

Monoidal Structures on the Category of Sets

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01NK This chapter contains some material on monoidal structures on Sets.

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01NL 5.1 The Monoidal Category of Sets and Products

01NM 5.1.1 Products of Sets

See [Constructions With Sets, Section 4.1.3](#).

01NN 5.1.2 The Internal Hom of Sets

See [Constructions With Sets, Section 4.3.5](#).

01NP 5.1.3 The Monoidal Unit

01NQ DEFINITION 5.1.3.1.1 ► THE MONOIDAL UNIT OF \times

The **monoidal unit of the product of sets** is the functor

$$\mathbb{1}^{\text{Sets}}: \text{pt} \rightarrow \text{Sets}$$

defined by

$$\mathbb{1}_{\text{Sets}} \stackrel{\text{def}}{=} \text{pt},$$

where pt is the terminal set of [Constructions With Sets, Definition 4.1.1.1.1](#).

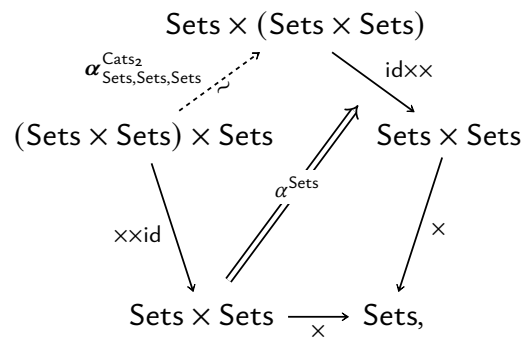
01NR 5.1.4 The Associator

01NS DEFINITION 5.1.4.1.1 ► THE ASSOCIATOR OF \times

The **associator of the product of sets** is the natural isomorphism

$$\alpha^{\text{Sets}}: \times \circ (\times \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}_2}_{\text{Sets}, \text{Sets}, \text{Sets}},$$

as in the diagram



whose component

$$\alpha^{\text{Sets}}_{X,Y,Z}: (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha^{\text{Sets}}_{X,Y,Z}((x, y), z) \stackrel{\text{def}}{=} (x, (y, z))$$

for each $((x, y), z) \in (X \times Y) \times Z$.

PROOF 5.1.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.4.1.1

Invertibility

The inverse of $\alpha^{\text{Sets}}_{X,Y,Z}$ is the morphism

$$\alpha^{\text{Sets}, -1}_{X,Y,Z}: X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \stackrel{\text{def}}{=} ((x, y), z)$$

for each $(x, (y, z)) \in X \times (Y \times Z)$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} \right]((x, y), z) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets},-1} \left(\alpha_{X,Y,Z}^{\text{Sets}}((x, y), z) \right) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \\ &\stackrel{\text{def}}{=} ((x, y), z) \\ &\stackrel{\text{def}}{=} [\text{id}_{(X \times Y) \times Z}]((x, y), z) \end{aligned}$$

for each $((x, y), z) \in (X \times Y) \times Z$, and therefore we have

$$\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} = \text{id}_{(X \times Y) \times Z}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[\alpha_{X,Y,Z}^{\text{Sets}} \circ \alpha_{X,Y,Z}^{\text{Sets},-1} \right](x, (y, z)) &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets}} \left(\alpha_{X,Y,Z}^{\text{Sets},-1}(x, (y, z)) \right) \\ &\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\text{Sets}}((x, y), z) \\ &\stackrel{\text{def}}{=} (x, (y, z)) \\ &\stackrel{\text{def}}{=} [\text{id}_{X \times (Y \times Z)}](x, (y, z)) \end{aligned}$$

for each $(x, (y, z)) \in X \times (Y \times Z)$, and therefore we have

$$\alpha_{X,Y,Z}^{\text{Sets},-1} \circ \alpha_{X,Y,Z}^{\text{Sets}} = \text{id}_{X \times (Y \times Z)}.$$

Therefore $\alpha_{X,Y,Z}^{\text{Sets}}$ is indeed an isomorphism.

Naturality

We need to show that, given functions

$$f: X \rightarrow X',$$

$$\begin{aligned} g: Y &\rightarrow Y', \\ h: Z &\rightarrow Z' \end{aligned}$$

the diagram


$$\begin{array}{ccc} (X \times Y) \times Z & \xrightarrow{(f \times g) \times h} & (X' \times Y') \times Z' \\ \alpha_{X,Y,Z}^{\text{Sets}} \downarrow & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}} \\ X \times (Y \times Z) & \xrightarrow{f \times (g \times h)} & X' \times (Y' \times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} ((x, y), z) & & ((x, y), z) \mapsto ((f(x), g(y)), h(z)) \\ \downarrow & & \downarrow \\ (x, (y, z)) \mapsto (f(x), (g(y), h(z))) & & (f(x), (g(y), h(z))) \end{array}$$

and hence indeed commutes, showing α^{Sets} to be a natural transformation.

Being a Natural Isomorphism

Since α^{Sets} is natural and $\alpha^{\text{Sets}, -1}$ is a componentwise inverse to α^{Sets} , it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that $\alpha^{\text{Sets}, -1}$ is also natural. Thus α^{Sets} is a natural isomorphism. 

01NT 5.1.5 The Left Unitor

01NU DEFINITION 5.1.5.1.1 ► THE LEFT UNITOR OF \times

The **left unitor of the product of sets** is the natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}} : \text{pt} \times X \xrightarrow{\sim} X$$

at $X \in \text{Obj}(\text{Sets})$ is given by

$$\lambda_X^{\text{Sets}}(\star, x) \stackrel{\text{def}}{=} x$$

for each $(\star, x) \in \text{pt} \times X$.

PROOF 5.1.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.5.1.1

Invertibility

The inverse of λ_X^{Sets} is the morphism

$$\lambda_X^{\text{Sets}, -1} : X \xrightarrow{\sim} \text{pt} \times X$$

defined by

$$\lambda_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\left[\lambda_X^{\text{Sets}, -1} \circ \lambda_X^{\text{Sets}} \right](\text{pt}, x) = \lambda_X^{\text{Sets}, -1} \left(\lambda_X^{\text{Sets}}(\text{pt}, x) \right)$$

$$\begin{aligned}
&= \lambda_X^{\text{Sets}, -1}(x) \\
&= (\text{pt}, x) \\
&= [\text{id}_{\text{pt} \times X}](\text{pt}, x)
\end{aligned}$$

for each $(\text{pt}, x) \in \text{pt} \times X$, and therefore we have

$$\lambda_X^{\text{Sets}, -1} \circ \lambda_X^{\text{Sets}} = \text{id}_{\text{pt} \times X}.$$

- *Invertibility II.* We have

$$\begin{aligned}
[\lambda_X^{\text{Sets}} \circ \lambda_X^{\text{Sets}, -1}](x) &= \lambda_X^{\text{Sets}}(\lambda_X^{\text{Sets}, -1}(x)) \\
&= \lambda_X^{\text{Sets}, -1}(\text{pt}, x) \\
&= x \\
&= [\text{id}_X](x)
\end{aligned}$$

for each $x \in X$, and therefore we have

$$\lambda_X^{\text{Sets}} \circ \lambda_X^{\text{Sets}, -1} = \text{id}_X.$$

Therefore λ_X^{Sets} is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \rightarrow Y$, the diagram


$$\begin{array}{ccc}
\text{pt} \times X & \xrightarrow{\text{id}_{\text{pt}} \times f} & \text{pt} \times Y \\
\lambda_X^{\text{Sets}} \downarrow & & \downarrow \lambda_Y^{\text{Sets}} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(\star, x) & & (\star, x) \mapsto (\star, f(x)) \\
\downarrow & & \downarrow \\
x & \mapsto & f(x)
\end{array}$$

and hence indeed commutes. Therefore λ^{Sets} is a natural transformation.

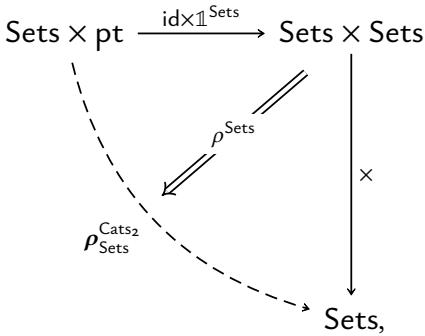
Being a Natural Isomorphism

Since λ^{Sets} is natural and $\lambda^{\text{Sets}, -1}$ is a componentwise inverse to λ^{Sets} , it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that $\lambda^{\text{Sets}, -1}$ is also natural. Thus λ^{Sets} is a natural isomorphism. 

01NV 5.1.6 The Right Unitor

01NW DEFINITION 5.1.6.1.1 ► THE RIGHT UNITOR OF \times

The **right unitor of the product of sets** is the natural isomorphism

$$\rho^{\text{Sets}}: \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2},$$


whose component

$$\rho_X^{\text{Sets}}: X \times \text{pt} \xrightarrow{\sim} X$$

at $X \in \text{Obj}(\text{Sets})$ is given by

$$\rho_X^{\text{Sets}}(x, \star) \stackrel{\text{def}}{=} x$$

for each $(x, \star) \in X \times \text{pt}$.

PROOF 5.1.6.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.6.1.1

Invertibility

The inverse of ρ_X^{Sets} is the morphism

$$\rho_X^{\text{Sets}, -1} : X \dashrightarrow X \times \text{pt}$$

defined by

$$\rho_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (x, \star)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\rho_X^{\text{Sets}, -1} \circ \rho_X^{\text{Sets}} \right](x, \star) &= \rho_X^{\text{Sets}, -1} \left(\rho_X^{\text{Sets}}(x, \star) \right) \\ &= \rho_X^{\text{Sets}, -1}(x) \\ &= (x, \star) \\ &= [\text{id}_{X \times \text{pt}}](x, \star) \end{aligned}$$

for each $(x, \star) \in X \times \text{pt}$, and therefore we have

$$\rho_X^{\text{Sets}, -1} \circ \rho_X^{\text{Sets}} = \text{id}_{X \times \text{pt}}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[\rho_X^{\text{Sets}} \circ \rho_X^{\text{Sets}, -1} \right](x) &= \rho_X^{\text{Sets}} \left(\rho_X^{\text{Sets}, -1}(x) \right) \\ &= \rho_X^{\text{Sets}, -1}(x, \star) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

for each $x \in X$, and therefore we have

$$\rho_X^{\text{Sets}} \circ \rho_X^{\text{Sets}, -1} = \text{id}_X.$$

Therefore ρ_X^{Sets} is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \rightarrow Y$, the diagram


$$\begin{array}{ccc} X \times \text{pt} & \xrightarrow{f \times \text{id}_{\text{pt}}} & Y \times \text{pt} \\ \rho_X^{\text{Sets}} \downarrow & & \downarrow \rho_Y^{\text{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x, \star) & & (x, \star) \mapsto (f(x), \star) \\ \downarrow & & \downarrow \\ x & \mapsto & f(x) \end{array}$$

and hence indeed commutes. Therefore ρ^{Sets} is a natural transformation.

Being a Natural Isomorphism

Since ρ^{Sets} is natural and $\rho^{\text{Sets}, -1}$ is a componentwise inverse to ρ^{Sets} , it follows from **Categories**, **Item 2** of **Proposition 11.9.7.1.2** that $\rho^{\text{Sets}, -1}$ is also natural. Thus ρ^{Sets} is a natural isomorphism. 

01NX 5.1.7 The Symmetry

01NY DEFINITION 5.1.7.1.1 ► THE SYMMETRY OF \times

The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}, \quad \begin{array}{ccc} \text{Sets} \times \text{Sets} & \xrightarrow{\times} & \text{Sets}, \\ & \Downarrow \sigma^{\text{Sets}} & \uparrow \times \\ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2} & & \text{Sets} \times \text{Sets} \end{array}$$

whose component

$$\sigma_{X,Y}^{\text{Sets}} : X \times Y \xrightarrow{\sim} Y \times X$$

at $X, Y \in \text{Obj}(\text{Sets})$ is defined by

$$\sigma_{X,Y}^{\text{Sets}}(x, y) \stackrel{\text{def}}{=} (y, x)$$

for each $(x, y) \in X \times Y$.

PROOF 5.1.7.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.7.1.1

Invertibility

The inverse of $\sigma_{X,Y}^{\text{Sets}}$ is the morphism

$$\sigma_{X,Y}^{\text{Sets}, -1} : Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets}, -1}(y, x) \stackrel{\text{def}}{=} (x, y)$$

for each $(y, x) \in Y \times X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\sigma_{X,Y}^{\text{Sets}, -1} \circ \sigma_{X,Y}^{\text{Sets}} \right](x, y) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets}, -1} \left(\sigma_{X,Y}^{\text{Sets}}(x, y) \right) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets}, -1}(y, x) \\ &\stackrel{\text{def}}{=} (x, y) \\ &\stackrel{\text{def}}{=} [\text{id}_{X \times Y}](x, y) \end{aligned}$$

for each $(x, y) \in X \times Y$, and therefore we have

$$\sigma_{X,Y}^{\text{Sets},-1} \circ \sigma_{X,Y}^{\text{Sets}} = \text{id}_{X \times Y}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[\sigma_{X,Y}^{\text{Sets}} \circ \sigma_{X,Y}^{\text{Sets},-1} \right] (y, x) &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets},-1} \left(\sigma_{X,Y}^{\text{Sets}} (y, x) \right) \\ &\stackrel{\text{def}}{=} \sigma_{X,Y}^{\text{Sets},-1} (x, y) \\ &\stackrel{\text{def}}{=} (y, x) \\ &\stackrel{\text{def}}{=} [\text{id}_{Y \times X}] (y, x) \end{aligned}$$

for each $(y, x) \in Y \times X$, and therefore we have

$$\sigma_{X,Y}^{\text{Sets}} \circ \sigma_{X,Y}^{\text{Sets},-1} = \text{id}_{Y \times X}.$$

Therefore $\sigma_{X,Y}^{\text{Sets}}$ is indeed an isomorphism.

Naturality

We need to show that, given functions

$$\begin{aligned} f &: X \rightarrow A, \\ g &: Y \rightarrow B \end{aligned}$$

the diagram


$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times g} & A \times B \\ \sigma_{X,Y}^{\text{Sets}} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}} \\ Y \times X & \xrightarrow{g \times f} & B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (x, y) & & (x, y) \longmapsto (f(x), g(y)) \\ \downarrow & & \downarrow \\ (y, x) \longmapsto (g(y), f(x)) & & (g(y), f(x)) \end{array}$$

and hence indeed commutes, showing $\sigma^{\mathbf{Sets}}$ to be a natural transformation.

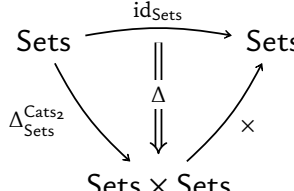
Being a Natural Isomorphism

Since $\sigma^{\mathbf{Sets}}$ is natural and $\sigma^{\mathbf{Sets}, -1}$ is a componentwise inverse to $\sigma^{\mathbf{Sets}}$, it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that $\sigma^{\mathbf{Sets}, -1}$ is also natural. Thus $\sigma^{\mathbf{Sets}}$ is a natural isomorphism. 

01NZ 5.1.8 The Diagonal

01P0 DEFINITION 5.1.8.1.1 ► THE DIAGONAL OF \times

The **diagonal of the product of sets** is the natural transformation

$$\Delta: \text{id}_{\mathbf{Sets}} \Rightarrow \times \circ \Delta_{\mathbf{Sets}}^{\mathbf{Cats}_2},$$


whose component

$$\Delta_X: X \rightarrow X \times X$$

at $X \in \text{Obj}(\mathbf{Sets})$ is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each $x \in X$.


PROOF 5.1.8.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.8.1.1

We need to show that, given a function $f: X \rightarrow Y$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Delta_X \downarrow & & \downarrow \Delta_Y \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} x & & x \xrightarrow{\quad} f(x) \\ \downarrow & & \downarrow \\ (x, x) \xrightarrow{\quad} (f(x), f(x)) & & (f(x), f(x)) \end{array}$$

and hence indeed commutes, showing Δ to be natural. 

01P1

PROPOSITION 5.1.8.1.3 ► PROPERTIES OF THE DIAGONAL MAP

Let X be a set.

01P2

1. *Monoidality.* The diagonal map

$$\Delta: \text{id}_{\text{Sets}} \Longrightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2},$$

is a monoidal natural transformation:

01P3

(a) *Compatibility With Strong Monoidality Constraints.* For each $X, Y \in \text{Obj}(\text{Sets})$, the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\Delta_X \times \Delta_Y} & (X \times X) \times (Y \times Y) \\ & \searrow \Delta_{X \times Y} & \downarrow \wr \\ & & (X \times Y) \times (X \times Y) \end{array}$$

commutes.

01P4

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{array}{ccc}
 \text{pt} & \xrightarrow{\Delta_{\text{pt}}} & \text{pt} \times \text{pt} \\
 \searrow & & \downarrow \lambda_{\text{pt}}^{\text{Sets}} \\
 & & \text{pt}
 \end{array}
 \quad
 \begin{array}{ccc}
 \text{pt} & \xrightarrow{\Delta_{\text{pt}}} & \text{pt} \times \text{pt} \\
 \searrow & & \downarrow \rho_{\text{pt}}^{\text{Sets}} \\
 & & \text{pt}
 \end{array}$$

commute, i.e. we have

$$\begin{aligned}
 \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets}, -1} \\
 &= \rho_{\text{pt}}^{\text{Sets}, -1},
 \end{aligned}$$

where we recall that the equalities

$$\begin{aligned}
 \lambda_{\text{pt}}^{\text{Sets}} &= \rho_{\text{pt}}^{\text{Sets}}, \\
 \lambda_{\text{pt}}^{\text{Sets}, -1} &= \rho_{\text{pt}}^{\text{Sets}, -1}
 \end{aligned}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

01P5

2. *The Diagonal of the Unit.* The component

$$\Delta_{\text{pt}} : \text{pt} \xrightarrow{\sim} \text{pt} \times \text{pt}$$

of Δ at pt is an isomorphism.

PROOF 5.1.8.1.4 ► PROOF OF PROPOSITION 5.1.8.1.3

Item 1: Monoidality

We claim that Δ is indeed monoidal:

024S

1. **Item 1a: Compatibility With Strong Monoidality Constraints:** We need to show that the diagram

$$\begin{array}{ccc}
 X \times Y & \xrightarrow{\Delta_X \times \Delta_Y} & (X \times X) \times (Y \times Y) \\
 & \searrow \Delta_{X \times Y} & \downarrow \wr \\
 & & (X \times Y) \times (X \times Y)
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 (x, y) & \longmapsto & ((x, x), (y, y)) & & (x, y) \\
 & & \downarrow & & \searrow \\
 & & ((x, y), (x, y)) & & ((x, y), (x, y))
 \end{array}$$

and hence indeed commutes.

024T

2. **Item 1b: Compatibility With Strong Unitality Constraints:** As shown in the proof of **Definition 5.1.5.1.1**, the inverse of the left unitor of Sets with respect to the product at $X \in \text{Obj}(\text{Sets})$ is given by

$$\lambda_X^{\text{Sets}, -1}(x) \stackrel{\text{def}}{=} (\star, x)$$

for each $x \in X$, so when $X = \text{pt}$, we have

$$\lambda_{\text{pt}}^{\text{Sets}, -1}(\star) \stackrel{\text{def}}{=} (\star, \star),$$


and also

$$\Delta_{\text{pt}}^{\text{Sets}}(\star) \stackrel{\text{def}}{=} (\star, \star),$$

so we have $\Delta_{\text{pt}} = \lambda_{\text{pt}}^{\text{Sets}, -1}$.

This finishes the proof.

Item 2: The Diagonal of the Unit

This follows from **Item 1** and the invertibility of the left/right unitor of Sets with respect to \times , proved in the proof of **Definition 5.1.5.1.1** for the left unitor or the proof of **Definition 5.1.6.1.1** for the right unitor. 

01P6 5.1.9 The Monoidal Category of Sets and Products

01P7

PROPOSITION 5.1.9.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO THE PRODUCT

The category **Sets** admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category **Sets** of pointed sets.
- *The Monoidal Product.* The product functor

$$\times: \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Proposition 4.1.3.1.4**.

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets}: \mathbf{Sets}^{\text{op}} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Proposition 4.3.5.1.2**.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}}: \text{pt} \rightarrow \mathbf{Sets}$$

of **Definition 5.1.3.1.1**.

- *The Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}}: \times \circ (\times \times \text{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \times \circ (\text{id}_{\mathbf{Sets}} \times \times) \circ \alpha_{\mathbf{Sets}, \mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}}$$

of **Definition 5.1.4.1.1**.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Sets}}: \times \circ (\mathbb{1}^{\mathbf{Sets}} \times \text{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}}^{\mathbf{Cats}_2}$$

of **Definition 5.1.5.1.1**.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.6.1.1.**

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.7.1.1.**

- *The Diagonals.* The monoidal natural transformation

$$\Delta : \text{id}_{\text{Sets}} \Longrightarrow \times \circ \Delta_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.1.8.1.1.**

PROOF 5.1.9.1.2 ► PROOF OF PROPOSITION 5.1.9.1.1

The Pentagon Identity

Let W, X, Y and Z be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \times (X \times Y)) \times Z & \\
 \alpha_{W,X,Y}^{\text{Sets}} \times \text{id}_Z \nearrow & & \searrow \alpha_{W,X \times Y,Z}^{\text{Sets}} \\
 ((W \times X) \times Y) \times Z & & W \times ((X \times Y) \times Z) \\
 \alpha_{W \times X,Y,Z}^{\text{Sets}} \searrow & & \swarrow \text{id}_W \times \alpha_{X,Y,Z}^{\text{Sets}} \\
 (W \times X) \times (Y \times Z) & \xrightarrow{\alpha_{W,X,Y \times Z}^{\text{Sets}}} & W \times (X \times (Y \times Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccccc}
 & & ((w, (x, y)), z) & & \\
 & \swarrow & & \searrow & \\
 (((w, x), y), z) & & & & ((w, (x, y)), z) \\
 \searrow & & & & \swarrow \\
 ((w, x), (y, z)) \mapsto (w, (x, (y, z))) & & & & (w, (x, (y, z)))
 \end{array}$$

and thus the pentagon identity is satisfied.

The Triangle Identity

Let X and Y be sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \times \text{pt}) \times Y & \xrightarrow{\alpha_{X,\text{pt},Y}^{\text{Sets}}} & X \times (\text{pt} \times Y) \\
 \searrow \rho_X^{\text{Sets}} \times \text{id}_Y & & \swarrow \text{id}_X \times \lambda_Y^{\text{Sets}} \\
 & X \times Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 ((x, \star), y) & \xrightarrow{\quad} & (x, (\star, y)) \\
 \searrow & & \swarrow \\
 (x, y) & & (x, y)
 \end{array}$$

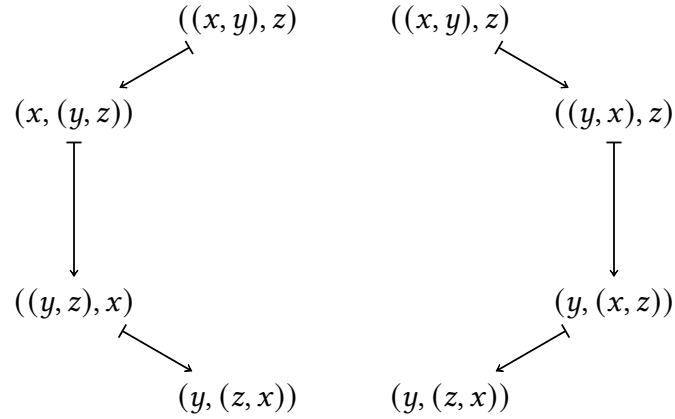
and thus the triangle identity is satisfied.

The Left Hexagon Identity

Let X , Y , and Z be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \times Y) \times Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}} \swarrow & & \searrow \sigma_{X,Y}^{\text{Sets}} \times \text{id}_Z \\
 X \times (Y \times Z) & & (Y \times X) \times Z \\
 \downarrow \sigma_{X,Y \times Z}^{\text{Sets}} & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}} \\
 (Y \times Z) \times X & & Y \times (X \times Z) \\
 \alpha_{Y,Z,X}^{\text{Sets}} \swarrow & & \swarrow \text{id}_Y \times \sigma_{X,Z}^{\text{Sets}} \\
 & Y \times (Z \times X) &
 \end{array}$$

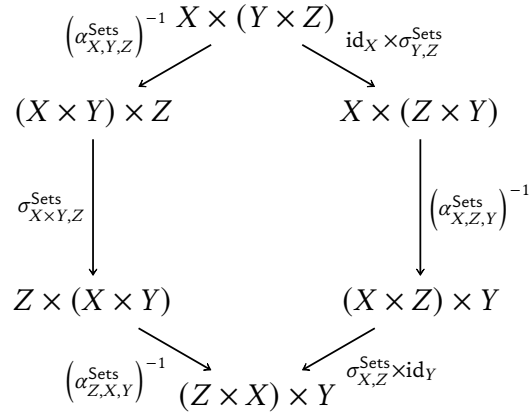
commutes. Indeed, this diagram acts on elements as



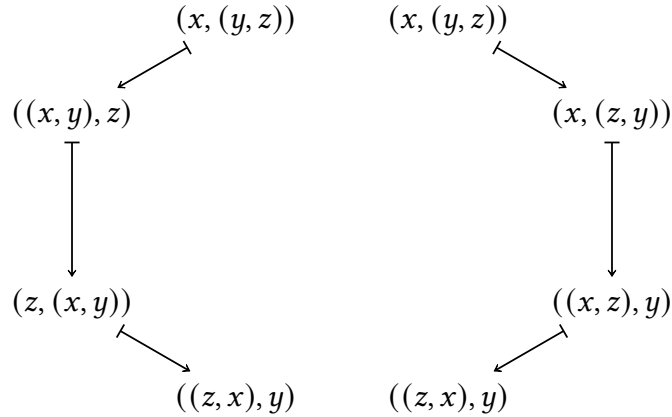
and thus the left hexagon identity is satisfied.

The Right Hexagon Identity

Let X , Y , and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as




and thus the right hexagon identity is satisfied.

Monoidal Closedness

This follows from **Constructions With Sets, Item 2** of **Proposition 4.3.5.1.2**

Existence of Monoidal Diagonals

This follows from **Items 1** and **2** of **Proposition 5.1.8.1.3**. 

01P8 5.1.10 The Universal Property of $(\mathbf{Sets}, \times, \text{pt})$

01P9 THEOREM 5.1.10.1.1 ► THE UNIVERSAL PROPERTY OF $(\mathbf{Sets}, \times, \text{pt})$

The symmetric monoidal structure on the category \mathbf{Sets} of **Proposition 5.1.9.1.1** is uniquely determined by the following requirements:

01PA 1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathbf{Sets}} : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of \mathbf{Sets} admits an internal Hom $[-_1, -_2]_{\mathbf{Sets}}$.

01PB 2. *The Unit Object Is pt.* We have $\mathbb{1}_{\mathbf{Sets}} \cong \text{pt}$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_{\infty}}^{\text{cld}}(\mathbf{Sets})$ of ?? spanned by the closed symmetric monoidal categories $(\mathbf{Sets}, \otimes_{\mathbf{Sets}},$

$[-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}}$ satisfying **Items 1** and **2** is contractible (i.e. equivalent to the punctual category).

PROOF 5.1.10.1.2 ► PROOF OF THEOREM 5.1.10.1.1

Unwinding the Statement

Let $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda', \rho', \sigma')$ be a closed symmetric monoidal category satisfying **Items 1** and **2**. We need to show that the identity functor

$$\text{id}_{\mathbf{Sets}} : \mathbf{Sets} \rightarrow \mathbf{Sets}$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \text{id}_{\mathbf{Sets}}^{\otimes} : A \otimes_{\mathbf{Sets}} B &\xrightarrow{\sim} A \times B, \\ \text{id}_{\mathbf{Sets}}^{\text{Hom}} : [A, B]_{\mathbf{Sets}} &\xrightarrow{\sim} \mathbf{Sets}(A, B), \\ \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes} : \mathbb{1}_{\mathbf{Sets}} &\xrightarrow{\sim} \text{pt}, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from $(\mathbf{Sets}, \otimes_{\mathbf{Sets}}, [-1, -2]_{\mathbf{Sets}}, \mathbb{1}_{\mathbf{Sets}}, \lambda', \rho', \sigma')$ to the closed symmetric monoidal category $(\mathbf{Sets}, \times, \mathbf{Sets}(-1, -2), \mathbb{1}_{\mathbf{Sets}}, \lambda^{\mathbf{Sets}}, \rho^{\mathbf{Sets}}, \sigma^{\mathbf{Sets}})$ of **Proposition 5.1.9.1.1**.

Constructing an Isomorphism $[-1, -2]_{\mathbf{Sets}} \cong \mathbf{Sets}(-1, -2)$

By ??, we have a natural isomorphism

$$\mathbf{Sets}(\text{pt}, [-1, -2]_{\mathbf{Sets}}) \cong \mathbf{Sets}(-1, -2).$$

By **Constructions With Sets**, **Item 3** of **Proposition 4.3.5.1.2**, we also have a natural isomorphism

$$\mathbf{Sets}(\text{pt}, [-1, -2]_{\mathbf{Sets}}) \cong [-1, -2]_{\mathbf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathbf{Sets}(-1, -2) \cong [-1, -2]_{\mathbf{Sets}}.$$

Given $A, B \in \text{Obj}(\mathbf{Sets})$, we will write

$$\text{id}_{A,B}^{\text{Hom}} : \mathbf{Sets}(A, B) \xrightarrow{\sim} [A, B]_{\mathbf{Sets}}$$

for the component of this isomorphism at (A, B) .

Constructing an Isomorphism $\otimes_{\mathbf{Sets}} \cong \times$

Since $\otimes_{\mathbf{Sets}}$ is adjoint in each variable to $[-_1, -_2]_{\mathbf{Sets}}$ by assumption and \times is adjoint in each variable to $\mathbf{Sets}(-_1, -_2)$ by **Constructions With Sets, Item 2** of **Proposition 4.3.5.1.2**, uniqueness of adjoints (??) gives us natural isomorphisms

$$\begin{aligned} A \otimes_{\mathbf{Sets}} - &\cong A \times -, \\ - \otimes_{\mathbf{Sets}} B &\cong B \times -. \end{aligned}$$

By ??, we then have $\otimes_{\mathbf{Sets}} \cong \times$. We will write

$$\text{id}_{\mathbf{Sets}|A,B}^{\otimes} : A \otimes_{\mathbf{Sets}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B) .

Alternative Construction of an Isomorphism $\otimes_{\mathbf{Sets}} \cong \times$

Alternatively, we may construct a natural isomorphism $\otimes_{\mathbf{Sets}} \cong \times$ as follows:

- 01PC 1. Let $A \in \text{Obj}(\mathbf{Sets})$.
- 01PD 2. Since $\otimes_{\mathbf{Sets}}$ is part of a closed monoidal structure, it preserves colimits in each variable by ??.
- 01PE 3. Since $A \cong \coprod_{a \in A} \text{pt}$ and $\otimes_{\mathbf{Sets}}$ preserves colimits in each variable, we have

$$\begin{aligned} A \otimes_{\mathbf{Sets}} B &\cong \left(\coprod_{a \in A} \text{pt} \right) \otimes_{\mathbf{Sets}} B \\ &\cong \coprod_{a \in A} (\text{pt} \otimes_{\mathbf{Sets}} B) \end{aligned}$$

$$\begin{aligned} &\cong \coprod_{a \in A} B \\ &\cong A \times B, \end{aligned}$$

naturally in $B \in \text{Obj}(\mathbf{Sets})$, where we have used that pt is the monoidal unit for $\otimes_{\mathbf{Sets}}$. Thus $A \otimes_{\mathbf{Sets}} - \cong A \times -$ for each $A \in \text{Obj}(\mathbf{Sets})$.

01PF

4. Similarly, $- \otimes_{\mathbf{Sets}} B \cong - \times B$ for each $B \in \text{Obj}(\mathbf{Sets})$.

01PG

5. By ??, we then have $\otimes_{\mathbf{Sets}} \cong \times$.

Below, we'll show that if a natural isomorphism $\otimes_{\mathbf{Sets}} \cong \times$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\text{id}_{\mathbf{Sets}|A,B}^{\otimes}: A \otimes_{\mathbf{Sets}} B \rightarrow A \times B$ from before.

Constructing an Isomorphism $\text{id}_{\mathbb{1}}^{\otimes}: \mathbb{1}_{\mathbf{Sets}} \rightarrow \text{pt}$

We define an isomorphism $\text{id}_{\mathbb{1}}^{\otimes}: \mathbb{1}_{\mathbf{Sets}} \rightarrow \text{pt}$ as the composition

$$\mathbb{1}_{\mathbf{Sets}} \xrightarrow[\sim]{\rho_{\mathbb{1}_{\mathbf{Sets}}}^{\mathbf{Sets}, -1}} \mathbb{1}_{\mathbf{Sets}} \times \text{pt} \xrightarrow[\sim]{\text{id}_{\mathbf{Sets}|\mathbb{1}_{\mathbf{Sets}}}^{\otimes}} \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt} \xrightarrow[\sim]{\lambda'_{\text{pt}}} \text{pt}$$

in \mathbf{Sets} .

Monoidal Left Unity of the Isomorphism $\otimes_{\mathbf{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{ccc} & \text{pt} \otimes_{\mathbf{Sets}} A & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},A}^{\otimes}} \text{pt} \times A \\ \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_A \nearrow & & \searrow \lambda_A^{\mathbf{Sets}} \\ \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A & \xrightarrow{\lambda'_A} & A \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} \text{pt} \times \text{pt} \\
 \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_{\text{pt}} \nearrow & & \searrow \lambda_{\text{pt}}^{\mathbf{Sets}} \\
 \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\lambda'_{\text{pt}}} & \text{pt}
 \end{array}$$

corresponding to the case $A = \text{pt}$, commutes by the terminality of pt (**Constructions With Sets, Construction 4.1.1.2**). Since this diagram commutes, so does the diagram

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes,-1}} \text{pt} \otimes_{\mathbf{Sets}} \text{pt} \\
 \lambda_{\text{pt}}^{\mathbf{Sets},-1} \nearrow & & \searrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes,-1} \otimes_{\mathbf{Sets}} \text{id}_{\text{pt}} \\
 \text{pt} & \xrightarrow{\lambda'_{\text{pt}}{}^{-1}} & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt}
 \end{array}
 \quad (\dagger)$$

Now, let $A \in \text{Obj}(\mathbf{Sets})$, let $a \in A$, and consider the diagram

$$\begin{array}{ccccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes,-1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes,-1} \times \text{id}_{\text{pt}}} \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} \text{pt} \\
 \lambda_{\text{pt}}^{\mathbf{Sets},-1} \nearrow & & & & \searrow \\
 \text{pt} & \xrightarrow{\lambda'_{\text{pt}}{}^{-1}} & & & \\
 \downarrow [a] & \downarrow \text{id}_{\text{pt}} \times [a] & (1) & \downarrow \text{id}_{\text{pt}} \otimes_{\mathbf{Sets}} [a] & \downarrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}} \times [a] \\
 & \text{pt} \times A & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},A}^{\otimes,-1}} & \text{pt} \otimes_{\mathbf{Sets}} A & \\
 \downarrow [a] & \downarrow \lambda_A^{\mathbf{Sets},-1} & (2) & \downarrow \text{id}_{\mathbb{1}_{\mathbf{Sets}}}^{\otimes,-1} \times \text{id}_A & \\
 A & \xrightarrow{\lambda'_A{}^{-1}} & & & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A
 \end{array}$$

(3) (4) (5)

Since:

- Subdiagram (5) commutes by the naturality of λ'^{-1} .
- Subdiagram (\dagger) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1}$.
- Subdiagram (1) commutes by the naturality of $\text{id}_{\mathbf{Sets}}^{\otimes, -1}$.
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathbf{Sets}, -1}$.

it follows that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times A & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} A \\
 & \nearrow \lambda_A^{\mathbf{Sets}, -1} & & & \searrow \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_A \\
 \text{pt} & \xrightarrow{[a]} & A & \xrightarrow{\lambda_A'^{-1}} & \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}
 \lambda_A'^{-1}(a) &= [\lambda_A'^{-1} \circ [a]](\star) \\
 &= \left[\left(\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \times \text{id}_A \right) \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\mathbf{Sets}, -1} \circ [a] \right](\star) \\
 &= \left[\left(\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \times \text{id}_A \right) \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\mathbf{Sets}, -1} \right](a)
 \end{aligned}$$

for each $a \in A$, and thus we have

$$\lambda_A'^{-1} = \left(\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes, -1} \times \text{id}_A \right) \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes, -1} \circ \lambda_A^{\mathbf{Sets}, -1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathbf{Sets}} \circ \text{id}_{\mathbf{Sets}|\text{pt}, A}^{\otimes} \circ \left(\text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \times \text{id}_A \right),$$

showing that the diagram

$$\begin{array}{ccc}
 & \text{pt} \otimes_{\mathbf{Sets}} A & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes} |_{\text{pt}, A}} \text{pt} \times A \\
 \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_A \nearrow & & \searrow \lambda_A^{\mathbf{Sets}} \\
 \mathbb{1}_{\mathbf{Sets}} \otimes_{\mathbf{Sets}} A & \xrightarrow{\lambda'_A} & A
 \end{array}$$

indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathbf{Sets}} \cong \times$

We can use the same argument we used to prove the monoidal left unity of the isomorphism $\otimes_{\mathbf{Sets}} \cong \times$ above. For completeness, we repeat it below.

We have to show that the diagram

$$\begin{array}{ccc}
 & A \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes} |_{A, \text{pt}}} A \times \text{pt} \\
 \text{id}_A \otimes_{\mathbf{Sets}} \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \nearrow & & \searrow \rho_A^{\mathbf{Sets}} \\
 A \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}} & \xrightarrow{\rho'_A} & A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}}^{\otimes} |_{\text{pt}, \text{pt}}} \text{pt} \times \text{pt} \\
 \text{id}_{\text{pt}} \otimes_{\mathbf{Sets}} \text{id}_{\mathbb{1}|\mathbf{Sets}}^{\otimes} \nearrow & & \searrow \rho_{\text{pt}}^{\mathbf{Sets}} \\
 \text{pt} \otimes_{\mathbf{Sets}} \mathbb{1}_{\mathbf{Sets}} & \xrightarrow{\rho'_{\text{pt}}} & \text{pt},
 \end{array}$$

corresponding to the case $A = \text{pt}$, commutes by the terminality of pt (**Constructions With Sets, Construction 4.1.1.2**). Since this diagram

commutes, so does the diagram

$$\begin{array}{ccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes,-1}} \text{pt} \otimes_{\text{Sets}} \text{pt} \\
 \nearrow \rho_{\text{pt}}^{\text{Sets},-1} & & \searrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes,-1} \\
 \text{pt} & \xrightarrow{\rho_{\text{pt}}'^{-1}} & \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}.
 \end{array}
 \quad (\dagger)$$

Now, let $A \in \text{Obj}(\text{Sets})$, let $a \in A$, and consider the diagram

$$\begin{array}{ccccc}
 & \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes,-1}} & \text{pt} \otimes_{\text{Sets}} \text{pt} & \\
 \nearrow \rho_{\text{pt}}^{\text{Sets},-1} & & & \searrow \text{id}_{\text{pt}} \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes,-1} & \\
 \text{pt} & & \xrightarrow{\rho_{\text{pt}}'^{-1}} & \text{pt} \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \\
 \downarrow [a] & \downarrow \text{id}_{\text{pt}} \times [a] & (1) & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} [a] & \downarrow \text{id}_{\mathbb{1}_{\text{Sets}}} \times [a] \\
 & & (3) & & (4) \\
 & & \downarrow & & \downarrow \\
 & & (5) & & \\
 & & A \times \text{pt} \xrightarrow{\text{id}_{\text{Sets}|A,\text{pt}}^{\otimes,-1}} A \otimes_{\text{Sets}} \text{pt} & & \\
 \nearrow \rho_A^{\text{Sets},-1} & & \searrow \text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes,-1} & & \\
 A & \xrightarrow{\rho_A'^{-1}} & A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & &
 \end{array}$$

Since:

- Subdiagram (5) commutes by the naturality of ρ'^{-1} .
- Subdiagram (\dagger) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes,-1}$.
- Subdiagram (1) commutes by the naturality of $\text{id}_{\text{Sets}}^{\otimes,-1}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\text{Sets},-1}$.

it follows that the diagram

$$\begin{array}{ccccc}
 & & A \times \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|A, \text{pt}}^{\otimes, -1}} & A \otimes_{\text{Sets}} \text{pt} \\
 & \nearrow \rho_A^{\text{Sets}, -1} & & & \searrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \\
 \text{pt} & \xrightarrow{[a]} & A & \xrightarrow{\rho_A'^{-1}} & A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}}
 \end{array}$$

Here's a step-by-step showcase of this argument: [\[Link\]](#). We then have

$$\begin{aligned}
 \rho_A'^{-1}(a) &= [\rho_A'^{-1} \circ [a]](\star) \\
 &= \left[\left(\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}| \text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1} \circ [a] \right](\star) \\
 &= \left[\left(\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}| \text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1} \right](a)
 \end{aligned}$$

for each $a \in A$, and thus we have

$$\rho_A'^{-1} = \left(\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}| \text{pt}, A}^{\otimes, -1} \circ \rho_A^{\text{Sets}, -1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\text{Sets}} \circ \text{id}_{\text{Sets}| \text{pt}, A}^{\otimes} \circ \left(\text{id}_A \times \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \right),$$

showing that the diagram

$$\begin{array}{ccccc}
 & & A \otimes_{\text{Sets}} \text{pt} & \xrightarrow{\text{id}_{\text{Sets}|A, \text{pt}}^{\otimes}} & A \times \text{pt} \\
 & \nearrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} & & & \searrow \rho_A^{\text{Sets}} \\
 A \otimes_{\text{Sets}} \mathbb{1}_{\text{Sets}} & \xrightarrow{\rho_A'} & & & A
 \end{array}$$

indeed commutes.

Monoidality of the Isomorphism $\otimes_{\text{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{ccc}
 & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & \\
 \text{id}_{\text{Sets}|A,B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \swarrow & & \searrow \alpha'_{A,B,C} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B,C}^{\otimes} & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B,C}^{\otimes} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \searrow \alpha_{A,B,C}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|A,B \times C}^{\otimes} \\
 & A \times (B \times C) &
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 & (\text{pt} \otimes_{\text{Sets}} \text{pt}) \otimes_{\text{Sets}} \text{pt} & \\
 \text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes} \otimes_{\text{Sets}} \text{id}_{\text{pt}} \swarrow & & \searrow \alpha'_{\text{pt},\text{pt},\text{pt}} \\
 (\text{pt} \times \text{pt}) \otimes_{\text{Sets}} \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \otimes_{\text{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\text{Sets}|\text{pt} \times \text{pt},\text{pt}}^{\otimes} & & \downarrow \text{id}_{\text{pt}} \otimes_{\text{Sets}} \text{id}_{\text{Sets}|\text{pt},\text{pt}}^{\otimes} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\text{Sets}} (\text{pt} \times \text{pt}) \\
 \searrow \alpha_{\text{pt},\text{pt},\text{pt}}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|\text{pt},\text{pt} \times \text{pt}}^{\otimes} \\
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 \downarrow !_{\text{pt} \times (\text{pt} \times \text{pt})} & & \\
 & \text{pt} &
 \end{array}$$

commutes by the terminality of pt (**Constructions With Sets, Construction 4.1.1.1.2**). Since the map $!_{\text{pt} \times (\text{pt} \times \text{pt})} : \text{pt} \times (\text{pt} \times \text{pt}) \rightarrow \text{pt}$ is an isomor-

phism (e.g. having inverse $\lambda_{\text{pt}}^{\mathbf{Sets}, -1} \circ \lambda_{\text{pt}}^{\mathbf{Sets}, -1}$), it follows that the diagram

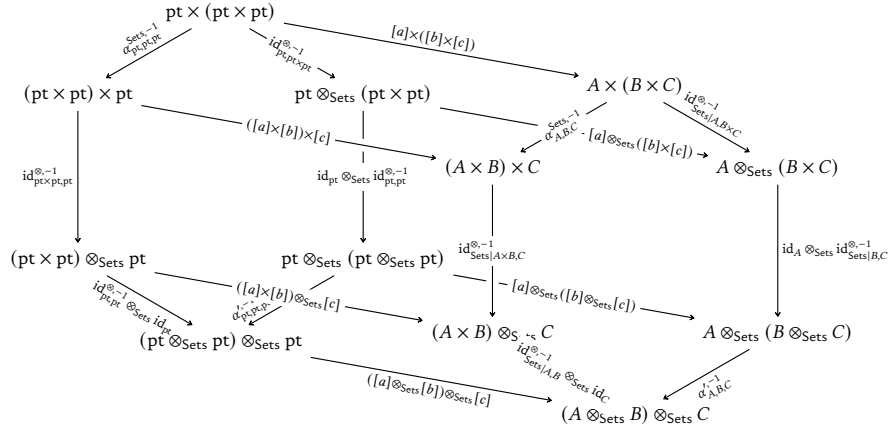
$$\begin{array}{ccc}
 & (\text{pt} \otimes_{\mathbf{Sets}} \text{pt}) \otimes_{\mathbf{Sets}} \text{pt} & \\
 \text{id}_{\mathbf{Sets}|\text{pt}, \text{pt}}^{\otimes} \otimes_{\mathbf{Sets}} \text{id}_{\text{pt}} \swarrow & & \searrow \alpha'_{\text{pt}, \text{pt}, \text{pt}} \\
 (\text{pt} \times \text{pt}) \otimes_{\mathbf{Sets}} \text{pt} & & \text{pt} \otimes_{\mathbf{Sets}} (\text{pt} \otimes_{\mathbf{Sets}} \text{pt}) \\
 \downarrow \text{id}_{\mathbf{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes} & & \downarrow \text{id}_{\text{pt}} \otimes_{\mathbf{Sets}} \text{id}_{\mathbf{Sets}|\text{pt}, \text{pt}}^{\otimes} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\mathbf{Sets}} (\text{pt} \times \text{pt}) \\
 \searrow \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\mathbf{Sets}} & & \swarrow \text{id}_{\mathbf{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes} \\
 & \text{pt} \times (\text{pt} \times \text{pt}) &
 \end{array}$$

also commutes. Taking inverses, we see that the diagram

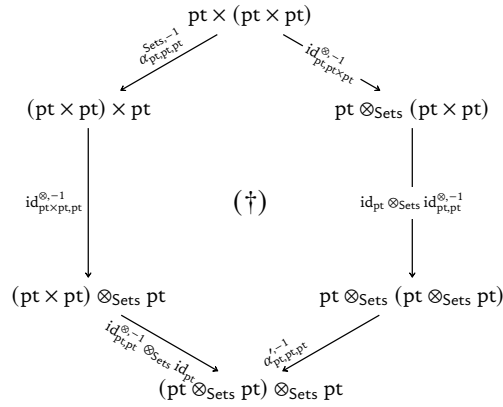
$$\begin{array}{ccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 \alpha_{\text{pt}, \text{pt}, \text{pt}}^{\mathbf{Sets}, -1} \swarrow & & \searrow \text{id}_{\mathbf{Sets}|\text{pt}, \text{pt} \times \text{pt}}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \times \text{pt} & & \text{pt} \otimes_{\mathbf{Sets}} (\text{pt} \times \text{pt}) \\
 \downarrow \text{id}_{\mathbf{Sets}|\text{pt} \times \text{pt}, \text{pt}}^{\otimes, -1} & (\dagger) & \downarrow \text{id}_{\text{pt}} \otimes_{\mathbf{Sets}} \text{id}_{\mathbf{Sets}|\text{pt}, \text{pt}}^{\otimes, -1} \\
 (\text{pt} \times \text{pt}) \otimes_{\mathbf{Sets}} \text{pt} & & \text{pt} \otimes_{\mathbf{Sets}} (\text{pt} \otimes_{\mathbf{Sets}} \text{pt}) \\
 \text{id}_{\mathbf{Sets}|\text{pt}, \text{pt}}^{\otimes, -1} \otimes_{\mathbf{Sets}} \text{id}_{\text{pt}} \swarrow & & \nwarrow \alpha'_{\text{pt}, \text{pt}, \text{pt}}{}^{-1} \\
 & (\text{pt} \otimes_{\mathbf{Sets}} \text{pt}) \otimes_{\mathbf{Sets}} \text{pt} &
 \end{array}$$

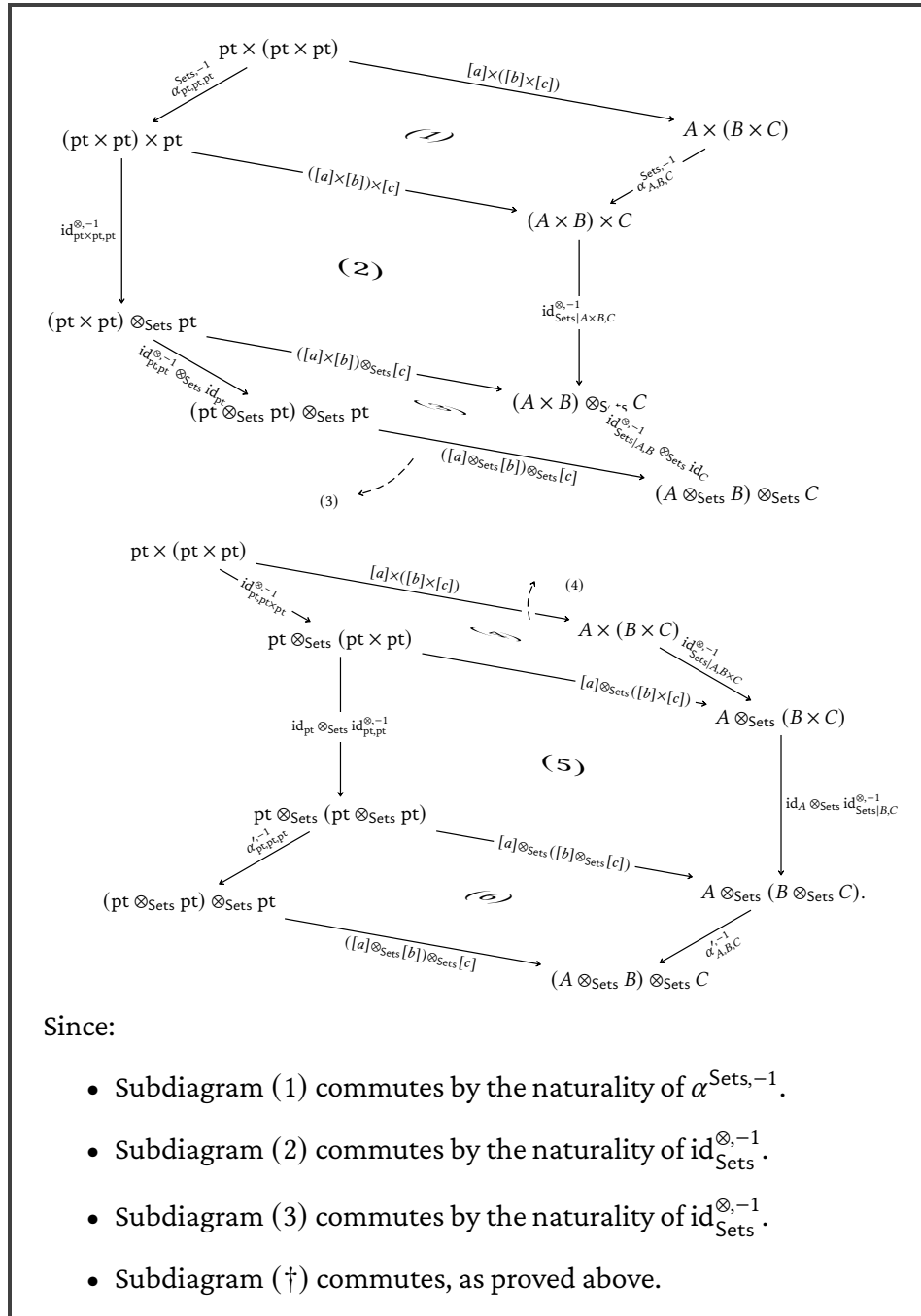
commutes as well. Now, let $A, B, C \in \text{Obj}(\mathbf{Sets})$, let $a \in A$, let $b \in B$, let

$c \in C$, and consider the diagram



which we partition into subdiagrams as follows:





- Subdiagram (4) commutes by the naturality of $\text{id}_{\text{Sets}}^{\otimes, -1}$.
- Subdiagram (5) commutes by the naturality of $\text{id}_{\text{Sets}}^{\otimes, -1}$.
- Subdiagram (6) commutes by the naturality of α'^{-1} .

it follows that the diagram

$$\begin{array}{ccc}
 & \text{pt} \times (\text{pt} \times \text{pt}) & \\
 & \downarrow [a] \times ([b] \times [c]) & \\
 & A \times (B \times C) & \\
 \alpha_{A,B,C}^{\text{Sets}, -1} \swarrow & & \searrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} & & \downarrow \text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \swarrow \text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C & & \swarrow \alpha'_{A,B,C}{}^{-1} \\
 (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & &
 \end{array}$$

also commutes. We then have

$$\begin{aligned}
 \left[\left(\text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C \right) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \right. \\
 \left. \circ \alpha_{A,B,C}^{\text{Sets}, -1} \right] (a, (b, c)) &= \left[\left(\text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C \right) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \right. \\
 &\quad \left. \circ \alpha_{A,B,C}^{\text{Sets}, -1} \circ ([a] \times ([b] \times [c])) \right] (\star, (\star, \star)) \\
 &= \left[\alpha'_{A,B,C}{}^{-1} \circ \left(\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \right) \right. \\
 &\quad \left. \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \circ ([a] \times ([b] \times [c])) \right] (\star, (\star, \star)) \\
 &= \left[\alpha'_{A,B,C}{}^{-1} \circ \left(\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1} \right] (a, (b, c))
 \end{aligned}$$

for each $(a, (b, c)) \in A \times (B \times C)$, and thus we have

$$\left(\text{id}_{\text{Sets}|A, B}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_C \right) \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes, -1} \circ \alpha_{A,B,C}^{\text{Sets}, -1} = \alpha'_{A,B,C}{}^{-1} \circ \left(\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes, -1} \right) \circ \text{id}_{\text{Sets}|A, B \times C}^{\otimes, -1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\text{Sets}} \circ \text{id}_{\text{Sets}|A \times B, C}^{\otimes} \circ \left(\text{id}_{\text{Sets}|A, B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \right) = \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \circ \left(\text{id}_A \times \text{id}_{\text{Sets}|B, C}^{\otimes} \right) \circ \alpha'_{A,B,C},$$

showing that the diagram

$$\begin{array}{ccc}
 & (A \otimes_{\text{Sets}} B) \otimes_{\text{Sets}} C & \\
 \text{id}_{\text{Sets}|A, B}^{\otimes} \otimes_{\text{Sets}} \text{id}_C \swarrow & & \searrow \alpha'_{A,B,C} \\
 (A \times B) \otimes_{\text{Sets}} C & & A \otimes_{\text{Sets}} (B \otimes_{\text{Sets}} C) \\
 \downarrow \text{id}_{\text{Sets}|A \times B, C}^{\otimes} & & \downarrow \text{id}_A \otimes_{\text{Sets}} \text{id}_{\text{Sets}|B, C}^{\otimes} \\
 (A \times B) \times C & & A \otimes_{\text{Sets}} (B \times C) \\
 \searrow \alpha_{A,B,C}^{\text{Sets}} & & \swarrow \text{id}_{\text{Sets}|A, B \times C}^{\otimes} \\
 & A \times (B \times C) &
 \end{array}$$

indeed commutes.

Braidedness of the Isomorphism $\otimes_{\text{Sets}} \cong \times$

We have to show that the diagram

$$\begin{array}{ccc}
 A \otimes_{\text{Sets}} B & \xrightarrow{\text{id}_{\text{Sets}|A, B}^{\otimes}} & A \times B \\
 \sigma'_{A, B} \downarrow & & \downarrow \sigma_{A, B}^{\text{Sets}} \\
 B \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|B, A}^{\otimes}} & B \times A
 \end{array}$$

commutes. First, note that the diagram

$$\begin{array}{ccc}
 \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 \sigma'_{\text{pt},\text{pt}} \downarrow & & \downarrow \sigma_{\text{pt},\text{pt}}^{\mathbf{Sets}} \\
 \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 & & \searrow !_{\text{pt} \times \text{pt}} \\
 & & \text{pt}
 \end{array}$$

commutes by the terminality of pt (**Constructions With Sets, Construction 4.1.1.1.2**). Since the map $!_{\text{pt} \times \text{pt}} : \text{pt} \times \text{pt} \rightarrow \text{pt}$ is invertible (e.g. with inverse $\lambda_{\text{pt}}^{\mathbf{Sets}, -1}$), the diagram

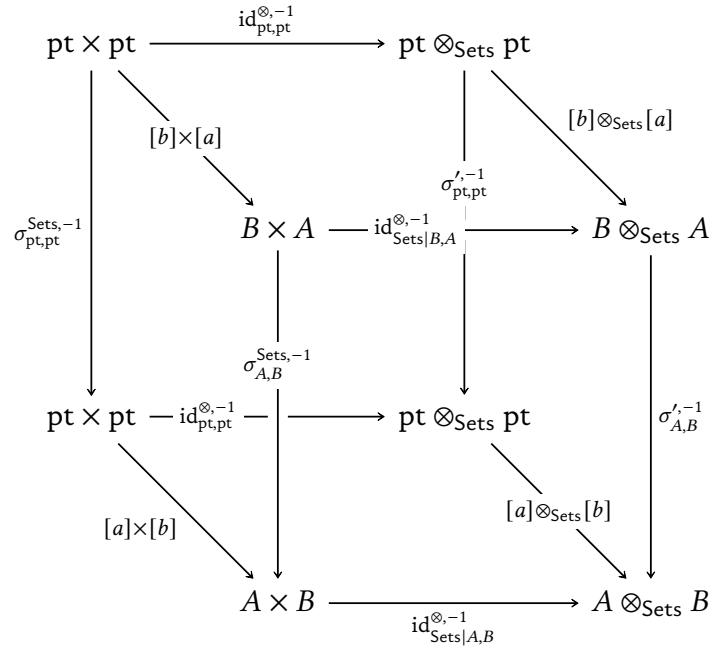
$$\begin{array}{ccc}
 \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt} \\
 \sigma'_{\text{pt},\text{pt}} \downarrow & & \downarrow \sigma_{\text{pt},\text{pt}}^{\mathbf{Sets}} \\
 \text{pt} \otimes_{\mathbf{Sets}} \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes}} & \text{pt} \times \text{pt}
 \end{array}$$

also commutes. Taking inverses, we see that the diagram

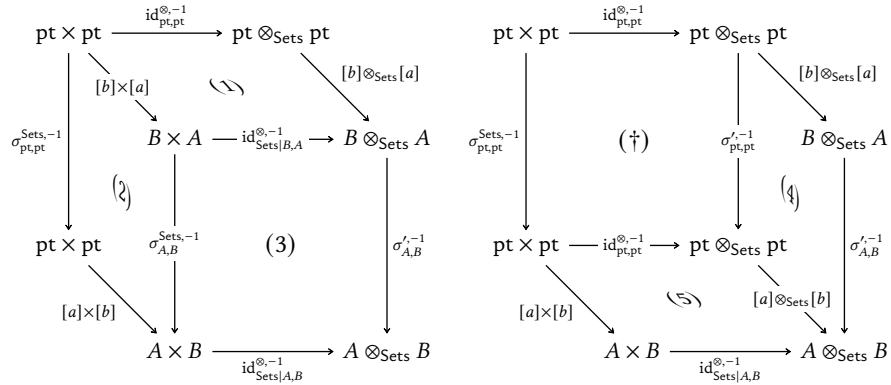
$$\begin{array}{ccc}
 \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt} \\
 \sigma_{\text{pt},\text{pt}}^{\mathbf{Sets}, -1} \downarrow & (\dagger) & \downarrow \sigma'_{\text{pt},\text{pt}}{}^{-1} \\
 \text{pt} \times \text{pt} & \xrightarrow{\text{id}_{\mathbf{Sets}|\text{pt},\text{pt}}^{\otimes, -1}} & \text{pt} \otimes_{\mathbf{Sets}} \text{pt}
 \end{array}$$

commutes as well. Now, let $A, B \in \text{Obj}(\mathbf{Sets})$, let $a \in A$, let $b \in B$, and

consider the diagram



which we partition into subdiagrams as follows:



Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\text{Sets}, -1}$.

- Subdiagram (5) commutes by the naturality of $\text{id}^{\otimes, -1}$.
- Subdiagram (\dagger) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of σ'^{-1} .
- Subdiagram (1) commutes by the naturality of $\text{id}^{\otimes, -1}$.

it follows that the diagram

$$\begin{array}{ccccc}
 & & \text{pt} \times \text{pt} & & \\
 & & \searrow [b] \times [a] & & \\
 & & B \times A & \xrightarrow{\text{id}_{\text{Sets}|B,A}^{\otimes}} & B \otimes_{\text{Sets}} A \\
 & \sigma_{A,B}^{\text{Sets}} \downarrow & & & \downarrow \sigma'_{A,B} \\
 & A \times B & \xrightarrow{\text{id}_{\text{Sets}|A,B}^{\otimes}} & & A \otimes_{\text{Sets}} B
 \end{array}$$

commutes. We then have

$$\begin{aligned}
 \left[\text{id}_{\text{Sets}|A,B}^{\otimes, -1} \circ \sigma_{A,B}^{\text{Sets}, -1} \right] (b, a) &= \left[\text{id}_{\text{Sets}|A,B}^{\otimes, -1} \circ \sigma_{A,B}^{\text{Sets}, -1} \circ ([b] \times [a]) \right] (\star, \star) \\
 &= \left[\sigma'_{A,B}{}^{-1} \circ \text{id}_{\text{Sets}|B,A}^{\otimes, -1} \circ ([b] \times [a]) \right] (\star, \star) \\
 &= \left[\sigma'_{A,B}{}^{-1} \circ \text{id}_{\text{Sets}|B,A}^{\otimes, -1} \right] (b, a)
 \end{aligned}$$

for each $(b, a) \in B \times A$, and thus we have

$$\text{id}_{\text{Sets}|A,B}^{\otimes, -1} \circ \sigma_{A,B}^{\text{Sets}, -1} = \sigma'_{A,B}{}^{-1} \circ \text{id}_{\text{Sets}|B,A}^{\otimes, -1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\text{Sets}} \circ \text{id}_{\text{Sets}|A,B}^{\otimes} = \text{id}_{\text{Sets}|B,A}^{\otimes} \circ \sigma'_{A,B},$$

showing that the diagram

$$\begin{array}{ccc} A \otimes_{\text{Sets}} B & \xrightarrow{\text{id}_{\text{Sets}|A,B}^{\otimes}} & A \times B \\ \sigma'_{A,B} \downarrow & & \downarrow \sigma_{A,B}^{\text{Sets}} \\ B \otimes_{\text{Sets}} A & \xrightarrow{\text{id}_{\text{Sets}|B,A}^{\otimes}} & B \times A \end{array}$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\text{Sets}} \cong \times$

Let $\phi, \psi: -_1 \otimes_{\text{Sets}} -_2 \Rightarrow -_1 \times -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to \times and \otimes (as shown above), we have

$$\lambda'_B = \lambda_B^{\text{Sets}} \circ \phi_{\text{pt},B} \circ \left(\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y \right),$$

$$\lambda'_B = \lambda_B^{\text{Sets}} \circ \psi_{\text{pt},B} \circ \left(\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y \right).$$

Postcomposing both sides with $\lambda_B^{\text{Sets}, -1}$ gives

$$\lambda_B^{\text{Sets}, -1} \circ \lambda'_B \circ \left(\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes, -1} \otimes_{\text{Sets}} \text{id}_Y \right) = \phi_{\text{pt},B},$$

$$\lambda_B^{\text{Sets}, -1} \circ \lambda'_B \circ \left(\text{id}_{\mathbb{1}|\text{Sets}}^{\otimes} \otimes_{\text{Sets}} \text{id}_Y \right) = \psi_{\text{pt},B},$$

and thus we have

$$\phi_{\text{pt},B} = \psi_{\text{pt},B}$$

for each $B \in \text{Obj}(\text{Sets})$. Now, let $a \in A$ and consider the naturality diagrams


$$\begin{array}{ccc} \text{pt} \times B & \xrightarrow{[a] \times \text{id}_B} & A \times B \\ \phi_{\text{pt},B} \downarrow & & \downarrow \phi_{A,B} \\ \text{pt} \otimes_{\text{Sets}} B & \xrightarrow{[a] \otimes_{\text{Sets}} \text{id}_B} & A \otimes_{\text{Sets}} B \end{array} \quad \begin{array}{ccc} \text{pt} \times B & \xrightarrow{[a] \times \text{id}_B} & A \times B \\ \psi_{\text{pt},B} \downarrow & & \downarrow \psi_{A,B} \\ \text{pt} \otimes_{\text{Sets}} B & \xrightarrow{[a] \otimes_{\text{Sets}} \text{id}_B} & A \otimes_{\text{Sets}} B \end{array}$$

for ϕ and ψ with respect to the morphisms $[a]$ and id_B . Having shown that $\phi_{\text{pt},B} = \psi_{\text{pt},B}$, we have

$$\begin{aligned}\phi_{A,B}(a, b) &= [\phi_{A,B} \circ ([a] \times \text{id}_B)](\star, b) \\ &= [([a] \otimes_{\text{Sets}} \text{id}_B) \circ \phi_{\text{pt},B}](\star, b) \\ &= [([a] \otimes_{\text{Sets}} \text{id}_B) \circ \psi_{\text{pt},B}](\star, b) \\ &= [\psi_{A,B} \circ ([a] \times \text{id}_B)](\star, b) \\ &= \psi_{A,B}(a, b)\end{aligned}$$

for each $(a, b) \in A \times B$. Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each $A, B \in \text{Obj}(\text{Sets})$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\text{Sets}} \cong \times$ to be unique. 

01PH

COROLLARY 5.1.10.1.3 ► A SECOND UNIVERSAL PROPERTY FOR $(\text{Sets}, \times, \text{pt})$

The symmetric monoidal structure on the category Sets of **Proposition 5.1.9.1.1** is uniquely determined by the following requirements:

01PJ

1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\text{Sets}} : \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$


of Sets preserves colimits separately in each variable.

01PK

2. *The Unit Object Is pt.* We have $\mathbb{1}_{\text{Sets}} \cong \text{pt}$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_{\infty}}(\text{Sets})$ of ?? spanned by the symmetric monoidal categories $(\text{Sets}, \otimes_{\text{Sets}}, \mathbb{1}_{\text{Sets}}, \lambda^{\text{Sets}}, \rho^{\text{Sets}}, \sigma^{\text{Sets}})$ satisfying **Items 1** and **2** is contractible.

PROOF 5.1.10.1.4 ► PROOF OF COROLLARY 5.1.10.1.3

Since **Sets** is locally presentable (??), it follows from ?? that **Item 1** is equivalent to the existence of an internal Hom as in **Item 1** of **Theorem 5.1.10.1.1**. The result then follows from **Theorem 5.1.10.1.1**. 

01PL 5.2 The Monoidal Category of Sets and Coproducts

01PM 5.2.1 Coproducts of Sets

See **Constructions With Sets**, Section 4.2.3.

01PN 5.2.2 The Monoidal Unit

01PP DEFINITION 5.2.2.1.1 ► THE MONOIDAL UNIT OF \coprod

The **monoidal unit of the coproduct of sets** is the functor

$$\mathbb{0}^{\text{Sets}} : \text{pt} \rightarrow \text{Sets}$$

defined by

$$\mathbb{0}_{\text{Sets}} \stackrel{\text{def}}{=} \emptyset,$$

where \emptyset is the empty set of **Constructions With Sets**, Definition 4.3.1.1.1.

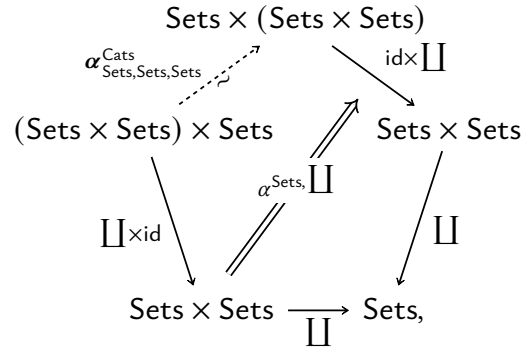
01PQ 5.2.3 The Associator

01PR DEFINITION 5.2.3.1.1 ► THE ASSOCIATOR OF \coprod

The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\text{Sets}, \coprod} : \coprod \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\text{id}_{\text{Sets}} \times \coprod) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg} : (X \amalg Y) \amalg Z \xrightarrow{\sim} X \amalg (Y \amalg Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg}(a) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } a = (0, (0, x)), \\ (1, (0, y)) & \text{if } a = (0, (1, y)), \\ (1, (1, a)) & \text{if } a = (1, z) \end{cases}$$

for each $a \in (X \amalg Y) \amalg Z$.

PROOF 5.2.3.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.3.1.1

Unwinding the Definitions of $(X \amalg Y) \amalg Z$ and $X \amalg (Y \amalg Z)$

Firstly, we unwind the expressions for $(X \amalg Y) \amalg Z$ and $X \amalg (Y \amalg Z)$.

We have

$$\begin{aligned} (X \amalg Y) \amalg Z &\stackrel{\text{def}}{=} \{(0, a) \in S \mid a \in X \amalg Y\} \cup \{(1, z) \in S \mid z \in Z\} \\ &= \{(0, (0, x)) \in S \mid x \in X\} \cup \{(0, (1, y)) \in S \mid y \in Y\} \\ &\quad \cup \{(1, z) \in S \mid z \in Z\}, \end{aligned}$$

where $S = \{0, 1\} \times ((X \amalg Y) \cup Z)$ and

$$\begin{aligned} X \amalg (Y \amalg Z) &\stackrel{\text{def}}{=} \{(0, x) \in S' \mid x \in X\} \cup \{(1, a) \in S' \mid a \in Y \amalg Z\} \\ &= \{(0, x) \in S' \mid x \in X\} \cup \{(1, (0, y)) \in S' \mid y \in Y\} \\ &\quad \cup \{(1, (1, z)) \in S' \mid z \in Z\}, \end{aligned}$$

where $S' = \{0, 1\} \times (X \cup (Y \amalg Z))$.

Invertibility

The inverse of $\alpha_{X,Y,Z}^{\text{Sets}, \amalg}$ is the map

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1} : X \amalg (Y \amalg Z) \rightarrow (X \amalg Y) \amalg Z$$

given by

$$\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1}(a) \stackrel{\text{def}}{=} \begin{cases} (0, (0, x)) & \text{if } a = (0, x), \\ (0, (1, y)) & \text{if } a = (1, (0, y)), \\ (1, z) & \text{if } a = (1, (1, z)) \end{cases}$$

for each $a \in X \amalg (Y \amalg Z)$. Indeed:

- *Invertibility I.* The map $\alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1} \circ \alpha_{X,Y,Z}^{\text{Sets}, \amalg}$ acts on elements as

$$\begin{aligned} (0, (0, x)) &\mapsto (0, x) \mapsto (0, (0, x)), \\ (0, (0, y)) &\mapsto (1, (0, y)) \mapsto (0, (0, y)), \\ (1, z) &\mapsto (1, (1, z)) \mapsto (1, z) \end{aligned}$$

and hence is equal to the identity map of $(X \amalg Y) \amalg Z$.

- *Invertibility II.* The map $\alpha_{X,Y,Z}^{\text{Sets}, \amalg} \circ \alpha_{X,Y,Z}^{\text{Sets}, \amalg, -1}$ acts on elements as

$$\begin{aligned} (0, x) &\mapsto (0, (0, x)) \mapsto (0, x), \\ (1, (0, y)) &\mapsto (0, (0, y)) \mapsto (1, (0, y)), \\ (1, (1, z)) &\mapsto (1, z) \mapsto (1, (1, z)) \end{aligned}$$

and hence is equal to the identity map of $X \amalg (Y \amalg Z)$.

Therefore $\alpha_{X,Y,Z}^{\text{Sets}, \coprod}$ is indeed an isomorphism.

Naturality

We need to show that, given functions

$$f: X \rightarrow X',$$

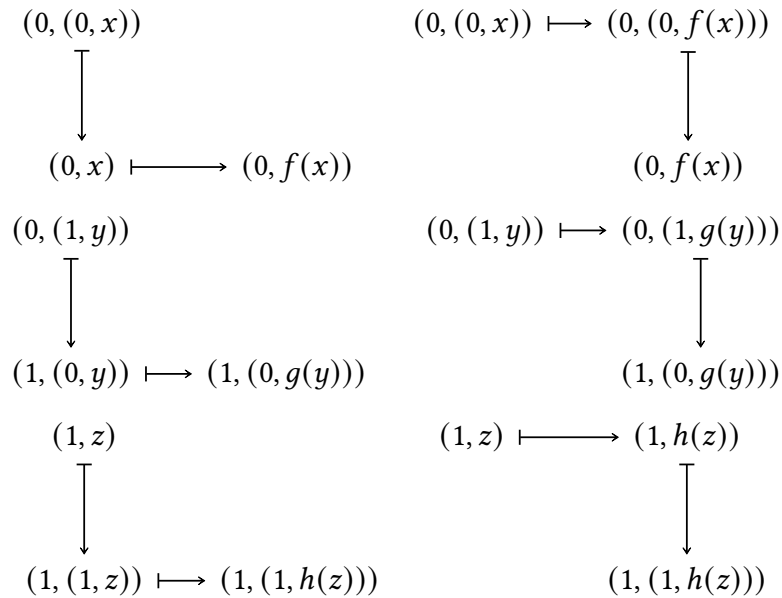
$$g: Y \rightarrow Y',$$

$$h: Z \rightarrow Z'$$

the diagram


$$\begin{array}{ccc}
 (X \coprod Y) \coprod Z & \xrightarrow{(f \coprod g) \coprod h} & (X' \coprod Y') \coprod Z' \\
 \downarrow \alpha_{X,Y,Z}^{\text{Sets}, \coprod} & & \downarrow \alpha_{X',Y',Z'}^{\text{Sets}, \coprod} \\
 X \coprod (Y \coprod Z) & \xrightarrow{f \coprod (g \coprod h)} & X' \coprod (Y' \coprod Z')
 \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing $\alpha^{\text{Sets}, \sqcup}$ to be a natural transformation.

Being a Natural Isomorphism

Since $\alpha^{\text{Sets}, \sqcup}$ is natural and $\alpha^{\text{Sets}, \sqcup, -1}$ is a componentwise inverse to $\alpha^{\text{Sets}, \sqcup}$, it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that $\lambda^{\text{Sets}, -1}$ is also natural. Thus $\alpha^{\text{Sets}, \sqcup}$ is a natural isomorphism. 

01PS 5.2.4 The Left Unitor

01PT DEFINITION 5.2.4.1.1 ► THE LEFT UNITOR OF \sqcup

The **left unitor of the coproduct of sets** is the natural isomorphism

$$\lambda^{\text{Sets}, \amalg} : \amalg \circ (\emptyset^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

whose component

$$\lambda_X^{\text{Sets}, \amalg} : \emptyset \amalg X \xrightarrow{\sim} X$$

at X is given by

$$\lambda_X^{\text{Sets}, \amalg}((1, x)) \stackrel{\text{def}}{=} x$$

for each $(1, x) \in \emptyset \amalg X$.

PROOF 5.2.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.4.1.1

Unwinding the Definition of $\emptyset \amalg X$

Firstly, we unwind the expressions for $\emptyset \amalg X$. We have

$$\begin{aligned} \emptyset \amalg X &\stackrel{\text{def}}{=} \{(0, z) \in S \mid z \in \emptyset\} \cup \{(1, x) \in S \mid x \in X\} \\ &= \emptyset \cup \{(1, x) \in S \mid x \in X\} \\ &= \{(1, x) \in S \mid x \in X\}, \end{aligned}$$

where $S = \{0, 1\} \times (\emptyset \cup X)$.

Invertibility

The inverse of $\lambda_X^{\text{Sets}, \amalg}$ is the map

$$\lambda_X^{\text{Sets}, \amalg, -1} : X \rightarrow \emptyset \amalg X$$

given by

$$\lambda_X^{\text{Sets}, \amalg, -1}(x) \stackrel{\text{def}}{=} (1, x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\lambda_X^{\text{Sets}, \amalg, -1} \circ \lambda_X^{\text{Sets}, \amalg} \right](1, x) &= \lambda_X^{\text{Sets}, \amalg, -1} \left(\lambda_X^{\text{Sets}, \amalg}(1, x) \right) \\ &= \lambda_X^{\text{Sets}, \amalg, -1}(x) \\ &= (1, x) \\ &= [\text{id}_{\emptyset \amalg X}](1, x) \end{aligned}$$

for each $(1, x) \in \emptyset \amalg X$, and therefore we have

$$\lambda_X^{\text{Sets}, \amalg, -1} \circ \lambda_X^{\text{Sets}, \amalg} = \text{id}_{\emptyset \amalg X}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[\lambda_X^{\text{Sets}, \amalg} \circ \lambda_X^{\text{Sets}, \amalg, -1} \right](x) &= \lambda_X^{\text{Sets}, \amalg} \left(\lambda_X^{\text{Sets}, \amalg, -1}(x) \right) \\ &= \lambda_X^{\text{Sets}, \amalg, -1}(1, x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

for each $x \in X$, and therefore we have

$$\lambda_X^{\text{Sets}, \amalg} \circ \lambda_X^{\text{Sets}, \amalg, -1} = \text{id}_X.$$

Therefore $\lambda_X^{\text{Sets}, \amalg}$ is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \rightarrow Y$, the diagram


$$\begin{array}{ccc} \emptyset \amalg X & \xrightarrow{\text{id}_\emptyset \amalg f} & \emptyset \amalg Y \\ \lambda_X^{\text{Sets}, \amalg} \downarrow & & \downarrow \lambda_Y^{\text{Sets}, \amalg} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (1, x) & \longmapsto & (1, f(x)) \\ \downarrow & & \downarrow \\ x & \longmapsto & f(x) \end{array}$$

and hence indeed commutes. Therefore $\lambda^{\text{Sets}, \amalg}$ is a natural transformation.

Being a Natural Isomorphism

Since $\lambda^{\text{Sets}, \amalg}$ is natural and $\lambda^{\text{Sets}, -1}$ is a componentwise inverse to $\lambda^{\text{Sets}, \amalg}$, it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that $\lambda^{\text{Sets}, -1}$ is also natural. Thus $\lambda^{\text{Sets}, \amalg}$ is a natural isomorphism. 

01PU 5.2.5 The Right Unitor

01PV DEFINITION 5.2.5.1.1 ► THE RIGHT UNITOR OF \amalg

The **right unitor of the coproduct of sets** is the natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \emptyset^{\text{Sets}}) \xrightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2},$$

whose component

$$\rho_X^{\text{Sets}, \amalg} : X \amalg \emptyset \xrightarrow{\sim} X$$

at X is given by

$$\rho_X^{\text{Sets}, \amalg}((0, x)) \stackrel{\text{def}}{=} x$$

for each $(0, x) \in X \amalg \emptyset$.

PROOF 5.2.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.5.1.1

Unwinding the Definition of $X \amalg \emptyset$

Firstly, we unwind the expression for $X \amalg \emptyset$. We have

$$\begin{aligned} X \amalg \emptyset &\stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, z) \in S \mid z \in \emptyset\} \\ &= \{(0, x) \in S \mid x \in X\} \cup \emptyset \\ &= \{(0, x) \in S \mid x \in X\}, \end{aligned}$$

where $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$.

Invertibility

The inverse of $\rho_X^{\text{Sets}, \amalg}$ is the map

$$\rho_X^{\text{Sets}, \amalg, -1} : X \rightarrow X \amalg \emptyset$$

given by

$$\rho_X^{\text{Sets}, \amalg, -1}(x) \stackrel{\text{def}}{=} (0, x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\rho_X^{\text{Sets}, \amalg, -1} \circ \rho_X^{\text{Sets}, \amalg} \right](0, x) &= \rho_X^{\text{Sets}, \amalg, -1} \left(\rho_X^{\text{Sets}, \amalg}(0, x) \right) \\ &= \rho_X^{\text{Sets}, \amalg, -1}(x) \\ &= (0, x) \\ &= [\text{id}_X \amalg \emptyset](0, x) \end{aligned}$$

for each $(0, x) \in \emptyset \amalg X$, and therefore we have

$$\rho_X^{\text{Sets}, \amalg, -1} \circ \rho_X^{\text{Sets}, \amalg} = \text{id}_{\emptyset \amalg X}.$$

- *Invertibility II.* We have

$$\begin{aligned} \left[\rho_X^{\text{Sets}, \amalg} \circ \rho_X^{\text{Sets}, \amalg, -1} \right](x) &= \rho_X^{\text{Sets}, \amalg} \left(\rho_X^{\text{Sets}, \amalg, -1}(x) \right) \\ &= \rho_X^{\text{Sets}, \amalg, -1}(0, x) \\ &= x \\ &= [\text{id}_X](x) \end{aligned}$$

for each $x \in X$, and therefore we have

$$\rho_X^{\text{Sets}, \amalg} \circ \rho_X^{\text{Sets}, \amalg, -1} = \text{id}_X.$$

Therefore $\rho_X^{\text{Sets}, \amalg}$ is indeed an isomorphism.

Naturality

We need to show that, given a function $f: X \rightarrow Y$, the diagram


$$\begin{array}{ccc} X \amalg \emptyset & \xrightarrow{f \amalg \text{id}_\emptyset} & Y \amalg \emptyset \\ \downarrow \rho_X^{\text{Sets}, \amalg} & & \downarrow \rho_Y^{\text{Sets}, \amalg} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc} (0, x) & & (0, x) \mapsto (1, f(x)) \\ \downarrow & & \downarrow \\ x & \xrightarrow{\quad} & f(x) \end{array}$$

and hence indeed commutes. Therefore $\rho^{\text{Sets}, \amalg}$ is a natural transformation.

Being a Natural Isomorphism

Since $\rho^{\text{Sets}, \amalg}$ is natural and $\rho^{\text{Sets}, -1}$ is a componentwise inverse to $\rho^{\text{Sets}, \amalg}$, it follows from [Categories, Item 2](#) of [Proposition 11.9.7.1.2](#) that $\rho^{\text{Sets}, -1}$ is also natural. Thus $\rho^{\text{Sets}, \amalg}$ is a natural isomorphism. 

01PW 5.2.6 The Symmetry

01PX DEFINITION 5.2.6.1.1 ► THE SYMMETRY OF \amalg

The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\text{Sets}, \amalg}: \amalg \xrightarrow{\sim} \amalg \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}, \quad \begin{array}{ccc} \text{Sets} \times \text{Sets} & \xrightarrow{\amalg} & \text{Sets}, \\ & \Downarrow \sigma^{\text{Sets}, \amalg} & \\ \text{Sets} \times \text{Sets} & & \end{array}$$

whose component

$$\sigma_{X,Y}^{\text{Sets}, \amalg} : X \amalg Y \xrightarrow{\sim} Y \amalg X$$

at $X, Y \in \text{Obj}(\text{Sets})$ is defined by

$$\sigma_{X,Y}^{\text{Sets}, \amalg}(x, y) \stackrel{\text{def}}{=} (y, x)$$

for each $(x, y) \in X \times Y$.

PROOF 5.2.6.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.6.1.1

Unwinding the Definitions of $X \amalg Y$ and $Y \amalg X$

Firstly, we unwind the expressions for $X \amalg Y$ and $Y \amalg X$. We have

$$X \amalg Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},$$

where $S = \{0, 1\} \times (X \cup Y)$ and

$$Y \amalg X \stackrel{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},$$

where $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$.

Invertibility

The inverse of $\sigma_{X,Y}^{\text{Sets}, \amalg}$ is the map

$$\sigma_{X,Y}^{\text{Sets}, \amalg, -1} : Y \amalg X \rightarrow X \amalg Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets}, \amalg, -1} \stackrel{\text{def}}{=} \sigma_{Y,X}^{\text{Sets}, \amalg}$$

and hence given by

$$\sigma_{X,Y}^{\text{Sets}, \amalg, -1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = (1, x), \\ (1, y) & \text{if } z = (0, y) \end{cases}$$

for each $z \in Y \amalg X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned}
 \left[\sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} \right] (0, x) &= \sigma_X^{\text{Sets}, \coprod, -1} \left(\sigma_X^{\text{Sets}, \coprod} (0, x) \right) \\
 &= \sigma_X^{\text{Sets}, \coprod, -1} (1, x) \\
 &= (0, x) \\
 &= \left[\text{id}_X \coprod_Y \right] (0, x)
 \end{aligned}$$

for each $(0, x) \in X \coprod Y$ and

$$\begin{aligned}
 \left[\sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} \right] (1, y) &= \sigma_X^{\text{Sets}, \coprod, -1} \left(\sigma_X^{\text{Sets}, \coprod} (1, y) \right) \\
 &= \sigma_X^{\text{Sets}, \coprod, -1} (0, y) \\
 &= (1, y) \\
 &= \left[\text{id}_X \coprod_Y \right] (1, y)
 \end{aligned}$$

for each $(1, y) \in X \coprod Y$, and therefore we have

$$\sigma_{X,Y}^{\text{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\text{Sets}, \coprod} = \text{id}_{X \coprod Y}.$$

- *Invertibility II.* We have

$$\begin{aligned}
 \left[\sigma_{X,Y}^{\text{Sets}, \coprod} \circ \sigma_{X,Y}^{\text{Sets}, \coprod, -1} \right] (0, y) &= \sigma_X^{\text{Sets}, \coprod} \left(\sigma_X^{\text{Sets}, \coprod, -1} (0, y) \right) \\
 &= \sigma_X^{\text{Sets}, \coprod, -1} (1, y) \\
 &= (0, y) \\
 &= \left[\text{id}_Y \coprod_X \right] (0, y)
 \end{aligned}$$

for each $(0, y) \in Y \coprod X$ and

$$\left[\sigma_{X,Y}^{\text{Sets}, \coprod} \circ \sigma_{X,Y}^{\text{Sets}, \coprod, -1} \right] (1, x) = \sigma_X^{\text{Sets}, \coprod} \left(\sigma_X^{\text{Sets}, \coprod, -1} (1, x) \right)$$

$$\begin{aligned}
&= \sigma_X^{\text{Sets}, \coprod, -1}(0, x) \\
&= (1, x) \\
&= [\text{id}_Y \coprod \text{id}_X](1, x)
\end{aligned}$$

for each $(1, x) \in Y \coprod X$, and therefore we have

$$\sigma_X^{\text{Sets}, \coprod} \circ \sigma_X^{\text{Sets}, \coprod, -1} = \text{id}_{Y \coprod X}.$$

Therefore $\sigma_{X,Y}^{\text{Sets}, \coprod}$ is indeed an isomorphism.

Naturality

We need to show that, given functions $f: A \rightarrow X$ and $g: B \rightarrow Y$, the diagram


$$\begin{array}{ccc}
A \coprod B & \xrightarrow{f \coprod g} & X \coprod Y \\
\sigma_{A,B}^{\text{Sets}, \coprod} \downarrow & & \downarrow \sigma_{X,Y}^{\text{Sets}, \coprod} \\
B \coprod A & \xrightarrow{g \coprod f} & Y \coprod X
\end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(0, a) & \longmapsto & (0, f(a)) \\
\downarrow & & \downarrow \\
(1, a) & \longmapsto & (1, f(a)) \\
\\
(1, b) & \longmapsto & (1, g(b)) \\
\downarrow & & \downarrow \\
(0, b) & \longmapsto & (0, g(b))
\end{array}$$

and hence indeed commutes. Therefore $\sigma^{\text{Sets}, \amalg}$ is a natural transformation.

Being a Natural Isomorphism

Since $\sigma^{\text{Sets}, \amalg}$ is natural and $\sigma^{\text{Sets}, -1}$ is a componentwise inverse to $\sigma^{\text{Sets}, \amalg}$, it follows from **Categories, Item 2** of **Proposition 11.9.7.1.2** that $\sigma^{\text{Sets}, -1}$ is also natural. Thus $\sigma^{\text{Sets}, \amalg}$ is a natural isomorphism. 

01PY 5.2.7 The Monoidal Category of Sets and Coproducts

01PZ PROPOSITION 5.2.7.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO \amalg

The category **Sets** admits a closed symmetric monoidal category structure consisting of:

- *The Underlying Category.* The category **Sets** of pointed sets.
- *The Monoidal Product.* The coproduct functor

$$\amalg : \text{Sets} \times \text{Sets} \rightarrow \text{Sets}$$

of **Constructions With Sets, Item 1** of **Proposition 4.2.3.1.4**.

- *The Monoidal Unit.* The functor

$$\mathbb{0}^{\text{Sets}} : \text{pt} \rightarrow \text{Sets}$$

of **Definition 5.2.2.1.1**.

- *The Associators.* The natural isomorphism

$$\alpha^{\text{Sets}, \amalg} : \amalg \circ (\amalg \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \amalg \circ (\text{id}_{\text{Sets}} \times \amalg) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of **Definition 5.2.3.1.1**.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}, \amalg} : \amalg \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of **Definition 5.2.4.1.1**.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \mathbb{0}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of Definition 5.2.5.1.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}, \amalg} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of Definition 5.2.6.1.1.

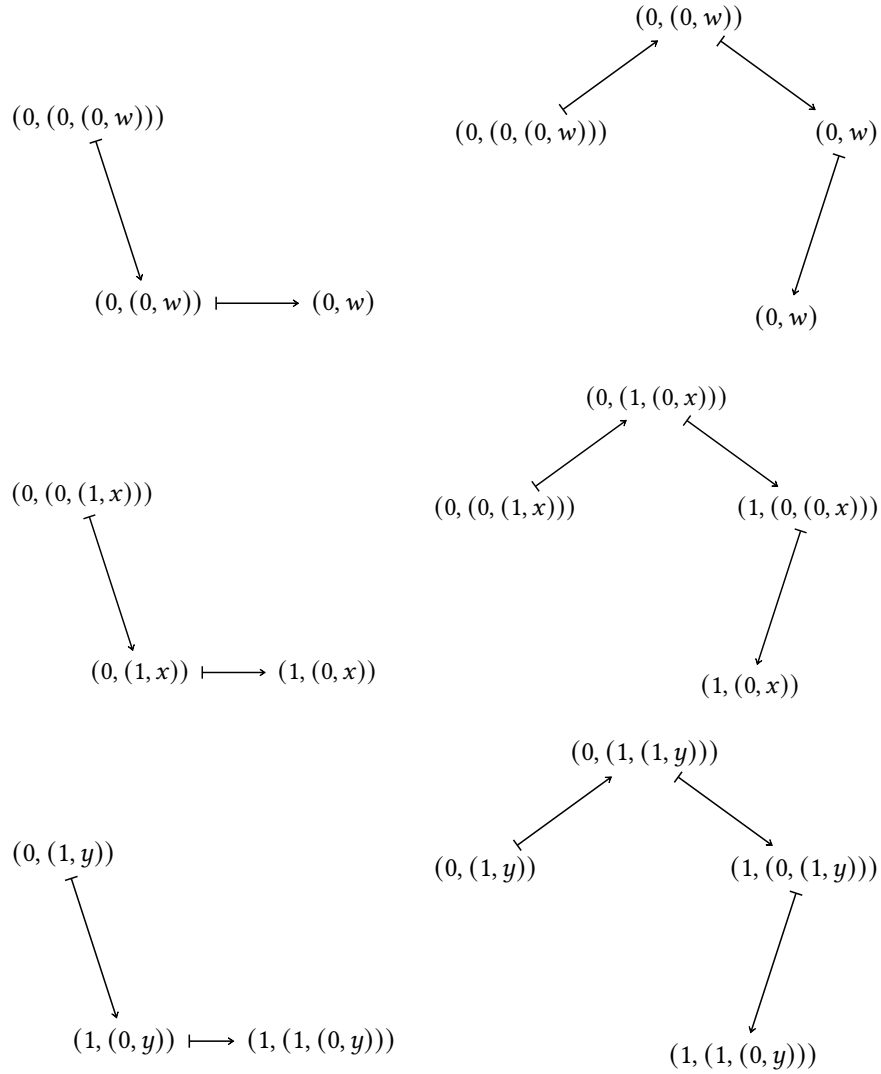
PROOF 5.2.7.1.2 ► PROOF OF PROPOSITION 5.2.7.1.1

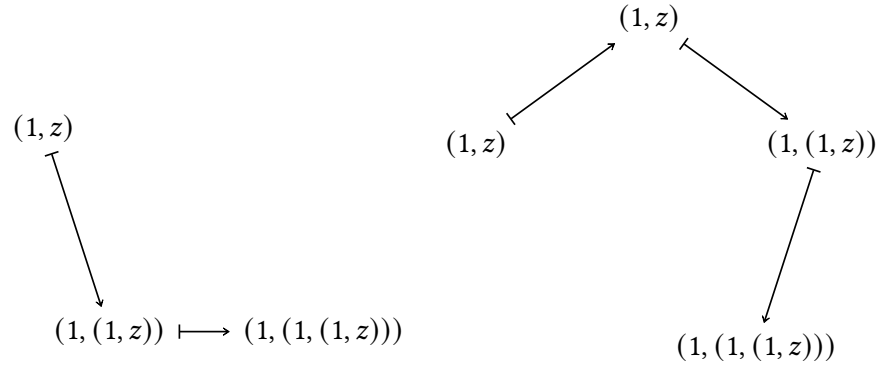
The Pentagon Identity

Let W, X, Y and Z be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (W \amalg (X \amalg Y)) \amalg Z & \\
 \alpha_{W,X,Y}^{\text{Sets}, \amalg} \amalg \text{id}_Z \nearrow & & \searrow \alpha_{W,X}^{\text{Sets}, \amalg} \amalg \text{id}_{Y,Z} \\
 ((W \amalg X) \amalg Y) \amalg Z & & W \amalg ((X \amalg Y) \amalg Z) \\
 \alpha_W^{\text{Sets}, \amalg} \amalg \alpha_{X,Y,Z}^{\text{Sets}, \amalg} \searrow & & \searrow \text{id}_W \amalg \alpha_{X,Y,Z}^{\text{Sets}, \amalg} \\
 (W \amalg X) \amalg (Y \amalg Z) & \xrightarrow{\alpha_{W,X,Y}^{\text{Sets}, \amalg} \amalg Z} & W \amalg (X \amalg (Y \amalg Z))
 \end{array}$$

commutes. Indeed, this diagram acts on elements as





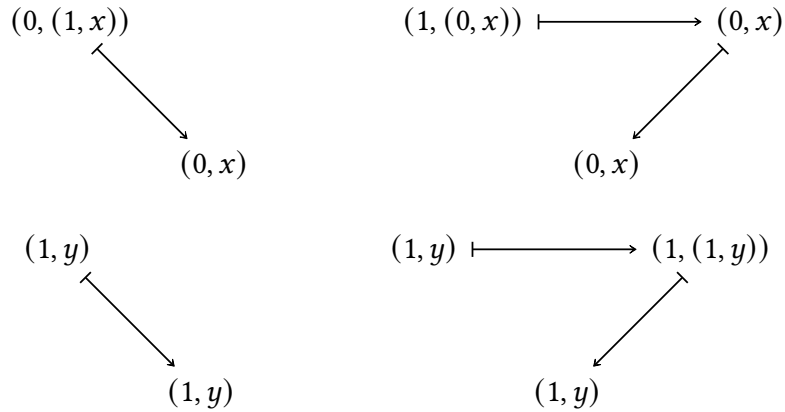
and therefore the pentagon identity is satisfied.

The Triangle Identity

Let X and Y be sets. We have to show that the diagram

$$\begin{array}{ccc}
 (X \amalg \emptyset) \amalg Y & \xrightarrow{\alpha_{X, \emptyset, Y}^{\text{Sets}, \amalg}} & X \amalg (\emptyset \amalg Y) \\
 \rho_X^{\text{Sets}, \amalg} \amalg \text{id}_Y \searrow & & \swarrow \text{id}_X \amalg \lambda_Y^{\text{Sets}, \amalg} \\
 & X \amalg Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as



and therefore the triangle identity is satisfied.

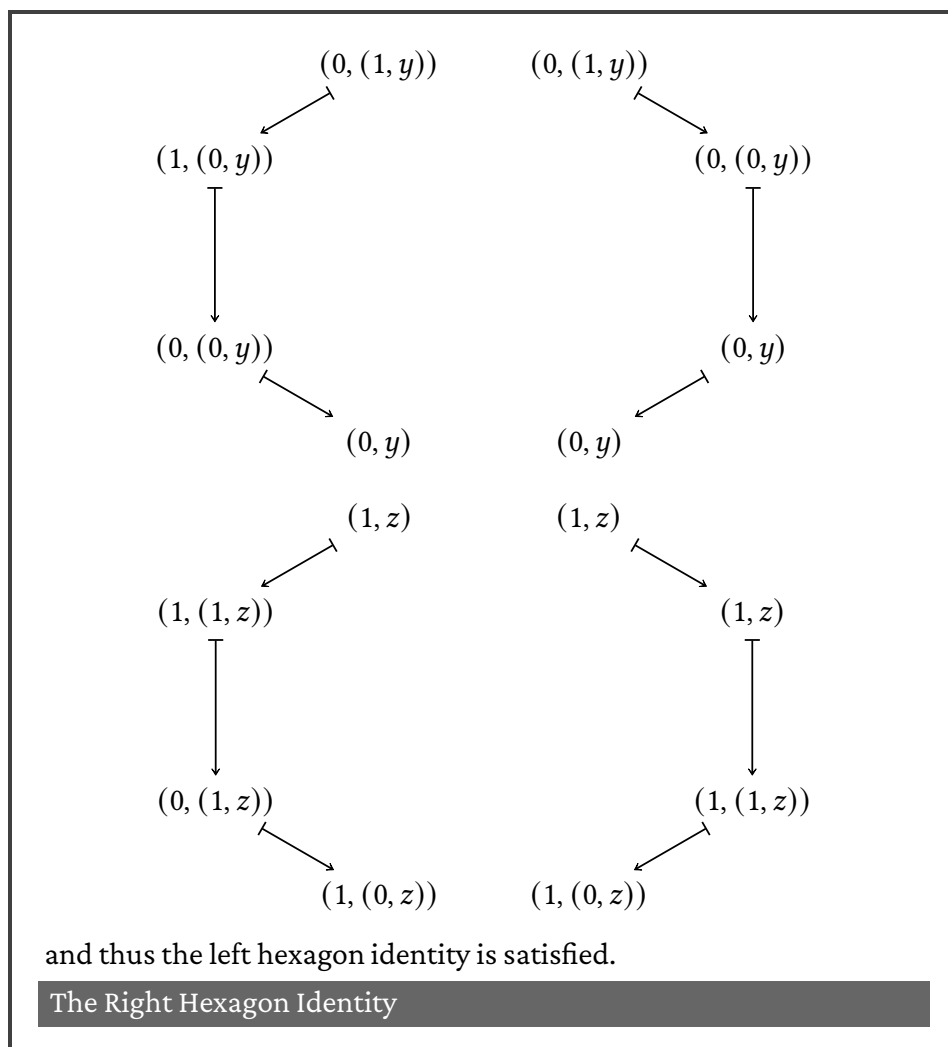
The Left Hexagon Identity

Let X , Y , and Z be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (X \amalg Y) \amalg Z & \\
 \alpha_{X,Y,Z}^{\text{Sets}} \swarrow & & \searrow \sigma_{X,Y}^{\text{Sets}} \amalg \text{id}_Z \\
 X \amalg (Y \amalg Z) & & (Y \amalg X) \amalg Z \\
 \downarrow \sigma_{X,Y}^{\text{Sets}} \amalg \text{id}_Z & & \downarrow \alpha_{Y,X,Z}^{\text{Sets}} \\
 (Y \amalg Z) \amalg X & & Y \amalg (X \amalg Z) \\
 \searrow \alpha_{Y,Z,X}^{\text{Sets}} & & \swarrow \text{id}_Y \amalg \sigma_{X,Z}^{\text{Sets}} \\
 & Y \amalg (Z \amalg X) &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (0, (0, x)) & (0, (0, x)) \\
 \swarrow & & \searrow \\
 (0, x) & & (0, (1, x)) \\
 \downarrow & & \downarrow \\
 (1, x) & & (1, (0, x)) \\
 \searrow & & \swarrow \\
 & (1, (1, x)) & (1, (1, x))
 \end{array}$$

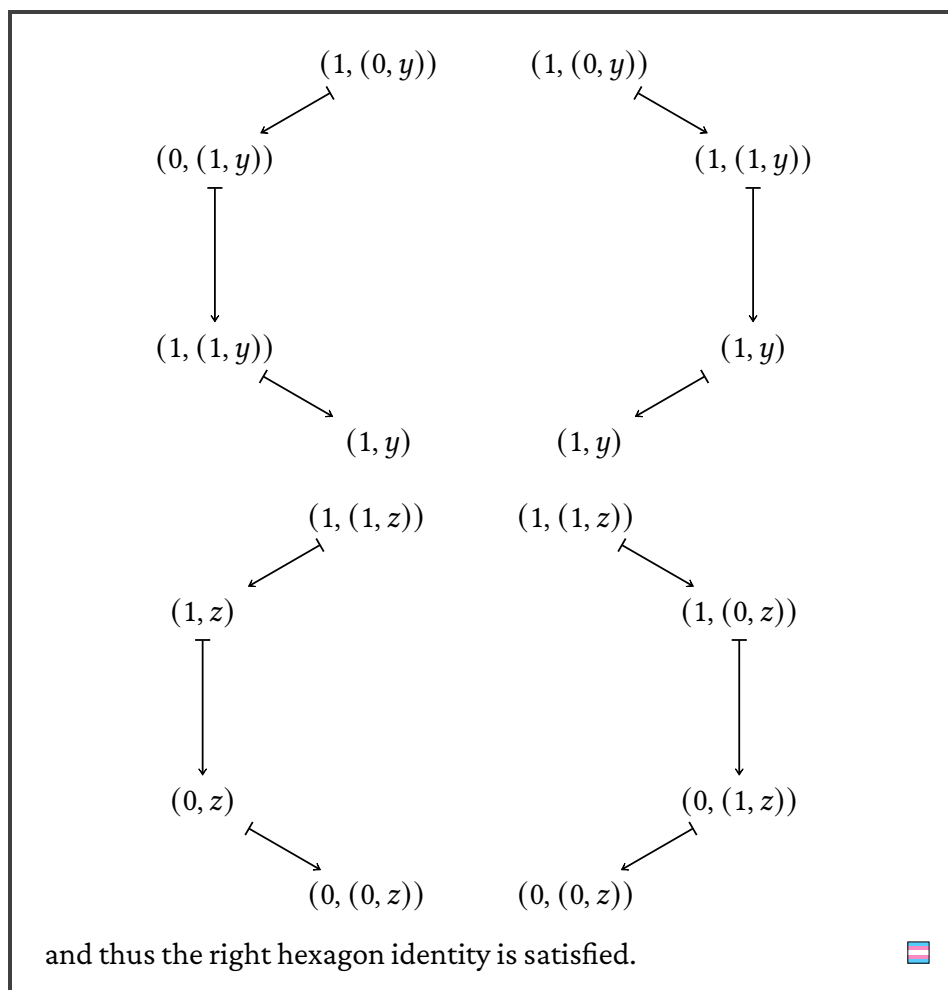


Let X , Y , and Z be sets. We have to show that the diagram

$$\begin{array}{ccc}
 & (\alpha_{X,Y,Z}^{\text{Sets}})^{-1} X \amalg (Y \amalg Z) & \\
 \swarrow \text{id}_X \amalg \sigma_{Y,Z}^{\text{Sets}} & & \searrow \\
 (X \amalg Y) \amalg Z & & X \amalg (Z \amalg Y) \\
 \downarrow \sigma_{X \amalg Y, Z}^{\text{Sets}} & & \downarrow (\alpha_{X,Z,Y}^{\text{Sets}})^{-1} \\
 Z \amalg (X \amalg Y) & & (X \amalg Z) \amalg Y \\
 \swarrow (\alpha_{Z,X,Y}^{\text{Sets}})^{-1} & & \searrow \sigma_{X,Z}^{\text{Sets}} \amalg \text{id}_Y \\
 & (Z \amalg X) \amalg Y &
 \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
 & (0, x) & \\
 \swarrow & & \searrow \\
 (0, (0, x)) & & (0, x) \\
 \downarrow & & \downarrow \\
 (1, (0, x)) & & (0, (0, x)) \\
 \swarrow & & \searrow \\
 & (0, (1, x)) &
 \end{array}$$



**01Q0 5.3 The Bimonoidal Category of Sets, Products,
and Coproducts**

01Q1 5.3.1 The Left Distributor

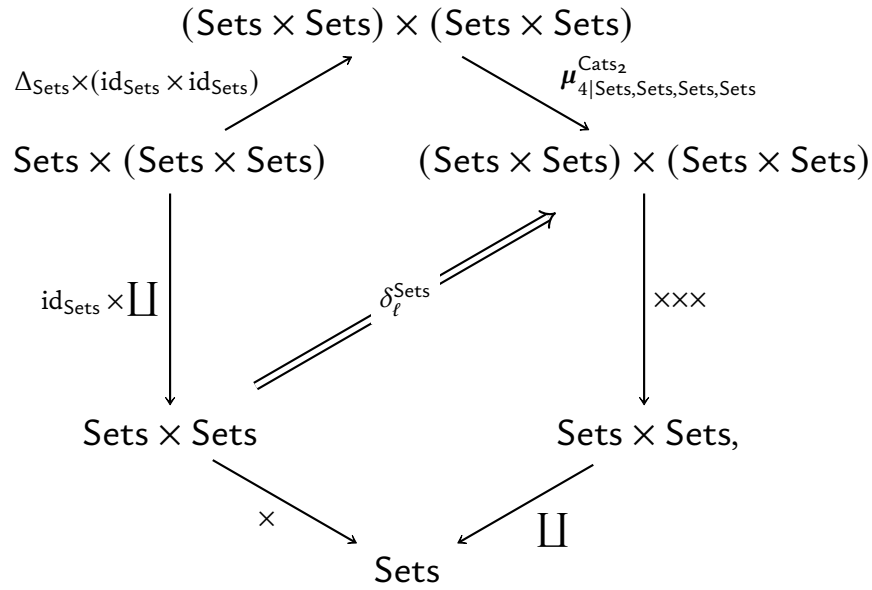
01Q2

DEFINITION 5.3.1.1 ► THE LEFT DISTRIBUTOR OF \times OVER \coprod

The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_{\ell}^{\text{Sets}} : \times \circ (\text{id}_{\text{Sets}} \times \coprod) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\Delta_{\text{Sets}} \times (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}))$$

as in the diagram



whose component

$$\delta_{\ell|X,Y,Z}^{\text{Sets}} : X \times (Y \coprod Z) \xrightarrow{\sim} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{\ell|X,Y,Z}^{\text{Sets}}(x, a) \stackrel{\text{def}}{=} \begin{cases} (0, (x, y)) & \text{if } a = (0, y), \\ (1, (x, z)) & \text{if } a = (1, z) \end{cases}$$

for each $(x, a) \in X \times (Y \coprod Z)$.

PROOF 5.3.1.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.1.1

Omitted.



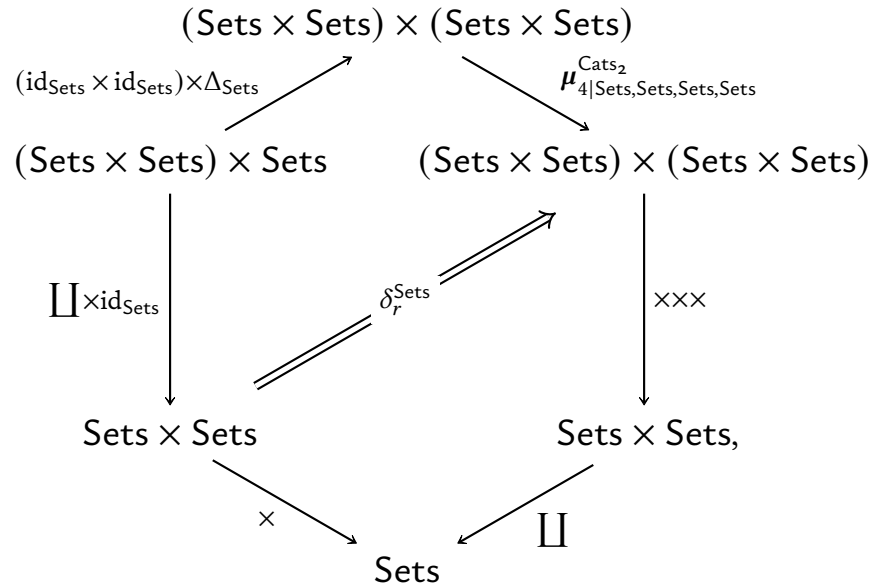
01Q3 5.3.2 The Right Distributor

01Q4 DEFINITION 5.3.2.1.1 ► THE RIGHT DISTRIBUTOR OF \times OVER \coprod

The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\text{Sets}} : \times \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ ((\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}})$$

as in the diagram



whose component


$$\delta_{r|X,Y,Z}^{\text{Sets}} : (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\text{Sets}}(a, z) \stackrel{\text{def}}{=} \begin{cases} (0, (x, z)) & \text{if } a = (0, x), \\ (1, (y, z)) & \text{if } a = (1, y) \end{cases}$$

for each $(a, z) \in (X \coprod Y) \times Z$.

PROOF 5.3.2.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.2.1.1

Omitted. 

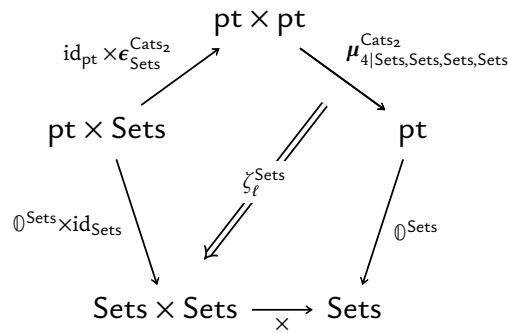
01Q5 5.3.3 The Left Annihilator

01Q6 DEFINITION 5.3.3.1.1 ► THE LEFT ANNIHILATOR OF \times

The **left annihilator of the product of sets** is the natural isomorphism

$$\zeta_{\ell}^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\text{id}_{\text{pt}} \times \epsilon_{\text{Sets}}^{\text{Cats}_2}) \xRightarrow{\sim} \times \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}})$$


as in the diagram



with components

$$\zeta_{\ell|A}^{\text{Sets}} : \emptyset \times A \xrightarrow{\sim} \emptyset.$$

PROOF 5.3.3.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.3.1.1

Omitted. For a partial proof, see [Pro25]. 

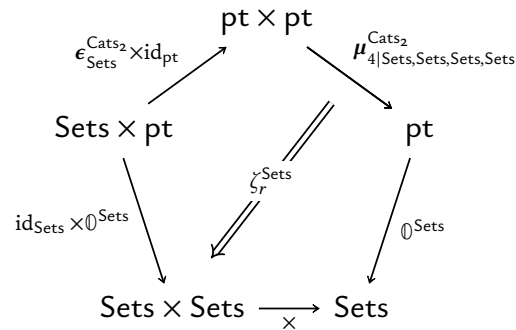
01Q7 5.3.4 The Right Annihilator

01Q8 DEFINITION 5.3.4.1.1 ► THE RIGHT ANNIHILATOR OF \times

The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta_r^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ \left(\epsilon_{\text{Sets}}^{\text{Cats}_2} \times \text{id}_{\text{pt}} \right) \xrightarrow{\sim} \times \circ \left(\text{id}_{\text{Sets}} \times \mathbb{0}^{\text{Sets}} \right)$$

as in the diagram



with components

$$\zeta_{r|A}^{\text{Sets}} : A \times \emptyset \xrightarrow{\sim} \emptyset.$$

PROOF 5.3.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.4.1.1

Omitted. For a partial proof, see [Pro25].



01Q9 5.3.5 The Bimonoidal Category of Sets, Products, and Co-products

PROPOSITION 5.3.5.1.1 ► THE BIMONOIDAL STRUCTURE ON SETS ASSOCIATED TO \times AND \amalg

The category **Sets** admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category.* The category **Sets** of pointed sets.

- *The Additive Monoidal Product.* The coproduct functor

$$\amalg : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Proposition 4.2.3.1.4**.

- *The Multiplicative Monoidal Product.* The product functor

$$\times : \mathbf{Sets} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, Item 1 of **Proposition 4.1.3.1.4**.

- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Sets}} : \mathbf{pt} \rightarrow \mathbf{Sets}$$

of **Definition 5.1.3.1.1**.

- *The Monoidal Zero.* The functor

$$\mathbb{0}^{\mathbf{Sets}} : \mathbf{pt} \rightarrow \mathbf{Sets}$$

of **Definition 5.1.3.1.1**.

- *The Internal Hom.* The internal Hom functor

$$\mathbf{Sets} : \mathbf{Sets}^{\mathbf{op}} \times \mathbf{Sets} \rightarrow \mathbf{Sets}$$

of **Constructions With Sets**, ?? of ??.

- *The Additive Associators.* The natural isomorphism

$$\alpha^{\mathbf{Sets}, \amalg} : \amalg \circ (\amalg \times \mathrm{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \amalg \circ (\mathrm{id}_{\mathbf{Sets}} \times \amalg) \circ \alpha_{\mathbf{Sets}, \mathbf{Sets}, \mathbf{Sets}}^{\mathbf{Cats}}$$

of **Definition 5.2.3.1.1**.

- *The Additive Left Unitors.* The natural isomorphism

$$\lambda^{\mathbf{Sets}, \amalg} : \amalg \circ (\mathbb{0}^{\mathbf{Sets}} \times \mathrm{id}_{\mathbf{Sets}}) \xrightarrow{\sim} \lambda_{\mathbf{Sets}}^{\mathbf{Cats}_2}$$

of **Definition 5.2.4.1.1**.

- *The Additive Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}, \amalg} : \amalg \circ (\text{id} \times \mathbb{0}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.5.1.1](#).

- *The Additive Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}, \amalg} : \amalg \xRightarrow{\sim} \amalg \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.2.6.1.1](#).

- *The Multiplicative Associators.* The natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha_{\text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}}$$

of [Definition 5.1.4.1.1](#).

- *The Multiplicative Left Unitors.* The natural isomorphism

$$\lambda^{\text{Sets}} : \times \circ (\mathbb{1}^{\text{Sets}} \times \text{id}_{\text{Sets}}) \xRightarrow{\sim} \lambda_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.1.5.1.1](#).

- *The Multiplicative Right Unitors.* The natural isomorphism

$$\rho^{\text{Sets}} : \times \circ (\text{id} \times \mathbb{1}^{\text{Sets}}) \xRightarrow{\sim} \rho_{\text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.1.6.1.1](#).

- *The Multiplicative Symmetry.* The natural isomorphism

$$\sigma^{\text{Sets}} : \times \xRightarrow{\sim} \times \circ \sigma_{\text{Sets}, \text{Sets}}^{\text{Cats}_2}$$

of [Definition 5.1.7.1.1](#).

- *The Left Distributor.* The natural isomorphism

$$\delta_\ell^{\text{Sets}} : \times \circ (\text{id}_{\text{Sets}} \times \amalg) \xRightarrow{\sim} \amalg \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\Delta_{\text{Sets}} \times (\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}))$$

of [Definition 5.3.1.1.1](#).

- *The Right Distributor.* The natural isomorphism

$$\delta_r^{\text{Sets}} : \times \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\times \times \times) \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ ((\text{id}_{\text{Sets}} \times \text{id}_{\text{Sets}}) \times \Delta_{\text{Sets}})$$

of **Definition 5.3.2.1.1.**

- *The Left Annihilator.* The natural isomorphism

$$\zeta_\ell^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\text{id}_{\text{pt}} \times \epsilon_{\text{Sets}}^{\text{Cats}_2}) \xrightarrow{\sim} \times \circ (\mathbb{0}^{\text{Sets}} \times \text{id}_{\text{Sets}})$$

of **Definition 5.3.3.1.1.**

- *The Right Annihilator.* The natural isomorphism

$$\zeta_r^{\text{Sets}} : \mathbb{0}^{\text{Sets}} \circ \mu_{4|\text{Sets}, \text{Sets}, \text{Sets}, \text{Sets}}^{\text{Cats}_2} \circ (\epsilon_{\text{Sets}}^{\text{Cats}_2} \times \text{id}_{\text{pt}}) \xrightarrow{\sim} \times \circ (\text{id}_{\text{Sets}} \times \mathbb{0}^{\text{Sets}})$$

of **Definition 5.3.4.1.1.**

PROOF 5.3.5.1.2 ► PROOF OF PROPOSITION 5.3.5.1.1

Omitted.



Appendices

A Other Chapters

Preliminaries

1. **Introduction**
2. **A Guide to the Literature**

Sets

3. **Sets**
4. **Constructions With Sets**

5. **Monoidal Structures on the Category of Sets**

6. **Pointed Sets**

7. **Tensor Products of Pointed Sets**

Relations

8. **Relations**
9. **Constructions With Relations**

10. Conditions on Relations

13. Constructions With Monoidal Categories

Categories

Bicategories

11. Categories

14. Types of Morphisms in Bicat-
egories

12. Presheaves and the Yoneda
Lemma

Extra Part

Monoidal Categories

15. Notes

References

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