

Conditions on Relations

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This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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10.1 Functional and Total Relations

10.1.1 Functional Relations

Let A and B be sets.

Definition 10.1.1.1.1. A relation $R: A \rightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 10.1.1.1.2. Let $R: A \rightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

- (a) The relation R is functional.
- (b) We have $R \diamond R^\dagger \subset \chi_B$.

Proof. Item 1, Characterisations: We claim that **Items 1a** and **1b** are indeed equivalent:

- **Item 1a** \implies **Item 1b**: Let $(b, b') \in B \times B$. We need to show that

$$[R \diamond R^\dagger](b, b') \preceq_{\{t, f\}} \chi_B(b, b'),$$

i.e. that if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- **Item 1b** \implies **Item 1a**: Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:
 - Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.
 - Since $R \diamond R^\dagger \subset \chi_B$, we have

$$[R \diamond R^\dagger](b, b') \preceq_{\{t, f\}} \chi_B(b, b'),$$

and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $[R \diamond R^\dagger](b, b') = \text{true}$, and thus $\chi_B(b, b') = \text{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

10.1.2 Total Relations

Let A and B be sets.

Definition 10.1.2.1.1. A relation $R: A \rightarrow B$ is **total** if, for each $a \in A$, we have

$R(a) \neq \emptyset$.

Proposition 10.1.2.1.2. Let $R: A \rightarrowtail B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

- (a) The relation R is total.
- (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. **Item 1**, *Characterisations:* We claim that **Items 1a** and **1b** are indeed equivalent:

- **Item 1a** \implies **Item 1b**: We have to show that, for each $(a, a') \in A$, we have

$$\chi_A(a, a') \preceq_{\{t, f\}} [R^\dagger \diamond R](a, a'),$$

i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R .

- **Item 1b** \implies **Item 1a**: Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have

$$\{a\} \subset [R^\dagger \diamond R](a),$$

implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

10.2 Reflexive Relations

10.2.1 Foundations

Let A be a set.

Definition 10.2.1.1.1. A **reflexive relation** is equivalently:¹

- An \mathbb{E}_0 -monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

Remark 10.2.1.1.2. In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R: \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

¹Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra

Definition 10.2.1.1.3. Let A be a set.

1. The **set of reflexive relations on A** is the subset $\text{Rel}^{\text{refl}}(A, A)$ of $\text{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on A** is the subposet $\mathbf{Rel}^{\text{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 10.2.1.1.4. Let R and S be relations on A .

1. *Interaction With Inverses.* If R is reflexive, then so is R^\dagger .
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. **Item 1**, *Interaction With Inverses*: Clear.

Item 2, *Interaction With Composition*: Clear. □

10.2.2 The Reflexive Closure of a Relation

Let R be a relation on A .

Definition 10.2.2.1.1. The **reflexive closure** of \sim_R is the relation \sim_R^{refl} ² satisfying the following universal property:³

- (\star) Given another reflexive relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{refl}} \subset \sim_S$.

Construction 10.2.2.1.2. Concretely, \sim_R^{refl} is the free pointed object on R in $(\text{Rel}(A, A), \chi_A)$ ⁴, being given by

$$\begin{aligned} R^{\text{refl}} &\stackrel{\text{def}}{=} R \coprod^{\text{Rel}(A, A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

Proof. Clear. □

Proposition 10.2.2.1.3. Let R be a relation on A .

structure.

²*Further Notation:* Also written R^{refl} .

³*Slogan:* The reflexive closure of R is the smallest reflexive relation containing R .

⁴Or, equivalently, the free \mathbb{E}_0 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{refl}} \dashv \overset{\circ}{\text{refl}} \right): \mathbf{Rel}(A, A) \overset{(-)^{\text{refl}}}{\underset{\overset{\circ}{\text{refl}}}{\rightleftarrows}} \mathbf{Rel}^{\text{refl}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{refl}}(R^{\text{refl}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{refl}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\text{refl}} = R$.

3. *Idempotency.* We have

$$(R^{\text{refl}})^{\text{refl}} = R^{\text{refl}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger} \right)^{\text{refl}} = \left(R^{\text{refl}} \right)^{\dagger}, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \mathbf{Rel}(A, A) \\ (-)^{\dagger} \downarrow & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \mathbf{Rel}(A, A). \end{array}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{refl}} = S^{\text{refl}} \diamond R^{\text{refl}}, \quad \begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ (-)^{\text{refl}} \times (-)^{\text{refl}} \downarrow & & \downarrow (-)^{\text{refl}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A). \end{array}$$

Proof. **Item 1, Adjointness:** This is a rephrasing of the universal property of the reflexive closure of a relation, stated in **Definition 10.2.2.1.1**.

Item 2, The Reflexive Closure of a Reflexive Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**.

Item 4, Interaction With Inverses: Clear.

Item 5, Interaction With Composition: This follows from **Item 2** of **Definition 10.2.1.1.4**.

□

10.3 Symmetric Relations

10.3.1 Foundations

Let A be a set.

Definition 10.3.1.1.1. A relation R on A is **symmetric** if we have $R^\dagger = R$.

Remark 10.3.1.1.2. In detail, a relation R is symmetric if it satisfies the following condition:

(★) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

Definition 10.3.1.1.3. Let A be a set.

1. The **set of symmetric relations on A** is the subset $\text{Rel}^{\text{symm}}(A, A)$ of $\text{Rel}(A, A)$ spanned by the symmetric relations.
2. The **poset of relations on A** is the subposet $\mathbf{Rel}^{\text{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 10.3.1.1.4. Let R and S be relations on A .

1. *Interaction With Inverses.* If R is symmetric, then so is R^\dagger .
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. **Item 1**, *Interaction With Inverses*: Clear.

Item 2, *Interaction With Composition*: Clear. □

10.3.2 The Symmetric Closure of a Relation

Let R be a relation on A .

Definition 10.3.2.1.1. The **symmetric closure** of \sim_R is the relation \sim_R^{symm} ⁵ satisfying the following universal property:⁶

(★) Given another symmetric relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{symm}} \subset \sim_S$.

Construction 10.3.2.1.2. Concretely, \sim_R^{symm} is the symmetric relation on A defined by

$$\begin{aligned} R^{\text{symm}} &\stackrel{\text{def}}{=} R \cup R^\dagger \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

⁵*Further Notation:* Also written R^{symm} .

⁶*Slogan:* The symmetric closure of R is the smallest symmetric relation containing R .

Proof. Clear. □

Proposition 10.3.2.1.3. Let R be a relation on A .

1. *Adjointness.* We have an adjunction

$$((-)^{\text{symm}} \dashv \overset{\circ}{\text{symm}}): \mathbf{Rel}(A, A) \overset{(-)^{\text{symm}}}{\underset{\overset{\circ}{\text{symm}}}{\rightleftarrows}} \mathbf{Rel}^{\text{symm}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{symm}}(R^{\text{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{symm}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\text{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\text{symm}})^{\text{symm}} = R^{\text{symm}}.$$

4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A) \\ \downarrow (-)^{\dagger} & & \downarrow (-)^{\dagger} \\ \mathbf{Rel}(A, A) & \xrightarrow{(-)^{\text{symm}}} & \mathbf{Rel}(A, A). \end{array} \quad (R^{\dagger})^{\text{symm}} = (R^{\text{symm}})^{\dagger},$$

5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A) \\ \downarrow (-)^{\text{symm}} \times (-)^{\text{symm}} & & \downarrow (-)^{\text{symm}} \\ \mathbf{Rel}(A, A) \times \mathbf{Rel}(A, A) & \xrightarrow{\diamond} & \mathbf{Rel}(A, A). \end{array} \quad (S \diamond R)^{\text{symm}} = S^{\text{symm}} \diamond R^{\text{symm}},$$

Proof. **Item 1, Adjointness:** This is a rephrasing of the universal property of the symmetric closure of a relation, stated in **Definition 10.3.2.1.1**.

Item 2, The Symmetric Closure of a Symmetric Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**.

Item 4, Interaction With Inverses: Clear.

Item 5, Interaction With Composition: This follows from **Item 2** of **Definition 10.3.1.1.4**. □

10.4 Transitive Relations

10.4.1 Foundations

Let A be a set.

Definition 10.4.1.1.1. A **transitive relation** is equivalently:⁷

- A non-unital \mathbb{E}_1 -monoid in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

Remark 10.4.1.1.2. In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R: R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

(\star) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 10.4.1.1.3. Let A be a set.

1. The **set of transitive relations from A to B** is the subset $\mathbf{Rel}^{\text{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.
2. The **poset of relations from A to B** is the subposet $\mathbf{Rel}^{\text{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

Proposition 10.4.1.1.4. Let R and S be relations on A .

1. *Interaction With Inverses.* If R is transitive, then so is R^\dagger .
2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ **may fail to be transitive**.

Proof. **Item 1, Interaction With Inverses:** Clear.

Item 2, Interaction With Composition: See [MSE 2096272].⁸ □

⁷Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

⁸*Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond R} e$, then:
 - There is some $b \in A$ such that:
 - * $a \sim_R b$;

10.4.2 The Transitive Closure of a Relation

Let R be a relation on A .

Definition 10.4.2.1.1. The **transitive closure** of \sim_R is the relation \sim_R^{trans} ⁹ satisfying the following universal property:¹⁰

- (★) Given another transitive relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{trans}} \subset \sim_S$.

Construction 10.4.2.1.2. Concretely, \sim_R^{trans} is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$ ¹¹, being given by

$$\begin{aligned} R^{\text{trans}} &\stackrel{\text{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\stackrel{\text{def}}{=} \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists some } (x_1, \dots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \dots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

Proof. Clear. □

Proposition 10.4.2.1.3. Let R be a relation on A .

1. *Adjointness.* We have an adjunction

$$((-)^{\text{trans}} \dashv \overset{\circlearrowleft}{\text{忘}}): \mathbf{Rel}(A, A) \overset{(-)^{\text{trans}}}{\underset{\overset{\circlearrowleft}{\text{忘}}}{\rightleftarrows}} \mathbf{Rel}^{\text{trans}}(A, A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{trans}}(R^{\text{trans}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{trans}}(A, A))$ and $S \in \text{Obj}(\mathbf{Rel}(A, B))$.

$$* \quad b \sim_S c;$$

– There is some $d \in A$ such that:

$$* \quad c \sim_R d;$$

$$* \quad d \sim_S e.$$

⁹*Further Notation:* Also written R^{trans} .

¹⁰*Slogan:* The transitive closure of R is the smallest transitive relation containing R .

¹¹Or, equivalently, the free non-unital \mathbb{E}_1 -monoid on R in $(\mathbf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\text{trans}} = R$.

3. *Idempotency.* We have

$$(R^{\text{trans}})^{\text{trans}} = R^{\text{trans}}.$$

4. *Interaction With Inverses.* We have

$$\begin{array}{ccc} \text{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \text{Rel}(A, A) \\ (-)^{\dagger} \downarrow & & \downarrow (-)^{\dagger} \\ \text{Rel}(A, A) & \xrightarrow{(-)^{\text{trans}}} & \text{Rel}(A, A) \end{array}$$

$$(R^{\dagger})^{\text{trans}} = (R^{\text{trans}})^{\dagger},$$

5. *Interaction With Composition.* We have

$$\begin{array}{ccc} \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A) \\ (-)^{\text{trans}} \times (-)^{\text{trans}} \downarrow & \text{X} & \downarrow (-)^{\text{trans}} \\ \text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A) \end{array}$$

$$(S \diamond R)^{\text{trans}} \stackrel{\text{poss.}}{\neq} S^{\text{trans}} \diamond R^{\text{trans}},$$

Proof. Item 1, Adjointness: This is a rephrasing of the universal property of the transitive closure of a relation, stated in **Definition 10.4.2.1.1**.

Item 2, The Transitive Closure of a Transitive Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**.

Item 4, Interaction With Inverses: We have

$$\begin{aligned} (R^{\dagger})^{\text{trans}} &= \bigcup_{n=1}^{\infty} (R^{\dagger})^{\diamond n} \\ &= \bigcup_{n=1}^{\infty} (R^{\diamond n})^{\dagger} \\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n} \right)^{\dagger} \\ &= (R^{\text{trans}})^{\dagger}, \end{aligned}$$

where we have used, respectively:

- **Definition 10.4.2.1.2.**

- **Constructions With Relations**, ?? of ??.
- **Constructions With Relations**, ?? of **Definition 9.2.3.1.2**.
- **Definition 10.4.2.1.2**.

This finishes the proof.

Item 5, Interaction With Composition: This follows from **Item 2** of **Definition 10.4.1.1.4**.

□

10.5 Equivalence Relations

10.5.1 Foundations

Let A be a set.

Definition 10.5.1.1.1. A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.¹²

Example 10.5.1.1.2. The **kernel of a function** $f: A \rightarrow B$ is the equivalence relation $\sim_{\text{Ker}(f)}$ on A obtained by declaring $a \sim_{\text{Ker}(f)} b$ iff $f(a) = f(b)$.¹³

Definition 10.5.1.1.3. Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\text{Rel}^{\text{eq}}(A, B)$ of $\text{Rel}(A, B)$ spanned by the equivalence relations.
2. The **poset of relations from A to B** is the subposet $\text{Rel}^{\text{eq}}(A, B)$ of $\text{Rel}(A, B)$ spanned by the equivalence relations.

10.5.2 The Equivalence Closure of a Relation

Let R be a relation on A .

Definition 10.5.2.1.1. The **equivalence closure**¹⁴ of \sim_R is the relation \sim_R^{eq} ¹⁵ satisfying the following universal property:¹⁶

¹²*Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

¹³The kernel $\text{Ker}(f): A \rightarrowtail A$ of f is the underlying functor of the monad induced by the adjunction $\text{Gr}(f) \dashv f^{-1}: A \rightleftarrows B$ in **Rel** of **Constructions With Relations**, ?? of ??.

¹⁴*Further Terminology:* Also called the **equivalence relation associated to \sim_R** .

¹⁵*Further Notation:* Also written R^{eq} .

¹⁶*Slogan:* The equivalence closure of R is the smallest equivalence relation containing R .

- (★) Given another equivalence relation \sim_S on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{eq}} \subset \sim_S$.

Construction 10.5.2.1.2. Concretely, \sim_R^{eq} is the equivalence relation on A defined by

$$\begin{aligned} R^{\text{eq}} &\stackrel{\text{def}}{=} ((R^{\text{refl}})^{\text{symm}})^{\text{trans}} \\ &= ((R^{\text{symm}})^{\text{trans}})^{\text{refl}} \\ &= \left\{ (a, b) \in A \times B \mid \begin{array}{l} \text{there exists } (x_1, \dots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \\ 1. \text{ The following conditions are satisfied:} \\ \\ \quad (a) \text{ We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \quad (b) \text{ We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \quad \quad \text{for each } 1 \leq i \leq n-1; \\ \quad (c) \text{ We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \\ 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive.
2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

Proposition 10.5.2.1.3. Let R be a relation on A .

1. *Adjointness.* We have an adjunction

$$((-)^{\text{eq}} \dashv \overset{\circlearrowleft}{\text{忘}}): \mathbf{Rel}(A, B) \overset{((-)^{\text{eq}})}{\underset{\overset{\circlearrowright}{\text{忘}}}{\rightleftarrows}} \mathbf{Rel}^{\text{eq}}(A, B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{eq}}(R^{\text{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{eq}}(A, B))$ and $S \in \text{Obj}(\mathbf{Rel}(A, B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\text{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\text{eq}})^{\text{eq}} = R^{\text{eq}}.$$

Proof. **Item 1, Adjointness:** This is a rephrasing of the universal property of the equivalence closure of a relation, stated in **Definition 10.5.2.1.1**.

Item 2, The Equivalence Closure of an Equivalence Relation: Clear.

Item 3, Idempotency: This follows from **Item 2**. □

10.6 Quotients by Equivalence Relations

10.6.1 Equivalence Classes

Let A be a set, let R be a relation on A , and let $a \in A$.

Definition 10.6.1.1.1. The **equivalence class associated to a** is the set $[a]$ defined by

$$\begin{aligned} [a] &\stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \end{aligned} \quad (\text{since } R \text{ is symmetric})$$

10.6.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A .

Definition 10.6.2.1.1. The **quotient of X by R** is the set X/\sim_R defined by

$$X/\sim_R \stackrel{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 10.6.2.1.2. The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalence classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

$$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

but we could equally well define

$$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.¹⁷

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \not\sim_R b$, and equal otherwise.

Proposition 10.6.2.1.3. Let $f: X \rightarrow Y$ be a function and let R be a relation on X .

1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\text{eq}} \cong \text{CoEq}(R \hookrightarrow X \times X \xrightarrow[\text{pr}_2]{\text{pr}_1} X),$$

where \sim_R^{eq} is the equivalence relation generated by \sim_R .

2. *As a Pushout.* We have an isomorphism of sets¹⁸

$$X/\sim_R^{\text{eq}} \cong X \amalg_{\text{Eq}(\text{pr}_1, \text{pr}_2)} X, \quad \begin{array}{ccc} X/\sim_R^{\text{eq}} & \longleftarrow & X \\ \uparrow \ulcorner & & \uparrow \\ X & \longleftarrow & \text{Eq}(\text{pr}_1, \text{pr}_2). \end{array}$$

where \sim_R^{eq} is the equivalence relation generated by \sim_R .

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets^{19,20}

$$X/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

¹⁷When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see *Constructions With Categories*, ??.

¹⁸Dually, we also have an isomorphism of sets

$$\text{Eq}(\text{pr}_1, \text{pr}_2) \cong X \times_{X/\sim_R^{\text{eq}}} X, \quad \begin{array}{ccc} \text{Eq}(\text{pr}_1, \text{pr}_2) & \longrightarrow & X \\ \downarrow \lrcorner & & \downarrow \\ X & \longrightarrow & X/\sim_R^{\text{eq}}. \end{array}$$

¹⁹*Further Terminology:* The set $X/\sim_{\text{Ker}(f)}$ is often called the **coimage** of f , and denoted by $\text{CoIm}(f)$.

²⁰In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f , as the kernel and image

$$\text{Ker}(f): X \dashv X,$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X . The following conditions are equivalent:

(a) There exists a map

$$\bar{f}: X/\sim_R \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists \nearrow & \\ X/\sim_R & \xrightarrow{\bar{f}} & \end{array}$$

commute.

(b) We have $R \subset \text{Ker}(f)$.

(c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then \bar{f} is the *unique* map making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists! \nearrow & \\ X/\sim_R & \xrightarrow{\bar{f}} & \end{array}$$

commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X . We have a bijection

$$\text{Hom}_{\text{Sets}}(X/\sim_R, Y) \cong \text{Hom}_{\text{Sets}}^R(X, Y),$$

$$\text{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left(\text{Gr}(f) \dashv f^{-1} \right): \begin{array}{ccc} & \text{Gr}(f) & \\ \uparrow & \text{---} & \downarrow \\ A & \xrightarrow{\quad} & B \\ \downarrow & \text{---} & \uparrow \\ & f^{-1} & \end{array}$$

of **Constructions With Relations**, ?? of ??.

natural in $X, Y \in \text{Obj}(\text{Sets})$, given by the assignment $f \mapsto \bar{f}$ of **Items 4** and **5**, where $\text{Hom}_{\text{Sets}}^R(X, Y)$ is the set defined by

$$\text{Hom}_{\text{Sets}}^R(X, Y) \stackrel{\text{def}}{=} \left\{ f \in \text{Hom}_{\text{Sets}}(X, Y) \left| \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right. \right\}.$$

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then the following conditions are equivalent:

- (a) The map \bar{f} is an injection.
- (b) We have $R = \text{Ker}(f)$.
- (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X . If the conditions of **Item 4** hold, then the following conditions are equivalent:

- (a) The map $f: X \rightarrow Y$ is surjective.
- (b) The map $\bar{f}: X/\sim_R \rightarrow Y$ is surjective.

9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let \sim_R^{eq} be the equivalence relation associated to R . The following conditions are equivalent:

- (a) The map f satisfies the equivalent conditions of **Item 4**:
 - There exists a map

$$\bar{f}: X/\sim_R^{\text{eq}} \rightarrow Y$$

making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \exists \nearrow \bar{f} & \\ X/\sim_R^{\text{eq}} & & \end{array}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\text{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. **Item 1**, As a Coequaliser: Omitted.

Item 2, As a Pushout: Omitted.

Item 3, The First Isomorphism Theorem for Sets: Clear.

Item 4, Descending Functions to Quotient Sets, I: See [Pro25c].

Item 5, Descending Functions to Quotient Sets, II: See [Pro25d].

Item 6, Descending Functions to Quotient Sets, III: This follows from **Items 5** and **6**.

Item 7, Descending Functions to Quotient Sets, IV: See [Pro25b].

Item 8, Descending Functions to Quotient Sets, V: See [Pro25a].

Item 9, Descending Functions to Quotient Sets, VI: The implication **Item 8a** \implies **Item 8b** is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R , we see that the condition $x \sim_R^{\text{eq}} y$ unwinds to the following:

- (★) There exist $(x_1, \dots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:
 - The following conditions are satisfied:
 - * We have $x \sim_R x_1$ or $x_1 \sim_R x$;
 - * We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
 - * We have $y \sim_R x_n$ or $x_n \sim_R y$;
 - We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned}
 f(x) &= f(x_1), \\
 f(x_1) &= f(x_2), \\
 &\vdots \\
 f(x_{n-1}) &= f(x_n), \\
 f(x_n) &= f(y),
 \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

Appendices

A Other Chapters

Preliminaries

1. [Introduction](#)
2. [A Guide to the Literature](#)

Sets

3. [Sets](#)
4. [Constructions With Sets](#)
5. [Monoidal Structures on the Category of Sets](#)
6. [Pointed Sets](#)
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Relations

8. [Relations](#)
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Categories

11. [Categories](#)
12. [Presheaves and the Yoneda Lemma](#)

Monoidal Categories

13. [Constructions With Monoidal Categories](#)

Bicategories

14. [Types of Morphisms in Bicategories](#)

Extra Part

15. [Notes](#)

References

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