

Pointed Sets

The Clowder Project Authors

July 22, 2025

0098 This chapter contains some foundational material on pointed sets.

Contents

6.1 Pointed Sets	2
6.1.1 Foundations	2
6.1.2 Morphisms of Pointed Sets	3
6.1.3 The Category of Pointed Sets	3
6.1.4 Elementary Properties of Pointed Sets	4
6.1.5 Active and Inert Morphisms of Pointed Sets	7
6.2 Limits of Pointed Sets	11
6.2.1 The Terminal Pointed Set	11
6.2.2 Products of Families of Pointed Sets	11
6.2.3 Products	13
6.2.4 Pullbacks	16
6.2.5 Equalisers	20
6.3 Colimits of Pointed Sets	23
6.3.1 The Initial Pointed Set	23
6.3.2 Coproducts of Families of Pointed Sets	23
6.3.3 Coproducts	25
6.3.4 Pushouts	29
6.3.5 Coequalisers	35

6.4	Constructions With Pointed Sets	37
6.4.1	Free Pointed Sets	37
6.4.2	Deleting Basepoints	45
A	Other Chapters	52

0099 6.1 Pointed Sets

009A 6.1.1 Foundations

009B **Definition 6.1.1.1.1.** A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_\bullet(\text{Sets}), \text{pt})$.
- A pointed object in (Sets, pt) .

009C **Remark 6.1.1.1.2.** In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X , called the **underlying set of** (X, x_0) .
- *The Basepoint.* A morphism

$$[x_0]: \text{pt} \rightarrow X$$

in Sets , determining an element $x_0 \in X$, called the **basepoint of** X .

009D **Example 6.1.1.1.3.** The **0-sphere**² is the pointed set $(S^0, 0)$ ³ consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0 .

009E **Example 6.1.1.1.4.** The **trivial pointed set** is the pointed set (pt, \star) consisting of:

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -**modules**.

²*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

³*Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also

- *The Underlying Set.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element \star of pt .

01QB Example 6.1.1.1.5. The **standard pointed set with $n + 1$ elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{\ast\} \cup \{1, \dots, n\}.$$

- *The Basepoint.* The element \ast of $\langle n \rangle$.

009H 6.1.2 Morphisms of Pointed Sets

009J Definition 6.1.2.1.1. A **morphism of pointed sets**^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A morphism of pointed objects in $(\mathbf{Sets}, \text{pt})$.

009K Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f: (X, x_0) \rightarrow (Y, y_0)$ is a morphism of sets $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

009L 6.1.3 The Category of Pointed Sets

009M Definition 6.1.3.1.1. The **category of pointed sets** is the category \mathbf{Sets}_\ast defined denoted $(\mathbb{R}_1, 0)$.

⁴*Further Terminology:* Also called a **pointed function**.

⁵*Further Terminology:* In the context of monoids with zero as models for \mathbb{R}_1 -algebras, morphisms of pointed sets are also called **morphism of \mathbb{R}_1 -modules**.

equivalently as:

- The homotopy category of the ∞ -category $\text{Mon}_{\mathbb{B}_0}(\mathbf{N}_\bullet(\text{Sets}), \text{pt})$ of ??, ??.
- The category Sets_* of Constructions With Categories, ??.

009N Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \text{Obj}(\text{Sets}_*)$, the unit map

$$\mathbb{1}_{(X, x_0)}^{\text{Sets}_*} : \text{pt} \rightarrow \text{Sets}_*((X, x_0), (X, x_0))$$

of Sets_* at (X, x_0) is defined by⁶

$$\text{id}_{(X, x_0)}^{\text{Sets}_*} \stackrel{\text{def}}{=} \text{id}_X.$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$, the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\text{Sets}_*} : \text{Sets}_*((Y, y_0), (Z, z_0)) \times \text{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \text{Sets}_*((X, x_0), (Z, z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by⁷

$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\text{Sets}_*} f \stackrel{\text{def}}{=} g \circ f.$$

009P 6.1.4 Elementary Properties of Pointed Sets

009Q Proposition 6.1.4.1.1. Let (X, x_0) be a pointed set.

⁶Note that id_X is indeed a morphism of pointed sets, as we have $\text{id}_X(x_0) = x_0$.

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$g(f(x_0)) = g(y_0) = z_0,$$

- 009R 1. *Completeness*. The category \mathbf{Sets}_* of pointed sets and morphisms between them is complete, having in particular:
- 009S (a) Products, described as in [Definition 6.2.3.1.1](#).
- 009T (b) Pullbacks, described as in [Definition 6.2.4.1.1](#).
- 009U (c) Equalisers, described as in [Definition 6.2.5.1.1](#).
- 009V 2. *Cocompleteness*. The category \mathbf{Sets}_* of pointed sets and morphisms between them is cocomplete, having in particular:
- 009W (a) Coproducts, described as in [Definition 6.3.3.1.1](#).
- 009X (b) Pushouts, described as in [Definition 6.3.4.1.1](#);
- 009Y (c) Coequalisers, described as in [Definition 6.3.5.1.1](#).
- 009Z 3. *Failure To Be Cartesian Closed*. The category \mathbf{Sets}_* is not Cartesian closed.⁸
- 00A0 4. *Morphisms From the Monoidal Unit*. We have a bijection of sets⁹

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

- 00A1 5. *Relation to Partial Functions*. We have an equivalence of categories¹⁰

$$\mathbf{Sets}_* \stackrel{\text{eq.}}{\cong} \mathbf{Sets}^{\text{part.}}$$


between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

⁸The category \mathbf{Sets}_* does admit a natural monoidal closed structure, however; see [Tensor Products of Pointed Sets](#).

⁹In other words, the forgetful functor

$$\text{忘}: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

¹⁰  *Warning*: This is not an isomorphism of categories, only an equivalence.

024U (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \mathbf{Sets}_* \xrightarrow{\cong} \mathbf{Sets}^{\text{part.}}$$

sends:

024V i. A pointed set (X, x_0) to X .

024W ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

024X (b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \mathbf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathbf{Sets}_*$$

sends:

024Y i. A set X is to the pointed set (X, \star) with \star an element that is not in X .

024Z ii. A partial function

$$f: X \rightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, Completeness: This follows from (the proofs) of [Definitions 6.2.3.1.1](#), [6.2.4.1.1](#) and [6.2.5.1.1](#) and ??.

Item 2, Cocompleteness: This follows from (the proofs) of [Definitions 6.3.3.1.1](#), [6.3.4.1.1](#) and [6.3.5.1.1](#) and ??.

Item 3, Failure To Be Cartesian Closed: See [\[MSE 2855868\]](#).

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X , we obtain a bijection between pointed maps $S^0 \rightarrow X$ and the elements of X .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} : S^0 \rightarrow X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \rightarrow X$ picking the element x_0 of X , and thus we see that the bijection between pointed maps $S^0 \rightarrow X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [\[MSE 884460\]](#). □

01QC 6.1.5 Active and Inert Morphisms of Pointed Sets

01QD **Definition 6.1.5.1.1.** Let $f : (X, x_0) \rightarrow (Y, y_0)$ be a morphism of pointed sets.

01QE 1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.

01QF 2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

01QG **Notation 6.1.5.1.2.** We write $\mathbf{Sets}_*^{\text{actv}}$ for the wide subcategory of \mathbf{Sets}_* spanned by pointed sets and the active maps between them.

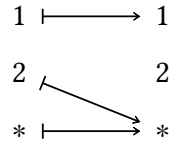
01QH **Example 6.1.5.1.3.** Here are some examples of active and inert maps of pointed sets.

01QJ 1. The map $\mu : \langle 2 \rangle \rightarrow \langle 1 \rangle$ given by

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \searrow & \\ * & \xrightarrow{\quad} & * \end{array}$$

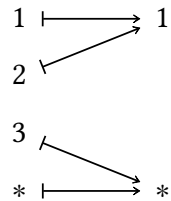
is active but not inert.

01QK 2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by



is inert but not active.

01QL 3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by

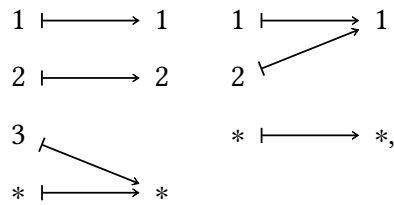


is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i: \langle 3 \rangle \rightarrow \langle 2 \rangle,$$

$$a: \langle 2 \rangle \rightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by



with i being inert and a being active.

01QM **Proposition 6.1.5.1.4.** Let (X, x_0) and (Y, y_0) be pointed sets.

01QN 1. *Active-Inert Factorisation.* Every morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

- 01QP (a) The map $i: (X, x_0) \rightarrow (K, k_0)$ is an inert morphism of pointed sets
- 01QQ (b) The map $a: (K, k_0) \rightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in \mathbf{Sets}_* .

Proof. Item 1, Active-Inert Factorisation: Let $f: X \rightarrow Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= (X \setminus f^{-1}(y_0)) \cup \{x_0\}; \end{aligned}$$

- $i: X \rightarrow K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

- $a: K \rightarrow Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{a} & B \end{array}$$

be a commutative diagram in \mathbf{Sets}_* . Consider the morphism $\phi: Y \rightarrow A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & \swarrow \text{\scriptsize $\exists!$} \phi & \downarrow g \\ A & \xrightarrow{a} & B. \end{array}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\stackrel{\text{def}}{=} \phi(i(x)) \\ &\stackrel{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [a \circ \phi](y) &\stackrel{\text{def}}{=} a(\phi(y)) \\ &\stackrel{\text{def}}{=} a(f(i^{-1}(y))) \\ &\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y)) \\ &= [g \circ i](i^{-1}(y)) \\ &\stackrel{\text{def}}{=} g(i(i^{-1}(y))) \\ &\stackrel{\text{def}}{=} g(y) \end{aligned}$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & \swarrow \psi & \downarrow g \\ A & \xrightarrow{a} & B \end{array}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned} \psi(y) &= \psi(i(x)) \\ &= f(x) \\ &= f(i^{-1}(y)) \\ &\stackrel{\text{def}}{=} \phi(y). \end{aligned}$$

This finishes the proof. □

00A2 6.2 Limits of Pointed Sets

00A3 6.2.1 The Terminal Pointed Set

00A4 **Definition 6.2.1.1.1.** The **terminal pointed set** is the terminal object of Sets_* as in Limits and Colimits, ??.

0250 **Construction 6.2.1.1.2.** Concretely, the **terminal pointed set** is the pair $\left((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\text{Sets}_*)} \right)$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\text{Sets})$.

Proof. We claim that (pt, \star) is the terminal object of Sets_* . Indeed, suppose we have a diagram of the form

$$(X, x_0) \quad (\text{pt}, \star)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely $!_X$. □

00A5 6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A6 **Definition 6.2.2.1.1.** The **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* as in Limits and Colimits, ??.

0251 Construction 6.2.2.1.2. Concretely, the **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\text{pr}_i\}_{i \in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- *The Cone.* The collection

$$\left\{ \text{pr}_i : \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i \left((x_j)_{j \in I} \right) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi : (P, *) \rightarrow \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$$

making the diagram

$$\begin{array}{ccc} (P, *) & & \\ \downarrow \phi \mid \exists! & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_i(*))_{i \in I} \\ &= (x_0^i)_{i \in I},\end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. \square

00A7 Proposition 6.2.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A8 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I}: \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??. \square

00A9 6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA Definition 6.2.3.1.1. The **product of (X, x_0) and (Y, y_0)** is the product of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, ??.

0252 Construction 6.2.3.1.2. Concretely, the **product of (X, x_0) and (Y, y_0)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned}\text{pr}_1: (X \times Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2: (X \times Y, (x_0, y_0)) &\rightarrow (Y, y_0)\end{aligned}$$

defined by

$$\begin{aligned}\text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y\end{aligned}$$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and

(Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccc} & (P, *) & \\ p_1 \swarrow & & \searrow p_2 \\ (X, x_0) & \xleftarrow{\text{pr}_1} (X \times Y, (x_0, y_0)) \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccc} & (P, *) & \\ p_1 \swarrow & \downarrow \phi \exists! & \searrow p_2 \\ (X, x_0) & \xleftarrow{\text{pr}_1} (X \times Y, (x_0, y_0)) \xrightarrow{\text{pr}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \text{pr}_1 \circ \phi &= p_1, \\ \text{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. \square

00AB Proposition 6.2.3.1.3. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00AC 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned} A \times - &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ - \times B &: \text{Sets}_* \rightarrow \text{Sets}_*, \\ -_1 \times -_2 &: \text{Sets}_* \times \text{Sets}_* \rightarrow \text{Sets}_*, \end{aligned}$$

defined in the same way as the functors of [Constructions With Sets, Item 1 of Definition 4.1.3.1.3](#).

01QR 2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

00AD 3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$.

00AE 4. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\text{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\text{pt}, \star) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.

00AF 5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\text{Sets}_*)$.

00AG 6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \times, (\text{pt}, \star))$ is a symmetric monoidal category.

Proof. [Item 1, Functoriality](#): This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

[Item 2, Lack of Adjointness](#): See [MSE 2855868].

[Item 3, Associativity](#): This follows from [Constructions With Sets, Item 4 of Definition 4.1.3.1.3](#).

[Item 4, Unitality](#): This follows from [Constructions With Sets, Item 5 of Definition 4.1.3.1.3](#).

[Item 5, Commutativity](#): This follows from [Constructions With Sets, Item 6 of Definition 4.1.3.1.3](#).

[Item 6, Symmetric Monoidality](#): This follows from [Constructions With Sets, Item 14 of Definition 4.1.3.1.3](#). \square

00AH 6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \rightarrow (Z, z_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ be morphisms of pointed sets.

00AJ Definition 6.2.4.1.1. The **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in \mathbf{Sets}_* as in Limits and Colimits, ??.

0253 Construction 6.2.4.1.2. Concretely, the **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1: (X \times_Z Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2: (X \times_Z Y, (x_0, y_0)) &\rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$\begin{array}{ccc} (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\ \text{pr}_1 \downarrow & & \downarrow g \\ (X, x_0) & \xrightarrow{f} & (Z, z_0). \end{array}$$

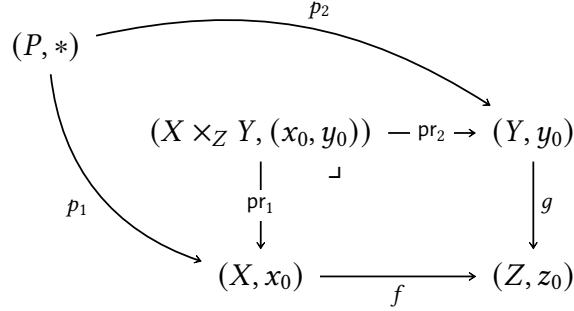
$f \circ \text{pr}_1 = g \circ \text{pr}_2$,

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \end{aligned}$$

$$\begin{aligned}
 &= g(\text{pr}_2(x, y)) \\
 &= [g \circ \text{pr}_2](x, y),
 \end{aligned}$$

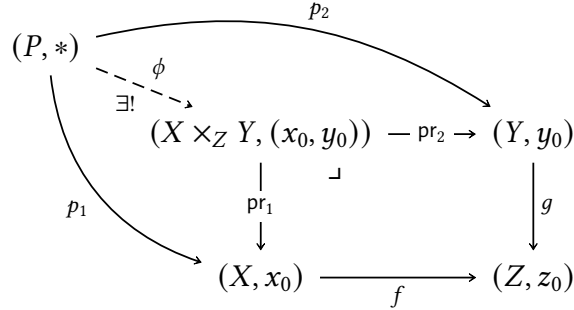
where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned}
 \text{pr}_1 \circ \phi &= p_1, \\
 \text{pr}_2 \circ \phi &= p_2
 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0),\end{aligned}$$

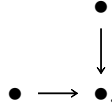
where we have used that p_1 and p_2 are morphisms of pointed sets. \square

00AK Proposition 6.2.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00AL 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}_*) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & & \\ \searrow \phi & & \searrow \chi & & \\ & X' & \xrightarrow{f'} & Z' & \end{array}$$

in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi : (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{array}{ccccc}
 X \times_Z Y & \xrightarrow{\quad} & Y & & \\
 \downarrow & \searrow \lrcorner & \downarrow g & \searrow \psi & \\
 & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' & \\
 & \downarrow & & \downarrow g' & \\
 X & \xrightarrow{f} & Z & \xrightarrow{\chi} & Z' \\
 \searrow \phi & & & & \\
 & X' & \xrightarrow{f'} & Z' &
 \end{array}$$

commute.

00AM 2. *Associativity.* Given a diagram

$$\begin{array}{ccccc}
 X & & Y & & Z \\
 & \searrow f & & \searrow h & \\
 & W & & V &
 \end{array}$$

in \mathbf{Sets}_* , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{array}{ccc}
 \begin{array}{c} (X \times_W Y) \times_Y Z \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} &
 \begin{array}{c} (X \times_W Y) \times_Y (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \times_W Y \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \quad Z \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array} &
 \begin{array}{c} X \times_W (Y \times_V Z) \\ \swarrow \quad \searrow \\ X \quad Y \times_V Z \\ \swarrow \quad \searrow \\ X \quad Y \\ \searrow f \quad \swarrow g \quad \searrow h \quad \swarrow k \\ W \quad V \end{array}
 \end{array}$$

00AN 3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 \begin{array}{c} A \xlongequal{\quad} A \\ \downarrow f \quad \lrcorner \quad \downarrow f \\ X \xlongequal{\quad} X \end{array} &
 \begin{array}{c} X \times_X A \cong A, \\ A \times_X X \cong A, \end{array} &
 \begin{array}{c} A \xrightarrow{f} X \\ \parallel \quad \lrcorner \quad \parallel \\ X \xrightarrow{f} X. \end{array}
 \end{array}$$

00AP 4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

00AQ 5. *Interaction With Products.* We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow \lrcorner & & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad X \times_{\text{pt}} Y \cong X \times Y,$$

00AR 6. *Symmetric Monoidality.* The triple $(\text{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from [Constructions With Sets, Item 4 of Definition 4.1.4.1.5.](#)

Item 3, Unitality: This follows from [Constructions With Sets, Item 6 of Definition 4.1.4.1.5.](#)

Item 4, Commutativity: This follows from [Constructions With Sets, Item 7 of Definition 4.1.4.1.5.](#)

Item 5, Interaction With Products: This follows from [Constructions With Sets, Item 10 of Definition 4.1.4.1.5.](#)

Item 6, Symmetric Monoidality: This follows from [Constructions With Sets, Item 11 of Definition 4.1.4.1.5.](#) \square

00AS 6.2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT **Definition 6.2.5.1.1.** The **equaliser of (f, g)** is the equaliser of f and g in Sets_* as in Limits and Colimits, ??.

0254 Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\text{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\text{eq}(f, g): (\text{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\text{eq}(f, g) \hookrightarrow \text{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[f]{f} & (Y, y_0) \\ & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightleftharpoons[f]{f} & (Y, y_0) \\ \uparrow \phi \exists! & \nearrow e & & & \\ (E, *) & & & & \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= e(*) \\ &= x_0,\end{aligned}$$

where we have used that e is a morphism of pointed sets. \square

00AU Proposition 6.2.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00AV 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))},$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[-g]{} \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in Sets_* , being explicitly given by

$$\text{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW 2. *Unitality.* We have an isomorphism of pointed sets

$$\text{Eq}(f, f) \cong X.$$

00AX 3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{Eq}(f, g) \cong \text{Eq}(g, f).$$

Proof. **Item 1, Associativity:** This follows from **Constructions With Sets, Item 1** of **Definition 4.1.5.1.3**.

Item 2, Unitality: This follows from **Constructions With Sets, Item 4** of **Definition 4.1.5.1.3**.

Item 3, Commutativity: This follows from **Constructions With Sets, Item 5** of **Definition 4.1.5.1.3**. \square

00AY 6.3 Colimits of Pointed Sets

00AZ 6.3.1 The Initial Pointed Set

00B0 **Definition 6.3.1.1.1.** The **initial pointed set** is the initial object of \mathbf{Sets}_* as in Limits and Colimits, ??.

0255 **Construction 6.3.1.1.2.** Concretely, the **initial pointed set** is the pair $\left((\text{pt}, \star), \{\iota_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)} \right)$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (\text{pt}, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$(\text{pt}, \star) \quad (X, x_0)$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi : (\text{pt}, \star) \rightarrow (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X . □

00B1 6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B2 **Definition 6.3.2.1.1.** The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ ¹¹ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* as in Limits and Colimits, ??.

¹¹*Further Terminology:* Also called the **wedge sum of the family** $\{(X_i, x_0^i)\}_{i \in I}$.

0256 Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $(\bigvee_{i \in I} X_i, p_0)$, $\{\text{inj}_i\}_{i \in I}$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:

- *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\coprod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{ \text{inj}_i : (X_i, x_0^i) \rightarrow \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow \iota_i & \uparrow \phi \uparrow \exists! \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. □

00B3 Proposition 6.3.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B4 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I}: \mathbf{Fun}(I_{\text{disc}}, \mathbf{Sets}_*) \rightarrow \mathbf{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??. □

00B5 6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6 Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0) ¹² is the coproduct of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* as in Limits and Colimits, ??.

0257 Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

- *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned} (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) \\ &\cong (X \amalg_{\text{pt}} Y, p_0) \\ &\cong (X \amalg Y / \sim, p_0), \end{aligned} \quad \begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow \ulcorner & & \uparrow [y_0] \\ X & \xleftarrow{[x_0]} & \text{pt}, \end{array}$$

where \sim is the equivalence relation on $X \amalg Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\ &= [(1, y_0)]. \end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1 &: (X, x_0) \rightarrow (X \vee Y, p_0), \\ \text{inj}_2 &: (Y, y_0) \rightarrow (X \vee Y, p_0), \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)], \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0)

¹²Further Terminology: Also called the **wedge sum of** (X, x_0) **and** (Y, y_0) .

in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_1 & & \nwarrow \iota_2 & \\ (X, x_0) & \xrightarrow{\text{inj}_1} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets} . Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_1 & \uparrow \phi \mid \exists! & \nwarrow \iota_2 & \\ (X, x_0) & \xrightarrow{\text{inj}_1} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \text{inj}_X &= \iota_X, \\ \phi \circ \text{inj}_Y &= \iota_Y \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0, x_0)]) \\ &= \iota_Y([(1, y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

00B7 **Proposition 6.3.3.1.3.** Let (X, x_0) and (Y, y_0) be pointed sets.

00B8 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*. \end{aligned}$$

00B9 2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$.

00BA 3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\text{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\text{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathbf{Sets}_*$.

00BB 4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$.

00BC 5. *Symmetric Monoidality.* The triple $(\mathbf{Sets}_*, \vee, \text{pt})$ is a symmetric monoidal category.

00BD 6. *The Fold Map.* We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\mathbf{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

Item 5, Symmetric Monoidality: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

$$\begin{array}{ccc} X \vee X & \xrightarrow{\nabla_X} & X \\ f \vee f \downarrow & & \downarrow f \\ Y \vee Y & \xrightarrow{\nabla_Y} & Y. \end{array}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. \square

00BE 6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \rightarrow (X, x_0)$ and $g: (Z, z_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BF Definition 6.3.4.1.1. The **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in \mathbf{Sets}_* as in Limits and Colimits, ??.

0258 Construction 6.3.4.1.2. Concretely, the **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Colimit.* The pointed set $(X \amalg_{f,Z,g} Y, p_0)$, where:
 - The set $X \amalg_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g ;
 - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\text{inj}_1: (X, x_0) \rightarrow (X \amalg_Z Y, p_0),$$

$$\text{inj}_2: (Y, y_0) \rightarrow (X \amalg_Z Y, p_0)$$

given by

$$\text{inj}_1(x) \stackrel{\text{def}}{=} [(0, x)]$$

$$\text{inj}_2(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \amalg_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \amalg_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

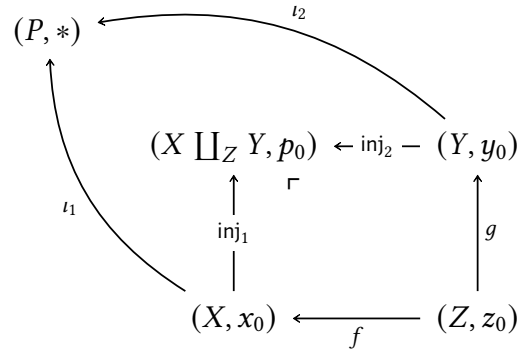
$$\begin{array}{ccc} (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ \text{inj}_1 \uparrow & & \uparrow g \\ (X, x_0) & \xleftarrow{f} & (Z, z_0) \end{array}$$

$\text{inj}_1 \circ f = \text{inj}_2 \circ g,$

Indeed, given $z \in Z$, we have

$$\begin{aligned}
 [\text{inj}_1 \circ f](z) &= \text{inj}_1(f(z)) \\
 &= [(0, f(z))] \\
 &= [(1, g(z))] \\
 &= \text{inj}_2(g(z)) \\
 &= [\text{inj}_2 \circ g](z),
 \end{aligned}$$

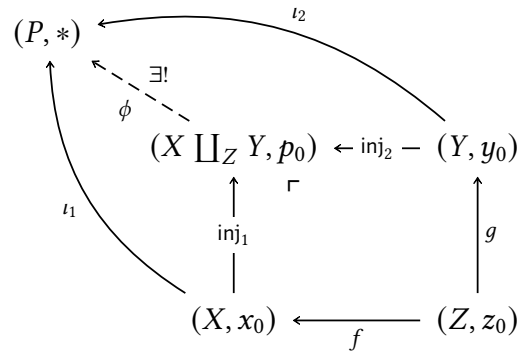
where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation \sim on $X \amalg Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \amalg_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of **Constructions With Sets, Definition 4.2.4.1.1**. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

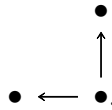
where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. \square

00BG Proposition 6.3.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

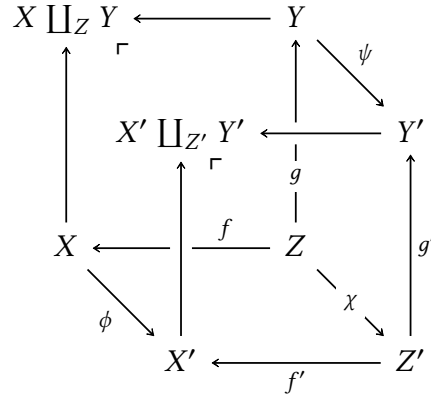
00BH 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \amalg_{f, Z, g} Y$ defines a functor

$$-_1 \amalg_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \amalg_{-3} -1$ is given by sending a morphism



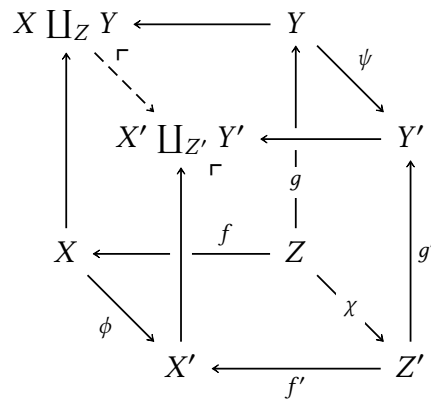
in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi: (X \amalg_Z Y, p_0) \xrightarrow{\exists!} (X' \amalg_{Z'} Y', p'_0)$$

given by

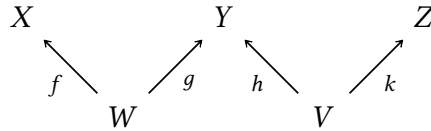
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

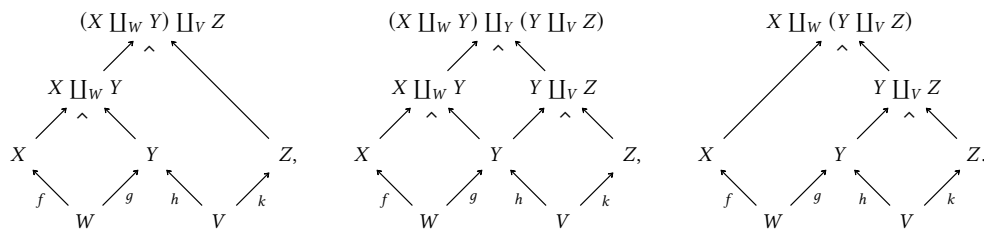
00BJ 2. *Associativity.* Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams



00BK 3. *Unitality.* We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \uparrow f & \lrcorner & \uparrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{c}
 X \amalg_X A \cong A, \\
 A \amalg_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xleftarrow{f} & X \\
 \parallel & \lrcorner & \parallel \\
 X & \xleftarrow{f} & X.
 \end{array}$$

00BL 4. *Commutativity.* We have an isomorphism of sets

$$\begin{array}{ccc}
 X \amalg_Z Y & \xleftarrow{\quad} & Y \\
 \uparrow & \lrcorner & \uparrow g \\
 X & \xleftarrow{f} & Z,
 \end{array}
 \quad
 X \amalg_Z Y \cong Y \amalg_Z X
 \quad
 \begin{array}{ccc}
 Y \amalg_Z X & \xleftarrow{\quad} & X \\
 \uparrow & \lrcorner & \uparrow f \\
 Y & \xleftarrow{g} & Z.
 \end{array}$$

00BM 5. *Interaction With Coproducts.* We have

$$\begin{array}{ccc}
 X \vee Y & \xleftarrow{\quad} & Y \\
 \uparrow & \lrcorner & \uparrow [y_0] \\
 X & \xleftarrow{[x_0]} & \text{pt.}
 \end{array}
 \quad
 X \amalg_{\text{pt}} Y \cong X \vee Y,$$

00BN 6. *Symmetric Monoidality*. The triple $(\mathbf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality**: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from **Constructions With Sets, Item 3 of Definition 4.2.4.1.6**.

Item 3, Unitality: This follows from **Constructions With Sets, Item 5 of Definition 4.2.4.1.6**.

Item 4, Commutativity: This follows from **Constructions With Sets, Item 6 of Definition 4.2.4.1.6**.

Item 5, Interaction With Coproducts: Omitted.

Item 6, Symmetric Monoidality: Omitted. □

00BP 6.3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ **Definition 6.3.5.1.1**. The **coequaliser of (f, g)** is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

0259 **Construction 6.3.5.1.2**. The **coequaliser of (f, g)** is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit*. The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in **Constructions With Sets, Definition 4.2.5.1.1**.
- *The Cocone*. The map

$$\mathrm{coeq}(f, g): Y \twoheadrightarrow (\mathrm{CoEq}(f, g), [y_0])$$

given by the quotient map, as in **Constructions With Sets, Item 2 of Definition 4.2.5.1.2**.

Proof. We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in \mathbf{Sets}_* . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$[\mathrm{coeq}(f, g) \circ f](x) \stackrel{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](f(x))$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\
&\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x)
\end{aligned}$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc}
(X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\
& & \searrow c \\
& & (C, *)
\end{array}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3** that there exists a unique map $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc}
(X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \xrightarrow{\text{coeq}(f, g)} (\text{CoEq}(f, g), [y_0]) \\
& & \searrow c \quad \downarrow \phi \exists! \\
& & (C, *)
\end{array}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}
\phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\
&= c([y_0]) \\
&= *,
\end{aligned}$$

where we have used that c is a morphism of pointed sets. \square

00BR Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BS 1. *Associativity.* We have isomorphisms of pointed sets

$$\begin{aligned}
\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{=\text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} &\cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{=\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},
\end{aligned}$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in Sets_* .

00BT 2. *Unitality*. We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

00BU 3. *Commutativity*. We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

Proof. **Item 1, Associativity**: This follows from **Constructions With Sets, Item 1** of **Definition 4.2.5.1.5**.

Item 2, Unitality: This follows from **Constructions With Sets, Item 4** of **Definition 4.2.5.1.5**.

Item 3, Commutativity: This follows from **Constructions With Sets, Item 5** of **Definition 4.2.5.1.5**. \square

00BV 6.4 Constructions With Pointed Sets

00BW 6.4.1 Free Pointed Sets

Let X be a set.

00BX **Definition 6.4.1.1.1.** The **free pointed set on X** is the pointed set X^+ consisting of:

- *The Underlying Set*. The set X^+ defined by¹³

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \coprod \text{pt} \\ &\stackrel{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint*. The element \star of X^+ .

00BY **Proposition 6.4.1.1.2.** Let X be a set.

¹³*Further Notation*: We sometimes write \star_X for the basepoint of X^+ for clarity, specially when

- 00BZ 1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of [Definition 6.4.1.1.1](#).

- *Action on Morphisms.* For each morphism $f : X \rightarrow Y$ of \mathbf{Sets} , the image

$$f^+ : X^+ \rightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

- 00C0 2. *Adjointness.* We have an adjunction

$$((-)^+ \dashv \mathbf{forget}) : \mathbf{Sets} \begin{array}{c} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\mathbf{forget}} \end{array} \mathbf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathbf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathbf{Sets}(X, Y),$$

natural in $X \in \mathbf{Obj}(\mathbf{Sets})$ and $(Y, y_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$.

- 00C1 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of [Item 1](#) has a symmetric strong monoidal structure

$$((-)^+, (-)^+, \amalg, (-)_{\mathbb{1}}^+ \amalg) : (\mathbf{Sets}, \amalg, \emptyset) \rightarrow (\mathbf{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \amalg : X^+ \vee Y^+ \xrightarrow{\sim} (X \amalg Y)^+,$$

$$(-)_{\mathbb{1}}^+ \amalg : \text{pt} \xrightarrow{\sim} \emptyset^+,$$

natural in $X, Y \in \mathbf{Obj}(\mathbf{Sets})$.

- 00C2 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)_{\mathbb{1}}^+): (\mathbf{Sets}, \times, \text{pt}) \rightarrow (\mathbf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+: X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$

$$(-)_{\mathbb{1}}^+: S^0 \xrightarrow{\sim} \text{pt}^+,$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

Proof. **Item 1**, *Functoriality*: We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \text{Obj}(\mathbf{Sets})$. We have

$$\text{id}_X^+(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\text{id}_X^+ = \text{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$f: X \rightarrow Y,$$

$$g: Y \rightarrow Z,$$

we have

$$\begin{aligned} [g^+ \circ f^+](x) &\stackrel{\text{def}}{=} g^+(f^+(x)) \\ &\stackrel{\text{def}}{=} g^+(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^+(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [g^+ \circ f^+](\star_X) &\stackrel{\text{def}}{=} g^+(f^+(\star_X)) \\ &\stackrel{\text{def}}{=} g^+(\star_Y) \\ &\stackrel{\text{def}}{=} \star_Z \\ &\stackrel{\text{def}}{=} [g \circ f]^+(\star_X), \end{aligned}$$

$$\text{so } (g \circ f)^+ = g^+ \circ f^+.$$

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

025A · *Map I.* We define a map

$$\Phi_{X,Y}: \text{Sets}_*(X^+, Y) \rightarrow \text{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger: X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

025B · *Map II.* We define a map

$$\Psi_{X,Y}: \text{Sets}(X, Y) \rightarrow \text{Sets}_*(X^+, Y)$$

given by sending a function $\xi: X \rightarrow Y$ to the morphism of pointed sets

$$\xi^\dagger: (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

025C · *Invertibility I.* Given a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0),$$

we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^\dagger) \end{aligned}$$

$$\begin{aligned}
&\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\
&= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\
&= \xi \\
&\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}_*(X^+, Y)}](\xi).
\end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}_*(X^+, Y)}.$$

025D · *Invertibility II.* Given a map of sets $\xi: X \rightarrow Y$, we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\
&= \Phi_{X,Y}(\xi^\dagger) \\
&= \Phi_{X,Y}\left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket\right) \\
&= \llbracket x \mapsto \xi(x) \rrbracket \\
&= \xi \\
&\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(X, Y)}](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X, Y)}.$$

025E · *Naturality for Φ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc}
\text{Sets}_*(X'^+, Y) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\
f^* \downarrow & & \downarrow f^* \\
\text{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y)
\end{array}$$

commutes. Indeed, given a morphism of pointed sets $\xi: X'^+ \rightarrow Y$, we have

$$\begin{aligned}
 [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
 &= \Phi_{X,Y}(\xi \circ f) \\
 &= \xi \circ f \\
 &= \Phi_{X',Y}(\xi) \circ f \\
 &= f^*(\Phi_{X',Y}(\xi)) \\
 &= f^*([\Phi_{X',Y}](\xi)) \\
 &= [f^* \circ \Phi_{X',Y}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

- 025F · *Naturality for Φ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
 \text{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y) \\
 g_* \downarrow & & \downarrow g_* \\
 \text{Sets}_*(X^+, Y') & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}(X, Y')
 \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger: X^+ \rightarrow Y,$$

we have

$$\begin{aligned}
 [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
 &= \Phi_{X,Y'}(g \circ \xi) \\
 &= g \circ \xi
 \end{aligned}$$

there are multiple free pointed sets involved in the current discussion.

$$\begin{aligned}
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*(\Phi_{X,Y'}(\xi)) \\
&= [g_* \circ \Phi_{X,Y'}](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- 025G · *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2 of Definition 11.9.7.1.2](#) that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to \amalg and \vee as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{+, \amalg} : X^+ \vee Y^+ \xrightarrow{\sim} (X \amalg Y)^+$$

is given by

$$(-)_{X,Y}^{+, \amalg}(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_{X \amalg Y} & \text{if } z = [(0, \star_X)], \\ \star_{X \amalg Y} & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+, \amalg, -1} : (X \amalg Y)^+ \xrightarrow{\sim} X^+ \vee Y^+$$

given by

$$(-)_{X,Y}^{+, \amalg, -1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(1, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_{X \amalg Y} \end{cases}$$

for each $z \in (X \amalg Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+, \coprod, \mathbb{1}}: \text{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_\emptyset .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+: X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)_{X,Y}^{+,-1}: (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+, \mathbb{1}}: S^0 \xrightarrow{\sim} \text{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $\text{pt}^+ = \{\star, \star_{\text{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. \square

01QS 6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

01QT **Definition 6.4.2.1.1.** The **set with deleted basepoint associated to X** is the set X^- defined by

$$X^- \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

01QU **Proposition 6.4.2.1.2.** Let (X, x_0) be a pointed set.

01QV 1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- : \mathbf{Sets}_*^{\text{actv}} \rightarrow \mathbf{Sets},$$

where:

· *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{Sets}_*^{\text{actv}})$, we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where X^- is the set of **Definition 6.4.2.1.1**.

· *Action on Morphisms.* For each morphism $f : X \rightarrow Y$ of $\mathbf{Sets}_*^{\text{actv}}$, the image

$$f^- : X^- \rightarrow Y^-$$

of f by $(-)^-$ is the map defined by

$$f^-(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

01QW 2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+) : \mathbf{Sets}_*^{\text{actv}} \begin{array}{c} \xrightarrow{(-)^-} \\ \text{eq} \\ \xleftarrow{(-)^+} \end{array} \mathbf{Sets},$$

witnessed by a bijection of sets

$$\mathbf{Sets}(X^-, Y) \cong \mathbf{Sets}_*(X, Y^+),$$

natural in $X \in \mathbf{Obj}(\mathbf{Sets}_*)$ and $Y \in \mathbf{Obj}(\mathbf{Sets})$, and by isomorphisms

$$(X^-)^+ \cong X,$$

$$(Y^+)^- \cong Y,$$

once again natural in $X \in \mathbf{Obj}(\mathbf{Sets}_*)$ and $Y \in \mathbf{Obj}(\mathbf{Sets})$.

- 01QX 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^-, (-)^{-, \vee}, (-)^{-, \vee}_{\mathbb{1}}): (\mathbf{Sets}_*^{\text{actv}}, \vee, \text{pt}) \rightarrow (\mathbf{Sets}, \coprod, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$(-)^{-, \vee}_{X, Y}: X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-,$$

$$(-)^{-, \vee}_{\mathbb{1}}: \emptyset \xrightarrow{\sim} \text{pt}^-,$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

- 01QY 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^-, (-)^{-, \times}, (-)^{-, \times}_{\mathbb{1}}): (\mathbf{Sets}_*^{\text{actv}}, \wedge, S^0) \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)^{-, \times}_{X, Y}: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-,$$

$$(-)^{-, \times}_{\mathbb{1}}: \text{pt} \xrightarrow{\sim} (S^0)^-,$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

Proof. **Item 1**, *Functoriality*: We claim that $(-)^-$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \text{Obj}(\mathbf{Sets})$. We have

$$\text{id}_X^-(x) \stackrel{\text{def}}{=} x$$

for each $x \in X^-$, so $\text{id}_X^- = \text{id}_{X^-}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f: (X, x_0) \rightarrow (Y, y_0),$$

$$g: (Y, y_0) \rightarrow (Z, z_0),$$

we have

$$\begin{aligned} [g^- \circ f^-](x) &\stackrel{\text{def}}{=} g^-(f^-(x)) \\ &\stackrel{\text{def}}{=} g^-(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^-(x) \end{aligned}$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2, Adjoint Equivalence: We proceed in a few steps:

025H 1. *Map I.* We define a map

$$\Phi_{X,Y}: \text{Sets}(X^-, Y) \rightarrow \text{Sets}_*^{\text{actv}}(X, Y^+)$$

by sending a map $\xi: X^- \rightarrow Y$ to the active morphism of pointed sets

$$\xi^\dagger: X \rightarrow Y^+$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

025J 2. *Map II.* We define a map

$$\Psi_{X,Y}: \text{Sets}_*^{\text{actv}}(X, Y^+) \rightarrow \text{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi: X \rightarrow Y^+$ to the map

$$\xi^\dagger: X^- \rightarrow Y$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

025K 3. *Invertibility I.* Given a map of sets $\xi: X^- \rightarrow Y$, we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}\left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket\right) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}(X^-, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}(X^-, Y)}.$$

025L 4. *Invertibility II.* Given a morphism of pointed sets

$$\xi: (X, x_0) \rightarrow (Y^+, \star_Y),$$

we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}.$$

025M 5. *Naturality for Φ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}(X'^-, Y) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}_*^{\text{actv}}(X', Y^+) \\ f^* \downarrow & & \downarrow f^* \\ \text{Sets}_*(X^-, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}_*^{\text{actv}}(X, Y^+) \end{array}$$

commutes. Indeed, given a map of sets $\xi: X' \rightarrow Y$, we have

$$\begin{aligned} [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \llbracket x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^- \\ \star_Y & \text{if } f(x) = x'_0 \end{cases} \rrbracket \\ &= f^* \left(\llbracket x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^- \\ \star_Y & \text{if } x' = x'_0 \end{cases} \rrbracket \right) \end{aligned}$$

$$\begin{aligned}
&= f^*(\Phi_{X',Y}(\xi)) \\
&= [f^* \circ \Phi_{X',Y}](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

025N 6. *Naturality for Φ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
\text{Sets}(X^-, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}_*^{\text{actv}}(X, Y^+) \\
\downarrow g_* & & \downarrow g_* \\
\text{Sets}(X^-, Y') & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}_*^{\text{actv}}(X, Y'^+)
\end{array}$$

commutes. Indeed, given a map of sets $\xi: X^- \rightarrow Y$, we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= \llbracket x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \rrbracket \\
&= g_* \left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \right) \\
&= g_*(\Phi_{X,Y}(\xi)) \\
&= [g_* \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y},$$

and the naturality diagram for Φ above indeed commutes.

- 025P 7. *Naturality for Ψ* . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2 of Definition 11.9.7.1.2](#) that Ψ is also natural in each argument.
- 025Q 8. *Fully Faithfulness of $(-)^-$* . We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{X,Y}^- : \text{Sets}_*^{\text{actv}}(X, Y) \xrightarrow{\sim} \text{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)_{X,Y}^{-,-1} : \text{Sets}(X^-, Y^-) \xrightarrow{\sim} \text{Sets}_*^{\text{actv}}(X, Y)$$

is given by sending a map of sets $f : X^- \rightarrow Y^-$ to the active morphism of pointed sets $f^\dagger : X \rightarrow Y$ defined by

$$f^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

- 025R 9. *Essential Surjectivity of $(-)^-$* . We need to show that, given an object $X \in \text{Obj}(\text{Sets})$, there exists some $X' \in \text{Obj}(\text{Sets}_*^{\text{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$\begin{aligned} (X^+)^- &\stackrel{\text{def}}{=} (X \cup \{\star_X\})^- \\ &\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

- 025S 10. *The Functor $(-)^-$ Is an Equivalence*. Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by [Categories, Item 1 of Definition 11.6.7.1.2](#).

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^-$ with respect to \vee and \coprod as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{-,\vee}: X^- \amalg Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0, x)] & \text{if } z = (0, x) \text{ with } x \in X, \\ [(1, y)] & \text{if } z = (1, y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \amalg Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1}: (X \vee Y)^- \xrightarrow{\sim} X^- \amalg Y^-$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = [(0, x)], \\ (1, y) & \text{if } z = [(1, y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\vee,\mathbb{1}}: \emptyset \xrightarrow{\sim} \text{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

Item 4. *Symmetric Strong Monoidality With Respect to Smash Products:* We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^-: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)_{X,Y}^-(x, y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1}: (X \wedge Y)^- \xrightarrow{\sim} X^- \times Y^-$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x, y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{-,1}_{X,Y} : \text{pt} \xrightarrow{\sim} (S^0)^{-}$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. \square

Appendices

A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations

10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

References

- [MSE 2855868] **Qiaochu Yuan**. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on pp. 7, 15).
- [MSE 884460] **Martin Brandenburg**. *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 7).