# Constructions With Monoidal Categories

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**O1UF** This chapter contains some material on constructions with monoidal categories.

## **C**ontents

	13	oduli Categories of Monoidal Structures	<b>1</b>
	Categor 13	y	
		oduli Categories of Closed Monoidal Structures	
	13.3 Moduli Categories of Refinements of Monoidal Structures		<b>15</b>
		her Chapters	
01UG	13.1	Moduli Categories of Monoidal Structures	
01UH	13.1.1	The Moduli Category of Monoidal Structures on a Cagory	te-

**Definition 13.1.1.1.1.** The moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{B}_1}(C)$  defined by

- 01UK Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category  $\mathcal{M}_{\mathbb{B}_1}(C)$  where:
  - Objects. The objects of  $\mathcal{M}_{\mathbb{B}_1}(C)$  are monoidal categories  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  whose underlying category is C.
  - *Morphisms*. A morphism from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  is a strong monoidal functor structure

$$\operatorname{id}_{C}^{\otimes} \colon A \boxtimes_{C} B \xrightarrow{\sim} A \otimes_{C} B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_{C} \xrightarrow{\sim} \mathbb{1}_{C}$$

on the identity functor  $id_C : C \to C$  of C.

• *Identities*. For each  $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$ , the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of  $\mathcal{M}_{\mathbb{E}_1}(C)$  at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{E}_{1}}(C)} \stackrel{\operatorname{def}}{=} \left(\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes}\right),$$

where  $\left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{1|C}^{\otimes}\right)$  is the identity monoidal functor of C of ??.

• Composition. For each  $M,N,P\in \mathrm{Obj}ig(\mathcal{M}_{\mathbb{E}_1}(C)ig)$  , the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{B}_{1}}(C)}(M,P)$$
of  $\mathcal{M}_{\mathbb{B}_{1}}(C)$  at  $(M,N,P)$  is defined by

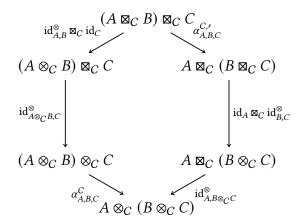
$$\left(\operatorname{id}_{\mathcal{C}}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{H}_{1}}(\mathcal{C})}\left(\operatorname{id}_{\mathcal{C}}^{\otimes},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right)\stackrel{\text{def}}{=}\left(\operatorname{id}_{\mathcal{C}}^{\otimes,\prime}\circ\operatorname{id}_{\mathcal{C}}^{\otimes},\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{1}|\mathcal{C}}^{\otimes}\right).$$

- **Q1UL** Remark 13.1.1.1.3. In particular, a morphism in  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  satisfies the following conditions:
- 01UM 1. *Naturality*. For each pair  $f: A \to X$  and  $g: B \to Y$  of morphisms of C, the diagram

$$\begin{array}{ccccc} A \boxtimes_C B & \xrightarrow{f\boxtimes_C g} & X \boxtimes_C Y \\ \operatorname{id}_{A,B}^{\otimes} & & & & & & & & & & & \\ A \otimes_C B & \xrightarrow{f\otimes_C g} & X \otimes_C Y & & & & & & \end{array}$$

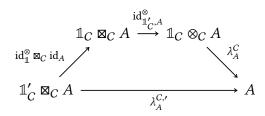
commutes.

**Olum** 2. *Monoidality.* For each  $A, B, C \in Obj(C)$ , the diagram



commutes.

**01UP** 3. Left Monoidal Unity. For each  $A \in Obj(C)$ , the diagram



commutes.

**01UQ** 4. *Right Monoidal Unity.* For each  $A \in Obj(C)$ , the diagram

$$A \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\otimes}} A \otimes_{C} \mathbb{1}_{C}$$

$$\operatorname{id}_{A} \boxtimes_{C} \operatorname{id}_{\mathbb{1}}^{\otimes} / \xrightarrow{\rho_{A}^{C}} A \otimes_{C} \mathbb{1}_{C}$$

$$A \boxtimes_{C} \mathbb{1}_{C}' \xrightarrow{\rho_{A}^{C,'}} A \otimes_{C} \mathbb{1}_{C}$$

commutes.

- **10 Proposition 13.1.1.1.4.** Let *C* be a category.
- 01US 1. Extra Monoidality Conditions. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .
- 01UT (a) The diagram

commutes.

01UU (b) The diagram

$$A\boxtimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\boxtimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\boxtimes_{C}(B\otimes_{C}C)$$

$$\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}\downarrow \qquad \qquad \qquad \downarrow \operatorname{id}_{A,B\otimes_{C}C}^{\otimes}$$

$$A\otimes_{C}(B\boxtimes_{C}C)\xrightarrow{\operatorname{id}_{A}\otimes_{C}\operatorname{id}_{B,C}^{\otimes}}A\otimes_{C}(B\otimes_{C}C)$$

commutes.

01WB 2. Extra Monoidal Unity Constraints. Let  $(id_C^{\otimes}, id_{1|C}^{\otimes})$  be a morphism of  $\mathcal{M}_{\mathbb{B}_1}(C)$  from  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  to  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ .

01WC (a) The diagram

commutes.

01WD (b) The diagram

commutes.

01WE (c) The diagram

commutes.

01WF (d) The diagram

commutes.

01UV 3. Mixed Associators. Let  $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$  and  $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$  be monoidal structures on C and let

$$\mathrm{id}_{-1,-2}^{\otimes} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \longrightarrow -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha_{ABC}^{\otimes}: (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{c|c} (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \\ id_{A \otimes_C B,C}^{\otimes} & & & \downarrow id_A \otimes_C id_{B,C}^{\otimes} \\ (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$\begin{array}{cccc} (A \boxtimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_C (B \boxtimes_C C) \\ \operatorname{id}_{A,B}^{\otimes} \boxtimes_C \operatorname{id}_C & & & & & \operatorname{id}_{A,B \boxtimes_C C} \\ (A \otimes_C B) \boxtimes_C C & \xrightarrow{\alpha_{A,B,C}^{\otimes}} A \otimes_C (B \boxtimes_C C) \end{array}$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

**01UX** (b) If there exists a natural transformation

$$\alpha_{A.B.C}^{\boxtimes} \colon (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_C (B \otimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & & & \operatorname{id}_{A,B \otimes_C C}^{\otimes} \\ & (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{C}} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$id_{A\boxtimes_{C}B,C}^{\otimes} \downarrow \qquad \qquad \downarrow id_{A}\boxtimes_{C} id_{B,C}^{\otimes}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes}} A \boxtimes_{C} (B \otimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{array}{cccc} (A \boxtimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_C (B \boxtimes_C C) \\ & \operatorname{id}_{A,B}^{\otimes} \otimes_C \operatorname{id}_C & & & & \operatorname{id}_{A,C}^{\otimes} \\ & (A \otimes_C B) \otimes_C C & \xrightarrow{\alpha_{A,B,C}^C} A \otimes_C (B \otimes_C C) \end{array}$$

and

$$(A \boxtimes_{C} B) \boxtimes_{C} C \xrightarrow{\alpha_{A,B,C}^{C,\prime}} A \boxtimes_{C} (B \boxtimes_{C} C)$$

$$\downarrow^{\operatorname{id}_{A\boxtimes_{C}B,C}^{\otimes}} \qquad \qquad \downarrow^{\operatorname{id}_{A,B\boxtimes_{C}C}^{\otimes}}$$

$$(A \boxtimes_{C} B) \otimes_{C} C \xrightarrow{\alpha_{A,B,C}^{\boxtimes,\otimes}} A \otimes_{C} (B \boxtimes_{C} C)$$

commute, then the natural transformation  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

*Proof. Item* **1**, *Extra Monoidality Conditions*: We claim that *Items* **1a** and **1b** are indeed true:

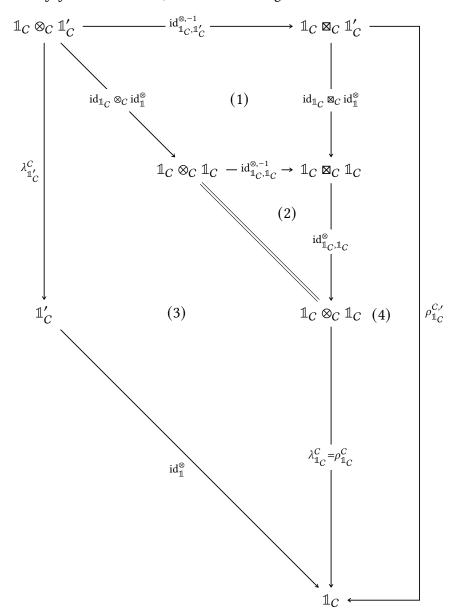
1. Proof of Item 1a: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_{A,B}^{\otimes}$  and  $id_{C}$ .

2. *Proof of Item 1b*: This follows from the naturality of  $id^{\otimes}$  with respect to the morphisms  $id_A$  and  $id_{RC}^{\otimes}$ .

This finishes the proof.

*Item* **2**, *Extra Monoidal Unity Constraints*: We claim that *Items* **2a** and **2b** are indeed true:

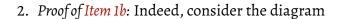
1. Proof of Item 1a: Indeed, consider the diagram

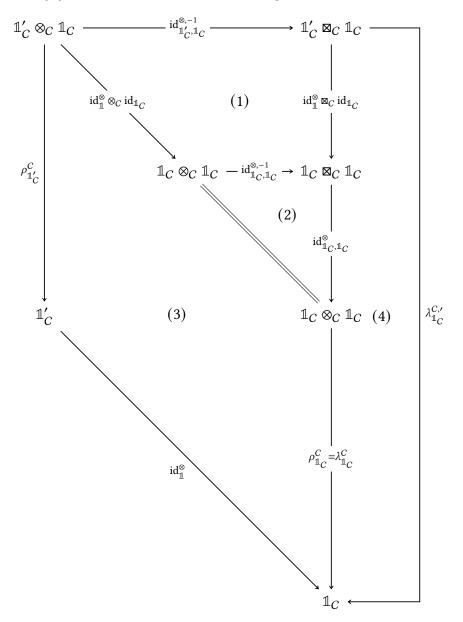


whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of  $\lambda^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from  $\ref{eq:composition}$ ;
- Subdiagram (4) commutes by the right monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ;

so does the boundary diagram, and we are done.





whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of  $\mathrm{id}_C^{\otimes,-1}$ ;
- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of  $\rho^C$ , where the equality  $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$  comes from  $\ref{eq:composition}$ ;
- Subdiagram (4) commutes by the left monoidal unity of  $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$ ; so does the boundary diagram, and we are done.
- 3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

commutes. But since  $\mathrm{id}_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \boxtimes_{C} \mathbb{1}'_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}_{C} \otimes_{C} \mathbb{1}'_{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

$$\downarrow \qquad \qquad \downarrow \lambda_{\mathbb{1}'_{C}}^{C}$$

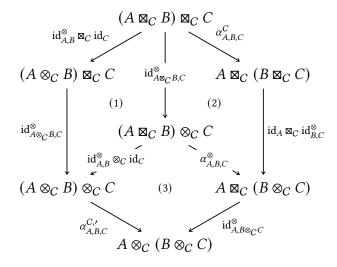
commutes. But since  $id_{1}^{\otimes,-1}$  is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

*Item 3*, *Mixed Associators*: We claim that *Items 3a* to *3c* are indeed true:

01UZ 1. Proof of Item 3a: We may partition the monoidality diagram for  $id^{\otimes}$  of



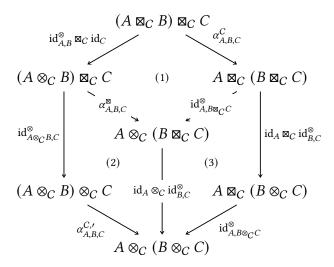


Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. Proof of Item 3b: We may partition the monoidality diagram for id<sup>®</sup> of Item 2 of Definition 13.1.1.1.3 as follows:

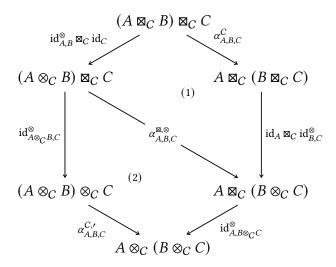


Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item 3c*: We may partition the monoidality diagram for id<sup>⊗</sup> of Item 2 of Definition 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e.  $id^{\otimes}$  satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- 01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- 01V4 13.2 Moduli Categories of Closed Monoidal Structures
- o1V5 13.3 Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

# **Appendices**

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- 1. Introduction
- 2. A Guide to the Literature

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- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets

7. Tensor Products of Pointed Sets

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### **Categories**

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

### **Monoidal Categories**

13. Constructions With Monoidal gories
Categories

## Bicategories

## Extra Part

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