Constructions With Monoidal Categories

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01UF This chapter contains some material on constructions with monoidal categories.

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1UH	13.1.1 The Moduli Category of Monoidal Structures on a Category	ory			
	Let <i>C</i> be a category.				

O1UJ **Definition 13.1.1.1.1.** The moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \stackrel{\mathrm{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, egin{pmatrix} \mathcal{M}_{\mathbb{E}_1}(C) & \longrightarrow & \mathsf{MonCats} \\ & & \downarrow & & \downarrow \\ & \mathsf{pt} & \xrightarrow{[C]} & \mathsf{Cats}. \end{pmatrix}$$

- O1UK Remark 13.1.1.1.2. In detail, the moduli category of monoidal structures on C is the category $\mathcal{M}_{\mathbb{B}_1}(C)$ where:
 - *Objects*. The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
 - *Morphisms*. A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\operatorname{id}_C^{\otimes} \colon A \boxtimes_C B \xrightarrow{\sim} A \otimes_C B,$$
$$\operatorname{id}_{\mathbb{1}|C}^{\otimes} \colon \mathbb{1}'_C \xrightarrow{\sim} \mathbb{1}_C$$

on the identity functor $id_C: C \to C$ of C.

• *Identities*. For each $M \stackrel{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}_{M,M}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \mathsf{pt} \to \mathsf{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,M)$$

of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\operatorname{id}_{M}^{\mathcal{M}_{\mathbb{B}_{1}}(C)} \stackrel{\text{def}}{=} \left(\operatorname{id}_{C}^{\otimes}, \operatorname{id}_{\mathbb{1}|C}^{\otimes}\right),$$

where $(id_C^{\otimes}, id_{1|C}^{\otimes})$ is the identity monoidal functor of C of ??.

• *Composition*. For each $M, N, P \in \text{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the composition map

$$\circ_{M,N,P}^{\mathcal{M}_{\mathbb{E}_1}(C)} \colon \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(N,P) \times \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,N) \to \operatorname{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M,P)$$
 of $\mathcal{M}_{\mathbb{E}_1}(C)$ at (M,N,P) is defined by

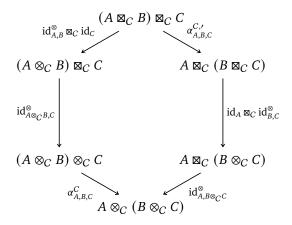
$$\left(\operatorname{id}_{C}^{\otimes,\prime},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\right)\circ_{M,N,P}^{\mathcal{M}_{\mathbb{B}_{1}}(C)}\left(\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right)\stackrel{\operatorname{def}}{=}\left(\operatorname{id}_{C}^{\otimes,\prime}\circ\operatorname{id}_{C}^{\otimes},\operatorname{id}_{\mathbb{1}|C}^{\otimes,\prime}\circ\operatorname{id}_{\mathbb{1}|C}^{\otimes}\right).$$

Remark 13.1.1.1.3. In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ satisfies the following conditions:

01UM 1. *Naturality*. For each pair $f: A \to X$ and $g: B \to Y$ of morphisms of C, the diagram

commutes.

Olum 2. *Monoidality*. For each $A, B, C \in Obj(C)$, the diagram



commutes.

01UP 3. *Left Monoidal Unity.* For each $A \in Obj(C)$, the diagram

$$\mathbb{1}_{C} \boxtimes_{C} A \xrightarrow{\operatorname{id}_{\mathbb{1}'_{C},^{A}}} \mathbb{1}_{C} \otimes_{C} A$$

$$\operatorname{id}_{\mathbb{1}}^{\otimes} \boxtimes_{C} \operatorname{id}_{A} \xrightarrow{\lambda_{A}^{C}} A$$

$$\mathbb{1}'_{C} \boxtimes_{C} A \xrightarrow{\lambda_{A}^{C,'}} A$$

commutes.

01UQ 4. Right Monoidal Unity. For each A ∈ Obj(C), the diagram

$$A\boxtimes_{C}\mathbb{1}_{C}\stackrel{\operatorname{id}_{A,\mathbb{1}_{C}'}^{\wedge}}{\longrightarrow}A\otimes_{C}\mathbb{1}_{C}$$

$$\operatorname{id}_{A}\boxtimes_{C}\operatorname{id}_{\mathbb{1}}^{\wedge} \qquad \qquad \bigwedge^{\rho_{A}^{C}}$$

$$A\boxtimes_{C}\mathbb{1}_{C}' \qquad \qquad \bigwedge^{\rho_{A}^{C,\prime}} A$$

commutes.

OUR Proposition 13.1.1.1.4. Let C be a category.

01US 1. Extra Monoidality Conditions. Let $(id_C^{\otimes}, id_{\mathbb{1}|C}^{\otimes})$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

01UT (a) The diagram

commutes.

01UU (b) The diagram

commutes.

01WB 2. Extra Monoidal Unity Constraints. Let $(\mathrm{id}_C^\otimes,\mathrm{id}_{\mathbb{1}|C}^\otimes)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C,\otimes_C,\mathbb{1}_C,\alpha^C,\lambda^C,\rho^C)$ to $(C,\boxtimes_C,\mathbb{1}_C',\alpha^{C,\prime},\lambda^{C,\prime},\rho^{C,\prime})$.

01WC (a) The diagram

commutes.

01WD (b) The diagram

commutes.

01WE (c) The diagram

commutes.

01WF (d) The diagram

commutes.

01UV 3. *Mixed Associators*. Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$id^{\otimes}_{-_1,-_2} \colon \mathrel{-_1} \boxtimes_{\mathcal{C}} \mathrel{-_2} \to \mathrel{-_1} \otimes_{\mathcal{C}} \mathrel{-_2}$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha_{A,B,C}^{\otimes} \colon (A \otimes_{C} B) \boxtimes_{C} C \to A \otimes_{C} (B \boxtimes_{C} C)$$

making the diagrams

and

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UX (b) If there exists a natural transformation

$$\alpha_{A,B,C}^{\boxtimes} : (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

and

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha_{ABC}^{\boxtimes,\otimes}: (A\boxtimes_C B)\otimes_C C \to A\otimes_C (B\boxtimes_C C)$$

making the diagrams

and

commute, then the natural transformation id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

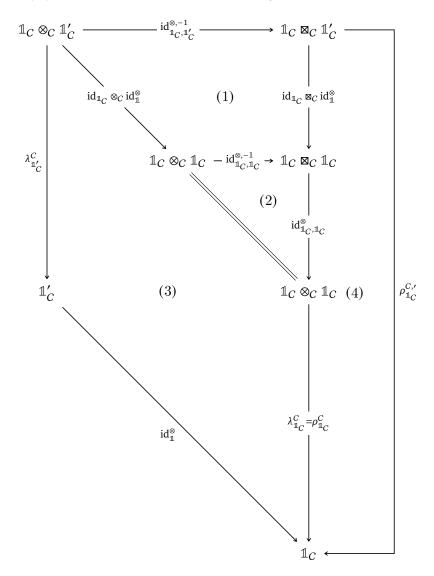
Proof. Item 1, Extra Monoidality Conditions: We claim that *Items 1a* and *1b* are indeed true:

- 1. *Proof of Item 1a*: This follows from the naturality of id^{\otimes} with respect to the morphisms $id_{A,B}^{\otimes}$ and id_{C} .
- 2. *Proof of Item 1b*: This follows from the naturality of id^{\otimes} with respect to the morphisms id_A and $id_{B,C}^{\otimes}$.

This finishes the proof.

Item 2, Extra Monoidal Unity Constraints: We claim that *Items 2a* and **2b** are indeed true:

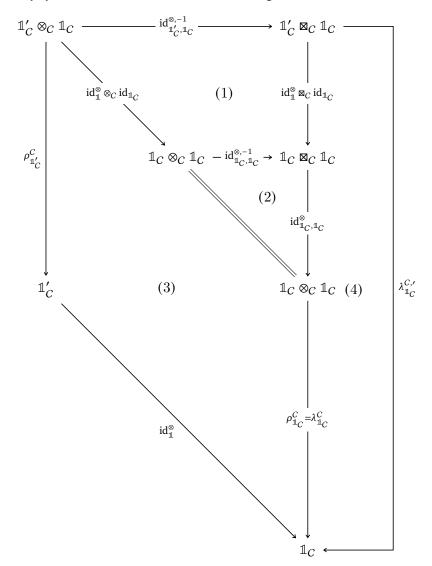
1. Proof of Item 1a: Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C , where the equality $\rho_{\mathbb{1}_C}^C = \lambda_{\mathbb{1}_C}^C$ comes from **??**;

- Subdiagram (4) commutes by the right monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$; so does the boundary diagram, and we are done.
- 2. Proof of Item 1b: Indeed, consider the diagram



whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_C^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;

- Subdiagram (3) commutes by the naturality of ρ^C , where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $(id_C, id_C^{\otimes}, id_{C|1}^{\otimes})$; so does the boundary diagram, and we are done.
- 3. Proof of Item 2c: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes,-1}} \mathbb{1}'_{C} \boxtimes_{C} \mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{1}_{C},\mathbb{1}'_{C}}^{\otimes}} \mathbb{1}'_{C} \otimes_{C} \mathbb{1}_{C}$$

$$\downarrow^{c,'}_{\mathbb{1}_{C}} \qquad \qquad (\dagger) \qquad \qquad \downarrow^{\rho^{c}_{\mathbb{1}'_{C}}}$$

$$\mathbb{1}_{C} \xrightarrow{\operatorname{id}_{\mathbb{S}^{-1}}^{\otimes,-1}} \mathbb{1}'_{C}$$

commutes. But since $id_{\mathbb{1}_C,\mathbb{1}'_C}^{\otimes,-1}$ is an isomorphism, it follows that the diagram (\dagger) also commutes, and we are done.

4. Proof of Item 2d: Indeed, consider the diagram

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

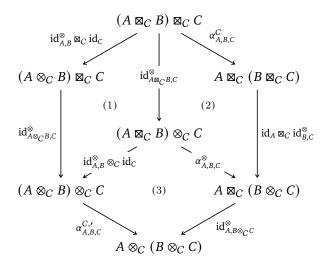
it follows that the diagram

commutes. But since $id_1^{\otimes,-1}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

01UZ 1. *Proof of Item 3a*: We may partition the monoidality diagram for id[⊗] of Item 2 of Definition 13.1.1.1.3 as follows:

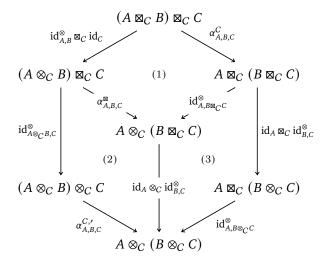


Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. id[®] satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01V0 2. *Proof of Item 3b*: We may partition the monoidality diagram for id[⊗] of Item 2 of Definition 13.1.1.1.3 as follows:



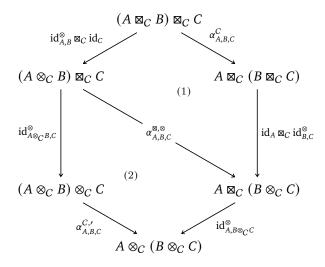
Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

01V1 3. Proof of Item 3c: We may partition the monoidality diagram for id^{\otimes} of

Item 2 of Definition 13.1.1.1.3 as follows:



Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. id^{\otimes} satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof.

- **01V2** 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category
- 01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category
- **13.2** Moduli Categories of Closed Monoidal Structures
- **13.3** Moduli Categories of Refinements of Monoidal Structures
- 01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters

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- 2. A Guide to the Literature

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- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

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- 11. Categories
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Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes