Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

Contents

13.1 Moduli Categories of Monoidal Structures

13.1.1 The Moduli Category of Monoidal Structures on a Category

Let $\mathcal{C}$ be a category.

DEFINITION 13.1.1.1.1 ▶ THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

The **moduli category of monoidal structures on $\mathcal{C}$** is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) & \longrightarrow & \mathsf{MonCats} \\ \downarrow & \lrcorner & \downarrow {\scriptstyle 忘} \\ \mathsf{pt} & \xrightarrow[{[\mathcal{C}]}]{} & \mathsf{Cats}. \end{array}$$

REMARK 13.1.1.1.2 ▶ UNWINDING DEFINITION 13.1.1.1.1, I

In detail, **the moduli category of monoidal structures on $\mathcal{C}$** is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ are monoidal categories $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ whose underlying category is $\mathcal{C}$.

- *Morphisms.* A morphism from $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ to $\left(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime}\right)$ is a strong monoidal functor structure

$$\mathrm{id}_{\mathcal{C}}^{\otimes} \colon A \boxtimes_{\mathcal{C}} B \dashrightarrow A \otimes_{\mathcal{C}} B,$$
$$\mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes} \colon \mathbb{1}'_{\mathcal{C}} \dashrightarrow \mathbb{1}_{\mathcal{C}}$$

  on the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Identities.* For each $M \overset{\text{def}}{=} \left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right) \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} \colon \mathsf{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, M)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at M is defined by

$$\mathrm{id}_M^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} \overset{\text{def}}{=} \left(\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right),$$

where $\left(\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)$ is the identity monoidal functor of $\mathcal{C}$ of **??**.

- *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} \colon \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(N, P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, P)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at (M, N, P) is defined by

$$\left(\mathrm{id}_{\mathcal{C}}^{\otimes,\prime}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})} \left(\mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right) \overset{\text{def}}{=} \left(\mathrm{id}_{\mathcal{C}}^{\otimes,\prime} \circ \mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime} \circ \mathrm{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right).$$

REMARK 13.1.1.1.3 ▶ UNWINDING DEFINITION 13.1.1.1.1, II

In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ to $\left(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime}\right)$ satisfies the following conditions:

1. *Naturality.* For each pair $f\colon A \to X$ and $g\colon B \to Y$ of morphisms of $\mathcal{C}$, the diagram

$$
\begin{array}{ccc}
A \boxtimes_{\mathcal{C}} B & \xrightarrow{f \boxtimes_{\mathcal{C}} g} & X \boxtimes_{\mathcal{C}} Y \\
{\scriptstyle \mathrm{id}_{A,B}^{\otimes}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{X,Y}^{\otimes}} \\
A \otimes_{\mathcal{C}} B & \xrightarrow[f \otimes_{\mathcal{C}} g]{} & X \otimes_{\mathcal{C}} Y
\end{array}
$$

commutes.

2. *Monoidality.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$(A \boxtimes_C B) \boxtimes_C C$$

$$\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C \qquad \alpha^{C,\prime}_{A,B,C}$$

$$(A \otimes_C B) \boxtimes_C C \qquad\qquad A \boxtimes_C (B \boxtimes_C C)$$

$$\mathrm{id}^{\otimes}_{A\otimes_C B,C} \qquad\qquad\qquad \mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}$$

$$(A \otimes_C B) \otimes_C C \qquad\qquad A \boxtimes_C (B \otimes_C C)$$

$$\alpha^C_{A,B,C} \qquad\qquad \mathrm{id}^{\otimes}_{A,B\otimes_C C}$$

$$A \otimes_C (B \otimes_C C)$$

commutes.

3. *Left Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\mathbb{1}_C \boxtimes_C A \xrightarrow{\ \mathrm{id}^{\otimes}_{\mathbb{1}'_C,A}\ } \mathbb{1}_C \otimes_C A$$

$$\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_C \mathrm{id}_A \qquad\qquad\qquad \lambda^C_A$$

$$\mathbb{1}'_C \boxtimes_C A \xrightarrow[\ \lambda^{C,\prime}_A\ ]{} A$$

commutes.

4. *Right Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$A \boxtimes_C \mathbb{1}_C \xrightarrow{\ \mathrm{id}^{\otimes}_{A,\mathbb{1}'_C}\ } A \otimes_C \mathbb{1}_C$$

$$\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}} \qquad\qquad\qquad \rho^C_A$$

$$A \boxtimes_C \mathbb{1}'_C \xrightarrow[\ \rho^{C,\prime}_A\ ]{} A$$

commutes.

<div style="background:#7a1515;color:white;padding:4px">

PROPOSITION 13.1.1.1.4 ▶ PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY

</div>

Let C be a category.

1. *Extra Monoidality Conditions.* Let $(\mathrm{id}_C^\otimes, \mathrm{id}_{\mathbb{1}|C}^\otimes)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

   (a) The diagram

$$
\begin{array}{ccc}
(A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\ \mathrm{id}^\otimes_{A,B}\,\boxtimes_C\,\mathrm{id}_C\ } & (A \otimes_C B) \boxtimes_C C \\[4pt]
{\scriptstyle \mathrm{id}^\otimes_{A\boxtimes_C B,C}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}^\otimes_{A\otimes_C B,C}} \\[4pt]
(A \boxtimes_C B) \otimes_C C & \xrightarrow[\ \mathrm{id}^\otimes_{A,B}\,\otimes_C\,\mathrm{id}_C\ ]{} & (A \otimes_C B) \otimes_C C
\end{array}
$$

   commutes.

   (b) The diagram

$$
\begin{array}{ccc}
A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\ \mathrm{id}_A\,\boxtimes_C\,\mathrm{id}^\otimes_{B,C}\ } & A \boxtimes_C (B \otimes_C C) \\[4pt]
{\scriptstyle \mathrm{id}^\otimes_{A,B\boxtimes_C C}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}^\otimes_{A,B\otimes_C C}} \\[4pt]
A \otimes_C (B \boxtimes_C C) & \xrightarrow[\ \mathrm{id}_A\,\otimes_C\,\mathrm{id}^\otimes_{B,C}\ ]{} & A \otimes_C (B \otimes_C C)
\end{array}
$$

   commutes.

2. *Extra Monoidal Unity Constraints.* Let $(\mathrm{id}_C^\otimes, \mathrm{id}_{\mathbb{1}|C}^\otimes)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

   (a) The diagram

$$
\begin{array}{ccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\ \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}\ } & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \\[4pt]
{\scriptstyle \lambda^C_{\mathbb{1}'_C}}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho^{C,\prime}_{\mathbb{1}_C}} \\[4pt]
\mathbb{1}'_C & \xrightarrow[\ \mathrm{id}^\otimes_{\mathbb{1}}\ ]{} & \mathbb{1}_C
\end{array}
$$

commutes.

(b) The diagram

$$
\begin{array}{ccc}
\mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\
{\scriptstyle \rho^C_{\mathbb{1}'_C}} \Big\downarrow & & \Big\downarrow {\scriptstyle \lambda^{C,'}_{\mathbb{1}_C}} \\
\mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}}} & \mathbb{1}_C
\end{array}
$$

commutes.

(c) The diagram

$$
\begin{array}{ccc}
\mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
{\scriptstyle \lambda^{C,'}_{\mathbb{1}_C}} \Big\downarrow & & \Big\downarrow {\scriptstyle \rho^C_{\mathbb{1}'_C}} \\
\mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}}} & \mathbb{1}'_C
\end{array}
$$

commutes.

(d) The diagram

$$
\begin{array}{ccc}
\mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
{\scriptstyle \rho^{C,'}_{\mathbb{1}_C}} \Big\downarrow & & \Big\downarrow {\scriptstyle \lambda^C_{\mathbb{1}'_C}} \\
\mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}}} & \mathbb{1}'_C
\end{array}
$$

commutes.

3. *Mixed Associators.* Let $\left(C,\, \otimes_C,\, \mathbb{1}_C,\, \alpha^C,\, \lambda^C,\, \rho^C\right)$ and $\left(C,\, \boxtimes_C,\, \mathbb{1}'_C,\right.$ $\left. \alpha^{C,'},\, \lambda^{C,'},\, \rho^{C,'}\right)$ be monoidal structures on C and let

$$
\mathrm{id}^{\otimes}_{-_1,-_2}\colon\ -_1 \boxtimes_C -_2 \to -_1 \otimes_C -_2
$$

be a natural transformation.

(a) If there exists a natural transformation

$$\alpha^{\otimes}_{A,B,C} \colon (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$
\begin{array}{ccc}
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow{\alpha^{\otimes}_{A,B,C}} & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}} \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow[\alpha^{C}_{A,B,C}]{} & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{array}
$$

and

$$
\begin{array}{ccc}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow{\alpha^{C,\prime}_{A,B,C}} & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{A,B \boxtimes_{\mathcal{C}} C}} \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow[\alpha^{\otimes}_{A,B,C}]{} & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)
\end{array}
$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(b) If there exists a natural transformation

$$\alpha^{\boxtimes}_{A,B,C} \colon (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

making the diagrams

$$
\begin{array}{ccc}
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow{\alpha^{\boxtimes}_{A,B,C}} & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C} \downarrow & & \downarrow {\scriptstyle \mathrm{id}^{\otimes}_{A,B \otimes_{\mathcal{C}} C}} \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow[\alpha^{C}_{A,B,C}]{} & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{array}
$$

and

$$
\begin{array}{ccc}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow{\alpha^{C,\prime}_{A,B,C}} & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A \boxtimes_{\mathcal{C}} B, C}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}} \\
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow[\alpha^{\boxtimes}_{A,B,C}]{} & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{array}
$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha^{\boxtimes,\otimes}_{A,B,C} \colon (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

making the diagrams

$$
\begin{array}{ccc}
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow{\alpha^{\boxtimes,\otimes}_{A,B,C}} & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}} \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow[\alpha^{\mathcal{C}}_{A,B,C}]{} & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{array}
$$

and

$$
\begin{array}{ccc}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C & \xrightarrow{\alpha^{\mathcal{C},\prime}_{A,B,C}} & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \\
{\scriptstyle \mathrm{id}^{\otimes}_{A\boxtimes_{\mathcal{C}} B,C}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}^{\otimes}_{A,B\boxtimes_{\mathcal{C}} C}} \\
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C & \xrightarrow[\alpha^{\boxtimes,\otimes}_{A,B,C}]{} & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)
\end{array}
$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

PROOF 13.1.1.1.5 ▶ PROOF OF PROPOSITION 13.1.1.1.4

Item 1: Extra Monoidality Conditions

We claim that Items 1a and 1b are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}^{\otimes}_{A,B}$ and id_C.

2. *Proof of Item 1b:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms id_A and $\mathrm{id}^{\otimes}_{B,C}$.
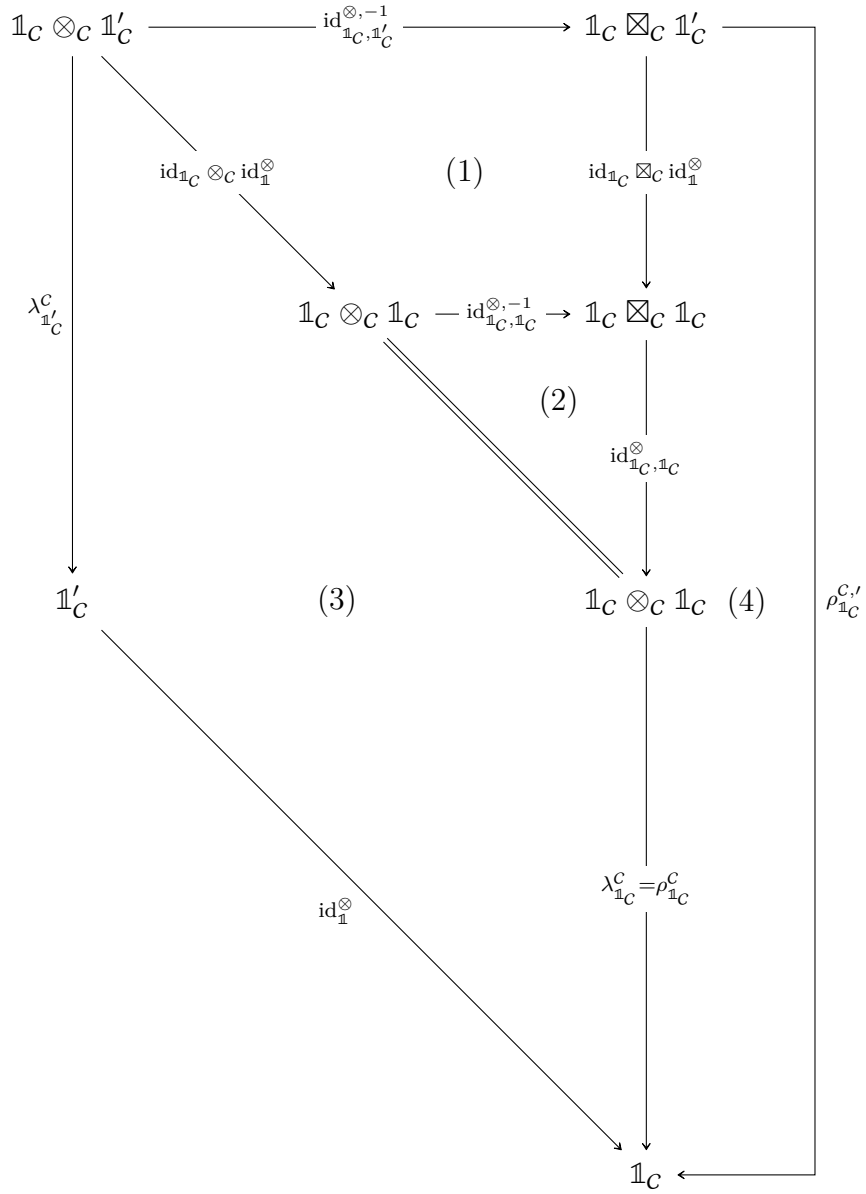
This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:

1. *Proof of Item 1a:* Indeed, consider the diagram

$$\mathbb{1}_C \otimes_C \mathbb{1}'_C \xrightarrow{\quad \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C} \quad} \mathbb{1}_C \boxtimes_C \mathbb{1}'_C$$

$\mathrm{id}_{\mathbb{1}_C} \otimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}$ (1) $\mathrm{id}_{\mathbb{1}_C} \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}$

$\lambda^C_{\mathbb{1}'_C}$

$$\mathbb{1}_C \otimes_C \mathbb{1}_C \xrightarrow{\quad \mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C} \quad} \mathbb{1}_C \boxtimes_C \mathbb{1}_C$$

(2)

$\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}$

$\mathbb{1}'_C$ (3) $\mathbb{1}_C \otimes_C \mathbb{1}_C$ (4) $\rho^{C,'}_{\mathbb{1}_C}$

$\mathrm{id}^{\otimes}_{\mathbb{1}}$

$\lambda^C_{\mathbb{1}_C} = \rho^C_{\mathbb{1}_C}$

$$\mathbb{1}_C \longleftarrow$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_{\mathcal{C}}^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathcal{C}}$, where the equality $\rho^{\mathcal{C}}_{\mathbb{1}_C} = \lambda^{\mathcal{C}}_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $\left(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}^{\otimes}, \mathrm{id}_{\mathcal{C}|\mathbb{1}}^{\otimes}\right)$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram

$$
\begin{array}{ccc}
\mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\;\mathrm{id}^{\otimes,-1}_{\mathbb{1}'_C,\mathbb{1}_C}\;} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C
\end{array}
$$

with labels $\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_C \mathrm{id}_{\mathbb{1}_C}$, (1), $\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_C \mathrm{id}_{\mathbb{1}_C}$

$\rho^C_{\mathbb{1}'_C}$

$$\mathbb{1}_C \otimes_C \mathbb{1}_C \;\xrightarrow{\;\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}\;}\; \mathbb{1}_C \boxtimes_C \mathbb{1}_C$$

(2)

$\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}$

$$\mathbb{1}'_C \qquad (3) \qquad \mathbb{1}_C \otimes_C \mathbb{1}_C \;\; (4) \qquad \lambda^{C,\prime}_{\mathbb{1}_C}$$

$\mathrm{id}^{\otimes}_{\mathbb{1}}$

$\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$

$$\mathbb{1}_C \longleftarrow$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^C, where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $\left(\mathrm{id}_C, \mathrm{id}_C^{\otimes}, \mathrm{id}_{C|\mathbb{1}}^{\otimes}\right)$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$
\begin{array}{ccccc}
\mathbb{1}_C' \otimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}} & \mathbb{1}_C' \boxtimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes}} & \mathbb{1}_C' \otimes_C \mathbb{1}_C \\
\Big\downarrow{\rho_{\mathbb{1}_C'}^C} & (1) & \Big\downarrow{\lambda_{\mathbb{1}_C}^{C,'}} & (\dagger) & \Big\downarrow{\rho_{\mathbb{1}_C'}^C} \\
\mathbb{1}_C' & \xrightarrow{\mathrm{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}}^{\otimes,-1}} & \mathbb{1}_C'.
\end{array}
$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$
\begin{array}{ccc}
\mathbb{1}_C' \otimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}} & \mathbb{1}_C' \boxtimes_C \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes}} & \mathbb{1}_C' \otimes_C \mathbb{1}_C \\
& & \Big\downarrow{\lambda_{\mathbb{1}_C}^{C,'}} & (\dagger) & \Big\downarrow{\rho_{\mathbb{1}_C'}^C} \\
& & \mathbb{1}_C & \xrightarrow{\mathrm{id}_{\mathbb{1}}^{\otimes,-1}} & \mathbb{1}_C'
\end{array}
$$

commutes. But since $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}_C'}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$
\begin{array}{ccccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
\Big\downarrow{\lambda^C_{\mathbb{1}'_C}} & (1) & \Big\downarrow{\rho^{C,'}_{\mathbb{1}_C}} & (\dagger) & \Big\downarrow{\lambda^C_{\mathbb{1}'_C}} \\
\mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes}_{\mathbb{1}}} & \mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}}} & \mathbb{1}_C
\end{array}
$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$
\begin{array}{ccccc}
\mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
& & \Big\downarrow{\rho^{C,'}_{\mathbb{1}_C}} & (\dagger) & \Big\downarrow{\lambda^C_{\mathbb{1}'_C}} \\
& & \mathbb{1}_C & \xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}}} & \mathbb{1}_C
\end{array}
$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}}$ is an isomorphism, it follows that the diagram $(\dagger)$ also commutes, and we are done.

This finishes the proof.

Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

1. *Proof of Item 3a:* We may partition the monoidality diagram for

$\mathrm{id}^{\otimes}$ of Item 2 of Remark 13.1.1.1.3 as follows:

$$
\begin{array}{c}
(A \boxtimes_C B) \boxtimes_C C
\end{array}
$$

with maps:
- $\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C$
- $\alpha^C_{A,B,C}$
- $(A \otimes_C B) \boxtimes_C C \quad \mathrm{id}^{\otimes}_{A \boxtimes_C B, C} \quad A \boxtimes_C (B \boxtimes_C C)$
- (1), (2)
- $\mathrm{id}^{\otimes}_{A \otimes_C B, C}$, $(A \boxtimes_C B) \otimes_C C$, $\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}$
- $\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C$, $\alpha^{\otimes}_{A,B,C}$
- $(A \otimes_C B) \otimes_C C \quad (3) \quad A \boxtimes_C (B \otimes_C C)$
- $\alpha^{C,\prime}_{A,B,C}$, $\mathrm{id}^{\otimes}_{A,B \otimes_C C}$
- $A \otimes_C (B \otimes_C C)$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

2. *Proof of Item 3b:* We may partition the monoidality diagram for

$\mathrm{id}^{\otimes}$ of Item 2 of Remark 13.1.1.1.3 as follows:

$$(A \boxtimes_C B) \boxtimes_C C$$

$\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C$ $\qquad\qquad$ $\alpha^C_{A,B,C}$

$$(A \otimes_C B) \boxtimes_C C \qquad (1) \qquad A \boxtimes_C (B \boxtimes_C C)$$

$\alpha^{\boxtimes}_{A,B,C}$ $\qquad\qquad$ $\mathrm{id}^{\otimes}_{A, B\boxtimes_C C}$

$\mathrm{id}^{\otimes}_{A\otimes_C B, C}$ $\qquad\qquad$ $A \otimes_C (B \boxtimes_C C)$ $\qquad\qquad$ $\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}$

$$(2) \qquad\qquad (3)$$

$$(A \otimes_C B) \otimes_C C \quad \mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C} \quad A \boxtimes_C (B \otimes_C C)$$

$\alpha^{C,\prime}_{A,B,C}$ $\qquad\qquad$ $\mathrm{id}^{\otimes}_{A, B\otimes_C C}$

$$A \otimes_C (B \otimes_C C)$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

3. *Proof of Item 3c:* We may partition the monoidality diagram for

id$^\otimes$ of Item 2 of Remark 13.1.1.1.3 as follows:

$$(A \boxtimes_C B) \boxtimes_C C$$

$$\mathrm{id}^\otimes_{A,B} \boxtimes_C \mathrm{id}_C \qquad \alpha^C_{A,B,C}$$

$$(A \otimes_C B) \boxtimes_C C \qquad\qquad A \boxtimes_C (B \boxtimes_C C)$$

$$(1)$$

$$\mathrm{id}^\otimes_{A \otimes_C B, C} \qquad \alpha^{\boxtimes,\otimes}_{A,B,C} \qquad \mathrm{id}_A \boxtimes_C \mathrm{id}^\otimes_{B,C}$$

$$(A \otimes_C B) \otimes_C C \qquad (2) \qquad A \boxtimes_C (B \otimes_C C)$$

$$\alpha^{C,\prime}_{A,B,C} \qquad \mathrm{id}^\otimes_{A, B \otimes_C C}$$

$$A \otimes_C (B \otimes_C C)$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id$^\otimes$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

13.1.2   The Moduli Category of Braided Monoidal Structures on a Category

13.1.3   The Moduli Category of Symmetric Monoidal Structures on a Category

13.2   Moduli Categories of Closed Monoidal Structures

13.3   Moduli Categories of Refinements of Monoidal Structures

13.3.1   The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A   Other Chapters