Pointed Sets

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July 21, 2025

This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

DEFINITION 6.1.1.1.1 ▶ POINTED SETS

A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$**-modules**.

REMARK 6.1.1.1.2 ▶ UNWINDING DEFINITION 6.1.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

EXAMPLE 6.1.1.1.3 ▶ THE ZERO SPHERE

The 0-**sphere**[1] is the pointed set $(S^0, 0)$[2] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[2] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

Example 6.1.1.1.4 ▶ The Trivial Pointed Set

The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

Example 6.1.1.1.5 ▶ The Standard Pointed Set With $n + 1$ Elements

The **standard pointed set with** $n + 1$ **elements** is the pointed set $\langle n\rangle$ consisting of

- *The Underlying Set.* The set $\langle n\rangle$ defined by

$$\langle n\rangle \overset{\mathrm{def}}{=} \{*\} \cup \{1, \dots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n\rangle$.

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1 ▶ Morphisms of Pointed Sets

A **morphism of pointed sets**[1,2] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **pointed function**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

Remark 6.1.2.1.2 ▶ Unwinding Definition 6.1.2.1.1

In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1 ▶ The Category of Pointed Sets

The **category of pointed sets** is the category Sets_* defined equivalently as:

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**.

- The category Sets_* of Constructions With Categories, **??**.

Remark 6.1.3.1.2 ▶ Unwinding Definition 6.1.3.1.1

In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.

- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.

- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

of Sets_* at (X, x_0) is defined by[1]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X .$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[2]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\text{def}}{=} g \circ f.$$

[1]Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[2]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

6.1.4 Elementary Properties of Pointed Sets

PROPOSITION 6.1.4.1.1 ▶ ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:
   (a) Products, described as in Definition 6.2.3.1.1.
   (b) Pullbacks, described as in Definition 6.2.4.1.1.
   (c) Equalisers, described as in Definition 6.2.5.1.1.
2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:
   (a) Coproducts, described as in Definition 6.3.3.1.1.
   (b) Pushouts, described as in Definition 6.3.4.1.1;
   (c) Coequalisers, described as in Definition 6.3.5.1.1.

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[1]

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[2]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[3]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

   (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

   $$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

   sends:

      i. A pointed set (X, x_0) to X.
      ii. A pointed function

      $$f \colon (X, x_0) \to (Y, y_0)$$

      to the partial function

      $$\xi_f \colon X \to Y$$

      defined on $f^{-1}(Y \setminus y_0)$ and given by

      $$\xi_f(x) \overset{\text{def}}{=} f(x)$$

      for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\mathrm{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

[1] The category Sets_* does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

[2] In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[3] *Warning:* This is not an isomorphism of categories, only an equivalence.

Proof 6.1.4.1.2 ▶ Proof of Proposition 6.1.4.1.1

Item 1: Completeness

This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.

Item 2: Cocompleteness

This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.
The isomorphism then

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*\left(S^0, X\right)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1 ▶ Active and Inert Morphisms of Pointed Sets

Let $f \colon (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2 ▶ The Category of Pointed Sets and Active Morphisms

We write $\mathsf{Sets}_*^{\mathrm{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3 ▶ Examples of Active and Inert Maps of Pointed Sets

Here are some examples of active and inert maps of pointed sets.

1. The map $\mu\colon \langle 2\rangle \longrightarrow \langle 1\rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

   is active but not inert.

2. The map $f\colon \langle 2\rangle \longrightarrow \langle 2\rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[dr, mapsto] & 2 \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

   is inert but not active.

3. The map $f\colon \langle 3\rangle \longrightarrow \langle 1\rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & \\
3 \arrow[dr, mapsto] & \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

   is neither inert nor active. However, it factors as $f = a \circ i$, where

$$\begin{aligned}
i&\colon \langle 3\rangle \longrightarrow \langle 2\rangle,\\
a&\colon \langle 2\rangle \longrightarrow \langle 1\rangle
\end{aligned}$$

are the morphisms of pointed sets given by

$$\begin{array}{ccc} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & * \\ * & \longmapsto & * \end{array} \qquad \begin{array}{ccc} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 1 \\ * & \longmapsto & *, \end{array}$$

with i being inert and a being active.

Proposition 6.1.5.1.4 ▶ Properties of Active and Inert Maps of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets $f: (X, x_0) \to (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

   (a) The map $i: (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets

   (b) The map $a: (K, k_0) \to (Y, y_0)$ is an active morphism of pointed sets.

   Moreover, this determines an orthogonal factorisation system in Sets_*.

Proof 6.1.5.1.5 ▶ Proof of Proposition 6.1.5.1.4

Item 1: Active-Inert Factorisation

Let $f: X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= \left(X \setminus f^{-1}(y_0)\right) \cup \{x_0\}; \end{aligned}$$

- $i\colon X \longrightarrow K$ is the inert morphism of pointed sets given by

$$i(x) \overset{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

  for each $x \in X$;

- $a\colon K \longrightarrow Y$ is the active morphism of pointed sets given by

$$a(x) \overset{\text{def}}{=} f(x)$$

  for each $x \in K$.

Next, let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y \longrightarrow A$ given by

$$\phi(y) = f\left(i^{-1}(y)\right)$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\phi", "\exists!"'] \\ A \arrow[r, "a"'] & B. \end{tikzcd}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned}[\phi \circ i](x) &\overset{\text{def}}{=} \phi(i(x)) \\ &\overset{\text{def}}{=} f\big(i^{-1}(i(x))\big) \\ &= f(x)\end{aligned}$$

for each $x \in X$ and

$$\begin{aligned}[a \circ \phi](y) &\overset{\text{def}}{=} a(\phi(y)) \\ &\overset{\text{def}}{=} a\big(f\big(i^{-1}(y)\big)\big) \\ &\overset{\text{def}}{=} [a \circ f]\big(i^{-1}(y)\big) \\ &= [g \circ i]\big(i^{-1}(y)\big) \\ &\overset{\text{def}}{=} g\big(i\big(i^{-1}(y)\big)\big) \\ &\overset{\text{def}}{=} g(y)\end{aligned}$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned}\psi(y) &= \psi(i(x)) \\ &= f(x) \\ &= f\big(i^{-1}(y)\big) \\ &\overset{\text{def}}{=} \phi(y).\end{aligned}$$

This finishes the proof. ∎

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1 ▶ The Terminal Pointed Set

The **terminal pointed set** is the terminal object of Sets_* as in Limits and Colimits, **??**.

Construction 6.2.1.1.2 ▶ Construction of the Terminal Pointed Set

Concretely, the **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{!_X : (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$!_X(x) \stackrel{\text{def}}{=} \star$$

  for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof 6.2.1.1.3 ▶ Proof of Construction 6.2.1.1.2

We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\mathrm{pt}, \star)$$

commute, namely $!_X$.

6.2.2 Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.2.2.1.1 ► The Product of a Family of Pointed Sets

The **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the product of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.2.1.2 ► Construction of the Product of a Family of Pointed Sets

Concretely, the **product of** $\left\{(X_i, x_0^i)\right\}_{i\in I}$ is the pair $\left(\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right), \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i \colon \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \to (X_i, x_0^i)\right\}_{i\in I}$$

  of maps given by

$$\mathrm{pr}_i\left((x_j)_{j\in I}\right) \overset{\mathrm{def}}{=} x_i$$

  for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

PROOF 6.2.2.1.3 ► **PROOF OF CONSTRUCTION 6.2.2.1.2**

We claim that $\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ is the categorical product of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd}
(P, *) \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \to \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd}
(P, *) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\
\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right)
\end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(*) &= (p_i(*))_{i\in I} \\
&= \left(x_0^i\right)_{i\in I},
\end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. ∎

Proposition 6.2.2.1.4 ▶ Properties of Products of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ defines a functor

$$\prod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \longrightarrow \mathsf{Sets}_*.$$

Proof 6.2.2.1.5 ▶ Proof of Proposition 6.2.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1 ▶ Products of Pointed Sets

The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.3.1.2 ▶ Construction of Products of Pointed Sets

Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \longrightarrow (X, x_0),\\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \longrightarrow (Y, y_0) \end{aligned}$$

  defined by

$$\mathrm{pr}_1(x, y) \stackrel{\text{def}}{=} x,$$

$$\mathrm{pr}_2(x, y) \stackrel{\text{def}}{=} y$$

for each $(x, y) \in X \times Y$.

PROOF 6.2.3.1.3 ▶ PROOF OF CONSTRUCTION 6.2.3.1.2

We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\
(X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y, y_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
 & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
(X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y, y_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets.

Proposition 6.2.3.1.4 ▶ Properties of Products of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

   $$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

   $$\begin{aligned} A \times - \colon & \quad \mathsf{Sets}_* \quad\ \ \to \mathsf{Sets}_*, \\ - \times B \colon & \quad \mathsf{Sets}_* \quad\ \ \to \mathsf{Sets}_*, \\ -_1 \times -_2 \colon & \ \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{aligned}$$

   defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

   $$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Unitality.* We have isomorphisms of pointed sets

   $$\begin{aligned}(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),\end{aligned}$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof 6.2.3.1.5 ► Proof of Proposition 6.2.3.1.4

Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.

Item 2: Lack of Adjointness

See [MSE 2855868].

Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

Item 4: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.

Item 5: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.

6.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1 ▶ Pullbacks of Pointed Sets

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.4.1.2 ▶ Construction of Pullbacks of Pointed Sets

Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned} \mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \longrightarrow (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \longrightarrow (Y, y_0) \end{aligned}$$

  defined by

  $$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times_Z Y$.

Proof 6.2.4.1.3 ▶ Proof of Construction 6.2.4.1.2

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned}
\left[f \circ \mathrm{pr}_1\right](x, y) &= f\big(\mathrm{pr}_1(x, y)\big) \\
&= f(x) \\
&= g(y) \\
&= g\big(\mathrm{pr}_2(x, y)\big) \\
&= \left[g \circ \mathrm{pr}_2\right](x, y),
\end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P, *) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

in Sets$_*$. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \longrightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd}
(P, *) \arrow[drr, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.4.1.4 ► Properties of Pullbacks of Pointed Sets

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor
$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$
In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by

sending a morphism

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \left(X' \times_{Z'} Y', \left(x_0', y_0'\right)\right)$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}[column sep=small]
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\vee" very near start] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned} X \times_X A &\cong A, \\ A \times_X X &\cong A, \end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y, \qquad \begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\ X \arrow[r, "!_X"'] & \mathrm{pt}. \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof 6.2.4.1.5 ▶ **Proof of Proposition 6.2.4.1.4**

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.4.1.7.

Item 3: Unitality

This follows from Constructions With Sets, Item 6 of Proposition 4.1.4.1.7.

Item 4: Commutativity

This follows from Constructions With Sets, Item 7 of Proposition 4.1.4.1.7.

Item 5: Interaction With Products

This follows from Constructions With Sets, Item 10 of Proposition 4.1.4.1.7.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 11 of Proposition 4.1.4.1.7.

□

6.2.5 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1 ► Equalisers of Pointed Sets

The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in Limits and Colimits, **??**.

Construction 6.2.5.1.2 ► Construction of Equalisers of Pointed Sets

Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof 6.2.5.1.3 ► Proof of Construction 6.2.5.1.2

We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= e(*) \\ &= x_0, \end{aligned}$$

where we have used that e is a morphism of pointed sets. ∎

Proposition 6.2.5.1.4 ▶ Properties of Equalisers of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \longrightarrow (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \underset{h}{\overset{f}{\underset{-g\rightarrow}{\rightrightarrows}}} (Y, y_0)$$

in Sets_*, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof 6.2.5.1.5 ► Proof of Proposition 6.2.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.1.5.1.4.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.1.5.1.4.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.1.5.1.4.

□

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1 ▶ The Initial Pointed Set

The **initial pointed set** is the initial object of Sets_* as in Limits and Colimits, ??.

Construction 6.3.1.1.2 ▶ Construction of the Initial Pointed Set

Concretely, the **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{\iota_X \colon (\mathrm{pt}, \star) \to (X, x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$\iota_X(\star) \stackrel{\mathrm{def}}{=} x_0.$$

Proof 6.3.1.1.3 ▶ Proof of Construction 6.3.1.1.2

We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathrm{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X. ∎

6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

Definition 6.3.2.1.1 ▶ Coproducts of Families of Pointed Sets

The **coproduct of the family** $\{(X_i, x_0^i)\}_{i\in I}$[1] is the coproduct of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_* as in Limits and Colimits, **??**.

[1] *Further Terminology:* Also called the **wedge sum of the family** $\{(X_i, x_0^i)\}_{i\in I}$.

Construction 6.3.2.1.2 ▶ Construction of Coproducts of Families of Pointed Sets

Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i\in I}$ is the pair $\left((\bigvee_{i\in I} X_i, p_0), \{\mathrm{inj}_i\}_{i\in I}\right)$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by

    $$\bigvee_{i\in I} X_i \overset{\mathrm{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring

    $$(i, x_0^i) \sim \left(j, x_0^j\right)$$

    for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by

    $$\begin{aligned} p_0 &\overset{\mathrm{def}}{=} \left[(i, x_0^i)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$

    for any $i, j \in I$.
- *The Cocone.* The collection

  $$\left\{\mathrm{inj}_i \colon (X_i, x_0^i) \to \left(\bigvee_{i\in I} X_i, p_0\right)\right\}_{i\in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

PROOF 6.3.2.1.3 ► PROOF OF CONSTRUCTION 6.3.2.1.2

We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C, *) \\ \left(X_i, x_0^i\right) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee\limits_{i\in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon \left(\bigvee_{i\in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & (C, *) \\ \left(X_i, x_0^i\right) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee\limits_{i\in I} X_i, p_0\right) \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i\left(\left[\left(i, x_0^i\right)\right]\right)$$

$$= *,$$

as ι_i is a morphism of pointed sets.

Proposition 6.3.2.1.4 ▶ Properties of Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor
$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \longrightarrow \mathsf{Sets}_*.$$

Proof 6.3.2.1.5 ▶ Proof of Proposition 6.3.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1 ▶ Coproducts of Pointed Sets

The **coproduct of** (X, x_0) **and** (Y, y_0)[1] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

[1] *Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

Construction 6.3.3.1.2 ▶ Construction of Coproducts of Pointed Sets

Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

- *The Underlying Set.* The set $X \vee Y$ defined by

$$
\begin{aligned}
(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \coprod (Y, y_0)\\
&\cong \left(X \textstyle\coprod_{\mathrm{pt}} Y, p_0\right)\\
&\cong (X \textstyle\coprod Y/{\sim}, p_0),
\end{aligned}
\qquad
\begin{tikzcd}
X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l]\\
X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"']
\end{tikzcd}
$$

  where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$
\begin{aligned}
p_0 &\overset{\text{def}}{=} [(0, x_0)]\\
&= [(1, y_0)].
\end{aligned}
$$

• *The Cocone.* The morphisms of pointed sets

$$
\begin{aligned}
\mathrm{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\
\mathrm{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),
\end{aligned}
$$

given by

$$
\begin{aligned}
\mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)],\\
\mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)],
\end{aligned}
$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.3.1.3 ▶ PROOF OF CONSTRUCTION 6.3.3.1.2

We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & (X \vee Y, p_0) & (Y,y_0) \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"]
\end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \vee Y, p_0) \longrightarrow (C, *)$$

making the diagram

$$\begin{tikzcd}
 & (C,*) & \\
(X,x_0) \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y,y_0) \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"]
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_X &= \iota_X,\\
\phi \circ \mathrm{inj}_Y &= \iota_Y
\end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0,x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1,y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

$$\begin{aligned} &= \iota_Y([(1, y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets.

Proposition 6.3.3.1.4 ► Properties of Wedge Sums of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee -&\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \vee Y&\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ -_1 \vee -_2&\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & {} & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof 6.3.3.1.5 ▶ Proof of Proposition 6.3.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, **??** of **??**.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation.

6.3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1 ► Pushouts of Pointed Sets

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

Construction 6.3.4.1.2 ▶ Construction of Pushouts of Pointed Sets

Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 \colon (X, x_0) &\longrightarrow (X \coprod_Z Y, p_0),\\ \mathrm{inj}_2 \colon (Y, y_0) &\longrightarrow (X \coprod_Z Y, p_0)\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)]\\ \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)]\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof 6.3.4.1.3 ▶ Proof of ??

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}x_0 &= f(z_0),\\ y_0 &= g(z_0)\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check

that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned} \left[\mathrm{inj}_1 \circ f\right](z) &= \mathrm{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \mathrm{inj}_2(g(z)) \\ &= \left[\mathrm{inj}_2 \circ g\right](z), \end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} (P, *) & & \\ & (X \coprod_Z Y, p_0) \arrow[dr, phantom, "\ulcorner"] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P,*) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!", "\phi"'] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)]) \\
&= \iota_1(x_0) \\
&= *,
\end{aligned}$$

or alternatively

$$\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)]) \\
&= \iota_2(y_0) \\
&= *,
\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. ∎

Proposition 6.3.4.1.4 ► Properties of Pushouts of Pointed Sets

Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:
$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$
In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism
$$\begin{tikzcd}[row sep=small, column sep=small]
X \coprod_Z Y \arrow[dr, phantom, "\ulcorner", very near start] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, "\ulcorner", very near start] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets
$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} \left(X' \coprod_{Z'} Y', p'_0\right)$$
given by
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$
for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed

sets making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, dashed, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \coprod_X A &\cong A, \\
A \coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[d, equal] & X \arrow[l, "f"'] \arrow[d, equal] \\
X & X. \arrow[l, "f"]
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} X \coprod_Z Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & Z, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad X \coprod_Z Y \cong Y \coprod_Z X \qquad \begin{tikzcd} Y \coprod_Z X \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[l] \\ Y \arrow[u] & Z. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts.* We have

$$X \coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\ X \arrow[u] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"'] \end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

PROOF 6.3.4.1.5 ▶ PROOF OF PROPOSITION 6.3.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.

Item 3: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.

Item 4: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.

Item 5: Interaction With Coproducts

Omitted.

Item 6: Symmetric Monoidality

Omitted.

6.3.5 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1 ► Coequalisers of Pointed Sets

The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Construction 6.3.5.1.2 ► Construction of Coequalisers of Pointed Sets

The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit*. The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
- *The Cocone*. The map

  $$\mathrm{coeq}(f, g)\colon Y \twoheadrightarrow (\mathrm{CoEq}(f, g), [y_0])$$

  given by the quotient map, as in Constructions With Sets, Item 2 of Construction 4.2.5.1.2.

Proof 6.3.5.1.3 ► Proof of Construction 6.3.5.1.2

We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned}[\mathrm{coeq}(f,g)\circ f](x) &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](f(x))\\ &\overset{\mathrm{def}}{=} [f(x)]\\ &= [g(x)]\\ &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](g(x))\\ &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)\circ g](x)\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \\ & & (C,*) \end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Proposition 10.6.2.1.3 that there exists a unique map $\phi\colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, two heads, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\ & & (C,*) \end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}\phi([y_0]) &= [\phi\circ\mathrm{coeq}(f,g)]([y_0])\\ &= c([y_0])\\ &= *,\end{aligned}$$

where we have used that c is a morphism of pointed sets. □

Proposition 6.3.5.1.4 ▶ Properties of Coequalisers of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} (X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, dashed, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0) \end{tikzcd}$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

Proof 6.3.5.1.5 ▶ Proof of Proposition 6.3.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.2.5.1.7.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.2.5.1.7.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.5.1.7.

∎

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let X be a set.

Definition 6.4.1.1.1 ▶ Free Pointed Sets

The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[1]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

[1] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

Proposition 6.4.1.1.2 ▶ Properties of Free Pointed Sets

Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ : \mathsf{Sets} \longrightarrow \mathsf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \overset{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

- *Action on Morphisms.* For each morphism $f : X \longrightarrow Y$ of Sets, the image

$$f^+ : X^+ \longrightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^{+} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection of sets

$$\mathsf{Sets}_{*}\left((X^{+}, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y}\colon X^{+} \vee Y^{+} \xrightarrow{\sim} (X \textstyle\coprod Y)^{+},$$
$$(-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+}, (-)^{+}, (-)^{+}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_{*}, \wedge, S^{0}\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+}_{X,Y}\colon X^{+} \wedge Y^{+} \xrightarrow{\sim} (X \times Y)^{+},$$
$$(-)^{+}_{\mathbb{1}}\colon S^{0} \xrightarrow{\sim} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.4.1.1.3 ► PROOF OF PROPOSITION 6.4.1.1.2

Item 1: Functoriality

We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}^+_X(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

  for each $x \in X^+$, so $\mathrm{id}^+_X = \mathrm{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$\begin{aligned} f &: X \to Y, \\ g &: Y \to Z, \end{aligned}$$

  we have

$$\begin{aligned} \left[g^+ \circ f^+\right](x) &\stackrel{\text{def}}{=} g^+\left(f^+(x)\right) \\ &\stackrel{\text{def}}{=} g^+(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^+(x) \end{aligned}$$

  for each $x \in X$ and

$$\begin{aligned} \left[g^+ \circ f^+\right](\star_X) &\stackrel{\text{def}}{=} g^+\left(f^+(\star_X)\right) \\ &\stackrel{\text{def}}{=} g^+(\star_Y) \\ &\stackrel{\text{def}}{=} \star_Z \\ &\stackrel{\text{def}}{=} [g \circ f]^+(\star_X), \end{aligned}$$

  so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2: Adjointness

We proceed in a few steps:

- *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_*\big(X^+, Y\big) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

to the function

$$\xi^\dagger \colon X \to Y$$

given by

$$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*\big(X^+, Y\big)$$

given by sending a function $\xi \colon X \to Y$ to the morphism of pointed sets

$$\xi^\dagger \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

defined by

$$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* Given a morphism of pointed sets

$$\xi \colon \big(X^+, \star_X\big) \to (Y, y_0),$$

we have

$$\begin{aligned} \big[\Psi_{X,Y} \circ \Phi_{X,Y}\big](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}\big(\Phi_{X,Y}(\xi)\big) \\ &= \Psi_{X,Y}\Big(\xi^\dagger\Big) \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \left[\!\left[x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right] \\
&= \left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right] \\
&= \xi \\
&\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}\right](\xi).
\end{aligned}
$$

Therefore we have

$$
\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)} .
$$

- *Invertibility II*. Given a map of sets $\xi\colon X \longrightarrow Y$, we have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right) \\
&= \Phi_{X,Y}\left(\xi^{\dagger}\right) \\
&= \Phi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\right]\right) \\
&= \left[\!\left[x \mapsto \xi(x) \right]\!\right] \\
&= \xi \\
&\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,Y)}\right](\xi).
\end{aligned}
$$

Therefore we have

$$
\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)} .
$$

- *Naturality for Φ, Part I*. We need to show that, given a morphism of pointed sets

$$
f\colon (X, x_0) \longrightarrow \left(X', x_0'\right),
$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(X'^{,+}, Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+} \to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Φ, Part II*. We need to show that, given a morphism of pointed sets

$$g\colon (Y, y_0) \to \left(Y', y_0'\right),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*(X^+, Y'), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger}\colon X^{+} \to Y,$$

we have

$$\begin{aligned}
\left[\Phi_{X,Y'} \circ g_{*}\right](\xi) &= \Phi_{X,Y'}(g_{*}(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_{*}\left(\Phi_{X,Y'}(\xi)\right) \\
&= \left[g_{*} \circ \Phi_{X,Y'}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_{*} = g_{*} \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for* Ψ*.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^{+}$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+,\coprod}_{X,Y}\colon X^{+} \vee Y^{+} \overset{\sim}{\dashrightarrow} \left(X \coprod Y\right)^{+}$$

is given by

$$(-)^{+,\coprod}_{X,Y}(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0, \star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)^{+,\coprod,-1}_{X,Y} \colon (X \coprod Y)^+ \overset{\sim}{\dashrightarrow} X^+ \vee Y^+$$

given by

$$(-)^{+,\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\coprod,\mathbb{1}}_{X,Y} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \emptyset^+$$

is given by sending $\star_X$ to $\star_\emptyset$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to $\times$ and $\wedge$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+}_{X,Y} \colon X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+$$

is given by

$$(-)^{+}_{X,Y}(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)^{+,-1}_{X,Y} \colon (X \times Y)^+ \overset{\sim}{\dashrightarrow} X^+ \wedge Y^+$$

given by

$$(-)^{+,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\mathbb{1}}_{X,Y} \colon S^0 \overset{\sim}{\dashrightarrow} \mathrm{pt}^+$$

is given by sending 0 to $\star_{\mathrm{pt}}$ and 1 to $\star$, where $\mathrm{pt}^+ = \{\star, \star_{\mathrm{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1 ► Sets With Deleted Basepoints

The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2 ► Properties of Sets With Deleted Basepoints

Let (X, x_0) be a pointed set.

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- \colon \mathsf{Sets}^{\mathrm{actv}}_* \longrightarrow \mathsf{Sets},$$

where:

- *Action on Objects.* For each $X \in \mathrm{Obj}\left(\mathsf{Sets}^{\mathrm{actv}}_*\right)$, we have

$$[(-)^-](X) \overset{\text{def}}{=} X^-,$$

where X^- is the set of Definition 6.4.2.1.1.

- *Action on Morphisms.* For each morphism $f\colon X \to Y$ of $\mathsf{Sets}_*^{\mathrm{actv}}$, the image
$$f^-\colon X^- \to Y^-$$
of f by $(-)^-$ is the map defined by
$$f^-(x) \overset{\mathrm{def}}{=} f(x)$$
for each $x \in X^-$.

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories
$$\big((-)^- \dashv (-)^+\big)\colon \quad \begin{tikzcd} \mathsf{Sets}_*^{\mathrm{actv}} \arrow[r, bend left, "(-)^-"] \arrow[r, phantom, "\perp_{\mathrm{eq}}"] & \mathsf{Sets}, \arrow[l, bend left, "(-)^+"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*\big(X, Y^+\big),$$
natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms
$$(X^-)^+ \cong X,$$
$$\big(Y^+\big)^- \cong Y,$$
once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure
$$\Big((-)^-, (-)^{-,\vee}, (-)^{-,\vee}_{\mathbb{1}}\Big)\colon \big(\mathsf{Sets}_*^{\mathrm{actv}}, \vee, \mathrm{pt}\big), \to \big(\mathsf{Sets}, \textstyle\coprod, \emptyset\big),$$
being equipped with isomorphisms of pointed sets
$$(-)^{-,\vee}_{X,Y}\colon X^- \textstyle\coprod Y^- \overset{\sim}{\dashrightarrow} (X \vee Y)^-,$$
$$(-)^{-,\vee}_{\mathbb{1}}\colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^-,$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-}, (-)^{-,\times}, (-)^{-,\times}_{\mathbb{1}}\right)\colon \left(\mathsf{Sets}_{*}^{\mathrm{actv}}, \wedge, S^{0}\right), \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned}(-)^{-}_{X,Y}\colon X^{-} \times Y^{-} &\overset{\sim}{\dashrightarrow} (X \wedge Y)^{-},\\ (-)^{-}_{\mathbb{1}}\colon \mathrm{pt} &\overset{\sim}{\dashrightarrow} \left(S^{0}\right)^{-},\end{aligned}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

PROOF 6.4.2.1.3 ▶ PROOF OF PROPOSITION 6.4.2.1.2

Item 1: Functoriality

We claim that $(-)^{-}$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

  $$\mathrm{id}^{-}_{X}(x) \overset{\mathrm{def}}{=} x$$

  for each $x \in X^{-}$, so $\mathrm{id}^{-}_{X} = \mathrm{id}_{X^{-}}$.

- *Preservation of Composition.* Given morphisms of pointed sets

  $$\begin{aligned}f\colon (X, x_0) &\to (Y, y_0),\\ g\colon (Y, y_0) &\to (Z, z_0),\end{aligned}$$

  we have

  $$\begin{aligned}[g^{-} \circ f^{-}](x) &\overset{\mathrm{def}}{=} g^{-}(f^{-}(x))\\ &\overset{\mathrm{def}}{=} g^{-}(f(x))\\ &\overset{\mathrm{def}}{=} g(f(x))\\ &\overset{\mathrm{def}}{=} [g \circ f]^{-}(x)\end{aligned}$$

  for each $x \in X$, so $(g \circ f)^{-} = g^{-} \circ f^{-}$.

This finishes the proof.

Item 2: Adjoint Equivalence

We proceed in a few steps:

1. *Map I.* We define a map
$$\Phi_{X,Y}\colon \mathsf{Sets}(X^-,Y) \to \mathsf{Sets}_*^{\mathrm{actv}}\left(X,Y^+\right)$$
by sending a map $\xi\colon X^- \to Y$ to the active morphism of pointed sets
$$\xi^\dagger\colon X \to Y^+$$
given by
$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$
for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. *Map II.* We define a map
$$\Psi_{X,Y}\colon \mathsf{Sets}_*^{\mathrm{actv}}\left(X,Y^+\right) \to \mathsf{Sets}(X^-,Y)$$
given by sending an active morphism of pointed sets $\xi\colon X \to Y^+$ to the map
$$\xi^\dagger\colon X^- \to Y$$
defined by
$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \xi(x)$$
for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi\colon X^- \to Y$, we have
$$\begin{aligned}\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\overset{\mathrm{def}}{=} \Psi_{X,Y}\left(\Phi_{X,Y}(\xi)\right) \\ &\overset{\mathrm{def}}{=} \Psi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]\right)\end{aligned}$$

$$
\begin{aligned}
&= [\![x \mapsto \xi(x)]\!] \\
&= \xi \\
&= \left[\mathrm{id}_{\mathsf{Sets}(X^-,Y)}\right](\xi).
\end{aligned}
$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}\,.$$

4. *Invertibility II.* Given a morphism of pointed sets

$$\xi\colon (X, x_0) \longrightarrow \left(Y^+, \star_Y\right),$$

we have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right) \\
&= \Phi_{X,Y}([\![x \mapsto \xi(x)]\!]) \\
&= [\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}]\!] \\
&= \xi \\
&= \left[\mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\right](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\,.$$

5. *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \longrightarrow \left(X', x_0'\right),$$

the diagram

$$
\begin{tikzcd}
\mathsf{Sets}\left(X'^{,-}, Y\right) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X', Y^+) \arrow[d, "f^*"] \\
\mathsf{Sets}_*(X^-, Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)
\end{tikzcd}
$$

commutes. Indeed, given a map of sets $\xi\colon X' \to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y}\circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi))\\
&= \Phi_{X,Y}(\xi\circ f)\\
&= \left[\!\left[x \mapsto \begin{cases}\xi(f(x)) & \text{if } f(x)\in X'^{,-}\\ \star_Y & \text{if } f(x) = x'_0\end{cases}\right]\!\right]\\
&= f^*\left(\left[\!\left[x' \mapsto \begin{cases}\xi(x') & \text{if } x'\in X'^{,-}\\ \star_Y & \text{if } x' = x'_0\end{cases}\right]\!\right]\right)\\
&= f^*\left(\Phi_{X',Y}(\xi)\right)\\
&= \left[f^*\circ\Phi_{X',Y}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y}\circ f^* = f^*\circ\Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g\colon (Y,y_0) \to \left(Y',y'_0\right),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}(X^-,Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+) \arrow[d, "g_*"]\\
\mathsf{Sets}(X^-,Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y'^{,+})
\end{tikzcd}$$

commutes. Indeed, given a map of sets $\xi\colon X^- \to Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y'}\circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi))\\
&= \Phi_{X,Y'}(g\circ\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \left[\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \right]\!\right] \\
&= g_*\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases} \right]\!\right] \right) \\
&= g_*\big(\Phi_{X,Y'}(\xi)\big) \\
&= \big[g_* \circ \Phi_{X,Y'}\big](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

7. *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

8. *Fully Faithfulness of* $(-)^-$. We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)^-_{X,Y} \colon \mathsf{Sets}_*^{\mathrm{actv}}(X, Y) \overset{\sim}{\dashrightarrow} \mathsf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)^{-,-1}_{X,Y} \colon \mathsf{Sets}(X^-, Y^-) \overset{\sim}{\dashrightarrow} \mathsf{Sets}_*^{\mathrm{actv}}(X, Y)$$

is given by sending a map of sets $f \colon X^- \longrightarrow Y^-$ to the active morphism of pointed sets $f^\dagger \colon X \longrightarrow Y$ defined by

$$f^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

9. *Essential Surjectivity of* $(-)^-$. We need to show that, given an object $X \in \mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}(\mathsf{Sets}_*^{\mathrm{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$\begin{aligned}(X^+)^- &\stackrel{\text{def}}{=} (X \cup \{\star_X\})^- \\ &\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X,\end{aligned}$$

   and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. *The Functor* $(-)^-$ *Is an Equivalence.* Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Proposition 11.6.7.1.2.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^-$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-,\vee}_{X,Y} \colon X^- \textstyle\coprod Y^- \xrightarrow{\sim} (X \vee Y)^-$$

  is given by

$$(-)^{-,\vee}_{X,Y}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

  for each $z \in X^- \coprod Y^-$, with inverse

$$(-)^{-,\vee,-1}_{X,Y} \colon (X \vee Y)^- \xrightarrow{\sim} X^- \textstyle\coprod Y^-$$

  given by

$$(-)^{-,\vee,-1}_{X,Y}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

  for each $z \in (X \vee Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\vee,\mathbb{1}}_{X,Y} \colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^{-}$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{-}$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^{+}$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \overset{\sim}{\dashrightarrow} (X \wedge Y)^{-}$$

is given by

$$(-)^{-}_{X,Y}(x,y) = x \wedge y$$

for each $(x,y) \in X^{-} \times Y^{-}$, with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \overset{\sim}{\dashrightarrow} X^{-} \times Y^{-}$$

given by

$$(-)^{-,-1}_{X,Y}(x \wedge y) \overset{\mathrm{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{-,\mathbb{1}}_{X,Y} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^{0}\right)^{-}$$

is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted.

Appendices

A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

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