Types of Morphisms in Bicategories

The Clowder Project Authors

July 29, 2025

In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

Contents

14.1 Monomorphisms in Bicategories

14.1.1 Representably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.1.1.1 ▶ Representably Faithful Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is faithful.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 14.1.1.1.3.

Remark 14.1.1.1.2 ▶ Unwinding Definition 14.1.1.1.1

In detail, f is representably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 14.1.1.1.3 ▶ Examples of Representably Faithful Morphisms

Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Categories, Item 2 of Proposition 11.6.1.1.2.

2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Relations, **??** of **??**.

14.1.2 Representably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.2.1.1 ▶ Representably Full Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full**[1] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

[1] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Example 14.1.2.1.3.

Remark 14.1.2.1.2 ▶ Unwinding Definition 14.1.2.1.1

In detail, f is representably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 14.1.2.1.3 ▶ Examples of Representably Full Morphisms

Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2*.* The representably full morphisms in Cats_2 are precisely the full functors; see Categories, ?? of Proposition 11.6.2.1.2.

2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Relations, ?? of ??.

14.1.3 Representably Fully Faithful Morphisms

Let C be a bicategory.

Definition 14.1.3.1.1 ▶ Representably Fully Faithful Morphisms

A 1-morphism $f : A \to B$ of C is **representably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1) and representably full (Definition 14.1.2.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

[1] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Example 14.1.3.1.3.

Remark 14.1.3.1.2 ▶ Unwinding Representably Fully Faithful Morphisms

In detail, f is representably fully faithful if the conditions in Remark 14.1.1.1.2 and Remark 14.1.2.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 14.1.3.1.3 ▶ Examples of Representably Fully Faithful Morphisms

Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Categories, Item 6 of Proposition 11.6.3.1.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Relations, ?? of ??) with the representably full morphisms in **Rel**, which are characterised in Relations, ?? of ??.

14.1.4 Morphisms Representably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.4.1.1 ▶ Morphisms Representably Faithful on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is faithful.

Remark 14.1.4.1.2 ▶ Unwinding Definition 14.1.4.1.1

In detail, f is representably faithful on cores if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

14.1.5 Morphisms Representably Full on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.5.1.1 ▶ Morphisms Representably Full on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is full.

Remark 14.1.5.1.2 ▶ Unwinding Definition 14.1.5.1.1

In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[r, "f"] & B \arrow[from=U, to=D, Rightarrow, "\alpha"'] \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.6 Morphisms Representably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is fully faithful.

Remark 14.1.6.1.2 ▶ Unwinding Definition 14.1.6.1.1

In detail, f is representably fully faithful on cores if the conditions in Remark 14.1.4.1.2 and Remark 14.1.5.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.7 Representably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.7.1.1 ▶ Representably Essentially Injective Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is essentially injective.

Remark 14.1.7.1.2 ▶ Unwinding Definition 14.1.7.1.1

In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

14.1.8 Representably Conservative Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.8.1.1 ▶ Representably Conservative Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably conservative** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is conservative.

Remark 14.1.8.1.2 ▶ Unwinding Definition 14.1.8.1.1

In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_f \star\alpha"] & B \end{tikzcd}$$

is a 2-isomorphism, then so is α.

14.1.9 Strict Monomorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.9.1.1 ▶ Strict Monomorphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

is injective.

Remark 14.1.9.1.2 ▶ Unwinding Definition 14.1.9.1.1

In detail, f is a strict monomorphism in $\mathcal{C}$ if, for each diagram in $\mathcal{C}$ of the form

$$X \overset{\phi}{\underset{\psi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 14.1.9.1.3 ▶ Examples of Strict Monomorphisms

Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Categories, Item 1 of Proposition 11.7.2.1.2.

2. *Strict Monomorphisms in* $\mathbf{Rel}$. The strict monomorphisms in $\mathbf{Rel}$ are characterised in Relations, ??.

14.1.10 Pseudomonic Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.10.1.1 ▶ Pseudomonic Morphisms

A 1-morphism $f: A \to B$ of $\mathcal{C}$ is **pseudomonic** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*: \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 14.1.10.1.2 ▶ Unwinding Definition 14.1.10.1.1

In detail, a 1-morphism $f: A \to B$ of $\mathcal{C}$ is pseudomonic if it satisfies the following conditions:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 14.1.10.1.3 ▶ Properties of Pseudomonic Morphisms

Let $f \colon A \to B$ be a 1-morphism of $\mathcal{C}$.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) The morphism f is representably full on cores and representably faithful.
   (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \\ A \arrow[r, "F"'] \arrow[ur, Leftrightarrow, dashed] & B \end{tikzcd}$$

   in $\mathcal{C}$ up to equivalence.

2. *Interaction With Cotensors.* If $\mathcal{C}$ has cotensors with $\mathbb{1}$, then the following conditions are equivalent:
   (a) The morphism f is pseudomonic.
   (b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1} \pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1} \pitchfork F"] \\ B \arrow[r, hook] \arrow[ur, Leftrightarrow, dashed] & \mathbb{1} \pitchfork B \end{tikzcd}$$

   in $\mathcal{C}$ up to equivalence.

Proof 14.1.10.1.4 ▶ Proof of Proposition 14.1.10.1.3

Item 1: Characterisations

Omitted.

Item 2: Interaction With Cotensors

Omitted.

14.2 Epimorphisms in Bicategories

14.2.1 Corepresentably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.1.1.1 ▶ Corepresentably Faithful Morphisms

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is faithful.

Remark 14.2.1.1.2 ▶ Unwinding Definition 14.2.1.1.1

In detail, f is corepresentably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 14.2.1.1.3 ▶ Examples of Corepresentably Faithful Morphisms

Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Categories, Item 5 of Proposition 11.6.1.1.2.

2. *Corepresentably Faithful Morphisms in* $\mathbf{Rel}$. Every morphism of $\mathbf{Rel}$ is corepresentably faithful; see Relations, **??** of **??**.

14.2.2 Corepresentably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.2.1.1 ▶ Corepresentably Full Morphisms

A 1-morphism $f: A \to B$ of $\mathcal{C}$ is **corepresentably full** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*: \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

Remark 14.2.2.1.2 ▶ Unwinding Definition 14.2.2.1.1

In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta: \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha: \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f\,.$$

Example 14.2.2.1.3 ▶ Examples of Corepresentably Full Morphisms

Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Categories, Item 7 of Proposition 11.6.2.1.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in Relations, ?? of ??.

14.2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 14.2.3.1.1 ▶ Corepresentably Fully Faithful Morphisms

A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[1] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is fully faithful.

[1] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

Remark 14.2.3.1.2 ▶ Unwinding Definition 14.2.3.1.1

In detail, f is corepresentably fully faithful if the conditions in Remark 14.2.1.1.2 and Remark 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f \,.$$

Example 14.2.3.1.3 ▶ Examples of Corepresentably Fully Faithful Morphisms

Here are some examples of corepresentably fully faithful morphisms.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Categories, Item 10 of Proposition 11.6.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Relations, ?? of ??) with the corepresentably full morphisms in **Rel**, which are characterised in Relations, ?? of ??.

14.2.4 Morphisms Corepresentably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.4.1.1 ▶ Morphisms Corepresentably Faithful on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is faithful.

Remark 14.2.4.1.2 ▶ Unwinding Definition 14.2.4.1.1

In detail, f is corepresentably faithful on cores if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\,\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

14.2.5 Morphisms Corepresentably Full on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.5.1.1 ▶ Morphisms Corepresentably Full on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is full.

Remark 14.2.5.1.2 ▶ Unwinding Definition 14.2.5.1.1

In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

14.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.6.1.1 ▶ Morphisms Corepresentably Fully Faithful on Cores

A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is fully faithful.

Remark 14.2.6.1.2 ▶ Unwinding Definition 14.2.6.1.1

In detail, f is corepresentably fully faithful on cores if the conditions in Remark 14.2.4.1.2 and Remark 14.2.5.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi \circ f"] \arrow[r, bend right, "\psi \circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi \circ f"] \arrow[r, bend right, "\psi \circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

14.2.7 Corepresentably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.7.1.1 ▶ Corepresentably Essentially Injective Morphisms

A 1-morphism $f\colon A\to B$ of $\mathcal{C}$ is **corepresentably essentially injective** if, for each $X\in\mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B,X)\to\mathsf{Hom}_{\mathcal{C}}(A,X)$$

given by precomposition by f is essentially injective.

Remark 14.2.7.1.2 ▶ Unwinding Definition 14.2.7.1.1

In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi,\psi\colon B\rightrightarrows X$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $\phi\circ f\cong\psi\circ f$, then $\phi\cong\psi$.

14.2.8 Corepresentably Conservative Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.8.1.1 ▶ Corepresentably Conservative Morphisms

A 1-morphism $f\colon A\to B$ of $\mathcal{C}$ is **corepresentably conservative** if, for each $X\in\mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B,X)\to\mathsf{Hom}_{\mathcal{C}}(A,X)$$

given by precomposition by f is conservative.

Remark 14.2.8.1.2 ▶ Unwinding Definition 14.2.8.1.1

In detail, f is corepresentably conservative if, for each pair of morphisms $\phi,\psi\colon B\rightrightarrows X$ and each 2-morphism

$$\alpha\colon\phi\overset{\sim}{\Longrightarrow}\psi,\qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, if the 2-morphism

$$\alpha \star \mathrm{id}_f \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\Downarrow \alpha\star\mathrm{id}_f"] & X \end{tikzcd}$$

is a 2-isomorphism, then so is α.

14.2.9 Strict Epimorphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.2.9.1.1 ▶ Strict Epimorphisms

A 1-morphism $f \colon A \to B$ is a **strict epimorphism in $\mathcal{C}$** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

is injective.

Remark 14.2.9.1.2 ▶ Unwinding Definition 14.2.9.1.1

In detail, f is a strict epimorphism if, for each diagram in $\mathcal{C}$ of the form

$$A \xrightarrow{f} B \overset{\phi}{\underset{\psi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 14.2.9.1.3 ▶ Examples of Strict Epimorphisms

Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Categories, Item 1 of Proposition 11.7.3.1.2.

2. *Strict Epimorphisms in* **Rel**. The strict epimorphisms in **Rel** are characterised in Relations, ??.

14.2.10 Pseudoepic Morphisms

Let C be a bicategory.

Definition 14.2.10.1.1 ▶ Pseudoepic Morphisms

A 1-morphism $f\colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^{*}\colon \mathsf{Hom}_{C}(B, X) \to \mathsf{Hom}_{C}(A, X)$$

given by precomposition by f is pseudomonic.

Remark 14.2.10.1.2 ▶ Unwinding Definition 14.2.10.1.1

In detail, a 1-morphism $f\colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_{f} = \beta \star \mathrm{id}_{f},$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f \, .$$

Proposition 14.2.10.1.3 ▶ Properties of Pseudoepic Morphisms

Let $f\colon A \to B$ be a 1-morphism of $\mathcal{C}$.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.

   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

   (c) We have an isococomma square of the form

   $$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \arrow[dl, phantom, "\Leftarrow\Rightarrow"] \\ B \arrow[u, "\mathrm{id}_B"] & A \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

   in $\mathcal{C}$ up to equivalence.

Proof 14.2.10.1.4 ▶ Proof of Proposition 14.2.10.1.3

Item 1: Characterisations

Omitted. ∎

Appendices

A Other Chapters

References

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on p. 17).