

Constructions With Monoidal Categories

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This chapter contains some material on constructions with monoidal categories.

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13.1 Moduli Categories of Monoidal Structures

13.1.1 The Moduli Category of Monoidal Structures on a Category

Let C be a category.

Definition 13.1.1.1.1. The **moduli category of monoidal structures on C** is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\text{def}}{=} \text{pt} \times_{\text{Cats}} \text{MonCats},$$

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \text{MonCats} \\ \downarrow & \lrcorner & \downarrow \text{忘} \\ \text{pt} & \xrightarrow{[C]} & \text{Cats.} \end{array}$$

Remark 13.1.1.1.2. In detail, the **moduli category of monoidal structures on C** is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C .
- *Morphisms.* A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$ is a strong monoidal functor structure

$$\text{id}_C^\otimes: A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$

$$\text{id}_{\mathbb{1}|C}^\otimes: \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor $\text{id}_C: C \rightarrow C$ of C .

- *Identities.* For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{pt} \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\text{id}_M^{\mathcal{M}_{\mathbb{E}_1}(C)} \stackrel{\text{def}}{=} (\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes),$$

where $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$ is the identity monoidal functor of C of??.

- *Composition.* For each $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)}: \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N, P) \times \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, N) \rightarrow \text{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, P)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at (M, N, P) is defined by

$$\left(\text{id}_C^{\otimes, \prime}, \text{id}_{\mathbb{1}|C}^{\otimes, \prime} \right) \circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \left(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes \right) \stackrel{\text{def}}{=} \left(\text{id}_C^{\otimes, \prime} \circ \text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^{\otimes, \prime} \circ \text{id}_{\mathbb{1}|C}^\otimes \right).$$

Remark 13.1.1.1.3. In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$ satisfies the following conditions:

1. *Naturality.* For each pair $f: A \rightarrow X$ and $g: B \rightarrow Y$ of morphisms of C , the diagram

$$\begin{array}{ccc} A \boxtimes_C B & \xrightarrow{f \boxtimes_C g} & X \boxtimes_C Y \\ \text{id}_{A,B}^\otimes \downarrow & & \downarrow \text{id}_{X,Y}^\otimes \\ A \otimes_C B & \xrightarrow{f \otimes_C g} & X \otimes_C Y \end{array}$$

commutes.

2. *Monoidality.* For each $A, B, C \in \text{Obj}(C)$, the diagram

$$\begin{array}{ccc} & (A \boxtimes_C B) \boxtimes_C C & \\ \text{id}_{A,B}^\otimes \boxtimes \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^{C'} \\ (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\ \text{id}_{A \otimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\ (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\ \alpha_{A,B,C}^C \searrow & & \swarrow \text{id}_{A,B \otimes_C C}^\otimes \\ & A \otimes_C (B \otimes_C C) & \end{array}$$

commutes.

3. *Left Monoidal Unity.* For each $A \in \text{Obj}(C)$, the diagram

$$\begin{array}{ccc} & \mathbb{1}_C \boxtimes_C A & \xrightarrow{\text{id}_{\mathbb{1}'_C, A}^\otimes} \mathbb{1}_C \otimes_C A \\ \text{id}_{\mathbb{1}}^\otimes \boxtimes \text{id}_A \nearrow & & \searrow \lambda_A^C \\ \mathbb{1}'_C \boxtimes_C A & \xrightarrow{\lambda_A^{C'}} & A \end{array}$$

commutes.

4. *Right Monoidal Unity.* For each $A \in \text{Obj}(C)$, the diagram

$$\begin{array}{ccc}
 & A \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_A^\otimes \text{id}_{\mathbb{1}_C}^\otimes} A \otimes_C \mathbb{1}_C \\
 \text{id}_A \boxtimes_C \text{id}_{\mathbb{1}_C}^\otimes \nearrow & & \searrow \rho_A^C \\
 A \boxtimes_C \mathbb{1}'_C & \xrightarrow{\rho_A^{C,'}} & A
 \end{array}$$

commutes.

Proposition 13.1.1.1.4. Let C be a category.

1. *Extra Monoidality Conditions.* Let $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$ be a morphism of $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$.

(a) The diagram

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C} & (A \otimes_C B) \boxtimes_C C \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A \otimes_C B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\text{id}_{A,B}^\otimes \otimes_C \text{id}_C} & (A \otimes_C B) \otimes_C C
 \end{array}$$

commutes.

(b) The diagram

$$\begin{array}{ccc}
 A \boxtimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes} & A \boxtimes_C (B \otimes_C C) \\
 \text{id}_{A, B \boxtimes_C C}^\otimes \downarrow & & \downarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 A \otimes_C (B \boxtimes_C C) & \xrightarrow{\text{id}_A \otimes_C \text{id}_{B,C}^\otimes} & A \otimes_C (B \otimes_C C)
 \end{array}$$

commutes.

2. *Extra Monoidal Unity Constraints.* Let $(\text{id}_C^\otimes, \text{id}_{\mathbb{1}|C}^\otimes)$ be a morphism of $\mathcal{M}_{\mathbb{B}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,'}, \lambda^{C,'}, \rho^{C,'})$.

(a) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & & \downarrow \rho_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

(b) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \\
 \rho_{\mathbb{1}'_C}^C \downarrow & & \downarrow \lambda_{\mathbb{1}_C}^{C, ' } \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C
 \end{array}$$

commutes.

(c) The diagram

$$\begin{array}{ccc}
 \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \lambda_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

(d) The diagram

$$\begin{array}{ccc}
 \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \rho_{\mathbb{1}_C}^{C, ' } \downarrow & & \downarrow \lambda_{\mathbb{1}'_C}^C \\
 \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}$$

commutes.

3. *Mixed Associators.* Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C'}, \lambda^{C'}, \rho^{C'})$ be monoidal structures on C and let

$$\mathrm{id}_{-1, -2}^{\otimes} : -_1 \boxtimes_C -_2 \rightarrow -_1 \otimes_C -_2$$

be a natural transformation.

- (a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} : (A \otimes_C B) \boxtimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \\ \mathrm{id}_{A \otimes_C B, C}^{\otimes} \downarrow & & \downarrow \mathrm{id}_A \otimes \mathrm{id}_{B, C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C'}} & A \boxtimes_C (B \boxtimes_C C) \\ \mathrm{id}_{A, B}^{\otimes} \boxtimes \mathrm{id}_C \downarrow & & \downarrow \mathrm{id}_{A, B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} & A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

- (b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{ccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} & A \boxtimes_C (B \otimes_C C) \\ \mathrm{id}_{A, B}^{\otimes} \otimes \mathrm{id}_C \downarrow & & \downarrow \mathrm{id}_{A, B \otimes_C C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_A \boxtimes_C \text{id}_{B, C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^\boxtimes} & A \boxtimes_C (B \otimes_C C)
 \end{array}$$

commute, then the natural transformation id^\otimes satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

(c) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes, \otimes} : (A \boxtimes_C B) \otimes_C C \rightarrow A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{ccc}
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C) \\
 \text{id}_{A, B}^\otimes \otimes \text{id}_C \downarrow & & \downarrow \text{id}_A \otimes_C \text{id}_{B, C}^\otimes \\
 (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} & A \otimes_C (B \otimes_C C)
 \end{array}$$

and

$$\begin{array}{ccc}
 (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} & A \boxtimes_C (B \boxtimes_C C) \\
 \text{id}_{A \boxtimes_C B, C}^\otimes \downarrow & & \downarrow \text{id}_{A, B \boxtimes_C C}^\otimes \\
 (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes, \otimes}} & A \otimes_C (B \boxtimes_C C)
 \end{array}$$

commute, then the natural transformation id^\otimes satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

Proof. **Item 1, Extra Monoidality Conditions:** We claim that **Items 1a** and **1b** are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of id^\otimes with respect to the morphisms $\text{id}_{A, B}^\otimes$ and id_C .

2. *Proof of Item 1b:* This follows from the naturality of id^\otimes with respect to the morphisms id_A and $\text{id}_{B,C}^\otimes$.

This finishes the proof.

Item 2, Extra Monoidal Unity Constraints: We claim that *Items 2a* and *2b* are indeed true:

1. *Proof of Item 1a:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 1_C \otimes_C 1'_C & \xrightarrow{\text{id}_{1_C, 1'_C}^{\otimes, -1}} & 1_C \boxtimes_C 1'_C & & \\
 \downarrow \lambda_{1'_C}^C & \searrow \text{id}_{1_C} \otimes_C \text{id}_1^\otimes & \downarrow \text{id}_{1_C} \boxtimes_C \text{id}_1^\otimes & & \\
 & & 1_C \otimes_C 1_C & \xrightarrow{\text{id}_{1_C, 1_C}^{\otimes, -1}} & 1_C \boxtimes_C 1_C \\
 & & \parallel & (2) & \downarrow \text{id}_{1_C, 1_C}^\otimes \\
 & & 1_C \otimes_C 1_C & & 1_C \otimes_C 1_C \\
 & (3) & & & (4) \\
 & & \downarrow \lambda_{1_C}^C = \rho_{1_C}^C & & \downarrow \rho_{1_C}^{C, \prime} \\
 & & 1_C & & 1_C
 \end{array}$$

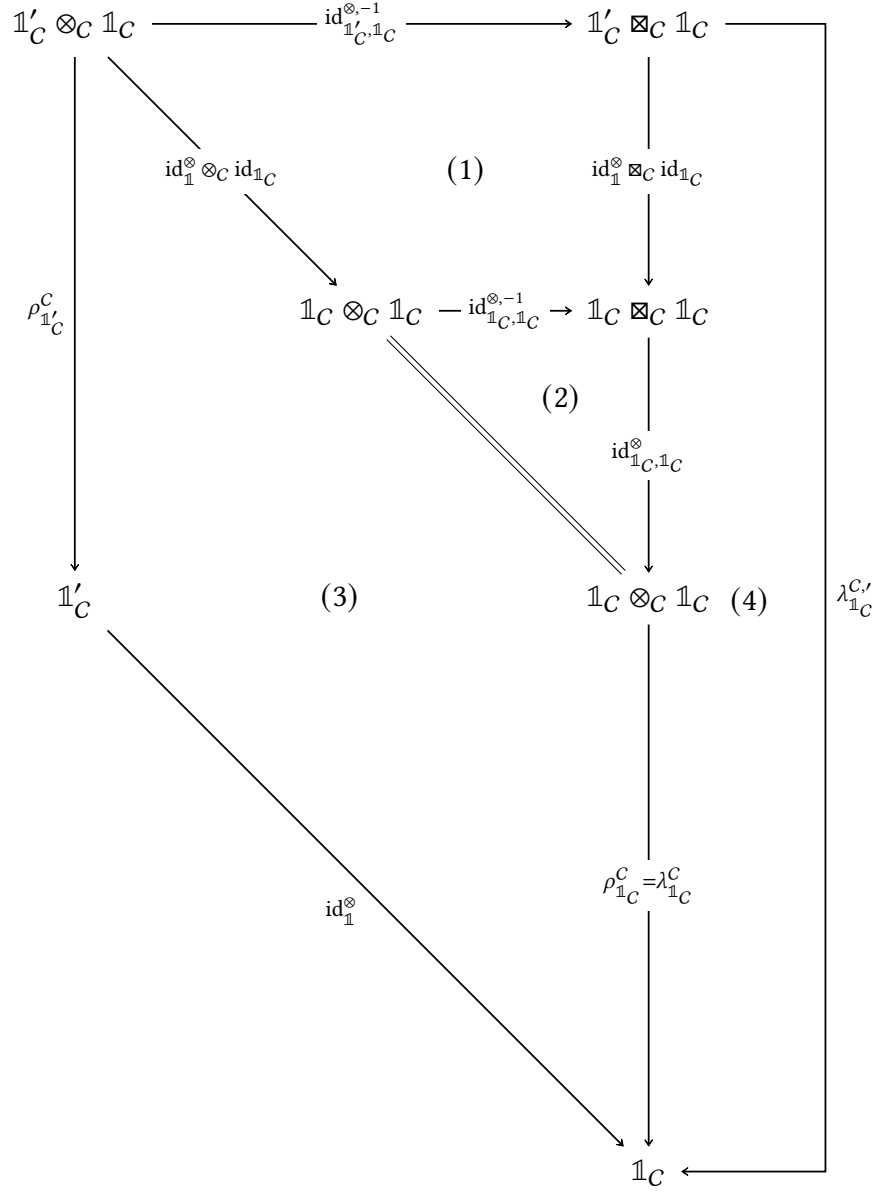
Diagram description: The diagram is a commutative diagram with nodes and arrows. The top row consists of $1_C \otimes_C 1'_C$, $1_C \boxtimes_C 1'_C$, and 1_C . The middle row consists of $1_C \otimes_C 1_C$ and $1_C \boxtimes_C 1_C$. The bottom row consists of 1_C . Arrows are labeled with various morphisms: $\lambda_{1'_C}^C$, $\text{id}_{1_C} \otimes_C \text{id}_1^\otimes$, $\text{id}_{1_C} \boxtimes_C \text{id}_1^\otimes$, $\text{id}_{1_C, 1'_C}^{\otimes, -1}$, $\text{id}_{1_C, 1_C}^{\otimes, -1}$, $\text{id}_{1_C, 1_C}^\otimes$, $\lambda_{1_C}^C = \rho_{1_C}^C$, $\rho_{1_C}^{C, \prime}$, and id_1^\otimes . The diagram is divided into four regions labeled (1), (2), (3), and (4).

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\text{id}_C^{\otimes, -1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{1_C}^C = \lambda_{1_C}^C$ comes from ??;
- Subdiagram (4) commutes by the right monoidal unity of $(\text{id}_C, \text{id}_C^{\otimes}, \text{id}_{C|1}^{\otimes})$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $id_C^{\otimes, -1}$;
- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from ??;
- Subdiagram (4) commutes by the left monoidal unity of $(\text{id}_C, \text{id}_C^\otimes, \text{id}_{C|\mathbb{1}}^\otimes)$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 \downarrow \rho_{\mathbb{1}'_C}^C & & \downarrow \lambda_{\mathbb{1}_C}^{C, \prime} & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^\otimes} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C.
 \end{array}
 \quad \begin{array}{c} (1) \end{array} \quad \begin{array}{c} (\dagger)$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1b**;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}'_C \otimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^\otimes} & \mathbb{1}'_C \otimes_C \mathbb{1}_C \\
 & & \downarrow \lambda_{\mathbb{1}_C}^{C, \prime} & & \downarrow \rho_{\mathbb{1}'_C}^C \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}'_C
 \end{array}
 \quad \begin{array}{c} (\dagger)$$

commutes. But since $\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 \lambda_{\mathbb{1}'_C}^C \downarrow & (1) & \rho_{\mathbb{1}_C}^{C, \prime} \downarrow & (\dagger) & \lambda_{\mathbb{1}'_C}^C \downarrow \\
 \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes}} & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by **Item 1a**;

it follows that the diagram

$$\begin{array}{ccccc}
 \mathbb{1}_C \otimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C & \xrightarrow{\text{id}_{\mathbb{1}_C, \mathbb{1}'_C}^{\otimes, -1}} & \mathbb{1}_C \otimes_C \mathbb{1}'_C \\
 & & \rho_{\mathbb{1}_C}^{C, \prime} \downarrow & (\dagger) & \lambda_{\mathbb{1}'_C}^C \downarrow \\
 & & \mathbb{1}_C & \xrightarrow{\text{id}_{\mathbb{1}}^{\otimes, -1}} & \mathbb{1}_C
 \end{array}$$

commutes. But since $\text{id}_{\mathbb{1}}^{\otimes, -1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that **Items 3a to 3c** are indeed true:

1. *Proof of Item 3a:* We may partition the monoidality diagram for id^{\otimes} of

Item 2 of Definition 13.1.1.3 as follows:

$$\begin{array}{ccccc}
 & (A \boxtimes_C B) \boxtimes_C C & & & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & \downarrow & \searrow \alpha_{A,B,C}^C & & \\
 (A \otimes_C B) \boxtimes_C C & \text{id}_{A \boxtimes_C B, C}^\otimes & A \boxtimes_C (B \boxtimes_C C) & & \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & (1) \downarrow & (2) \downarrow & & \\
 (A \otimes_C B) \otimes_C C & (A \boxtimes_C B) \otimes_C C & A \boxtimes_C (B \otimes_C C) & & \\
 \swarrow \text{id}_{A,B}^\otimes \otimes_C \text{id}_C & \searrow \alpha_{A,B,C}^\otimes & \downarrow \text{id}_A \boxtimes_C \text{id}_{B,C}^\otimes & & \\
 (A \otimes_C B) \otimes_C C & (3) & A \boxtimes_C (B \otimes_C C) & & \\
 \searrow \alpha_{A,B,C}^{C'} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & &
 \end{array}$$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id^\otimes satisfies the monoidality condition of Item 2 of Definition 13.1.1.3.

2. *Proof of Item 3b:* We may partition the monoidality diagram for id^\otimes of Item 2 of Definition 13.1.1.3 as follows:

$$\begin{array}{ccccc}
 & (A \boxtimes_C B) \boxtimes_C C & & & \\
 \text{id}_{A,B}^\otimes \boxtimes_C \text{id}_C \swarrow & \downarrow & \searrow \alpha_{A,B,C}^C & & \\
 (A \otimes_C B) \boxtimes_C C & (1) & A \boxtimes_C (B \boxtimes_C C) & & \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \searrow \alpha_{A,B,C}^\otimes & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 (A \otimes_C B) \otimes_C C & A \otimes_C (B \boxtimes_C C) & & & \\
 \downarrow \text{id}_A \otimes_C \text{id}_{B,C}^\otimes & (2) \downarrow & (3) \downarrow & & \\
 (A \otimes_C B) \otimes_C C & \text{id}_A \otimes_C \text{id}_{B,C}^\otimes & A \boxtimes_C (B \otimes_C C) & & \\
 \searrow \alpha_{A,B,C}^{C'} & \downarrow & \swarrow \text{id}_{A, B \otimes_C C}^\otimes & & \\
 & A \otimes_C (B \otimes_C C) & & &
 \end{array}$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by **Item 1b** of **Item 1**.

it follows that the boundary diagram also commutes, i.e. id^\otimes satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

3. *Proof of Item 3c:* We may partition the monoidality diagram for id^\otimes of **Item 2** of **Definition 13.1.1.1.3** as follows:

$$\begin{array}{ccc}
 & (A \boxtimes_C B) \boxtimes_C C & \\
 \text{id}_{A,B}^\otimes \boxtimes \text{id}_C \swarrow & & \searrow \alpha_{A,B,C}^C \\
 (A \otimes_C B) \boxtimes_C C & & A \boxtimes_C (B \boxtimes_C C) \\
 \downarrow \text{id}_{A \otimes_C B, C}^\otimes & \searrow \alpha_{A,B,C}^{\boxtimes, \otimes} & \downarrow \text{id}_A \boxtimes \text{id}_{B,C}^\otimes \\
 (A \otimes_C B) \otimes_C C & & A \boxtimes_C (B \otimes_C C) \\
 \searrow \alpha_{A,B,C}^{C, \otimes} & & \swarrow \text{id}_{A, B \otimes_C C}^\otimes \\
 & A \otimes_C (B \otimes_C C) &
 \end{array}$$

(1) (2)

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^\otimes satisfies the monoidality condition of **Item 2** of **Definition 13.1.1.1.3**.

This finishes the proof. □

13.1.2 The Moduli Category of Braided Monoidal Structures on a Category

13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category

13.2 Moduli Categories of Closed Monoidal Structures

13.3 Moduli Categories of Refinements of Monoidal Structures

13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

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