptyset^{+}$$

is given by sending $\star_X$ to $\star_\emptyset$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\times$ and $\wedge$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+}_{X,Y}\colon X^{+}\wedge Y^{+} \dashrightarrow{\sim} (X\times Y)^{+}$$

is given by

$$(-)^{+}_{X,Y}(x\wedge y) = \begin{cases} (x,y) & \text{if } x\neq \star_X \text{ and } y\neq \star_Y \\ \star_{X\times Y} & \text{otherwise} \end{cases}$$

for each $x\wedge y\in X^{+}\wedge Y^{+}$, with inverse

$$(-)^{+,-1}_{X,Y}\colon (X\times Y)^{+} \dashrightarrow{\sim} X^{+}\wedge Y^{+}$$

given by

$$(-)^{+,-1}_{X,Y}(z) \stackrel{\text{def}}{=} \begin{cases} x\wedge y & \text{if } z=(x,y) \text{ with } (x,y)\in X\times Y, \\ \star_X\wedge\star_Y & \text{if } z=\star_{X\times Y}, \end{cases}$$

for each $z\in (X\times Y)^{+}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\mathbb{1}}_{X,Y}\colon S^{0} \dashrightarrow{\sim} \mathrm{pt}^{+}$$

is given by sending 0 to $\star_{\mathrm{pt}}$ and 1 to $\star$, where $\mathrm{pt}^{+}=\{\star,\star_{\mathrm{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted. □

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1. The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2. Let (X, x_0) be a pointed set.

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

   $$X^- \colon \mathsf{Sets}_*^{\mathsf{actv}} \to \mathsf{Sets},$$

   where:

   - *Action on Objects.* For each $X \in \mathrm{Obj}\big(\mathsf{Sets}_*^{\mathsf{actv}}\big)$, we have

     $$[(-)^-](X) \overset{\text{def}}{=} X^-,$$

     where X^- is the set of Definition 6.4.2.1.1.

   - *Action on Morphisms.* For each morphism $f \colon X \to Y$ of $\mathsf{Sets}_*^{\mathsf{actv}}$, the image

     $$f^- \colon X^- \to Y^-$$

     of f by $(-)^-$ is the map defined by

     $$f^-(x) \overset{\text{def}}{=} f(x)$$

     for each $x \in X^-$.

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

   $$\big((-)^- \dashv (-)^+\big)\colon \quad \begin{tikzcd} \mathsf{Sets}_*^{\mathsf{actv}} \arrow[r, bend left, "(-)^-"] \arrow[r, phantom, "\perp_{\mathsf{eq}}"] & \mathsf{Sets}, \arrow[l, bend left, "(-)^+"] \end{tikzcd}$$

   witnessed by a bijection of sets

   $$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*\big(X, Y^+\big),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms

   $$\begin{aligned} (X^-)^+ &\cong X, \\ \big(Y^+\big)^- &\cong Y, \end{aligned}$$

   once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-}, (-)^{-,\vee}, (-)^{-,\vee}_{\mathbb{1}}\right) \colon \left(\mathsf{Sets}^{\mathsf{actv}}_{*}, \vee, \mathrm{pt}\right), \longrightarrow \left(\mathsf{Sets}, \coprod, \emptyset\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{-,\vee}_{X,Y} \colon X^{-} \coprod Y^{-} \overset{\sim}{\dashrightarrow} (X \vee Y)^{-},$$
$$(-)^{-,\vee}_{\mathbb{1}} \colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^{-},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-}, (-)^{-,\times}, (-)^{-,\times}_{\mathbb{1}}\right) \colon \left(\mathsf{Sets}^{\mathsf{actv}}_{*}, \wedge, S^{0}\right), \longrightarrow (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \overset{\sim}{\dashrightarrow} (X \wedge Y)^{-},$$
$$(-)^{-}_{\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^{0}\right)^{-},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: We claim that $(-)^{-}$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}^{-}_{X}(x) \overset{\text{def}}{=} x$$

for each $x \in X^{-}$, so $\mathrm{id}^{-}_{X} = \mathrm{id}_{X^{-}}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f \colon (X, x_0) \longrightarrow (Y, y_0),$$
$$g \colon (Y, y_0) \longrightarrow (Z, z_0),$$

we have

$$\begin{aligned} [g^{-} \circ f^{-}](x) &\overset{\text{def}}{=} g^{-}(f^{-}(x)) \\ &\overset{\text{def}}{=} g^{-}(f(x)) \\ &\overset{\text{def}}{=} g(f(x)) \\ &\overset{\text{def}}{=} [g \circ f]^{-}(x) \end{aligned}$$

for each $x \in X$, so $(g \circ f)^{-} = g^{-} \circ f^{-}$.

This finishes the proof.

Item 2, *Adjoint Equivalence*: We proceed in a few steps:

1. *Map I*. We define a map

$$\Phi_{X,Y}\colon \mathsf{Sets}(X^-, Y) \to \mathsf{Sets}_*^{\mathsf{actv}}(X, Y^+)$$

by sending a map $\xi\colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^\dagger\colon X \to Y^+$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. *Map II*. We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}_*^{\mathsf{actv}}(X, Y^+) \to \mathsf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi\colon X \to Y^+$ to the map

$$\xi^\dagger\colon X^- \to Y$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I*. Given a map of sets $\xi\colon X^- \to Y$, we have

$$\begin{aligned} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]\right) \\ &= [\![x \mapsto \xi(x)]\!] \\ &= \xi \\ &= \left[\mathrm{id}_{\mathsf{Sets}(X^-,Y)}\right](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}.$$

4. *Invertibility II.* Given a morphism of pointed sets
$$\xi\colon (X,x_0) \to \bigl(Y^+,\star_Y\bigr),$$
we have
$$\begin{aligned}\bigl[\Phi_{X,Y}\circ\Psi_{X,Y}\bigr](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\bigl(\Psi_{X,Y}(\xi)\bigr)\\ &= \Phi_{X,Y}(\llbracket x\mapsto \xi(x)\rrbracket)\\ &= \left\llbracket x\mapsto \begin{cases}\xi(x) & \text{if } x\in X^-\\ \star_Y & \text{if } x=x_0\end{cases}\right\rrbracket\\ &= \xi\\ &= \left[\mathrm{id}_{\mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+)}\right](\xi).\end{aligned}$$
Therefore we have
$$\Phi_{X,Y}\circ\Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+)}.$$

5. *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets
$$f\colon (X,x_0) \to \bigl(X',x'_0\bigr),$$
the diagram
$$\begin{tikzcd}\mathsf{Sets}\bigl(X^{\prime,-},Y\bigr) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}^{\mathrm{actv}}_*(X',Y^+) \arrow[d, "f^*"] \\ \mathsf{Sets}_*(X^-,Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+)\end{tikzcd}$$
commutes. Indeed, given a map of sets $\xi\colon X'\to Y$, we have
$$\begin{aligned}\bigl[\Phi_{X,Y}\circ f^*\bigr](\xi) &= \Phi_{X,Y}(f^*(\xi))\\ &= \Phi_{X,Y}(\xi\circ f)\\ &= \left\llbracket x\mapsto \begin{cases}\xi(f(x)) & \text{if } f(x)\in X^{\prime,-}\\ \star_Y & \text{if } f(x)=x'_0\end{cases}\right\rrbracket\\ &= f^*\left(\left\llbracket x'\mapsto \begin{cases}\xi(x') & \text{if } x'\in X^{\prime,-}\\ \star_Y & \text{if } x'=x'_0\end{cases}\right\rrbracket\right)\end{aligned}$$

$$
\begin{aligned}
&= f^*\big(\Phi_{X',Y}(\xi)\big)\\
&= \big[f^* \circ \Phi_{X',Y}\big](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g\colon (Y, y_0) \to \big(Y', y_0'\big),$$

the diagram

$$
\begin{tikzcd}
\mathsf{Sets}(X^-, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}_*^{\mathsf{actv}}(X, Y^+) \arrow[d, "g_*"] \\
\mathsf{Sets}(X^-, Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}_*^{\mathsf{actv}}(X, Y'^{,+})
\end{tikzcd}
$$

commutes. Indeed, given a map of sets $\xi\colon X^- \to Y$, we have

$$
\begin{aligned}
\big[\Phi_{X,Y'} \circ g_*\big](\xi) &= \Phi_{X,Y'}(g_*(\xi))\\
&= \Phi_{X,Y'}(g \circ \xi)\\
&= \left[\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}\right]\!\right]\\
&= g_*\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases}\right]\!\right]\right)\\
&= g_*\big(\Phi_{X,Y'}(\xi)\big)\\
&= \big[g_* \circ \Phi_{X,Y'}\big](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

7. *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

8. *Fully Faithfulness of* $(-)^{-}$. We aim to show that the assignment $f \mapsto f^{-}$ sets up a bijection

$$(-)^{-}_{X,Y} \colon \mathsf{Sets}^{\mathsf{actv}}_{*}(X, Y) \overset{\sim}{\dashrightarrow} \mathsf{Sets}(X^{-}, Y^{-}).$$

Indeed, the inverse map

$$(-)^{-,-1}_{X,Y} \colon \mathsf{Sets}(X^{-}, Y^{-}) \overset{\sim}{\dashrightarrow} \mathsf{Sets}^{\mathsf{actv}}_{*}(X, Y)$$

is given by sending a map of sets $f \colon X^{-} \longrightarrow Y^{-}$ to the active morphism of pointed sets $f^{\dagger} \colon X \longrightarrow Y$ defined by

$$f^{\dagger}(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

9. *Essential Surjectivity of* $(-)^{-}$. We need to show that, given an object $X \in \mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}\big(\mathsf{Sets}^{\mathsf{actv}}_{*}\big)$ such that $(X')^{-} \cong X$. Indeed, taking $X' = X^{+}$, we have

$$\begin{aligned} \big(X^{+}\big)^{-} &\overset{\text{def}}{=} (X \cup \{\star_X\})^{-} \\ &\overset{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^{+})^{-} = X$, showing $(-)^{-}$ to be essentially surjective.

10. *The Functor* $(-)^{-}$ *Is an Equivalence.* Since $(-)^{-}$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.
Item 3, *Symmetric Strong Monoidality With Respect to Wedge Sums*: We construct the strong monoidal structure on $(-)^{-}$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-,\vee}_{X,Y} \colon X^{-} \coprod Y^{-} \xrightarrow{\sim} (X \vee Y)^{-}$$

is given by

$$(-)^{-,\vee}_{X,Y}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^{-} \coprod Y^{-}$, with inverse

$$(-)^{-,\vee,-1}_{X,Y} \colon (X \vee Y)^{-} \xrightarrow{\sim} X^{-} \coprod Y^{-}$$

given by

$$(-)^{-,\vee,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\vee,\mathbb{1}}_{X,Y} \colon \emptyset \xrightarrow{\sim} \mathrm{pt}^{-}$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{-}$ into a symmetric strong monoidal functor is omitted.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-}$$

is given by

$$(-)^{-}_{X,Y}(x,y) = x \wedge y$$

for each $(x,y) \in X^{-} \times Y^{-}$, with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)^{-,-1}_{X,Y}(x \wedge y) \overset{\text{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{-,\mathbb{1}}_{X,Y} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} (S^0)^{-}$$

is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted. □

Appendices

A Other Chapters

References

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