

# Tensor Products of Pointed Sets

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**00C3** In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

introduced in **Section 7.5.1**, defined via a universal property as inducing a bijection between the following data:

- Pointed maps  $f : X \wedge Y \rightarrow Z$ .
- Maps of sets  $f : X \times Y \rightarrow Z$  satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ .

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \triangleleft : \mathbf{Sets}_* \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ \triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to  $\wedge$ , which turns out to endow  $\mathbf{Sets}_*$  with a monoidal category structure (**Definition 7.5.9.1.1**),

these do not admit invertible associators and unitors, but do endow  $\mathbf{Sets}_*$  with the structure of a skew monoidal category, however ([Definitions 7.3.8.1.1](#) and [7.4.8.1.1](#)).

Finally, in addition to the tensor products  $\triangleleft$ ,  $\triangleleft$ , and  $\wedge$ , we also have a “tensor product” of the form

$$\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\otimes_{k,\ell} : \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{B}_\ell}(\mathbf{Sets}) \rightarrow \mathbf{Mon}_{\mathbb{B}_{k+\ell}}(\mathbf{Sets}),$$

$$\triangleleft_{i,k} : \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}) \rightarrow \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}),$$

$$\triangleright_{i,k} : \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}) \times \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}) \rightarrow \mathbf{Mon}_{\mathbb{B}_k}(\mathbf{Sets}),$$

where  $k, \ell, i \in \mathbb{N}$  with  $i \leq k - 1$ . Together with the Cartesian product  $\times$  of  $\mathbf{Sets}$ , the tensor products studied in this chapter form the cases:

- $(k, \ell) = (-1, -1)$  for the Cartesian product of  $\mathbf{Sets}$ ;
- $(k, \ell) = (0, -1)$  and  $(-1, 0)$  for the tensor of sets with pointed sets of [Definition 7.2.1.1.1](#);
- $(i, k) = (-1, 0)$  for the left and right tensor products of pointed sets of [Sections 7.3](#) and [7.4](#);
- $(k, \ell) = (-1, -1)$  for the smash product of pointed sets of [Section 7.5](#).

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in [??](#), we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to  $k, \ell \leq 2$ , and hence *all* cases since  $\mathbb{B}_k$ -monoids on  $\mathbf{Sets}$  are the same as  $\mathbb{B}_2$ -monoids on  $\mathbf{Sets}$  when  $k \geq 2$ .

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## 00C4 7.1 Bilinear Morphisms of Pointed Sets

### 00C5 7.1.1 Left Bilinear Morphisms of Pointed Sets

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

00C6 **Definition 7.1.1.1.** A **left bilinear morphism of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  **is a map of sets**

$$f: X \times Y \rightarrow Z$$

satisfying the following condition:<sup>1,2</sup>

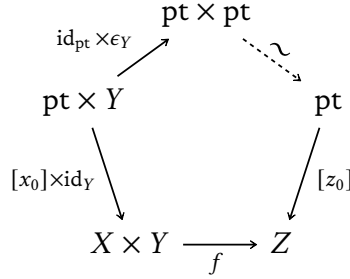
<sup>1</sup>*Slogan:* The map  $f$  is left bilinear if it preserves basepoints in its first argument.

<sup>2</sup>Succinctly,  $f$  is bilinear if we have

$$f(x_0, y) = z_0$$

for each  $y \in Y$ .

(★) *Left Unital Bilinearity*. The diagram



commutes, i.e. for each  $y \in Y$ , we have

$$f(x_0, y) = z_0.$$

**00C7 Definition 7.1.1.1.2.** The **set of left bilinear morphisms of pointed sets from  $(X \times Y, (x_0, y_0))$  to  $(Z, z_0)$**  is the set  $\text{Hom}_{\text{Sets}_*}^{\otimes, L}(X \times Y, Z)$  defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, L}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

### 00C8 7.1.2 Right Bilinear Morphisms of Pointed Sets

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

**00C9 Definition 7.1.2.1.1.** A **right bilinear morphism of pointed sets from  $(X \times Y, (x_0, y_0))$  to  $(Z, z_0)$**  is a map of sets

$$f: X \times Y \rightarrow Z$$

satisfying the following condition:<sup>3,4</sup>

<sup>3</sup>*Slogan:* The map  $f$  is right bilinear if it preserves basepoints in its second argument.

<sup>4</sup>Succinctly,  $f$  is bilinear if we have

$$f(x, y_0) = z_0$$

for each  $x \in X$ .

(★) *Right Unital Bilinearity.* The diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & \nearrow \epsilon_X \times \text{id}_{\text{pt}} & & \searrow \sim & \\
 X \times \text{pt} & & & & \text{pt} \\
 \downarrow \text{id}_X \times [y_0] & & & & \downarrow [z_0] \\
 X \times Y & \xrightarrow{f} & Z & & 
 \end{array}$$

commutes, i.e. for each  $x \in X$ , we have

$$f(x, y_0) = z_0.$$

**00CA Definition 7.1.2.1.2.** The **set of right bilinear morphisms of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  is the set  $\text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z)$  defined by

$$\text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

### 00CB 7.1.3 Bilinear Morphisms of Pointed Sets

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.

**00CC Definition 7.1.3.1.1.** A **bilinear morphism of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  is a map of sets

$$f: X \times Y \rightarrow Z$$

that is both left bilinear and right bilinear.

**00CD Remark 7.1.3.1.2.** In detail, a **bilinear morphism of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  is a map of sets

$$f: (X \times Y, (x_0, y_0)) \rightarrow (Z, z_0)$$

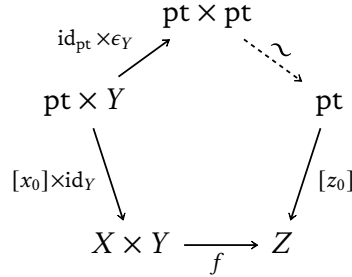
satisfying the following conditions:<sup>5,6</sup>

<sup>5</sup>*Slogan:* The map  $f$  is bilinear if it preserves basepoints in each argument.

<sup>6</sup>Succinctly,  $f$  is bilinear if we have

$$f(x_0, y) = z_0,$$

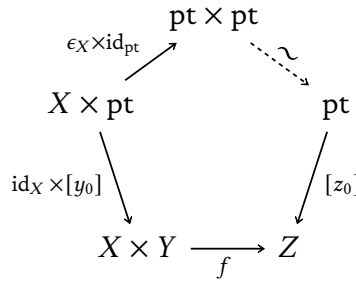
025T 1. *Left Unital Bilinearity.* The diagram



commutes, i.e. for each  $y \in Y$ , we have

$$f(x_0, y) = z_0.$$

025U 2. *Right Unital Bilinearity.* The diagram



commutes, i.e. for each  $x \in X$ , we have

$$f(x, y_0) = z_0.$$

00CE **Definition 7.1.3.1.3.** The **set of bilinear morphisms of pointed sets from**  $(X \times Y, (x_0, y_0))$  **to**  $(Z, z_0)$  **is the set**  $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$  **defined by**

$$\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \text{Hom}_{\text{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

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$$f(x, y_0) = z_0$$

for each  $x \in X$  and each  $y \in Y$ .

## 00CF 7.2 Tensors and Cotensors of Pointed Sets by Sets

### 00CG 7.2.1 Tensors of Pointed Sets by Sets

Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

00CH **Definition 7.2.1.1.1.** The **tensor of  $(X, x_0)$  by  $A$** <sup>7</sup> is the tensor  $A \odot (X, x_0)$ <sup>8</sup> of  $(X, x_0)$  by  $A$  as in Limits and Colimits, ??.

00CJ **Remark 7.2.1.1.2.** In detail, the **tensor of  $(X, x_0)$  by  $A$**  is the pointed set  $A \odot (X, x_0)$  satisfying the following universal property:

(★) We have a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K)),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ .

This universal property is in turn equivalent to the following one:

(★) We have a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K),$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ , where  $\text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$  is the set defined by

$$\text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \text{Sets}(A \times X, K) \left| \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, x_0) = k_0 \end{array} \right. \right\}.$$

*Proof.* We claim that we have a bijection

$$\text{Sets}(A, \text{Sets}_*(X, K)) \cong \text{Sets}_{\mathbb{E}_0}^{\otimes}(A \times X, K)$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ . Indeed, this bijection is a restriction of the bijection

$$\text{Sets}(A, \text{Sets}(X, K)) \cong \text{Sets}(A \times X, K)$$

of **Constructions With Sets, Item 2** of **Definition 4.1.3.1.3**:

<sup>7</sup>*Further Terminology:* Also called the **copower of  $(X, x_0)$  by  $A$** .

<sup>8</sup>*Further Notation:* Often written  $A \odot X$  for simplicity.



- A map

$$\xi: A \longrightarrow \text{Sets}_*(X, K),$$

$$a \mapsto (\xi_a: X \rightarrow K),$$

in  $\text{Sets}(A, \text{Sets}_*(X, K))$  gets sent to the map

$$\xi^\dagger: A \times X \rightarrow K$$

defined by

$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each  $(a, x) \in A \times X$ , which indeed lies in  $\text{Sets}_{\mathbb{E}_0}^\otimes(A \times X, K)$ , as we have

$$\begin{aligned} \xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &\stackrel{\text{def}}{=} k_0 \end{aligned}$$

for each  $a \in A$ , where we have used that  $\xi_a \in \text{Sets}_*(X, K)$  is a morphism of pointed sets.

- Conversely, a map

$$\xi: A \times X \rightarrow K$$

in  $\text{Sets}_{\mathbb{E}_0}^\otimes(A \times X, K)$  gets sent to the map

$$\xi^\dagger: A \longrightarrow \text{Sets}_*(X, K),$$

$$a \mapsto (\xi_a^\dagger: X \rightarrow K),$$

where

$$\xi_a^\dagger: X \rightarrow K$$

is the map defined by

$$\xi_a^\dagger(x) \stackrel{\text{def}}{=} \xi(a, x)$$

for each  $x \in X$ , and indeed lies in  $\text{Sets}_*(X, K)$ , as we have

$$\begin{aligned} \xi_a^\dagger(x_0) &\stackrel{\text{def}}{=} \xi(a, x_0) \\ &\stackrel{\text{def}}{=} k_0. \end{aligned}$$

This finishes the proof.  $\square$

**00CK Construction 7.2.1.1.3.** Concretely, the **tensor of**  $(X, x_0)$  **by**  $A$  is the pointed set  $A \odot (X, x_0)$  consisting of:

- *The Underlying Set.* The set  $A \odot X$  given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0),$$

where  $\bigvee_{a \in A} (X, x_0)$  is the wedge product of the  $A$ -indexed family  $((X, x_0))_{a \in A}$  of **Pointed Sets**, **Definition 6.3.2.1.1**.

- *The Basepoint.* The point  $[(a, x_0)] = [(a', x_0)]$  of  $\bigvee_{a \in A} (X, x_0)$ .

*Proof.* (Proven below in a bit.)  $\square$

**00CL Notation 7.2.1.1.4.** We write  $a \odot x$  for the element  $[(a, x)]$  of

$$\begin{aligned} A \odot X &\cong \bigvee_{a \in A} (X, x_0) \\ &\stackrel{\text{def}}{=} \left( \coprod_{i \in I} X_i \right) / \sim. \end{aligned}$$

**00CM Remark 7.2.1.1.5.** Taking the tensor of any element of  $A$  with the basepoint  $x_0$  of  $X$  leads to the same element in  $A \odot X$ , i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each  $a, a' \in A$ . This is due to the equivalence relation  $\sim$  on

$$\bigvee_{a \in A} (X, x_0) \stackrel{\text{def}}{=} \coprod_{a \in A} X / \sim$$

identifying  $(a, x_0)$  with  $(a', x_0)$ , so that the equivalence class  $a \odot x_0$  is independent from the choice of  $a \in A$ .

*Proof.* We claim we have a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K))$$

natural in  $(K, k_0) \in \text{Obj}(\text{Sets}_*)$ .

025V 1. *Map I.* We define a map

$$\Phi_K: \text{Sets}_*(A \odot X, K) \rightarrow \text{Sets}(A, \text{Sets}_*(X, K))$$

by sending a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0)$$

to the map of sets

$$\begin{aligned} \xi^\dagger: A &\rightarrow \text{Sets}_*(X, K), \\ a &\mapsto (\xi_a: X \rightarrow K), \end{aligned}$$

where

$$\xi_a: (X, x_0) \rightarrow (K, k_0)$$

is the morphism of pointed sets defined by

$$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

for each  $x \in X$ . Note that we have

$$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

so that  $\xi_a$  is indeed a morphism of pointed sets, where we have used that  $\xi$  is a morphism of pointed sets.

025W 2. *Map II.* We define a map

$$\Psi_K: \text{Sets}(A, \text{Sets}_*(X, K)) \rightarrow \text{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi: A &\rightarrow \text{Sets}_*(X, K), \\ a &\mapsto (\xi_a: X \rightarrow K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger: (A \odot X, a \odot x_0) \rightarrow (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each  $a \odot x \in A \odot X$ . Note that  $\xi^\dagger$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that  $\xi(a) \in \text{Sets}_*(X, K)$  is a morphism of pointed sets.

**025X** 3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \text{id}_{\text{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \llbracket x \mapsto \xi(a \odot x) \rrbracket \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket) \\ &= \llbracket a \odot x \mapsto \text{ev}_x(\text{ev}_a(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket a \odot x \mapsto \text{ev}_x(\llbracket x' \mapsto \xi(a \odot x') \rrbracket) \rrbracket \\ &= \llbracket a \odot x \mapsto \xi(a \odot x) \rrbracket \\ &= \xi. \end{aligned}$$

**025Y** 4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \text{id}_{\text{Sets}(A, \text{Sets}_*(X, K))}.$$

Indeed, given a morphism  $\xi: A \rightarrow \text{Sets}_*(X, K)$ , we have

$$\begin{aligned} [\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket a \odot x \mapsto \xi_a(x) \rrbracket) \\ &= \llbracket a \mapsto \llbracket x \mapsto \xi_a(x) \rrbracket \rrbracket \\ &= \llbracket a \mapsto \xi(a) \rrbracket \\ &= \xi. \end{aligned}$$

- 025Z 5. *Naturality of  $\Phi$* . We need to show that, given a morphism of pointed sets

$$\phi: (K, k_0) \rightarrow (K', k'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}_*(A \odot X, K) & \xrightarrow{\Phi_K} & \text{Sets}(A, \text{Sets}_*(X, K)) \\ \phi_* \downarrow & & \downarrow (\phi_*)_* \\ \text{Sets}_*(A \odot X, K') & \xrightarrow{\Phi_{K'}} & \text{Sets}(A, \text{Sets}_*(X, K')) \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi: (A \odot X, a \odot x_0) \rightarrow (K, k_0),$$

we have

$$\begin{aligned} [\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\ &= \Phi_{K'}(\phi \circ \xi) \\ &= (\phi \circ \xi)^\dagger \\ &= \llbracket a \mapsto \phi \circ \xi(a \odot -) \rrbracket \\ &= \llbracket a \mapsto \phi_*(\xi(a \odot -)) \rrbracket \\ &= (\phi_*)_*(\llbracket a \mapsto \xi(a \odot -) \rrbracket) \\ &= (\phi_*)_*(\Phi_K(\xi)) \\ &= [(\phi_*)_* \circ \Phi_K](\xi). \end{aligned}$$

- 0260 6. *Naturality of  $\Psi$* . Since  $\Phi$  is natural and  $\Phi$  is a componentwise inverse to  $\Psi$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\Psi$  is also natural.

This finishes the proof.  $\square$

00CN **Proposition 7.2.1.1.6.** Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

- 00CP 1. *Functoriality*. The assignments  $A, (X, x_0), (A, (X, x_0))$  define functors

$$\begin{aligned} A \odot -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ - \odot X: \quad \text{Sets} &\rightarrow \text{Sets}_*, \\ -_1 \odot -_2: \text{Sets} \times \text{Sets}_* &\rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given:

- A map of sets  $f: A \rightarrow B$ ;
- A pointed map  $\phi: (X, x_0) \rightarrow (Y, y_0)$ ;

the induced map

$$f \odot \phi: A \odot X \rightarrow B \odot Y$$

is given by

$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$

for each  $a \odot x \in A \odot X$ .

**00CQ** 2. *Adjointness I.* We have an adjunction

$$(- \odot X \dashv \text{Sets}_*(X, -)): \text{Sets} \begin{array}{c} \xrightarrow{- \odot X} \\ \perp \\ \xleftarrow{\text{Sets}_*(X, -)} \end{array} \text{Sets}_*$$

witnessed by a bijection

$$\text{Sets}_*(A \odot X, K) \cong \text{Sets}(A, \text{Sets}_*(X, K)),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

**00CR** 3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -): \text{Sets}_* \begin{array}{c} \xrightarrow{A \odot -} \\ \perp \\ \xleftarrow{A \pitchfork -} \end{array} \text{Sets}_*$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets}_*}(A \odot X, Y) \cong \text{Hom}_{\text{Sets}_*}(X, A \pitchfork Y),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

**00CS** 4. *As a Weighted Colimit.* We have

$$A \odot X \cong \text{colim}^{[A]}(X),$$

where in the right hand side we write:

- $A$  for the functor  $A: \mathbf{pt} \rightarrow \mathbf{Sets}$  picking  $A \in \mathbf{Obj}(\mathbf{Sets})$ ;
- $X$  for the functor  $X: \mathbf{pt} \rightarrow \mathbf{Sets}_*$  picking  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

**00CT** 5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in  $A, B \in \mathbf{Obj}(\mathbf{Sets})$  and  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

**00CU** 6. *Interaction With Homs.* We have a natural isomorphism

$$\mathbf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathbf{Sets}_*(X, -).$$

**00CV** 7. *The Tensor Evaluation Map.* For each  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , we have a map

$$\mathrm{ev}_{X,Y}^\odot: \mathbf{Sets}_*(X, Y) \odot X \rightarrow Y,$$

natural in  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , and given by

$$\mathrm{ev}_{X,Y}^\odot(f \odot x) \stackrel{\mathrm{def}}{=} f(x)$$

for each  $f \odot x \in \mathbf{Sets}_*(X, Y) \odot X$ .

**00CW** 8. *The Tensor Coevaluation Map.* For each  $A \in \mathbf{Obj}(\mathbf{Sets})$  and each  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$ , we have a map

$$\mathrm{coev}_{A,X}^\odot: A \rightarrow \mathbf{Sets}_*(X, A \odot X),$$

natural in  $A \in \mathbf{Obj}(\mathbf{Sets})$  and  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$ , and given by

$$\mathrm{coev}_{A,X}^\odot(a) \stackrel{\mathrm{def}}{=} \llbracket x \mapsto a \odot x \rrbracket$$

for each  $a \in A$ .

*Proof.* **Item 1, Functoriality:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 2, Adjointness I:** This is simply a rephrasing of **Definition 7.2.1.1.1**.

**Item 3, Adjointness II:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 4, As a Weighted Colimit:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

*Item 5, Iterated Tensors:* This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

*Item 6, Interaction With Homs:* This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

*Item 7, The Tensor Evaluation Map:* This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

*Item 8, The Tensor Coevaluation Map:* This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .  $\square$

## 00CX 7.2.2 Cotensors of Pointed Sets by Sets

Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

00CY **Definition 7.2.2.1.1.** The **cotensor of  $(X, x_0)$  by  $A$** <sup>9</sup> is the cotensor  $A \pitchfork (X, x_0)$ <sup>10</sup> of  $(X, x_0)$  by  $A$  as in Limits and Colimits, ??.

00CZ **Remark 7.2.2.1.2.** In detail, the **cotensor of  $(X, x_0)$  by  $A$**  is the pointed set  $A \pitchfork (X, x_0)$  satisfying the following universal property:

(★) We have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in  $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

This universal property is in turn equivalent to the following one:

(★) We have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X),$$

natural in  $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ , where  $\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X)$  is the set defined by

$$\mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Sets}(A \times K, X) \left| \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right. \right\}.$$

<sup>9</sup>*Further Terminology:* Also called the **power of  $(X, x_0)$  by  $A$** .

<sup>10</sup>*Further Notation:* Often written  $A \pitchfork X$  for simplicity.



*Proof.* This follows from the bijection

$$\mathbf{Sets}(A, \mathbf{Sets}_*(K, X)) \cong \mathbf{Sets}_{\mathbb{E}_0}^{\otimes}(A \times K, X),$$

natural in  $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$  constructed in the proof of **Definition 7.2.1.1.2.**

□

**00D0 Construction 7.2.2.1.3.** Concretely, the **cotensor of**  $(X, x_0)$  **by**  $A$  is the pointed set  $A \pitchfork (X, x_0)$  consisting of:

- *The Underlying Set.* The set  $A \pitchfork X$  given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

where  $\bigwedge_{a \in A} (X, x_0)$  is the smash product of the  $A$ -indexed family  $((X, x_0))_{a \in A}$  of **Definition 7.6.1.1.1.**

- *The Basepoint.* The point  $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \dots)]$  of  $\bigwedge_{a \in A} (X, x_0)$ .

*Proof.* We claim we have a bijection

$$\mathbf{Sets}_*(K, A \pitchfork X) \cong \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

natural in  $(K, k_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$ .

**0261** 1. *Map I.* We define a map

$$\Phi_K: \mathbf{Sets}_*(K, A \pitchfork X) \rightarrow \mathbf{Sets}(A, \mathbf{Sets}_*(K, X)),$$

by sending a morphism of pointed sets

$$\xi: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

to the map of sets

$$\begin{aligned} \xi^\dagger: A &\rightarrow \mathbf{Sets}_*(K, X), \\ a &\mapsto (\xi_a: K \rightarrow X), \end{aligned}$$

where

$$\xi_a: (K, k_0) \rightarrow (X, x_0)$$

is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x_a^k & \text{if } \xi(k) \neq [(x_0)_{a \in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a \in A}] \end{cases}$$

for each  $k \in K$ , where  $x_a^k$  is the  $a$ th component of  $\xi(k) = [(x_a^k)_{a \in A}]$ . Note that:

- 0262 (a) The definition of  $\xi_a(k)$  is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned} \xi(k) &= [(x_a^k)_{a \in A}] \\ &= [(y_a^k)_{a \in A}] \end{aligned}$$

with  $x_a^k \neq y_a^k$  for some  $a \in A$ . Then there exist  $a_x, a_y \in A$  such that  $x_{a_x}^k = y_{a_y}^k = x_0$ . The equivalence relation  $\sim$  on  $\prod_{a \in A} X$  then forces

$$\begin{aligned} [(x_a^k)_{a \in A}] &= [(x_0)_{a \in A}], \\ [(y_a^k)_{a \in A}] &= [(x_0)_{a \in A}], \end{aligned}$$

however, and  $\xi_a(k)$  is defined to be  $x_0$  in this case.

- 0263 (b) The map  $\xi_a$  is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since  $\xi(k_0) = [(x_0)_{a \in A}]$  as  $\xi$  is a morphism of pointed sets and  $\xi_a(k_0)$ , defined to be the  $a$ th component of  $[(x_0)_{a \in A}]$ , is equal to  $x_0$ .

- 0264 2. *Map II.* We define a map

$$\Psi_K: \text{Sets}(A, \text{Sets}_*(K, X)) \rightarrow \text{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned} \xi: A &\longrightarrow \text{Sets}_*(K, X), \\ a &\longmapsto (\xi_a: K \longrightarrow X), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

defined by

$$\xi^\dagger(k) \stackrel{\text{def}}{=} [(\xi_a(k))_{a \in A}]$$

for each  $k \in K$ . Note that  $\xi^\dagger$  is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(k_0) &\stackrel{\text{def}}{=} [(\xi_a(k_0))_{a \in A}] \\ &= x_0, \end{aligned}$$

where we have used that  $\xi_a \in \text{Sets}_*(K, X)$  is a morphism of pointed sets for each  $a \in A$ .

**0265** 3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \text{id}_{\text{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi: (K, k_0) \rightarrow (A \pitchfork X, [(x_0)_{a \in A}])$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \xi_a \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \xi_{a'} \rrbracket) \\ &= \llbracket k \mapsto [(\text{ev}_a(\llbracket a' \mapsto \xi_{a'}(k) \rrbracket))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket. \end{aligned}$$

Now, we have two cases:

**0266** (a) If  $\xi(k) = [(x_0)_{a \in A}]$ , we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(x_0)_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto \xi(k) \rrbracket \\ &= \xi. \end{aligned}$$

0267 (b) If  $\xi(k) \neq [(x_0)_{a \in A}]$  and  $\xi(k) = [(x_a^k)_{a \in A}]$  instead, we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto [(x_a^k)_{a \in A}] \rrbracket \\ &= \llbracket k \mapsto \xi(k) \rrbracket \\ &= \xi. \end{aligned}$$

In both cases, we have  $[\Psi_K \circ \Phi_K](\xi) = \xi$ , and thus we are done.

0268 4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \text{id}_{\text{Sets}(A, \text{Sets}_*(K, X))}.$$

Indeed, given a morphism  $\xi: A \rightarrow \text{Sets}_*(K, X)$ , we have

$$\begin{aligned} [\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket) \\ &= \llbracket a \mapsto \llbracket k \mapsto \xi_a(k) \rrbracket \rrbracket \\ &= \xi \end{aligned}$$

0269 5. *Naturality of  $\Psi$ .* We need to show that, given a morphism of pointed sets

$$\phi: (K, k_0) \rightarrow (K', k'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}(A, \text{Sets}_*(K', X)) & \xrightarrow{\Psi_{K'}} & \text{Sets}_*(K', A \pitchfork X) \\ (\phi^*)_* \downarrow & & \downarrow \phi^* \\ \text{Sets}(A, \text{Sets}_*(K, X)) & \xrightarrow{\Psi_K} & \text{Sets}_*(K, A \pitchfork X) \end{array}$$

commutes. Indeed, given a map of sets

$$\begin{aligned} \xi: A &\rightarrow \text{Sets}_*(K', X), \\ a &\mapsto (\xi_a: K' \rightarrow X), \end{aligned}$$

we have

$$\begin{aligned}
 [\Psi_K \circ (\phi^*)_*](\xi) &= \Psi_K((\phi^*)_*(\xi)) \\
 &= \Psi_K((\phi^*)_*([a \mapsto \xi_a])) \\
 &= \Psi_K([a \mapsto \phi^*(\xi_a)]) \\
 &= \Psi_K([a \mapsto [k \mapsto \xi_a(\phi(k))]]) \\
 &= [k \mapsto [(\xi_a(\phi(k)))_{a \in A}]] \\
 &= \phi^*([k' \mapsto [(\xi_a(k'))_{a \in A}]]) \\
 &= \phi^*(\Psi_{K'}(\xi)) \\
 &= [\phi^* \circ \Psi_{K'}](\xi).
 \end{aligned}$$

- 026A** 6. *Naturality of  $\Phi$ .* Since  $\Psi$  is natural and  $\Psi$  is a componentwise inverse to  $\Phi$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\Phi$  is also natural.

This finishes the proof.  $\square$

- 00D1** **Proposition 7.2.2.1.4.** Let  $(X, x_0)$  be a pointed set and let  $A$  be a set.

- 00D2** 1. *Functoriality.* The assignments  $A, (X, x_0), (A, (X, x_0))$  define functors

$$\begin{aligned}
 A \pitchfork - &: \text{Sets}_* \rightarrow \text{Sets}_*, \\
 - \pitchfork X &: \text{Sets}^{\text{op}} \rightarrow \text{Sets}_*, \\
 -_1 \pitchfork -_2 &: \text{Sets}^{\text{op}} \times \text{Sets}_* \rightarrow \text{Sets}_*.
 \end{aligned}$$

In particular, given:

- A map of sets  $f: A \rightarrow B$ ;
- A pointed map  $\phi: (X, x_0) \rightarrow (Y, y_0)$ ;

the induced map

$$f \odot \phi: A \pitchfork X \rightarrow B \pitchfork Y$$

is given by

$$[f \odot \phi]([(x_a)_{a \in A}]) \stackrel{\text{def}}{=} [(\phi(x_{f(a)}))_{a \in A}]$$

for each  $[(x_a)_{a \in A}] \in A \pitchfork X$ .

00D3 2. *Adjointness I.* We have an adjunction

$$(- \pitchfork X \dashv \text{Sets}_*(-, X)) : \text{Sets}^{\text{op}} \begin{array}{c} \xrightarrow{- \pitchfork X} \\ \perp \\ \xleftarrow{\text{Sets}_*(-, X)} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Sets}_*^{\text{op}}(A \pitchfork X, K) \cong \text{Sets}(A, \text{Sets}_*(K, X)),$$

i.e. by a bijection

$$\text{Sets}_*(K, A \pitchfork X) \cong \text{Sets}(A, \text{Sets}_*(K, X)),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

00D4 3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -) : \text{Sets}_* \begin{array}{c} \xrightarrow{A \odot -} \\ \perp \\ \xleftarrow{A \pitchfork -} \end{array} \text{Sets}_*,$$

witnessed by a bijection

$$\text{Hom}_{\text{Sets}_*}(A \odot X, Y) \cong \text{Hom}_{\text{Sets}_*}(X, A \pitchfork Y),$$

natural in  $A \in \text{Obj}(\text{Sets})$  and  $X, Y \in \text{Obj}(\text{Sets}_*)$ .

00D5 4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- $A$  for the functor  $A : \text{pt} \rightarrow \text{Sets}$  picking  $A \in \text{Obj}(\text{Sets})$ ;
- $X$  for the functor  $X : \text{pt} \rightarrow \text{Sets}_*$  picking  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

00D6 5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in  $A, B \in \text{Obj}(\text{Sets})$  and  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

00D7 6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \mathbf{Sets}_*(X, -) &\cong \mathbf{Sets}_*(A \odot X, -), \\ A \pitchfork \mathbf{Sets}_*(-, Y) &\cong \mathbf{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

00D8 7. *The Cotensor Evaluation Map.* For each  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , we have a map

$$\mathrm{ev}_{X,Y}^{\pitchfork}: X \rightarrow \mathbf{Sets}_*(X, Y) \pitchfork Y,$$

natural in  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , and given by

$$\mathrm{ev}_{X,Y}^{\pitchfork}(x) \stackrel{\mathrm{def}}{=} [(f(x))_{f \in \mathbf{Sets}_*(X,Y)}]$$

for each  $x \in X$ .

00D9 8. *The Cotensor Coevaluation Map.* For each  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$  and each  $A \in \mathbf{Obj}(\mathbf{Sets})$ , we have a map

$$\mathrm{coev}_{A,X}^{\pitchfork}: A \rightarrow \mathbf{Sets}_*(A \pitchfork X, X),$$

natural in  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$  and  $A \in \mathbf{Obj}(\mathbf{Sets})$ , and given by

$$\mathrm{coev}_{A,X}^{\pitchfork}(a) \stackrel{\mathrm{def}}{=} \llbracket [(x_b)_{b \in A}] \mapsto x_a \rrbracket$$

for each  $a \in A$ .

*Proof.* **Item 1, Functoriality:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 2, Adjointness I:** This is simply a rephrasing of **Definition 7.2.2.1.1**.

**Item 3, : Adjointness II:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 4, As a Weighted Limit:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 5, Iterated Cotensors:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 6, Commutativity With Homs:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 7, The Cotensor Evaluation Map:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .

**Item 8, The Cotensor Coevaluation Map:** This is the special case of Limits and Colimits, ?? of ?? for  $C = \mathbf{Sets}_*$ .  $\square$

## 00DA 7.3 The Left Tensor Product of Pointed Sets

### 00DB 7.3.1 Foundations

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00DC **Definition 7.3.1.1.1.** The **left tensor product of pointed sets** is the functor<sup>11</sup>

$$\triangleleft : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_* \times \mathbf{Sets}_* \xrightarrow{\text{id} \times \omega} \mathbf{Sets}_* \times \mathbf{Sets} \xrightarrow{\beta_{\mathbf{Sets}_*, \mathbf{Sets}}^{\mathbf{Cats}_2}} \mathbf{Sets} \times \mathbf{Sets}_* \xrightarrow{\odot} \mathbf{Sets}_*,$$

where:

- $\omega : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.
- $\beta_{\mathbf{Sets}_*, \mathbf{Sets}}^{\mathbf{Cats}_2} : \mathbf{Sets}_* \times \mathbf{Sets} \xrightarrow{\sim} \mathbf{Sets} \times \mathbf{Sets}_*$  is the braiding of  $\mathbf{Cats}_2$ , i.e. the functor witnessing the isomorphism

$$\mathbf{Sets}_* \times \mathbf{Sets} \cong \mathbf{Sets} \times \mathbf{Sets}_*.$$

- $\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the tensor functor of **Item 1** of **Definition 7.2.1.1.6**.

00DD **Remark 7.3.1.1.2.** The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}^{\otimes, L}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

026B 1. Pointed maps  $f : X \triangleleft Y \rightarrow Z$ .

026C 2. Maps of sets  $f : X \times Y \rightarrow Z$  satisfying  $f(x_0, y) = z_0$  for each  $y \in Y$ .

00DE **Remark 7.3.1.1.3.** The left tensor product of pointed sets may be described as follows:

<sup>11</sup>*Further Notation:* Also written  $\triangleleft_{\mathbf{Sets}_*}$ .



- The left tensor product of  $(X, x_0)$  and  $(Y, y_0)$  is the pair  $((X \triangleleft Y, x_0 \triangleleft y_0), \iota)$  consisting of
  - A pointed set  $(X \triangleleft Y, x_0 \triangleleft y_0)$ ;
  - A left bilinear morphism of pointed sets  $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \triangleleft Y$ ;

satisfying the following universal property:

- (★) Given another such pair  $((Z, z_0), f)$  consisting of
- \* A pointed set  $(Z, z_0)$ ;
  - \* A left bilinear morphism of pointed sets  $f: (X \times Y, (x_0, y_0)) \rightarrow Z$ ;

there exists a unique morphism of pointed sets  $X \triangleleft Y \xrightarrow{\exists!} Z$  making the diagram

$$\begin{array}{ccc}
 & X \triangleleft Y & \\
 \iota \nearrow & & \downarrow \exists! \\
 X \times Y & \xrightarrow{f} & Z
 \end{array}$$

commute.

**00DF Construction 7.3.1.1.4.** In detail, the **left tensor product of  $(X, x_0)$  and  $(Y, y_0)$**  is the pointed set  $(X \triangleleft Y, [x_0])$  consisting of:

- *The Underlying Set.* The set  $X \triangleleft Y$  defined by

$$\begin{aligned}
 X \triangleleft Y &\stackrel{\text{def}}{=} |Y| \odot X \\
 &\cong \bigvee_{y \in Y} (X, x_0),
 \end{aligned}$$

where  $|Y|$  denotes the underlying set of  $(Y, y_0)$ .

- *The Underlying Basepoint.* The point  $[(y_0, x_0)]$  of  $\bigvee_{y \in Y} (X, x_0)$ , which is equal to  $[(y, x_0)]$  for any  $y \in Y$ .

*Proof.* Since  $\bigvee_{y \in Y} (X, x_0)$  is defined as the quotient of  $\coprod_{y \in Y} X$  by the equivalence relation  $R$  generated by declaring  $(y, x) \sim (y', x')$  if  $x = x' = x_0$ , we have, by **Conditions on Relations, ??**, a natural bijection

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \mathbf{Hom}_{\mathbf{Sets}}^R\left(\coprod_{y \in Y} X, Z\right),$$

where  $\mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z)$  is the set

$$\mathbf{Hom}_{\mathbf{Sets}}^R\left(\coprod_{y \in Y} X, Z\right) \stackrel{\text{def}}{=} \left\{ f \in \mathbf{Hom}_{\mathbf{Sets}}\left(\coprod_{y \in Y} X, Z\right) \left| \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (y, x) \sim_R (y', x'), \text{ then} \\ f(y, x) = f(y', x') \end{array} \right. \right\}.$$

However, the condition  $(y, x) \sim_R (y', x')$  only holds when:

- 026D 1. We have  $x = x'$  and  $y = y'$ .
- 026E 2. We have  $x = x' = x_0$ .

So, given  $f \in \mathbf{Hom}_{\mathbf{Sets}}(\coprod_{y \in Y} X, Z)$  with a corresponding  $\bar{f}: X \triangleleft Y \rightarrow Z$ , the latter case above implies

$$\begin{aligned} f([(y, x_0)]) &= f([(y', x_0)]) \\ &= f([(y_0, x_0)]), \end{aligned}$$

and since  $\bar{f}: X \triangleleft Y \rightarrow Z$  is a pointed map, we have

$$\begin{aligned} f([(y_0, x_0)]) &= \bar{f}([(y_0, x_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements  $f$  in  $\mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z)$  are precisely those functions  $f: X \times Y \rightarrow Z$  satisfying the equality

$$f(x_0, y) = z_0$$

for each  $y \in Y$ , giving an equality

$$\mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z) = \mathbf{Hom}_{\mathbf{Sets}_*}^{\otimes, L}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \mathbf{Hom}_{\mathbf{Sets}}^R(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof.  $\square$

**00DG Notation 7.3.1.1.5.** We write<sup>12</sup>  $x \triangleleft y$  for the element  $[(y, x)]$  of

$$X \triangleleft Y \cong |Y| \odot X.$$

**00DH Remark 7.3.1.1.6.** Employing the notation introduced in **Definition 7.3.1.1.5**, we have

$$x_0 \triangleleft y_0 = x_0 \triangleleft y$$

for each  $y \in Y$ , and

$$x_0 \triangleleft y = x_0 \triangleleft y'$$

for each  $y, y' \in Y$ .

**00DJ Proposition 7.3.1.1.7.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00DK** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto X \triangleleft Y$  define functors

$$\begin{aligned} X \triangleleft -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ - \triangleleft Y: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ -_1 \triangleleft -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f: (X, x_0) &\rightarrow (A, a_0), \\ g: (Y, y_0) &\rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \triangleleft g: X \triangleleft Y \rightarrow A \triangleleft B$$

is given by

$$[f \triangleleft g](x \triangleleft y) \stackrel{\text{def}}{=} f(x) \triangleleft g(y)$$

for each  $x \triangleleft y \in X \triangleleft Y$ .

**00DL** 2. *Adjointness I.* We have an adjunction

$$\left( - \triangleleft Y \dashv [Y, -]_{\text{Sets}_*}^{\triangleleft} \right): \text{Sets}_* \begin{array}{c} \xrightarrow{- \triangleleft Y} \\ \perp \\ \xleftarrow{[Y, -]_{\text{Sets}_*}^{\triangleleft}} \end{array} \text{Sets}_*$$

<sup>12</sup>*Further Notation:* Also written  $x \triangleleft_{\text{Sets}_*} y$ .

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft})$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ , where  $[X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$  is the pointed set of [Definition 7.3.2.1.1](#).

**00DM** 3. *Adjointness II.* The functor

$$X \triangleleft -: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

**00DN** 4. *Adjointness III.* We have a  $\overline{\mathbf{Set}}$ -relative adjunction

$$(X \triangleleft - \dashv \mathbf{Sets}_*(X, -)) : \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \triangleleft -} \\ \perp_{\overline{\mathbf{Set}}} \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|Y|, \mathbf{Sets}_*(X, Z))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ .

*Proof.* **Item 1, Functoriality:** This follows from the definition of  $\triangleleft$  as a composition of functors ([Definition 7.3.1.1.1](#)).

**Item 2, Adjointness I:** This follows from **Item 3** of [Definition 7.2.1.1.6](#).

**Item 3, Adjointness II:** For  $X \triangleleft -$  to admit a right adjoint would require it to preserve colimits by [??, ??](#) of [??](#). However, we have

$$\begin{aligned} X \triangleleft \mathrm{pt} &\stackrel{\mathrm{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\neq \mathrm{pt}, \end{aligned}$$

and thus we see that  $X \triangleleft -$  does not have a right adjoint.

**Item 4, Adjointness III:** This follows from **Item 2** of [Definition 7.2.1.1.6](#). □

**00DP Remark 7.3.1.1.8.** Here is some intuition on why  $X \triangleleft -$  fails to be a left adjoint. **Item 4** of **Definition 7.3.1.1.7** states that we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|Y|, \mathbf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give  $X \triangleleft - \dashv \mathbf{Sets}_*(X, -)$ . However, such a bijection would require every map

$$f: X \triangleleft Y \rightarrow Z$$

to satisfy

$$f(x \triangleleft y_0) = z_0$$

for each  $x \in X$ , whereas we are imposing such a basepoint preservation condition only for elements of the form  $x_0 \triangleleft y$ . Thus  $\mathbf{Sets}_*(X, -)$  can't be a right adjoint for  $X \triangleleft -$ , and as shown by **Item 3** of **Definition 7.3.1.1.7**, no functor can.<sup>13</sup>

## 00DQ 7.3.2 The Left Internal Hom of Pointed Sets

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00DR Definition 7.3.2.1.1.** The **left internal Hom**<sup>14</sup> of pointed sets is the functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\overline{\omega} \times \mathrm{id}} \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\pitchfork} \mathbf{Sets}_*,$$

where:

- $\overline{\omega}: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.

<sup>13</sup>The functor  $\mathbf{Sets}_*(X, -)$  is instead right adjoint to  $X \wedge -$ , the smash product of pointed sets of **Definition 7.5.1.1.1**. See **Item 2** of **Definition 7.5.1.1.10**.

<sup>14</sup>For a proof that  $[-, -]_{\mathbf{Sets}_*}^{\triangleleft}$  is indeed the left internal Hom of  $\mathbf{Sets}_*$  with respect to the left tensor product of pointed sets, see **Item 2** of **Definition 7.3.1.1.7**.

- $\pitchfork : \mathbf{Sets}^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the cotensor functor of **Item 1** of **Definition 7.2.2.1.4**.

**00DS Remark 7.3.2.1.2.** The left internal Hom of pointed sets satisfies the following universal property:

$$\mathbf{Sets}_*(X \triangleleft Y, Z) \cong \mathbf{Sets}_*(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft})$$

That is to say, the following data are in bijection:

- 026F** 1. Pointed maps  $f : X \triangleleft Y \rightarrow Z$ .
- 026G** 2. Pointed maps  $f : X \rightarrow [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft}$ .

**00DT Remark 7.3.2.1.3.** In detail, the **left internal Hom of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pointed set  $([X, Y]_{\mathbf{Sets}_*}^{\triangleleft}, [(y_0)_{x \in X}])$  consisting of:

- *The Underlying Set.* The set  $[X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$  defined by

$$\begin{aligned} [X, Y]_{\mathbf{Sets}_*}^{\triangleleft} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0), \end{aligned}$$

where  $|X|$  denotes the underlying set of  $(X, x_0)$ .

- *The Underlying Basepoint.* The point  $[(y_0)_{x \in X}]$  of  $\bigwedge_{x \in X} (Y, y_0)$ .

**00DU Proposition 7.3.2.1.4.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

- 00DV** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto [X, Y]_{\mathbf{Sets}_*}^{\triangleleft}$  define functors

$$\begin{aligned} [X, -]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ [-, Y]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_*^{\text{op}} &\rightarrow \mathbf{Sets}_*, \\ [-, -]_{\mathbf{Sets}_*}^{\triangleleft} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f : (X, x_0) &\rightarrow (A, a_0), \\ g : (Y, y_0) &\rightarrow (B, b_0), \end{aligned}$$

the induced map

$$[f, g]_{\mathbf{Sets}_*}^{\triangleleft} : [A, Y]_{\mathbf{Sets}_*}^{\triangleleft} \rightarrow [X, B]_{\mathbf{Sets}_*}^{\triangleleft}$$

is given by

$$[f, g]_{\mathbf{Sets}_*}^{\triangleleft}([(y_a)_{a \in A}]) \stackrel{\text{def}}{=} [(g(y_{f(x)}))_{x \in X}]$$

for each  $[(y_a)_{a \in A}] \in [A, Y]_{\mathbf{Sets}_*}^{\triangleleft}$ .

**00DW** 2. *Adjointness I.* We have an adjunction

$$\left( - \triangleleft Y \dashv [Y, -]_{\mathbf{Sets}_*}^{\triangleleft} \right): \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleleft Y} \\ \perp \\ \xleftarrow{[Y, -]_{\mathbf{Sets}_*}^{\triangleleft}} \end{array} \mathbf{Sets}_*$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleleft Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, [Y, Z]_{\mathbf{Sets}_*}^{\triangleleft})$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$

**00DX** 3. *Adjointness II.* The functor

$$X \triangleleft -: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

*Proof. Item 1, Functoriality:* This follows from the definition of  $[-, -]_{\mathbf{Sets}_*}^{\triangleleft}$  as a composition of functors (**Definition 7.3.2.1.1**).

*Item 2, Adjointness I:* This is a repetition of **Item 2** of **Definition 7.3.1.1.7**, and is proved there.

*Item 3, Adjointness II:* This is a repetition of **Item 3** of **Definition 7.3.1.1.7**, and is proved there.  $\square$

### **00DY** 7.3.3 The Left Skew Unit

**00DZ** **Definition 7.3.3.1.1.** The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}_{\mathbf{Sets}_*, \triangleleft}: \text{pt} \rightarrow \mathbf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathbf{Sets}_*}^{\triangleleft} \stackrel{\text{def}}{=} S^0.$$

### 7.3.4 The Left Skew Associator

**Definition 7.3.4.1.1.** The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\text{Sets}_*, \triangleleft} : \triangleleft \circ (\triangleleft \times \text{id}_{\text{Sets}_*}) \Longrightarrow \triangleleft \circ (\text{id}_{\text{Sets}_*} \times \triangleleft) \circ \alpha^{\text{Cats}}_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}$$

as in the diagram

$$\begin{array}{ccc}
 & \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & \\
 \alpha^{\text{Cats}}_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*} \nearrow & & \searrow \text{id} \times \triangleleft \\
 (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & \text{Sets}_* \times \text{Sets}_* \\
 \triangleleft \times \text{id} \searrow & \alpha^{\text{Sets}_*, \triangleleft} \nearrow & \searrow \triangleleft \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\triangleleft} & \text{Sets}_*
 \end{array}$$

whose component

$$\alpha^{\text{Sets}_*, \triangleleft}_{X, Y, Z} : (X \triangleleft Y) \triangleleft Z \rightarrow X \triangleleft (Y \triangleleft Z)$$

at  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned}
 (X \triangleleft Y) \triangleleft Z &\stackrel{\text{def}}{=} |Z| \odot (X \triangleleft Y) \\
 &\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X) \\
 &\cong \bigvee_{z \in Z} |Y| \odot X \\
 &\cong \bigvee_{z \in Z} \left( \bigvee_{y \in Y} X \right) \\
 &\rightarrow \bigvee_{[(z, y)] \in \bigvee_{z \in Z} Y} X \\
 &\cong \bigvee_{[(z, y)] \in |Z| \odot Y} X \\
 &\cong ||Z| \odot Y| \odot X \\
 &\stackrel{\text{def}}{=} |Y \triangleleft Z| \odot X
 \end{aligned}$$



$$\stackrel{\text{def}}{=} X \triangleleft (Y \triangleleft Z),$$

where the map

$$\bigvee_{z \in Z} \bigvee_{y \in Y} X \rightarrow \bigvee_{(z,y) \in \bigvee_{z \in Z} Y} X$$

is given by  $[(z, [(y, x)])] \mapsto [([(z, y)], x)]$ .

*Proof.* (Proven below in a bit.) □

**00E2 Remark 7.3.4.1.2.** Unwinding the notation for elements, we have

$$\begin{aligned} [(z, [(y, x)])] &\stackrel{\text{def}}{=} [(z, x \triangleleft y)] \\ &\stackrel{\text{def}}{=} (x \triangleleft y) \triangleleft z \end{aligned}$$

and

$$\begin{aligned} [([(z, y)], x)] &\stackrel{\text{def}}{=} [(y \triangleleft z, x)] \\ &\stackrel{\text{def}}{=} x \triangleleft (y \triangleleft z). \end{aligned}$$

So, in other words,  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$  acts on elements via

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} ((x \triangleleft y) \triangleleft z) \stackrel{\text{def}}{=} x \triangleleft (y \triangleleft z)$$

for each  $(x \triangleleft y) \triangleleft z \in (X \triangleleft Y) \triangleleft Z$ .

**00E3 Remark 7.3.4.1.3.** Taking  $y = y_0$ , we see that the morphism  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$  acts on elements as

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} ((x \triangleleft y_0) \triangleleft z) \stackrel{\text{def}}{=} x \triangleleft (y_0 \triangleleft z).$$

However, by the definition of  $\triangleleft$ , we have  $y_0 \triangleleft z = y_0 \triangleleft z'$  for all  $z, z' \in Z$ , preventing  $\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft}$  from being non-invertible.

*Proof.* Firstly, note that, given  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} : (X \triangleleft Y) \triangleleft Z \rightarrow X \triangleleft (Y \triangleleft Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} ((x_0 \triangleleft y_0) \triangleleft z_0) = x_0 \triangleleft (y_0 \triangleleft z_0).$$

Next, we claim that  $\alpha^{\text{Sets}_*, \triangleleft}$  is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} (X \triangleleft Y) \triangleleft Z & \xrightarrow{(f \triangleleft g) \triangleleft h} & (X' \triangleleft Y') \triangleleft Z' \\ \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}_*, \triangleleft} \\ X \triangleleft (Y \triangleleft Z) & \xrightarrow{f \triangleleft (g \triangleleft h)} & X' \triangleleft (Y' \triangleleft Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x \triangleleft y) \triangleleft z & \longmapsto & (f(x) \triangleleft g(y)) \triangleleft h(z) \\ \downarrow & & \downarrow \\ x \triangleleft (y \triangleleft z) & \longmapsto & f(x) \triangleleft (g(y) \triangleleft h(z)) \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}_*, \triangleleft}$  to be a natural transformation. This finishes the proof.  $\square$

### 00E4 7.3.5 The Left Skew Left Unitor

00E5 **Definition 7.3.5.1.1.** The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\text{Sets}_*, \triangleleft} : \triangleleft \circ (\mathbb{1}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \lambda_{\text{Sets}_*}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}_*, \triangleleft} : S^0 \triangleleft X \rightarrow X$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} S^0 \triangleleft X &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X, \end{aligned}$$

where  $\bigvee_{x \in X} S^0 \rightarrow X$  is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x \end{aligned}$$

for each  $x \in X$ .

*Proof.* (Proven below in a bit.)

□

**00E6 Remark 7.3.5.1.2.** In other words,  $\lambda_X^{\text{Sets}_*, \triangleleft}$  acts on elements as

$$\begin{aligned} \lambda_X^{\text{Sets}_*, \triangleleft} (0 \triangleleft x) &\stackrel{\text{def}}{=} x_0, \\ \lambda_X^{\text{Sets}_*, \triangleleft} (1 \triangleleft x) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each  $1 \triangleleft x \in S^0 \triangleleft X$ .

**00E7 Remark 7.3.5.1.3.** The morphism  $\lambda_X^{\text{Sets}_*, \triangleleft}$  is almost invertible, with its would-be-inverse

$$\phi_X : X \rightarrow S^0 \triangleleft X$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} 1 \triangleleft x$$

for each  $x \in X$ . Indeed, we have

$$\begin{aligned} [\lambda_X^{\text{Sets}_*, \triangleleft} \circ \phi](x) &= \lambda_X^{\text{Sets}_*, \triangleleft} (\phi(x)) \\ &= \lambda_X^{\text{Sets}_*, \triangleleft} (1 \triangleleft x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

so that

$$\lambda_X^{\text{Sets}_*, \triangleleft} \circ \phi = \text{id}_X$$

and

$$\begin{aligned} [\phi \circ \lambda_X^{\text{Sets}_*, \triangleleft}](1 \triangleleft x) &= \phi(\lambda_X^{\text{Sets}_*, \triangleleft}(1 \triangleleft x)) \\ &= \phi(x) \\ &= 1 \triangleleft x \\ &= [\text{id}_{S^0 \triangleleft X}](1 \triangleleft x), \end{aligned}$$

but

$$\begin{aligned} [\phi \circ \lambda_X^{\text{Sets}_*, \triangleleft}](0 \triangleleft x) &= \phi(\lambda_X^{\text{Sets}_*, \triangleleft}(0 \triangleleft x)) \\ &= \phi(x_0) \\ &= 1 \triangleleft x_0, \end{aligned}$$

where  $0 \triangleleft x \neq 1 \triangleleft x_0$ . Thus

$$\phi \circ \lambda_X^{\text{Sets}_*, \triangleleft} \stackrel{?}{=} \text{id}_{S^0 \triangleleft X}$$

holds for all elements in  $S^0 \triangleleft X$  except one.

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\lambda_X^{\text{Sets}_*, \triangleleft} : S^0 \triangleleft X \rightarrow X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\text{Sets}_*, \triangleleft}(0 \triangleleft x_0) = x_0.$$

Next, we claim that  $\lambda^{\text{Sets}_*, \triangleleft}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} S^0 \triangleleft X & \xrightarrow{\text{id}_{S^0} \triangleleft f} & S^0 \triangleleft Y \\ \lambda_X^{\text{Sets}_*, \triangleleft} \downarrow & & \downarrow \lambda_Y^{\text{Sets}_*, \triangleleft} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 \triangleleft x & & 0 \triangleleft x \mapsto 0 \triangleleft f(x) \\
 \downarrow & & \downarrow \\
 x_0 \mapsto f(x_0) & & y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 \triangleleft x \mapsto 1 \triangleleft f(x) \\
 \downarrow \quad \quad \downarrow \\
 x \mapsto f(x)
 \end{array}$$

and hence indeed commutes, showing  $\lambda^{\text{Sets}_*, \triangleleft}$  to be a natural transformation. This finishes the proof.  $\square$

### 00E8 7.3.6 The Left Skew Right Unitor

00E9 **Definition 7.3.6.1.1.** The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\text{Sets}_*, \triangleleft} : \rho_{\text{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleleft \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}),$$

whose component

$$\rho_X^{\text{Sets}_*, \triangleleft} : X \rightarrow X \triangleleft S^0$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$X \rightarrow X \vee X$$

$$\begin{aligned} &\cong |S^0| \odot X \\ &\cong X \triangleleft S^0, \end{aligned}$$

where  $X \rightarrow X \vee X$  is the map sending  $X$  to the second factor of  $X$  in  $X \vee X$ .

*Proof.* (Proven below in a bit.)  $\square$

**00EA Remark 7.3.6.1.2.** In other words,  $\rho_X^{\text{Sets}_*, \triangleleft}$  acts on elements as

$$\rho_X^{\text{Sets}_*, \triangleleft}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho_X^{\text{Sets}_*, \triangleleft}(x) \stackrel{\text{def}}{=} x \triangleleft 1$$

for each  $x \in X$ .

**00EB Remark 7.3.6.1.3.** The morphism  $\rho_X^{\text{Sets}_*, \triangleleft}$  is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements  $x \triangleleft 0$  of  $X \triangleleft S^0$  with  $x \neq x_0$  are outside the image of  $\rho_X^{\text{Sets}_*, \triangleleft}$ , which sends  $x$  to  $x \triangleleft 1$ .

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\rho_X^{\text{Sets}_*, \triangleleft}: X \rightarrow X \triangleleft S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho_X^{\text{Sets}_*, \triangleleft}(x_0) &= x_0 \triangleleft 1 \\ &= x_0 \triangleleft 0. \end{aligned}$$

Next, we claim that  $\rho^{\text{Sets}_*, \triangleleft}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho_X^{\text{Sets}_*, \triangleleft} \downarrow & & \downarrow \rho_Y^{\text{Sets}_*, \triangleleft} \\ X \triangleleft S^0 & \xrightarrow{f \triangleleft \text{id}_{S^0}} & Y \triangleleft S^0 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \xrightarrow{\quad} & f(x) \\ \downarrow & & \downarrow \\ x \triangleleft 0 & \xrightarrow{\quad} & f(x) \triangleleft 0 \end{array}$$

and hence indeed commutes, showing  $\rho^{\text{Sets}_*, \triangleleft}$  to be a natural transformation. This finishes the proof.  $\square$

### 7.3.7 The Diagonal

**Definition 7.3.1.1.** The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleleft} : \text{id}_{\text{Sets}_*} \Rightarrow \triangleleft \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^{\triangleleft} : (X, x_0) \rightarrow (X \triangleleft X, x_0 \triangleleft x_0)$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\Delta_X^{\triangleleft}(x) \stackrel{\text{def}}{=} x \triangleleft x$$

for each  $x \in X$ .

*Proof. Being a Morphism of Pointed Sets:* We have

$$\Delta_X^{\triangleleft}(x_0) \stackrel{\text{def}}{=} x_0 \triangleleft x_0,$$

and thus  $\Delta_X^{\triangleleft}$  is a morphism of pointed sets.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\triangleleft \downarrow & & \downarrow \Delta_Y^\triangleleft \\ X \triangleleft X & \xrightarrow{f \triangleleft f} & Y \triangleleft Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \xrightarrow{\quad} & f(x) \\ \downarrow & & \downarrow \\ x \triangleleft x & \xrightarrow{\quad} & f(x) \triangleleft f(x) \end{array}$$

and hence indeed commutes, showing  $\Delta^\triangleleft$  to be natural.  $\square$

### 00EE 7.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\triangleleft$

00EF **Proposition 7.3.8.1.1.** The category  $\mathbf{Sets}_*$  admits a left-closed left skew monoidal category structure consisting of:

- *The Underlying Category.* The category  $\mathbf{Sets}_*$  of pointed sets.
- *The Left Skew Monoidal Product.* The left tensor product functor

$$\triangleleft : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 7.3.1.1.1.**

- *The Left Internal Skew Hom.* The left internal Hom functor

$$[-, -]_{\mathbf{Sets}_*}^\triangleleft : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 7.3.2.1.1.**

- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleleft} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of **Definition 7.3.3.1.1.**



- *The Left Skew Associators.* The natural transformation

$$\alpha^{\text{Sets}_*, \triangleleft} : \triangleleft \circ (\triangleleft \times \text{id}_{\text{Sets}_*}) \Longrightarrow \triangleleft \circ (\text{id}_{\text{Sets}_*} \times \triangleleft) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}}$$

of [Definition 7.3.4.1.1](#).

- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\text{Sets}_*, \triangleleft} : \triangleleft \circ (\mathbb{1}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \lambda_{\text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.3.5.1.1](#).

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\text{Sets}_*, \triangleleft} : \rho_{\text{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleleft \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*})$$

of [Definition 7.3.6.1.1](#).

*Proof. The Pentagon Identity:* Let  $(W, w_0)$ ,  $(X, x_0)$ ,  $(Y, y_0)$  and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & (W \triangleleft (X \triangleleft Y)) \triangleleft Z & & \\
 & \nearrow \alpha_{W,X,Y}^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Z & & \nwarrow \alpha_{W,X \triangleleft Y,Z}^{\text{Sets}_*, \triangleleft} & \\
 ((W \triangleleft X) \triangleleft Y) \triangleleft Z & & & & W \triangleleft ((X \triangleleft Y) \triangleleft Z) \\
 \searrow \alpha_{W \triangleleft X,Y,Z}^{\text{Sets}_*, \triangleleft} & & & & \swarrow \text{id}_W \triangleleft \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleleft} \\
 (W \triangleleft X) \triangleleft (Y \triangleleft Z) & \xrightarrow{\alpha_{W,X,Y \triangleleft Z}^{\text{Sets}_*, \triangleleft}} & W \triangleleft (X \triangleleft (Y \triangleleft Z)) & & 
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (w \triangleleft (x \triangleleft y)) \triangleleft z & \\
 \swarrow & & \searrow \\
 ((w \triangleleft x) \triangleleft y) \triangleleft z & & w \triangleleft ((x \triangleleft y) \triangleleft z) \\
 \searrow & & \swarrow \\
 (w \triangleleft x) \triangleleft (y \triangleleft z) & \longmapsto & w \triangleleft (x \triangleleft (y \triangleleft z))
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

*The Left Skew Left Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 (S^0 \triangleleft X) \triangleleft Y & \xrightarrow{\alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleleft}} & S^0 \triangleleft (X \triangleleft Y) \\
 \searrow \lambda_X^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Y & & \downarrow \lambda_{X \triangleleft Y}^{\text{Sets}_*, \triangleleft} \\
 & & X \triangleleft Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (0 \triangleleft x) \triangleleft y & \longmapsto & 0 \triangleleft (x \triangleleft y) \\
 \searrow & & \downarrow \\
 & & x_0 \triangleleft y = x_0 \triangleleft y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 (1 \triangleleft x) \triangleleft y & \longmapsto & 1 \triangleleft (x \triangleleft y) \\
 \searrow & & \downarrow \\
 & & x \triangleleft y
 \end{array}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

*The Left Skew Right Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleleft Y & & \\
 \downarrow \rho_{X \triangleleft Y}^{\text{Sets}_*, \triangleleft} & \searrow \text{id}_X \triangleleft \rho_Y^{\text{Sets}_*, \triangleleft} & \\
 (X \triangleleft Y) \triangleleft S^0 & \xrightarrow{\alpha_{X, Y, S^0}^{\text{Sets}_*, \triangleleft}} & X \triangleleft (Y \triangleleft S^0)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleleft y & & \\
 \downarrow & \searrow & \\
 (x \triangleleft y) \triangleleft 1 & \longmapsto & x \triangleleft (y \triangleleft 1)
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Left Skew Middle Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleleft Y & \xlongequal{\quad} & X \triangleleft Y \\
 \downarrow \rho_X^{\text{Sets}_*, \triangleleft} \triangleleft \text{id}_Y & & \uparrow \text{id}_A \triangleleft \lambda_Y^{\text{Sets}_*, \triangleleft} \\
 (X \triangleleft S^0) \triangleleft Y & \xrightarrow{\alpha_{X, S^0, Y}^{\text{Sets}_*, \triangleleft}} & X \triangleleft (S^0 \triangleleft Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleleft y & \longmapsto & x \triangleleft y \\
 \downarrow & & \uparrow \\
 (x \triangleleft 1) \triangleleft y & \longmapsto & x \triangleleft (1 \triangleleft y)
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Zig-Zag Identity:* We have to show that the diagram

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\rho_{S^0}^{\text{Sets}_*, \triangleleft}} & S^0 \triangleleft S^0 \\
 & \searrow & \downarrow \lambda_{S^0}^{\text{Sets}_*, \triangleleft} \\
 & & S^0
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 0 \triangleleft 1 \\
 & \searrow & \downarrow \\
 & & 0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1 \triangleleft 1 \\
 & \searrow & \downarrow \\
 & & 1
 \end{array}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

*Left Skew Monoidal Left-Closedness:* This follows from **Item 2** of **Definition 7.3.1.1.7**.

□

## 00EG 7.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

**00EH Proposition 7.3.9.1.1.** The category of monoids on  $(\text{Sets}_*, \triangleleft, S^0)$  is isomorphic to the category of “monoids with left zero”<sup>15</sup> and morphisms between them.

*Proof.* *Monoids on  $(\text{Sets}_*, \triangleleft, S^0)$ :* A monoid on  $(\text{Sets}_*, \triangleleft, S^0)$  consists of:

<sup>15</sup>A monoid with left zero is defined similarly as the monoids with zero of ???. Succinctly, they are monoids  $(A, \mu_A, \eta_A)$  with a special element  $0_A$  satisfying

$$0_A a = 0_A$$

for each  $a \in A$ .

- *The Underlying Object.* A pointed set  $(A, 0_A)$ .
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A: A \triangleleft A \rightarrow A,$$

determining a left bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A: S^0 \rightarrow A$$

picking an element  $1_A$  of  $A$ .

satisfying the following conditions:

- 026H 1. *Associativity.* The diagram

$$\begin{array}{ccc} & A \triangleleft (A \triangleleft A) & \\ \alpha_{A,A,A}^{\text{Sets}, \triangleleft} \nearrow & & \searrow \text{id}_A \triangleleft \mu_A \\ (A \triangleleft A) \triangleleft A & & A \triangleleft A \\ \mu_A \triangleleft \text{id}_A \searrow & & \swarrow \mu_A \\ A \triangleleft A & \xrightarrow{\mu_A} & A \end{array}$$

- 026J 2. *Left Unitality.* The diagram

$$\begin{array}{ccc} S^0 \triangleleft A & \xrightarrow{\eta_A \times \text{id}_A} & A \triangleleft A \\ & \searrow \lambda_A^{\text{Sets}, \triangleleft} & \downarrow \mu_A \\ & & A \end{array}$$

commutes.

026K 3. *Right Unitality*. The diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\rho_A^{\text{Sets}_{\ast}, \triangleleft}} & A \triangleleft S^0 \\
 \parallel & & \downarrow \text{id}_A \times \eta_A \\
 A & \xleftarrow{\mu_A} & A \triangleleft A
 \end{array}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each  $a \in A$ . Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

026L 1. *Associativity*. The associativity condition acts as

$$\begin{array}{ccc}
 & & a \triangleleft (b \triangleleft c) \\
 & \swarrow & \searrow \\
 (a \triangleleft b) \triangleleft c & & (a \triangleleft b) \triangleleft c \\
 \searrow & & \swarrow \\
 ab \triangleleft c & \mapsto & (ab)c
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & a \triangleleft (b \triangleleft c) \\
 & \swarrow & \searrow \\
 (a \triangleleft b) \triangleleft c & & a \triangleleft bc \\
 \searrow & & \swarrow \\
 ab \triangleleft c & \mapsto & a(bc)
 \end{array}$$

This gives

$$(ab)c = a(bc)$$

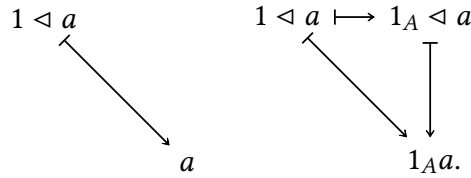
for each  $a, b, c \in A$ .

026M 2. *Left Unitality*. The left unitality condition acts:

026N (a) On  $0 \triangleleft a$  as

$$\begin{array}{ccc}
 0 \triangleleft a & \xrightarrow{\quad} & 0_A \triangleleft a \\
 \searrow & & \downarrow \\
 & & 0_A a
 \end{array}$$

026P (b) On  $1 \triangleleft a$  as

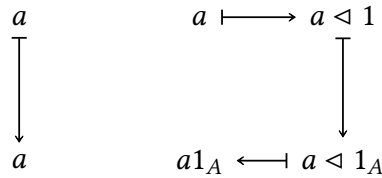


This gives

$$\begin{aligned}
 1_A a &= a, \\
 0_A a &= 0_A
 \end{aligned}$$

for each  $a \in A$ .

026Q 3. *Right Unitality*. The right unitality condition acts as



This gives

$$a 1_A = a$$

for each  $a \in A$ .

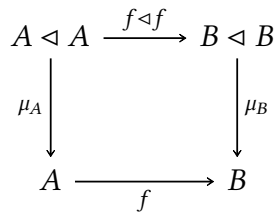
Thus we see that monoids with respect to  $\triangleleft$  are exactly monoids with left zero.

*Morphisms of Monoids on  $(\text{Sets}_*, \triangleleft, S^0)$* : A morphism of monoids on  $(\text{Sets}_*, \triangleleft, S^0)$  from  $(A, \mu_A, \eta_A, 0_A)$  to  $(B, \mu_B, \eta_B, 0_B)$  is a morphism of pointed sets

$$f: (A, 0_A) \rightarrow (B, 0_B)$$

satisfying the following conditions:

026R 1. *Compatibility With the Multiplication Morphisms*. The diagram



commutes.

026S 2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_A} & A \\ & \searrow \eta_B & \downarrow f \\ & & B \end{array}$$

commutes.

These act on elements as

$$\begin{array}{ccc} a \triangleleft b & & a \triangleleft b \mapsto f(a) \triangleleft f(b) \\ \downarrow & & \downarrow \\ ab \mapsto f(ab) & & f(a)f(b) \end{array}$$

and

$$\begin{array}{ccc} 0 & \mapsto & 0_A \\ & \searrow & \downarrow \\ & & 0_B \end{array} \quad \begin{array}{ccc} 0 \mapsto 0_A & & \\ \downarrow & & \\ f(0_A) & & \end{array}$$

and

$$\begin{array}{ccc} 1 & \mapsto & 1_A \\ & \searrow & \downarrow \\ & & 1_B \end{array} \quad \begin{array}{ccc} 1 \mapsto 1_A & & \\ \downarrow & & \\ f(1_A) & & \end{array}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each  $a, b \in A$ , which is exactly a morphism of monoids with left zero.

*Identities and Composition:* Similarly, the identities and composition of  $\text{Mon}(\text{Sets}_*, \triangleleft, S^0)$  can be easily seen to agree with those of monoids with left zero, which finishes the proof.  $\square$



## 00EJ 7.4 The Right Tensor Product of Pointed Sets

### 00EK 7.4.1 Foundations

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00EL **Definition 7.4.1.1.1.** The **right tensor product of pointed sets** is the functor<sup>16</sup>

$$\triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_* \times \mathbf{Sets}_* \xrightarrow{\text{忘} \times \text{id}} \mathbf{Sets} \times \mathbf{Sets}_* \xrightarrow{\odot} \mathbf{Sets}_*,$$

where:

- $\text{忘} : \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.
- $\odot : \mathbf{Sets} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the tensor functor of **Item 1** of **Definition 7.2.1.1.6**.

00EM **Remark 7.4.1.1.2.** The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathbf{Sets}_*(X \triangleright Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}^{\otimes, \mathbf{R}}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

- 026T 1. Pointed maps  $f : X \triangleright Y \rightarrow Z$ .
- 026U 2. Maps of sets  $f : X \times Y \rightarrow Z$  satisfying  $f(x, y_0) = z_0$  for each  $x \in X$ .

00EN **Remark 7.4.1.1.3.** The right tensor product of pointed sets may be described as follows:

- The right tensor product of  $(X, x_0)$  and  $(Y, y_0)$  is the pair  $((X \triangleright Y, x_0 \triangleright y_0), \iota)$  consisting of
  - A pointed set  $(X \triangleright Y, x_0 \triangleright y_0)$ ;

<sup>16</sup>*Further Notation:* Also written  $\triangleright_{\mathbf{Sets}_*}$ .

- A right bilinear morphism of pointed sets  $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \triangleright Y$ ;

satisfying the following universal property:

(★) Given another such pair  $((Z, z_0), f)$  consisting of

- \* A pointed set  $(Z, z_0)$ ;
- \* A right bilinear morphism of pointed sets  $f: (X \times Y, (x_0, y_0)) \rightarrow Z \triangleright Y$ ;

there exists a unique morphism of pointed sets  $X \triangleright Y \xrightarrow{\exists!} Z$  making the diagram

$$\begin{array}{ccc} & & X \triangleright Y \\ & \nearrow \iota & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

commute.

**00EP Construction 7.4.1.1.4.** In detail, the **right tensor product** of  $(X, x_0)$  and  $(Y, y_0)$  is the pointed set  $(X \triangleright Y, [y_0])$  consisting of:

- *The Underlying Set.* The set  $X \triangleright Y$  defined by

$$\begin{aligned} X \triangleright Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

where  $|X|$  denotes the underlying set of  $(X, x_0)$ .

- *The Underlying Basepoint.* The point  $[(x_0, y_0)]$  of  $\bigvee_{x \in X} (Y, y_0)$ , which is equal to  $[(x, y_0)]$  for any  $x \in X$ .

*Proof.* Since  $\bigvee_{y \in Y} (X, x_0)$  is defined as the quotient of  $\coprod_{x \in X} Y$  by the equivalence relation  $R$  generated by declaring  $(x, y) \sim (x', y')$  if  $y = y' = y_0$ , we have, by **Conditions on Relations**, ??, a natural bijection

$$\text{Sets}_*(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}}^R\left(\coprod_{x \in X} Y, Z\right),$$

where  $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$  is the set

$$\text{Hom}_{\text{Sets}}^R\left(\coprod_{x \in X} Y, Z\right) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}\left(\coprod_{x \in X} Y, Z\right) \left| \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right. \right\}.$$

However, the condition  $(x, y) \sim_R (x', y')$  only holds when:

026V 1. We have  $x = x'$  and  $y = y'$ .

026W 2. We have  $y = y' = y_0$ .

So, given  $f \in \text{Hom}_{\text{Sets}}(\coprod_{x \in X} Y, Z)$  with a corresponding  $\bar{f}: X \triangleright Y \rightarrow Z$ , the latter case above implies

$$\begin{aligned} f([(x, y_0)]) &= f([(x', y_0)]) \\ &= f([(x_0, y_0)]), \end{aligned}$$

and since  $\bar{f}: X \triangleright Y \rightarrow Z$  is a pointed map, we have

$$\begin{aligned} f([(x_0, y_0)]) &= \bar{f}([(x_0, y_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements  $f$  in  $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$  are precisely those functions  $f: X \times Y \rightarrow Z$  satisfying the equality

$$f(x, y_0) = z_0$$

for each  $y \in Y$ , giving an equality

$$\text{Hom}_{\text{Sets}}^R(X \times Y, Z) = \text{Hom}_{\text{Sets}_*}^{\otimes, R}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\text{Sets}_*(X \triangleright Y, Z) \cong \text{Hom}_{\text{Sets}}^R(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof.  $\square$

**00EQ Notation 7.4.1.1.5.** We write<sup>17</sup>  $x \triangleright y$  for the element  $[(x, y)]$  of

$$X \triangleright Y \cong |X| \odot Y.$$

**00ER Remark 7.4.1.1.6.** Employing the notation introduced in **Definition 7.4.1.1.5**, we have

$$x_0 \triangleright y_0 = x \triangleright y_0$$

for each  $x \in X$ , and

$$x \triangleright y_0 = x' \triangleright y_0$$

for each  $x, x' \in X$ .

**00ES Proposition 7.4.1.1.7.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00ET** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto X \triangleright Y$  define functors

$$\begin{aligned} X \triangleright -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ - \triangleright Y: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ -_1 \triangleright -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f: (X, x_0) &\rightarrow (A, a_0), \\ g: (Y, y_0) &\rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \triangleright g: X \triangleright Y \rightarrow A \triangleright B$$

is given by

$$[f \triangleright g](x \triangleright y) \stackrel{\text{def}}{=} f(x) \triangleright g(y)$$

for each  $x \triangleright y \in X \triangleright Y$ .

**00EU** 2. *Adjointness I.* We have an adjunction

$$\left( X \triangleright - \dashv [X, -]_{\text{Sets}_*}^{\triangleright} \right): \text{Sets}_* \begin{array}{c} \xrightarrow{X \triangleright -} \\ \perp \\ \xleftarrow{[X, -]_{\text{Sets}_*}^{\triangleright}} \end{array} \text{Sets}_*,$$

<sup>17</sup>*Further Notation:* Also written  $x \triangleright_{\text{Sets}_*} y$ .

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(Y, [X, Z]_{\mathbf{Sets}_*}^{\triangleright})$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ , where  $[X, Y]_{\mathbf{Sets}_*}^{\triangleright}$  is the pointed set of [Definition 7.4.2.1.1](#).

**00EV** 3. *Adjointness II.* The functor

$$- \triangleright Y : \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

**00EW** 4. *Adjointness III.* We have a  $\overline{\mathbf{Set}}$ -relative adjunction

$$(- \triangleright Y \dashv \mathbf{Sets}_*(Y, -)) : \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \triangleright Y} \\ \perp_{\overline{\mathbf{Set}}} \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|X|, \mathbf{Sets}_*(Y, Z))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ .

*Proof.* **Item 1, Functoriality:** This follows from the definition of  $\triangleright$  as a composition of functors ([Definition 7.4.1.1.1](#)).

**Item 2, Adjointness I:** This follows from **Item 3** of [Definition 7.2.1.1.6](#).

**Item 3, Adjointness II:** For  $- \triangleright Y$  to admit a right adjoint would require it to preserve colimits by [??, ??](#) of [??](#). However, we have

$$\begin{aligned} \mathrm{pt} \triangleright X &\stackrel{\mathrm{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\neq \mathrm{pt}, \end{aligned}$$

and thus we see that  $- \triangleright Y$  does not have a right adjoint.

**Item 4, Adjointness III:** This follows from **Item 2** of [Definition 7.2.1.1.6](#). □

**00EX Remark 7.4.1.1.8.** Here is some intuition on why  $- \triangleright Y$  fails to be a left adjoint. **Item 4** of **Definition 7.3.1.1.7** states that we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}}(|X|, \mathbf{Sets}_*(Y, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

also holds, which would give  $- \triangleright Y \dashv \mathbf{Sets}_*(Y, -)$ . However, such a bijection would require every map

$$f: X \triangleright Y \rightarrow Z$$

to satisfy

$$f(x_0 \triangleright y) = z_0$$

for each  $x \in X$ , whereas we are imposing such a basepoint preservation condition only for elements of the form  $x \triangleright y_0$ . Thus  $\mathbf{Sets}_*(Y, -)$  can't be a right adjoint for  $- \triangleright Y$ , and as shown by **Item 3** of **Definition 7.4.1.1.7**, no functor can.<sup>18</sup>

## 00EY 7.4.2 The Right Internal Hom of Pointed Sets

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00EZ Definition 7.4.2.1.1.** The **right internal Hom**<sup>19</sup> of pointed sets is the functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

defined as the composition

$$\mathbf{Sets}_*^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\overline{\omega} \times \mathrm{id}} \mathbf{Sets}^{\mathrm{op}} \times \mathbf{Sets}_* \xrightarrow{\pitchfork} \mathbf{Sets}_*,$$

where:

- $\overline{\omega}: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$  is the forgetful functor from pointed sets to sets.

<sup>18</sup>The functor  $\mathbf{Sets}_*(Y, -)$  is instead right adjoint to  $- \wedge Y$ , the smash product of pointed sets of **Definition 7.5.1.1.1**. See **Item 2** of **Definition 7.5.1.1.10**.

<sup>19</sup>For a proof that  $[-, -]_{\mathbf{Sets}_*}^{\triangleright}$  is indeed the right internal Hom of  $\mathbf{Sets}_*$  with respect to the right tensor product of pointed sets, see **Item 2** of **Definition 7.4.1.1.7**.

- $\pitchfork : \mathbf{Sets}^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$  is the cotensor functor of **Item 1** of **Definition 7.2.2.1.4**.

**00F0 Remark 7.4.2.1.2.** We have

$$[-, -]_{\mathbf{Sets}_*}^{\triangleleft} = [-, -]_{\mathbf{Sets}_*}^{\triangleright}.$$

**00F1 Remark 7.4.2.1.3.** The right internal Hom of pointed sets satisfies the following universal property:

$$\mathbf{Sets}_*(X \triangleright Y, Z) \cong \mathbf{Sets}_*(Y, [X, Z]_{\mathbf{Sets}_*}^{\triangleright})$$

That is to say, the following data are in bijection:

- 026X** 1. Pointed maps  $f : X \triangleright Y \rightarrow Z$ .
- 026Y** 2. Pointed maps  $f : Y \rightarrow [X, Z]_{\mathbf{Sets}_*}^{\triangleright}$ .

**00F2 Remark 7.4.2.1.4.** In detail, the **right internal Hom of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pointed set  $([X, Y]_{\mathbf{Sets}_*}^{\triangleright}, [(y_0)_{x \in X}])$  consisting of:

- *The Underlying Set.* The set  $[X, Y]_{\mathbf{Sets}_*}^{\triangleright}$  defined by

$$\begin{aligned} [X, Y]_{\mathbf{Sets}_*}^{\triangleright} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0), \end{aligned}$$

where  $|X|$  denotes the underlying set of  $(X, x_0)$ .

- *The Underlying Basepoint.* The point  $[(y_0)_{x \in X}]$  of  $\bigwedge_{x \in X} (Y, y_0)$ .

**00F3 Proposition 7.4.2.1.5.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00F4** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto [X, Y]_{\mathbf{Sets}_*}^{\triangleright}$  define functors

$$\begin{aligned} [X, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ [-, Y]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\text{op}} &\rightarrow \mathbf{Sets}_*, \\ [-_1, -_2]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$f : (X, x_0) \rightarrow (A, a_0),$$

$$g: (Y, y_0) \rightarrow (B, b_0),$$

the induced map

$$[f, g]_{\mathbf{Sets}_*}^\triangleright : [A, Y]_{\mathbf{Sets}_*}^\triangleright \rightarrow [X, B]_{\mathbf{Sets}_*}^\triangleright$$

is given by

$$[f, g]_{\mathbf{Sets}_*}^\triangleright ((y_a)_{a \in A}) \stackrel{\text{def}}{=} [(g(y_{f(x)}))_{x \in X}]$$

for each  $[(y_a)_{a \in A}] \in [A, Y]_{\mathbf{Sets}_*}^\triangleright$ .

**00F5** 2. *Adjointness I.* We have an adjunction

$$\left( X \triangleright - \dashv [X, -]_{\mathbf{Sets}_*}^\triangleright \right): \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \triangleright -} \\ \perp \\ \xleftarrow{[X, -]_{\mathbf{Sets}_*}^\triangleright} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Hom}_{\mathbf{Sets}_*}(X \triangleright Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(Y, [X, Z]_{\mathbf{Sets}_*}^\triangleright)$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ , where  $[X, Y]_{\mathbf{Sets}_*}^\triangleright$  is the pointed set of **Definition 7.4.2.1.1**.

**00F6** 3. *Adjointness II.* The functor

$$- \triangleright Y: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

does not admit a right adjoint.

*Proof.* **Item 1, Functoriality:** This follows from the definition of  $[-, -]_{\mathbf{Sets}_*}^\triangleright$  as a composition of functors (**Definition 7.4.2.1.1**).

**Item 2, Adjointness I:** This is a repetition of **Item 2** of **Definition 7.4.1.1.7**, and is proved there.

**Item 3, Adjointness II:** This is a repetition of **Item 3** of **Definition 7.4.1.1.7**, and is proved there.  $\square$



### 00F7 7.4.3 The Right Skew Unit

00F8 **Definition 7.4.3.1.1.** The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\text{Sets}_*, \triangleright} : \text{pt} \rightarrow \text{Sets}_*$$

defined by

$$\mathbb{1}_{\text{Sets}_*}^{\triangleright} \stackrel{\text{def}}{=} S^0.$$

### 00F9 7.4.4 The Right Skew Associator

00FA **Definition 7.4.4.1.1.** The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\text{Sets}_*, \triangleright} : \triangleright \circ (\text{id}_{\text{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \text{id}_{\text{Sets}_*}) \circ \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}, -1}$$

as in the diagram

$$\begin{array}{ccccc}
 & & (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & \\
 & \nearrow \alpha_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*}^{\text{Cats}, -1} & & \searrow \triangleright \times \text{id} & \\
 \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & & & & \text{Sets}_* \times \text{Sets}_* \\
 \downarrow \text{id} \times \triangleright & \nearrow \alpha^{\text{Sets}_*, \triangleright} & & \searrow \triangleright & \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\triangleright} & \text{Sets}_* & & 
 \end{array}$$

whose component

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} : X \triangleright (Y \triangleright Z) \rightarrow (X \triangleright Y) \triangleright Z$$

at  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned}
 X \triangleright (Y \triangleright Z) &\stackrel{\text{def}}{=} |X| \odot (Y \triangleright Z) \\
 &\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
 &\cong \bigvee_{x \in X} (|Y| \odot Z)
 \end{aligned}$$

$$\begin{aligned}
&\cong \bigvee_{x \in X} \left( \bigvee_{y \in Y} Z \right) \\
&\rightarrow \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\
&\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\
&\cong ||X| \odot Y| \odot Z \\
&\stackrel{\text{def}}{=} |X \triangleright Y| \odot Z \\
&\stackrel{\text{def}}{=} (X \triangleright Y) \triangleright Z,
\end{aligned}$$

where the map

$$\bigvee_{x \in X} \left( \bigvee_{y \in Y} Z \right) \rightarrow \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z$$

is given by  $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$ .

*Proof.* (Proven below in a bit.) □

**00FB Remark 7.4.4.1.2.** Unwinding the notation for elements, we have

$$\begin{aligned}
[(x, [(y, z)])] &\stackrel{\text{def}}{=} [(x, y \triangleright z)] \\
&\stackrel{\text{def}}{=} x \triangleright (y \triangleright z)
\end{aligned}$$

and

$$\begin{aligned}
[([(x, y)], z)] &\stackrel{\text{def}}{=} [(x \triangleright y, z)] \\
&\stackrel{\text{def}}{=} (x \triangleright y) \triangleright z.
\end{aligned}$$

So, in other words,  $\alpha_{X,Y,Z}^{\text{Sets}, \triangleright}$  acts on elements via

$$\alpha_{X,Y,Z}^{\text{Sets}, \triangleright} (x \triangleright (y \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y) \triangleright z$$

for each  $x \triangleright (y \triangleright z) \in X \triangleright (Y \triangleright Z)$ .

**00FC Remark 7.4.4.1.3.** Taking  $y = y_0$ , we see that the morphism  $\alpha_{X,Y,Z}^{\text{Sets}, \triangleright}$  acts on elements as

$$\alpha_{X,Y,Z}^{\text{Sets}, \triangleright} (x \triangleright (y_0 \triangleright z)) \stackrel{\text{def}}{=} (x \triangleright y_0) \triangleright z.$$

However, by the definition of  $\triangleright$ , we have  $x \triangleright y_0 = x' \triangleright y_0$  for all  $x, x' \in X$ , preventing  $\alpha_{X,Y,Z}^{\text{Sets}, \triangleright}$  from being non-invertible.

*Proof.* Firstly, note that, given  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} : X \triangleright (Y \triangleright Z) \rightarrow (X \triangleright Y) \triangleright Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} (x_0 \triangleright (y_0 \triangleright z_0)) = (x_0 \triangleright y_0) \triangleright z_0.$$

Next, we claim that  $\alpha^{\text{Sets}_*, \triangleright}$  is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ h &: (Z, z_0) \rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc} X \triangleright (Y \triangleright Z) & \xrightarrow{f \triangleright (g \triangleright h)} & X' \triangleright (Y' \triangleright Z') \\ \alpha_{X,Y,Z}^{\text{Sets}_*, \triangleright} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}_*, \triangleright} \\ (X \triangleright Y) \triangleright Z & \xrightarrow{(f \triangleright g) \triangleright h} & (X' \triangleright Y') \triangleright Z' \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \triangleright (y \triangleright z) & \longmapsto & f(x) \triangleright (g(y) \triangleright h(z)) \\ \downarrow & & \downarrow \\ (x \triangleright y) \triangleright z & \longmapsto & (f(x) \triangleright g(y)) \triangleright h(z) \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}_*, \triangleright}$  to be a natural transformation. This finishes the proof.  $\square$

### 7.4.5 The Right Skew Left Unit

**Definition 7.4.5.1.1.** The **skew left unit of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\text{Sets}_*, \triangleright} : \lambda_{\text{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleright \circ (\mathbb{1}_{\text{Sets}_*} \times \text{id}_{\text{Sets}_*})$$

whose component

$$\lambda_X^{\text{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright X$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} X &\rightarrow X \vee X \\ &\cong |S^0| \odot X \\ &\cong S^0 \triangleright X, \end{aligned}$$

where  $X \rightarrow X \vee X$  is the map sending  $X$  to the second factor of  $X$  in  $X \vee X$ .

*Proof.* (Proven below in a bit.) □

**Remark 7.4.5.1.2.** In other words,  $\lambda_X^{\text{Sets}_*, \triangleright}$  acts on elements as

$$\lambda_X^{\text{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\lambda_X^{\text{Sets}_*, \triangleright}(x) \stackrel{\text{def}}{=} 1 \triangleright x$$

for each  $x \in X$ .

**Remark 7.4.5.1.3.** The morphism  $\lambda_X^{\text{Sets}_*, \triangleright}$  is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements  $0 \triangleright x$  of  $S^0 \triangleright X$  with  $x \neq x_0$  are outside the image of  $\lambda_X^{\text{Sets}_*, \triangleright}$ , which sends  $x$  to  $1 \triangleright x$ .

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\lambda_X^{\text{Sets}_*, \triangleright} : X \rightarrow S^0 \triangleright X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \lambda_X^{\text{Sets}_*, \triangleright}(x_0) &= 1 \triangleright x_0 \\ &= 0 \triangleright x_0. \end{aligned}$$

Next, we claim that  $\lambda^{\text{Sets}_*, \triangleright}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \lambda_X^{\text{Sets}_*, \triangleright} \downarrow & & \downarrow \lambda_Y^{\text{Sets}_*, \triangleright} \\ S^0 \triangleright X & \xrightarrow{\text{id}_{S^0} \triangleright f} & S^0 \triangleright Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ 1 \triangleright x & \longmapsto & 1 \triangleright f(x) \end{array}$$

and hence indeed commutes, showing  $\lambda^{\text{Sets}_*, \triangleright}$  to be a natural transformation. This finishes the proof.  $\square$

## 00FH 7.4.6 The Right Skew Right Unitor

00FJ **Definition 7.4.6.1.1.** The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\text{Sets}_*, \triangleright} : \triangleright \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}) \xRightarrow{\sim} \rho_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}_*, \triangleright} : X \triangleright S^0 \rightarrow X$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} X \triangleright S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\rightarrow X, \end{aligned}$$

where  $\bigvee_{x \in X} S^0 \rightarrow X$  is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x \end{aligned}$$

for each  $x \in X$ .

*Proof.* (Proven below in a bit.) □

00FK **Remark 7.4.6.1.2.** In other words,  $\rho_X^{\text{Sets}_*, \triangleright}$  acts on elements as

$$\begin{aligned} \rho_X^{\text{Sets}_*, \triangleright} (x \triangleright 0) &\stackrel{\text{def}}{=} x_0, \\ \rho_X^{\text{Sets}_*, \triangleright} (x \triangleright 1) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each  $x \triangleright 1 \in X \triangleright S^0$ .

**00FL Remark 7.4.6.1.3.** The morphism  $\rho_X^{\text{Sets}_*, \triangleright}$  is almost invertible, with its would-be-inverse

$$\phi_X: X \rightarrow X \triangleright S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \triangleright 1$$

for each  $x \in X$ . Indeed, we have

$$\begin{aligned} [\rho_X^{\text{Sets}_*, \triangleright} \circ \phi](x) &= \rho_X^{\text{Sets}_*, \triangleright}(\phi(x)) \\ &= \rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 1) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

so that

$$\rho_X^{\text{Sets}_*, \triangleright} \circ \phi = \text{id}_X$$

and

$$\begin{aligned} [\phi \circ \rho_X^{\text{Sets}_*, \triangleright}](x \triangleright 1) &= \phi(\rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 1)) \\ &= \phi(x) \\ &= x \triangleright 1 \\ &= [\text{id}_{X \triangleright S^0}](x \triangleright 1), \end{aligned}$$

but

$$\begin{aligned} [\phi \circ \rho_X^{\text{Sets}_*, \triangleright}](x \triangleright 0) &= \phi(\rho_X^{\text{Sets}_*, \triangleright}(x \triangleright 0)) \\ &= \phi(x_0) \\ &= 1 \triangleright x_0, \end{aligned}$$

where  $x \triangleright 0 \neq 1 \triangleright x_0$ . Thus

$$\phi \circ \rho_X^{\text{Sets}_*, \triangleright} \stackrel{?}{=} \text{id}_{X \triangleright S^0}$$

holds for all elements in  $X \triangleright S^0$  except one.

*Proof.* Firstly, note that, given  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ , the map

$$\rho_X^{\text{Sets}_*, \triangleright}: X \triangleright S^0 \rightarrow X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\text{Sets}_*, \triangleright}(x_0 \triangleright 0) = x_0.$$

Next, we claim that  $\rho^{\text{Sets}_*, \triangleright}$  is a natural transformation. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X \triangleright S^0 & \xrightarrow{f \triangleright \text{id}_{S^0}} & Y \triangleright S^0 \\ \rho_X^{\text{Sets}_*, \triangleright} \downarrow & & \downarrow \rho_Y^{\text{Sets}_*, \triangleright} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \triangleright 0 & & x \triangleright 0 \mapsto f(x) \triangleright 0 \\ \downarrow & & \downarrow \\ x_0 & \mapsto & f(x_0) \\ & & y_0 \end{array}$$

and

$$\begin{array}{ccc} x \triangleright 1 & \mapsto & f(x) \triangleright 1 \\ \downarrow & & \downarrow \\ x & \mapsto & f(x) \end{array}$$

and hence indeed commutes, showing  $\rho^{\text{Sets}_*, \triangleright}$  to be a natural transformation. This finishes the proof.  $\square$



### 00FM 7.4.7 The Diagonal

00FN **Definition 7.4.7.1.1.** The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleright} : \text{id}_{\text{Sets}_*} \Rightarrow \triangleright \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^{\triangleright} : (X, x_0) \rightarrow (X \triangleright X, x_0 \triangleright x_0)$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\Delta_X^{\triangleright}(x) \stackrel{\text{def}}{=} x \triangleright x$$

for each  $x \in X$ .

*Proof. Being a Morphism of Pointed Sets:* We have

$$\Delta_X^{\triangleright}(x_0) \stackrel{\text{def}}{=} x_0 \triangleright x_0,$$

and thus  $\Delta_X^{\triangleright}$  is a morphism of pointed sets.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^{\triangleright} \downarrow & & \downarrow \Delta_Y^{\triangleright} \\ X \triangleright X & \xrightarrow{f \triangleright f} & Y \triangleright Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \triangleright x & \longmapsto & f(x) \triangleright f(x) \end{array}$$

and hence indeed commutes, showing  $\Delta^{\triangleright}$  to be natural.  $\square$

## 00FP 7.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

00FQ **Proposition 7.4.8.1.1.** The category  $\mathbf{Sets}_*$  admits a right-closed right skew monoidal category structure consisting of:

- *The Underlying Category.* The category  $\mathbf{Sets}_*$  of pointed sets.
- *The Right Skew Monoidal Product.* The right tensor product functor

$$\triangleright : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 7.4.1.1.1.**

- *The Right Internal Skew Hom.* The right internal Hom functor

$$[-, -]_{\mathbf{Sets}_*}^{\triangleright} : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of **Definition 7.4.2.1.1.**

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*, \triangleright} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of **Definition 7.4.3.1.1.**

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\text{id}_{\mathbf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \text{id}_{\mathbf{Sets}_*}) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\text{Cats}, -1}$$

of **Definition 7.4.4.1.1.**

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Sets}_*, \triangleright} : \lambda_{\mathbf{Sets}_*}^{\text{Cats}_2} \xrightarrow{\sim} \triangleright \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*})$$

of **Definition 7.4.5.1.1.**

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Sets}_*, \triangleright} : \triangleright \circ (\text{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xrightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\text{Cats}_2}$$

of **Definition 7.4.6.1.1.**

*Proof. The Pentagon Identity:* Let  $(W, w_0)$ ,  $(X, x_0)$ ,  $(Y, y_0)$  and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & W \triangleright ((X \triangleright Y) \triangleright Z) & & \\
 & \nearrow \alpha_{W, X, Y}^{\text{Sets}, \triangleright} \triangleright \text{id}_Z & & \nwarrow \alpha_{W, X \triangleright Y, Z}^{\text{Sets}, \triangleright} & \\
 W \triangleright (X \triangleright (Y \triangleright Z)) & & & & (W \triangleright (X \triangleright Y)) \triangleright Z \\
 \searrow \alpha_{W \triangleright X, Y, Z}^{\text{Sets}, \triangleright} & & & & \swarrow \text{id}_W \triangleright \alpha_{X, Y, Z}^{\text{Sets}, \triangleright} \\
 (W \triangleright X) \triangleright (Y \triangleright Z) & \xrightarrow{\alpha_{W, X, Y \triangleright Z}^{\text{Sets}, \triangleright}} & ((W \triangleright X) \triangleright Y) \triangleright Z & & 
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 & & w \triangleright ((x \triangleright y) \triangleright z) & & \\
 & \nearrow & & \nwarrow & \\
 w \triangleright (x \triangleright (y \triangleright z)) & & & & (w \triangleright (x \triangleright y)) \triangleright z \\
 \searrow & & & & \swarrow \\
 (w \triangleright x) \triangleright (y \triangleright z) & \longmapsto & ((w \triangleright x) \triangleright y) \triangleright z & & 
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

*The Right Skew Left Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright Y & & \\
 \lambda_{X \triangleright Y}^{\text{Sets}_*, \triangleright} \downarrow & \searrow \lambda_X^{\text{Sets}_*, \triangleright} \triangleright \text{id}_Y & \\
 S^0 \triangleright (X \triangleright Y) & \xrightarrow{\alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleright}} & (S^0 \triangleright X) \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright y & & \\
 \downarrow & \swarrow & \\
 1 \triangleright (x \triangleright y) & \longmapsto & (1 \triangleright x) \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

*The Right Skew Right Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright (Y \triangleright S^0) & \xrightarrow{\text{id}_X \triangleright \rho_Y^{\text{Sets}_*, \triangleright}} & (X \triangleright Y) \triangleright S^0 \\
 \searrow \alpha_{S^0, X, Y}^{\text{Sets}_*, \triangleright} & & \downarrow \rho_{X \triangleright Y}^{\text{Sets}_*, \triangleright} \\
 & & X \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright (y \triangleright 0) & \longmapsto & (x \triangleright y) \triangleright 0 \\
 \swarrow & & \downarrow \\
 & & x \triangleright y_0 = x_0 \triangleright y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 x \triangleright (y \triangleright 1) & \longmapsto & (x \triangleright y) \triangleright 1 \\
 \swarrow & & \downarrow \\
 & & x \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Right Skew Middle Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 X \triangleright Y & \xlongequal{\quad} & X \triangleright Y \\
 \text{id}_X \triangleright \lambda_Y^{\text{Sets}_*, \triangleright} \downarrow & & \uparrow \rho_X^{\text{Sets}_*, \triangleright} \triangleright \text{id}_Y \\
 X \triangleright (S^0 \triangleright Y) & \xrightarrow[\alpha_{X, S^0, Y}^{\text{Sets}_*, \triangleright}]{} & (X \triangleright S^0) \triangleright Y
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 x \triangleright y & \xrightarrow{\quad} & x \triangleright y \\
 \downarrow & & \uparrow \\
 x \triangleright (1 \triangleright y) & \xrightarrow{\quad} & (x \triangleright 1) \triangleright y
 \end{array}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

*The Zig-Zag Identity:* We have to show that the diagram

$$\begin{array}{ccc}
 S^0 & \xrightarrow{\lambda_{S^0}^{\text{Sets}_*, \triangleright}} & S^0 \triangleright S^0 \\
 \searrow & & \downarrow \rho_{S^0}^{\text{Sets}_*, \triangleright} \\
 & & S^0
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad} & 1 \triangleright 0 \\
 \swarrow & & \downarrow \\
 & & 0
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 1 \triangleright 1 \\
 \swarrow & & \downarrow \\
 & & 1
 \end{array}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

*Right Skew Monoidal Right-Closedness:* This follows from **Item 2** of **Definition 7.4.1.1.7**.  $\square$

## 00FR 7.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

00FS **Proposition 7.4.9.1.1.** The category of monoids on  $(\text{Sets}_*, \triangleright, S^0)$  is isomorphic to the category of “monoids with right zero”<sup>20</sup> and morphisms between them.

*Proof.* *Monoids on  $(\text{Sets}_*, \triangleright, S^0)$ :* A monoid on  $(\text{Sets}_*, \triangleright, S^0)$  consists of:

- *The Underlying Object.* A pointed set  $(A, 0_A)$ .
- *The Multiplication Morphism.* A morphism of pointed sets

$$\mu_A: A \triangleright A \rightarrow A,$$

determining a right bilinear morphism of pointed sets

$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

$$\eta_A: S^0 \rightarrow A$$

picking an element  $1_A$  of  $A$ .

satisfying the following conditions:

---

<sup>20</sup>A monoid with right zero is defined similarly as the monoids with zero of ???. Succinctly, they are monoids  $(A, \mu_A, \eta_A)$  with a special element  $0_A$  satisfying

$$0_A a = 0_A$$

for each  $a \in A$ .

026Z 1. *Associativity*. The diagram

$$\begin{array}{ccc}
 & A \triangleright (A \triangleright A) & \\
 \alpha_{A,A,A}^{\text{Sets}, \triangleright} \nearrow & & \searrow \text{id}_A \triangleright \mu_A \\
 (A \triangleright A) \triangleright A & & A \triangleright A \\
 \mu_A \triangleright \text{id}_A \searrow & & \swarrow \mu_A \\
 A \triangleright A & \xrightarrow{\mu_A} & A
 \end{array}$$

0270 2. *Left Unitality*. The diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\lambda_A^{\text{Sets}, \triangleright}} & S^0 \triangleright A \\
 \parallel & & \downarrow \eta_A \times \text{id}_A \\
 A & \xleftarrow{\mu_A} & A \triangleright A
 \end{array}$$

commutes.

0271 3. *Right Unitality*. The diagram

$$\begin{array}{ccc}
 A \triangleright S^0 & \xrightarrow{\text{id}_A \times \eta_A} & A \triangleright A \\
 \searrow \rho_A^{\text{Sets}, \triangleright} & & \downarrow \mu_A \\
 & & A
 \end{array}$$

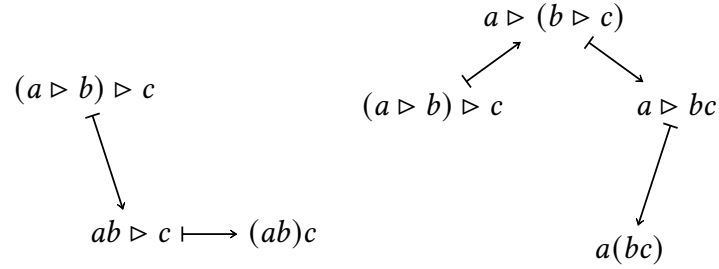
commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each  $a \in A$ . Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

0272 1. *Associativity*. The associativity condition acts as

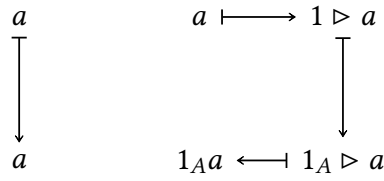


This gives

$$(ab)c = a(bc)$$

for each  $a, b, c \in A$ .

0273 2. *Left Unitality*. The left unitality condition acts as



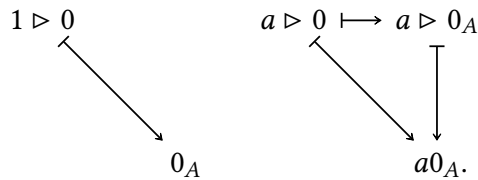
This gives

$$1_A a = a$$

for each  $a \in A$ .

0274 3. *Right Unitality*. The right unitality condition acts:

0275 (a) On  $1 \triangleright 0$  as





0276 (b) On  $a \triangleright 1$  as

$$\begin{array}{ccc} a \triangleright 1 & & a \triangleright 1 \mapsto a \triangleright 1_A \\ & \searrow & \downarrow \\ & a & a1_A. \end{array}$$

This gives

$$\begin{aligned} a1_A &= a, \\ a0_A &= 0_A \end{aligned}$$

for each  $a \in A$ .

Thus we see that monoids with respect to  $\triangleright$  are exactly monoids with right zero.

*Morphisms of Monoids on  $(\text{Sets}_*, \triangleright, S^0)$* : A morphism of monoids on  $(\text{Sets}_*, \triangleright, S^0)$  from  $(A, \mu_A, \eta_A, 0_A)$  to  $(B, \mu_B, \eta_B, 0_B)$  is a morphism of pointed sets

$$f: (A, 0_A) \rightarrow (B, 0_B)$$

satisfying the following conditions:

0277 1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{array}{ccc} A \triangleright A & \xrightarrow{f \triangleright f} & B \triangleright B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

0278 2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta_A} & A \\ & \searrow \eta_B & \downarrow f \\ & & B \end{array}$$

commutes.

These act on elements as

$$\begin{array}{ccc}
 a \triangleright b & & a \triangleright b \mapsto f(a) \triangleright f(b) \\
 \downarrow & & \downarrow \\
 ab \mapsto f(ab) & & f(a)f(b)
 \end{array}$$

and

$$\begin{array}{ccc}
 0 & \mapsto & 0_A \\
 \searrow & & \downarrow \\
 & & f(0_A) \\
 & & \downarrow \\
 & & 0_B
 \end{array}$$

and

$$\begin{array}{ccc}
 1 & \mapsto & 1_A \\
 \searrow & & \downarrow \\
 & & f(1_A) \\
 & & \downarrow \\
 & & 1_B
 \end{array}$$

giving

$$\begin{aligned}
 f(ab) &= f(a)f(b), \\
 f(0_A) &= 0_B, \\
 f(1_A) &= 1_B,
 \end{aligned}$$

for each  $a, b \in A$ , which is exactly a morphism of monoids with right zero.

*Identities and Composition:* Similarly, the identities and composition of  $\text{Mon}(\text{Sets}_*, \triangleright, S^0)$  can be easily seen to agree with those of monoids with right zero, which finishes the proof.  $\square$

## 00FT 7.5 The Smash Product of Pointed Sets

### 00FU 7.5.1 Foundations

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

**00FV Definition 7.5.1.1.1.** The **smash product of**  $(X, x_0)$  **and**  $(Y, y_0)$ <sup>21</sup> is the pointed set  $X \wedge Y$ <sup>22</sup> satisfying the bijection

$$\text{Sets}_*(X \wedge Y, Z) \cong \text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z),$$

naturally in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$ .

**00FW Remark 7.5.1.1.2.** That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps  $f: X \wedge Y \rightarrow Z$ .
- Maps of sets  $f: X \times Y \rightarrow Z$  satisfying

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ .

**00FX Remark 7.5.1.1.3.** The smash product of pointed sets may be described as follows:

- The smash product of  $(X, x_0)$  and  $(Y, y_0)$  is the pair  $((X \wedge Y, x_0 \wedge y_0), \iota)$  consisting of
  - A pointed set  $(X \wedge Y, x_0 \wedge y_0)$ ;
  - A bilinear morphism of pointed sets  $\iota: (X \times Y, (x_0, y_0)) \rightarrow X \wedge Y$ ;

satisfying the following universal property:

- (★) Given another such pair  $((Z, z_0), f)$  consisting of
- \* A pointed set  $(Z, z_0)$ ;
  - \* A bilinear morphism of pointed sets  $f: (X \times Y, (x_0, y_0)) \rightarrow X \wedge Y$ ;

<sup>21</sup>*Further Terminology:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the smash product  $X \wedge Y$  is also called the **tensor product of  $\mathbb{F}_1$ -modules of**  $(X, x_0)$  **and**  $(Y, y_0)$  or the **tensor product of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $\mathbb{F}_1$ .

<sup>22</sup>*Further Notation:* In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the smash product  $X \wedge Y$  is also denoted  $X \otimes_{\mathbb{F}_1} Y$ .

$$\begin{array}{ccc} & X \wedge Y & \\ \nearrow \iota & & \downarrow \exists! \\ X \times Y & \xrightarrow{f} & Z \end{array}$$

However, the condition  $(x, y) \sim_R (x', y')$  only holds when:

- 0279 1. We have  $x = x'$  and  $y = y'$ .
- 027A 2. The following conditions are satisfied:
- 027B (a) We have  $x = x_0$  or  $y = y_0$ .
- 027C (b) We have  $x' = x_0$  or  $y' = y_0$ .

So, given  $f \in \text{Hom}_{\text{Sets}}(X \times Y, Z)$  with a corresponding  $\bar{f}: X \wedge Y \rightarrow Z$ , the latter case above implies

$$\begin{aligned} f(x_0, y) &= f(x, y_0) \\ &= f(x_0, y_0), \end{aligned}$$

and since  $\bar{f}: X \wedge Y \rightarrow Z$  is a pointed map, we have

$$\begin{aligned} f(x_0, y_0) &= \bar{f}(x_0, y_0) \\ &= z_0. \end{aligned}$$

Thus the elements  $f$  in  $\text{Hom}_{\text{Sets}}^R(X \times Y, Z)$  are precisely those functions  $f: X \times Y \rightarrow Z$  satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ , giving an equality

$$\text{Hom}_{\text{Sets}}^R(X \times Y, Z) = \text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\text{Sets}_*(X \wedge Y, Z) \cong \text{Hom}_{\text{Sets}}^R(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof.  $\square$

00FZ **Remark 7.5.1.1.5.** It is also somewhat common to write

$$X \wedge Y \stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying  $X \vee Y$  with the subspace  $(\{x_0\} \times Y) \cup (X \times \{y_0\})$  of  $X \times Y$ , and having the quotient be defined by declaring  $(x, y) \sim (x', y')$  iff we have  $(x, y), (x', y') \in X \vee Y$ .

**01VX Construction 7.5.1.1.6.** Alternatively, the smash product of  $(X, x_0)$  and  $(Y, y_0)$  may be constructed as the pointed set  $X \wedge Y$  given by

$$\begin{aligned} X \wedge Y &\cong \bigvee_{x \in X^-} Y \\ &\cong \bigvee_{y \in Y^-} X. \end{aligned}$$

*Proof.* Indeed, since  $X \cong \bigvee_{x \in X^-} S^0$ , we have

$$\begin{aligned} X \wedge Y &\cong \left( \bigvee_{x \in X^-} S^0 \right) \wedge Y \\ &\cong \bigvee_{x \in X^-} S^0 \wedge Y \\ &\cong \bigvee_{x \in X^-} Y, \end{aligned}$$

where we have used that  $\wedge$  preserves colimits in both variables via ?? for the second isomorphism above, since it has right adjoints in both variables by **Item 2**.

A similar proof applies to the isomorphism  $X \wedge Y \cong \bigvee_{y \in Y^-} X$ .  $\square$

**00G0 Notation 7.5.1.1.7.** We write  $x \wedge y$  for the element  $[(x, y)]$  of  $X \wedge Y \cong X \times Y / \sim$ .

**00G1 Remark 7.5.1.1.8.** Employing the notation introduced in **Definition 7.5.1.1.7**, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each  $x \in X$  and each  $y \in Y$ , and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each  $x, x' \in X$  and each  $y, y' \in Y$ .

**00G2 Example 7.5.1.1.9.** Here are some examples of smash products of pointed sets.

- 00G3** 1. *Smashing With pt.* For any pointed set  $X$ , we have isomorphisms of pointed sets

$$\begin{aligned} \text{pt} \wedge X &\cong \text{pt}, \\ X \wedge \text{pt} &\cong \text{pt}. \end{aligned}$$

- 00G4** 2. *Smashing With  $S^0$ .* For any pointed set  $X$ , we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

**00G5** **Proposition 7.5.1.1.10.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

- 00G6** 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto X \wedge Y$  define functors

$$\begin{aligned} X \wedge -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ - \wedge Y: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\ -_1 \wedge -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f: (X, x_0) &\rightarrow (A, a_0), \\ g: (Y, y_0) &\rightarrow (B, b_0), \end{aligned}$$

the induced map

$$f \wedge g: X \wedge Y \rightarrow A \wedge B$$

is given by

$$[f \wedge g](x \wedge y) \stackrel{\text{def}}{=} f(x) \wedge g(y)$$

for each  $x \wedge y \in X \wedge Y$ .

- 00G7** 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)): \quad \text{Sets}_* &\begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \text{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)): \quad \text{Sets}_* &\begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \text{Sets}_*, \end{aligned}$$





*Proof.* **Item 1, Functoriality:** The map  $f \wedge g$  comes from **Conditions on Relations, Item 4** of **Definition 10.6.2.1.3** via the map

$$f \wedge g: X \times Y \rightarrow A \wedge B$$

sending  $(x, y)$  to  $f(x) \wedge g(y)$ , which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each  $(x, y), (x', y') \in X \times Y$  with  $(x, y) \sim_R (x', y')$ , where  $\sim_R$  is the relation constructing  $X \wedge Y$  as

$$X \wedge Y \cong (X \times Y) / \sim_R$$

in **Definition 7.5.1.1.4**. The condition defining  $\sim$  is that at least one of the following conditions is satisfied:

- 027D 1. We have  $x = x'$  and  $y = y'$ ;
- 027E 2. Both of the following conditions are satisfied:
  - 027F (a) We have  $x = x_0$  or  $y = y_0$ .
  - 027G (b) We have  $x' = x_0$  or  $y' = y_0$ .

We have five cases:

- 027H 1. In the first case, we clearly have

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

since  $x = x'$  and  $y = y'$ .

- 027J 2. If  $x = x_0$  and  $x' = x_0$ , we have

$$\begin{aligned} [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y'). \end{aligned}$$

027K 3. If  $x = x_0$  and  $y' = y_0$ , we have

$$\begin{aligned}
 [f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\
 &= a_0 \wedge g(y) \\
 &= a_0 \wedge b_0 \\
 &= f(x') \wedge b_0 \\
 &= f(x') \wedge g(y_0) \\
 &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).
 \end{aligned}$$

027L 4. If  $y = y_0$  and  $x' = x_0$ , we have

$$\begin{aligned}
 [f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\
 &= f(x) \wedge b_0 \\
 &= a_0 \wedge b_0 \\
 &= a_0 \wedge g(y') \\
 &= f(x_0) \wedge g(y') \\
 &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').
 \end{aligned}$$

027M 5. If  $y = y_0$  and  $y' = y_0$ , we have

$$\begin{aligned}
 [f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\
 &= f(x) \wedge b_0 \\
 &= f(x') \wedge b_0 \\
 &= f(x) \wedge g(y_0) \\
 &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).
 \end{aligned}$$

Thus  $f \wedge g$  is well-defined. Next, we claim that  $\wedge$  preserves identities and composition:

- *Preservation of Identities.* We have

$$\begin{aligned}
 [\text{id}_X \wedge \text{id}_Y](x \wedge y) &\stackrel{\text{def}}{=} \text{id}_X(x) \wedge \text{id}_Y(y) \\
 &= x \wedge y \\
 &= [\text{id}_{X \wedge Y}](x \wedge y)
 \end{aligned}$$

for each  $x \wedge y \in X \wedge Y$ , and thus

$$\text{id}_X \wedge \text{id}_Y = \text{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

$$\begin{aligned} f &: (X, x_0) \rightarrow (X', x'_0), \\ h &: (X', x'_0) \rightarrow (X'', x''_0), \\ g &: (Y, y_0) \rightarrow (Y', y'_0), \\ k &: (Y', y'_0) \rightarrow (Y'', y''_0), \end{aligned}$$

we have

$$\begin{aligned} [(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\ &\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\ &\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\ &\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y) \end{aligned}$$

for each  $x \wedge y \in X \wedge Y$ , and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.

**Item 2, Adjointness:** We prove only the adjunction  $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$ , witnessed by a natural bijection

$$\text{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction  $X \wedge - \dashv \mathbf{Sets}_*(X, -)$  is similar. We claim we have a bijection

$$\text{Hom}_{\mathbf{Sets}_*}^{\otimes}(X \times Y, Z) \cong \text{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\mathbf{Sets}_*)$ , implying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathbf{Sets}(X \times Y, Z) \cong \mathbf{Sets}(X, \mathbf{Sets}(Y, Z))$$

of **Constructions With Sets, Item 2** of **Definition 4.1.3.1.3**:

- A map

$$\xi: X \times Y \rightarrow Z$$

in  $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$  gets sent to the pointed map

$$\begin{aligned}\xi^{\dagger} &: (X, x_0) \rightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x^{\dagger} : Y \rightarrow Z),\end{aligned}$$

where  $\xi_x^{\dagger} : Y \rightarrow Z$  is the map defined by

$$\xi_x^{\dagger}(y) \stackrel{\text{def}}{=} \xi(x, y)$$

for each  $y \in Y$ , where:

- The map  $\xi^{\dagger}$  is indeed pointed, as we have

$$\begin{aligned}\xi_{x_0}^{\dagger}(y) &\stackrel{\text{def}}{=} \xi(x_0, y) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each  $y \in Y$ . Thus  $\xi_{x_0}^{\dagger} = \Delta_{z_0}$  and  $\xi^{\dagger}$  is pointed.

- The map  $\xi_x^{\dagger}$  indeed lies in  $\mathbf{Sets}_*(Y, Z)$ , as we have

$$\begin{aligned}\xi_x^{\dagger}(y_0) &\stackrel{\text{def}}{=} \xi(x, y_0) \\ &\stackrel{\text{def}}{=} z_0.\end{aligned}$$

- Conversely, a map

$$\begin{aligned}\xi &: (X, x_0) \rightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x : Y \rightarrow Z),\end{aligned}$$

in  $\text{Hom}_{\text{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$  gets sent to the map

$$\xi^{\dagger} : X \times Y \rightarrow Z$$

defined by

$$\xi^{\dagger}(x, y) \stackrel{\text{def}}{=} \xi_x(y)$$

for each  $(x, y) \in X \times Y$ , which indeed lies in  $\text{Hom}_{\text{Sets}_*}^{\otimes}(X \times Y, Z)$ , as:

– *Left Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x_0, y) &\stackrel{\text{def}}{=} \xi_{x_0}(y) \\ &\stackrel{\text{def}}{=} \Delta_{z_0}(y) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each  $y \in Y$ , since  $\xi_{x_0} = \Delta_{z_0}$  as  $\xi$  is assumed to be a pointed map.

– *Right Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x, y_0) &\stackrel{\text{def}}{=} \xi_x(y_0) \\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

for each  $x \in X$ , since  $\xi_x \in \mathbf{Sets}_*(Y, Z)$  is a morphism of pointed sets.

This finishes the proof.

*Item 3, Enriched Adjointness:* This follows from *Item 2* and Monoidal Categories, ?? of ??.

*Item 4, As a Pushout:* Following the description of *Constructions With Sets*, *Definition 4.2.4.1.3*, we have

$$\text{pt} \coprod_{X \vee Y} (X \times Y) \cong (\text{pt} \times (X \times Y)) / \sim,$$

where  $\sim$  identifies the element  $\star$  in  $\text{pt}$  with all elements of the form  $(x_0, y)$  and  $(x, y_0)$  in  $X \times Y$ . Thus *Conditions on Relations*, *Item 4* of *Definition 10.6.2.1.3* coupled with *Definition 7.5.1.1.8* then gives us a well-defined map

$$\text{pt} \coprod_{X \vee Y} (X \times Y) \rightarrow X \wedge Y$$

via  $[(\star, (x, y))] \mapsto x \wedge y$ , with inverse

$$X \wedge Y \rightarrow \text{pt} \coprod_{X \vee Y} (X \times Y)$$

given by  $x \wedge y \mapsto [(\star, (x, y))]$ .

*Item 5, Distributivity Over Wedge Sums:* This follows from *Definition 7.5.9.1.1*, Monoidal Categories, ?? of ??, and the fact that  $\vee$  is the coproduct in  $\mathbf{Sets}_*$  (*Pointed Sets*, *Definition 6.3.3.1.1*).  $\square$

## 00GB 7.5.2 The Internal Hom of Pointed Sets

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00GC **Definition 7.5.2.1.1.** The **internal Hom**<sup>23</sup> of pointed sets from  $(X, x_0)$  to  $(Y, y_0)$  is the pointed set  $\mathbf{Sets}_*((X, x_0), (Y, y_0))$ <sup>24</sup> consisting of:

- *The Underlying Set.* The set  $\mathbf{Sets}_*((X, x_0), (Y, y_0))$  of morphisms of pointed sets from  $(X, x_0)$  to  $(Y, y_0)$ .
- *The Basepoint.* The element

$$\Delta_{y_0} : (X, x_0) \rightarrow (Y, y_0)$$

of  $\mathbf{Sets}_*((X, x_0), (Y, y_0))$  given by

$$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

for each  $x \in X$ .

00GD **Proposition 7.5.2.1.2.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00GE 1. *Functoriality.* The assignments  $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$  define functors

$$\begin{aligned} \mathbf{Sets}_*(X, -) : \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*, \\ \mathbf{Sets}_*(-, Y) : \mathbf{Sets}_*^{\text{op}} &\rightarrow \mathbf{Sets}_*, \\ \mathbf{Sets}_*(-_1, -_2) : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* &\rightarrow \mathbf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f : (X, x_0) &\rightarrow (A, a_0), \\ g : (Y, y_0) &\rightarrow (B, b_0), \end{aligned}$$

the induced map

$$\mathbf{Sets}_*(f, g) : \mathbf{Sets}_*(A, Y) \rightarrow \mathbf{Sets}_*(X, B)$$

is given by

$$[\mathbf{Sets}_*(f, g)](\phi) \stackrel{\text{def}}{=} g \circ \phi \circ f$$

for each  $\phi \in \mathbf{Sets}_*(A, Y)$ .

<sup>23</sup>For a proof that  $\mathbf{Sets}_*$  is indeed the internal Hom of  $\mathbf{Sets}_*$  with respect to the smash product of pointed sets, see Item 2 of Definition 7.5.1.1.10.

<sup>24</sup>*Further Notation:* Also written  $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$ .

00GF 2. *Adjointness.* We have adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) : \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) : \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)), \\ \mathrm{Hom}_{\mathbf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathbf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ .

00GG 3. *Enriched Adjointness.* We have  $\mathbf{Sets}_*$ -enriched adjunctions

$$\begin{aligned} (X \wedge - \dashv \mathbf{Sets}_*(X, -)) : \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{X \wedge -} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(X, -)} \end{array} \mathbf{Sets}_*, \\ (- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) : \quad & \mathbf{Sets}_* \begin{array}{c} \xrightarrow{- \wedge Y} \\ \perp \\ \xleftarrow{\mathbf{Sets}_*(Y, -)} \end{array} \mathbf{Sets}_*, \end{aligned}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned} \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)), \\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)), \end{aligned}$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$ .

*Proof.* **Item 1, Functoriality:** This follows from **Constructions With Sets, Item 1** of **Definition 4.3.5.1.2** and from the equalities

$$\begin{aligned} g \circ \Delta_{y_0} &= \Delta_{z_0}, \\ \Delta_{y_0} \circ f &= \Delta_{y_0} \end{aligned}$$

for morphisms  $f: (K, k_0) \rightarrow (X, x_0)$  and  $g: (Y, y_0) \rightarrow (Z, z_0)$ , which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

*Item 2, Adjointness:* This is a repetition of *Item 2* of [Definition 7.5.1.1.10](#), and is proved there.

*Item 3, Enriched Adjointness:* This is a repetition of *Item 3* of [Definition 7.5.1.1.10](#), and is proved there.  $\square$

### 00GH 7.5.3 The Monoidal Unit

00GJ **Definition 7.5.3.1.1.** The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\text{Sets}_*} : \text{pt} \rightarrow \text{Sets}_*$$

defined by

$$\mathbb{1}_{\text{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

### 00GK 7.5.4 The Associator

00GL **Definition 7.5.4.1.1.** The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\text{Sets}_*} : \wedge \circ (\wedge \times \text{id}_{\text{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\text{id}_{\text{Sets}_*} \times \wedge) \circ \alpha^{\text{Cats}}_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*},$$

as in the diagram

$$\begin{array}{ccccc}
 & & \text{Sets}_* \times (\text{Sets}_* \times \text{Sets}_*) & & \\
 & \nearrow \alpha^{\text{Cats}}_{\text{Sets}_*, \text{Sets}_*, \text{Sets}_*} & \searrow \text{id} \times \wedge & & \\
 (\text{Sets}_* \times \text{Sets}_*) \times \text{Sets}_* & & & \text{Sets}_* \times \text{Sets}_* & \\
 \downarrow \wedge \times \text{id} & \nearrow \alpha^{\text{Sets}_*} & & \downarrow \wedge & \\
 \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\wedge} & \text{Sets}_* & & 
 \end{array}$$

whose component

$$\alpha^{\text{Sets}_*}_{X,Y,Z} : (X \wedge Y) \wedge Z \xrightarrow{\sim} X \wedge (Y \wedge Z)$$



at  $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$  is given by

$$\alpha_{X,Y,Z}^{\text{Sets}_*}((x \wedge y) \wedge z) \stackrel{\text{def}}{=} x \wedge (y \wedge z)$$

for each  $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$ .

*Proof. Well-Definedness:* Let  $[((x, y), z)] = [((x', y'), z')]$  be an element in  $(X \wedge Y) \wedge Z$ . Then either:

- 027N 1. We have  $x = x', y = y',$  and  $z = z'$ .
- 027P 2. Both of the following conditions are satisfied:
  - 027Q (a) We have  $x = x_0$  or  $y = y_0$  or  $z = z_0$ .
  - 027R (b) We have  $x' = x_0$  or  $y' = y_0$  or  $z' = z_0$ .

In the first case,  $\alpha_{X,Y,Z}^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X \wedge (Y \wedge Z)$ . Meanwhile, in the latter case both elements are equal to the basepoint  $(x_0 \wedge y_0) \wedge z_0$  of  $(X \wedge Y) \wedge Z$ , which gets sent to the basepoint  $x_0 \wedge (y_0 \wedge z_0)$  of  $X \wedge (Y \wedge Z)$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\alpha_{X,Y,Z}^{\text{Sets}_*}((x_0 \wedge y_0) \wedge z_0) \stackrel{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus  $\alpha_{X,Y,Z}^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* The inverse of  $\alpha_{X,Y,Z}^{\text{Sets}_*}$  is given by the morphism

$$\alpha_{X,Y,Z}^{\text{Sets}_*, -1}: X \wedge (Y \wedge Z) \xrightarrow{\sim} (X \wedge Y) \wedge Z$$

defined by

$$\alpha_{X,Y,Z}^{\text{Sets}_*, -1}(x \wedge (y \wedge z)) \stackrel{\text{def}}{=} (x \wedge y) \wedge z$$

for each  $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$ .

*Naturality:* We need to show that, given morphisms of pointed sets

$$\begin{aligned} f: (X, x_0) &\rightarrow (X', x'_0), \\ g: (Y, y_0) &\rightarrow (Y', y'_0), \\ h: (Z, z_0) &\rightarrow (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{array}{ccc}
 (X \wedge Y) \wedge Z & \xrightarrow{(f \wedge g) \wedge h} & (X' \wedge Y') \wedge Z' \\
 \downarrow \alpha_{X,Y,Z}^{\text{Sets}_*} & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}_*} \\
 X \wedge (Y \wedge Z) & \xrightarrow{f \wedge (g \wedge h)} & X' \wedge (Y' \wedge Z')
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x \wedge y) \wedge z & \longmapsto & (f(x) \wedge g(y)) \wedge h(z) \\
 \downarrow & & \downarrow \\
 x \wedge (y \wedge z) & \longmapsto & f(x) \wedge (g(y) \wedge h(z))
 \end{array}$$

and hence indeed commutes, showing  $\alpha^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\alpha^{\text{Sets}_*}$  is natural and  $\alpha^{\text{Sets}_*, -1}$  is a componentwise inverse to  $\alpha^{\text{Sets}_*}$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\alpha^{\text{Sets}_*, -1}$  is also natural. Thus  $\alpha^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

## 00GM 7.5.5 The Left Unitor

00GN **Definition 7.5.5.1.1.** The left unitor of the smash product of pointed sets is the natural isomorphism

$$\lambda^{\text{Sets}_*} : \wedge \circ (\mathbb{1}^{\text{Sets}_*} \times \text{id}_{\text{Sets}_*}) \xRightarrow{\sim} \lambda_{\text{Sets}_*}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}_*} : S^0 \wedge X \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\text{Sets}_*)$  is given by

$$\begin{aligned} 0 \wedge x &\mapsto x_0, \\ 1 \wedge x &\mapsto x \end{aligned}$$

for each  $x \in X$ .

*Proof. Well-Definedness:* Let  $[(x, y)] = [(x', y')]$  be an element in  $S^0 \wedge X$ . Then either:

- 027S 1. We have  $x = x'$  and  $y = y'$ .
- 027T 2. Both of the following conditions are satisfied:
  - 027U (a) We have  $x = 0$  or  $y = x_0$ .
  - 027V (b) We have  $x' = 0$  or  $y' = x_0$ .

In the first case,  $\lambda_X^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X$ . Meanwhile, in the latter case both elements are equal to the basepoint  $0 \wedge x_0$  of  $S^0 \wedge X$ , which gets sent to the basepoint  $x_0$  of  $X$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\lambda_X^{\text{Sets}_*}(0 \wedge x_0) \stackrel{\text{def}}{=} x_0,$$

and thus  $\lambda_X^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* The inverse of  $\lambda_X^{\text{Sets}_*}$  is the morphism

$$\lambda_X^{\text{Sets}_*, -1} : X \xrightarrow{\sim} S^0 \wedge X$$

defined by

$$\lambda_X^{\text{Sets}_*, -1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each  $x \in X$ . Indeed:

- 027W 1. *Invertibility I.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*}](0 \wedge x) &= \lambda_X^{\text{Sets}_*, -1}(\lambda_X^{\text{Sets}_*}(0 \wedge x)) \\ &= \lambda_X^{\text{Sets}_*, -1}(x_0) \\ &= 1 \wedge x_0 \end{aligned}$$

$$= 0 \wedge x,$$

and

$$\begin{aligned} [\lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*}](1 \wedge x) &= \lambda_X^{\text{Sets}_*, -1}(\lambda_X^{\text{Sets}_*}(1 \wedge x)) \\ &= \lambda_X^{\text{Sets}_*, -1}(x) \\ &= 1 \wedge x \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\lambda_X^{\text{Sets}_*, -1} \circ \lambda_X^{\text{Sets}_*} = \text{id}_{S^0 \wedge X}.$$

**027X** 2. *Invertibility II.* We have

$$\begin{aligned} [\lambda_X^{\text{Sets}_*} \circ \lambda_X^{\text{Sets}_*, -1}](x) &= \lambda_X^{\text{Sets}_*}(\lambda_X^{\text{Sets}_*, -1}(x)) \\ &= \lambda_X^{\text{Sets}_*, -1}(1 \wedge x) \\ &= x \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\lambda_X^{\text{Sets}_*} \circ \lambda_X^{\text{Sets}_*, -1} = \text{id}_X.$$

This shows  $\lambda_X^{\text{Sets}_*}$  to be invertible.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} S^0 \wedge X & \xrightarrow{\text{id}_{S^0} \wedge f} & S^0 \wedge Y \\ \lambda_X^{\text{Sets}_*} \downarrow & & \downarrow \lambda_Y^{\text{Sets}_*} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} 0 \wedge x & & 0 \wedge x \longmapsto 0 \wedge f(x) \\ \downarrow & & \downarrow \\ x_0 & \longmapsto & f(x_0) \\ & & y_0 \end{array}$$

and

$$\begin{array}{ccc} 1 \wedge x & \longmapsto & 1 \wedge f(x) \\ \downarrow & & \downarrow \\ x & \longmapsto & f(x) \end{array}$$

and hence indeed commutes, showing  $\lambda^{\mathbf{Sets}_*}$  to be a natural transformation. *Being a Natural Isomorphism:* Since  $\lambda^{\mathbf{Sets}_*}$  is natural and  $\lambda^{\mathbf{Sets}_*, -1}$  is a componentwise inverse to  $\lambda^{\mathbf{Sets}_*}$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\lambda^{\mathbf{Sets}_*, -1}$  is also natural. Thus  $\lambda^{\mathbf{Sets}_*}$  is a natural isomorphism.  $\square$

## 00GP 7.5.6 The Right Unitor

00GQ **Definition 7.5.6.1.1.** The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathbf{Sets}_*} : \wedge \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Sets}_*}) \xRightarrow{\sim} \rho_{\mathbf{Sets}_*}^{\mathbf{Cats}_2},$$

whose component

$$\rho_X^{\mathbf{Sets}_*} : X \wedge S^0 \xrightarrow{\sim} X$$

at  $X \in \mathrm{Obj}(\mathbf{Sets}_*)$  is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x \end{aligned}$$

for each  $x \in X$ .

*Proof. Well-Definedness:* Let  $[(x, y)] = [(x', y')]$  be an element in  $X \wedge S^0$ . Then either:

- 027Y 1. We have  $x = x'$  and  $y = y'$ .
- 027Z 2. Both of the following conditions are satisfied:
- 0280 (a) We have  $x = x_0$  or  $y = 0$ .
- 0281 (b) We have  $x' = x_0$  or  $y' = 0$ .

In the first case,  $\rho_X^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X$ . Meanwhile, in the latter case both elements are equal to the basepoint  $x_0 \wedge 0$  of  $X \wedge S^0$ , which gets sent to the basepoint  $x_0$  of  $X$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\rho_X^{\text{Sets}_*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus  $\rho_X^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* The inverse of  $\rho_X^{\text{Sets}_*}$  is the morphism

$$\rho_X^{\text{Sets}_*, -1} : X \xrightarrow{\sim} X \wedge S^0$$

defined by

$$\rho_X^{\text{Sets}_*, -1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each  $x \in X$ . Indeed:

- 0282 1. *Invertibility I.* We have

$$\begin{aligned} [\rho_X^{\text{Sets}_*, -1} \circ \rho_X^{\text{Sets}_*}](x \wedge 0) &= \rho_X^{\text{Sets}_*, -1}(\rho_X^{\text{Sets}_*}(x \wedge 0)) \\ &= \rho_X^{\text{Sets}_*, -1}(x_0) \\ &= x_0 \wedge 1 \\ &= x \wedge 0, \end{aligned}$$

and

$$\begin{aligned} [\rho_X^{\text{Sets}_*, -1} \circ \rho_X^{\text{Sets}_*}](x \wedge 1) &= \rho_X^{\text{Sets}_*, -1}(\rho_X^{\text{Sets}_*}(x \wedge 1)) \\ &= \rho_X^{\text{Sets}_*, -1}(x) \\ &= x \wedge 1 \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\rho_X^{\text{Sets}_*, -1} \circ \rho_X^{\text{Sets}_*} = \text{id}_{X \wedge S^0}.$$

0283 2. *Invertibility II.* We have

$$\begin{aligned} [\rho_X^{\text{Sets}_*} \circ \rho_X^{\text{Sets}_*, -1}](x) &= \rho_X^{\text{Sets}_*}(\rho_X^{\text{Sets}_*, -1}(x)) \\ &= \rho_X^{\text{Sets}_*, -1}(x \wedge 1) \\ &= x \end{aligned}$$

for each  $x \in X$ , and thus we have

$$\rho_X^{\text{Sets}_*} \circ \rho_X^{\text{Sets}_*, -1} = \text{id}_X.$$

This shows  $\rho_X^{\text{Sets}_*}$  to be invertible.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X \wedge S^0 & \xrightarrow{f \wedge \text{id}_{S^0}} & Y \wedge S^0 \\ \rho_X^{\text{Sets}_*} \downarrow & & \downarrow \rho_Y^{\text{Sets}_*} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \wedge 0 & & x \wedge 0 \longmapsto f(x) \wedge 0 \\ \downarrow & & \downarrow \\ x_0 & \longmapsto & f(x_0) \\ & & y_0 \end{array}$$

and

$$\begin{array}{ccc} x \wedge 1 & \longmapsto & f(x) \wedge 1 \\ \downarrow & & \downarrow \\ x & \longmapsto & f(x) \end{array}$$

and hence indeed commutes, showing  $\rho^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\rho^{\text{Sets}_*}$  is natural and  $\rho^{\text{Sets}_*, -1}$  is a component-wise inverse to  $\rho^{\text{Sets}_*}$ , it follows from [Categories, Item 2](#) of [Definition 11.9.7.1.2](#) that  $\rho^{\text{Sets}_*, -1}$  is also natural. Thus  $\rho^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

### 00GR 7.5.7 The Symmetry

00GS **Definition 7.5.7.1.1.** The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\text{Sets}_*} : \wedge \xrightarrow{\sim} \wedge \circ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2}, \quad \begin{array}{ccc} \text{Sets}_* \times \text{Sets}_* & \xrightarrow{\wedge} & \text{Sets}_* \\ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2} \searrow & \Downarrow \sigma^{\text{Sets}_*} & \nearrow \wedge \\ & \text{Sets}_* \times \text{Sets}_* & \end{array}$$

whose component

$$\sigma_{X,Y}^{\text{Sets}_*} : X \wedge Y \xrightarrow{\sim} Y \wedge X$$

at  $X, Y \in \text{Obj}(\text{Sets}_*)$  is defined by

$$\sigma_{X,Y}^{\text{Sets}_*}(x \wedge y) \stackrel{\text{def}}{=} y \wedge x$$

for each  $x \wedge y \in X \wedge Y$ .

*Proof. Well-Definedness:* Let  $[(x, y)] = [(x', y')]$  be an element in  $X \wedge Y$ . Then either:

- 0284 1. We have  $x = x'$  and  $y = y'$ .
- 0285 2. Both of the following conditions are satisfied:
  - 0286 (a) We have  $x = x_0$  or  $y = y_0$ .
  - 0287 (b) We have  $x' = x_0$  or  $y' = y_0$ .

In the first case,  $\sigma_X^{\text{Sets}_*}$  clearly sends both elements to the same element in  $X$ . Meanwhile, in the latter case both elements are equal to the basepoint  $x_0 \wedge y_0$  of  $X \wedge Y$ , which gets sent to the basepoint  $y_0 \wedge x_0$  of  $Y \wedge X$ .

*Being a Morphism of Pointed Sets:* As just mentioned, we have

$$\sigma_X^{\text{Sets}_*}(x_0 \wedge y_0) \stackrel{\text{def}}{=} y_0 \wedge x_0,$$

and thus  $\sigma_X^{\text{Sets}_*}$  is a morphism of pointed sets.

*Invertibility:* The inverse of  $\sigma_{X,Y}^{\text{Sets}_*}$  is given by the morphism

$$\sigma_{X,Y}^{\text{Sets}_*, -1} : Y \wedge X \xrightarrow{\sim} X \wedge Y$$



defined by

$$\sigma_{X,Y}^{\text{Sets}_*, -1}(y \wedge x) \stackrel{\text{def}}{=} x \wedge y$$

for each  $y \wedge x \in Y \wedge X$ .

*Naturality:* We need to show that, given morphisms of pointed sets

$$f: (X, x_0) \rightarrow (A, a_0),$$

$$g: (Y, y_0) \rightarrow (B, b_0)$$

the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{f \wedge g} & A \wedge B \\ \sigma_{X,Y}^{\text{Sets}_*} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}_*} \\ Y \wedge X & \xrightarrow{g \wedge f} & B \wedge A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \wedge y & \longmapsto & f(x) \wedge g(y) \\ \downarrow & & \downarrow \\ y \wedge x & \longmapsto & g(y) \wedge f(x) \end{array}$$

and hence indeed commutes, showing  $\sigma^{\text{Sets}_*}$  to be a natural transformation.

*Being a Natural Isomorphism:* Since  $\sigma^{\text{Sets}_*}$  is natural and  $\sigma^{\text{Sets}_*, -1}$  is a component-wise inverse to  $\sigma^{\text{Sets}_*}$ , it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that  $\sigma^{\text{Sets}_*, -1}$  is also natural. Thus  $\sigma^{\text{Sets}_*}$  is a natural isomorphism.  $\square$

## 00GT 7.5.8 The Diagonal

00GU **Definition 7.5.8.1.1.** The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^\wedge: \text{id}_{\text{Sets}_*} \Rightarrow \wedge \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

whose component

$$\Delta_X^\wedge : (X, x_0) \rightarrow (X \wedge X, x_0 \wedge x_0)$$

at  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$  is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta_X^\wedge} (X \times X, (x_0, x_0)) \\ &\longrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\ &\stackrel{\text{def}}{=} (X \wedge X, x_0 \wedge x_0) \end{aligned}$$

in  $\text{Sets}_*$ , and thus by

$$\Delta_X^\wedge(x) \stackrel{\text{def}}{=} x \wedge x$$

for each  $x \in X$ .

*Proof. Being a Morphism of Pointed Sets:* We have

$$\Delta_X^\wedge(x_0) \stackrel{\text{def}}{=} x_0 \wedge x_0,$$

and thus  $\Delta_X^\wedge$  is a morphism of pointed sets.

*Naturality:* We need to show that, given a morphism of pointed sets

$$f : (X, x_0) \rightarrow (Y, y_0),$$

the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X^\wedge \downarrow & & \downarrow \Delta_Y^\wedge \\ X \wedge X & \xrightarrow{f \wedge f} & Y \wedge Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & \longmapsto & f(x) \\ \downarrow & & \downarrow \\ x \wedge x & \longmapsto & f(x) \wedge f(x) \end{array}$$

and hence indeed commutes, showing  $\Delta^\wedge$  to be natural. □

**00GV Proposition 7.5.8.1.2.** Let  $(X, x_0) \in \text{Obj}(\text{Sets}_*)$ .

**00GW** 1. *Monoidality.* The diagonal

$$\Delta^\wedge : \text{id}_{\text{Sets}_*} \Longrightarrow \wedge \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2},$$

of the smash product of pointed sets is a monoidal natural transformation:

**00GX** (a) *Compatibility With Strong Monoidality Constraints.* For each  $(X, x_0), (Y, y_0) \in \text{Obj}(\text{Sets}_*)$ , the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\Delta_X^\wedge \wedge \Delta_Y^\wedge} & (X \wedge X) \wedge (Y \wedge Y) \\ & \searrow \Delta_{X \wedge Y}^\wedge & \downarrow \wr \\ & & (X \wedge Y) \wedge (X \wedge Y) \end{array}$$

commutes.

**00GY** (b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{array}{ccc} S^0 & \xrightarrow{\Delta_{S^0}^\wedge} & S^0 \wedge S^0 \\ \parallel & \searrow \lambda_{S^0}^{\text{Sets}_*} & \downarrow \rho_{S^0}^{\text{Sets}_*} \\ & & S^0 \end{array} \quad \begin{array}{ccc} S^0 & \xrightarrow{\Delta_{S^0}^\wedge} & S^0 \wedge S^0 \\ \parallel & \searrow \rho_{S^0}^{\text{Sets}_*} & \downarrow \lambda_{S^0}^{\text{Sets}_*} \\ & & S^0 \end{array}$$

commute, i.e. we have

$$\begin{aligned} \Delta_{S^0}^\wedge &= \lambda_{S^0}^{\text{Sets}_*, -1} \\ &= \rho_{S^0}^{\text{Sets}_*, -1}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned} \lambda_{S^0}^{\text{Sets}_*} &= \rho_{S^0}^{\text{Sets}_*}, \\ \lambda_{S^0}^{\text{Sets}_*, -1} &= \rho_{S^0}^{\text{Sets}_*, -1} \end{aligned}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

00GZ 2. *The Diagonal of the Unit.* The component

$$\Delta_{S^0}^\wedge : S^0 \xrightarrow{\sim} S^0 \wedge S^0$$

of  $\Delta^\wedge$  at  $S^0$  is an isomorphism.

*Proof.* **Item 1, Monoidality:** We claim that  $\Delta^\wedge$  is indeed monoidal:

0288 1. **Item 1a: Compatibility With Strong Monoidality Constraints:** We need to show that the diagram

$$\begin{array}{ccc} X \wedge Y & \xrightarrow{\Delta_X^\wedge \wedge \Delta_Y^\wedge} & (X \wedge X) \wedge (Y \wedge Y) \\ & \searrow \Delta_{X \wedge Y}^\wedge & \downarrow \wr \\ & & (X \wedge Y) \wedge (X \wedge Y) \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x \wedge y & \xrightarrow{\quad} & (x \wedge x) \wedge (y \wedge y) \\ & \searrow & \downarrow \\ & & (x \wedge y) \wedge (x \wedge y) \end{array}$$

and hence indeed commutes.

0289 2. **Item 1b: Compatibility With Strong Unitality Constraints:** As shown in the proof of **Definition 7.5.5.1.1**, the inverse of the left unitor of  $\mathbf{Sets}_*$  with respect to the smash product of pointed sets at  $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$  is given by

$$\lambda_X^{\mathbf{Sets}_*, -1}(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each  $x \in X$ , so when  $X = S^0$ , we have

$$\lambda_{S^0}^{\mathbf{Sets}_*, -1}(0) \stackrel{\text{def}}{=} 1 \wedge 0,$$

$$\lambda_{S^0}^{\mathbf{Sets}_*, -1}(1) \stackrel{\text{def}}{=} 1 \wedge 1.$$

But since  $1 \wedge 0 = 0 \wedge 0$  and

$$\Delta_{S^0}^\wedge(0) \stackrel{\text{def}}{=} 0 \wedge 0,$$

$$\Delta_{S^0}^\wedge(1) \stackrel{\text{def}}{=} 1 \wedge 1,$$

it follows that we indeed have  $\Delta_{S^0}^\wedge = \lambda_{S^0}^{\mathbf{Sets}_*, -1}$ .

This finishes the proof.

*Item 2, The Diagonal of the Unit:* This follows from *Item 1* and the invertibility of the left/right unitor of  $\mathbf{Sets}_*$  with respect to  $\wedge$ , proved in the proof of *Definition 7.5.5.1.1* for the left unitor or the proof of *Definition 7.5.6.1.1* for the right unitor.  $\square$

## 00H0 7.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

00H1 **Proposition 7.5.9.1.1.** The category  $\mathbf{Sets}_*$  admits a closed monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category  $\mathbf{Sets}_*$  of pointed sets.
- *The Monoidal Product.* The smash product functor

$$\wedge : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of *Item 1* of *Definition 7.5.1.1.10*.

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}_* : \mathbf{Sets}_*^{\text{op}} \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of *Item 1* of *Definition 7.5.2.1.2*.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}_*} : \text{pt} \rightarrow \mathbf{Sets}_*$$

of *Definition 7.5.3.1.1*.

- *The Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}_*} : \wedge \circ (\wedge \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \wedge \circ (\text{id}_{\mathbf{Sets}_*} \times \wedge) \circ \alpha_{\mathbf{Sets}_*, \mathbf{Sets}_*, \mathbf{Sets}_*}^{\mathbf{Cats}}$$

of *Definition 7.5.4.1.1*.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Sets}_*} : \wedge \circ (\mathbb{1}^{\mathbf{Sets}_*} \times \text{id}_{\mathbf{Sets}_*}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}_*}^{\mathbf{Cats}_2}$$

of *Definition 7.5.5.1.1*.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}_*} : \wedge \circ (\text{id} \times \mathbb{1}^{\text{Sets}_*}) \xRightarrow{\sim} \rho_{\text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.6.1.1](#).

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}_*} : \wedge \xRightarrow{\sim} \wedge \circ \sigma_{\text{Sets}_*, \text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.7.1.1](#).

- *The Diagonals.* The monoidal natural transformation

$$\Delta^\wedge : \text{id}_{\text{Sets}_*} \Longrightarrow \wedge \circ \Delta_{\text{Sets}_*}^{\text{Cats}_2}$$

of [Definition 7.5.8.1.1](#).

*Proof. The Pentagon Identity:* Let  $(W, w_0)$ ,  $(X, x_0)$ ,  $(Y, y_0)$  and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccccc}
 & & (W \wedge (X \wedge Y)) \wedge Z & & \\
 & \nearrow \alpha_{W,X,Y}^{\text{Sets}_*} \wedge \text{id}_Z & & \nwarrow \alpha_{W,X \wedge Y,Z}^{\text{Sets}_*} & \\
 ((W \wedge X) \wedge Y) \wedge Z & & & & W \wedge ((X \wedge Y) \wedge Z) \\
 \searrow \alpha_{W \wedge X,Y,Z}^{\text{Sets}_*} & & & & \swarrow \text{id}_W \wedge \alpha_{X,Y,Z}^{\text{Sets}_*} \\
 (W \wedge X) \wedge (Y \wedge Z) & \xrightarrow{\alpha_{W,X,Y \wedge Z}^{\text{Sets}_*}} & W \wedge (X \wedge (Y \wedge Z)) & & 
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (w \wedge (x \wedge y)) \wedge z & \\
 \swarrow & & \searrow \\
 ((w \wedge x) \wedge y) \wedge z & & w \wedge ((x \wedge y) \wedge z) \\
 \searrow & & \swarrow \\
 (w \wedge x) \wedge (y \wedge z) & \longmapsto & w \wedge (x \wedge (y \wedge z))
 \end{array}$$

and thus we see that the pentagon identity is satisfied.

*The Triangle Identity:* Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \wedge S^0) \wedge Y & \xrightarrow{\alpha_{X, S^0, Y}^{\text{Sets}_*}} & X \wedge (S^0 \wedge Y) \\
 \searrow \rho_X^{\text{Sets}_*} \wedge \text{id}_Y & & \swarrow \text{id}_X \wedge \lambda_Y^{\text{Sets}_*} \\
 & X \wedge Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 (x \wedge 0) \wedge y & \longmapsto & x \wedge (0 \wedge y) \\
 \searrow & & \swarrow \\
 x_0 \wedge y & & x \wedge y_0
 \end{array}$$

and

$$\begin{array}{ccc}
 (x \wedge 1) \wedge y & \longmapsto & x \wedge (1 \wedge y) \\
 \searrow & & \swarrow \\
 & x \wedge y, &
 \end{array}$$

and thus we see that the triangle identity is satisfied.

*The Left Hexagon Identity:* Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \wedge Y) \wedge Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}_*} \swarrow & & \searrow \beta_{X,Y}^{\text{Sets}_*} \wedge \text{id}_Z \\
 X \wedge (Y \wedge Z) & & (Y \wedge X) \wedge Z \\
 \downarrow \beta_{X,Y \wedge Z}^{\text{Sets}_*} & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}_*} \\
 (Y \wedge Z) \wedge X & & Y \wedge (X \wedge Z) \\
 \searrow \alpha_{Y,Z,X}^{\text{Sets}_*} & & \swarrow \text{id}_Y \wedge \beta_{X,Z}^{\text{Sets}_*} \\
 & Y \wedge (Z \wedge X) &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (x \wedge y) \wedge z & \\
 \swarrow & & \searrow \\
 x \wedge (y \wedge z) & & (y \wedge x) \wedge z \\
 \downarrow & & \downarrow \\
 (y \wedge z) \wedge x & & y \wedge (x \wedge z) \\
 \swarrow & & \searrow \\
 & y \wedge (z \wedge x) &
 \end{array}$$

and thus we see that the left hexagon identity is satisfied.

*The Right Hexagon Identity:* Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.



We have to show that the diagram

$$\begin{array}{ccc}
 & X \wedge (Y \wedge Z) & \\
 (\alpha_{X,Y,Z}^{\text{Sets}_*})^{-1} \swarrow & & \searrow \text{id}_X \wedge \beta_{Y,Z}^{\text{Sets}_*} \\
 (X \wedge Y) \wedge Z & & X \wedge (Z \wedge Y) \\
 \downarrow \beta_{X \wedge Y, Z}^{\text{Sets}_*} & & \downarrow (\alpha_{X,Z,Y}^{\text{Sets}_*})^{-1} \\
 Z \wedge (X \wedge Y) & & (X \wedge Z) \wedge Y \\
 & \swarrow (\alpha_{Z,X,Y}^{\text{Sets}_*})^{-1} \quad \nwarrow \beta_{X,Z}^{\text{Sets}_*} \wedge \text{id}_Y & \\
 & (Z \wedge X) \wedge Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & x \wedge (y \wedge z) & \\
 \swarrow & & \searrow \\
 (x \wedge y) \wedge z & & x \wedge (z \wedge y) \\
 \downarrow & & \downarrow \\
 z \wedge (x \wedge y) & & (x \wedge z) \wedge y \\
 & \swarrow \quad \nwarrow & \\
 & (z \wedge x) \wedge y &
 \end{array}$$

and thus we see that the right hexagon identity is satisfied.

*Monoidal Closedness:* This follows from **Item 2** of **Definition 7.5.1.1.10**.

*Existence of Monoidal Diagonals:* This follows from **Items 1** and **2** of **Definition 7.5.8.1.2**.  $\square$

## 01VY 7.5.10 The Universal Property of $(\text{Sets}_*, \wedge, S^0)$

01VZ **Theorem 7.5.10.1.1.** The symmetric monoidal structure on the category  $\text{Sets}_*$  of **Definition 7.5.9.1.1** is uniquely determined by the following requirements:

- 01W0 1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathbf{Sets}_*} : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of  $\mathbf{Sets}_*$  admits an internal Hom  $[-1, -2]_{\mathbf{Sets}_*}$ .

- 01W1 2. *The Unit Object Is  $S^0$ .* We have  $\mathbb{1}_{\mathbf{Sets}_*} \cong S^0$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{B}_\infty}^{\text{cl}}(\mathbf{Sets}_*)$  of ?? spanned by the closed symmetric monoidal categories  $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, [-1, -2]_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$  satisfying **Items 1** and **2** is contractible (i.e. equivalent to the punctual category).

*Proof. Unwinding the Statement:* Let  $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, [-1, -2]_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda', \rho', \sigma')$  be a closed symmetric monoidal category satisfying **Items 1** and **2**. We need to show that the identity functor

$$\text{id}_{\mathbf{Sets}_*} : \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \text{id}_{\mathbf{Sets}_*}^{\otimes} : X \otimes_{\mathbf{Sets}_*} Y &\xrightarrow{\sim} X \wedge Y, \\ \text{id}_{\mathbf{Sets}_*}^{\text{Hom}} : [X, Y]_{\mathbf{Sets}_*} &\xrightarrow{\sim} \mathbf{Sets}_*(X, Y), \\ \text{id}_{\mathbb{1}_{\mathbf{Sets}_*}}^{\otimes} : \mathbb{1}_{\mathbf{Sets}_*} &\xrightarrow{\sim} S^0, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from  $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, [-1, -2]_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda', \rho', \sigma')$  to the closed symmetric monoidal category  $(\mathbf{Sets}_*, \times, \mathbf{Sets}_*(-1, -2), \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$  of **Definition 7.5.9.1.1**.

*Constructing an Isomorphism  $[-1, -2]_{\mathbf{Sets}_*} \cong \mathbf{Sets}_*(-1, -2)$ :* By ??, we have a natural isomorphism

$$\mathbf{Sets}_*(S^0, [-1, -2]_{\mathbf{Sets}_*}) \cong \mathbf{Sets}_*(-1, -2).$$

By **Pointed Sets**, **Item 4** of **Definition 6.1.4.1.1**, we also have a natural isomorphism

$$\mathbf{Sets}_*(S^0, [-1, -2]_{\mathbf{Sets}_*}) \cong [-1, -2]_{\mathbf{Sets}_*}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathbf{Sets}_*(-1, -2) \cong [-1, -2]_{\mathbf{Sets}_*}.$$

Given  $X, Y \in \mathbf{Obj}(\mathbf{Sets}_*)$ , we will write

$$\mathrm{id}_{X,Y}^{\mathrm{Hom}} : \mathbf{Sets}_*(X, Y) \xrightarrow{\sim} [X, Y]_{\mathbf{Sets}_*}$$

for the component of this isomorphism at  $(X, Y)$ .

*Constructing an Isomorphism  $\otimes_{\mathbf{Sets}_*} \cong \wedge$ :* Since  $\otimes_{\mathbf{Sets}_*}$  is adjoint in each variable to  $[-1, -2]_{\mathbf{Sets}_*}$  by assumption and  $\wedge$  is adjoint in each variable to  $\mathbf{Sets}_*(-1, -2)$  by **Constructions With Sets, Item 2 of Definition 4.3.5.1.2**, uniqueness of adjoints (??) gives us natural isomorphisms

$$\begin{aligned} X \otimes_{\mathbf{Sets}_*} - &\cong X \wedge -, \\ - \otimes_{\mathbf{Sets}_*} Y &\cong Y \wedge -. \end{aligned}$$

By ??, we then have  $\otimes_{\mathbf{Sets}_*} \cong \wedge$ . We will write

$$\mathrm{id}_{\mathbf{Sets}_*|X,Y}^{\otimes} : X \otimes_{\mathbf{Sets}_*} Y \xrightarrow{\sim} X \wedge Y$$

for the component of this isomorphism at  $(X, Y)$ .

*Alternative Construction of an Isomorphism  $\otimes_{\mathbf{Sets}_*} \cong \wedge$ :* Alternatively, we may construct a natural isomorphism  $\otimes_{\mathbf{Sets}_*} \cong \wedge$  as follows:

- 01W2 1. Let  $X \in \mathbf{Obj}(\mathbf{Sets}_*)$ .
- 01W3 2. Since  $\otimes_{\mathbf{Sets}_*}$  is part of a closed monoidal structure, it preserves colimits in each variable by ??.
- 01W4 3. Since  $X \cong \bigvee_{x \in X^-} S^0$  and  $\otimes_{\mathbf{Sets}_*}$  preserves colimits in each variable, we have

$$\begin{aligned} X \otimes_{\mathbf{Sets}_*} Y &\cong \left( \bigvee_{x \in X^-} S^0 \right) \otimes_{\mathbf{Sets}_*} Y \\ &\cong \bigvee_{x \in X^-} (S^0 \otimes_{\mathbf{Sets}_*} Y) \\ &\cong \bigvee_{x \in X^-} Y \\ &\cong \bigvee_{x \in X^-} S^0 \wedge Y \\ &\cong \left( \bigvee_{x \in X^-} S^0 \right) \wedge Y \end{aligned}$$

$$\cong X \wedge Y,$$

naturally in  $Y \in \text{Obj}(\text{Sets}_*)$ , where we have used that  $S^0$  is the monoidal unit for  $\otimes_{\text{Sets}_*}$ . Thus  $X \otimes_{\text{Sets}_*} - \cong X \wedge -$  for each  $X \in \text{Obj}(\text{Sets}_*)$ .

01W5 4. Similarly,  $- \otimes_{\text{Sets}_*} Y \cong - \wedge Y$  for each  $Y \in \text{Obj}(\text{Sets}_*)$ .

01W6 5. By ??, we then have  $\otimes_{\text{Sets}_*} \cong \wedge$ .

Below, we'll show that if a natural isomorphism  $\otimes_{\text{Sets}_*} \cong \wedge$  exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\text{id}_{\text{Sets}_*|X,Y}^\otimes: X \otimes_{\text{Sets}_*} Y \rightarrow X \wedge Y$  from before.

*Constructing an Isomorphism  $\text{id}_\mathbb{1}^\otimes: \mathbb{1}_{\text{Sets}_*} \rightarrow S^0$ :* We define an isomorphism  $\text{id}_\mathbb{1}^\otimes: \mathbb{1}_{\text{Sets}_*} \rightarrow S^0$  as the composition

$$\mathbb{1}_{\text{Sets}_*} \xrightarrow[\sim]{\rho_{\text{Sets}_*, -1}^{\text{Sets}_*, -1}} \mathbb{1}_{\text{Sets}_*} \wedge S^0 \xrightarrow[\sim]{\text{id}_{\text{Sets}_*| \mathbb{1}_{\text{Sets}_*}}^{\otimes, -1}} \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0 \xrightarrow[\sim]{\lambda'_{S^0}} S^0$$

in  $\text{Sets}_*$ .

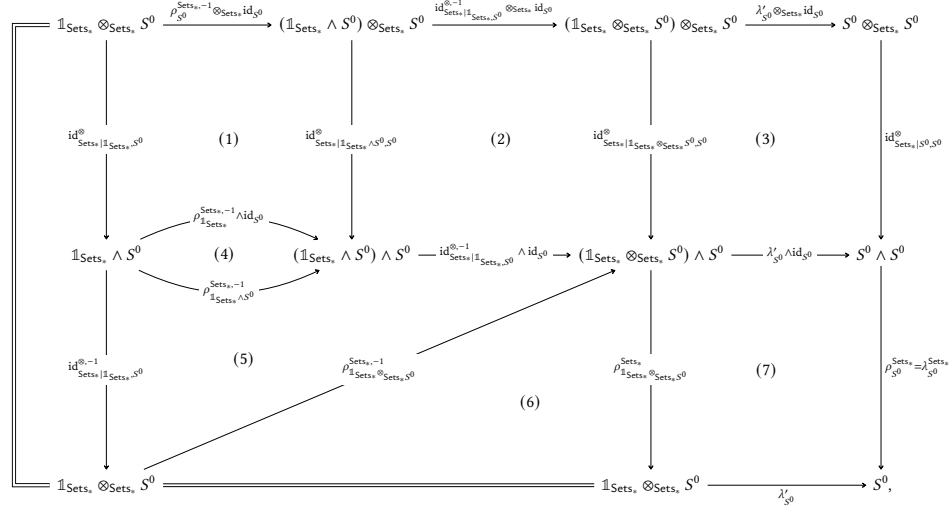
*Monoidal Left Unity of the Isomorphism  $\otimes_{\text{Sets}_*} \cong \wedge$ :* We have to show that the diagram

$$\begin{array}{ccc} & S^0 \otimes_{\text{Sets}_*} X & \xrightarrow{\text{id}_{\text{Sets}_*|S^0,X}^\otimes} S^0 \wedge X \\ \text{id}_\mathbb{1}^\otimes|_{\text{Sets}_*} \otimes_{\text{Sets}_*} \text{id}_X \nearrow & & \searrow \lambda_X^{\text{Sets}_*} \\ \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X & \xrightarrow{\lambda'_X} & X \end{array}$$

commutes. To this end, we will first show that the diagram

$$\begin{array}{ccc} & S^0 \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0,S^0}^\otimes} S^0 \wedge S^0 \\ \text{id}_\mathbb{1}^\otimes|_{\text{Sets}_*} \otimes_{\text{Sets}_*} \text{id}_{S^0} \nearrow & & \searrow \lambda_{S^0}^{\text{Sets}_*} \\ \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\lambda'_{S^0}} & S^0, \end{array} \quad (\dagger)$$

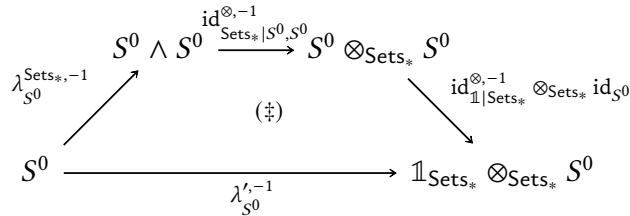
corresponding to the case  $X = S^0$ , commutes. Indeed, consider the diagram



whose boundary diagram corresponds to the diagram  $(\dagger)$  above. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of  $\text{id}_{\mathbf{Sets}_*}^{\otimes}$ .
- Subdiagram (4) commutes by ??.
- Subdiagram (5) commutes by the naturality of  $\rho^{\mathbf{Sets}_*, -1}$ .
- Subdiagram (6) commutes trivially.
- Subdiagram (7) commutes by the naturality of  $\rho^{\mathbf{Sets}_*}$ , where the equality  $\rho_{S^0}^{\mathbf{Sets}_*} = \lambda_{S^0}^{\mathbf{Sets}_*}$  comes from ??.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram  $(\dagger)$  above. As a result, the diagram



also commutes. Now, let  $X \in \text{Obj}(\text{Sets}_*)$ , let  $x \in X$ , and consider the diagram

$$\begin{array}{ccccc}
 & & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \wedge \text{id}_{S^0}} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0 \\
 & \nearrow \lambda_{S^0}^{\text{Sets}_*, -1} & \downarrow & \text{(\ddagger)} & \downarrow & \searrow & \downarrow \\
 S^0 & \xrightarrow{\lambda_{S^0}^{\text{Sets}_*, -1}} & S^0 \wedge S^0 & \xrightarrow{\lambda_{S^0}^{\prime, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{S^0} \otimes_{\text{Sets}_*} [x]} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0 \\
 & \downarrow \text{id}_{S^0} \wedge [x] & \downarrow & (1) & \downarrow \text{id}_{S^0} \otimes_{\text{Sets}_*} [x] & \downarrow & \downarrow \text{id}_{\mathbb{1}_{\text{Sets}_*}} \wedge [x] \\
 & (3) & S^0 \wedge X & \xrightarrow{\text{id}_{S^0|S^0, X}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} X & \xrightarrow{\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \wedge \text{id}_X} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X \\
 & & \downarrow & (5) & \downarrow & & \downarrow \\
 X & \xrightarrow{\lambda_X^{\text{Sets}_*, -1}} & S^0 \wedge X & \xrightarrow{\lambda_X^{\prime, -1}} & S^0 \otimes_{\text{Sets}_*} X & \xrightarrow{\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \wedge \text{id}_X} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X \\
 & & \downarrow & (2) & \downarrow & & \downarrow \\
 & & X & \xrightarrow{\lambda_X^{\prime, -1}} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X & & 
 \end{array}$$

Since:

- Subdiagram (5) commutes by the naturality of  $\lambda^{\prime, -1}$ .
- Subdiagram  $(\ddagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}_{\text{Sets}_*}^{\otimes, -1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\text{Sets}_*, -1}$ .

it follows that the diagram

$$\begin{array}{ccccc}
 & & S^0 \wedge X & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} X \\
 & \nearrow \lambda_X^{\text{Sets}_*, -1} & & & \searrow \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \otimes_{\text{Sets}_*} \text{id}_X \\
 S^0 & \xrightarrow{[x]} & X & \xrightarrow{\lambda_X^{\prime, -1}} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\lambda_X^{\prime, -1}(x) = [\lambda_X^{\prime, -1} \circ [x]](1)$$

$$\begin{aligned}
&= [(\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \wedge \text{id}_X) \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1} \circ \lambda_X^{\text{Sets}_*, -1} \circ [x]](1) \\
&= [(\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \wedge \text{id}_X) \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1} \circ \lambda_X^{\text{Sets}_*, -1}](x)
\end{aligned}$$

for each  $x \in X$ , and thus we have

$$\lambda_X'^{-1} = (\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \wedge \text{id}_X) \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1} \circ \lambda_X^{\text{Sets}_*, -1}.$$

Taking inverses then gives

$$\lambda_X' = \lambda_X^{\text{Sets}_*} \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes} \circ (\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes} \wedge \text{id}_X),$$

showing that the diagram

$$\begin{array}{ccc}
& S^0 \otimes_{\text{Sets}_*} X & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, X}^{\otimes}} S^0 \wedge X \\
\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes} \otimes_{\text{Sets}_*} \text{id}_X \nearrow & & \searrow \lambda_X^{\text{Sets}_*} \\
\mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X & \xrightarrow{\lambda_X'} & X
\end{array}$$

indeed commutes.

*Braidedness of the Isomorphism  $\otimes_{\text{Sets}_*} \cong \wedge$ :* We have to show that the diagram

$$\begin{array}{ccc}
X \otimes_{\text{Sets}_*} Y & \xrightarrow{\text{id}_{\text{Sets}_*|X, Y}^{\otimes}} & X \wedge Y \\
\sigma'_{X, Y} \downarrow & & \downarrow \sigma_{X, Y}^{\text{Sets}_*} \\
Y \otimes_{\text{Sets}_*} X & \xrightarrow{\text{id}_{\text{Sets}_*|Y, X}^{\otimes}} & Y \wedge X
\end{array}$$

commutes. To this end, we will first show that the diagram

$$\begin{array}{ccc}
S^0 \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes}} & S^0 \wedge S^0 \\
\sigma'_{S^0, S^0} \downarrow & (\dagger) & \downarrow \sigma_{S^0, S^0}^{\text{Sets}_*} \\
S^0 \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes}} & S^0 \wedge S^0
\end{array}$$

commutes. To that end, we will first show that the diagram

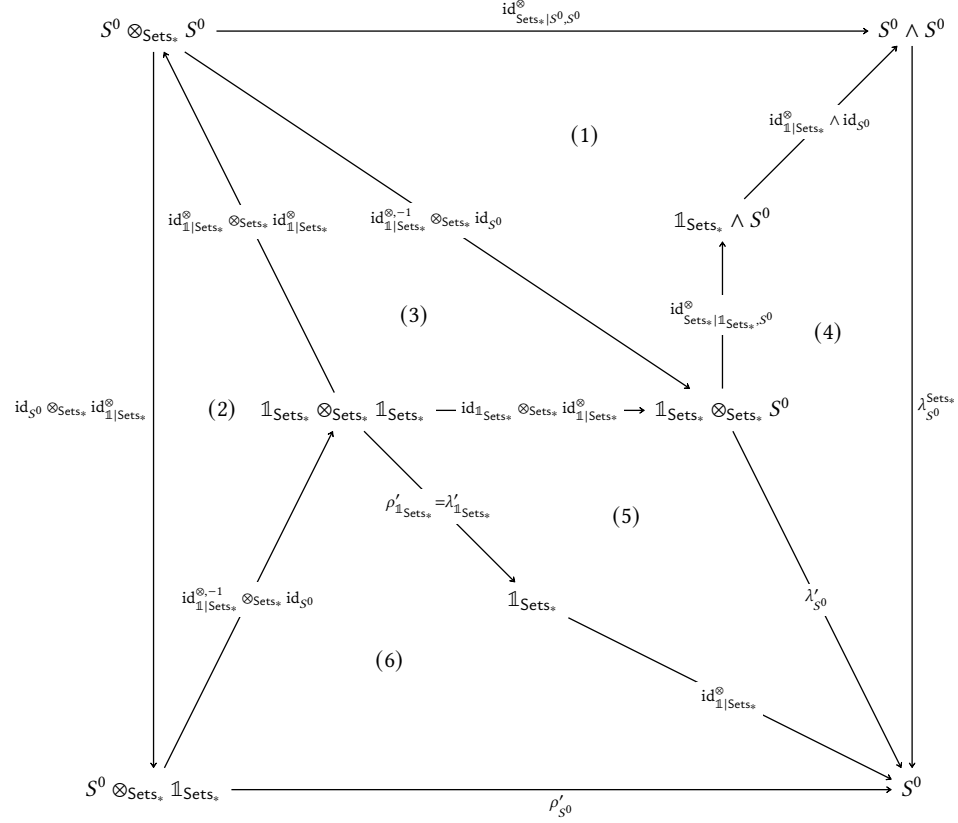
$$\begin{array}{ccc}
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\mathrm{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*}}^\otimes} & S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} \\
 \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} \downarrow & (\dagger) & \downarrow \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} \\
 \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\mathrm{id}_{\mathbf{Sets}_*| \mathbb{1}_{\mathbf{Sets}_*}, S^0}^\otimes} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0
 \end{array}$$

commutes, and, to this end, we will first show that the diagram

$$\begin{array}{ccc}
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\mathrm{id}_{\mathbf{Sets}_*|S^0, S^0}^\otimes} & S^0 \wedge S^0 \\
 \mathrm{id}_{S^0} \otimes_{\mathbf{Sets}_*} \mathrm{id}_{\mathbf{Sets}_*| \mathbb{1}}^\otimes \uparrow & (\S) & \downarrow \lambda_{S^0}^{\mathbf{Sets}_*} \\
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho'_{S^0}} & S^0
 \end{array}$$



commutes. Indeed, consider the diagram

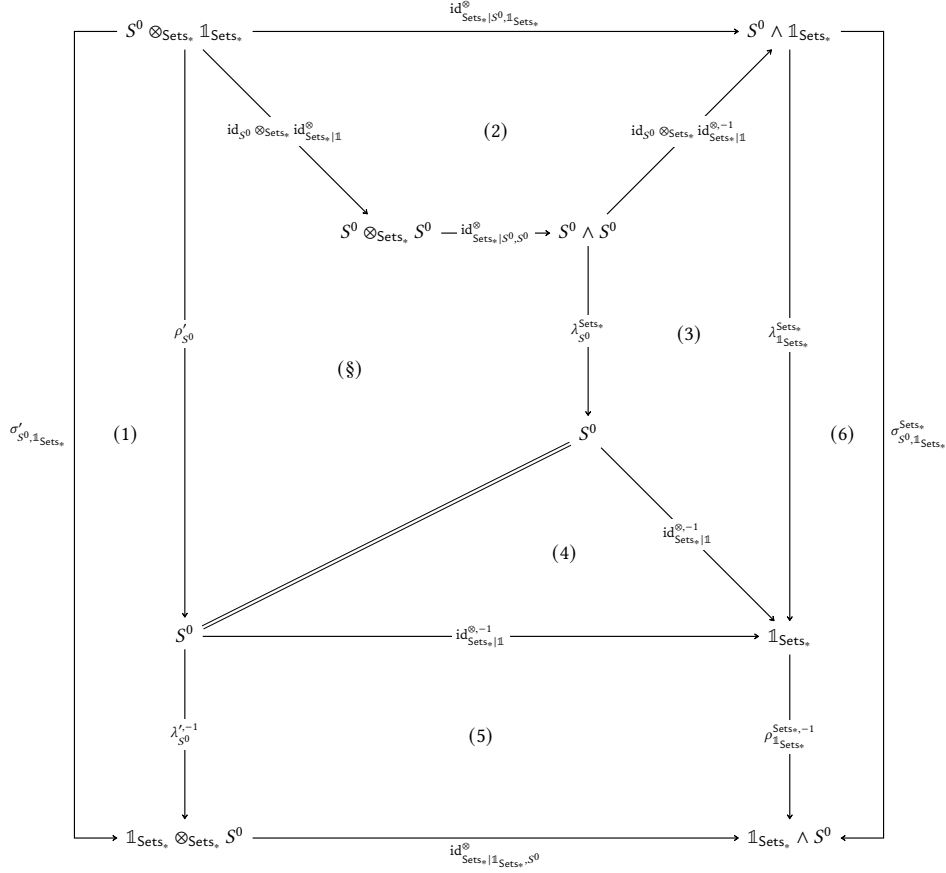


whose boundary diagram corresponds to diagram (§) above. Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_{\mathbf{Sets}_*}^{\otimes}$ ;
- Subdiagrams (2) and (3) commute by the functoriality of  $\otimes$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(\text{id}^{\otimes}, \text{id}_{\mathbb{1}}^{\otimes})$ , which we proved above;
- Subdiagram (5) commutes by the naturality of  $\lambda'$ ;
- Subdiagram (6) commutes by the naturality of  $\rho'$ , where the equality  $\rho'_{\mathbb{1}_{\mathbf{Sets}_*}} = \lambda'_{\mathbb{1}_{\mathbf{Sets}_*}}$  comes from ??;

it follows that the boundary diagram, i.e. diagram (§), also commutes. Next,

consider the diagram



whose boundary diagram corresponds to the diagram  $(\ddagger)$  above. Since:

- Subdiagrams (1) and (6) commute by ??;
- Subdiagram (2) commutes by the naturality of  $\text{id}_{\mathbf{Sets}_*}^{\otimes}$ ;
- Subdiagram (§) commutes, as was shown above;
- Subdiagram (3) commutes by the naturality of  $\lambda^{\mathbf{Sets}_*}$ ;
- Subdiagram (4) commutes trivially;
- Subdiagram (5) commutes by **Constructions With Monoidal Categories, Item 2c** of **Item 2** of **Definition 13.1.1.4**, whose proof uses only the left monoidal unity of  $(\text{id}^{\otimes}, \text{id}_{\mathbb{1}}^{\otimes})$ , which has been proven above;

it follows that the boundary diagram, i.e. diagram  $(\dagger)$ , also commutes. Next, consider the diagram

$$\begin{array}{ccccc}
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{S^0, S^0}^\otimes} & S^0 \wedge S^0 & & \\
 \downarrow \sigma'_{S^0, S^0} & \searrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}}^{\otimes, -1} & & \swarrow \text{id}_{S^0} \wedge \text{id}_{\mathbb{1}}^{\otimes, -1} & \downarrow \sigma_{S^0, S^0}^{\mathbf{Sets}_*} \\
 & & (1) & & \\
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^\otimes} & S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} & & \\
 \downarrow \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} & & \downarrow \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} & & \\
 \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{\mathbb{1}_{\mathbf{Sets}_*}, S^0}^\otimes} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 & & \\
 \downarrow \text{id}_{\mathbb{1}}^\otimes \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} & & \downarrow \text{id}_{\mathbb{1}}^\otimes \wedge \text{id}_{S^0} & & \\
 S^0 \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\text{id}_{S^0, S^0}^\otimes} & S^0 \wedge S^0 & & \\
 & & (4) & & 
 \end{array}$$

$(2) \quad \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} \quad (\dagger) \quad \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} \quad (3)$

whose boundary diagram corresponds to the diagram  $(\dagger)$ . Since:

- Subdiagram (1) commutes by the naturality of  $\text{id}_{\mathbf{Sets}_*}^\otimes$ ;
- Subdiagram (2) commutes by the naturality of  $\sigma'$  and the fact that  $\text{id}_{\mathbb{1}}^\otimes$  is invertible;
- Subdiagram  $(\dagger)$  commutes as proved above;
- Subdiagram (3) commutes by the naturality of  $\sigma^{\mathbf{Sets}_*}$  and the fact that  $\text{id}_{\mathbb{1}}^\otimes$  is invertible;
- Subdiagram (4) commutes by the naturality of  $\text{id}_{\mathbf{Sets}_*}^\otimes$ ;

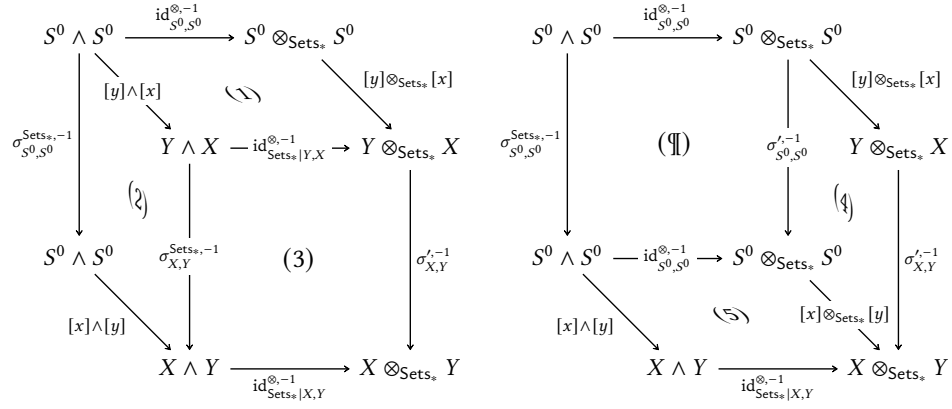
it follows that the boundary diagram, i.e. diagram  $(\dagger)$  also commutes. Taking inverses for the diagram  $(\dagger)$ , we see that the diagram

$$\begin{array}{ccc}
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 \\
 \sigma_{S^0, S^0}^{\text{Sets}_*, -1} \downarrow & (\mathbb{Q}) & \downarrow \sigma'_{S^0, S^0}{}^{-1} \\
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0
 \end{array}$$

commutes as well. Now, let  $X, Y \in \text{Obj}(\text{Sets}_*)$ , let  $x \in X$ , let  $y \in Y$ , and consider the diagram

$$\begin{array}{ccccc}
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 & & \\
 \downarrow \sigma_{S^0, S^0}^{\text{Sets}_*, -1} & \searrow [y] \wedge [x] & \downarrow \sigma'_{S^0, S^0}{}^{-1} & \searrow [y] \otimes_{\text{Sets}_*} [x] & \\
 & Y \wedge X & \xrightarrow{\text{id}_{\text{Sets}_*|Y, A}^{\otimes, -1}} & Y \otimes_{\text{Sets}_*} X & \\
 & \downarrow \sigma_{A, Y}^{\text{Sets}_*, -1} & \downarrow & \downarrow \sigma'_{A, Y}{}^{-1} & \\
 S^0 \wedge S^0 & \xrightarrow{\text{id}_{S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 & & \\
 \downarrow \sigma_{S^0, S^0}^{\text{Sets}_*, -1} & \searrow [x] \wedge [y] & \downarrow \sigma'_{S^0, S^0}{}^{-1} & \searrow [x] \otimes_{\text{Sets}_*} [y] & \\
 & X \wedge Y & \xrightarrow{\text{id}_{\text{Sets}_*|A, Y}^{\otimes, -1}} & X \otimes_{\text{Sets}_*} Y &
 \end{array}$$

which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of  $\sigma^{\text{Sets}_*, -1}$ .
- Subdiagram (5) commutes by the naturality of  $\text{id}^{\otimes, -1}$ .
- Subdiagram (5) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{-1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}^{\otimes, -1}$ .

it follows that the diagram

$$\begin{array}{ccc}
 S^0 \wedge S^0 & \xrightarrow{[y] \wedge [x]} & Y \wedge X \\
 & \searrow & \downarrow \sigma_{X, Y}^{\text{Sets}_*} \\
 & & X \wedge Y \\
 & & \downarrow \sigma_{X, Y}' \\
 & & X \otimes_{\text{Sets}_*} Y
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \downarrow \sigma_{X, Y}' \\
 & & X \otimes_{\text{Sets}_*} Y
 \end{array}$$

$\xrightarrow{\text{id}_{\text{Sets}_* | Y, X}^{\otimes}} Y \otimes_{\text{Sets}_*} X \xrightarrow{\sigma_{X, Y}'} X \otimes_{\text{Sets}_*} Y$

commutes. We then have

$$[\text{id}_{\text{Sets}_* | X, Y}^{\otimes, -1} \circ \sigma_{X, Y}^{\text{Sets}_*, -1}](y, x) = [\text{id}_{\text{Sets}_* | X, Y}^{\otimes, -1} \circ \sigma_{X, Y}^{\text{Sets}_*, -1} \circ ([y] \wedge [x])](1, 1)$$

$$\begin{aligned}
&= [\sigma'_{X,Y}{}^{-1} \circ \text{id}_{\text{Sets}_*|Y,X}^{\otimes,-1} \circ ([y] \wedge [x])](1, 1) \\
&= [\sigma'_{X,Y}{}^{-1} \circ \text{id}_{\text{Sets}_*|Y,X}^{\otimes,-1}](y, x)
\end{aligned}$$

for each  $(y, x) \in Y \wedge X$ , and thus we have

$$\text{id}_{\text{Sets}_*|X,Y}^{\otimes,-1} \circ \sigma_{X,Y}^{\text{Sets}_*, -1} = \sigma'_{X,Y}{}^{-1} \circ \text{id}_{\text{Sets}_*|Y,X}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{X,Y}^{\text{Sets}_*} \circ \text{id}_{\text{Sets}_*|X,Y}^{\otimes} = \text{id}_{\text{Sets}_*|Y,X}^{\otimes} \circ \sigma'_{X,Y},$$

showing that the diagram

$$\begin{array}{ccc}
A \otimes_{\text{Sets}_*} B & \xrightarrow{\text{id}_{\text{Sets}_*|A,B}^{\otimes}} & A \wedge B \\
\sigma'_{A,B} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}_*} \\
B \otimes_{\text{Sets}_*} A & \xrightarrow{\text{id}_{\text{Sets}_*|B,A}^{\otimes}} & B \wedge A
\end{array}$$

indeed commutes.

*Monoidal Right Unity of the Isomorphism  $\otimes_{\text{Sets}_*} \cong \wedge$ :* We have to show that the diagram

$$\begin{array}{ccc}
& X \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|X,S^0}^{\otimes}} X \wedge S^0 \\
\text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\mathbb{1}}^{\otimes} \nearrow & & \searrow \rho_X^{\text{Sets}_*} \\
X \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*} & \xrightarrow{\rho'_X} & X
\end{array}$$

commutes. To this end, we will first show that the diagram

$$\begin{array}{ccc}
& S^0 \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0,S^0}^{\otimes}} S^0 \wedge S^0 \\
\text{id}_{\mathbb{1}}^{\otimes} \otimes_{\text{Sets}_*} \text{id}_{S^0} \nearrow & & \searrow \rho_{S^0}^{\text{Sets}_*} \\
S^0 \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*} & \xrightarrow{\rho'_{S^0}} & S^0,
\end{array} \quad (\dagger)$$

corresponding to the case  $X = S^0$ , commutes. First, notice that we may write

$$\sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} : S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} \rightarrow \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0$$

as the composition

$$\begin{aligned} S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} &\xrightarrow{\mathrm{id}_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes}} S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} \\ &\xrightarrow{\lambda_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*}} \mathbb{1}_{\mathbf{Sets}_*} \\ &\xrightarrow{\rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 \\ &\xrightarrow{\mathrm{id}_{\mathbb{1}_{\mathbf{Sets}_*, S^0}}^{\otimes, -1}} \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0. \end{aligned}$$

Indeed, we may write this composition as part of the diagram

$$\begin{array}{ccccc} S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\mathrm{id}_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes}} & S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\lambda_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*}} & \mathbb{1}_{\mathbf{Sets}_*} \\ \downarrow \sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}} & & \downarrow \sigma_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} & \nearrow \rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1} & \\ & (1) & & (2) & \\ \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\mathrm{id}_{\mathbb{1}_{\mathbf{Sets}_*, S^0}}^{\otimes}} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 & \xrightarrow{\mathrm{id}_{\mathbb{1}_{\mathbf{Sets}_*, S^0}}^{\otimes, -1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0, \end{array}$$

which commutes since:

- Subdiagram (1) commutes by the braidedness of  $\mathrm{id}^{\otimes}$ , as proved above.
- Subdiagram (2) commutes by ??.

Next, consider the diagram

$$\begin{array}{ccccccc}
 S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \rho_{S^0}^{\mathbf{Sets}_*, -1}} & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) & \xrightarrow{\text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*|S^0}^{\otimes, -1}} & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) & \xrightarrow{\text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \lambda'_{S^0}} & S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \downarrow \text{id}_{\mathbf{Sets}_*|S^0}^{\otimes} & & \downarrow \text{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0}^{\otimes} & & \downarrow \text{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0}^{\otimes} & & \downarrow \text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes} \\
 S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\text{id}_{S^0} \wedge \rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} & S^0 \wedge (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) & \xrightarrow{\text{id}_{S^0} \wedge \text{id}_{\mathbf{Sets}_*|S^0}^{\otimes, -1}} & S^0 \wedge (\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) & \xrightarrow{\text{id}_{S^0} \wedge \lambda'_{S^0}} & S^0 \wedge S^0 \\
 \downarrow \lambda_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*} & & \downarrow \lambda_{\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0}^{\mathbf{Sets}_*} & & \downarrow \lambda_{\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0}^{\mathbf{Sets}_*} & & \downarrow \lambda_{S^0}^{\mathbf{Sets}_*} = \rho_{S^0}^{\mathbf{Sets}_*} \\
 \mathbb{1}_{\mathbf{Sets}_*} & \xrightarrow{\rho_{\mathbb{1}_{\mathbf{Sets}_*}}^{\mathbf{Sets}_*, -1}} & \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0}^{\otimes, -1}} & \mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0 & \xrightarrow{\lambda'_{S^0}} & S^0,
 \end{array}$$

whose boundary diagram corresponds to the diagram  $(\dagger)$  above, since the composition in **red** is equal to  $\sigma'_{S^0, \mathbb{1}_{\mathbf{Sets}_*}}$  as proved above, and then the composition in **red** composed with  $\lambda'_{S^0}$  is equal to  $\rho'_{S^0}$  by **??**. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of  $\text{id}_{\mathbf{Sets}_*}^{\otimes}$ .
- Subdiagrams (4), (5), and (6) commute by the naturality of  $\lambda_{\mathbf{Sets}_*}^{\mathbf{Sets}_*}$ , where the equality  $\lambda_{S^0}^{\mathbf{Sets}_*} = \rho_{S^0}^{\mathbf{Sets}_*}$  comes from **??**.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram  $(\dagger)$  above. As a result, the diagram

$$\begin{array}{ccc}
 & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\mathbf{Sets}_*|S^0}^{\otimes, -1}} S^0 \otimes_{\mathbf{Sets}_*} S^0 \\
 \rho_{S^0}^{\mathbf{Sets}_*, -1} \nearrow & & \searrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbb{1}_{\mathbf{Sets}_*}}^{\otimes, -1} \\
 S^0 & \xrightarrow{\rho_{S^0}^{\mathbf{Sets}_*, -1}} & S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*}
 \end{array}
 \quad (\dagger)$$



also commutes. Now, let  $X \in \text{Obj}(\text{Sets}_*)$ , let  $x \in X$ , and consider the diagram

$$\begin{array}{ccccc}
 & & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 \\
 & \nearrow \rho_{S^0}^{\text{Sets}_*, -1} & \downarrow & \text{(\ddagger)} & \downarrow \text{id}_{S^0} \wedge \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \\
 S^0 & \xrightarrow{\rho_{S^0}^{\text{Sets}_*, -1}} & S^0 \wedge S^0 & \xrightarrow{\rho_{S^0}^{\prime, -1}} & S^0 \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*} \\
 \downarrow [x] & & \downarrow \text{id}_{S^0} \wedge [x] & (1) & \downarrow \text{id}_{S^0} \otimes_{\text{Sets}_*} [x] \\
 & & X \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|X, S^0}^{\otimes, -1}} & X \otimes_{\text{Sets}_*} S^0 \\
 & \nearrow \rho_X^{\text{Sets}_*, -1} & \downarrow & (5) & \downarrow \text{id}_X \wedge \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \\
 X & \xrightarrow{\rho_X^{\text{Sets}_*, -1}} & X \wedge S^0 & \xrightarrow{\rho_X^{\prime, -1}} & X \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*} \\
 & & \downarrow & (2) & \downarrow \text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \\
 & & X & \xrightarrow{\rho_X^{\text{Sets}_*, -1}} & X \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*}
 \end{array}$$

Since:

- Subdiagram (5) commutes by the naturality of  $\rho^{\prime, -1}$ .
- Subdiagram  $(\ddagger)$  commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1}$ .
- Subdiagram (1) commutes by the naturality of  $\text{id}_{\text{Sets}_*}^{\otimes, -1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\text{Sets}_*, -1}$ .

it follows that the diagram

$$\begin{array}{ccccc}
 & & X \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|X, S^0}^{\otimes, -1}} & X \otimes_{\text{Sets}_*} S^0 \\
 & \nearrow \rho_X^{\text{Sets}_*, -1} & & & \downarrow \text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1} \\
 S^0 & \xrightarrow{[x]} & X & \xrightarrow{\rho_X^{\prime, -1}} & X \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*}
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\rho_X^{\prime, -1}(a) = [\rho_X^{\prime, -1} \circ [x]](1)$$

$$\begin{aligned}
&= [(\text{id}_X \wedge \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1}) \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1} \circ \rho_X^{\text{Sets}_*, -1} \circ [x]](1) \\
&= [(\text{id}_X \wedge \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1}) \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1} \circ \rho_X^{\text{Sets}_*, -1}](a)
\end{aligned}$$

for each  $a \in X$ , and thus we have

$$\rho_X'^{-1} = (\text{id}_X \wedge \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes, -1}) \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1} \circ \rho_X^{\text{Sets}_*, -1}.$$

Taking inverses then gives

$$\rho_X' = \rho_X^{\text{Sets}_*} \circ \text{id}_{\text{Sets}_*|S^0, X}^{\otimes} \circ (\text{id}_X \wedge \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes}),$$

showing that the diagram

$$\begin{array}{ccc}
& X \otimes_{\text{Sets}_*} S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|X, S^0}^{\otimes}} X \wedge S^0 \\
\text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\mathbb{1}|\text{Sets}_*}^{\otimes} \nearrow & & \searrow \rho_X^{\text{Sets}_*} \\
X \otimes_{\text{Sets}_*} \mathbb{1}_{\text{Sets}_*} & \xrightarrow{\rho_X'} & X
\end{array}$$

indeed commutes.

*Monoidality of the Isomorphism  $\otimes_{\text{Sets}_*} \cong \wedge$ :* We have to show that the diagram

$$\begin{array}{ccc}
& (X \otimes_{\text{Sets}_*} Y) \otimes_{\text{Sets}_*} Z & \\
\text{id}_{\text{Sets}_*|X, Y}^{\otimes} \otimes_{\text{Sets}_*} \text{id}_Z \swarrow & & \searrow \alpha'_{X, Y, Z} \\
(X \wedge Y) \otimes_{\text{Sets}_*} Z & & X \otimes_{\text{Sets}_*} (Y \otimes_{\text{Sets}_*} Z) \\
\downarrow \text{id}_{\text{Sets}_*|X \wedge Y, Z}^{\otimes} & & \downarrow \text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\text{Sets}_*|Y, Z}^{\otimes} \\
(X \wedge Y) \wedge Z & & X \otimes_{\text{Sets}_*} (Y \wedge Z) \\
\searrow \alpha_{X, Y, Z}^{\text{Sets}_*} & & \swarrow \text{id}_{\text{Sets}_*|X, Y \wedge Z}^{\otimes} \\
& X \wedge (Y \wedge Z) &
\end{array}$$

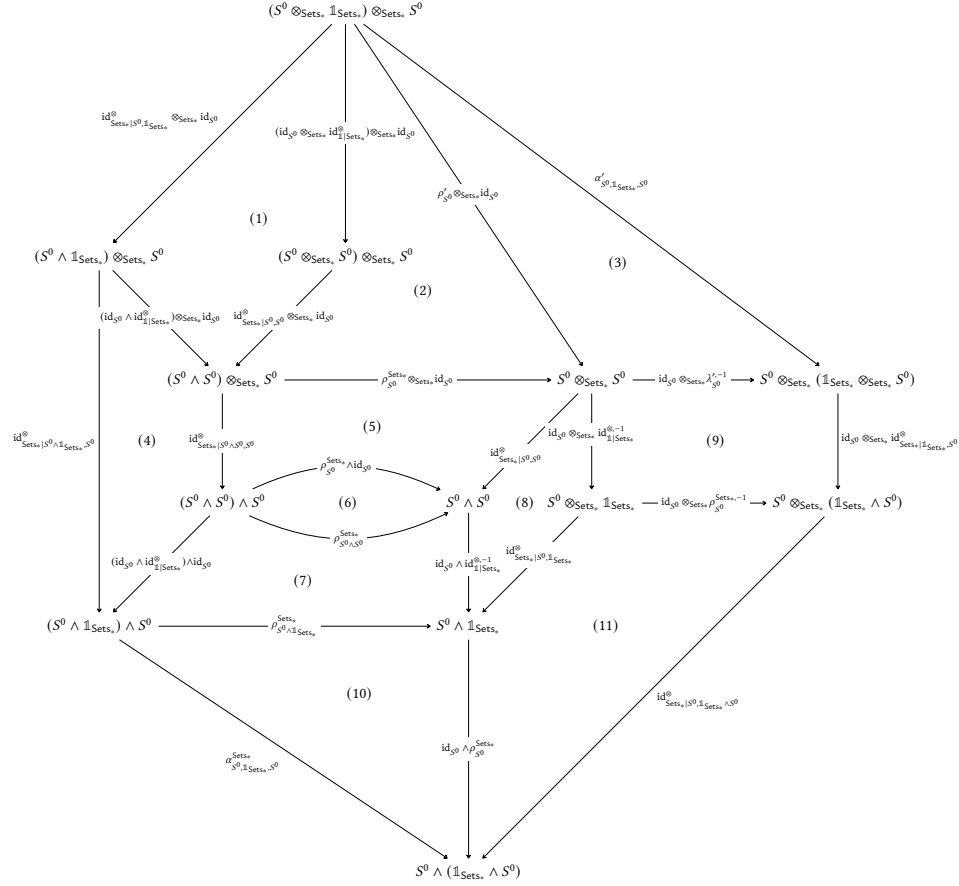
commutes. To this end, we will first prove that the diagram

$$\begin{array}{ccc}
 \text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} & \begin{array}{c} (S^0 \otimes_{\mathbf{Sets}_*} S^0) \otimes_{\mathbf{Sets}_*} S^0 \\ \searrow \alpha_{S^0, S^0, S^0} \end{array} & \\
 & \begin{array}{cc} (S^0 \wedge S^0) \otimes_{\mathbf{Sets}_*} S^0 & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \otimes_{\mathbf{Sets}_*} S^0) \end{array} & \\
 \downarrow \text{id}_{\mathbf{Sets}_*|S^0 \wedge S^0, S^0}^{\otimes} & (\dagger) & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*|S^0, S^0}^{\otimes} \\
 (S^0 \wedge S^0) \wedge S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (S^0 \wedge S^0) \\
 \searrow \alpha_{S^0, S^0, S^0}^{\mathbf{Sets}_*} & & \swarrow \text{id}_{\mathbf{Sets}_*|S^0, S^0 \wedge S^0}^{\otimes} \\
 & S^0 \wedge (S^0 \wedge S^0) &
 \end{array}$$

commutes, and, to that end, we will first show that the diagram

$$\begin{array}{ccc}
 \text{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*}}^{\otimes} \otimes_{\mathbf{Sets}_*} \text{id}_{S^0} & \begin{array}{c} (S^0 \otimes_{\mathbf{Sets}_*} \mathbb{1}_{\mathbf{Sets}_*}) \otimes_{\mathbf{Sets}_*} S^0 \\ \searrow \alpha_{S^0, \mathbb{1}_{\mathbf{Sets}_*}, S^0} \end{array} & \\
 & \begin{array}{cc} (S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*}) \otimes_{\mathbf{Sets}_*} S^0 & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \otimes_{\mathbf{Sets}_*} S^0) \end{array} & \\
 \downarrow \text{id}_{\mathbf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes} & (\ddagger) & \downarrow \text{id}_{S^0} \otimes_{\mathbf{Sets}_*} \text{id}_{\mathbf{Sets}_*|\mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\otimes} \\
 (S^0 \wedge \mathbb{1}_{\mathbf{Sets}_*}) \wedge S^0 & & S^0 \otimes_{\mathbf{Sets}_*} (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) \\
 \searrow \alpha_{S^0, \mathbb{1}_{\mathbf{Sets}_*}, S^0}^{\mathbf{Sets}_*} & & \swarrow \text{id}_{\mathbf{Sets}_*|S^0, \mathbb{1}_{\mathbf{Sets}_*} \wedge S^0}^{\otimes} \\
 & S^0 \wedge (\mathbb{1}_{\mathbf{Sets}_*} \wedge S^0) &
 \end{array}$$

commutes. Indeed, consider the diagram

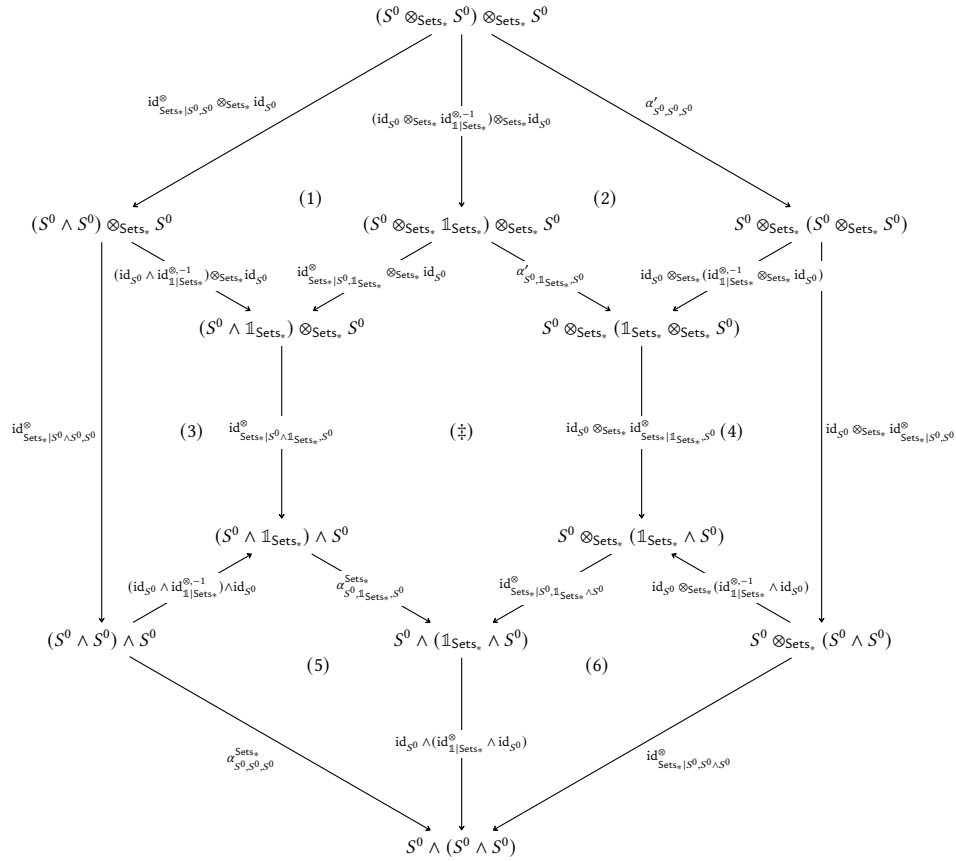


whose boundary diagram corresponds to diagram  $(\ddagger)$  above. Since:

- Subdiagrams (1), (4), (5), (8), and (11) commute by the naturality of  $\text{id}_{\text{Sets}_*}^{\otimes}$ ;
- Subdiagram (2) commutes by the right monoidal unity of  $(\text{id}_{\text{Sets}_*}^{\otimes}, \text{id}_{\mathbb{1}_{\text{Sets}_*}}^{\otimes})$ ;
- Subdiagram (3) commutes by the triangle identity for  $(\alpha', \lambda', \rho')$ ;
- Subdiagram (6) commutes by ??;
- Subdiagram (7) commutes by the naturality of  $\rho^{\text{Sets}_*}$ ;
- Subdiagram (9) commutes by ??;

- Subdiagram (10) commutes by ??;

it follows that the boundary diagram, i.e. diagram  $(\ddagger)$ , also commutes. Consider now the diagram



whose boundary corresponds to diagram  $(\ddagger)$  above. Since:

- Subdiagrams (1), (3), (4), and (6) commute by the naturality of  $\text{id}_{\text{Sets}_*}^{\otimes}$ ;
- Subdiagram  $(\ddagger)$  commutes, as proved above;
- Subdiagram (2) commutes by the naturality of  $\alpha'$ ;
- Subdiagram (5) commutes by the naturality of  $\alpha^{\text{Sets}_*}$ ;

it follows that the boundary diagram, i.e. diagram  $(\ddagger)$ , also commutes. Taking

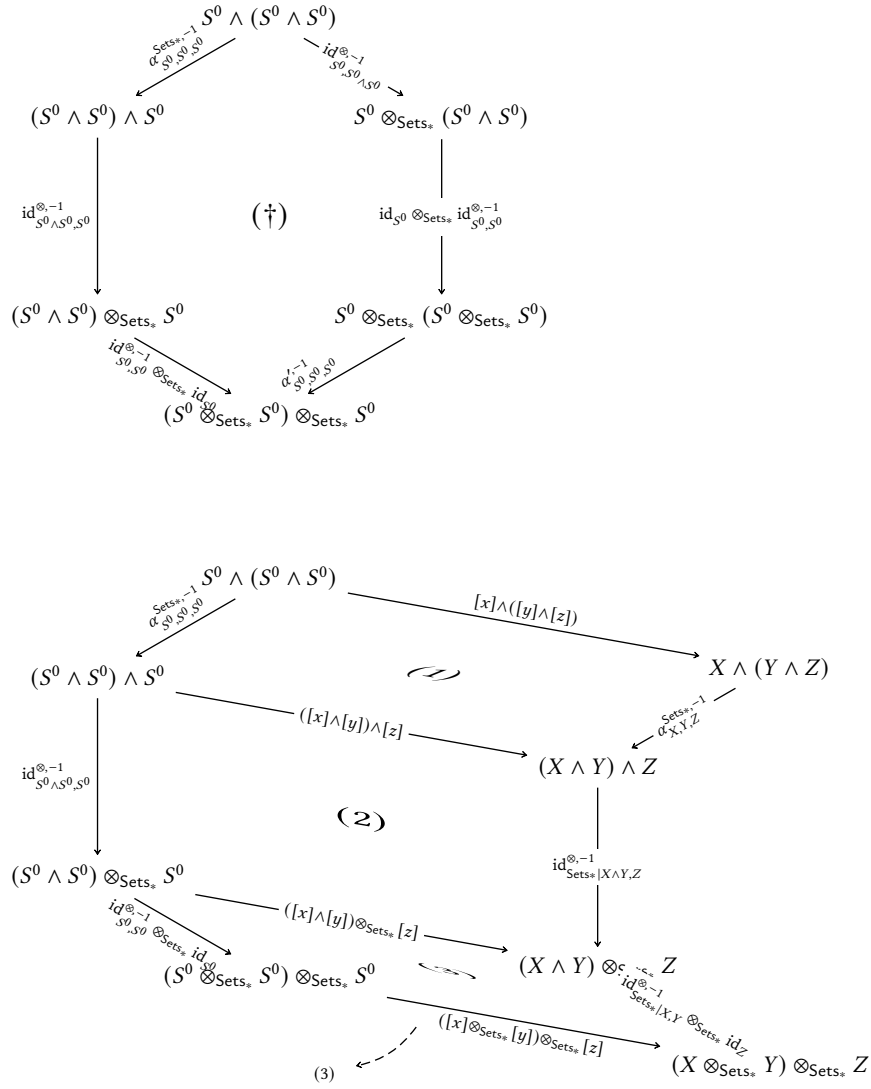
inverses on the diagram  $(\dagger)$ , we see that the diagram

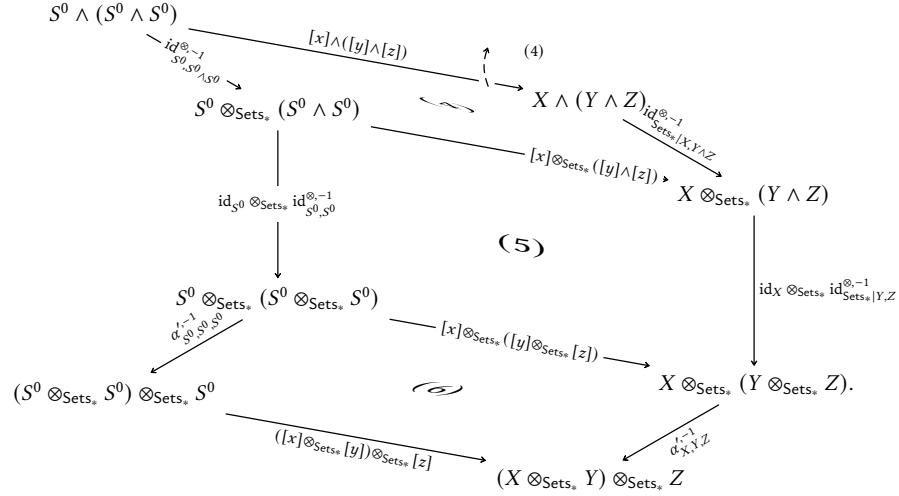
$$\begin{array}{ccc}
 & S^0 \wedge (S^0 \wedge S^0) & \\
 \alpha_{S^0, S^0, S^0}^{\text{Sets}_*, -1} \swarrow & & \searrow \text{id}_{\text{Sets}_* | S^0, S^0 \wedge S^0}^{\otimes, -1} \\
 (S^0 \wedge S^0) \wedge S^0 & & S^0 \otimes_{\text{Sets}_*} (S^0 \wedge S^0) \\
 \downarrow \text{id}_{\text{Sets}_* | S^0 \wedge S^0, S^0}^{\otimes, -1} & (\dagger) & \downarrow \text{id}_{S^0} \otimes_{\text{Sets}_*} \text{id}_{\text{Sets}_* | S^0, S^0}^{\otimes, -1} \\
 (S^0 \wedge S^0) \otimes_{\text{Sets}_*} S^0 & & S^0 \otimes_{\text{Sets}_*} (S^0 \otimes_{\text{Sets}_*} S^0) \\
 \downarrow \text{id}_{\text{Sets}_* | S^0, S^0}^{\otimes, -1} \otimes_{\text{Sets}_*} \text{id}_{S^0} & & \downarrow \alpha_{S^0, S^0, S^0}'^{\otimes, -1} \\
 (S^0 \otimes_{\text{Sets}_*} S^0) \otimes_{\text{Sets}_*} S^0 & & S^0 \otimes_{\text{Sets}_*} (S^0 \otimes_{\text{Sets}_*} S^0)
 \end{array}$$

commutes as well. Now, let  $X, Y, Z \in \text{Obj}(\text{Sets}_*)$ , let  $x \in X$ , let  $y \in Y$ , let  $z \in Z$ , and consider the diagram

$$\begin{array}{ccccccc}
 & S^0 \wedge (S^0 \wedge S^0) & & & & & \\
 \alpha_{S^0, S^0, S^0}^{\text{Sets}_*, -1} \swarrow & & \searrow \text{id}_{\text{Sets}_* | S^0, S^0 \wedge S^0}^{\otimes, -1} & & [x] \wedge ([y] \wedge [z]) & & \\
 (S^0 \wedge S^0) \wedge S^0 & & S^0 \otimes_{\text{Sets}_*} (S^0 \wedge S^0) & & X \wedge (Y \wedge Z) & & \\
 \downarrow \text{id}_{\text{Sets}_* | S^0 \wedge S^0, S^0}^{\otimes, -1} & & \downarrow \text{id}_{S^0} \otimes_{\text{Sets}_*} \text{id}_{S^0, S^0}^{\otimes, -1} & & \downarrow \alpha_{X, Y, Z}^{\text{Sets}_*, -1} & & \downarrow \text{id}_{\text{Sets}_* | X, Y \wedge Z}^{\otimes, -1} \\
 (S^0 \wedge S^0) \otimes_{\text{Sets}_*} S^0 & & (X \wedge Y) \wedge Z & & X \otimes_{\text{Sets}_*} (Y \wedge Z) & & \\
 \downarrow \text{id}_{\text{Sets}_* | S^0, S^0}^{\otimes, -1} \otimes_{\text{Sets}_*} \text{id}_{S^0} & & \downarrow \text{id}_{\text{Sets}_* | X \wedge Y, Z}^{\otimes, -1} & & \downarrow \text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\text{Sets}_* | Y, Z}^{\otimes, -1} & & \\
 (S^0 \otimes_{\text{Sets}_*} S^0) \otimes_{\text{Sets}_*} S^0 & & (X \wedge Y) \otimes_{\text{Sets}_*} Z & & X \otimes_{\text{Sets}_*} (Y \otimes_{\text{Sets}_*} Z) & & \\
 \downarrow \alpha_{S^0, S^0, S^0}'^{\otimes, -1} & & \downarrow \text{id}_{\text{Sets}_* | X, Y, Z}^{\otimes, -1} & & \downarrow \alpha_{X, Y, Z}'^{\otimes, -1} & & \\
 (S^0 \otimes_{\text{Sets}_*} S^0) \otimes_{\text{Sets}_*} S^0 & & (X \otimes_{\text{Sets}_*} Y) \otimes_{\text{Sets}_*} Z & & & & \\
 \downarrow ([x] \wedge [y]) \otimes_{\text{Sets}_*} [z] & & \downarrow ([x] \otimes_{\text{Sets}_*} [y]) \otimes_{\text{Sets}_*} [z] & & & & \\
 & & & & & & 
 \end{array}$$

which we partition into subdiagrams as follows:





Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\text{Sets}_*, -1}$ .
- Subdiagram (2) commutes by the naturality of  $\text{id}^{0, -1}_{\text{Sets}_*}$ .
- Subdiagram (3) commutes by the naturality of  $\text{id}^{0, -1}_{\text{Sets}_*}$ .
- Subdiagram ( $\dagger$ ) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\text{id}^{0, -1}_{\text{Sets}_*}$ .
- Subdiagram (5) commutes by the naturality of  $\text{id}^{0, -1}_{\text{Sets}_*}$ .
- Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .



it follows that the diagram

$$\begin{array}{c}
 S^0 \wedge (S^0 \wedge S^0) \\
 \downarrow \\
 [x] \wedge ([y] \wedge [z]) \\
 \downarrow \\
 X \wedge (Y \wedge Z) \\
 \swarrow \alpha_{X,Y,Z}^{\text{Sets}_*, -1} \quad \searrow \text{id}_{\text{Sets}_*|X,Y \wedge Z}^{\otimes, -1} \\
 (X \wedge Y) \wedge Z \quad X \otimes_{\text{Sets}_*} (Y \wedge Z) \\
 \downarrow \text{id}_{\text{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \quad \downarrow \text{id}_X \wedge \text{id}_{\text{Sets}_*|Y, Z}^{\otimes, -1} \\
 (X \wedge Y) \otimes_{\text{Sets}_*} Z \quad X \otimes_{\text{Sets}_*} (Y \otimes_{\text{Sets}_*} Z) \\
 \swarrow \text{id}_{\text{Sets}_*|X, Y}^{\otimes, -1} \otimes \text{id}_Z \quad \searrow \alpha'_{X,Y,Z}{}^{-1} \\
 (X \otimes_{\text{Sets}_*} Y) \otimes_{\text{Sets}_*} Z
 \end{array}$$

also commutes. We then have

$$\begin{aligned}
 & \left[ (\text{id}_{\text{Sets}_*|X,Y}^{\otimes, -1} \otimes \text{id}_Z) \circ \text{id}_{\text{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{X,Y,Z}^{\text{Sets}_*, -1} \right] (x, (y, z)) = \left[ (\text{id}_{\text{Sets}_*|X,Y}^{\otimes, -1} \otimes \text{id}_Z) \circ \text{id}_{\text{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \right. \\
 & \quad \left. \circ \alpha_{X,Y,Z}^{\text{Sets}_*, -1} \circ ([x] \wedge ([y] \wedge [z])) \right] (1, (1, 1)) \\
 & = \left[ \alpha'_{X,Y,Z}{}^{-1} \circ (\text{id}_X \wedge \text{id}_{\text{Sets}_*|Y, Z}^{\otimes, -1}) \right. \\
 & \quad \left. \circ \text{id}_{\text{Sets}_*|X, Y \wedge Z}^{\otimes, -1} \circ ([x] \wedge ([y] \wedge [z])) \right] (1, (1, 1)) \\
 & = [\alpha'_{X,Y,Z}{}^{-1} \circ (\text{id}_X \wedge \text{id}_{\text{Sets}_*|Y, Z}^{\otimes, -1}) \circ \text{id}_{\text{Sets}_*|X, Y \wedge Z}^{\otimes, -1}] (x, (y, z))
 \end{aligned}$$

for each  $(x, (y, z)) \in X \wedge (Y \wedge Z)$ , and thus we have

$$(\text{id}_{\text{Sets}_*|X,Y}^{\otimes, -1} \otimes \text{id}_Z) \circ \text{id}_{\text{Sets}_*|X \wedge Y, Z}^{\otimes, -1} \circ \alpha_{X,Y,Z}^{\text{Sets}_*, -1} = \alpha'_{X,Y,Z}{}^{-1} \circ (\text{id}_X \wedge \text{id}_{\text{Sets}_*|Y, Z}^{\otimes, -1}) \circ \text{id}_{\text{Sets}_*|X, Y \wedge Z}^{\otimes, -1}.$$

Taking inverses then gives

$$\alpha_{X,Y,Z}^{\text{Sets}_*} \circ \text{id}_{\text{Sets}_*|X \wedge Y, Z}^{\otimes} \circ (\text{id}_{\text{Sets}_*|X, Y}^{\otimes} \otimes \text{id}_Z) = \text{id}_{\text{Sets}_*|X, Y \wedge Z}^{\otimes} \circ (\text{id}_X \wedge \text{id}_{\text{Sets}_*|Y, Z}^{\otimes}) \circ \alpha'_{X,Y,Z}.$$

showing that the diagram

$$\begin{array}{ccc}
 & (X \otimes_{\text{Sets}_*} Y) \otimes_{\text{Sets}_*} Z & \\
 \text{id}_{\text{Sets}_*|X,Y}^{\otimes} \otimes_{\text{Sets}_*} \text{id}_Z \swarrow & & \searrow \alpha'_{X,Y,Z} \\
 (X \wedge Y) \otimes_{\text{Sets}_*} Z & & X \otimes_{\text{Sets}_*} (Y \otimes_{\text{Sets}_*} Z) \\
 \downarrow \text{id}_{\text{Sets}_*|X \wedge Y,Z}^{\otimes} & & \downarrow \text{id}_X \otimes_{\text{Sets}_*} \text{id}_{\text{Sets}_*|Y,Z}^{\otimes} \\
 (X \wedge Y) \wedge Z & & X \otimes_{\text{Sets}_*} (Y \wedge Z) \\
 \searrow \alpha_{X,Y,Z}^{\text{Sets}_*} & & \swarrow \text{id}_{\text{Sets}_*|X,Y \wedge Z}^{\otimes} \\
 & X \wedge (Y \wedge Z) &
 \end{array}$$

indeed commutes.

*Uniqueness of the Isomorphism  $\otimes_{\text{Sets}_*} \cong \wedge$ :* Let  $\phi, \psi: -_1 \otimes_{\text{Sets}_*} -_2 \Rightarrow -_1 \wedge -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of  $\text{Sets}_*$  with respect to  $\wedge$  and  $\otimes$  (as shown above), we have

$$\begin{aligned}
 \lambda'_Y &= \lambda_Y^{\text{Sets}_*} \circ \phi_{S^0,Y} \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y), \\
 \lambda'_Y &= \lambda_Y^{\text{Sets}_*} \circ \psi_{S^0,Y} \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y).
 \end{aligned}$$

Postcomposing both sides with  $\lambda_Y^{\text{Sets}_*, -1}$  and then precomposing both sides with  $\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_Y$  gives

$$\begin{aligned}
 \lambda_Y^{\text{Sets}_*, -1} \circ \lambda'_Y \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_Y) &= \phi_{S^0,Y}, \\
 \lambda_Y^{\text{Sets}_*, -1} \circ \lambda'_Y \circ (\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_Y) &= \psi_{S^0,Y},
 \end{aligned}$$

and thus we have

$$\phi_{S^0,Y} = \psi_{S^0,Y}$$

for each  $Y \in \text{Obj}(\text{Sets}_*)$ . Now, let  $x \in X$  and consider the naturality diagrams

$$\begin{array}{ccc}
 S^0 \wedge Y & \xrightarrow{[x] \wedge \text{id}_Y} & X \wedge Y \\
 \phi_{S^0,Y} \downarrow & & \downarrow \phi_{X,Y} \\
 S^0 \otimes_{\text{Sets}_*} Y & \xrightarrow{[x] \otimes_{\text{Sets}_*} \text{id}_Y} & X \otimes_{\text{Sets}_*} Y
 \end{array}
 \quad
 \begin{array}{ccc}
 S^0 \wedge Y & \xrightarrow{[x] \wedge \text{id}_Y} & X \wedge Y \\
 \psi_{S^0,Y} \downarrow & & \downarrow \psi_{X,Y} \\
 S^0 \otimes_{\text{Sets}_*} Y & \xrightarrow{[x] \otimes_{\text{Sets}_*} \text{id}_Y} & X \otimes_{\text{Sets}_*} Y
 \end{array}$$

for  $\phi$  and  $\psi$  with respect to the morphisms  $[x]$  and  $\text{id}_Y$ . Having shown that  $\phi_{S^0, Y} = \psi_{S^0, Y}$ , we have

$$\begin{aligned}\phi_{X, Y}(x, y) &= [\phi_{X, Y} \circ ([x] \wedge \text{id}_Y)](1, y) \\ &= [([x] \otimes_{\mathbf{Sets}_*} \text{id}_Y) \circ \phi_{S^0, Y}](1, y) \\ &= [([x] \otimes_{\mathbf{Sets}_*} \text{id}_Y) \circ \psi_{S^0, Y}](1, y) \\ &= [\psi_{X, Y} \circ ([x] \wedge \text{id}_Y)](1, y) \\ &= \psi_{X, Y}(x, y)\end{aligned}$$

for each  $(x, y) \in X \wedge Y$ . Therefore we have

$$\phi_{X, Y} = \psi_{X, Y}$$

for each  $X, Y \in \text{Obj}(\mathbf{Sets}_*)$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\mathbf{Sets}_*} \cong \times$  to be unique.  $\square$

**01W7 Corollary 7.5.10.1.2.** The symmetric monoidal structure on the category  $\mathbf{Sets}_*$  of **Definition 7.5.9.1.1** is uniquely determined by the following requirements:

**01W8** 1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\mathbf{Sets}_*} : \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*$$

of  $\mathbf{Sets}_*$  preserves colimits separately in each variable.

**01W9** 2. *The Unit Object Is  $S^0$ .* We have  $\mathbb{1}_{\mathbf{Sets}_*} \cong S^0$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{B}_\infty}(\mathbf{Sets}_*)$  of ?? spanned by the symmetric monoidal categories  $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$  satisfying **Items 1** and **2** is contractible.

*Proof.* Since  $\mathbf{Sets}_*$  is locally presentable (??), it follows from ?? that **Definition 7.5.10.1.2** is equivalent to the existence of an internal Hom as in **Item 1** of **Definition 7.5.10.1.1**. The result then follows from **Definition 7.5.10.1.1**.  $\square$

**01WA Corollary 7.5.10.1.3.** The symmetric monoidal structure on the category  $\mathbf{Sets}_*$  is the unique symmetric monoidal structure on  $\mathbf{Sets}_*$  such that the free pointed set functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*$$

admits a symmetric monoidal structure, i.e. the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathbf{Sets}_*)$  of ?? spanned by the symmetric monoidal categories  $(\mathbf{Sets}_*, \otimes_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$  with respect to which  $(-)^+$  admits a symmetric monoidal structure is contractible.

*Proof.* Let  $(\otimes_{\mathbf{Sets}_*}, \mathbb{1}_{\mathbf{Sets}_*}, \lambda^{\mathbf{Sets}_*}, \rho^{\mathbf{Sets}_*}, \sigma^{\mathbf{Sets}_*})$  be a symmetric monoidal structure on  $\mathbf{Sets}_*$  such that  $(-)^+$  admits a symmetric monoidal structure with respect to  $\otimes_{\mathbf{Sets}_*}$  and  $\wedge$ . We have isomorphisms

$$\begin{aligned} X \otimes_{\mathbf{Sets}_*} Y &\cong (X^-)^+ \otimes_{\mathbf{Sets}_*} (Y^-)^+ \\ &\cong (X^- \times Y^-)^+ \\ &\cong (X^-)^+ \wedge (Y^-)^+ \\ &\cong X \wedge Y, \end{aligned}$$

all natural in  $X$  and  $Y$ . Now, since  $\wedge$  preserves colimits in both variables and  $\otimes_{\mathbf{Sets}_*} \cong \wedge$ , it follows that  $\otimes_{\mathbf{Sets}_*}$  also preserves colimits in both variables, so the result then follows from [Definition 7.5.10.1.2](#).  $\square$

## 00H6 7.5.11 Monoids With Respect to the Smash Product of Pointed Sets

00H7 **Proposition 7.5.11.1.1.** The category of monoids on  $(\mathbf{Sets}_*, \wedge, S^0)$  is isomorphic to the category of monoids with zero and morphisms between them.

*Proof.* See ??, in particular ??, ??, ??, and ??.  $\square$

## 00H8 7.5.12 Comonoids With Respect to the Smash Product of Pointed Sets

00H9 **Proposition 7.5.12.1.1.** The symmetric monoidal functor

$$((-)^+, (-)^{+, \times}, (-)^{+, \times}_{\mathbb{1}}): (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

of [Pointed Sets](#), [Item 4](#) of [Definition 6.4.1.1.2](#) lifts to an equivalence of categories

$$\begin{aligned} \text{CoMon}(\mathbf{Sets}_*, \wedge, S^0) &\stackrel{\text{eq.}}{\cong} \text{CoMon}(\mathbf{Sets}, \times, \text{pt}) \\ &\cong \mathbf{Sets}. \end{aligned}$$

*Proof.* We follow [PS19, Lemma 2.4].

*Faithfulness:* Given morphisms  $f, g: X \rightarrow Y$ , if  $f^+ = g^+$ , then we have

$$\begin{aligned} f(x) &\stackrel{\text{def}}{=} f^+(x) \\ &= g^+(x) \\ &\stackrel{\text{def}}{=} g(x) \end{aligned}$$

for each  $x \in X^+$ , and thus  $f = g$ , showing  $(-)^+$  to be faithful.

*Fullness:* Let  $f: X^+ \rightarrow Y^+$  be a morphism of comonoids in  $\mathbf{Sets}_*$ . By counitality, the diagram

$$\begin{array}{ccc} X^+ & \xrightarrow{f} & Y^+ \\ \epsilon_X^+ \searrow & & \swarrow \epsilon_Y^+ \\ & S^0 & \end{array}$$

commutes. If  $f(x) = \star_Y$  for  $x \neq \star_X$ , the commutativity of this diagram then gives

$$\begin{aligned} 1 &= \epsilon_X^+(x) \\ &= \epsilon_Y^+(f(x)) \\ &= \epsilon_Y^+(\star_Y) \\ &= 0, \end{aligned}$$

which is a contradiction. Thus  $f$  is an active morphism of pointed sets, so there exists a map  $f^-$  such that  $(f^-)^+ = f$  (**Pointed Sets, Item 1 of Definition 6.4.2.1.2**).

*Essential Surjectivity:* Let  $(X, \Delta_X, \epsilon_X)$  be a comonoid in  $\mathbf{Sets}_*$ . We claim that

$$\Delta_X(x) = x \wedge x$$

for each  $x \in X$  with  $x \neq \star_X$ . Indeed:

- Suppose that  $x \neq \star_X$  and write  $\Delta_X(x) = x_1 \wedge x_2$ .
- Since  $\text{id}_X \wedge \epsilon_X$  is pointed, we have

$$[\text{id}_X \wedge \epsilon_X](x_1 \wedge x_2) = \star_{X \wedge S^0}.$$

- The counitality condition for  $\Delta_X$ , corresponding to the commutativity of the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\Delta_X} & X \wedge X \\
 & \searrow \rho_X^{\text{Sets}_*, -1} & \downarrow \text{id}_X \wedge \epsilon_X \\
 & & X \wedge S^0
 \end{array}$$

gives

$$\begin{aligned}
 x \wedge 1 &= \rho_X^{\text{Sets}_*, -1}(x) \\
 &= [\text{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\
 &= [\text{id}_X \wedge \epsilon_X](\Delta_X(x)) \\
 &= [\text{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\
 &= \star_{X \wedge S^0},
 \end{aligned}$$

which is a contradiction. Thus  $x_1 \neq \star_X$ .

- Similarly, if  $x \neq \star_X$ , then  $x_2 \neq \star_X$ .
- Next, we claim that  $\epsilon_X(x_2) = 1$ , as otherwise we would have

$$\begin{aligned}
 \star_{X \wedge S^0} &= x_1 \wedge 0 \\
 &= [\text{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\
 &= [\text{id}_X \wedge \epsilon_X](\Delta_X(x)) \\
 &= [\text{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\
 &= \rho_X^{\text{Sets}_*, -1}(x) \\
 &= x \wedge 1,
 \end{aligned}$$

a contradiction. Thus  $\epsilon_X(x_2) = 1$ .

- Similarly, if  $x \neq \star_X$ , then  $\epsilon_X(x_1) = 1$ .
- Now, since  $\Delta_X$  is counital, we have

$$\begin{aligned}
 x \wedge 1 &= \rho_X^{\text{Sets}_*, -1}(x) \\
 &= [\text{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\
 &= [\text{id}_X \wedge \epsilon_X](\Delta_X(x))
 \end{aligned}$$

$$\begin{aligned}
&= [\text{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\
&= x_1 \wedge 1,
\end{aligned}$$

so  $x = x_1$ .

- Similarly,  $x = x_2$ , and we are done.

Next, notice that  $X \cong \epsilon_X^{-1}(0) \coprod \epsilon_X^{-1}(1)$ , and let  $x \in \epsilon_X^{-1}(0)$ . We then have

$$\begin{aligned}
[(\text{id}_X \wedge \epsilon_X) \circ \Delta_X](x) &= [\text{id}_X \wedge \epsilon_X](x \wedge x) \\
&= x \wedge 0 \\
&= \star_{X \wedge S^0}.
\end{aligned}$$

The counitality condition for  $\Delta_X$  then gives  $x = \star_X$ , so  $\epsilon_X^{-1}(0) = \{\star_X\}$ . Thus we have  $(\epsilon_X^{-1}(1))^+ \cong X$ , and this isomorphism is compatible with the comonoid structures when equipping  $\epsilon_X^{-1}(1)$  with its unique comonoid structure. This shows that  $(-)^+$  is essentially surjective.

*Equivalence:* Since  $(-)^+$  is fully faithful and essentially surjective, it is an equivalence by [Categories, Item 1b](#) of [Item 1](#) of [Definition 11.6.7.1.2](#).  $\square$

## 00HA 7.6 Miscellany

### 00HB 7.6.1 The Smash Product of a Family of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

**00HC Definition 7.6.1.1.1.** The **smash product of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pointed set  $\bigwedge_{i \in I} X_i$  consisting of:

- *The Underlying Set.* The set  $\bigwedge_{i \in I} X_i$  defined by

$$\bigwedge_{i \in I} X_i \stackrel{\text{def}}{=} \left( \prod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\prod_{i \in I} X_i$  obtained by declaring

$$(x_i)_{i \in I} \sim (y_i)_{i \in I}$$

if there exist  $i_0 \in I$  such that  $x_{i_0} = x_0$  and  $y_{i_0} = y_0$ , for each  $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i$ .

- *The Basepoint.* The element  $[(x_0)_{i \in I}]$  of  $\bigwedge_{i \in I} X_i$ .

# Appendices

## A Other Chapters

### Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

### Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
7. [Tensor Products of Pointed Sets](#)

### Relations

8. [Relations](#)
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10. [Conditions on Relations](#)

### Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

### Monoidal Categories

13. [Constructions With Monoidal Categories](#)

### Bicategories

14. [Types of Morphisms in Bicategories](#)

### Extra Part

15. [Notes](#)

## References

- [PS19] Maximilien Péroux and Brooke Shipley. “Coalgebras in Symmetric Monoidal Categories of Spectra”. In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. ISSN: 1532-0073. DOI: [10.4310/HHA.2019.v21.n1.a1](https://doi.org/10.4310/HHA.2019.v21.n1.a1). URL: <https://doi.org/10.4310/HHA.2019.v21.n1.a1> (cit. on p. 133).