Pointed Sets

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This chapter contains some foundational material on pointed sets.

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0099 6.1 Pointed Sets

009A 6.1.1 Foundations

009B DEFINITION 6.1.1.1.1 ▶ POINTED SETS

A **pointed set¹** is equivalently:

- An \mathbb{E}_0 -monoid in $(N_{\bullet}(Sets), pt)$.
- A pointed object in (Sets, pt).

009C REMARK 6.1.1.1.2 ➤ UNWINDING DEFINITION 6.1.1.1.1

In detail, a **pointed set** is a pair (X, x_0) consisting of:

- The Underlying Set. A set X, called the **underlying set of** (X, x_0) .
- The Basepoint. A morphism

$$[x_0]: pt \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

009D EXAMPLE 6.1.1.1.3 ► THE ZERO SPHERE

The 0-**sphere**¹ is the pointed set $(S^0, 0)^2$ consisting of:

• The Underlying Set. The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

¹Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -modules.

• The Basepoint. The element 0 of S^0 .

¹Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

² Further Notation: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

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EXAMPLE 6.1.1.1.4 ► THE TRIVIAL POINTED SET

The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- The Underlying Set. The punctual set pt $\stackrel{\text{def}}{=} \{ \star \}$.
- *The Basepoint.* The element \star of pt.

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EXAMPLE 6.1.1.1.5 \blacktriangleright The Standard Pointed Set With n+1 Elements

The **standard pointed set with** n + 1 **elements** is the pointed set $\langle n \rangle$ consisting of

• *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{ * \} \cup \{ 1, \dots, n \}.$$

• *The Basepoint.* The element * of $\langle n \rangle$.

009H 6.1.2 Morphisms of Pointed Sets

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DEFINITION 6.1.2.1.1 ► MORPHISMS OF POINTED SETS

A **morphism of pointed sets**^{1,2} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$.
- A morphism of pointed objects in (Sets, pt).

¹Further Terminology: Also called a **pointed function**.

²Further Terminology: In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of** \mathbb{F}_1 -**modules**.

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REMARK 6.1.2.1.2 ▶ Unwinding Definition 6.1.2.1.1

In detail, a **morphism of pointed sets** $f: (X, x_0) \to (Y, y_0)$ is a morphism of sets $f: X \to Y$ such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & [y_0] \\
X & \xrightarrow{f} Y
\end{array}$$

commutes, i.e. such that

$$f(x_0)=y_0.$$

009L 6.1.3 The Category of Pointed Sets

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DEFINITION 6.1.3.1.1 ► THE CATEGORY OF POINTED SETS

The **category of pointed sets** is the category Sets* defined equivalently as:

- The homotopy category of the ∞ -category $\mathsf{Mon}_{\mathbb{E}_0}(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$ of $\ref{eq:Normalize}$??
- The category Sets* of Constructions With Categories, ??.

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REMARK 6.1.3.1.2 ► Unwinding Definition 6.1.3.1.1

In detail, the **category of pointed sets** is the category Sets* where:

- Objects. The objects of $Sets_*$ are pointed sets.
- Morphisms. The morphisms of $Sets_*$ are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets_{*} at (X, x_0) is defined by¹

$$id_{(X,x_0)}^{Sets_*} \stackrel{\text{def}}{=} id_X$$
.

• *Composition*. For each (X, x_0) , (Y, y_0) , $(Z, z_0) \in Obj(Sets_*)$, the composition map

$$\circ_{(X,x_0),(Y,y_0),(Z,z_0)}^{\mathsf{Sets}_*} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

of Sets_{*} at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by²

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\mathrm{def}}{=} g \circ f.$$

²Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

009P 6.1.4 Elementary Properties of Pointed Sets

0090 PROPOSITION 6.1.4.1.1 ► ELEMENTARY PROPERTIES OF POINTED SETS

Let (X, x_0) be a pointed set.

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- 1. *Completeness*. The category Sets_{*} of pointed sets and morphisms between them is complete, having in particular:
 - (a) Products, described as in Definition 6.2.3.1.1.
 - (b) Pullbacks, described as in Definition 6.2.4.1.1.
 - (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets_{*} of pointed sets and morphisms between them is cocomplete, having in particular:
 - (a) Coproducts, described as in Definition 6.3.3.1.1.
 - (b) Pushouts, described as in Definition 6.3.4.1.1;
 - (c) Coequalisers, described as in Definition 6.3.5.1.1.

¹Note that id_X is indeed a morphism of pointed sets, as we have $id_X(x_0) = x_0$.

009Z

3. Failure To Be Cartesian Closed. The category $Sets_*$ is not Cartesian closed.

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4. Morphisms From the Monoidal Unit. We have a bijection of sets²

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathsf{Sets}_*(S^0,X)\cong (X,x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

00A1

5. Relation to Partial Functions. We have an equivalence of categories³

$$\mathsf{Sets}_* \overset{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

024U

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part}.}$$

sends:

- 024V
- 024W

- i. A pointed set (X, x_0) to X.
- ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f\colon X\to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

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(b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1}$$
: Sets^{part.} $\stackrel{\cong}{\rightarrow}$ Sets_{*}

sends:

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- i. A set *X* is to the pointed set (X, \star) with \star an element that is not in *X*.
- ii. A partial function

$$f: X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

**Warning: This is not an isomorphism of categories, only an equivalence.

PROOF 6.1.4.1.2 ▶ Proof of Proposition 6.1.4.1.1

Item 1: Completeness

This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

 $^{^1}$ The category Sets $_*$ does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

²In other words, the forgetful functor

Item 2: Cocompleteness

This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

Item 3: Failure To Be Cartesian Closed

See [MSE 2855868].

Item 4: Morphisms From the Monoidal Unit

Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathsf{Sets}_*\big(S^0,X\big)\cong (X,x_0)$$

follows by noting that $\Delta_{x_0} \colon S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5: Relation to Partial Functions

See [MSE 884460].

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01QC 6.1.5 Active and Inert Morphisms of Pointed Sets

DEFINITION 6.1.5.1.1 ► ACTIVE AND INERT MORPHISMS OF POINTED SETS

Let $f: (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

- 1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
- 2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

01QG NOTATION 6.1.5.1.2 ► THE CATEGORY OF POINTED SETS AND ACTIVE MORPHISMS

We write Sets** for the wide subcategory of Sets* spanned by pointed sets and the active maps between them.

01QH EXAMPLE 6.1.5.1.3 ► EXAMPLES OF ACTIVE AND INERT MAPS OF POINTED SETS

Here are some examples of active and inert maps of pointed sets.

01QJ 1. The map $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$ given by

$$1 \longmapsto 1$$

$$2$$

is active but not inert.

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01QL

2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by

$$1 \longmapsto 1$$

$$2 \longmapsto 3$$

is inert but not active.

3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by

is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i:\langle 3\rangle \to \langle 2\rangle,$$

$$a \colon \langle 2 \rangle \to \langle 1 \rangle$$

are the morphisms of pointed sets given by

with *i* being inert and *a* being active.

PROPOSITION 6.1.5.1.4 ► PROPERTIES OF ACTIVE AND INERT MAPS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

1. Active-Inert Factorisation. Every morphism of pointed sets $f: (X, x_0) \to (Y, y_0)$ factors uniquely as

$$f = a \circ i$$
,

where:

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- (a) The map $i \colon (X, x_0) \to (K, k_0)$ is an inert morphism of pointed sets
- (b) The map $a: (K, k_0) \rightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_{*}.

PROOF 6.1.5.1.5 ► PROOF OF PROPOSITION 6.1.5.1.4

Item 1: Active-Inert Factorisation

Let $f \colon X \to Y$ be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\longrightarrow} K \stackrel{a}{\longrightarrow} Y$$
,

where:

• *K* is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$

= $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$

• $i: X \to K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

• $a: K \to Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$X \xrightarrow{i} Y$$

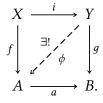
$$f \downarrow \qquad \qquad \downarrow g$$

$$A \xrightarrow{g} B$$

be a commutative diagram in $\mathsf{Sets}_*.$ Consider the morphism $\phi\colon Y\to A$ given by

$$\phi(y)=f\bigl(i^{-1}(y)\bigr)$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram



Indeed, this diagram commutes, as we have

$$[\phi \circ i](x) \stackrel{\text{def}}{=} \phi(i(x))$$
$$\stackrel{\text{def}}{=} f(i^{-1}(i(x)))$$
$$= f(x)$$

for each $x \in X$ and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$X \xrightarrow{i} Y$$

$$f \downarrow \qquad \qquad \downarrow \downarrow \qquad \downarrow g$$

$$A \xrightarrow{g} B$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

00A2 6.2 Limits of Pointed Sets

00A3 6.2.1 The Terminal Pointed Set

00A4 DEFINITION 6.2.1.1.1 ► THE TERMINAL POINTED SET

The **terminal pointed set** is the terminal object of Sets_{*} as in Limits and Colimits, ??.

0250 CONSTRUCTION 6.2.1.1.2 ➤ CONSTRUCTION OF THE TERMINAL POINTED SET

Concretely, the **terminal pointed set** is the pair $(pt, \star), \{!_X\}_{(X,x_0) \in Obj(Sets_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\operatorname{pt}, \star)\}_{(X, x_0) \in \operatorname{Obi}(\operatorname{\mathsf{Sets}})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathsf{Sets})$.

PROOF 6.2.1.1.3 ► PROOF OF CONSTRUCTION 6.2.1.1.2

We claim that (pt, \star) is the terminal object of Sets_{*}. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
 (pt, \star)

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathsf{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists !}} (pt, \star)$$

commute, namely $!_X$.

00A5 6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

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DEFINITION 6.2.2.1.1 ► THE PRODUCT OF A FAMILY OF POINTED SETS

The **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets* as in Limits and Colimits, ??.

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CONSTRUCTION 6.2.2.1.2 ► CONSTRUCTION OF THE PRODUCT OF A FAMILY OF POINTED Sets

Concretely, the **product of** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$ consisting of:

- The Limit. The pointed set $\left(\prod_{i\in I}X_i,\left(x_0^i\right)_{i\in I}\right)$.
- *The Cone*. The collection

$$\left\{\operatorname{pr}_i : \left(\prod_{i \in I} X_i, \left(x_0^i\right)_{i \in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_i\Big(\big(x_j\big)_{j\in I}\Big)\stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j\in I}\in\prod_{i\in I}X_i$ and each $i\in I$.

PROOF 6.2.2.1.3 ➤ PROOF OF CONSTRUCTION 6.2.2.1.2

We claim that $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_{*}. Indeed, suppose we have, for each $i\in I$, a diagram of the form

$$(P,*)$$

$$p_{i}$$

$$(\prod_{i\in I}X_{i},\left(x_{0}^{i}\right)_{i\in I})\xrightarrow{\operatorname{pt}_{i}}\left(X_{i},x_{0}^{i}\right)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to \left(\prod_{i \in I} X_i, \left(x_0^i \right)_{i \in I} \right)$$

making the diagram

$$(P, *)$$

$$\phi \mid \exists !$$

$$(\prod_{i \in I} X_i, (x_0^i)_{i \in I}) \xrightarrow{\operatorname{pr}_i} (X_i, x_0^i)$$

commute, being uniquely determined by the condition $\operatorname{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I}$$
$$= (x_0^i)_{i \in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$.

6.2.3 Products

00A7

PROPOSITION 6.2.2.1.4 ► PROPERTIES OF PRODUCTS OF FAMILIES OF POINTED SETS

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A8

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

16

$$\prod_{i \in I} : \operatorname{Fun}(I_{\operatorname{disc}}, \operatorname{Sets}_*) \to \operatorname{Sets}_*.$$

PROOF 6.2.2.1.5 ▶ PROOF OF PROPOSITION 6.2.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

00A9 **6.2.3** Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA

DEFINITION 6.2.3.1.1 ► PRODUCTS OF POINTED SETS

The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

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CONSTRUCTION 6.2.3.1.2 ► CONSTRUCTION OF PRODUCTS OF POINTED SETS

Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- The Limit. The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 \colon (X \times Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_1(x,y) \stackrel{\text{def}}{=} x,$$

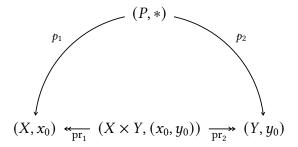
6.2.3 Products 17

$$\operatorname{pr}_2(x,y) \stackrel{\text{def}}{=} y$$

for each $(x, y) \in X \times Y$.

PROOF 6.2.3.1.3 ► PROOF OF CONSTRUCTION 6.2.3.1.2

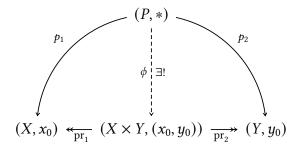
We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$

$$\operatorname{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

00AB

PROPOSITION 6.2.3.1.4 ▶ PROPERTIES OF PRODUCTS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00AC

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{array}{ll} A \times -\colon & \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\ - \times B\colon & \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\ -_1 \times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{array}$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

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2. *Lack of Adjointness*. The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

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3. Associativity. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in (X, x_0) , (Y, y_0) , $(Z, z_0) \in \text{Obj}(\mathsf{Sets}_*)$.

00AE

4. Unitality. We have isomorphisms of pointed sets

$$(pt, \star) \times (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \times (pt, \star) \cong (X, x_0),$

natural in $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$.

00AF

5. *Commutativity*. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in (X, x_0) , $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

00AG

6. Symmetric Monoidality. The triple (Sets $_*$, \times , (pt, \star)) is a symmetric monoidal category.

PROOF 6.2.3.1.5 ▶ PROOF OF PROPOSITION 6.2.3.1.4

Item 1: Functoriality

This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2: Lack of Adjointness

See [MSE 2855868].

Item 3: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.3.1.4.

Item 4: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.1.3.1.4.

Item 5: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.1.3.1.4.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 14 of Proposition 4.1.3.1.4.

00AH 6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

00AJ DEFINITION 6.2.4.1.1 ▶ PULLBACKS OF POINTED SETS

The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

0253 CONSTRUCTION 6.2.4.1.2 ➤ CONSTRUCTION OF PULLBACKS OF POINTED SETS

Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- The Limit. The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$

 $\operatorname{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$

 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$

for each $(x, y) \in X \times_Z Y$.

PROOF 6.2.4.1.3 ► PROOF OF CONSTRUCTION 6.2.4.1.2

We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad (X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0})$$

$$\downarrow g \qquad \qquad \downarrow g \qquad \qquad \downarrow g \qquad \qquad \downarrow g \qquad \qquad (X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$[f \circ \operatorname{pr}_1](x, y) = f(\operatorname{pr}_1(x, y))$$

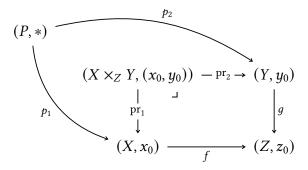
$$= f(x)$$

$$= g(y)$$

$$= g(\operatorname{pr}_2(x, y))$$

$$= [g \circ \operatorname{pr}_2](x, y),$$

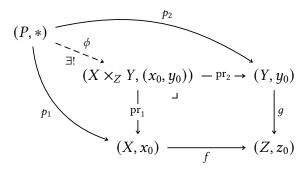
where f(x) = g(y) since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
$$\operatorname{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f\circ p_1=g\circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

= $(x_0, y_0),$

where we have used that p_1 and p_2 are morphisms of pointed sets.

00AK

PROPOSITION 6.2.4.1.4 ► PROPERTIES OF PULLBACKS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00AL

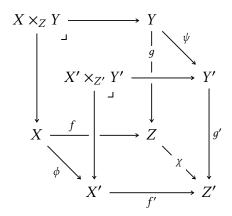
1. Functoriality. The assignment $(X,Y,Z,f,g)\mapsto X\times_{f,Z,g}Y$ defines a functor

$$\mathsf{-_1} \times_{\mathsf{-_3}} \mathsf{-_1} \colon \mathsf{Fun}(\mathcal{P},\mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \times_{-3} -1$ is given by sending a morphism



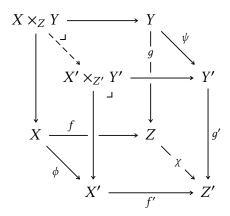
in $\operatorname{Fun}(\mathcal{P},\operatorname{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x,y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

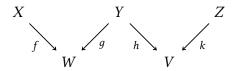
for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

00AM

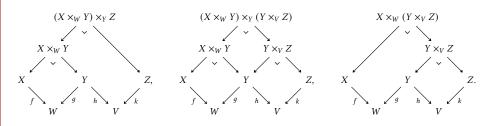
2. Associativity. Given a diagram



in Sets*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



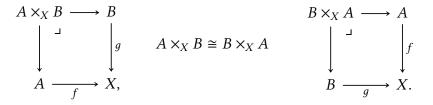
00AN

3. Unitality. We have isomorphisms of pointed sets



00AP

4. Commutativity. We have an isomorphism of pointed sets



00A0

5. *Interaction With Products*. We have an isomorphism of pointed sets

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \xrightarrow{} Y$$

$$\downarrow !_{Y}$$

$$X \xrightarrow{} pt.$$

00AR

6. *Symmetric Monoidality.* The triple (Sets_{*}, \times_X , X) is a symmetric monoidal category.

PROOF 6.2.4.1.5 ► PROOF OF PROPOSITION 6.2.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, $\ref{eq:condition}$ of $\ref{eq:condition}$, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 4 of Proposition 4.1.4.1.7.

Item 3: Unitality

This follows from Constructions With Sets, Item 6 of Proposition 4.1.4.1.7.

Item 4: Commutativity

This follows from Constructions With Sets, Item 7 of Proposition 4.1.4.1.7.

Item 5: Interaction With Products

This follows from Constructions With Sets, Item 10 of Proposition 4.1.4.1.7.

Item 6: Symmetric Monoidality

This follows from Constructions With Sets, Item 11 of Proposition 4.1.4.1.7.

00AS 6.2.5 Equalisers

Let $f, g: (X, x_0) \Rightarrow (Y, y_0)$ be morphisms of pointed sets.

00AT

The **equaliser of** (f, g) is the equaliser of f and g in Sets_{*} as in Limits and Colimits, ??.

0254 CONSTRUCTION 6.2.5.1.2 ► CONSTRUCTION OF EQUALISERS OF POINTED SETS

Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(Eq(f, g), x_0)$.
- The Cone. The morphism of pointed sets

$$\operatorname{eq}(f,g) \colon (\operatorname{Eq}(f,g),x_0) \hookrightarrow (X,x_0)$$

given by the canonical inclusion $eq(f,g) \hookrightarrow Eq(f,g) \hookrightarrow X$.

PROOF 6.2.5.1.3 ► PROOF OF CONSTRUCTION 6.2.5.1.2

We claim that $(\text{Eq}(f,g),x_0)$ is the categorical equaliser of f and g in Sets $_*$. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ eq(f,g) = g \circ eq(f,g),$$

which indeed holds by the definition of the set ${\rm Eq}(f,g)$. Next, we prove that ${\rm Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$(E,*)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E,*)\to (\mathrm{Eq}(f,g),x_0)$$

making the diagram

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow \phi \mid \exists! \qquad e$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in Eq(f,g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

00AU

PROPOSITION 6.2.5.1.4 ► PROPERTIES OF EQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h \colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00AV

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \underbrace{\mathrm{Eq}(f,g,h)}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))} = \underbrace{\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}$$

00AW

00AX

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets*, being explicitly given by

$$Eq(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. Unitality. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, f) \cong X$$
.

3. Commutativity. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,g) \cong \operatorname{Eq}(g,f).$$

PROOF 6.2.5.1.5 ► PROOF OF PROPOSITION 6.2.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.1.5.1.4.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.1.5.1.4.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.1.5.1.4.

ODAY 6.3 Colimits of Pointed Sets

00AZ 6.3.1 The Initial Pointed Set

00B0 DEFINITION 6.3.1.1.1 ► THE INITIAL POINTED SET

The **initial pointed set** is the initial object of Sets_{*} as in Limits and Colimits, ??.

0255 CONSTRUCTION 6.3.1.1.2 ➤ CONSTRUCTION OF THE INITIAL POINTED SET

Concretely, the **initial pointed set** is the pair (pt, \star) , $\{\iota_X\}_{(X,x_0) \in \text{Obj}(\mathsf{Sets}_*)}$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathrm{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathrm{Obi}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

PROOF 6.3.1.1.3 ► PROOF OF CONSTRUCTION 6.3.1.1.2

We claim that (pt, \star) is the initial object of Sets $_*$. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
 (X, x_0)

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt},\star) \xrightarrow{-\frac{\phi}{\exists !}} (X,x_0)$$

commute, namely ι_X .

6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B2 DEFINITION 6.3.2.1.1 ➤ COPRODUCTS OF FAMILIES OF POINTED SETS

The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets* as in Limits and Colimits, ??.

¹Further Terminology: Also called the **wedge sum of the family** $\{(X_i, x_0^i)\}_{i \in I}$.

0256 CONSTRUCTION 6.3.2.1.2 ➤ CONSTRUCTION OF COPRODUCTS OF FAMILIES OF POINTED SETS

Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \{\inf_i\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:
 - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

- *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \stackrel{\text{def}}{=} \left[\left(i, x_0^i \right) \right] \\ = \left[\left(j, x_0^j \right) \right]$$

for any $i, j \in I$.

• *The Cocone*. The collection

$$\left\{\operatorname{inj}_i\colon \left(X_i, x_0^i\right) \to \left(\bigvee_{i \in I} X_i, p_0\right)\right\}_{i \in I}$$

of morphism of pointed sets given by

$$\operatorname{inj}_{i}(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

PROOF 6.3.2.1.3 ► PROOF OF CONSTRUCTION 6.3.2.1.2

We claim that $(\bigvee_{i\in I}X_i,p_0)$ is the categorical coproduct of $\{(X_i,x_0^i)\}_{i\in I}$ in Sets_{*}. Indeed, suppose we have, for each $i\in I$, a diagram of the form

$$(X_i, x_0^i) \xrightarrow[\inf_i]{l_i} \left(\bigvee_{i \in I} X_i, p_0\right)$$

in Sets*. Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i\in I} X_i, p_0\right) \to (C, *)$$

making the diagram

$$(X_i, x_0^i) \xrightarrow[\text{inj}_i]{(C, *)} \begin{pmatrix} (C, *) \\ \downarrow \downarrow \\ \downarrow \downarrow X_i, p_0 \end{pmatrix}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i(\left[\left(i, x_0^i\right)\right])$$

= *,

as ι_i is a morphism of pointed sets.

00B3

PROPOSITION 6.3.2.1.4 ▶ PROPERTIES OF COPRODUCTS OF FAMILIES OF POINTED SETS

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B4

1. Functoriality. The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \operatorname{Fun}(I_{\operatorname{disc}}, \operatorname{Sets}_*) \to \operatorname{Sets}_*.$$

PROOF 6.3.2.1.5 ► PROOF OF PROPOSITION 6.3.2.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

00B5 6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6

DEFINITION 6.3.3.1.1 ► COPRODUCTS OF POINTED SETS

The **coproduct of** (X, x_0) **and** $(Y, y_0)^1$ is the coproduct of (X, x_0) and (Y, y_0) in Sets_{*} as in Limits and Colimits, ??.

¹ Further Terminology: Also called the **wedge sum of** (X, x_0) **and** (Y, y_0) .

0257

CONSTRUCTION 6.3.3.1.2 ► CONSTRUCTION OF COPRODUCTS OF POINTED SETS

Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0) , also called their **wedge sum**, is the pair consisting of:

• *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

- *The Underlying Set.* The set $X \vee Y$ defined by

$$(X \lor Y, p_0) \stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \qquad X \lor Y \longleftarrow Y$$

$$\cong \left(X \coprod_{pt} Y, p_0 \right) \qquad \qquad \uparrow \qquad \qquad \uparrow \qquad \qquad \downarrow [y_0]$$

$$\cong (X \coprod Y/\sim, p_0), \qquad X \longleftarrow_{[x_0]} pt,$$

where \sim is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

- *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$

= $[(1, y_0)].$

• The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \vee Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \vee Y, p_0),$

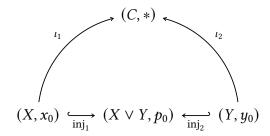
given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)],
inj_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.3.1.3 ► PROOF OF CONSTRUCTION 6.3.3.1.2

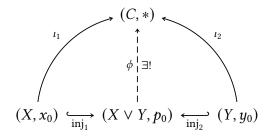
We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_{*}. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$

$$\phi \circ \operatorname{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$

=
$$\iota_Y([(1, y_0)])$$

= *,

as ι_X and ι_Y are morphisms of pointed sets.

00B7

PROPOSITION 6.3.3.1.4 ▶ PROPERTIES OF WEDGE SUMS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets.

00B8

1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$

00B9

2. Associativity. We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in Sets_*$.

00BA

3. *Unitality*. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$

 $(X, x_0) \lor (pt, *) \cong (X, x_0),$

natural in $(X, x_0) \in Sets_*$.

00BB

4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$
.

natural in (X, x_0) , $(Y, y_0) \in Sets_*$.

00BC

5. Symmetric Monoidality. The triple (Sets $_*$, \vee , pt) is a symmetric monoidal category.

00BD

6. The Fold Map. We have a natural transformation



called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at *X* is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

PROOF 6.3.3.1.5 ► PROOF OF PROPOSITION 6.3.3.1.4

Item 1: Functoriality

This follows from Limits and Colimits, ?? of ??.

Item 2: Associativity

Omitted.

Item 3: Unitality

Omitted.

Item 4: Commutativity

Omitted.

Item 5: Symmetric Monoidality

Omitted.

Item 6: The Fold Map

Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f:(X,x_0)\to (Y,y_0)$, we have

$$\nabla_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad f \vee f \downarrow \qquad \qquad \downarrow f$$

$$Y \vee Y \xrightarrow{\nabla_{X}} Y.$$

Indeed, we have

$$\begin{split} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{split}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation.

00BE 6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \to (X, x_0)$ and $g: (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BF DEFINITION 6.3.4.1.1 ▶ PUSHOUTS OF POINTED SETS

The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_{*} as in Limits and Colimits, ??.

0258 CONSTRUCTION 6.3.4.1.2 ➤ CONSTRUCTION OF PUSHOUTS OF POINTED SETS

Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:
 - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;

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- We have $p_0 = [x_0] = [y_0]$.
- The Cocone. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$

 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$

given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)]
inj_2(y) \stackrel{\text{def}}{=} [(1, y)]$$

for each $x \in X$ and each $y \in Y$.

PROOF 6.3.4.1.3 ► PROOF OF ??

Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$

$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_{*}. First we need to check

that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \stackrel{\operatorname{inj}_{2}}{\longleftarrow} (Y, y_{0})$$

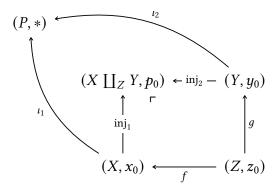
$$\operatorname{inj}_{1} \circ f = \operatorname{inj}_{2} \circ g, \qquad \operatorname{inj}_{1} \qquad \qquad \int_{g} g$$

$$(X, x_{0}) \stackrel{\longleftarrow}{\longleftarrow} (Z, z_{0}).$$

Indeed, given $z \in Z$, we have

$$\begin{aligned} \left[\inf_{1} \circ f \right](z) &= \inf_{1} (f(z)) \\ &= \left[(0, f(z)) \right] \\ &= \left[(1, g(z)) \right] \\ &= \inf_{2} (g(z)) \\ &= \left[\inf_{2} \circ g \right](z), \end{aligned}$$

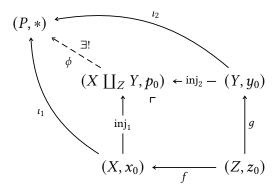
where [(0, f(z))] = [(1, g(z))] by the definition of the relation \sim on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X\coprod_Z Y, p_0) \to (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1,$$

$$\phi \circ \operatorname{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$

= $\iota_1(x_0)$
= *,

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$

= $\iota_2(y_0)$
= *

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets.

00BG

PROPOSITION 6.3.4.1.4 ▶ PROPERTIES OF PUSHOUTS OF POINTED SETS

Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00BH

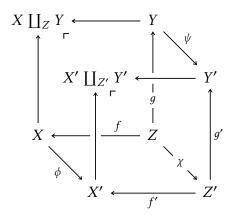
1. Functoriality. The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 : \operatorname{Fun}(\mathcal{P}, \operatorname{Sets}) \to \operatorname{Sets}_*,$$

where \mathcal{P} is the category that looks like this:



In particular, the action on morphisms of $-1 \coprod_{-3} -1$ is given by sending a morphism



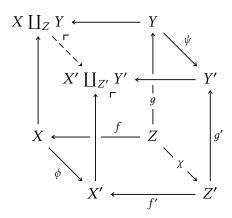
in $\operatorname{Fun}(\mathcal{P},\operatorname{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

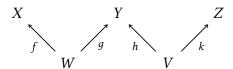
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

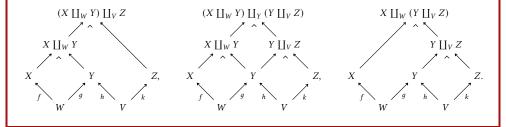
2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams



00BJ

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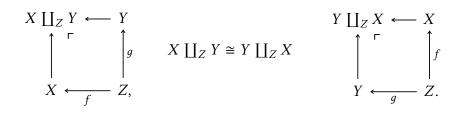
00BK

3. *Unitality*. We have isomorphisms of sets



00BL

4. Commutativity. We have an isomorphism of sets



00BM

5. Interaction With Coproducts. We have

$$X \coprod_{\text{pt}} Y \cong X \vee Y, \qquad \uparrow \qquad \downarrow [y_0]$$

$$X \longleftarrow_{[x_0]} Y \cong X \vee Y, \qquad \uparrow \qquad \downarrow [y_0]$$

00BN

6. *Symmetric Monoidality.* The triple (Sets_{*}, \coprod_X , (X, x_0)) is a symmetric monoidal category.

PROOF 6.3.4.1.5 ► PROOF OF PROPOSITION 6.3.4.1.4

Item 1: Functoriality

This is a special case of functoriality of co/limits, Limits and Colimits, $\ref{eq:condition}$ of $\ref{eq:condition}$, with the explicit expression for ξ following from the commutativity

of the cube pushout diagram.

Item 2: Associativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.4.1.8.

Item 3: Unitality

This follows from Constructions With Sets, Item 5 of Proposition 4.2.4.1.8.

Item 4: Commutativity

This follows from Constructions With Sets, Item 6 of Proposition 4.2.4.1.8.

Item 5: Interaction With Coproducts

Omitted.

Item 6: Symmetric Monoidality

Omitted.

00BP 6.3.5 Coequalisers

Let $f, g: (X, x_0) \Rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BQ DEFINITION 6.3.5.1.1 ➤ COEQUALISERS OF POINTED SETS

The **coequaliser of** (f, g) is the pointed set $(CoEq(f, g), [y_0])$.

0259 CONSTRUCTION 6.3.5.1.2 ► CONSTRUCTION OF COEQUALISERS OF POINTED SETS

The **coequaliser of** (f,g) is the pair $((CoEq(f,g),[y_0]), coeq(f,g))$ consisting of:

- The Colimit. The pointed set $(CoEq(f, g), [y_0])$, where CoEq(f, g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
- The Cocone. The map

$$coeq(f, q): Y \rightarrow (CoEq(f, q), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Construction 4.2.5.1.2.

PROOF 6.3.5.1.3 ► PROOF OF CONSTRUCTION 6.3.5.1.2

We claim that $(CoEq(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_{*}. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f, g) \circ f = coeq(f, g) \circ g.$$

Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$

$$\stackrel{\text{def}}{=} [f(x)]$$

$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f,g) \circ g](x)$$

for each $x \in X$. Next, we prove that CoEq(f, g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{\operatorname{coeq}(f,g)} (\operatorname{CoEq}(f,g), [y_0])$$

$$(C, *)$$

in Sets. Then, since c(f(a)) = c(g(a)) for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Proposition 10.6.2.1.3 that there exists a unique map $\phi \colon \operatorname{CoEq}(f,g) \stackrel{\exists!}{\longrightarrow} C$ making the diagram

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ coeq(f, g)]([y_0])$$
= $c([y_0])$
= *,

where we have used that c is a morphism of pointed sets.

00BR

PROPOSITION 6.3.5.1.4 ▶ PROPERTIES OF COEQUALISERS OF POINTED SETS

Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BS

1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)}\cong \underbrace{\mathrm{CoEq}(f,g,h)\cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g),}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f \atop -g \xrightarrow{h}} (Y, y_0)$$

in Sets_{*}.

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2. Unitality. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

00BU

3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f, g) \cong CoEq(g, f)$$
.

PROOF 6.3.5.1.5 ► PROOF OF PROPOSITION 6.3.5.1.4

Item 1: Associativity

This follows from Constructions With Sets, Item 1 of Proposition 4.2.5.1.7.

Item 2: Unitality

This follows from Constructions With Sets, Item 2 of Proposition 4.2.5.1.7.

Item 3: Commutativity

This follows from Constructions With Sets, Item 3 of Proposition 4.2.5.1.7.

6.4 Constructions With Pointed Sets

00BW 6.4.1 Free Pointed Sets

Let *X* be a set.

00BX DEFINITION 6.4.1.1.1 ► FREE POINTED SETS

The **free pointed set on** X is the pointed set X^+ consisting of:

• The Underlying Set. The set X^+ defined by I^-

$$X^{+} \stackrel{\text{def}}{=} X \coprod \text{pt}$$
$$\stackrel{\text{def}}{=} X \coprod \{ \star \}.$$

• *The Basepoint.* The element \star of X^+ .

PROPOSITION 6.4.1.1.2 ► PROPERTIES OF FREE POINTED SETS

Let *X* be a set.

00BY

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1. Functoriality. The assignment $X \mapsto X^+$ defines a functor

$$(-)^+$$
: Sets \rightarrow Sets_{*},

where:

¹Further Notation: We sometimes write \star_X for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

• Action on Objects. For each $X \in \text{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

• Action on Morphisms. For each morphism $f: X \to Y$ of Sets, the image

$$f^+\colon X^+\to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. Adjointness. We have an adjunction

$$((-)^+ \dashv \stackrel{\leftarrow}{\sim}): \operatorname{Sets} \stackrel{(-)^+}{\underset{\leftarrow}{\smile}} \operatorname{Sets}_*,$$

witnessed by a bijection of sets

$$\operatorname{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \operatorname{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$.

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+,(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathsf{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} \colon X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod} \colon \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

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00C2

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)^+_{\mathbb{1}}) \colon (\mathsf{Sets}, \mathsf{x}, \mathsf{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_{1}^+ \colon S^0 \xrightarrow{\sim} \mathrm{pt}^+,$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

PROOF 6.4.1.1.3 ► PROOF OF PROPOSITION 6.4.1.1.2

Item 1: Functoriality

We claim that $(-)^+$ is indeed a functor:

• Preservation of Identities. Let $X \in \text{Obj}(\mathsf{Sets})$. We have

$$\operatorname{id}_{X}^{+}(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_{X} & \text{if } x = \star_{X}, \end{cases}$$

for each $x \in X^+$, so $id_X^+ = id_{X^+}$.

• Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$
, $g: Y \to Z$,

we have

$$[g^+ \circ f^+](x) \stackrel{\text{def}}{=} g^+(f^+(x))$$
$$\stackrel{\text{def}}{=} g^+(f(x))$$
$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^+(x)$$

for each $x \in X$ and

$$[g^{+} \circ f^{+}](\star_{X}) \stackrel{\text{def}}{=} g^{+}(f^{+}(\star_{X}))$$

$$\stackrel{\text{def}}{=} g^{+}(\star_{Y})$$

$$\stackrel{\text{def}}{=} \star_{Z}$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(\star_{X}),$$

so
$$(g \circ f)^+ = g^+ \circ f^+$$
.

This finishes the proof.

Item 2: Adjointness

We proceed in a few steps:

• Map I. We define a map

$$\Phi_{XY} : \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger} \colon X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

• Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function $\xi \colon X \to Y$ to the morphism of pointed sets

$$\xi^{\dagger} \colon \left(X^+, \star_X \right) \to (Y, y_0)$$

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defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

• Invertibility I. Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \left(\Phi_{X,Y}(\xi)\right) \\ &= \Psi_{X,Y} \left(\xi^{\dagger}\right) \\ &\stackrel{\text{def}}{=} \left[\!\!\left[x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\!\!\right] \\ &= \left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\!\!\right] \\ &= \xi \\ \stackrel{\text{def}}{=} \left[\text{idsets}\left(y + y_1\right)\right](\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}$$
.

• *Invertibility II.* Given a map of sets $\xi \colon X \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} \left(\Psi_{X,Y}(\xi)\right) \\ &= \Phi_{X,Y} \left(\xi^{\dagger}\right) \\ &= \Phi_{X,Y} \left(\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\right]\right) \\ &= \left[\left[x \mapsto \xi(x)\right]\right] \end{split}$$

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$$= \xi$$

$$\stackrel{\text{def}}{=} \left[id_{\text{Sets}(X,Y)} \right] (\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}$$
.

• Naturality for Φ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\operatorname{Sets}_*(X'^{,+}, Y) \xrightarrow{\Phi_{X',Y}} \operatorname{Sets}(X', Y)$$

$$f^* \downarrow \qquad \qquad \downarrow f^*$$

$$\operatorname{Sets}_*(X^+, Y) \xrightarrow{\Phi_{X,Y}} \operatorname{Sets}(X, Y)$$

commutes. Indeed, given a morphism of pointed sets $\xi \colon X'^{+} \to Y$, we have

$$\begin{split} \big[\Phi_{X,Y} \circ f^* \big] (\xi) &= \Phi_{X,Y} (f^*(\xi)) \\ &= \Phi_{X,Y} (\xi \circ f) \\ &= \xi \circ f \\ &= \Phi_{X',Y} (\xi) \circ f \\ &= f^* \big(\Phi_{X',Y} (\xi) \big) \\ &= f^* \big(\Phi_{X',Y} (\xi) \big) \\ &= \big[f^* \circ \Phi_{X',Y} \big] (\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for $\boldsymbol{\Phi}$ above indeed commutes.

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• Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$q: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\begin{split} \mathsf{Sets}_*(X^+,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}(X,Y) \\ g_* & & \downarrow g_* \\ \mathsf{Sets}_*(X^+,Y'), & \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}(X,Y') \end{split}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger} : X^+ \to Y,$$

we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_* \left(\Phi_{X,Y'}(\xi)\right) \\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'} \circ q_* = q_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

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• Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^+$ with respect to \coprod and \vee as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{+,\coprod}_{X,Y}: X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+,\coprod,-1}\colon (X\coprod Y)^+\stackrel{\sim}{\dashrightarrow} X^+\vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} \colon \operatorname{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_\emptyset .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \land y \in X^+ \land Y^+$, with inverse

$$(-)_{X,Y}^{+,-1} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \operatorname{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $pt^+ = \{\star, \star_{pt}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

01QS 6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

01QT DEFINITION 6.4.2.1.1 ► SETS WITH DELETED BASEPOINTS

The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^{-} \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

PROPOSITION 6.4.2.1.2 ▶ PROPERTIES OF SETS WITH DELETED BASEPOINTS

Let (X, x_0) be a pointed set.

01QV 1. Functoriality. The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- \colon \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets},$$

where:

01QW

• Action on Objects. For each $X \in Obj(Sets^{actv}_*)$, we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where X^- is the set of Definition 6.4.2.1.1.

• Action on Morphisms. For each morphism $f: X \to Y$ of $\mathsf{Sets}^{\mathsf{actv}}_*$, the image

$$f^-\colon X^-\to Y^-$$

of f by $(-)^-$ is the map defined by

$$f^{-}(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

2. Adjoint Equivalence. We have an adjoint equivalence of categories

$$((-)^- + (-)^+)$$
: Sets actv $\stackrel{(-)^-}{\underset{(-)^+}{\longleftarrow}}$ Sets,

witnessed by a bijection of sets

$$Sets(X^-, Y) \cong Sets_*(X, Y^+),$$

natural in $X \in \text{Obj}(\mathsf{Sets}_*)$ and $Y \in \text{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^{-})^{+} \cong X,$$
$$(Y^{+})^{-} \cong Y,$$

once again natural in $X \in \text{Obj}(\mathsf{Sets}_*)$ and $Y \in \text{Obj}(\mathsf{Sets})$.

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-},(-)^{-,\vee},(-)_{\mathbb{1}}^{-,\vee}\right)\colon \left(\mathsf{Sets}^{\mathrm{actv}}_{*},\vee,\mathsf{pt}\right), \to (\mathsf{Sets}, \coprod, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee}\colon X^-\coprod Y^-\stackrel{\sim}{\dashrightarrow} (X\vee Y)^-,$$
$$(-)_{\mathbb{1}}^{-,\vee}\colon \not O\stackrel{\sim}{\dashrightarrow} \mathsf{pt}^-,$$

natural in $X, Y \in Obj(Sets)$.

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)^{-,\times}_{\mathbb{1}}) : (\operatorname{Sets}^{\operatorname{actv}}_*, \wedge, S^0), \to (\operatorname{Sets}, \times, \operatorname{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-},$$
$$(-)_{1}^{-} \colon \operatorname{pt} \xrightarrow{\sim} (S^{0})^{-},$$

natural in $X, Y \in \text{Obj}(\mathsf{Sets})$.

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PROOF 6.4.2.1.3 ► PROOF OF PROPOSITION 6.4.2.1.2

Item 1: Functoriality

We claim that $(-)^-$ is indeed a functor:

• Preservation of Identities. Let $X \in \text{Obj}(\mathsf{Sets})$. We have

$$id_X^-(x) \stackrel{\text{def}}{=} x$$

for each $x \in X^-$, so $id_X^- = id_{X^-}$.

• Preservation of Composition. Given morphisms of pointed sets

$$f: (X, x_0) \to (Y, y_0),$$

 $q: (Y, y_0) \to (Z, z_0),$

we have

$$[g^{-} \circ f^{-}](x) \stackrel{\text{def}}{=} g^{-}(f^{-}(x))$$

$$\stackrel{\text{def}}{=} g^{-}(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^{-}(x)$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2: Adjoint Equivalence

We proceed in a few steps:

1. Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-, Y) \to \mathsf{Sets}^{\mathsf{actv}}_*(X, Y^+)$$

by sending a map $\xi\colon X^-\to Y$ to the active morphism of pointed sets

$$\xi^{\dagger} \colon X \to Y^{+}$$

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given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in$ $Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$

given by sending an active morphism of pointed sets $\xi\colon X\to Y^+$ to the map

$$\xi^{\dagger} \colon X^{-} \to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi \colon X^- \to Y$, we have

$$\begin{split} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right] (\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y} \Big(\Phi_{X,Y}(\xi)\Big) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y} \Bigg(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \Bigg) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &= \left[\text{id}_{\mathsf{Sets}(X^-,Y)} \right] (\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)} .$$

4. Invertibility II. Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

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we have

$$\begin{split} \left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}\big(\Psi_{X,Y}(\xi)\big) \\ &= \Phi_{X,Y}\big(\llbracket x \mapsto \xi(x) \rrbracket\big) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= \left[\text{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} \right](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} .$$

5. *Naturality for* Φ , *Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x_0'),$$

the diagram

$$\operatorname{Sets}(X^{',-},Y) \xrightarrow{\Phi_{X',Y}} \operatorname{Sets}^{\operatorname{actv}}_*(X',Y^+)$$

$$f^* \downarrow \qquad \qquad \downarrow f^* \downarrow$$

$$\operatorname{Sets}_*(X^-,Y) \xrightarrow{\Phi_{X,Y}} \operatorname{Sets}^{\operatorname{actv}}_*(X,Y^+)$$

commutes. Indeed, given a map of sets $\xi \colon X' \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \left[\!\left[x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases}\right] \right] \\ &= f^* \left(\left[\!\left[x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x'_0 \end{cases}\right] \right) \end{split}$$

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$$= f^* (\Phi_{X',Y}(\xi))$$

= $[f^* \circ \Phi_{X',Y}](\xi).$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. Naturality for Φ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y_0'),$$

the diagram

$$\begin{array}{c|c} \mathsf{Sets}(X^-,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y^+) \\ & g_* & & \downarrow g_* \\ \\ \mathsf{Sets}(X^-,Y') & \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}^{\mathrm{actv}}_*(X,Y'^{,+}) \end{array}$$

commutes. Indeed, given a map of sets $\xi \colon X^- \to Y$, we have

$$\begin{split} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= \left[\!\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \right]\!\!\right] \\ &= g_* \left(\!\!\left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases} \right]\!\!\right) \\ &= g_* \left(\Phi_{X,Y'}(\xi)\right) \\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'} \circ q_* = q_* \circ \Phi_{X,Y'},$$

and the naturality diagram for $\boldsymbol{\Phi}$ above indeed commutes.

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7. Naturality for Ψ . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Proposition 11.9.7.1.2 that Ψ is also natural in each argument.

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8. Fully Faithfulness of $(-)^-$. We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{X,Y}^- \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y) \xrightarrow{\sim} \mathsf{Sets}(X^-,Y^-).$$

Indeed, the inverse map

$$(-)_{XY}^{-,-1} : \mathsf{Sets}(X^-, Y^-) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X, Y)$$

is given by sending a map of sets $f\colon X^-\to Y^-$ to the active morphism of pointed sets $f^\dag\colon X\to Y$ defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_{0} & \text{if } x = x_{0} \end{cases}$$

for each $x \in X$.

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9. Essential Surjectivity of $(-)^-$. We need to show that, given an object $X \in \text{Obj}(\mathsf{Sets})$, there exists some $X' \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}}_*)$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$(X^{+})^{-} \stackrel{\text{def}}{=} (X \cup \{\star_{X}\})^{-}$$
$$\stackrel{\text{def}}{=} (X \cup \{\star_{X}\}) \setminus \{\star_{X}\}$$
$$= X,$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

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10. The Functor $(-)^-$ Is an Equivalence. Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Proposition 11.6.7.1.2.

This finishes the proof.

Item 3: Symmetric Strong Monoidality With Respect to Wedge Sums

We construct the strong monoidal structure on $(-)^-$ with respect to \vee and [] as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{-,\vee}_{XY} \colon X^- \coprod Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \coprod Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1} \colon (X \vee Y)^- \xrightarrow{\sim} X^- \coprod Y^-$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\vee,\mathbb{1}} \colon \emptyset \xrightarrow{\sim} \mathsf{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

Item 4: Symmetric Strong Monoidality With Respect to Smash Product

We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^-_{X,Y} \colon X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{X,Y}(x,y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1} \colon (X \wedge Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x,y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{-,1} \colon \operatorname{pt} \xrightarrow{\sim} (S^0)^-$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Appendices

A Other Chapters

Preliminaries

- 1. Introduction
- 2. A Guide to the Literature

Sets

3. Sets

- 4. Constructions With Sets
- Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

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Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

References

[MSE 2855868] Qiaochu Yuan. Is the category of pointed sets Cartesian closed?

Mathematics Stack Exchange. URL: https://math.stackexchange.

com/q/2855868 (cit. on pp. 8, 19).

[MSE 884460] Martin Brandenburg. Why are the category of pointed sets and the

category of sets and partial functions "essentially the same"? Mathematics Stack Exchange. URL: https://math.stackexchange.

com/q/884460 (cit. on p. 8).