Relations

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This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 8.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 8.1.2).
3. The various kinds of categories that relations form, namely:
    (a) A category (Section 8.3.2).
    (b) A monoidal category (Section 8.3.3).
    (c) A 2-category (Section 8.3.4).
    (d) A double category (Section 8.3.5).
4. The various categorical properties of the 2-category of relations, including:
    (a) The self-duality of Rel and $\mathbf{Rel}$ (Definition 8.5.1.1.1).
    (b) Identifications of equivalences and isomorphisms in $\mathbf{Rel}$ with bijections (Definition 8.5.2.1.2).
    (c) Identifications of adjunctions in $\mathbf{Rel}$ with functions (Definition 8.5.3.1.1).
    (d) Identifications of monads in $\mathbf{Rel}$ with preorders (**??**).
    (e) Identifications of comonads in $\mathbf{Rel}$ with subsets (**??**).
    (f) A description of the monoids and comonoids in $\mathbf{Rel}$ with respect to the Cartesian product (Definition 8.5.9.1.1).

(g) Characterisations of monomorphisms in Rel (**??**).

(h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (**??**).

(i) Characterisations of epimorphisms in Rel (**??**).

(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (**??**).

(k) The partial co/completeness of Rel (Definition 8.5.12.1.1).

(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (**??**).

(m) The closedness of **Rel** (Definition 8.5.17.1.1).

(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Definition 8.5.18.1.1).

5. The adjoint pairs

$$R_! \dashv R_{-1} : \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_* : \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, as well as the properties of $R_!$, R_{-1}, R^{-1}, and R_* (Section 8.7).

Of particular note are the following points:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_! \dashv f^{-1} \dashv f_*$ induced by a function $f\colon A \to B$ studied in Constructions With Sets, Section 4.6.

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Definition 8.7.2.1.3).

(c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

$$R_! \dashv R_{-1} = R^{-1} \dashv R_*$$

reduces to the triple adjunction

$$f_! \dashv f^{-1} \dashv f_*$$

from Constructions With Sets, Section 4.6.

(d) The pairs $R_! \dashv R_{-1}$ and $R^{-1} \dashv R_*$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (Topological Spaces, **??**).

6. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ appearing in the definition of a relative monad (Sections 8.8 and 8.9).

This chapter is under revision. TODO:

1. Replicate Section 8.5 for apartness composition
2. Revise Section 8.7
3. Add subsection "A Six Functor Formalism for Sets, Part 2", now with relations, building upon Section 8.7.
4. Replicate Section 8.7 for apartness composition
5. Revise sections on skew monoidal structures on $\mathbf{Rel}(A, B)$
6. Replicate the sections on skew monoidal structures on $\mathbf{Rel}(A, B)$ for apartness composition.
7. Explore relative co/monads in $\mathbf{Rel}$, defined to be co/monoids in $\mathbf{Rel}(A, B)$ with its left/right skew monoidal structures of Relations, Sections 8.8 and 8.9
8. functional total relations defined with "satisfying the following equivalent conditions:"

Contents

8.1 Relations

8.1.1 Foundations

Let A and B be sets.

Definition 8.1.1.1.1. A **relation** $R \colon A \nrightarrow B$ **from** A **to** B[1,2] is equivalently:

1. A subset R of $A \times B$.
2. A function from $A \times B$ to $\{\text{true}, \text{false}\}$.
3. A function from A to $\mathcal{P}(B)$.
4. A function from B to $\mathcal{P}(A)$.
5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.
6. A continuous morphism of posets from $(\mathcal{P}(B), \supset)$ to $(\mathcal{P}(A), \supset)$.

Proof. (We will prove that Items 1 to 6 are indeed equivalent in a bit.) □

Remark 8.1.1.1.2. We may think of a relation $R \colon A \nrightarrow B$ as a function from A to B that is *multivalued*, assigning to each element a in A a set $R(a)$ of elements of B, thought of as the *set of values of R at a*.

Note that this includes also the possibility of R having no value at all on a given $a \in A$ when $R(a) = \emptyset$.

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B.
[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

Remark 8.1.1.1.3. Another way of stating the equivalence between Items 1 to 5 of Definition 8.1.1.1.1 is by saying that we have bijections of sets

$$
\begin{aligned}
\{\text{relations from } A \text{ to } B\} &\cong \mathcal{P}(A \times B)\\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\})\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B))\\
&\cong \mathsf{Sets}(B, \mathcal{P}(A))\\
&\cong \mathsf{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B))\\
&\cong \mathsf{Pos}^{\mathcal{C}}(\mathcal{P}(B), \mathcal{P}(A))
\end{aligned}
$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are endowed with the poset structure given by inclusion.

Proof. We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This is a special case of Constructions With Sets, Items 2 and 3 of Definition 4.5.1.1.4.
- *Item 2* $\iff$ *Item 3*: This follows from the bijections

  $$
  \begin{aligned}
  \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
  &\cong \mathsf{Sets}(A, \mathcal{P}(B)),
  \end{aligned}
  $$

  where the last bijection is from Constructions With Sets, Items 2 and 3 of Definition 4.5.1.1.4.
- *Item 2* $\iff$ *Item 4*: This follows from the bijections

  $$
  \begin{aligned}
  \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
  &\cong \mathsf{Sets}(B, \mathcal{P}(A)),
  \end{aligned}
  $$

  where again the last bijection is from Constructions With Sets, Items 2 and 3 of Definition 4.5.1.1.4.
- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

  $$\chi_X : X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$, as in Constructions With Sets, Definition 4.4.5.1.1. In particular, the bijection

$$\mathsf{Sets}(A, \mathcal{P}(B)) \cong \mathsf{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B))$$

is given by extending each $f \colon A \to \mathcal{P}(B)$ in $\mathsf{Sets}(A, \mathcal{P}(B))$ from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X, recovering the direct image function $f_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$ of f of Constructions With Sets, Definition 4.6.1.1.1.

- *Item 5* $\iff$ *Item 6*: Omitted.

This finishes the proof. □

Notation 8.1.1.1.4. Let A and B be sets and let $R \colon \;\, \nrightarrow B$ be a relation from A to B.

1. We write $\mathrm{Rel}(A, B)$ for the set of relations from A to B.
2. We write $\mathbf{Rel}(A, B)$ for the sub-poset of $(\mathcal{P}(A \times B), \subset)$ spanned by the relations from A to B.
3. Given $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.
4. When viewing R as a function

$$R \colon A \times B \to \{\mathsf{t}, \mathsf{f}\},$$

we write R^b_a for the value of R at (a, b).[3]

Proposition 8.1.1.1.5. Let A and B be sets and let $R, S \colon A \nrightarrow B$ be relations.

1. *End Formula for the Set of Inclusions of Relations.* We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, S^b_a).$$

Proof. *Item 1, End Formula for the Set of Inclusions of Relations*: Unwinding the

[3] The choice to write R^b_a in place of R^a_b is to keep the notation consistent with the notation we will later employ for profunctors in **??**.

expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a,S^b_a)\cong\begin{cases}\mathrm{pt} & \text{if, for each } a\in A \text{ and each } b\in B,\\ & \text{we have } \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a,S^b_a)\cong\mathrm{pt}\\ \emptyset & \text{otherwise.}\end{cases}$$

Since we have $\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a,S^b_a) = \{\mathsf{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathsf{false}$ or $R^b_a = S^b_a = \mathsf{true}$, we get

$$\int_{a\in A}\int_{b\in B}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a,S^b_a)\cong\begin{cases}\mathrm{pt} & \text{if, for each } a\in A \text{ and each } b\in B,\\ & \text{if } a\sim_R b\text{, then } a\sim_S b,\\ \emptyset & \text{otherwise.}\end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S)\cong\begin{cases}\mathrm{pt} & \text{if } R\subset S,\\ \emptyset & \text{otherwise.}\end{cases}$$

Since $(a\sim_R b \implies a\sim_S b)$ iff $R\subset S$, the two sets above are isomorphic. This finishes the proof. □

8.1.2 Relations as Decategorifications of Profunctors

Remark 8.1.2.1.1. The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}}\times\mathcal{C}\to\mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A\times B\to\{\mathsf{true},\mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats}\to\mathsf{Cats}$ restricts to the identity endofunctor on Sets.

- The values that profunctors and relations take are analogous:
  - A category is enriched over the category
    $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$
    of sets, with profunctors taking values on it.
  - A set is enriched over the set
    $$\{\mathsf{true}, \mathsf{false}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$
    of classical truth values, with relations taking values on it.

Remark 8.1.2.1.2. Extending Definition 8.1.2.1.1, the equivalent definitions of relations in Definition 8.1.1.1.1 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$.
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{CoPSh}(\mathcal{C})$.
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.
5. A limit-preserving functor $\mathfrak{p}\colon \mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}} \to \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms
  $$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\end{aligned}$$
  and
  $$\begin{aligned}\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$
- The equivalence between Items 2 and 4 follows from the universal properties of:

- The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$, as stated and proved in Constructions With Sets, Definition 4.4.5.1.1.

- The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in Presheaves and the Yoneda Lemma, **??** of Definition 12.1.4.1.3.

• The equivalence between Items 3 and 5 follows from the universal properties of:

- The powerset $\mathcal{P}(X)$ of a set X as the free completion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$, as stated and proved in Constructions With Sets, Definition 4.4.6.1.1.

- The category $\mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$ of copresheaves on a category $\mathcal{D}$ as the free completion of $\mathcal{D}$ via the dual Yoneda embedding

$$\text{よ}^{\text{op}}\colon \mathcal{D} \hookrightarrow \mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$$

of $\mathcal{D}$ into $\mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$, as stated and proved in Presheaves and the Yoneda Lemma, **??** of Definition 12.1.4.1.3.

8.1.3 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 8.1.3.1.1. The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

1. Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

2. Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\begin{aligned} (S \diamond R)^{-1}_{-2} &\overset{\text{def}}{=} \int^{b \in B} S^{-1}_{b} \times R^{b}_{-2} \\ &= \bigvee_{b \in B} S^{-1}_{b} \times R^{b}_{-2}, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

3. Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] \arrow[u, phantom, shift left=4, "\Swarrow"] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned} [\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{b \in B} \chi_{\mathcal{P}(B)}(\chi_b, V) \odot S(b) \\ &\cong \int^{b \in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b \in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b \in V} S(b) \end{aligned}$$

for each $V \in \mathcal{P}(B)$, so we have[4]

$$
\begin{aligned}
[S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\
&\overset{\text{def}}{=} \bigcup_{b \in R(a)} S(b).
\end{aligned}
$$

for each $a \in A$.

Remark 8.1.3.1.2. You might wonder what happens if we instead define an alternative composition of relations $\diamond'$ via right Kan extensions. In this case, we would take the right Kan extension of S along the dual characteristic embedding $B \hookrightarrow \mathcal{P}(B)^{\mathsf{op}}$:

$$
S \diamond' R \overset{\text{def}}{=} \mathrm{Ran}_{\chi_B}(S) \circ R, \qquad
\begin{tikzcd}
 & B \arrow[r, "S"] \arrow[d, hook', "\chi_B"'] & \mathcal{P}(C). \\
A \arrow[r, "R"'] & \mathcal{P}(B)^{\mathsf{op}} \arrow[ur, "\mathrm{Ran}_{\chi'_B}(S)"'] \arrow[u, phantom, shift left=3, "\Uparrow"] &
\end{tikzcd}
$$

In this case, we would have[5]

$$
[S \diamond' R](a) \overset{\text{def}}{=} \bigcap_{b \in R(a)} S(b).
$$

This alternative composition turns out to actually be a different kind of structure: it's an internal right Kan extension in **Rel**, namely $\mathrm{Ran}_{R^\dagger}(S)$ — see Section 8.5.15.

Example 8.1.3.1.3. Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$
\begin{aligned}
\leq \diamond \geq &= \sim_{\text{triv}}, \\
\geq \diamond \leq &= \sim_{\text{triv}}.
\end{aligned}
$$

[4]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\{c_{j_i}\}_{j_i \in J_i}\right\}_{i \in I}$ in C.

[5]If we replace $R(a)$ with $B \setminus R(a)$, defining

$$
S \,\square\, R \overset{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b),
$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq. \end{aligned}$$

Proposition 8.1.3.1.4. Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Functoriality.* The assignments $R, S, (R, S) \mapsto S \diamond R$ define functors

$$\begin{aligned} S \diamond -\colon &\quad \mathbf{Rel}(A, B) &&\to \mathbf{Rel}(A, C), \\ - \diamond R\colon &\quad \mathbf{Rel}(B, C) &&\to \mathbf{Rel}(A, C), \\ -_1 \diamond -_2\colon &\quad \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) &&\to \mathbf{Rel}(A, C). \end{aligned}$$

   In particular, given relations

$$\begin{tikzcd} A \arrow[r, shift left, "R_1"] \arrow[r, shift right, "R_2"'] & B \arrow[r, shift left, "S_1"] \arrow[r, shift right, "S_2"'] & C, \end{tikzcd}$$

   if $R_1 \subset R_2$ and $S_1 \subset S_2$, then $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

   That is, we have

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)$$

   for each $a \in A$.

3. *Unitality.* We have

$$\begin{aligned} \Delta_B \diamond R &= R, \\ R \diamond \Delta_A &= R. \end{aligned}$$

   That is, we have

$$\bigcup_{b \in R(a)} \{b\} = R(a),$$

$$\bigcup_{a\in\{a\}} R(a) = R(a)$$

for each $a \in A$.

4. *Relation to Apartness Composition of Relations.* We have

$$\begin{aligned}(S \diamond R)^{\mathsf{c}} &= S^{\mathsf{c}} \mathbin{\square} R^{\mathsf{c}},\\ (S \mathbin{\square} R)^{\mathsf{c}} &= S^{\mathsf{c}} \diamond R^{\mathsf{c}},\end{aligned}$$

where $(-)^{\mathsf{c}}$ is the complement functor of Constructions With Sets, Section 4.3.11. In particular, $\diamond$ is a special case of apartness composition of relations, as we have

$$S \diamond R = (S^{\mathsf{c}} \mathbin{\square} R^{\mathsf{c}})^{\mathsf{c}}.$$

This is also compatible with units, as we have $\Delta_A^{\mathsf{c}} = \nabla_A$.

5. *Linear Distributivity.* We have inclusions of relations

$$\begin{aligned}T \diamond (S \mathbin{\square} R) &\subset (T \diamond S) \mathbin{\square} R,\\ (T \mathbin{\square} S) \diamond R &\subset T \mathbin{\square} (S \diamond R).\end{aligned}$$

That is, we have

$$\begin{aligned}T\Bigg(\bigcap_{b\in B\setminus R(a)} S(b)\Bigg) &\subset \bigcap_{b\in B\setminus R(a)} T(S(b))\\ \bigcup_{b\in R(a)} \bigcap_{c\in C\setminus S(b)} T(c) &\subset \bigcap_{c\in C\setminus S(R(a))} T(c)\end{aligned}$$

or, unwinding the expression for $S(R(a))$, we have

$$\begin{aligned}\bigcup_{c\in\bigcap_{b\in B\setminus R(a)} S(b)} T(c) &\subset \bigcap_{b\in B\setminus R(a)} \bigcup_{c\in S(b)} T(c)\\ \bigcup_{b\in R(a)} \bigcap_{c\in C\setminus S(b)} T(c) &\subset \bigcap_{c\in C\setminus\bigcup_{b\in R(a)} S(b)} T(c)\end{aligned}$$

for each $a \in A$.

we instead obtain the apartness composition of relations; see Section 8.1.4.

6. *Interaction With Converses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

7. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R),\\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

Proof. *Item 1*, *Functoriality*: We have

$$\begin{aligned} S_1 \diamond R_1 &\overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B, \text{ such}\\ \text{that } a \sim_{R_1} b \text{ or } b \sim_{S_1} c\end{array} \right\} \\ &\subset \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B, \text{ such}\\ \text{that } a \sim_{R_2} b \text{ or } b \sim_{S_2} c\end{array} \right\} \\ &\overset{\text{def}}{=} S_2 \diamond R_2. \end{aligned}$$

This finishes the proof.

Item 2, *Associativity, Proof I*: Indeed, we have

$$\begin{aligned} (T \diamond S) \diamond R &\overset{\text{def}}{=} \left(\int^{c \in C} T^{-1}_{c} \times S^{c}_{-_2}\right) \diamond R \\ &\overset{\text{def}}{=} \int^{b \in B} \left(\int^{c \in C} T^{-1}_{c} \times S^{c}_{b}\right) \times R^{b}_{-_2} \\ &= \int^{b \in B} \int^{c \in C} (T^{-1}_{c} \times S^{c}_{b}) \times R^{b}_{-_2} \\ &= \int^{c \in C} \int^{b \in B} (T^{-1}_{c} \times S^{c}_{b}) \times R^{b}_{-_2} \\ &= \int^{c \in C} \int^{b \in B} T^{-1}_{c} \times (S^{c}_{b} \times R^{b}_{-_2}) \\ &= \int^{c \in C} T^{-1}_{c} \times \left(\int^{b \in B} S^{c}_{b} \times R^{b}_{-_2}\right) \\ &\overset{\text{def}}{=} \int^{c \in C} T^{-1}_{c} \times (S \diamond R)^{c}_{-_2} \\ &\overset{\text{def}}{=} T \diamond (S \diamond R). \end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   - We have $a \sim_R b$;
   - We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
     - We have $b \sim_S c$;
     - We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   - We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
     - We have $a \sim_R b$;
     - We have $b \sim_S c$;
   - We have $c \sim_T d$;

both of which are equivalent to the statement

($\star$) There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 2, Associativity, Proof II: Using Item 3 of Definition 8.1.3.1.1, we have

$$\begin{aligned}[(T \diamond S) \diamond R](a) &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} (T \diamond S)(b) \\ &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c)\end{aligned}$$

on the one hand and

$$\begin{aligned}[T \diamond (S \diamond R)](a) &\stackrel{\text{def}}{=} \bigcup_{c \in [S \diamond R](a)} T(c) \\ &\stackrel{\text{def}}{=} \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)\end{aligned}$$

on the other, so we want to prove an equality of the form

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c).$$

This then follows from an application of Constructions With Sets, Item 2 of Definition 4.3.6.1.2 in which we consider $X = D$, consider $\mathcal{P}(\mathcal{P}(\mathcal{P}(D)))$, take $U = U_c = T(c)$, take A to be

$$A_b \stackrel{\text{def}}{=} \{T(c) \in \mathcal{P}(D) \mid c \in S(b)\},$$

and then finally take

$$\begin{aligned}\mathcal{A} &\overset{\text{def}}{=} \{A_b \in \mathcal{P}(\mathcal{P}(D)) \mid b \in R(a)\}\\ &\overset{\text{def}}{=} \{\{T(c) \in \mathcal{P}(D) \mid c \in S(b)\} \mid b \in R(a)\}.\end{aligned}$$

Indeed, we have

$$\begin{aligned}\bigcup_{A\in\mathcal{A}}\Big(\bigcup_{U\in A} U\Big) &= \bigcup_{A_b\in\mathcal{A}}\Big(\bigcup_{c\in S(b)} T(c)\Big)\\ &= \bigcup_{b\in R(a)}\Big(\bigcup_{c\in S(b)} T(c)\Big)\end{aligned}$$

and

$$\begin{aligned}\bigcup_{U\in\bigcup_{A\in\mathcal{A}} A} U &= \bigcup_{U_c\in\bigcup_{b\in R(a)} A_b} U_c\\ &= \bigcup_{T(c)\in\bigcup_{b\in R(a)} A_b} T(c)\\ &= \bigcup_{c\in\bigcup_{b\in R(a)} S(b)} T(c).\end{aligned}$$

This finishes the proof.

Item 3, *Unitality*: Indeed, we have

$$\begin{aligned}\Delta_B \diamond R &\overset{\text{def}}{=} \int^{b\in B} (\Delta_B)^{-_1}_{b} \times R^{b}_{-_2}\\ &= \bigvee_{b\in B} (\Delta_B)^{-_1}_{b} \times R^{b}_{-_2}\\ &= \bigvee_{\substack{b\in B\\ b=-_1}} R^{b}_{-_2}\\ &= R^{-_1}_{-_2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \Delta_A &\overset{\text{def}}{=} \int^{a\in A} R^{-_1}_{a} \times (\Delta_A)^{a}_{-_2}\\ &= \bigvee_{a\in B} R^{-_1}_{a} \times (\Delta_A)^{a}_{-_2}\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\substack{a\in B\\ a=-2}} R_a^{-1} \\
&= R_{-2}^{-1}.
\end{aligned}
$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\Delta_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  - We have $a \sim_b B$.
  - There exists some $b' \in B$ such that:
    * We have $a \sim_R b'$
    * We have $b' \sim_{\Delta_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \Delta_A = R$$
witnesses the equivalence of the following two statements:
  - There exists some $a' \in A$ such that:
    * We have $a \sim_{\Delta_B} a'$, i.e. $a = a'$.
    * We have $a' \sim_R b$
  - We have $a \sim_b B$.

Item 4, Relation to Apartness Composition of Relations: This is a repetition of Item 4 of Definition 8.1.4.1.3 and is proved there.
Item 5, Linear Distributivity: This is a repetition of Item 5 of Definition 8.1.4.1.3 and is proved there.
Item 6, Interaction With Converses: This is a repetition of Item 3 of Definition 8.1.5.1.3 and is proved there.
Item 7, Interaction With Ranges and Domains: We have

$$
\begin{aligned}
\mathrm{dom}(S \diamond R) &\stackrel{\text{def}}{=} \{a \in A \mid a \sim_{S\diamond R} c \text{ for some } c \in C\}, \\
&= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists some } b \in B \text{ and } c \in C \\ \text{such that } a \sim_R b \text{ and } b \sim_R c \end{array} \right\},
\end{aligned}
$$

$$
\begin{aligned}
&\subset \left\{ a \in A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B\\ \text{such that } a \sim_R b\end{array} \right\},\\
&\overset{\text{def}}{=} \mathrm{dom}(R)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{range}(S \diamond R) &\overset{\text{def}}{=} \{c \in C \mid a \sim_{S \diamond R} c \text{ for some } a \in A\},\\
&= \left\{ c \in C \,\middle|\, \begin{array}{l}\text{there exists some } a \in A \text{ and } b \in B\\ \text{such that } a \sim_R b \text{ and } b \sim_R c\end{array} \right\},\\
&\subset \left\{ c \in C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B\\ \text{such that } b \sim_S c\end{array} \right\},\\
&\overset{\text{def}}{=} \mathrm{range}(S).
\end{aligned}
$$

This finishes the proof. □

8.1.4 Apartness Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 8.1.4.1.1. The **apartness composition of** R **and** S is the relation $S \mathbin{\square} R$ defined as follows:

- Viewing relations as subsets of $A \times C$, we define

$$
S \mathbin{\square} R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have}\\ a \sim_R b \text{ or } b \sim_S c\end{array} \right\}.
$$

- Viewing relations as functions $A \times C \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$
\begin{aligned}
(S \mathbin{\square} R)^{-1}_{-2} &\overset{\text{def}}{=} \int_{b \in B} S^{-1}_{b} \coprod R^{b}_{-2}\\
&= \bigwedge_{b \in B} S^{-1}_{b} \coprod R^{b}_{-2},
\end{aligned}
$$

where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

- Viewing relations as functions $A \to \mathcal{P}(C)$, we define

$$[S \mathbin{\square} R](a) \overset{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b)$$

for each $a \in A$.

Example 8.1.4.1.2. Here are some examples of apartness composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \mathbin{\square} \geq &= \emptyset, \\ \geq \mathbin{\square} \leq &= \emptyset. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \mathbin{\square} \leq &= \emptyset, \\ \geq \mathbin{\square} \geq &= \emptyset. \end{aligned}$$

3. *Equality and Inequality.* Let $A = B = C = \mathbb{Z}$. We have

$$\begin{aligned} = \mathbin{\square} \neq &= =, \\ \neq \mathbin{\square} = &= =. \end{aligned}$$

4. *Subset Inclusion.* Let X be a set with at least three elements and consider the relations $\subset$ and $\supset$ in $\mathcal{P}(X)$. We have

$$\supset \mathbin{\square} \subset = \{(U, V) \in \mathcal{P}(X) \mid U = \emptyset \text{ or } V = \emptyset\}.$$

Proposition 8.1.4.1.3. Let $R \colon A \nrightarrow B$, $S \colon B \nrightarrow C$, and $T \colon C \nrightarrow D$ be relations.

1. *Functoriality.* The assignments $R, S, (R, S) \mapsto S \mathbin{\square} R$ define functors

$$\begin{aligned} S \mathbin{\square} - \colon & \quad \mathbf{Rel}(A, B) & \to \mathbf{Rel}(A, C), \\ - \mathbin{\square} R \colon & \quad \mathbf{Rel}(B, C) & \to \mathbf{Rel}(A, C), \\ -_1 \mathbin{\square} -_2 \colon & \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) & \to \mathbf{Rel}(A, C). \end{aligned}$$

In particular, given relations

$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \underset{S_2}{\overset{S_1}{\rightrightarrows}} C,$$

if $R_1 \subset R_2$ and $S_1 \subset S_2$, then $S_1 \mathbin{\Box} R_1 \subset S_2 \mathbin{\Box} R_2$.

2. *Associativity.* We have

$$(T \mathbin{\Box} S) \mathbin{\Box} R = T \mathbin{\Box} (S \mathbin{\Box} R).$$

3. *Unitality.* We have

$$\begin{aligned} \nabla_B \mathbin{\Box} R &= R, \\ R \mathbin{\Box} \nabla_A &= R. \end{aligned}$$

4. *Relation to Composition of Relations.* We have

$$\begin{aligned} (S \mathbin{\Box} R)^{\mathsf{c}} &= S^{\mathsf{c}} \diamond R^{\mathsf{c}}, \\ (S \diamond R)^{\mathsf{c}} &= S^{\mathsf{c}} \mathbin{\Box} R^{\mathsf{c}}, \end{aligned}$$

where $(-)^{\mathsf{c}}$ is the complement functor of Constructions With Sets, Section 4.3.11. In particular, $\Box$ is a special case of composition of relations, as we have

$$S \mathbin{\Box} R = (S^{\mathsf{c}} \diamond R^{\mathsf{c}})^{\mathsf{c}}.$$

This is also compatible with units, as we have $\nabla_A^{\mathsf{c}} = \Delta_A$.

5. *Linear Distributivity.* We have inclusions of relations

$$\begin{aligned} T \diamond (S \mathbin{\Box} R) &\subset (T \diamond S) \mathbin{\Box} R, \\ (T \mathbin{\Box} S) \diamond R &\subset T \mathbin{\Box} (S \diamond R). \end{aligned}$$

6. *Interaction With Converses.* We have

$$(S \mathbin{\Box} R)^{\dagger} = R^{\dagger} \mathbin{\Box} S^{\dagger}.$$

Proof. *Item 1*, *Functoriality*: We have

$$\begin{aligned} S_1 \mathbin{\square} R_1 &\overset{\text{def}}{=} \left\{ (a,c) \in A \times C \,\middle|\, \begin{array}{l} \text{for each } b \in B, \text{we have} \\ a \sim_{R_1} b \text{ or } b \sim_{S_1} c \end{array} \right\} \\ &\subset \left\{ (a,c) \in A \times C \,\middle|\, \begin{array}{l} \text{for each } b \in B, \text{we have} \\ a \sim_{R_2} b \text{ or } b \sim_{S_2} c \end{array} \right\} \\ &\overset{\text{def}}{=} S_2 \mathbin{\square} R_2. \end{aligned}$$

This finishes the proof.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned} (T \mathbin{\square} S) \mathbin{\square} R &\overset{\text{def}}{=} \left(\int_{c \in C} T^{-1}_{c} \amalg S^{c}_{-_2}\right) \mathbin{\square} R \\ &\overset{\text{def}}{=} \int_{b \in B} \left(\int_{c \in C} T^{-1}_{c} \amalg S^{c}_{b}\right) \amalg R^{b}_{-_2} \\ &= \int_{b \in B} \int_{c \in C} (T^{-1}_{c} \amalg S^{c}_{b}) \amalg R^{b}_{-_2} \\ &= \int_{c \in C} \int_{b \in B} (T^{-1}_{c} \amalg S^{c}_{b}) \amalg R^{b}_{-_2} \\ &= \int_{c \in C} \int_{b \in B} T^{-1}_{c} \amalg (S^{c}_{b} \amalg R^{b}_{-_2}) \\ &= \int_{c \in C} T^{-1}_{c} \amalg \left(\int_{b \in B} S^{c}_{b} \amalg R^{b}_{-_2}\right) \\ &\overset{\text{def}}{=} \int_{c \in C} T^{-1}_{c} \amalg (S \mathbin{\square} R)^{c}_{-_2} \\ &\overset{\text{def}}{=} T \mathbin{\square} (S \mathbin{\square} R). \end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

- We have $a \sim_{(T \square S) \square R} d$, i.e. there exists some $b \in B$ such that:
  - We have $a \sim_{R} b$;
  - We have $b \sim_{T \square S} d$, i.e. there exists some $c \in C$ such that:
    - We have $b \sim_{S} c$;
    - We have $c \sim_{T} d$;

- We have $a \sim_{T \square (S \square R)} d$, i.e. there exists some $c \in C$ such that:
  - We have $a \sim_{S \square R} c$, i.e. there exists some $b \in B$ such that:
    - We have $a \sim_R b$;
    - We have $b \sim_S c$;
  - We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, *Unitality*: Indeed, we have

$$\begin{aligned}
\nabla_B \square R &\stackrel{\text{def}}{=} \int_{b \in B} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2} \\
&= \bigwedge_{b \in B} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2} \\
&= \Big(\bigwedge_{\substack{b \in B \\ b = -1}} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2}\Big) \wedge \Big(\bigwedge_{\substack{b \in B \\ b \neq -1}} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2}\Big) \\
&= \Big((\nabla_B)^{-1}_{-1} \coprod R^{-1}_{-2}\Big) \wedge \Big(\bigwedge_{\substack{b \in B \\ b \neq -1}} \mathrm{t} \coprod R^{b}_{-2}\Big) \\
&= \Big(\mathrm{f} \coprod R^{-1}_{-2}\Big) \wedge \Big(\bigwedge_{\substack{b \in B \\ b \neq -1}} \mathrm{t}\Big) \\
&= R^{-1}_{-2} \wedge \mathrm{t} \\
&= R^{-1}_{-2},
\end{aligned}$$

and

$$\begin{aligned}
R \square \nabla_A &\stackrel{\text{def}}{=} \int_{a \in A} R^{-1}_{a} \coprod (\nabla_A)^{a}_{-2} \\
&= \bigwedge_{a \in A} R^{-1}_{a} \coprod (\nabla_A)^{a}_{-2} \\
&= \Big(\bigwedge_{\substack{a \in A \\ a = -2}} R^{-1}_{a} \coprod (\nabla_A)^{a}_{-2}\Big) \wedge \Big(\bigwedge_{\substack{a \in A \\ a \neq -2}} R^{-1}_{a} \coprod (\nabla_A)^{a}_{-2}\Big) \\
&= \Big(R^{-1}_{-2} \coprod (\nabla_A)^{-2}_{-2}\Big) \wedge \Big(\bigwedge_{\substack{a \in A \\ a \neq -2}} R^{-1}_{a} \coprod \mathrm{t}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= (R^{-1}_{-_2} \coprod \mathrm{f}) \wedge \Big(\bigwedge_{\substack{a\in A\\ a\neq -_2}} \mathrm{t}\Big)\\
&= R^{-1}_{-_2} \wedge \mathrm{t}\\
&= R^{-1}_{-_2},
\end{aligned}
$$

This finishes the proof.

Item 4, Relation to Composition of Relations: We proceed in a few steps.

- We have $a \sim_{(S\,\square\,R)^{\mathsf{c}}} b$ iff $a \nsim_{S\,\square\,R} b$.
- We have $a \nsim_{S\,\square\,R} b$ iff the assertion "for each $b \in B$, we have $a \sim_R b$ or $b \sim_S c$" is false.
- That happens iff there exists some $b \in B$ such that $a \nsim_R b$ and $b \nsim_S c$.
- That happens iff there exists some $b \in B$ such that $a \sim_{R^{\mathsf{c}}} b$ and $b \sim_{S^{\mathsf{c}}} c$.

The second equality then follows from the first one by Constructions With Sets, Item 3 of Definition 4.3.11.1.2.

Item 5, Linear Distributivity: We have

$$
\begin{aligned}
T \diamond (S \mathbin{\square} R) &\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{there exists some } c \in C \text{ such}\\ \text{that } a \sim_{S\,\square\,R} c \text{ and } c \sim_T d\end{array} \right\}\\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{there exists some } c \in C \text{ such that}\\ c \sim_T d \text{ and, for each } b \in B,\\ \text{we have } a \sim_R b \text{ or } b \sim_S c\end{array} \right\}\\
&= \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{the following conditions are satisfied:}\\ \quad 1.\ \text{For each } b \in B, \text{ we have } a \sim_R b \text{ or } b \sim_S c.\\ \quad 2.\ \text{There exists } c \in C \text{ such that } c \sim_T d.\end{array} \right\}\\
&\subset \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{for each } b \in B, \text{ at least one of the}\\ \text{following conditions is satisfied:}\\ \quad 1.\ \text{We have } a \sim_R b.\\ \quad 2.\ \text{There exists } c \in C \text{ such that } b \sim_S c\\ \qquad \text{and } c \sim_T d.\end{array} \right\}\\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{for each } b \in B, \text{ we have}\\ a \sim_R b \text{ or there exists some } c \in C\\ \text{such that } b \sim_S c \text{ and } c \sim_T d\end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B\text{, we have} \\ a \sim_R b \text{ or } b \sim_{T \diamond S} d \end{array} \right\} \\
&\stackrel{\text{def}}{=} (T \diamond S) \mathbin{\square} R
\end{aligned}
$$

and

$$
\begin{aligned}
(T \mathbin{\square} S) \diamond R &\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_{T \square S} d \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and, for each } c \in C\text{,} \\ \text{we have } b \sim_S c \text{ or } c \sim_T d \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ satisfying} \\ \text{the following conditions:} \\ \quad 1.\ \text{We have } a \sim_R b. \\ \quad 2.\ \text{For each } c \in C\text{, we have } b \sim_S c \\ \quad\quad \text{or } c \sim_T d. \end{array} \right\} \\
&\subset \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C\text{, at least one of the} \\ \text{following conditions is satisfied:} \\ \quad 1.\ \text{We have } c \sim_T d. \\ \quad 2.\ \text{There exists some } b \in B \text{ such that} \\ \quad\quad \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C\text{, we have } c \sim_T d \\ \text{or there exists some } b \in B\text{, such that} \\ \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C\text{, we have} \\ a \sim_{S \diamond R} c \text{ or } c \sim_T d \end{array} \right\} \\
&\subset T \mathbin{\square} (S \diamond R).
\end{aligned}
$$

This finishes the proof.

Item 6, Interaction With Converses: This is a repetition of Item 4 of Definition 8.1.5.1.3 and is proved there. □

8.1.5 The Converse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 8.1.5.1.1. The **converse of** R[6] is the relation $R^{\dagger}$ defined as follows:

- Viewing relations as subsets, we define

$$R^{\dagger} \overset{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } a \sim_R b\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$[R^{\dagger}]^{a}_{b} \overset{\text{def}}{=} R^{b}_{a}$$

  for each $(b,a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define[7]

$$R^{\dagger}(b) \overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}$$

  for each $b \in B$.

Example 8.1.5.1.2. Here are some examples of converses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^{\dagger} = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^{\dagger} = \leq$.
3. *Functions.* Let $f \colon A \to B$ be a function. We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ (f^{-1})^{\dagger} &= \mathrm{Gr}(f), \end{aligned}$$

   where $\mathrm{Gr}(f)$ and f^{-1} are the relations of Sections 8.2.2 and 8.2.3.

Proposition 8.1.5.1.3. Let $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$ be relations.

1. *Functoriality.* The assignment $R \mapsto R^{\dagger}$ defines a functor (i.e. morphism of posets)

$$(-)^{\dagger} \colon \mathbf{Rel}(A,B) \to \mathbf{Rel}(B,A).$$

   In other words, given relations $R, S \colon A \rightrightarrows B$, we have:

   ($\star$) If $R \subset S$, then $R^{\dagger} \subset S^{\dagger}$.

[6]*Further Terminology:* Also called the **opposite of** R or the **transpose of** R.
[7]Note that $R^{\dagger}(b) = R^{-1}(\{b\})$.

2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(R^\dagger) &= \mathrm{range}(R),\\ \mathrm{range}(R^\dagger) &= \mathrm{dom}(R).\end{aligned}$$

3. *Interaction With Composition.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

4. *Interaction With Apartness Composition.* We have

$$(S \mathbin{\square} R)^\dagger = R^\dagger \mathbin{\square} S^\dagger.$$

5. *Invertibility.* We have

$$(R^\dagger)^\dagger = R.$$

6. *Identity I.* We have

$$\Delta_A^\dagger = \Delta_A.$$

7. *Identity II.* We have

$$\nabla_A^\dagger = \nabla_A.$$

Proof. *Item 1*, *Functoriality*: We have

$$\begin{aligned}R^\dagger &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}\\ &\subset \{a \in A \mid b \in S(a)\}\\ &\overset{\text{def}}{=} S^\dagger.\end{aligned}$$

This finishes the proof.
Item 2, *Interaction With Ranges and Domains*: We have

$$\begin{aligned}\mathrm{dom}(R^\dagger) &\overset{\text{def}}{=} \{b \in B \mid b \sim_{R^\dagger} a \text{ for some } a \in A\}\\ &= \{b \in B \mid a \sim_R b \text{ for some } a \in A\}\\ &\overset{\text{def}}{=} \mathrm{range}(R)\end{aligned}$$

and

$$\begin{aligned}\mathrm{range}(R^\dagger) &\overset{\text{def}}{=} \{a \in A \mid b \sim_{R^\dagger} a \text{ for some } b \in B\}\\ &= \{a \in A \mid a \sim_R b \text{ for some } b \in B\}\end{aligned}$$

$$\stackrel{\text{def}}{=} \operatorname{dom}(R).$$

This finishes the proof.
Item 3, Interaction With Composition: We have

$$\begin{aligned}
(S \diamond R)^\dagger &\stackrel{\text{def}}{=} \left\{(c,a) \in C \times A \,\middle|\, c \sim_{(S \diamond R)^\dagger} a\right\} \\
&= \{(c,a) \in C \times A \mid a \sim_{S \diamond R} c\} \\
&= \left\{(c,a) \in C \times A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array}\right\} \\
&= \left\{(c,a) \in C \times A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } b \sim_{R^\dagger} a \text{ and } c \sim_{S^\dagger} b\end{array}\right\} \\
&= \left\{(c,a) \in C \times A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } c \sim_{S^\dagger} b \text{ and } b \sim_{R^\dagger} a\end{array}\right\} \\
&\stackrel{\text{def}}{=} R^\dagger \diamond S^\dagger.
\end{aligned}$$

This finishes the proof.
Item 4, Interaction With Apartness Composition: We have

$$\begin{aligned}
(S \mathbin{\square} R)^\dagger &\stackrel{\text{def}}{=} \left\{(c,a) \in C \times A \,\middle|\, c \sim_{(S \square R)^\dagger} a\right\} \\
&= \{(c,a) \in C \times A \mid a \sim_{S \square R} c\} \\
&= \left\{(c,a) \in C \times A \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_S c\end{array}\right\} \\
&= \left\{(c,a) \in C \times A \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have} \\ b \sim_{R^\dagger} a \text{ or } c \sim_{S^\dagger} b\end{array}\right\} \\
&= \left\{(c,a) \in C \times A \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have} \\ c \sim_{S^\dagger} b \text{ or } b \sim_{R^\dagger} a\end{array}\right\} \\
&\stackrel{\text{def}}{=} R^\dagger \mathbin{\square} S^\dagger.
\end{aligned}$$

This finishes the proof.
Item 5, Invertibility: We have

$$\begin{aligned}
(R^\dagger)^\dagger &\stackrel{\text{def}}{=} \{(a,b) \in A \times B \mid b \sim_{R^\dagger} a\} \\
&= \{(a,b) \in A \times B \mid a \sim_R b\} \\
&\stackrel{\text{def}}{=} R.
\end{aligned}$$

This finishes the proof.
Item 6, Identity I: We have

$$\begin{aligned}\Delta_A^\dagger &\overset{\text{def}}{=} \left\{(a,b) \in A \times A \,\middle|\, a \sim_{\Delta_A} b\right\}\\ &= \{(a,b) \in A \times A \mid a = b\}\\ &= \Delta_A.\end{aligned}$$

This finishes the proof.
Item 7, Identity II: We have

$$\begin{aligned}\nabla_A^\dagger &\overset{\text{def}}{=} \left\{(a,b) \in A \times A \,\middle|\, a \sim_{\nabla_A} b\right\}\\ &= \{(a,b) \in A \times A \mid a \neq b\}\\ &= \nabla_A.\end{aligned}$$

This finishes the proof. □

8.2 Examples of Relations

8.2.1 Elementary Examples of Relations

Example 8.2.1.1.1. The **trivial relation on** A **and** B is the relation $\sim_{\text{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\text{triv}} \overset{\text{def}}{=} A \times B.$$

2. As a function from $A \times B$ to $\{\text{true}, \text{false}\}$, the relation $\sim_{\text{triv}}$ is the constant function

$$\Delta_{\text{true}} \colon A \times B \longrightarrow \{\text{true}, \text{false}\}$$

   from $A \times B$ to $\{\text{true}, \text{false}\}$ taking the value true.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{triv}}$ is the function

$$\Delta_{\text{true}} \colon A \longrightarrow \mathcal{P}(B)$$

   defined by

$$\Delta_{\text{true}}(a) \overset{\text{def}}{=} B$$

   for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

Example 8.2.1.1.2. The **cotrivial relation on** A **and** B is the relation $\sim_{\text{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\text{cotriv}} \overset{\text{def}}{=} \emptyset.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\text{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}} \colon A \times B \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value false.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\text{cotriv}}$ is the function

$$\Delta_{\mathsf{false}} \colon A \longrightarrow \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{false}}(a) \overset{\text{def}}{=} \emptyset$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \nsim_R b$ for $a \in A$ and each $b \in B$.

Example 8.2.1.1.3. The characteristic relation χ_X on X of Constructions With Sets, Definition 4.5.3.1.1:

1. As a subset of $X \times X$, we have

$$\begin{aligned} \sim_{\chi_X} &\overset{\text{def}}{=} \Delta_X \\ &\overset{\text{def}}{=} \{(x, x) \in X \times X\}. \end{aligned}$$

2. As a function from $X \times X$ to $\{\mathsf{true}, \mathsf{false}\}$, we have

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

3. As a function from X to $\mathcal{P}(X)$, we have

$$\chi_X(x) \stackrel{\text{def}}{=} \{x\}$$

for each $x \in X$.

Example 8.2.1.1.4. The **antidiagonal relation on** X is the relation ∇_X defined equivalently as follows:

1. As a subset of $X \times X$, we have

$$\begin{aligned} \sim_{\nabla_X} &\stackrel{\text{def}}{=} \nabla_X \\ &\stackrel{\text{def}}{=} X \setminus \Delta_X \\ &= \{(x,y) \in X \times X \mid x \neq y\}. \end{aligned}$$

2. As a function from $X \times X$ to $\{\mathsf{true}, \mathsf{false}\}$, we have

$$\nabla_X(x,y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a \neq b, \\ \mathsf{false} & \text{if } a = b \end{cases}$$

for each $x, y \in X$.

3. As a function from X to $\mathcal{P}(X)$, we have

$$\nabla_X(x) \stackrel{\text{def}}{=} X \setminus \{x\}$$

for each $x \in X$.

Example 8.2.1.1.5. Partial functions may be viewed (or defined) as being exactly those relations which are functional; see Conditions on Relations, Section 10.1.1.

Example 8.2.1.1.6. Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in $\mathbb{Q}$.* Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-}\colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 8.2.1.1.7. The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \overset{\text{def}}{=} \left\{\log(\sqrt{a^2+b^2}) + i\arg(a+bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z}\right\}$$

for each $a+bi \in \mathbb{C}$.

Example 8.2.1.1.8. See [Wik25] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

8.2.2 The Graph of a Function

Let $f\colon A \to B$ be a function.

Definition 8.2.2.1.1. The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[8]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\mathrm{Gr}(f)^b_a \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

[8] *Further Terminology and Notation:* When $f = \mathrm{id}_A$, we write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, calling it the

for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \overset{\chi_B}{\hookrightarrow} \mathcal{P}(B).$$

Proposition 8.2.2.1.2. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \overset{\mathrm{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\overset{\mathrm{def}}{=}\mathrm{Rel}(A,B)}$$

  of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \overset{\mathrm{def}}{=} \mathrm{Gr}(f),$$

  where $\mathrm{Gr}(f)$ is the graph of f as in Definition 8.2.2.1.1.

In particular, the following statements are true:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

  for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

  for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_!)\colon \quad \begin{tikzcd} \mathrm{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_!"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathrm{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathrm{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

3. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathrm{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

4. *Adjointness Inside* **Rel**. We have an internal adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 8.2.3.1.1.

5. *Interaction With Converses.* We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ (f^{-1})^{\dagger} &= \mathrm{Gr}(f). \end{aligned}$$

6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The inverse and coinverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: This is a repetition of Constructions With Sets, Defini-

graph of A.

tion 4.4.4.1.1, and is proved there.
Item 3, *Cocontinuity*: This follows from Item 2 and **??**.
Item 4, *Adjointness Inside* **Rel**: We need to check that there are inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.
2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.
Item 5, *Interaction With Converses*: Omitted.
Item 6, *Characterisations*: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of Definition 8.5.2.1.2.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton. Since the conditions
  - $R(a) \cap V \neq \emptyset$;
  - $R(a) \subset V$;

  are equivalent when $R(a)$ is a singleton, it follows that the sets

  $$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\},$$
  $$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

  are equal for all $V \in \mathcal{P}(B)$.
- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality*. We proceed in a few steps:
    * If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$.
    * But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction.

        * Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.

    – *Functionality*. If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned}\{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\})\end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, so R is functional.

- *Item 6a* $\iff$ *Item 6d*. This follows from Relations, Definition 8.5.3.1.1.

This finishes the proof. □

8.2.3 The Inverse of a Function

Let $f : A \to B$ be a function.

Definition 8.2.3.1.1. The **inverse of** f is the relation $f^{-1} : B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \overset{\text{def}}{=} \left\{(b, f^{-1}(b)) \in B \times A \,\middle|\, a \in A\right\},$$

  where

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$[f^{-1}]^b_a \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \overset{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

  for each $b \in B$.

Proposition 8.2.3.1.2. Let $f\colon A\to B$ be a function.

1. *Functoriality.* The assignment $A\mapsto A$, $f\mapsto f^{-1}$ defines a functor

$$(-)^{-1}\colon \mathsf{Sets}\to\mathrm{Rel}$$

where

- *Action on Objects.* For each $A\in\mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A)\stackrel{\text{def}}{=}A.$$

- *Action on Morphisms.* For each $A,B\in\mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A,B)\to\mathrm{Rel}(A,B)$$

of $(-)^{-1}$ at (A,B) is defined by

$$(-)^{-1}_{A,B}(f)\stackrel{\text{def}}{=}\left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 8.2.3.1.1.

In particular, the following statements are true:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1}=\chi_A$$

for each $A\in\mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g\circ f)^{-1}=g^{-1}\diamond f^{-1}$$

for pair of functions $f\colon A\to B$ and $g\colon B\to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f)\dashv f^{-1}\right)\colon\quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "\shortmid"{marking}] \end{tikzcd}$$

in **Rel**.

3. *Interaction With Converses of Relations.* We have

$$\begin{aligned}(f^{-1})^{\dagger} &= \mathrm{Gr}(f),\\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}.\end{aligned}$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness Inside **Rel**: This is a repetition of Item 4 of Definition 8.2.2.1.2 and is proved there.
Item 3, Interaction With Converses of Relations: This is a repetition of Item 5 of Definition 8.2.2.1.2 and is proved there. □

8.2.4 Representable Relations

Let A and B be sets.

Definition 8.2.4.1.1. Let $f\colon A \to B$ and $g\colon B \to A$ be functions.[9]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f\times\mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

   i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times\mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

   i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

[9] More generally, given functions

$$\begin{aligned}f &\colon A \to C,\\ g &\colon B \to D\end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f\times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

8.3 Categories of Relations

8.3.1 The Category of Relations Between Two Sets

Definition 8.3.1.1.1. The **category of relations from** A **to** B is the category $\mathbf{Rel}(A, B)$ defined by[10]

$$\mathbf{Rel}(A, B) \overset{\text{def}}{=} \mathbf{Rel}(A, B)_{\mathsf{pos}},$$

where $\mathbf{Rel}(A, B)_{\mathsf{pos}}$ is the posetal category associated to the poset $\mathbf{Rel}(A, B)$ of Item 2 of Definition 8.1.1.1.4 and Categories, Definition 11.2.7.1.1.

8.3.2 The Category of Relations

Definition 8.3.2.1.1. The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

  $$\mathsf{Rel}(A, B) \overset{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

  $$\mathbb{1}^{\mathsf{Rel}}_{A} \colon \mathrm{pt} \to \mathrm{Rel}(A, A)$$

  of Rel at A is defined by

  $$\mathrm{id}^{\mathsf{Rel}}_{A} \overset{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Definition 8.2.1.1.3.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

  $$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

  $$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

[10] Here we choose to abuse notation by writing $\mathbf{Rel}(A, B)$ instead of $\mathbf{Rel}(A, B)_{\mathsf{pos}}$ for the

8.3.3 The Closed Symmetric Monoidal Category of Relations

8.3.3.1 The Monoidal Product

Definition 8.3.3.1.1. The **monoidal product of** Rel is the functor

$$\times\colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

  where $A \times B$ is the Cartesian product of sets of Constructions With Sets, Definition 4.1.3.1.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)}\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \longrightarrow \mathrm{Rel}(A \times C, B \times D)$$

  of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$\begin{aligned} R&\colon A \nrightarrow B, \\ S&\colon C \nrightarrow D \end{aligned}$$

  to the relation

$$R \times S\colon A \times C \nrightarrow B \times D$$

  of Constructions With Relations, Definition 9.2.6.1.1.

8.3.3.2 The Monoidal Unit

Definition 8.3.3.2.1. The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}}\colon \mathrm{pt} \longrightarrow \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \stackrel{\text{def}}{=} \mathrm{pt}$$

of Rel.

8.3.3.3 The Associator

Definition 8.3.3.3.1. The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathrm{id}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Rel} \times (\mathsf{Rel} \times \mathsf{Rel}) \arrow[dr, "\mathrm{id}\times(\times)", "\shortmid"{marking}] & \\
(\mathsf{Rel} \times \mathsf{Rel}) \times \mathsf{Rel} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\sim"', "\shortmid"{marking}] \arrow[d, "(\times)\times\mathrm{id}"', "\shortmid"{marking}] & & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times", "\shortmid"{marking}] \\
\mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"', "\shortmid"{marking}] \arrow[urr, phantom, "\Nearrow\alpha^{\mathsf{Rel}}"] & & \mathsf{Rel},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} \colon (A \times B) \times C \nrightarrow A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

8.3.3.4 The Left Unitor

Definition 8.3.3.4.1. The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}} \colon \times \circ (\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd}
\mathrm{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Rel}}"] \\
 & \mathsf{Rel}
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_A \colon \mathbb{1}_{\mathsf{Rel}} \times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a = b$.

8.3.3.5 The Right Unitor

Definition 8.3.3.5.1. The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ (\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

8.3.3.6 The Symmetry

Definition 8.3.3.6.1. The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\Downarrow \sigma^{\mathsf{Rel}}"] & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B}\colon A\times B\longrightarrow B\times A$$

at (A,B) is defined by declaring

$$(a,b)\sim_{\sigma^{\mathsf{Rel}}_{A,B}}(b',a')$$

iff $a=a'$ and $b=b'$.

8.3.3.7 The Internal Hom

Definition 8.3.3.7.1. The **internal Hom of** Rel is the functor

$$\mathrm{Rel}\colon \mathsf{Rel}^{\mathsf{op}}\times\mathsf{Rel}\longrightarrow\mathsf{Rel}$$

defined

- On objects by sending $A,B\in\mathrm{Obj}(\mathsf{Rel})$ to the set $\mathrm{Rel}(A,B)$ of **??** of **??**.
- On morphisms by pre/post-composition defined as in Definition 8.1.3.1.1.

Proposition 8.3.3.7.2. Let $A,B,C\in\mathrm{Obj}(\mathsf{Rel})$.

1. *Adjointness.* We have adjunctions

$$(A\times - \dashv \mathrm{Rel}(A,-))\colon\quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(-\times B \dashv \mathrm{Rel}(B,-))\colon\quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Rel}(A\times B,C)&\cong\mathrm{Rel}(A,\mathrm{Rel}(B,C)),\\ \mathrm{Rel}(A\times B,C)&\cong\mathrm{Rel}(B,\mathrm{Rel}(A,C)),\end{aligned}$$

natural in $A,B,C\in\mathrm{Obj}(\mathsf{Rel})$.

posetal category of relations from A to B, even though the same notation is used for the poset of relations from A to B.

Proof. *Item 1*, *Adjointness*: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A \times B \times C, \{\mathsf{true}, \mathsf{false}\})\\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C)\\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, \mathrm{Rel}(B, C)),\end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathrm{Rel}(A, C))$. □

8.3.3.8 The Closed Symmetric Monoidal Category of Relations

Proposition 8.3.3.8.1. The category Rel admits a closed symmetric monoidal category structure consisting of[11]

- *The Underlying Category.* The category Rel of sets and relations of Definition 8.3.2.1.1.

- *The Monoidal Product.* The functor

  $$\times \colon \mathrm{Rel} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 8.3.3.1.1.

- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Rel} \colon \mathrm{Rel}^{\mathsf{op}} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 8.3.3.7.1.

- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathrm{Rel}} \colon \mathrm{pt} \longrightarrow \mathrm{Rel}$$

  of Definition 8.3.3.2.1.

- *The Associators.* The natural isomorphism

  $$\alpha^{\mathrm{Rel}} \colon \times \circ (\times \times \mathrm{id}_{\mathrm{Rel}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathrm{Rel}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 8.3.3.3.1.

[11] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in fact

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathrm{Rel}} \colon \times \circ (\mathbb{1}^{\mathrm{Rel}} \times \mathrm{id}_{\mathrm{Rel}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 8.3.3.4.1.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathrm{Rel}} \colon \times \circ (\mathsf{id} \times \mathbb{1}^{\mathrm{Rel}}) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$

  of Definition 8.3.3.5.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathrm{Rel}} \colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathrm{Rel},\mathrm{Rel}}$$

  of Definition 8.3.3.6.1.

Proof. Omitted. □

8.3.4 The 2-Category of Relations

Definition 8.3.4.1.1. The **2-category of relations** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets.

- **Hom**-*Objects.* For each $A, B \in \mathrm{Obj}(\mathrm{Sets})$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Rel}}(A, B) &\overset{\mathrm{def}}{=} \mathbf{Rel}(A, B) \\ &\overset{\mathrm{def}}{=} (\mathrm{Rel}(A, B), \subset). \end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_A \colon \mathrm{pt} \to \mathbf{Rel}(A, A)$$

  of **Rel** at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_A \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Definition 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[12]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

  of $\mathbf{Rel}$ at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

8.3.5 The Double Category of Relations

8.3.5.1 The Double Category of Relations

Definition 8.3.5.1.1. The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets.
- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f \colon A \to B$.
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R \colon A \nrightarrow X$.
- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "|" marking] & Y \end{tikzcd}$$

with α written as $\begin{smallmatrix}\|\\ \alpha \\ \Downarrow\end{smallmatrix}$ in the middle of the square,

  of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

given by the disjoint union of sets; see Constructions With Relations, ??.

[12] That this is indeed a morphism of posets is proven in ?? of Definition 8.1.3.1.4.

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 8.3.5.2.1.

- *Vertical Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel}^{\mathsf{dbl}})$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \overset{\text{def}}{=} \mathrm{id}_A .$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\substack{\Vert\\ \mathrm{id}_R \\ \Downarrow}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "|"{marking}] & B \end{tikzcd}$$

of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true}, \mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 8.3.5.3.1.

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \overset{\text{def}}{=} g \circ f .$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 8.3.5.4.1.

- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.5.1.

- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.6.1.

- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.7.1.

8.3.5.2 Horizontal Identities

Definition 8.3.5.2.1. The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel}^{\mathsf{dbl}}_0)$, we have

$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2).$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

  of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

  of Constructions With Sets, Item 1 of Definition 4.5.3.1.3.

8.3.5.3 Horizontal Composition

Definition 8.3.5.3.1. The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\text{def}}{=} S \diamond R,$$

where $S \diamond R$ is the composition of R and S of Definition 8.1.3.1.1.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "|" marking] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S \odot R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "|" marking] & Z \end{tikzcd}$$

of α and β is the inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R)$$

follows from the fact that the statement

- We have $a \sim_{(U \diamond T)\circ(f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  - We have $f(a) \sim_T y$.
  - We have $y \sim_U h(c)$.

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  - We have $a \sim_R b$.
  - We have $b \sim_S c$.

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

This finishes the proof. $\square$

8.3.5.4 Vertical Composition of 2-Morphisms

Definition 8.3.5.4.1. The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow \beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h\circ f"'] \arrow[dr, phantom, "\Downarrow \beta\circ\alpha"] & X \arrow[d, "k\circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"{description}] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion

$$T \circ [(h \circ f) \times (k \circ g)] \subset R$$

follows from the fact that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

This finishes the proof. □

8.3.5.5 The Associators

Definition 8.3.5.5.1. For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "|"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "|"] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "|"] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "|"] & B \arrow[r, "S"', "|"] & C \arrow[r, "T"', "|"] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[13]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T\diamond S)\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{45}{=}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T\diamond(S\diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[13] As proved in Item 2 of Definition 8.1.3.1.4.

8.3.5.6 The Left Unitors

Definition 8.3.5.6.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon \mathbb{1}_B \odot R \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mapsto" marking] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "\mapsto" marking] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\mapsto" marking] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[14]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

8.3.5.7 The Right Unitors

Definition 8.3.5.7.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \mathbb{1}_A \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\mapsto" marking] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "\mapsto" marking] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\mapsto" marking] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[15]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[14]As proved in Item 3 of Definition 8.1.3.1.4.
[15]As proved in Item 3 of Definition 8.1.3.1.4.

8.4 Categories of Relations With Apartness Composition

8.4.1 The Category of Relations With Apartness Composition

Definition 8.4.1.1.1. The **category of relations with apartness composition** is the category $\mathsf{Rel}^{\square}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\square}$ are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}^{\square}(A, B) \overset{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel}^{\square})$, the unit map

$$\mathbb{1}_{A}^{\mathsf{Rel}^{\square}} : \mathrm{pt} \longrightarrow \mathrm{Rel}(A, A)$$

  of $\mathsf{Rel}^{\square}$ at A is defined by

$$\mathrm{id}_{A}^{\mathsf{Rel}^{\square}} \overset{\text{def}}{=} \nabla_{A}(-_1, -_2),$$

  where $\nabla_{A}(-_1, -_2)$ is the antidiagonal relation of A of Definition 8.2.1.1.4.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel}^{\square})$, the composition map

$$\circ_{A,B,C}^{\mathsf{Rel}^{\square}} : \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, C)$$

  of $\mathsf{Rel}^{\square}$ at (A, B, C) is defined by

$$S \circ_{A,B,C}^{\mathsf{Rel}^{\square}} R \overset{\text{def}}{=} S \mathbin{\square} R$$

  for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.4.1.1.

Proposition 8.4.1.1.2. The functor

$$(-)^{\mathsf{c}} : \mathsf{Rel} \longrightarrow \mathsf{Rel}^{\square}$$

given by the identity on objects and by $R \mapsto R^{\mathsf{c}}$ on morphisms is an isomorphism of categories.

Proof. By Item 4 of Definition 8.1.4.1.3, we see that $(-)^{\mathsf{c}}$ is indeed a functor.

By Categories, Item 1 of Definition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\mathsf{c}} \colon \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, B)$$

defined by the assignment $R \mapsto R^{\mathsf{c}}$ is a bijection by Constructions With Sets, Item 3 of Definition 4.3.11.1.2. Thus $(-)^{\mathsf{c}}$ is an isomorphism of categories. □

8.4.2 The 2-Category of Relations With Apartness Composition

Definition 8.4.2.1.1. The **2-category of relations with apartness composition** is the locally posetal 2-category $\mathbf{Rel}$ where

- *Objects.* The objects of $\mathbf{Rel}$ are sets.
- **Hom**-*Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A, B) &\overset{\text{def}}{=} \mathbf{Rel}(A, B)\\ &\overset{\text{def}}{=} (\mathrm{Rel}(A, B), \subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_A \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A, A)$$

  of $\mathbf{Rel}$ at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_A \overset{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Definition 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[16]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C)$$

  of $\mathbf{Rel}$ at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \overset{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

[16] That this is indeed a morphism of posets is proven in **??** of Definition 8.1.4.1.3.

Proposition 8.4.2.1.2. The functor

$$(-)^{\mathsf{c}} \colon \mathbf{Rel} \to \mathbf{Rel}^{\square,\mathsf{co}}$$

given by the identity on objects and by $R \mapsto R^{\mathsf{c}}$ on 1-morphisms is a 2-isomorphism of 2-categories.

Proof. By Item 4 of Definition 8.1.4.1.3, we see that $(-)^{\mathsf{c}}$ is indeed a functor. By Constructions With Sets, Item 1 of Definition 4.3.11.1.2, it is also a 2-functor.

By **??**, it suffices to show that $(-)^{\mathsf{c}}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Definition 8.4.1.1.1.
- Bijective on 2-morphisms, which follows from Constructions With Sets, Item 1 of Definition 4.3.11.1.2.

Thus $(-)^{\mathsf{c}}$ is indeed a 2-isomorphism of categories. □

8.4.3 The Linear Bicategory of Relations

Definition 8.4.3.1.1. The **linear bicategory of relations** is the linear bicategory consisting of:

- *The Underlying Bicategory I.* The bicategory Rel of Definition 8.3.4.1.1.
- *The Underlying Bicategory II.* The bicategory Rel of Definition 8.4.2.1.1.
- *Linear Distributors.* The inclusions

  $$\begin{aligned}\delta^{\ell}_{R,S,T} &\colon T \diamond (S \mathbin{\square} R) \hookrightarrow (T \diamond S) \mathbin{\square} R,\\ \delta^{r}_{R,S,T} &\colon (T \mathbin{\square} S) \diamond R \hookrightarrow T \mathbin{\square} (S \diamond R)\end{aligned}$$

  of Item 5 of Definition 8.1.4.1.3.

Proof. Since Rel and Rel$^{\square}$ are locally posetal, the commutativity of the coherence conditions for linear bicategories follows automatically (Categories, Item 4 of Definition 11.2.7.1.2). □

8.4.4 Other Categorical Structures With Apartness Composition

Remark 8.4.4.1.1. It seems apartness composition fails to form the following categorical structures:

- *Monoidal Category With Products.* Products don't seem to endow $\mathsf{Rel}^{\square}$ with a monoidal structure.
- *Monoidal Category With Coproducts.* Coproducts also don't seem to endow $\mathsf{Rel}^{\square}$ with a monoidal structure.
- *Double Categorical Structure.* It seems the apartness composition of relations doesn't form a double category in a natural[17] way.

8.5 Properties of the 2-Category of Relations

8.5.1 Self-Duality

Proposition 8.5.1.1.1. The 2-/category of relations is self-dual:

1. *Self-Duality I.* We have an isomorphism
$$\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$$
of categories.
2. *Self-Duality II.* We have a 2-isomorphism
$$\mathbf{Rel}^{\mathsf{op}} \cong \mathbf{Rel}$$
of 2-categories.

Proof. *Item 1, Self-Duality I*: We claim that the functor
$$(-)^{\dagger} \colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Rel}$$
given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories. Note that this is indeed a functor by Items 3 and 6 of Definition 8.1.5.1.3.

[17] I.e. such that the composition of vertical morphisms is the usual composition of functions,

By Categories, Item 1 of Definition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\dagger}\colon \mathrm{Rel}(A,B) \to \mathrm{Rel}(B,A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Definition 8.1.5.1.3, showing $(-)^{\dagger}$ to be fully faithful.

Item 2, Self-Duality II: We claim that the 2-functor

$$(-)^{\dagger}\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Definition 8.1.5.1.3, is an isomorphism of categories.

By **??**, it suffices to show that $(-)^{\dagger}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Definition 8.1.5.1.3.

Thus $(-)^{\dagger}$ is indeed a 2-isomorphism of categories. □

8.5.2 Isomorphisms and Equivalences

Let $R\colon A \nrightarrow B$ be a relation from A to B.

Lemma 8.5.2.1.1. The conditions below are row-wise equivalent:

Condition	Inclusion
R is functional	$R \diamond R^{\dagger} \subset \Delta_B$
R is total	$\Delta_A \subset R^{\dagger} \diamond R$
R is injective	$R^{\dagger} \diamond R \subset \Delta_A$
R is surjective	$\Delta_B \subset R \diamond R^{\dagger}$

Proof. Functionality Is Equivalent to $R \diamond R^{\dagger} \subset \Delta_B$: The condition $R \diamond R^{\dagger} \subset \Delta_B$ unwinds to

as in Sets.

($\star$) For each $b, b' \in B$, if there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$, then $b = b'$.

Since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, the condition says that $a \sim_R b$ and $a \sim_R b'$ imply $b = b'$. This is precisely the condition for R to be functional.
Totality Is Equivalent to $\Delta_A \subset R^\dagger \diamond R$: The condition $\Delta_A \subset R^\dagger \diamond R$ unwinds to

($\star$) For each $a, a' \in A$, if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$.

Since $b \sim_{R^\dagger} a'$ is the same as $a' \sim_R b$, the condition says that for each $a \in A$, there is some $b \in B$ with $b \in R(a)$, so $R(a) \neq \emptyset$. This is precisely the condition for R to be total.
Injectivity Is Equivalent to $R^\dagger \diamond R \subset \Delta_A$: The condition $R^\dagger \diamond R \subset \Delta_A$ unwinds to

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$, then $a = a'$.

Since $b \sim_{R^\dagger} a'$ is the same as $a' \sim_R b$, the condition says that for each $b \in B$, if $a \sim_R b$ and $a' \sim_R b$, then $a = a'$. This is precisely the condition for R to be injective.
Surjectivity Is Equivalent to $\Delta_B \subset R \diamond R^\dagger$: The condition $\Delta_B \subset R \diamond R^\dagger$ unwinds to

($\star$) For each $b, b' \in B$, if $b = b'$, then there exists some $a \in A$ such that $b \sim_{R^\dagger} a$ and $a \sim_R b'$.

Since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, the condition says that for each $b \in B$, there is some $a \in A$ with $b \in R(a)$, so $R^{-1}(b) \neq \emptyset$. This is precisely the condition for R to be surjective. □

Proposition 8.5.2.1.2. The following conditions are equivalent:

1. The relation $R \colon A \nrightarrow B$ is an equivalence in **Rel**, i.e.:

   ($\star$) There exists a relation $R^{-1} \colon B \nrightarrow A$ from B to A together with isomorphisms

$$
\begin{aligned}
R^{-1} \diamond R &\cong \Delta_A, \\
R \diamond R^{-1} &\cong \Delta_B.
\end{aligned}
$$

2. The relation $R \colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

(★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$
\begin{aligned}
R^{-1} \diamond R &= \Delta_A,\\
R \diamond R^{-1} &= \Delta_B.
\end{aligned}
$$

3. There exists a bijection $f\colon A \xrightarrow{\sim} B$ with $R = \mathrm{Gr}(f)$.

Proof. We claim that Items 1 to 3 are indeed equivalent:

- *Item 1* $\iff$ *Item 2*: This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.
- *Item 2* $\implies$ *Item 3*: We proceed in a few steps:
  - First, note that the equalities in Item 2 imply $R \dashv R^{-1}$ and thus, by Definition 8.5.3.1.1, there exists a function $f_R\colon A \to B$ associated to R.
  - By Definition 8.5.2.1.1, f_R is a bijection.
- *Item 3* $\implies$ *Item 2*: By Item 4 of Definition 8.2.2.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$
\begin{aligned}
\Delta_A &\subset f^{-1} \diamond \mathrm{Gr}(f),\\
\mathrm{Gr}(f) \diamond f^{-1} &\subset \Delta_B.
\end{aligned}
$$

  If f is bijective, then the reverse inclusions are also true by Definition 8.5.2.1.1.

This finishes the proof. □

8.5.3 Internal Adjunctions

Let A and B be sets.

Proposition 8.5.3.1.1. We have a natural bijection

$$
\left\{\begin{gathered}\text{Adjunctions in } \mathbf{Rel}\\ \text{from } A \text{ to } B\end{gathered}\right\} \cong \left\{\begin{gathered}\text{Functions}\\ \text{from } A \text{ to } B\end{gathered}\right\},
$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof. We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$\begin{aligned} R\colon A &\nrightarrow B, \\ S\colon B &\nrightarrow A, \end{aligned}$$

together with inclusions

$$\begin{aligned} \Delta_A &\subset S \diamond R, \\ R \diamond S &\subset \Delta_B. \end{aligned}$$

By Definition 8.5.2.1.1, R is total and functional. In particular, $R(a)$ is a singleton for all $a \in A$. Defining f_R such that $f_R(a)$ is the unique element of $R(a)$ then gives us our desired function, forming a map

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

Moreover, by uniqueness of adjoints (**??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 4 of Definition 8.2.2.1.2, every function $f\colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This follows form the fact that we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}\colon A \to B$, we have

$$\begin{aligned} \mathrm{Gr}(f_{R,S}) &= R, \\ f^{-1}_{R,S} &= S. \end{aligned}$$

We check these explicitly:

- $\mathrm{Gr}(f_{R,S}) = R$. We have

$$\begin{aligned}\mathrm{Gr}(f_{R,S}) &\overset{\text{def}}{=} \left\{(a, f_{R,S}(a)) \in A \times B \,\middle|\, a \in A\right\}\\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\}\\ &= R.\end{aligned}$$

- $f^{-1}_{R,S} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* We proceed in a few steps.
    * Since $\Delta_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
    * Since $a \sim_R b$ and R is functional, we have $k = b$.
    * Thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* We proceed in a few steps.
    * First note that, since R is total, we have $a \sim_R b'$ for some $b' \in B$.
    * Since $R \diamond S \subset \Delta_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$.
    * Thus $a \sim_R b$.

  Having show this, we now have

$$\begin{aligned}f^{-1}_{R,S}(b) &\overset{\text{def}}{=} \left\{a \in A \,\middle|\, f_{R,S}(a) = b\right\}\\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\}\\ &= \{a \in A \mid b \sim_S a\}\\ &\overset{\text{def}}{=} S(b).\end{aligned}$$

  for each $b \in B$, and thus $f^{-1}_{R,S} = S$.

This finishes the proof. □

8.5.4 Internal Monads

Let X be a set.

Proposition 8.5.4.1.1. We have a natural identification[18]

$$\begin{Bmatrix}\text{Monads in}\\ \textbf{Rel}\text{ on }X\end{Bmatrix} \cong \{\text{Preorders on } X\}.$$

Proof. A monad in **Rel** on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\mu_R &\colon R \diamond R \subset R,\\ \eta_R &\colon \Delta_X \subset R\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
\Delta_X \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R", "|" marking] \arrow[dr, equal, "\lambda^{\mathsf{Rel}}_R"'] & R \diamond R \arrow[d, "\mu_R", "|" marking] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \diamond (R \diamond R) \arrow[dr, "\mathrm{id}_R \diamond \mu_R", "|" marking] & \\
(R \diamond R) \diamond R \arrow[ur, equal, "\alpha^{\mathsf{Rel}}_{R,R,R}"] \arrow[d, "\mu_R \diamond \mathrm{id}_R"', "|" marking] & & R \diamond R \arrow[d, "\mu_R", "|" marking] \\
R \diamond R \arrow[rr, "\mu_R"', "|" marking] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \Delta_X \arrow[r, "\mathrm{id}_R \diamond \eta_R", "|" marking] \arrow[dr, equal, "\rho^{\mathsf{Rel}}_R"'] & R \diamond R \arrow[d, "\mu_R", "|" marking] \\
& R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $x, z \in X$, if there exists some $y \in Y$ such that $x \sim_R y$ and $y \sim_R z$, then $x \sim_R z$.
2. For each $x \in X$, we have $x \sim_R x$.

These are exactly the requirements for R to be a preorder (**??**). Conversely, any preorder $\preceq$ gives rise to a pair of maps $\mu_\preceq$ and $\eta_\preceq$, forming a monad on X. □

Example 8.5.4.1.2. Let $R\colon A \nrightarrow B$ be a relation.

[18] See also **??** for an extension of this correspondence to "relative monads in **Rel**".

1. The codensity monad $\mathrm{Ran}_R(R) \colon B \nrightarrow B$ is given by

$$[\mathrm{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b) \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(R)", "\shortmid" marking] \\ A \arrow[ru, "R", "\shortmid" marking] \arrow[r, "R"', "\shortmid" marking] & B \end{tikzcd}$$

   for each $b \in B$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Ran}_R(R)} \colon B \times B \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

   on B obtained by declaring $b \preceq_{\mathrm{Ran}_R(R)} b'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a \sim_R b'$.

   (b) We have $R^{-1}(b) \subset R^{-1}(b')$.

2. The dual codensity monad $\mathrm{Rift}_R(R) \colon A \nrightarrow A$ is given by

$$[\mathrm{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\} \qquad \begin{tikzcd} & A \arrow[d, "R", "\shortmid" marking] \\ A \arrow[ru, dashed, "\mathrm{Rift}_R(R)", "\shortmid" marking] \arrow[r, "R"', "\shortmid" marking] & B \end{tikzcd}$$

   for each $a \in A$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Rift}_R(R)} \colon A \times A \longrightarrow \{\mathsf{t}, \mathsf{f}\}$$

   on A obtained by declaring $a \preceq_{\mathrm{Rift}_R(R)} a'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a' \sim_R b$.

   (b) We have $R(a') \subset R(a)$.

8.5.5 Internal Comonads

Let X be a set.

Proposition 8.5.5.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \textsf{Rel}\text{ on } X\end{matrix}\right\} \cong \{\text{Subsets of } X\}.$$

Proof. A comonad in **Rel** on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\Delta_R\colon R &\subset R \diamond R,\\ \epsilon_R\colon R &\subset \Delta_X\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\textsf{Rel},-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\
& \Delta_X \diamond R
\end{tikzcd}
\quad
\begin{tikzcd}
& R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & \\
R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \diamond (R \diamond R) \arrow[d, equal, "\alpha^{\textsf{Rel},-1}_{R,R,R}"] \\
R \diamond R \arrow[rr, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & & (R \diamond R) \diamond R
\end{tikzcd}
\quad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\textsf{Rel},-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\
& R \diamond \Delta_X
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $x, y \in X$, if $x \sim_R y$, then there exists some $k \in X$ such that $x \sim_R k$ and $k \sim_R y$.
2. For each $x, y \in X$, if $x \sim_R y$, then $x = y$.

The second condition implies that $R \subset \Delta_X$, so R must be a subset of X. Taking $k = y$ in the first condition above then shows it to be trivially satisfied. Conversely, any subset U of X satisfies $U \subset \Delta_X$, defining a comonad as above. □

Example 8.5.5.1.2. Let $f\colon A \to B$ be a function.

1. The density comonad $\mathrm{Lan}_f(f)\colon B \nrightarrow B$ is given by

$$[\mathrm{Lan}_f(f)](b) = \bigcup_{a \in f^{-1}(b)} f(a) \qquad \begin{tikzcd} & B \\ A \arrow[ur, "f" description, "/" marking] \arrow[r, "f"', "|" marking] & B \arrow[u, dashed, "\mathrm{Lan}_f(f)"', "|" marking] \arrow[u, phantom, shift right=4, "\Uparrow"] \end{tikzcd}$$

for each $b \in B$. Thus, it corresponds to the image $\mathrm{Im}(f)$ of f as a subset of B.

2. The dual density comonad $\mathrm{Lift}_{f^\dagger}(f^\dagger)\colon A \nrightarrow A$ is given by

$$[\mathrm{Lift}_{f^\dagger}(f^\dagger)](b) = \bigcup_{a \in f^{-1}(b)} f(a) \qquad \begin{tikzcd} & B \arrow[d, "f^\dagger", "|" marking] \\ B \arrow[ur, dashed, "\mathrm{Lift}_{f^\dagger}(f^\dagger)", "/" marking] \arrow[r, "f^\dagger"', "|" marking] & A \arrow[u, phantom, shift right=4, "\Uparrow"] \end{tikzcd}$$

for each $b \in B$. Thus, it also corresponds to the image $\mathrm{Im}(f)$ of f as a subset of B.

8.5.6 Modules Over Internal Monads

Let A be a set.

Proposition 8.5.6.1.1. Let $\preceq_A$ be a preorder on A, viewed also as an internal monad on A via Definition 8.5.4.1.1.

1. *Left Modules.* We have a natural identification

$$\{\text{Left modules over } \preceq_A\} \cong \left\{ \begin{array}{l} \text{Relations } R\colon B \nrightarrow A \text{ such that,} \\ \text{for each } b \in B \text{, the set } R(b) \text{ is} \\ \text{upward-closed in } A \end{array} \right\}.$$

2. *Right Modules.* We have a natural identification

$$\{\text{Right modules over } \preceq_A\} \cong \left\{ \begin{array}{l} \text{Relations } R\colon A \nrightarrow B \text{ such that,} \\ \text{for each } b \in B \text{, the set } R^{-1}(b) \text{ is} \\ \text{downward-closed in } A \end{array} \right\}.$$

3. *Bimodules.* We have a natural identification

$$\{\text{Bimodules over } \preceq_A\} \cong \left\{\begin{array}{l}\text{Quadruples } (B, C, R, S) \text{ such that:}\\ \text{1. For each } b \in B, \text{ the set } R(b) \text{ is}\\ \quad \text{upward-closed in } A.\\ \text{2. For each } c \in C, \text{ the set } S^{-1}(c) \text{ is}\\ \quad \text{downward-closed in } A.\end{array}\right\}.$$

Proof. *Item 1*, *Left Modules*: A left module over $\preceq_A$ in **Rel** consists of a relation $R\colon B \nrightarrow A$ together with an inclusion

$$\alpha_B \colon \preceq_A \diamond R \subset R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and hence all that is left is the data of the inclusion α_B. This corresponds to the following condition:

(★) For each $a, a' \in A$, if there exists some $b \in B$ such that $b \sim_R a$ and $a \preceq_a a'$, then $b \sim_R a'$.

This condition is equivalent to $R(b)$ being downward-closed for all $b \in B$.
Item 2, *Right Modules*: The proof is dual to Item 1, and is therefore omitted.
Item 3, *Bimodules*: Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (Categories, Item 4 of Definition 11.2.7.1.2), and hence a bimodule is just a left module along with a right module. □

8.5.7 Comodules Over Internal Comonads

Let A be a set.

Proposition 8.5.7.1.1. Let U be a subset of A, viewed also as an internal comonad on A via Definition 8.5.5.1.1.

1. *Left Comodules.* We have a natural identification

$$\{\text{Left comodules over } U\} \cong \left\{\begin{array}{l}\text{Relations } R\colon B \nrightarrow A \text{ such that,}\\ \text{for each } b \in B, \text{ we have } R(b) \subset U\end{array}\right\}.$$

2. *Right Comodules.* We have a natural identification

$$\{\text{Right comodules over } U\} \cong \left\{\begin{array}{l}\text{Relations } R\colon A \nrightarrow B \text{ such that,}\\ \text{for each } b \in B, \text{ we have } R^{-1}(b) \subset U\end{array}\right\}.$$

3. *Bicomodules.* We have a natural identification

$$\{\text{Bicomodules over } U\} \cong \left\{\begin{array}{l}\quad\text{Quadruples } (B, C, R, S) \text{ such that:}\\ \text{1. For each } b \in B, \text{ we have } R(b) \subset U\\ \text{2. For each } c \in C, \text{ we have } S^{-1}(c) \subset U\end{array}\right\}.$$

Proof. *Item 1*, *Left Comodules*: A left comodule over U in **Rel** consists of a relation $R\colon B \nrightarrow A$ together with an inclusion

$$R \subset U \diamond R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

(★) For each $b \in B$, if $b \sim_R a$, then there exists some $a' \in A$ such that $b \sim_R a'$ and $a' \sim_U a$.

Since $a' \sim_U a$ is true if $a = a'$ and $a \in U$, this condition ends up being equivalent to $R(b) \subset U$.

Item 2, *Right Comodules*: A right comodule over U in **Rel** consists of a relation $R\colon A \nrightarrow B$ together with an inclusion

$$R \subset R \diamond U$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

(★) For each $a \in A$, if $a \sim_R b$, then there exists some $x \in A$ such that $a \sim_U x$ and $x \sim_R b$.

Since $a \sim_U x$ is true if $a = x$ and $a \in U$, this condition ends up being equivalent to $R^{-1}(b) \subset U$.

Item 3, *Bicomodules*: Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (Categories, Item 4 of Definition 11.2.7.1.2), and hence a bicomodule is just a left comodule along with a right comodule. □

8.5.8 Eilenberg–Moore and Kleisli Objects

Let X be a set.

Proposition 8.5.8.1.1. Let R be a preorder on X, viewed as an internal monad on X via Definition 8.5.4.1.1.

1. *Eilenberg–Moore Objects in* **Rel**. The Eilenberg–Moore object for R exists iff it is an equivalence relation, in which case it is the quotient $X/{\sim_R}$ of X by R.

2. *Kleisli Objects in* **Rel**. [...]

Proof. Omitted. □

8.5.9 Co/Monoids

Remark 8.5.9.1.1. The monoids in **Rel** with respect to the Cartesian monoidal structure of Definition 8.3.3.8.1 are called *hypermonoids*, and their theory is explored in **??**. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in **??**.

8.5.10 Monomorphisms and 2-Categorical Monomorphisms

Explanation 8.5.10.1.1. In this section, we characterise:

- The 1-categorical monomorphisms in Rel, following **??**, **??**.

- The 2-categorical monomorphisms in **Rel**, following Types of Morphisms in Bicategories, Section 14.1.

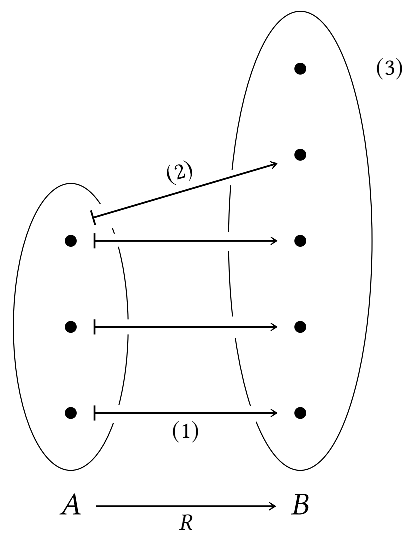
More specifically:

- Definition 8.5.10.1.2 gives *conceptual* characterisations of the monomorphisms in Rel.

- Definition 8.5.10.1.3 gives *point-set* characterisations of the monomorphisms in Rel.

- Definitions 8.5.10.1.8 and 8.5.10.1.9 characterise the 2-categorical monomorphisms in **Rel**.[19]

Essentially, a monomorphism $R: A \nrightarrow B$ in Rel is a relation that is total and injective. Therefore, it looks like this:



In particular:

1. R should contain an injection $f: A \hookrightarrow B$ embedding a copy of A into B.
2. R can be non-functional, mapping elements of A to multiple elements of B (but not to more than one in $\mathrm{Im}(f)$).
3. R doesn't need to be surjective, so B can have elements that aren't in the image of R.

Proposition 8.5.10.1.2. Let $R: A \nrightarrow B$ be a relation. The following conditions are equivalent:[20]

[19] *Summary:* As it turns out, every 1-morphism in **Rel** is representably faithful and most other notions of 2-categorical monomorphism agree with the usual (1-categorical) notion of monomorphism.

[20] Items 3 to 6 unwind respectively to the following statements:

- For each $U, V \in \mathcal{P}(A)$, if $R_!(U) = R_!(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $R_*(U) = R_*(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.
- For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

1. The relation R is a monomorphism in Rel.
2. The relation R is total and injective.
3. The direct image function
$$R_! \colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
4. The codirect image function
$$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
5. The direct image functor
$$R_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
associated to R is full.
6. The codirect image functor
$$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
associated to R is full.
7. For each pair of relations $S, T \colon X \rightrightarrows A$, the following condition is satisfied:
   (★) If $R \diamond S \subset R \diamond T$, then $S \subset T$.
8. There exists an injective function $f \colon A \hookrightarrow B$ satisfying the following conditions:[21]
   (a) We have $\mathrm{Gr}(f) \subset R$.[22]

[21] We are assuming the axiom of choice for this item (Item 8).
[22] In other words, for each $a \in A$, we have $f(a) \in R(a)$.

(b) The diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, hook, "\chi_A"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A) \end{tikzcd}$$

commutes.[23]

9. The diagram

$$\begin{tikzcd} A \arrow[r, "R"] \arrow[dr, hook, "\chi_A"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A) \end{tikzcd}$$

commutes. In other words, we have

$$R^{-1}(R(a)) = \{a\}$$

for each $a \in A$.

10. We have

$$R_{-1} \circ R_{!} = \mathrm{id}_{\mathcal{P}(A)} \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_{!}"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

In other words, we have

$$U = \underbrace{\{a \in A \mid R(a) \subset R(U)\}}_{=R_{-1}(R_{!}(U))}$$

for each $U \in \mathcal{P}(A)$.

[23] In other words, for each $a \in A$, we have $R^{-1}(f(a)) = \{a\}$.

11. We have

$$
R^{-1} \circ R_! = \mathrm{id}_{\mathcal{P}(A)} \qquad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\
& \mathcal{P}(A).
\end{tikzcd}
$$

In other words, we have

$$
U = \underbrace{\{a \in A \mid R(a) \cap R(U) \neq \emptyset\}}_{=R^{-1}(R_!(U))}
$$

for each $U \in \mathcal{P}(A)$.

12. We have

$$
R^{-1} \circ R_* = \mathrm{id}_{\mathcal{P}(A)} \qquad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\
& \mathcal{P}(A).
\end{tikzcd}
$$

In other words, we have

$$
U = \underbrace{\left\{a \in A \;\middle|\; \begin{gathered}\text{there exists some } b \in R(a)\\ \text{such that we have } R^{-1}(b) \subset U\end{gathered}\right\}}_{=R^{-1}(R_*(U))}
$$

for each $U \in \mathcal{P}(A)$.

13. We have

$$
R_{-1} \circ R_* = \mathrm{id}_{\mathcal{P}(A)} \qquad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, equal] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\
& \mathcal{P}(A).
\end{tikzcd}
$$

In other words, we have

$$U = \underbrace{\left\{a \in A \,\middle|\, R^{-1}(R(a)) \subset U\right\}}_{=R_{-1}(R_*(U))}$$

for each $U \in \mathcal{P}(A)$.

Proof. We will prove this by showing:

- Step 1: Item 1 $\iff$ Item 2.
- Step 2: Item 3 $\iff$ Item 4 and Item 4 $\iff$ Item 6.
- Step 3: Item 1 $\iff$ Item 3.
- Step 4: Item 3 $\iff$ Item 5.
- Step 5: Item 5 $\iff$ Item 7.
- Step 6: Item 1 $\iff$ Item 8.
- Step 7: Item 1 $\iff$ Item 9.
- Step 8: Item 1 $\iff$ Item 10.
- Step 9: Item 1 $\iff$ Item 11.
- Step 10: Item 1 $\iff$ Item 12.
- Step 11: Item 1 $\iff$ Item 13.

Step 1: Item 1 $\iff$ Item 2: We defer this proof to Definition 8.5.10.1.5.
Step 2: Item 3 $\iff$ Item 4 and Item 4 $\iff$ Item 6: This follows from Item 7 of Definition 8.7.1.1.4.
Step 3: First Proof of Item 1 $\iff$ Item 3: We claim that Items 1 and 3 are equivalent:

- *Item 1 $\implies$ Item 3*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "U", "|" marking] \arrow[r, shift right, "V"', "|" marking] & A \arrow[r, "R", "|" marking] & B. \end{tikzcd}$$

  By Definition 8.7.1.1.3, we have

$$R_!(U) = R \diamond U,$$

$$R_!(V) = R \diamond V.$$

Now, if $R \diamond U = R \diamond V$, i.e. $R_!(U) = R_!(V)$, then $U = V$ since R is assumed to be a monomorphism, showing $R_!$ to be injective.

- *Item 3* $\implies$ *Item 1*: Conversely, suppose that $R_!$ is injective, consider the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "S", "\shortmid"{marking}] \arrow[r, shift right, "T"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

and suppose that $R \diamond S = R \diamond T$. Note that, since $R_!$ is injective, given a diagram of the form

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "U", "\shortmid"{marking}] \arrow[r, shift right, "V"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

if $R_!(U) = R \diamond U = R \diamond V = R_!(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[x]}", "\shortmid"{marking}] & X \arrow[r, shift left, "S", "\shortmid"{marking}] \arrow[r, shift right, "T"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

where we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each $x \in X$. Thus $S = T$ and R is a monomorphism.

Step 3.5: Second Proof of Item 1 $\iff$ *Item 3*: A more abstract proof can also be given, following [MSE 350788]:

- *Definition 8.5.10.1.2* $\implies$ *Definition 8.5.10.1.3*: Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -) \colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to $R_!$ by Definition 8.7.1.1.3.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by Limits and Colimits, **??** of **??**, it follows by **??**, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.

  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to $R_!$, it follows that $R_!$ is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that $R_!$ is injective.

- *Definition 8.5.10.1.3* $\implies$ *Definition 8.5.10.1.2*: Assume that $R_!$ is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to $R_!$ by Definition 8.7.1.1.3.
  - Since the monomorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that $R_!$ is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??**, **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since $R_!$ is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to $R_!$, it follows that R is also a monomorphism.

Step 4: Item 3 $\iff$ *Item 5*: We claim that Items 3 and 5 are equivalent:

- *Item 3* $\implies$ *Item 5*: We proceed in a few steps:
  - Let $U, V \in \mathcal{P}(A)$ such that $R_!(U) \subset R_!(V)$, assume $R_!$ to be injective, and consider the set $U \cup V$.
  - Since $R_!(U) \subset R_!(V)$, we have
  $$\begin{aligned} R_!(U \cup V) &= R_!(U) \cup R_!(V) \\ &= R_!(V), \end{aligned}$$
  where we have used Item 5 of Definition 8.7.1.1.4 for the first equality.
  - Since $R_!$ is injective, this implies $U \cup V = V$.
  - Thus $U \subset V$, as we wished to show.

- *Item 3* $\implies$ *Item 5*: We proceed in a few steps:
  - Suppose Item 5 holds, and let $U, V \in \mathcal{P}(A)$ such that $R_!(U) = R_!(V)$.
  - Since $R_!(U) = R_!(V)$, we have $R_!(U) \subset R_!(V)$ and $R_!(V) \subset R_!(U)$.
  - By assumption, this implies $U \subset V$ and $V \subset U$.

- Thus $U = V$, showing $R_!$ to be injective.

Step 5: Item 5 $\iff$ *Item 7*: We claim that Items 5 and 7 are equivalent:

- *Item 5* $\implies$ *Item 7*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U", shift left, "\shortmid"{marking}] \arrow[r, "V"', shift right, "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B. \end{tikzcd}$$

By Definition 8.7.1.1.3, we have

$$\begin{aligned} R_!(U) &= R \diamond U, \\ R_!(V) &= R \diamond V. \end{aligned}$$

Now, if $R \diamond U \subset R \diamond V$, then $R_!(U) \subset R_!(V)$. By assumption, we then have $U \subset V$.

- *Item 7* $\implies$ *Item 5*: Consider the diagram

$$\begin{tikzcd} X \arrow[r, "S", shift left, "\shortmid"{marking}] \arrow[r, "T"', shift right, "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U", shift left, "\shortmid"{marking}] \arrow[r, "V"', shift right, "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

if $R_!(U) = R \diamond U \subset R \diamond V = R_!(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[x]}", "\shortmid"{marking}] & X \arrow[r, "S", shift left, "\shortmid"{marking}] \arrow[r, "T"', shift right, "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each $x \in X$, implying $S \subset T$.

This finishes the proof.

Step 6: Item 1 $\iff$ *Item 8*: We defer this proof to Definition 8.5.10.1.4.

Step 7: Item 1 $\iff$ *Item 9*: We defer this proof to Definition 8.5.10.1.6.

Step 8: Item 1 $\iff$ *Item 10*: We defer this proof to Definition 8.5.10.1.4.

Step 9: Item 1 $\iff$ *Item 11*: We defer this proof to Definition 8.5.10.1.6.

Step 10: Item 1 $\iff$ *Item 12*: We defer this proof to Definition 8.5.10.1.6.

Step 11: Item 1 $\iff$ *Item 13*: We defer this proof to Definition 8.5.10.1.6. □

Proposition 8.5.10.1.3. Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is a monomorphism in Rel.
2. For each $a \in A$ and each $U \in \mathcal{P}(A)$, if $R(a) \subset R(U)$, then $a \in U$.
3. For each $a \in A$, there exists some $b \in B$ such that $R^{-1}(b) = \{a\}$.

Proof. We will prove this by showing:

- Step 1: Item 1 $\implies$ Item 2.
- Step 2: Item 2 $\implies$ Item 3.
- Step 3: Item 3 $\implies$ Item 1.

Step 1: Item 1 $\implies$ *Item 2*: We proceed in a few steps:

- If R is a monomorphism, then, by Item 3 of Definition 8.5.10.1.2, the functor
$$R_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
is full.
- As a result, given $a \in A$ and $U \in \mathcal{P}(A)$ such that $R(a) \subset R(U)$, it follows that $\{a\} \subset U$.
- Thus, we have $a \in U$.

Step 2: Item 2 $\implies$ *Item 3*: We proceed in a few steps:

- Let $a \in A$ and consider the subset $U = A \setminus \{a\}$.
- Since $a \notin U$, we have $R(a) \not\subset R(U)$ by the contrapositve of Item 2.

- As a result, there must exist some $b \in R(a)$ with $b \notin R(U)$.
- In particular, we have $a \in R^{-1}(b)$.
- Moreover, the condition $b \notin R(U) = R(A \setminus \{a\})$ means that, if $a' \in A \setminus \{a\}$, then $a' \notin R^{-1}(b)$.
- Thus $R^{-1}(b) = \{a\}$.

Step 3: Item 3 $\implies$ Item 1: We proceed in a few steps:

- By the equivalence between Items 1 and 5 of Definition 8.5.10.1.2, to show that R is a monomorphism it suffices to prove that, for each $U, V \in \mathcal{P}(A)$, if $R(U) \subset R(V)$, then $U \subset V$.
- So let $u \in U$ and assume $R(U) \subset R(V)$.
- By assumption, there exists some $b \in B$ with $R^{-1}(b) = \{u\}$.
- In particular, $b \in R(U)$.
- Since $R(U) \subset R(V)$, we also have $b \in R(V)$.
- Thus, there exists some $v \in V$ with $b \in R(v)$.
- However, $R^{-1}(b) = \{u\}$, so we must in fact have $v = u$.
- Therefore $u \in V$, showing that $U \subset V$.

This finishes the proof. □

Corollary 8.5.10.1.4. Items 1, 8 and 10 of Definition 8.5.10.1.2 are indeed equivalent.

Proof. *Item 1 $\iff$ Item 8*: We claim that Item 3 of Definition 8.5.10.1.3 is equivalent to Item 8 of Definition 8.5.10.1.2:

- *Item 3 of Definition 8.5.10.1.3 $\implies$ Item 8*: By assumption, given $a \in A$, there exists some $b \in B$ such that $R^{-1}(b) = \{a\}$. Invoking the axiom of choice, we may pick one such b for each $a \in A$, giving us our desired function $f \colon A \to B$. All the requirements listed in Item 8 of Definition 8.5.10.1.2 then follow by construction.

- *Item 8* $\implies$ *Item 3* of *Definition 8.5.10.1.3*: Given $a \in A$, we may pick $b = f(a)$, in which case $R^{-1}(f(a))$ will be equal to $\{a\}$ by assumption.

By Definition 8.5.10.1.3, Item 3 of Definition 8.5.10.1.3 is equivalent to Item 1 of Definition 8.5.10.1.3. Since Item 1 of Definition 8.5.10.1.3 is exactly the same condition as Item 1 of Definition 8.5.10.1.2, the result follows.
Item 1 $\iff$ *Item 10*: Indeed, we have

$$\begin{aligned}[R_{-1} \circ R_!](U) &\stackrel{\text{def}}{=} R_{-1}(R_!(U)) \\ &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset R(U)\}\end{aligned}$$

for each $U \in \mathcal{P}(A)$. As a result, the condition $R_{-1} \circ R_! = \mathrm{id}_{\mathcal{P}(A)}$ becomes

$$\{a \in A \mid R(a) \subset R(U)\} = U,$$

which holds precisely when Item 2 of Definition 8.5.10.1.3 does. By Definition 8.5.10.1.3, that in turn holds precisely if Item 1 of Definition 8.5.10.1.3 holds. Since Item 1 of Definition 8.5.10.1.3 is exactly the same condition as Item 1 of Definition 8.5.10.1.2, the result follows. □

Corollary 8.5.10.1.5. Items 1 and 2 of Definition 8.5.10.1.2 are indeed equivalent.[24]

Proof. We claim that Items 1 and 2 of Definition 8.5.10.1.2 are indeed equivalent:

- *Item 1* $\implies$ *Item 2*: First, note that R is total by Item 3 of Definition 8.5.10.1.3. Next, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$ and consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "{[a]}", "|" marking] \arrow[r, shift right, "{[a']}"', "|" marking] & A \arrow[r, "R", "|" marking] & B. \end{tikzcd}$$

  Then:

  - Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$.
  - Similarly, $\star \sim_{R \diamond [a']} b$.
  - Thus $R \diamond [a] = R \diamond [a']$.

[24] I.e. a relation is a monomorphism in Rel iff it is total and injective.

    - Since R is a monomorphism, we have $[a] = [a']$, so $a = a'$.

- *Item 2* $\implies$ *Item 1*: Consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\longrightarrow} B,$$

where $R \diamond S = R \diamond T$, and let $x \in X$ and $a \in A$ such that $x \sim_S a$.

    - Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$.
    - In this case, we have $x \sim_{R\diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $x \sim_{R\diamond T} b$.
    - Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$.
    - However, since $a \sim_R b$ and $a' \sim_R b$, we must have $a = a'$ by condition ($\star$).
    - Thus $x \sim_T a$ as well.
    - A similar argument shows that if $x \sim_T a$, then $x \sim_S a$.
    - Thus $S = T$, showing R to be a monomorphism.

This finishes the proof. □

Corollary 8.5.10.1.6. Items 1, 9 and 11 to 13 of Definition 8.5.10.1.2 are indeed equivalent.

Proof. We will prove this by showing:

- Step 1: Item 1 $\implies$ Item 11.
- Step 2: Item 11 $\implies$ Item 1.
- Step 3: Item 1 $\implies$ Item 12.
- Step 4: Item 12 $\implies$ Item 1.
- Step 5: Item 9 $\iff$ Item 13.
- Step 6: Item 9 $\iff$ Item 2.

Step 1: Item 1 $\implies$ *Item 11*: Assume that R is a monomorphism, which is equivalent to R being total and injective by Definition 8.5.10.1.2. Let $S(U) \overset{\text{def}}{=} \{a \in A \mid R(a) \cap R(U) \neq \emptyset\}$. We need to show that $U = S(U)$ for any $U \in \mathcal{P}(A)$ by proving double inclusion.

- *$U \subset S(U)$:* Let $u \in U$.
  - Since R is total, we have $R(u) \neq \emptyset$.
  - By definition, $R(U) = \bigcup_{x \in U} R(x)$, so $R(u) \subset R(U)$.
  - Therefore, $R(u) \cap R(U) = R(u)$.
  - Since $R(u) \neq \emptyset$, we have $R(u) \cap R(U) \neq \emptyset$.
  - By the definition of $S(U)$, it follows that $u \in S(U)$.
- *$S(U) \subset U$:* Let $a \in S(U)$.
  - By assumption, $R(a) \cap R(U) \neq \emptyset$.
  - This means $R(a) \cap \bigcup_{u \in U} R(u) \neq \emptyset$.
  - Using the distributivity of intersection over union, this is equivalent to $\bigcup_{u \in U} (R(a) \cap R(u)) \neq \emptyset$.
  - For this union of sets to be non-empty, at least one of the sets in the union must be non-empty. Thus, there must exist some $u \in U$ such that $R(a) \cap R(u) \neq \emptyset$.
  - Since R is injective, the images of distinct elements are disjoint. For the intersection $R(a) \cap R(u)$ to be non-empty, we must therefore have $a = u$.
  - Since $u \in U$, it follows that $a \in U$.

As both inclusions hold, we conclude that $U = S(U)$.
Step 2: Item 11 $\implies$ Item 1: Assume that for every $U \in \mathcal{P}(A)$, we have $U = \{a \in A \mid R(a) \cap R(U) \neq \emptyset\}$. We must show that R is both total and injective.

- *Totality:* Let $a \in A$. We must show that $R(a) \neq \emptyset$.
  - Consider the singleton set $U = \{a\}$.
  - By assumption, $U = \{x \in A \mid R(x) \cap R(U) \neq \emptyset\}$.
  - Since $a \in U$, we must have $R(a) \cap R(U) \neq \emptyset$.
  - Substituting $U = \{a\}$, we get $R(a) \cap R(\{a\}) \neq \emptyset$.
  - Since $R(\{a\}) = R(a)$, this simplifies to $R(a) \cap R(a) = R(a) \neq \emptyset$.
  - Thus $R(a) \neq \emptyset$ for all $a \in A$, showing R to be total.

- *Injectivity:* Let $a, a' \in A$ such that $a \neq a'$. We must show that $R(a) \cap R(a') = \emptyset$.
  - Consider the singleton set $U = \{a\}$.
  - By assumption, $U = \{x \in A \mid R(x) \cap R(U) \neq \emptyset\}$.
  - Since $a \neq a'$, we have $a' \notin U$.
  - Therefore, a' cannot satisfy the membership condition for U. This means $R(a') \cap R(U) = \emptyset$.
  - Substituting $U = \{a\}$, we get $R(a') \cap R(\{a\}) = \emptyset$, which simplifies to $R(a') \cap R(a) = \emptyset$.
  - As this holds for any pair of distinct elements, the relation R is injective.

This completes the proof.
Step 3: Item 1 $\implies$ *Item 12*: We proceed by taking a specific choice of subset U:

- Let a be an arbitrary element of A. By our assumption, the condition $R^{-1}(R_*(U)) = U$ must hold for the singleton set $U = \{a\}$.
- From $R^{-1}(R_*(\{a\})) = \{a\}$, it follows that $a \in R^{-1}(R_*(\{a\}))$.
- This means there must exist some $b \in R(a)$ such that $R^{-1}(b) \subset \{a\}$.
- The condition $b \in R(a)$ implies that $a \in R^{-1}(b)$. Therefore, $R^{-1}(b)$ is a non-empty subset of $\{a\}$.
- The only non-empty subset of $\{a\}$ is $\{a\}$ itself.
- Thus, we must have $R^{-1}(b) = \{a\}$.

Step 4: Item 12 $\implies$ *Item 1*: By Item 2 of Definition 8.5.10.1.3, for each $a \in A$, there exists some $b \in B$ such that $R^{-1}(b) = \{a\}$. We need to show that $R^{-1}(R_*(U)) = U$ for any $U \in \mathcal{P}(A)$, which requires proving two set inclusions.

- $R^{-1}(R_*(U)) \subset U$: We proceed in a few steps:
  - Let $a \in R^{-1}(R_*(U))$.
  - By definition, there exists some $b \in R(a)$ such that $R^{-1}(b) \subset U$.
  - Since $b \in R(a)$ implies $a \in R^{-1}(b)$, it follows immediately that $a \in U$.

  - Thus, $R^{-1}(R_*(U)) \subset U$.

- $U \subset R^{-1}(R_*(U))$: Let $a \in U$. By assumption, there exists an element $b \in B$ such that $R^{-1}(b) = \{a\}$. Thus $R^{-1}(b) \subset U$, so $a \in R^{-1}(R_*(U))$.

Combining both inclusions gives $R^{-1}(R_*(U)) = U$.
Step 5: Item 9 $\iff$ Item 13: We claim that Items 9 and 13 are equivalent:

- *Item 13 $\implies$ Item 9*: Let $a \in A$.

  - First, let $U = \{a\}$. By assumption, we have
$$\{a\} = \left\{a' \in A \,\middle|\, R^{-1}(R(a')) \subset \{a\}\right\}.$$
    Since a is in the set on the left-hand side, it must also be in the set on the right-hand side. Thus $R^{-1}(R(a')) \subset \{a\}$ must be true.
  - Next, consider the complement $U = A \setminus \{a\}$. By assumption, we have
$$A \setminus \{a\} = \left\{a' \in A \,\middle|\, R^{-1}(R(a')) \subset A \setminus \{a\}\right\}$$
    Since a is not in the set on the left-hand side, it cannot be in the set on the right-hand side. Thus $R^{-1}(R(a)) \not\subset A \setminus \{a\}$.
  - The statement $R^{-1}(R(a)) \not\subset A \setminus \{a\}$ implies that there exists an element $x \in R^{-1}(R(a))$ such that $x \notin A \setminus \{a\}$. The only such element is a, so we must have $a \in R^{-1}(R(a))$.
  - Combining these two results, namely $R^{-1}(R(a)) \subset \{a\}$ and $a \in R^{-1}(R(a))$, we conclude that $R^{-1}(R(a)) = \{a\}$, as we wished to show.

- *Item 9 $\implies$ Item 13*: We have
$$\begin{aligned} R^{-1}(R_*(U)) &= \left\{a \in A \,\middle|\, R^{-1}(R(a)) \subset U\right\} \\ &= \{a \in A \mid \{a\} \subset U\} \\ &= U. \end{aligned}$$

Step 6: Item 9 $\iff$ Item 2: We claim that Items 2 and 9 are equivalent:

- *Item 9 $\implies$ Item 2*: By definition,
$$R^{-1}(R(a)) = \{x \in A \mid R(x) \cap R(a) \neq \emptyset\}.$$
  The condition $R^{-1}(R(a)) = \{a\}$ implies two facts:

  - The element a must belong to the set $\{x \in A \mid R(x) \cap R(a) \neq \emptyset\}$. For this to be true, the condition must hold for $x = a$, so $R(a) \cap R(a) \neq \emptyset$. This is equivalent to $R(a) \neq \emptyset$. Since this must hold for all $a \in A$, the relation R is total.
  - Any element $x \in A$ such that $x \neq a$ must not belong to the set. This means that for any $x \neq a$, we must have $R(x) \cap R(a) = \emptyset$. This means the image sets of distinct elements of A are pairwise disjoint.

  Thus, R is total and injective.

- *Item 2* $\implies$ *Item 9*: Let $a \in A$. We wish to show $R^{-1}(R(a)) = \{a\}$.
  - Let $x \in R^{-1}(R(a))$. By definition, this means $R(x) \cap R(a) \neq \emptyset$.
  - Since the image sets are pairwise disjoint, this can only be true if $x = a$.
  - Therefore, $R^{-1}(R(a)) \subset \{a\}$.
  - Since R is total, $R(a)$ is non-empty.
  - Thus $R(a) \cap R(a) \neq \emptyset$, which implies $a \in R^{-1}(R(a))$.
  - Therefore, $\{a\} \subset R^{-1}(R(a))$.

  Combining both inclusions, we have $R^{-1}(R(a)) = \{a\}$.

This finishes the proof. □

Remark 8.5.10.1.7. Taking the contrapositive of Item 2 of Definition 8.5.10.1.3 and letting $U = \{a'\}$ shows that the subset

$$\{R(a) \in \mathcal{P}(B) \mid a \in A\}$$

of $\mathcal{P}(B)$ forms an antichain in $\mathcal{P}(B)$. The converse however, fails.

Proposition 8.5.10.1.8. Every 1-morphism in **Rel** is representably faithful.

Proof. A relation $R\colon A \nrightarrow B$ will be representably faithful in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R_*\colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

given by postcomposition by R is faithful. This happens iff the morphism

$$R_{*|S,T}\colon \operatorname{Hom}_{\mathbf{Rel}(X,A)}(S,T) \to \operatorname{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each $S, T \in \operatorname{Obj}(\mathbf{Rel}(X,A))$.

However, since **Rel** is locally posetal, the Hom-set $\operatorname{Hom}_{\mathbf{Rel}(X,A)}(S,T)$ is either empty or a singleton. As a result, the map $R_{*|S,T}$ will necessarily be injective in either of these cases. □

Proposition 8.5.10.1.9. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The morphism $R\colon A \nrightarrow B$ is a monomorphism in Rel.
2. The 1-morphism $R\colon A \nrightarrow B$ is representably full in **Rel**.
3. The 1-morphism $R\colon A \nrightarrow B$ is representably fully faithful in **Rel**.
4. The 1-morphism $R\colon A \nrightarrow B$ is pseudomonic in **Rel**.
5. The 1-morphism $R\colon A \nrightarrow B$ is representably essentially injective in **Rel**.
6. The 1-morphism $R\colon A \nrightarrow B$ is representably conservative in **Rel**.

Proof. We will prove this by showing:

- Step 1: Item 1 $\iff$ Item 2.
- Step 2: Item 2 $\iff$ Item 3.
- Step 3: Item 3 $\iff$ Item 4 $\iff$ Item 5 $\iff$ Item 6.

Step 1: Item 1 $\iff$ Item 2: The condition that R is representably full corresponds precisely to Item 7 of Definition 8.5.10.1.2, so this follows by Definition 8.5.10.1.2.
Step 2: Item 2 $\iff$ Item 3: This follows from Step 1 and Definition 8.5.10.1.8.
Step 3: Item 3 $\iff$ Item 4 $\iff$ Item 5 $\iff$ Item 6: Since **Rel** is locally posetal, the conditions in Items 4 to 6 all collapse to the one of Item 3. □

8.5.11 Epimorphisms and 2-Categorical Epimorphisms

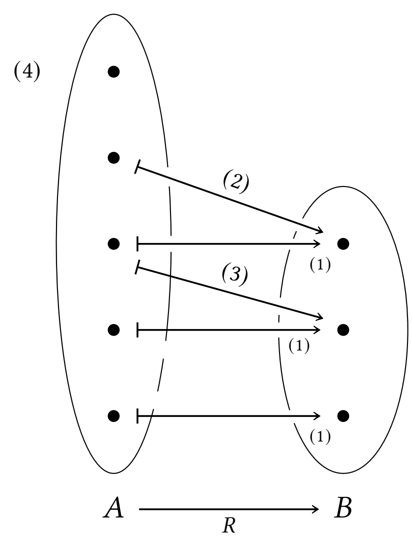
Explanation 8.5.11.1.1. In this section, we characterise:

- The 1-categorical epimorphisms in Rel, following **??**, **??**.
- The 2-categorical epimorphisms in **Rel**, following Types of Morphisms in Bicategories, Section 14.2.

More specifically:

- Definition 8.5.11.1.2 gives *conceptual* characterisations of the epimorphisms in Rel.
- Definition 8.5.11.1.3 gives *point-set* characterisations of the epimorphisms in Rel.
- Definition 8.5.11.1.6 lists a few conditions that look natural but fail to characterise epimorphisms in Rel.
- Definitions 8.5.11.1.8 and 8.5.11.1.9 characterise the 2-categorical epimorphisms in **Rel**.[25]

Essentially, an epimorphism $R\colon A \nrightarrow B$ in Rel looks like this:



In particular:

[25] *Summary:* As it turns out, every 1-morphism in **Rel** is representably faithful and most other notions of 2-categorical epimorphism agree with the usual (1-categorical) notion of epimorphism.

1. R should contain a surjection $f\colon A \twoheadrightarrow B$.
2. R doesn't need to be injective, so R can map different elements of A to the same element of B.
3. R can be non-functional, mapping elements of A to multiple elements of B.
4. R can be non-total, so R doesn't need to be defined on all of A.
5. For each $b \in B$, there must exist some $a \in A$ with $R(a) = \{b\}$.

Moreover, if R is functional, then being an epimorphism is equivalent to being surjective.

Proposition 8.5.11.1.2. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:[26]

1. The relation R is an epimorphism in Rel.
2. The inverse image function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to R is injective.

3. The coinverse image function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

associated to R is injective.

4. The inverse image functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

associated to R is full.

[26] Items 2 to 5 unwind respectively to the following statements:

- For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) = R^{-1}(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) = R_{-1}(V)$, then $U = V$.
- For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

5. The coinverse image functor
$$R_{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$
associated to R is full.

6. For each pair of relations $S, T \colon B \rightrightarrows X$, the following condition is satisfied:
   (★) If $S \diamond R \subset T \diamond R$, then $S \subset T$.

7. There exists an injective function $f \colon B \hookrightarrow A$ satisfying the following conditions:[27]
   (a) We have $\mathrm{Gr}(f) \subset R^{\dagger}$.[28]
   (b) The diagram
   $$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, hook, "\chi_B"'] & A \arrow[d, "R"] \\ & \mathcal{P}(B) \end{tikzcd}$$
   commutes.[29]

8. We have
$$R_* \circ R^{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "R^{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "R_*"] \\ & \mathcal{P}(B). \end{tikzcd}$$
   In other words, we have
$$U = \underbrace{\left\{ b \in B \,\middle|\, R^{-1}(b) \subset R^{-1}(U) \right\}}_{=R_*(R^{-1}(U))}$$
   for each $U \in \mathcal{P}(B)$.

- For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

[27] We are assuming the axiom of choice for this item (Item 7).
[28] In other words, for each $b \in B$, we have $f(b) \in R^{-1}(b)$.
[29] In other words, for each $b \in B$, we have $R(f(b)) = \{b\}$.

9. We have

$$R_! \circ R_{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "R_{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "R_!"] \\ & \mathcal{P}(B). \end{tikzcd}$$

In other words, we have

$$U = \underbrace{\left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in R^{-1}(b) \\ \text{such that we have } R(a) \subset U \end{array} \right\}}_{=R_!(R_{-1}(U))}$$

for each $U \in \mathcal{P}(B)$.

Proof. We will prove this by showing:

- Step 1: Item 2 $\iff$ Item 3 and Item 3 $\iff$ Item 5.
- Step 2: Item 1 $\iff$ Item 2.
- Step 3: Item 2 $\iff$ Item 4.
- Step 4: Item 4 $\iff$ Item 6.
- Step 5: Item 1 $\iff$ Item 7.
- Step 6: Item 1 $\iff$ Item 8.
- Step 7: Item 1 $\iff$ Item 9.

Step 1: Item 2 $\iff$ Item 3 and Item 3 $\iff$ Item 5: This follows from Item 7 of Definition 8.7.3.1.3.
Step 2: First Proof of Item 1 $\iff$ Item 2: We claim that Items 1 and 2 are equivalent:

- *Item 1 $\implies$ Item 2*: Let $U, V \in \mathcal{P}(B)$ and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

By Definition 8.7.1.1.3, we have

$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$

Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\implies$ *Item 1*: Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

for each $x \in X$. Thus $S = T$ and R is an epimorphism.

Step 2.5: Second Proof of Item 1 $\iff$ *Item 2*: A more abstract proof can also be given, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2*: Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathrm{op}} \to \mathrm{Sets}$ maps R to R^{-1} by Definition 8.7.3.1.2.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by Limits and Colimits, **??** of **??**, it follows by **??**, **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves epimorphisms.

- That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends epimorphisms in $\mathrm{Rel}^{\mathrm{op}}$ to epimorphisms in Sets.
- The epimorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??**, **??** of **??**.
- Since R is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is an epimorphism.
- Since the epimorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R^{-1} is injective.

- *Item 2* $\Longrightarrow$ *Item 1*: Assume that R^{-1} is injective.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathrm{op}} \to \mathrm{Sets}$ maps R to R^{-1} by Definition 8.7.3.1.2.
  - Since the epimorphisms in Sets are precisely the injections (**??**, **??** of **??**), it follows that R^{-1} is an epimorphism.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??**, **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects epimorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects epimorphisms in Sets to epimorphisms in $\mathrm{Rel}^{\mathrm{op}}$.
  - The epimorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??**, **??** of **??**.
  - Since R^{-1} is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

Step 3: Item 2 $\Longleftrightarrow$ *Item 4*: We claim that Items 2 and 4 are equivalent:

- *Item 2* $\Longrightarrow$ *Item 4*: We proceed in a few steps:
  - Let $U, V \in \mathcal{P}(B)$ such that $R^{-1}(U) \subset R^{-1}(V)$, assume R^{-1} to be injective, and consider the set $U \cup V$.
  - Since $R^{-1}(U) \subset R^{-1}(V)$, we have

    $$\begin{aligned} R^{-1}(U \cup V) &= R^{-1}(U) \cup R^{-1}(V) \\ &= R^{-1}(V), \end{aligned}$$

    where we have used Item 5 of Definition 8.7.3.1.3 for the first equality.

- Since R^{-1} is injective, this implies $U \cup V = V$.
- Thus $U \subset V$, as we wished to show.

- *Item 2* $\implies$ *Item 4*: We proceed in a few steps:
  - Suppose Item 4 holds, and let $U, V \in \mathcal{P}(B)$ such that $R^{-1}(U) = R^{-1}(V)$.
  - Since $R^{-1}(U) = R^{-1}(V)$, we have $R^{-1}(U) \subset R^{-1}(V)$ and $R^{-1}(V) \subset R^{-1}(U)$.
  - By assumption, this implies $U \subset V$ and $V \subset U$.
  - Thus $U = V$, showing R^{-1} to be injective.

Step 4: Item 4 $\iff$ *Item 6*: We claim that Items 4 and 6 are equivalent:

- *Item 4* $\implies$ *Item 6*: Let $U, V \in \mathcal{P}(B)$ and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

  By Definition 8.7.3.1.2, we have

$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$

  Now, if $U \diamond R \subset V \diamond R$, then $R^{-1}(U) \subset R^{-1}(V)$. By assumption, we then have $U \subset V$.

- *Item 6* $\implies$ *Item 4*: Consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

  and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\begin{tikzcd} A \arrow[r, "R", "\mid" marking] & B \arrow[r, shift left, "S", "\mid" marking] \arrow[r, shift right, "T"', "\mid" marking] & X \arrow[r, "{[x]}", "\mid" marking] & \mathrm{pt}, \end{tikzcd}$$

for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

for each $x \in X$, implying $S \subset T$.

This finishes the proof.
Step 5: Item 1 $\iff$ Item 7: We defer this proof to Definition 8.5.11.1.4.
Step 6: Item 1 $\iff$ Item 8: We defer this proof to Definition 8.5.11.1.4.
Step 6: Item 1 $\iff$ Item 9: We defer this proof to Definition 8.5.11.1.5. □

Proposition 8.5.11.1.3. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is an epimorphism in Rel.
2. For each $b \in B$ and each $U \in \mathcal{P}(B)$, if $R^{-1}(b) \subset R^{-1}(U)$, then $b \in U$.
3. For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is an epimorphism, then it is surjective, and the converse holds if R is functional.

Proof. We will prove this by showing:

- Step 1: Item 1 $\implies$ Item 2.
- Step 2: Item 2 $\implies$ Item 3.
- Step 3: Item 3 $\implies$ Item 1.
- Step 4: The second half of the statement of Definition 8.5.11.1.2.

Step 1: Item 1 $\implies$ Item 2: We proceed in a few steps:

- If R is an epimorphism, then, by Item 2 of Definition 8.5.11.1.2, the functor

$$R^{-1}\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

is full.

- As a result, given $b \in B$ and $U \in \mathcal{P}(B)$ such that $R^{-1}(b) \subset R^{-1}(U)$, it follows that $\{b\} \subset U$.
- Thus, we have $b \in U$.

Step 2: Item 2 $\implies$ *Item 3*: We proceed in a few steps:

- Let $b \in B$ and consider the subset $U = B \setminus \{b\}$.
- Since $b \notin U$, we have $R^{-1}(b) \not\subset R^{-1}(U)$ by the contrapositve of Item 2.
- As a result, there must exist some $a \in R^{-1}(b)$ with $a \notin R^{-1}(U)$.
- In particular, we have $b \in R(a)$.
- Moreover, the condition $a \notin R^{-1}(U) = R^{-1}(B \setminus \{b\})$ means that, if $b' \in B \setminus \{b\}$, then $b' \notin R(a)$.
- Thus $R(a) = \{b\}$.

Step 3: Item 3 $\implies$ *Item 1*: We proceed in a few steps:

- By the equivalence between Items 1 and 4 of Definition 8.5.11.1.2, to show that R is an epimorphism it suffices to prove that, for each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.
- So let $u \in U$ and assume $R^{-1}(U) \subset R^{-1}(V)$.
- By assumption, there exists some $a \in A$ with $R(a) = \{u\}$.
- In particular, $a \in R^{-1}(U)$.
- Since $R^{-1}(U) \subset R^{-1}(V)$, we also have $a \in R^{-1}(V)$.
- Thus, there exists some $v \in V$ with $a \in R(v)$.
- However, $R(a) = \{u\}$, so we must in fact have $v = u$.
- Therefore $u \in V$, showing that $U \subset V$.

Step 4: Proof of the Second Half of Definition 8.5.11.1.3: We claim that R being an epimorphism implies surjectivity, and the converse holds if R is functional:

- *If R Is an Epimorphism, Then R Is Surjective:* Consider the diagram

$$A \xrightarrow{R} B \overset{S}{\underset{T}{\rightrightarrows}} \{0, 1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$.

  - We claim that $S \diamond R = T \diamond R$:
    - If $R(a) = \emptyset$, then

$$\begin{aligned} [S \diamond R](a) &= \emptyset \\ [T \diamond R](a) &= \emptyset \end{aligned}$$

      by the definition of relational composition, so $[S \diamond R](a) = [T \diamond R](a)$.
    - If $R(a) \neq \emptyset$, then we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ by the definition of S and T, with no element of A being related to 1 by $S \diamond R$ or $T \diamond R$.
  - Now, since R is an epimorphism, we have $S = T$.
  - However, by the definition of T, this implies $\mathrm{Im}(R) = B$.
  - Thus R is surjective.

- *If R Is Functional and Surjective, Then R Is an Epimorphism:* Let $U, V \in \mathcal{P}(B)$, consider the diagram

$$A \xrightarrow{R} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt},$$

where $R_{-1}(U) = R_{-1}(V)$, and let $b \in U$.

  - By surjectivity, there exists some $a \in A$ such that $a \in R^{-1}(b)$.

- Since $R_{-1}(U) = R_{-1}(V)$, if $R(a) \subset U$, then $R(a) \subset V$.
- Since R is functional, we have $R(a) = \{b\}$, so $R(a) \subset U$.
- Thus, $R(a) \subset V$, and $b \in V$.
- A similar argument shows that if $b \in V$, then $b \in U$.
- Thus $U = V$, showing R_{-1} to be injective.
- By the equivalence between Items 1 and 3 of Definition 8.5.11.1.2, this shows R to be an epimorphism.

This finishes the proof. □

Corollary 8.5.11.1.4. Items 1, 7 and 8 of Definition 8.5.11.1.2 are indeed equivalent.

Proof. *Item 1* $\iff$ *Item 7*: We claim that Item 3 of Definition 8.5.11.1.3 is equivalent to Item 7 of Definition 8.5.11.1.2:

- *Item 3 of Definition 8.5.11.1.3* $\implies$ *Item 7*: By assumption, given $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$. Invoking the axiom of choice, we may pick one such a for each $b \in B$, giving us our desired function $f : B \to A$. All the requirements listed in Item 7 then follow by construction.

- *Item 7* $\implies$ *Item 3 of Definition 8.5.11.1.3*: Given $b \in B$, we may pick $a = f(b)$, in which case $R(f(b))$ will be equal to $\{b\}$ by assumption.

By Definition 8.5.11.1.3, Item 3 of Definition 8.5.11.1.3 is equivalent to Item 1 of Definition 8.5.11.1.3. Since Item 1 of Definition 8.5.11.1.3 is exactly the same condition as Item 1 of Definition 8.5.11.1.2, the result follows.
Item 1 $\iff$ *Item 8*: Indeed, we have

$$\begin{aligned} [R_* \circ R^{-1}](U) &\stackrel{\text{def}}{=} R_*(R^{-1}(U)) \\ &\stackrel{\text{def}}{=} \left\{b \in B \,\middle|\, R^{-1}(b) \subset R^{-1}(U)\right\} \end{aligned}$$

for each $U \in \mathcal{P}(B)$. As a result, the condition $R_* \circ R^{-1} = \mathrm{id}_{\mathcal{P}(B)}$ becomes

$$\left\{b \in B \,\middle|\, R^{-1}(b) \subset R^{-1}(U)\right\} = U,$$

which holds precisely when Item 2 of Definition 8.5.11.1.3 does. By Definition 8.5.11.1.3, that in turn holds precisely if Item 1 of Definition 8.5.11.1.3 holds. Since Item 1 of Definition 8.5.11.1.3 is exactly the same condition as Item 1 of Definition 8.5.11.1.2, the result follows. □

Corollary 8.5.11.1.5. Items 1 and 9 of Definition 8.5.11.1.2 are indeed equivalent.

Proof. *Item 1* $\implies$ *Item 9*: To show that R is an epimorphism, we will prove that for each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$. The will then follow from Item 3 of Definition 8.5.11.1.3.

- Let $b \in B$ and consider $U = \{b\}$.
- By assumption, we have $U = R_!(R_{-1}(U))$.
- In particular, this means that $b \in R_!(R_{-1}(\{b\}))$.
- Unwinding the definition, this means there exists some $a \in R^{-1}(b)$ such that $R(a) \subset \{b\}$.
  - The condition $a \in R^{-1}(b)$ implies that $b \in R(a)$.
  - The condition $R(a) \subset \{b\}$ implies that every element of $R(a)$ must be b.
- For $R(a)$ to be a non-empty subset of $\{b\}$, it must be the case that $R(a) = \{b\}$.

This completes the proof.
Item 9 $\implies$ *Item 1*: We wish to show that for any $U \in \mathcal{P}(B)$, we have $U = R_!(R_{-1}(U))$. This requires proving two set inclusions.

- *$R_!(R_{-1}(U)) \subset U$:* Let $b \in R_!(R_{-1}(U))$.
  - By definition, there exists an $a \in A$ such that $a \in R^{-1}(b)$ and $R(a) \subset U$.
  - The condition $a \in R^{-1}(b)$ means that $b \in R(a)$.
  - Since $b \in R(a)$ and $R(a) \subset U$, it follows directly that $b \in U$.
  - Therefore, $R_!(R_{-1}(U)) \subset U$.
- *$U \subset R_!(R_{-1}(U))$:* Let $b \in U$.
  - By Item 3 of Definition 8.5.11.1.3, there exists an element $a \in A$ such that $R(a) = \{b\}$.
  - We must verify that this choice of a places b into the set $R_!(R_{-1}(U))$. This requires checking two conditions:

      * $a \in R^{-1}(b)$: Since $R(a) = \{b\}$, we have $b \in R(a)$, which is equivalent to $a \in R^{-1}(b)$.
      * $R(a) \subset U$: Since $R(a) = \{b\}$ and we assumed $b \in U$, we have $\{b\} \subset U$, so the condition holds.
   - As both conditions are met, it follows that $b \in R_!(R_{-1}(U))$.
   - Therefore, $U \subset R_!(R_{-1}(U))$.

As both inclusions hold, we conclude that $U = R_!(R_{-1}(U))$, which is precisely the statement of Item 9. □

Warning 8.5.11.1.6. The following conditions are equivalent and imply R is an epimorphism, but the converse may fail. Thus they are ***not*** equivalent to R being an epimorphism:

1. The relation R is a surjective partial function.

2. The diagram

$$\begin{tikzcd} B \arrow[r, "R^{-1}"] \arrow[dr, hook, "\chi_B"'] & \mathcal{P}(A) \arrow[d, "R_!"] \\ & \mathcal{P}(B) \end{tikzcd}$$

   commutes. In other words, we have

$$R_!(R^{-1}(b)) = \{b\}$$

   for each $b \in B$.

3. We have

$$R_! \circ R^{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "R^{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "R_!"] \\ & \mathcal{P}(B). \end{tikzcd}$$

   In other words, we have

$$U = \underbrace{\left\{b \in B \,\middle|\, R^{-1}(b) \cap R^{-1}(U) \neq \emptyset\right\}}_{=R_!(R^{-1}(U))}$$

   for each $U \in \mathcal{P}(B)$.

4. We have

$$R_* \circ R_{-1} = \mathrm{id}_{\mathcal{P}(B)} \qquad \begin{tikzcd} \mathcal{P}(B) \arrow[r, "R_{-1}"] \arrow[dr, equal] & \mathcal{P}(A) \arrow[d, "R_*"] \\ & \mathcal{P}(B). \end{tikzcd}$$

In other words, we have

$$U = \underbrace{\left\{b \in B \,\middle|\, R(R^{-1}(b)) \subset U\right\}}_{=R_*(R_{-1}(U))}$$

for each $U \in \mathcal{P}(B)$.

Proof. First, note that the relation depicted in Definition 8.5.11.1.1 is not a surjective partial function, but it is an epimorphism in Rel by Definition 8.5.11.1.3, the next proposition. Moreover, partial surjective functions are epimorphisms by Definition 8.5.11.1.3. For the rest of the proposition, we proceed by showing:

- Step 1: Item 1 $\Longleftrightarrow$ Item 2.
- Step 2: Item 2 $\Longrightarrow$ Item 3.
- Step 3: Item 3 $\Longrightarrow$ Item 2.
- Step 4: Item 2 $\Longrightarrow$ Item 4.
- Step 5: Item 4 $\Longrightarrow$ Item 1.

Step 1: Item 1 $\Longleftrightarrow$ *Item 2*: Note that we have

$$R_!(R^{-1}(b)) \overset{\text{def}}{=} \left\{b' \in B \,\middle|\, R^{-1}(b') \cap R^{-1}(b) \neq \emptyset\right\}.$$

We now claim Items 1 and 2 are equivalent:

- *Item 1* $\Longrightarrow$ *Item 2*: We proceed in a few steps:
  - Since R is functional, $R^{-1}(b)$ has at most one element.
  - Since R is surjective, $R^{-1}(b)$ has at least one element.
  - Thus, $R^{-1}(b)$ is a singleton.

  - The set $R(R^{-1}(b))$ will then be precisely $\{b\}$.

- *Item 2* $\implies$ *Item 1*: We claim R is functional and surjective.

  - *Functionality*. The inclusion

$$R_!(R^{-1}(b)) \subset \{b\}$$

    implies that if $a \in R(b')$ and $a \in R(b)$, then $b = b'$. Thus R must be functional.

  - *Surjectivity*. The inclusion

$$\{b\} \subset R_!(R^{-1}(b))$$

    implies $R^{-1}(b) \neq \emptyset$, so R must be surjective.

  Since R is functional and surjective, it is a surjective partial function.

Step 2: Item 2 $\implies$ *Item 3*: We have

$$\begin{aligned}
[R_! \circ R^{-1}](U) &\stackrel{\text{def}}{=} R_!(R^{-1}(U)) \\
&= R_!\left(R^{-1}\left(\bigcup_{u \in U}\{u\}\right)\right) \\
&= R_!\left(\bigcup_{u \in U} R^{-1}(\{u\})\right) \\
&= \bigcup_{u \in U} R_!(R^{-1}(\{u\})) \\
&= \bigcup_{u \in U} \{u\} \\
&= U
\end{aligned}$$

for each $U \in \mathcal{P}(B)$, where we have used:

- **??** of Definition 8.7.3.1.3 for the third equality.

- **??** of Definition 8.7.1.1.4 for the fourth equality.

- Item 2 of this proposition for the fifth equality.

Step 3: Item 3 $\implies$ *Item 2*: Taking $U = \{b\}$ gives $R_!(R^{-1}(b)) = \{b\}$.
Step 4: Item 2 $\implies$ *Item 4*: We have

$$\begin{aligned} R_*(R_{-1}(U)) &= \left\{b \in B \,\middle|\, R(R^{-1}(b)) \subset U\right\} \\ &= \{b \in B \mid \{b\} \subset U\} \\ &= U. \end{aligned}$$

Step 5: Item 4 $\implies$ *Item 1*: Suppose that for each $U \in \mathcal{P}(B)$, we have $R_*(R_{-1}(U)) = U$. We must show that R is functional and surjective.

- *Functionality:* We show that if $b, b' \in R(a)$, then $b = b'$.
  - Consider the singleton set $U = \{b\}$. By the assumed identity, we have
    $$\{b\} = \left\{b \in B \,\middle|\, R(R^{-1}(b)) \subset \{b\}\right\}.$$
  - Since b is an element of the set on the left-hand side, it must satisfy the membership condition on the right-hand side. Thus, we have $R(R^{-1}(b)) \subset \{b\}$.
  - By assumption, $b \in R(a)$, which implies $a \in R^{-1}(b)$.
  - By assumption, we also have $b' \in R(a)$.
  - Since $a \in R^{-1}(b)$, it follows that the image of a is contained in the image of the set $R^{-1}(b)$, i.e., $R(a) \subset R(R^{-1}(b))$.
  - Combining these steps, we have $b' \in R(a) \subset R(R^{-1}(b))$.
  - As we established that $R(R^{-1}(b)) \subset \{b\}$, we must have $b' \in \{b\}$.
  - Therefore, $b' = b$, which shows R to be functional.
- *Surjectivity:* We show that for each $b \in B$, the preimage set $R^{-1}(b)$ is non-empty.
  - Consider the empty set $U = \emptyset$. By the assumed identity, we have
    $$\emptyset = \left\{b \in B \,\middle|\, R(R^{-1}(b)) \subset \emptyset\right\}.$$
  - The identity thus states that there is no element $b \in B$ for which $R(R^{-1}(b))$ is the empty set.
  - In other words, for each $b \in B$, we must have $R(R^{-1}(b)) \neq \emptyset$.
  - The image of a set $R(S)$ is empty iff the set S is empty.

- Therefore, the condition $R(R^{-1}(b)) \neq \emptyset$ is equivalent to the condition $R^{-1}(b) \neq \emptyset$.
- Thus, R is surjective.

Since R is both functional and surjective, it is a surjective partial function. This finishes the proof. □

Remark 8.5.11.1.7. Taking the contrapositive of Item 2 of Definition 8.5.11.1.3 and letting $U = \{b'\}$ shows that the subset

$$\left\{R^{-1}(b) \in \mathcal{P}(A) \,\middle|\, b \in B\right\}$$

of $\mathcal{P}(A)$ forms an antichain in $\mathcal{P}(A)$. The converse however, fails.

Proposition 8.5.11.1.8. Every 1-morphism in **Rel** is corepresentably faithful.

Proof. A relation $R\colon A \nrightarrow B$ will be corepresentably faithful in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

given by precomposition by R is faithful. This happens iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$.

However, since **Rel** is locally posetal, the Hom-set $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, As a result, the map $R^*_{S,T}$ will necessarily be injective in either of these cases. □

Proposition 8.5.11.1.9. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The morphism $R\colon A \nrightarrow B$ is an epimorphism in Rel.
2. The 1-morphism $R\colon A \nrightarrow B$ is corepresentably full in **Rel**.
3. The 1-morphism $R\colon A \nrightarrow B$ is corepresentably fully faithful in **Rel**.
4. The 1-morphism $R\colon A \nrightarrow B$ is pseudoepic in **Rel**.
5. The 1-morphism $R\colon A \nrightarrow B$ is corepresentably essentially injective in **Rel**.

6. The 1-morphism $R\colon A \nrightarrow B$ is corepresentably conservative in **Rel**.

Proof. We will prove this by showing:

- Step 1: Item 1 $\iff$ Item 2.
- Step 2: Item 2 $\iff$ Item 3.
- Step 3: Item 3 $\iff$ Item 4 $\iff$ Item 5 $\iff$ Item 6.

Step 1: Item 1 $\iff$ Item 2: The condition that R is representably full corresponds precisely to Item 6 of Definition 8.5.11.1.2, so this follows by Definition 8.5.11.1.2.
Step 2: Item 2 $\iff$ Item 3: This follows from Step 1 and Definition 8.5.11.1.8.
Step 3: Item 3 $\iff$ Item 4 $\iff$ Item 5 $\iff$ Item 6: Since **Rel** is locally posetal, the conditions in Items 4 to 6 all collapse to the one of Item 3. □

8.5.12 Co/Limits

Proposition 8.5.12.1.1. This will be properly written later on.

Proof. Omitted. □

8.5.13 Internal Left Kan Extensions

Proposition 8.5.13.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Internal Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Internal Left Kan Extensions Along Them*. The following conditions are equivalent:

   (a) The left Kan extension
   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$
   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 8.2.2.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Internal Left Kan Extensions in* **Rel**: By Item 2, it suffices to take a relation that doesn't have a left adjoint.
Item 2, Characterisation of Relations Admitting Left Kan Extensions Along Them: This proof is mostly due to Tim Campion, via [`MO 460693`].

- We may view precomposition
$$- \diamond R \colon \mathrm{Rel}(B, C) \to \mathrm{Rel}(A, C)$$
with $R \colon A \nrightarrow B$ as a cocontinuous functor from $\mathcal{P}(B \times C)$ to $\mathcal{P}(A \times C)$ (via Item 5 of Definition 8.1.1.1.1).

- By the adjoint functor theorem (**??**), this map has a left adjoint iff it preserves limits.

- If $C = \emptyset$, this holds trivially.

- Otherwise, C admits pt as a retract, and we reduce to the case $C = \mathrm{pt}$ via **??**.

- For the case $C = \mathrm{pt}$, a relation $T \colon B \nrightarrow \mathrm{pt}$ is the same as a subset of B, and $- \diamond R$ becomes the inverse image functor R^{-1} of Section 8.7.3.

- Now, again by the adjoint functor theorem, R^{-1} preserves limits exactly when it has a left adjoint.

- Finally R^{-1} has a left adjoint precisely when $R = \mathrm{Gr}(f)$ for f a function (Item 8 of Definition 8.7.3.1.3).

This finishes the proof. □

Example 8.5.13.1.2. Given a function $f \colon A \to B$, the left Kan extension
$$\mathrm{Lan}_f \colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$
along f exists by Item 2 of Definition 8.5.13.1.1. Explicitly, given a relation $R \colon A \nrightarrow X$, the left Kan extension

$$\mathrm{Lan}_f(R) \colon B \nrightarrow X, \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Lan}_f(R)", "|"] \\ A \arrow[ur, "f", "/"] \arrow[r, "R"', "|"] \arrow[ur, phantom, shift left=3ex, "\Nearrow"] & X \end{tikzcd}$$

may be described as follows:

1. We declare $b \sim_{\mathrm{Lan}_f(R)} x$ iff there exists some $a \in R$ such that $b = f(a)$ and $a \sim_R x$.

2. We have[30]

$$[\mathrm{Lan}_f(R)](b) = \bigcup_{a \in f^{-1}(b)} R(a)$$

for each $b \in B$.

Remark 8.5.13.1.3. Following Definition 8.5.13.1.2, given a relation $R\colon A \nrightarrow B$ and a relation $F\colon A \nrightarrow X$, we could perhaps try to define an "honorary" left Kan extension

$$\mathrm{Lan}'_R(F)\colon B \nrightarrow X$$

by

$$[\mathrm{Lan}'_F(F)](b) \overset{\mathrm{def}}{=} \bigcup_{a \in R^{-1}(b)} F(a)$$

for each $b \in B$.

The failure of $\mathrm{Lan}'_R(F)$ to be a Kan extension can then be seen as follows. Let $G\colon B \nrightarrow X$ be a relation. If $\mathrm{Lan}'_R(F)$ were a left Kan extension, then the following conditions ***would be*** equivalent:

1. For each $b \in B$, we have $\bigcup_{a \in R^{-1}(b)} F(a) \subset G(b)$.

2. For each $a \in A$, we have $F(a) \subset \bigcup_{b \in R(a)} G(b)$.

The issue is two-fold:

- *Totality.* If R isn't total, then the implication Item 1 $\Longrightarrow$ Item 2 fails.

- *Functionality.* If R isn't functional, then the implication Item 2 $\Longrightarrow$ Item 1 fails.

Question 8.5.13.1.4. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension[31]

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

[30] Cf. Item 3 of Definition 8.5.15.1.2.

[31] Specifically for R and S, not Lan_S the functor.

8.5.14 Internal Left Kan Lifts

Proposition 8.5.14.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Internal Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Internal Left Kan Lifts Along Them.* The following conditions are equivalent:
   (a) The left Kan lift
   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$
   along R exists.
   (b) The relation R admits a right adjoint in **Rel**.
   (c) The relation R is of the form f^{-1} (as in Definition 8.2.3.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Internal Left Kan Lifts in* **Rel**: By Item 2, it suffices to take a relation that doesn't have a right adjoint.
Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them: This proof is dual to that of Item 2 of Definition 8.5.13.1.1, and is therefore omitted. □

Example 8.5.14.1.2. Given a function $f\colon A \to B$, the left Kan lift

$$\mathrm{Lift}_{f^\dagger}\colon \mathbf{Rel}(X,A) \to \mathbf{Rel}(X,B)$$

along $f^\dagger$ exists by Item 2 of Definition 8.5.14.1.1. Explicitly, given a relation $R\colon X \nrightarrow A$, the left Kan lift

$$\mathrm{Lift}_{f^\dagger}(R)\colon X \nrightarrow B, \qquad \begin{tikzcd} & B \arrow[d, "f^\dagger"] \\ X \arrow[ru, dashed, "\mathrm{Lift}_{f^\dagger}(R)"] \arrow[r, "R"'] \arrow[ru, phantom, bend right, "\Uparrow"] & A. \end{tikzcd}$$

is given by

$$\begin{aligned} [\mathrm{Lift}_f(R)](x) &= [\mathrm{Gr}(f) \diamond R](a) \\ &= \bigcup_{a \in R(x)} f(a) \end{aligned}$$

for each $x \in X$.

Question 8.5.14.1.3. Given relations $S\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} X$ and $R\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} B$, is there a characterisation of when the left Kan lift[32]

$$\mathrm{Lift}_S(R)\colon X \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} A$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

8.5.15 Internal Right Kan Extensions

Let A, B, and X be sets and let $R\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} B$ and $F\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} X$ be relations.

Motivation 8.5.15.1.1. We want to understand internal right Kan extensions in **Rel**, which look like this:

$$\begin{tikzcd}
 & B \arrow[d, dashed, "\shortmid" marking, "\mathrm{Ran}_R(F)"] \arrow[dl, phantom, "\Swarrow"] \\
A \arrow[ur, "R", "\shortmid" marking] \arrow[r, "F"', "\shortmid" marking] & X.
\end{tikzcd}$$

Note in particular here that $F\colon A \mathrel{\mkern3mu\mapstochar\mkern-9mu\rightarrow} X$ is a relation from A to X. These will form a functor

$$\mathrm{Ran}_R\colon \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X)$$

that is right adjoint to the precomposition by R functor

$$R^*\colon \mathbf{Rel}(B,X) \to \mathbf{Rel}(A,X).$$

Proposition 8.5.15.1.2. The internal right Kan extension of F along R is the relation $\mathrm{Ran}_R(F)$ described as follows:

1. Viewing relations from B to X as subsets of $B \times X$, we have

$$\mathrm{Ran}_R(F) = \left\{ (b,x) \in B \times X \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if } a \sim_R b, \\ \text{then we have } a \sim_F x \end{array} \right\}.$$

[32] Specifically for R and S, not Lift_S the functor.

2. Viewing relations as functions $B \times X \longrightarrow \{\mathsf{true}, \mathsf{false}\}$, we have

$$\begin{aligned}(\mathrm{Ran}_R(F))^{-1}_{-2} &= \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-2}_a, F^{-1}_a) \\ &= \bigwedge_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-2}_a, F^{-1}_a),\end{aligned}$$

where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

3. Viewing relations as functions $B \to \mathcal{P}(X)$, we have

$$\mathrm{Ran}_R(F) = \mathrm{Ran}_{\chi'_A}(F) \circ R^{-1}, \qquad \begin{tikzcd} & A \arrow[r, "F"] \arrow[d, hook, "\chi_A"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{P}(X), \\ B \arrow[r, "R^{-1}"'] & \mathcal{P}(A)^{\mathsf{op}} \arrow[ur, "\mathrm{Ran}_{\chi_A}(F)"'] & \end{tikzcd}$$

where $\mathrm{Ran}_{\chi'_B}(F)$ is computed by the formula

$$\begin{aligned}[\mathrm{Ran}_{\chi'_A}(F)](V) &\cong \int_{a \in A} \chi_{\mathcal{P}(A)^{\mathsf{op}}}(V, \chi_a) \pitchfork F(a) \\ &\cong \int_{a \in A} \chi_{\mathcal{P}(A)}(\chi_a, V) \pitchfork F(a) \\ &\cong \int_{a \in A} \chi_V(a) \pitchfork F(a) \\ &\cong \bigcap_{a \in A} \chi_V(a) \pitchfork F(a) \\ &\cong \bigcap_{a \in V} F(a)\end{aligned}$$

for each $V \in \mathcal{P}(B)$, so we have

$$[\mathrm{Ran}_R(F)](b) = \bigcap_{a \in R^{-1}(b)} F(a)$$

for each $b \in B$.

Proof. We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,X)}(F \diamond R, T) \cong \int_{a \in A} \int_{x \in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((F \diamond R)^x_a, T^x_a)$$

$$
\begin{aligned}
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} F^x_b \times R^b_a\right), T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(F^x_b \times R^b_a, T^x_a)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(F^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(F^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^x_b, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(F, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-_2}_a, T^{-_1}_a)\right)
\end{aligned}
$$

naturally in each $F \in \mathbf{Rel}(B, X)$ and each $T \in \mathbf{Rel}(A, X)$, showing that

$$\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-_2}_a, T^{-_1}_a)$$

is right adjoint to the precomposition functor $- \diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Definition 8.1.1.1.5.
2. Definition 8.1.3.1.1.
3. Ends and Coends, **??** of **??**.
4. Sets, Definition 3.2.2.1.5.
5. Ends and Coends, **??** of **??**.
6. Ends and Coends, **??** of **??**.
7. Relations, Item 1 of Definition 8.1.1.1.5.

This finishes the proof. □

Example 8.5.15.1.3. Here are some examples of internal right Kan extensions of relations.

1. *Orthogonal Complements.* Let $A = B = X = \mathcal{V}$ be an inner product space,

and let $R = F = \perp$ be the orthogonality relation, so that we have

$$\begin{aligned} R(v) &= v^{\perp} \\ F(u) &= u^{\perp}, \end{aligned}$$

for each $u, v \in \mathcal{V}$, where

$$v^{\perp} \overset{\text{def}}{=} \{u \in V \mid v \perp u\}$$

is the orthogonal complement of v. The right Kan extension $\mathrm{Ran}_R(F)$ is then given by

$$\begin{aligned} [\mathrm{Ran}_R(F)](v) &= \bigcap_{u \in R^{-1}(v)} F(u) \\ &= \bigcap_{\substack{u \in V \\ u \perp v}} u^{\perp} \\ &= (v^{\perp})^{\perp}, \end{aligned}$$

the double orthogonal complement. In particular:

- If $\mathcal{V}$ is finite-dimensional, then $[\mathrm{Ran}_R(F)](v) = \mathrm{Span}(v)$.
- If $\mathcal{V}$ is a Hilbert space, then $[\mathrm{Ran}_R(F)](v) = \overline{\mathrm{Span(v)}}$.

2. *Galois Connections and Closure Operators.* Let:
   - $B = X = (P, \preceq_P)$ and $A = (Q, \preceq_Q)$ be posets;
   - (f, g) be a Galois connection (adjunction) between P and Q;
   - $R, F \colon Q \nrightarrow P$ be the relations defined by

   $$\begin{aligned} R(q) &\overset{\text{def}}{=} \left\{p \in P \,\middle|\, q \preceq_Q f(p)\right\}, \\ F(q) &\overset{\text{def}}{=} \{p \in P \mid p \preceq_P g(q)\} \end{aligned}$$

   for each $q \in Q$.

   We have

   $$[\mathrm{Ran}_R(F)](p) = \bigcap_{q \in R^{-1}(p)} F(q)$$

$$
\begin{aligned}
&= \bigcap_{\substack{q \in Q \\ q \preceq_Q f(p)}} \{p \in P \mid p \preceq_P g(q)\} \\
&= \{p \in P \mid p \preceq_P g(f(q))\} \\
&= \downarrow g(f(p)),
\end{aligned}
$$

the down set of $g(f(p))$. In other words, $\mathrm{Ran}_R(F)$ is the closure operator on P associated with the Galois connection (f, g).

Proposition 8.5.15.1.4. Let A, B, C and X be sets and let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $F\colon A \nrightarrow X$ be relations.

1. *Functoriality.* The assignments $R, F, (R, F) \mapsto \mathrm{Ran}_R(F)$ define functors

$$
\begin{aligned}
\mathrm{Ran}_{(-)}(F)\colon \quad & \mathbf{Rel}(A, B)^{\mathsf{op}} \to \mathbf{Rel}(B, X), \\
\mathrm{Ran}_R\colon \quad & \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X), \\
\mathrm{Ran}_{(-_1)}(-_2)\colon \quad & \mathbf{Rel}(A, X) \times \mathbf{Rel}(A, B)^{\mathsf{op}} \to \mathbf{Rel}(B, X).
\end{aligned}
$$

In other words, given relations

$$
\begin{tikzcd}
A \arrow[r, shift left, "R_1"] \arrow[r, shift right, "R_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, shift left, "F_1"] \arrow[r, shift right, "F_2"'] & X,
\end{tikzcd}
$$

if $R_1 \subset R_2$ and $F_1 \subset F_2$, then $\mathrm{Ran}_{R_2}(F_1) \subset \mathrm{Ran}_{R_1}(F_2)$.

2. *Interaction With Composition.* We have

$$
\mathrm{Ran}_{S \diamond R}(F) = \mathrm{Ran}_S(\mathrm{Ran}_R(F))
$$

and an equality

$$
\begin{tikzcd}
 & & C \arrow[dd, dashed, "\mathrm{Ran}_S(\mathrm{Ran}_R(F))"] \\
 & B \arrow[ur, "S"] \arrow[dr, dashed, "\mathrm{Ran}_R(F)"'] \arrow[d, phantom, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] & {} \\
A \arrow[ur, "R"] \arrow[rr, "F"'] & {} & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & C \arrow[dd, dashed, "\mathrm{Ran}_{S \diamond R}(F)"] \\
 & B \arrow[ur, "S"] \arrow[d, phantom, "\Downarrow"] & \\
A \arrow[ur, "R"] \arrow[rr, "F"'] & {} & X
\end{tikzcd}
$$

of pasting diagrams in **Rel**.

3. *Interaction With Converses.* We have

$$\mathrm{Ran}_R(F)^{\dagger} = \mathrm{Rift}_{R^{\dagger}}(F^{\dagger}).$$

4. *Interaction With Inverse Images.* We have

$$[\mathrm{Ran}_R(F)]^{-1}(x) = \left\{b \in B \,\middle|\, R^{-1}(b) \subset F^{-1}(x)\right\}$$

for each $x \in X$.

Proof. *Item 1, Functoriality*: We have

$$\begin{aligned}[\mathrm{Ran}_{R_2}(F_1)](b) &= \bigcap_{a \in R_2^{-1}(b)} F_1(a)\\ &\subset \bigcap_{a \in R_1^{-1}(b)} F_1(a)\\ &\subset \bigcap_{a \in R_1^{-1}(b)} F_2(a)\\ &= [\mathrm{Ran}_{R_1}(F_2)](b)\end{aligned}$$

for each $b \in B$, so we therefore have $\mathrm{Ran}_{R_2}(F_1) \subset \mathrm{Ran}_{R_1}(F_2)$.
Item 2, Interaction With Composition: This holds in a general bicategory with the necessary right Kan extensions, being therefore a special case of **??**.
Item 3, Interaction With Converses: We have

$$\begin{aligned}[\mathrm{Rift}_{R^{\dagger}}(F^{\dagger})](x) &= \left\{b \in B \,\middle|\, R^{\dagger}(b) \subset F^{\dagger}(x)\right\}\\ &= \left\{b \in B \,\middle|\, R^{-1}(b) \subset F^{-1}(x)\right\}\\ &= \mathrm{Ran}_R(F)^{-1}(x)\\ &= \mathrm{Ran}_R(F)^{\dagger}(x)\end{aligned}$$

where we have used Definition 8.5.16.1.2 and Item 4.
Item 4, Interaction With Inverse Images: We proceed in a few steps.

- We have $b \in [\mathrm{Ran}_R(F)]^{-1}(x)$ iff, for each $a \in R^{-1}(b)$, we have $b \in F(a)$.
- This holds iff, for each $a \in R^{-1}(b)$, we have $a \in F^{-1}(b)$.
- This holds iff $R^{-1}(b) \subset F^{-1}(b)$.

This finishes the proof. □

8.5.16 Internal Right Kan Lifts

Let A, B, and X be sets and let $R\colon A \nrightarrow B$ and $F\colon X \nrightarrow B$ be relations.

Motivation 8.5.16.1.1. We want to understand internal right Kan lifts in **Rel**, which look like this:

$$\begin{tikzcd} & A \arrow[d, "R"] \arrow[dl, phantom, "\Leftarrow"] \\ X \arrow[ur, dashed, "\mathrm{Rift}_R(F)"] \arrow[r, "F"'] & B. \end{tikzcd}$$

Note in particular here that $F\colon B \nrightarrow X$ is a relation from B to X. These will form a functor

$$\mathrm{Rift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$

that is right adjoint to the postcomposition by R functor

$$R_*\colon \mathbf{Rel}(X,A) \to \mathbf{Rel}(X,B).$$

Proposition 8.5.16.1.2. The internal right Kan lift of F along R is the relation $\mathrm{Rift}_R(F)$ described as follows:

1. Viewing relations from X to A as subsets of $X \times A$, we have

$$\mathrm{Rift}_R(F) = \left\{ (x,a) \in X \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B, \text{ if } a \sim_R b, \\ \text{then we have } x \sim_F b \end{array} \right\}.$$

2. Viewing relations as functions $X \times A \to \{\mathsf{true}, \mathsf{false}\}$, we have

$$\begin{aligned} (\mathrm{Rift}_R(F))^{-1}_{-2} &= \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-1}, F^b_{-2}) \\ &= \bigwedge_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-1}, F^b_{-2}), \end{aligned}$$

   where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Sets, Definition 3.2.2.1.3.

3. Viewing relations as functions $X \to \mathcal{P}(A)$, we have

$$[\mathrm{Rift}_R(F)](x) = \{a \in A \mid R(a) \subset F(x)\}$$

   for each $a \in A$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond F, T) &\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((R \diamond F)^b_x, T^b_x) \\
&\cong \int_{x \in X} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((\int^{a \in A} R^b_a \times F^a_x), T^b_x) \\
&\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a \times F^a_x, T^b_x) \\
&\cong \int_{x \in X} \int_{b \in B} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(F^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x)) \\
&\cong \int_{x \in X} \int_{a \in A} \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(F^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x)) \\
&\cong \int_{x \in X} \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(F^a_x, \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^b_x)) \\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}(F, \int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, T^b_{-_2}))
\end{aligned}$$

naturally in each $F \in \mathbf{Rel}(X, A)$ and each $T \in \mathbf{Rel}(X, B)$, showing that

$$\int_{b \in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, F^b_{-_2})$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Relations, Item 1 of Definition 8.1.1.1.5.
2. Definition 8.1.3.1.1.
3. Ends and Coends, **??** of **??**.
4. Sets, Definition 3.2.2.1.5.
5. Ends and Coends, **??** of **??**.
6. Ends and Coends, **??** of **??**.
7. Relations, Item 1 of Definition 8.1.1.1.5.

This finishes the proof. □

Example 8.5.16.1.3. Here are some examples of internal right Kan lifts of relations.

1. *Pullbacks.* Let $p\colon A \to B$ and $f\colon X \to B$ be functions. We have
$$\begin{aligned}[\mathrm{Rift}_{\mathrm{Gr}(p)}(\mathrm{Gr}(f))](x) &= \{a \in A \mid [\mathrm{Gr}(p)](a) \subset [\mathrm{Gr}(f)](x)\}\\ &= \{a \in A \mid p(a) = f(x)\}.\end{aligned}$$
Thus, as a subset of $X \times A$, the right Kan lift $\mathrm{Rift}_{\mathrm{Gr}(p)}(\mathrm{Gr}(f))$ corresponds precisely to the pullback $X \times_B A$ of X and A along p and f of Constructions With Sets, Section 4.1.4.

Proposition 8.5.16.1.4. Let A, B, C and X be sets and let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $F\colon X \nrightarrow B$ be relations.

1. *Functoriality.* The assignments $R, F, (R, F) \mapsto \mathrm{Rift}_R(F)$ define functors
$$\begin{aligned}\mathrm{Rift}_{(-)}(F)\colon &\qquad \mathbf{Rel}(A,B)^{\mathrm{op}} \to \mathbf{Rel}(B,X),\\ \mathrm{Rift}_R\colon &\qquad \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X),\\ \mathrm{Rift}_{(-_1)}(-_2)\colon &\ \mathbf{Rel}(A,X) \times \mathbf{Rel}(A,B)^{\mathrm{op}} \to \mathbf{Rel}(B,X).\end{aligned}$$
In other words, given relations
$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \qquad A \underset{F_2}{\overset{F_1}{\rightrightarrows}} X,$$
if $R_1 \subset R_2$ and $F_1 \subset F_2$, then $\mathrm{Rift}_{R_2}(F_1) \subset \mathrm{Rift}_{R_1}(F_2)$.

2. *Interaction With Composition.* We have
$$\mathrm{Rift}_{S\diamond R}(F) = \mathrm{Rift}_R(\mathrm{Ran}_S(F))$$
and an equality
$$\begin{tikzcd} & A \arrow[d, "R"] \\ & B \arrow[d, "S"] \\ X \arrow[uur, dashed, "\mathrm{Rift}_R(\mathrm{Rift}_S(F))"] \arrow[ur, dashed, "\mathrm{Rift}_S(F)"'] \arrow[r, "F"'] & C \end{tikzcd} = \begin{tikzcd} & A \arrow[d, "R"] \\ & B \arrow[d, "S"] \\ X \arrow[uur, dashed, "\mathrm{Rift}_{S\diamond R}(F)"] \arrow[r, "F"'] & C \end{tikzcd}$$
of pasting diagrams in $\mathbf{Rel}$.

3. *Interaction With Converses.* We have

$$\mathrm{Rift}_R(F)^\dagger = \mathrm{Ran}_{R^\dagger}(F^\dagger).$$

4. *Interaction With Inverse Images.* We have

$$\mathrm{Rift}_R(F)^\dagger = \mathrm{Ran}_{\chi'_B}\left(F^\dagger\right) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "F^\dagger"] \arrow[d, "\chi_B"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{P}(X), \\ A \arrow[r, "R"'] & \mathcal{P}(B)^{\mathsf{op}} \arrow[ur, bend right, "\mathrm{Ran}_{\chi_A}\left(F^{-1}\right)"'] & \end{tikzcd}$$

where $\mathrm{Ran}_{\chi_A}\left(F^\dagger\right)$ is computed by the formula

$$\begin{aligned} [\mathrm{Ran}_{\chi_A}\left(F^\dagger\right)](U) &\cong \int_{a\in A} \chi_{\mathcal{P}(B)^{\mathsf{op}}}(U, \chi_a) \pitchfork F^\dagger(a) \\ &\cong \int_{a\in A} \chi_{\mathcal{P}(B)}(\chi_a, U) \pitchfork F^{-1}(a) \\ &\cong \int_{a\in A} \chi_U(a) \pitchfork F(a) \\ &\cong \bigcap_{a\in A} \chi_U(a) \pitchfork F(a) \\ &\cong \bigcap_{a\in U} F(a) \end{aligned}$$

for each $U \in \mathcal{P}(A)$, so we have

$$[\mathrm{Rift}_R(F)]^{-1}(a) = \bigcap_{b\in R(a)} F^{-1}(b)$$

for each $a \in A$.

Proof. *Item 1, Functoriality*: We have

$$\begin{aligned} [\mathrm{Rift}_{R_2}(F_1)](x) &= \{a \in A \mid R_2(a) \subset F_1(x)\} \\ &\subset \{a \in A \mid R_1(a) \subset F_1(x)\} \\ &\subset \{a \in A \mid R_1(a) \subset F_2(x)\} \\ &= \mathrm{Rift}_{R_1}(F_2) \end{aligned}$$

for each $x \in X$, so we therefore have $\mathrm{Rift}_{R_2}(F_1) \subset \mathrm{Rift}_{R_1}(F_2)$.
Item 2, Interaction With Composition: This holds in a general bicategory with the necessary right Kan lifts, being therefore a special case of **??**.
Item 3, Interaction With Converses: This follows from Item 3 of Definition 8.5.15.1.4 by duality.
Item 4, Interaction With Inverse Images: We proceed in a few steps.

- We have $x \in \mathrm{Rift}_R(F)^\dagger(a)$ iff $a \in \mathrm{Rift}_R(F)(x)$.
- This holds iff $R(a) \subset F(x)$.
- This holds iff, for each $b \in R(a)$, we have $b \in F(x)$.
- This holds iff, for each $b \in R(a)$, we have $x \in F^{-1}(b)$.
- This holds iff $x \in \bigcap_{b \in R(a)} F^{-1}(b)$.

This finishes the proof. □

8.5.17 Closedness

Proposition 8.5.17.1.1. The 2-category **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_! \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),$$

natural in $S \in \mathrm{Rel}(B,X)$, $T \in \mathrm{Rel}(A,X)$, $U \in \mathrm{Rel}(X,A)$, and $V \in \mathrm{Rel}(X,B)$.

Proof. This follows from Constructions With Relations, **????**. □

8.5.18 Rel as a Category of Free Algebras

Proposition 8.5.18.1.1. We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}_!}(\mathsf{Sets}),$$

where $\mathcal{P}_!$ is the powerset monad of **??**, **??**.

Proof. Omitted. □

8.6 Properties of the 2-Category of Relations With Apartness Composition

8.6.1 Self-Duality

Proposition 8.6.1.1.1. The 2-/category of relations with apartness-composition-is self-dual:

1. *Self-Duality I.* We have an isomorphism

   $$(\mathsf{Rel}^{\square})^{\mathsf{op}} \cong \mathsf{Rel}^{\square}$$

   of categories.

2. *Self-Duality II.* We have a 2-isomorphism

   $$(\mathbf{Rel}^{\square})^{\mathsf{op}} \cong \mathbf{Rel}^{\square}$$

   of 2-categories.

Proof. *Item 1, Self-Duality I*: We claim that the functor

$$(-)^{\dagger} \colon (\mathsf{Rel}^{\square})^{\mathsf{op}} \longrightarrow \mathsf{Rel}^{\square}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories. Note that this is indeed a functor by Items 4 and 7 of Definition 8.1.5.1.3.

By Categories, Item 1 of Definition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\dagger} \colon \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(B, A)$$

defined by the assignment $R \mapsto R^\dagger$ is a bijection by Item 5 of Definition 8.1.5.1.3, showing $(-)^\dagger$ to be fully faithful.
Item 2, Self-Duality II: We claim that the 2-functor

$$(-)^\dagger \colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^\dagger$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Definition 8.1.5.1.3, is an isomorphism of categories.

By **??**, it suffices to show that $(-)^\dagger$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Definition 8.1.5.1.3.

Thus $(-)^\dagger$ is indeed a 2-isomorphism of categories. □

8.6.2 Isomorphisms and Equivalences

Let $R \colon A \nrightarrow B$ be a relation from A to B, and recall that $R^{\mathsf{c}} \stackrel{\text{def}}{=} B \times A \setminus R$.

Lemma 8.6.2.1.1. The conditions below are row-wise equivalent:

Condition	Inclusion
R^{c} is functional	$\nabla_B \subset R \mathbin{\square} R^\dagger$
R^{c} is total	$R \mathbin{\square} R^\dagger \subset \nabla_A$
R^{c} is injective	$\nabla_A \subset R^\dagger \mathbin{\square} R$
R^{c} is surjective	$R^\dagger \mathbin{\square} R \subset \nabla_B$

Proof. This follows from Definition 8.5.2.1.1 and Item 4 of Definition 8.1.4.1.3. For instance:

- Suppose we have $R \mathbin{\square} R^\dagger \subset \nabla_B$.
- Taking complements, we obtain $\nabla_B^{\mathsf{c}} \subset (R \mathbin{\square} R^\dagger)^{\mathsf{c}}$.
- Applying Item 4 of Definition 8.1.4.1.3, this becomes $\Delta_B \subset R^{\mathsf{c}} \diamond (R^\dagger)^{\mathsf{c}}$.
- Then, by Definition 8.5.2.1.1, this is equivalent to R^{c} being total.

The proof of the other equivalences is similar, and thus omitted. □

Remark 8.6.2.1.2. The statements in Definition 8.6.2.1.1 unwind to the following:

Inclusion	Quantifier	Condition
$\nabla_B \subset R \mathbin{\square} R^\dagger$	For each $b_1, b_2 \in B$	If $b_1 \neq b_2$, then, for each $a \in A$, we have $a \sim_R b_1$ or $a \sim_R b_2$.
$R \mathbin{\square} R^\dagger \subset \nabla_B$	For each $b_1, b_2 \in B$	If, for each $a \in A$, $a \sim_R b_1$ or $a \sim_R b_2$, then $b_1 \neq b_2$.
$\nabla_A \subset R^\dagger \mathbin{\square} R$	For each $a_1, a_2 \in A$	If $a_1 \neq a_2$, then, for each $b \in B$, we have $a_1 \sim_R b$ or $a_2 \sim_R b$.
$R^\dagger \mathbin{\square} R \subset \nabla_A$	For each $a_1, a_2 \in A$	If, for each $b \in B$, $a_1 \sim_R b$ or $a_2 \sim_R b$, then $a_1 \neq a_2$.

Equivalently:

Inclusion	Quantifier	If	Then
$\nabla_B \subset R \mathbin{\square} R^\dagger$	For each $b_1, b_2 \in B$	$b_1 \neq b_2$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$
$R \mathbin{\square} R^\dagger \subset \nabla_B$	For each $b_1, b_2 \in B$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$	$b_1 \neq b_2$
$\nabla_A \subset R^\dagger \mathbin{\square} R$	For each $a_1, a_2 \in A$	$a_1 \neq a_2$	$R(a_1) \cup R(a_2) = B$
$R^\dagger \mathbin{\square} R \subset \nabla_A$	For each $a_1, a_2 \in A$	$R(a_1) \cup R(a_2) = B$	$a_1 \neq a_2$

Proposition 8.6.2.1.3. The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in $\mathbf{Rel}^{\square}$, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

$$R^{-1} \mathbin{\square} R \cong \nabla_A,$$
$$R \mathbin{\square} R^{-1} \cong \nabla_B.$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

(★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$\begin{aligned} R^{-1} \diamond R &= \nabla_A, \\ R \diamond R^{-1} &= \nabla_B. \end{aligned}$$

3. There exists a bijection $f\colon B \xrightarrow{\sim} A$ with $R^{\mathsf{c}} = f^{-1}$.

Proof. This follows from Definition 8.5.2.1.2 and Item 4 of Definition 8.1.4.1.3. □

8.6.3 Internal Adjunctions

Let A and B be sets.

Proposition 8.6.3.1.1. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel}^{\square} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } B \text{ to } A\end{matrix}\right\},$$

with every adjunction in $\mathbf{Rel}^{\square}$ being of the form $(f^{-1})^{\mathsf{c}} \dashv \mathrm{Gr}(f)^{\mathsf{c}}$ for some function $f\colon B \longrightarrow A$.

Proof. This follows from Definition 8.5.3.1.1 and Item 4 of Definition 8.1.4.1.3. □

8.6.4 Internal Monads

Let X be a set.

Proposition 8.6.4.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Monads in} \\ \mathbf{Rel}^{\square} \text{ on } X\end{matrix}\right\} \cong \{\text{Subsets of } X\}.$$

Proof. This follows from Definition 8.6.4.1.1 and Item 4 of Definition 8.1.4.1.3. □

8.6.5 Internal Comonads

Let X be a set.

Proposition 8.6.5.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in} \\ \mathsf{Rel}^{\square} \text{ on } X\end{matrix}\right\} \cong \{\text{Strict total orders on } X\}.$$

Proof. A comonad in $\mathsf{Rel}^{\square}$ on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\Delta_R &\colon R \subset R \mathbin{\square} R, \\ \epsilon_R &\colon R \subset \nabla_X\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\lambda^{\mathsf{Rel}^{\square},-1}_R"'] & R \mathbin{\square} R \arrow[d, "\epsilon_R \mathbin{\square} \mathrm{id}_R", "\shortmid"{marking}] \\
& \nabla_X \mathbin{\square} R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \mathbin{\square} R \arrow[dr, "\mathrm{id}_R \mathbin{\square} \Delta_R", "\shortmid"{marking}] & \\
R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[d, "\Delta_R"', "\shortmid"{marking}] & & R \mathbin{\square} (R \mathbin{\square} R) \arrow[d, equal, "\alpha^{\mathsf{Rel}^{\square},-1}_{R,R,R}"] \\
R \mathbin{\square} R \arrow[rr, "\Delta_R \mathbin{\square} \mathrm{id}_R"', "\shortmid"{marking}] & & (R \mathbin{\square} R) \mathbin{\square} R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equal, "\rho^{\mathsf{Rel}^{\square},-1}_R"'] & R \mathbin{\square} R \arrow[d, "\mathrm{id}_R \mathbin{\square} \epsilon_R", "\shortmid"{marking}] \\
& R \mathbin{\square} \nabla_X
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $x, z \in X$, if $x \sim_R z$, then, for each $y \in X$, we have $x \sim_R y$ or $y \sim_R z$.

2. For each $x, y \in X$, if $x \sim_R y$, then $x \neq y$.

Replacing $\sim_R$ with $<_R$ and taking the contrapositive of each condition, we obtain:

1. For each $x, z \in X$, if there exists some $y \in X$ such that $x <_R y$ and $y <_R z$, then $x <_R z$.

2. For each $x \in X$, we have $x \not<_R x$.

These are exactly the requirements for R to be a strict linear order (**??**). Conversely, any strict linear order $<_R$ gives rise to a pair of maps $\Delta_{<_R}$ and $\epsilon_{<_R}$, forming a comonad on X. □

Example 8.6.5.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. The codensity monad $\mathrm{Ran}_R(R)\colon B \nrightarrow B$ is given by

$$[\mathrm{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b) \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(R)", "|"{marking}] \\ A \arrow[ur, "R", "/"{marking}] \arrow[r, "R"', "|"{marking}] \arrow[r, phantom, shift left=4, "\Leftarrow"] & B \end{tikzcd}$$

for each $b \in B$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Ran}_R(R)}\colon B \times B \to \{\mathrm{t}, \mathrm{f}\}$$

on B obtained by declaring $b \preceq_{\mathrm{Ran}_R(R)} b'$ iff the following equivalent conditions are satisfied:

(a) For each $a \in A$, if $a \sim_R b$, then $a \sim_R b'$.

(b) We have $R^{-1}(b) \subset R^{-1}(b')$.

2. The dual codensity monad $\mathrm{Rift}_R(R)\colon A \nrightarrow A$ is given by

$$[\mathrm{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\} \qquad \begin{tikzcd} & A \arrow[d, "R", "|"{marking}] \\ A \arrow[ur, dashed, "\mathrm{Rift}_R(R)", "/"{marking}] \arrow[r, "R"', "|"{marking}] \arrow[r, phantom, shift left=4, "\Leftarrow"] & B \end{tikzcd}$$

for each $a \in A$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Rift}_R(R)}\colon A \times A \to \{\mathrm{t}, \mathrm{f}\}$$

on A obtained by declaring $a \preceq_{\mathrm{Rift}_R(R)} a'$ iff the following equivalent conditions are satisfied:

(a) For each $a \in A$, if $a \sim_R b$, then $a' \sim_R b$.

(b) We have $R(a') \subset R(a)$.

8.6.6 Modules Over Internal Monads

8.6.7 Comodules Over Internal Comonads

8.6.8 Eilenberg–Moore and Kleisli Objects

8.6.9 Monomorphisms

8.6.10 2-Categorical Monomorphisms

8.6.11 Epimorphisms

8.6.12 2-Categorical Epimorphisms

8.6.13 Co/Limits

This will be expanded later on.

8.6.14 Internal Left Kan Extensions

8.6.15 Internal Left Kan Lifts

8.6.16 Internal Right Kan Extensions

8.6.17 Internal Right Kan Lifts

8.6.18 Coclosedness

8.7 The Adjoint Pairs $R_! \dashv R_{-1}$ and $R^{-1} \dashv R_*$

8.7.1 Direct Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.1.1.1. The **direct image function associated to** R is the function[33]

$$R_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

[33] *Further Notation:* Also written simply $R\colon \mathcal{P}(X) \to \mathcal{P}(Y)$.

defined by[34]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \bigcup_{a \in U} R(a) \\ &= \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Warning 8.7.1.1.2. Notation for direct images between powersets is tricky; see Constructions With Sets, Definition 4.6.1.1.3. Here we'll try to align our notation for relations with that for functions.

Remark 8.7.1.1.3. Identifying subsets of X with relations from pt to X via Constructions With Sets, Item 3 of Definition 4.4.1.1.4, we see that the direct image function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(\mathrm{pt}, X)} \to \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(\mathrm{pt}, Y)}$$

defined by

$$R_!(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(X)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} X \overset{R}{\nrightarrow} Y.$$

Proposition 8.7.1.1.4. Let $R \colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

   $$R_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

     $$[R_!](U) \overset{\text{def}}{=} R_!(U).$$

[34] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(X)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by:

(a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow R_{-1} \circ R_!, \\ R_! \circ R_{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset R_{-1}(R_!(U)), \\ R_!(R_{-1}(V)) &\subset V \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$

(b) A bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$. In particular:

($\star$) The following conditions are equivalent:
  - We have $R_!(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_!(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R_!(U) \cup R_!(V) &= R_!(U \cup V), \\ R_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_!(\bigcap_{i\in I} U_i) \subset \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R_!(U \cap V) &\subset R_!(U) \cap R_!(V), \\ R_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_!, R_!^{\otimes}, R_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_!(U) \cup R_!(V) &\xrightarrow{=} R_!(U \cup V), \\ R_{*|\mathbb{1}}^{\otimes} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(R_!, R_!^{\otimes}, R_{*|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} R_{*|U,V}^{\otimes} \colon R_!(U \cap V) &\subset R_!(U) \cap R_!(V), \\ R_{*|\mathbb{1}}^{\otimes} \colon R_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

7. *Relation to Codirect Images.* We have

$$R_!(U) = Y \setminus R_*(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and **??**, **??** of **??**.
Item 4, *Oplax Preservation of Limits*: Omitted.
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Relation to Codirect Images*: The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 17 of Definition 4.6.1.1.5): applying Item 7 of Definition 8.7.4.1.3 to $A \setminus U$, we have

$$\begin{aligned} R_*(X \setminus U) &= Y \setminus R_!(X \setminus (X \setminus U)) \\ &= Y \setminus R_!(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_!(U) &= Y \setminus (Y \setminus R_!(U)), \\ &= Y \setminus R_*(X \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 8.7.1.1.5. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathrm{Rel}(X,Y) \to \mathsf{Sets}(\mathcal{P}(X),\mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_!\colon \mathrm{Rel}(X,Y) \to \mathsf{Pos}((\mathcal{P}(X),\subset),(\mathcal{P}(Y),\subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have[35]

$$(\chi_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

[35]That is, the postcomposition function

$$(\chi_X)_!\colon \mathrm{Rel}(\mathrm{pt},X) \to \mathrm{Rel}(\mathrm{pt},X)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},X)}$.

4. *Interaction With Composition*. For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have[36]

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(Y) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: Indeed, we have

$$\begin{aligned} (\chi_X)_!(U) &\overset{\text{def}}{=} \bigcup_{a\in U} \chi_X(a) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} \{a\} \\ &= U \\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{P}(X)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_! = \mathrm{id}_{\mathcal{P}(X)}$.
Item 4, *Interaction With Composition*: Indeed, we have

$$\begin{aligned} (S \diamond R)_!(U) &\overset{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{a\in U} S_!(R(a)) \\ &= S_!\Big(\bigcup_{a\in U} R(a)\Big) \end{aligned}$$

[36]That is, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, X) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathrm{Rel}(\mathrm{pt}, Y) \arrow[d, "S_!"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} S_!(R_!(U)) \\
&\stackrel{\text{def}}{=} [S_! \circ R_!](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we used Item 3 of Definition 8.7.1.1.4. Thus $(S \diamond R)_! = S_! \circ R_!$. □

8.7.2 Coinverse Images

Let X and Y be sets and let $R \colon X \nrightarrow Y$ be a relation.

Definition 8.7.2.1.1. The **coinverse image function associated to** R is the function

$$R_{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[37]

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in X \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 8.7.2.1.2. Identifying subsets of Y with relations from pt to Y via Constructions With Sets, Item 3 of Definition 4.4.1.1.4, we see that the inverse image function associated to R is equivalently the function

$$R_{-1} \colon \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(\mathrm{pt},Y)} \to \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(\mathrm{pt},X)}$$

defined by

$$
R_{-1}(V) \stackrel{\text{def}}{=} \mathrm{Rift}_R(V), \qquad
\begin{tikzcd}
 & X \arrow[d, "R", "\shortmid"{marking}] \\
\mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\shortmid"{marking}] \arrow[r, "V"', "\shortmid"{marking}] & Y, \arrow[ul, phantom, shift left=2, "\Swarrow"]
\end{tikzcd}
$$

and being explicitly computed by

$$
\begin{aligned}
R_{-1}(V) &\stackrel{\text{def}}{=} \mathrm{Rift}_R(V) \\
&\cong \int_{b \in Y} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^b_{-1}, V^b_{-2}),
\end{aligned}
$$

where we have used **??**.

[37] *Further Terminology:* The set $R_{-1}(V)$ is called the **coinverse image of** V **by** R.

Proof. We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_{-_1}, V^b_{-_2}) \\
&= \left\{ a \in X \;\middle|\; \int_{b\in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, V^b_\star) = \mathsf{true} \right\} \\
&= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{for each } b \in Y\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathsf{true} \\ \qquad \text{(b) We have } V^b_\star = \mathsf{true} \end{array} \right\} \\
&= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{for each } b \in Y\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(a) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } b \in V \end{array} \right\} \\
&= \{a \in X \mid \text{for each } b \in R(a)\text{, we have } b \in V\} \\
&= \{a \in X \mid R(a) \subset V\} \\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof. □

Proposition 8.7.2.1.3. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(Y)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(Y)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1}) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_!(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} R_{-1}(U_i) \subset R_{-1}(\bigcup_{i\in I} U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have inclusions

$$R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$\emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(Y)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(Y) = Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(Y)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon R_{-1}(X) \xrightarrow{=} Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Inverse Images I.* We have

$$R_{-1}(V) = X \setminus R^{-1}(Y \setminus V)$$

for each $V \in \mathcal{P}(Y)$.

8. *Interaction With Inverse Images II.* Let $R \colon X \nrightarrow Y$ be a relation from X to Y.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(Y)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Lax Preservation of Colimits*: Omitted.
Item 4, *Preservation of Limits*: This follows from Item 2 and **??**, **??** of **??**.
Item 5, *Symmetric Lax Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Strict Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Inverse Images I*: We claim we have an equality

$$R_{-1}(Y \setminus V) = X \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(Y \setminus V) &= \{a \in X \mid R(a) \subset Y \setminus V\}, \\ X \setminus R^{-1}(V) &= \{a \in X \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = Y \setminus V$ then implies the original statement.
Item 8, *Interaction With Inverse Images II*: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Definition 8.2.2.1.2. □

Proposition 8.7.2.1.4. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(Y) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_X)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in X \mid \chi_X(a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_{-1} = \mathrm{id}_{\mathcal{P}(X)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in X \mid [S \diamond R](a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid S(R(a)) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in X \mid S_!(R(a)) \subset U\}\\ &= \{a \in X \mid R(a) \subset S_{-1}(U)\}\\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U))\\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Definition 8.7.2.1.3, which implies that the conditions

- We have $S_!(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

8.7.3 Inverse Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.3.1.1. The **inverse image function associated to** R[38] is the function

$$R^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[39]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in X \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(Y)$.

[38] *Further Terminology:* Also called simply the **inverse image function associated to** R.
[39] *Further Terminology:* The set $R^{-1}(V)$ is called the **inverse image of** V **by** R or simply the

Remark 8.7.3.1.2. Identifying subsets of Y with relations from Y to pt via Constructions With Sets, Item 3 of Definition 4.4.1.1.4, we see that the inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(Y)}_{\cong\mathrm{Rel}(Y,\mathrm{pt})} \to \underbrace{\mathcal{P}(X)}_{\cong\mathrm{Rel}(X,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \overset{\mathrm{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(X)$, where $R \diamond V$ is the composition

$$X \overset{R}{\nrightarrow} Y \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\overset{\mathrm{def}}{=} V \diamond R \\ &\overset{\mathrm{def}}{=} \int^{b\in Y} V_b^{-1} \times R_{-_2}^b. \end{aligned}$$

Proof. We have

$$\begin{aligned} V \diamond R &\overset{\mathrm{def}}{=} \int^{b\in Y} V_b^{-1} \times R_{-_2}^b \\ &= \left\{ a \in X \;\middle|\; \int^{b\in Y} V_b^{\star} \times R_a^b = \mathsf{true} \right\} \\ &= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V_b^{\star} = \mathsf{true} \\ \quad 2.\ \text{We have } R_a^b = \mathsf{true} \end{array} \right\} \\ &= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \in V \\ \quad 2.\ \text{We have } b \in R(a) \end{array} \right\} \end{aligned}$$

$$\begin{aligned}
&= \{a \in X \mid \text{there exists } b \in V \text{ such that } b \in R(a)\}\\
&= \{a \in X \mid R(a) \cap V \neq \emptyset\}\\
&\overset{\text{def}}{=} R^{-1}(V)
\end{aligned}$$

This finishes the proof. □

Proposition 8.7.3.1.3. Let $R \colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

   $$R^{-1} \colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(Y)$, we have

     $$[R^{-1}](V) \overset{\text{def}}{=} R^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(Y)$:
     - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

   $$\left(R^{-1} \dashv R_*\right)\colon \quad \begin{tikzcd} \mathcal{P}(Y) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "R_*"] \end{tikzcd}$$

   witnessed by a bijections of sets

   $$\mathrm{Hom}_{\mathcal{P}(X)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

   natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

   (★) The following conditions are equivalent:
   - We have $R^{-1}(U) \subset V$.
   - We have $U \subset R_*(V)$.

inverse image of V **by** R.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\Big(\bigcap_{i\in I} U_i\Big) \subset \bigcap_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) &\xrightarrow{=} R^{-1}(U \cup V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$\begin{aligned} R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Coinverse Images I.* We have

$$R^{-1}(V) = X \setminus R_{-1}(Y \setminus V)$$

for each $V \in \mathcal{P}(Y)$.

8. *Interaction With Coinverse Images II.* Let $R\colon X \nrightarrow Y$ be a relation from X to Y.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(Y)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness*: This follows from Kan Extensions, **??** of **??**.
Item 3, *Preservation of Colimits*: This follows from Item 2 and **??**, **??** of **??**.
Item 4, *Oplax Preservation of Limits*: Omitted.
Item 5, *Symmetric Strict Monoidality With Respect to Unions*: This follows from Item 3.
Item 6, *Symmetric Oplax Monoidality With Respect to Intersections*: This follows from Item 4.
Item 7, *Interaction With Coinverse Images I*: This follows from Item 7 of Definition 8.7.2.1.3.
Item 8, *Interaction With Coinverse Images II*: This was proved in Item 8 of Definition 8.7.2.1.3. □

Proposition 8.7.3.1.4. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathsf{Rel}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathsf{Rel}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have[40]

$$(\chi_X)^{-1} = \mathrm{id}_{\mathcal{P}(X)}\,.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \mathrel{\rightarrow\mkern-14mu\mapstochar\mkern14mu} Y$ and $S\colon Y \mathrel{\rightarrow\mkern-14mu\mapstochar\mkern14mu} C$, we have[41]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(Y) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Categories, Item 5 of Definition 11.1.4.1.2.
Item 4, Interaction With Composition: This follows from Categories, Item 2 of Definition 11.1.4.1.2. □

8.7.4 Codirect Images

Let X and Y be sets and let $R\colon X \mathrel{\rightarrow\mkern-14mu\mapstochar\mkern14mu} Y$ be a relation.

Definition 8.7.4.1.1. The **codirect image function associated to** R is the function

$$R_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

[40]That is, the postcomposition

$$(\chi_X)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, X) \to \mathrm{Rel}(\mathrm{pt}, X)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},X)}$.

[41]That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, Y) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, X). \end{tikzcd}$$

defined by[42,43]

$$
\begin{aligned}
R_*(U) &\stackrel{\text{def}}{=} \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{for each } a \in X, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\
&= \left\{ b \in Y \;\middle|\; R^{-1}(b) \subset U \right\}
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

Remark 8.7.4.1.2. Identifying subsets of Y with relations from pt to Y via Constructions With Sets, Item 3 of Definition 4.4.1.1.4, we see that the codirect image function associated to R is equivalently the function

$$
R_* \colon \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(X,\mathrm{pt})} \to \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(Y,\mathrm{pt})}
$$

defined by

$$
R_*(U) \stackrel{\text{def}}{=} \mathrm{Ran}_R(U), \qquad
\begin{tikzcd}
 & Y \arrow[d, dashed, "\mathrm{Ran}_R(U)", "|"{marking}] \arrow[dl, phantom, "\Downarrow"] \\
X \arrow[ur, "R", "|"{marking}] \arrow[r, "U"', "|"{marking}] & \mathrm{pt},
\end{tikzcd}
$$

being explicitly computed by

$$
\begin{aligned}
R_*(U) &\stackrel{\text{def}}{=} \mathrm{Ran}_R(U) \\
&\cong \int_{a \in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-2}_a, U^{-1}_a),
\end{aligned}
$$

where we have used **??**.

Proof. We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a \in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{-2}_a, U^{-1}_a) \\
&= \left\{ b \in Y \;\middle|\; \int_{a \in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^{b}_a, U^{\star}_a) = \mathsf{true} \right\}
\end{aligned}
$$

[42] *Further Terminology:* The set $R_*(U)$ is called the **codirect image of** U **by** R.

[43] We also have

$$R_*(U) = Y \setminus R_!(X \setminus U);$$

$$= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{for each } a \in X\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathrm{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathrm{true} \\ \qquad \text{(b) We have } U^\star_a = \mathrm{true} \end{array} \right\}$$

$$= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{for each } a \in X\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(X) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\}$$

$$\begin{aligned} &= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{for each } a \in X\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in Y \,\middle|\, R^{-1}(b) \subset U \right\} \\ &\overset{\mathrm{def}}{=} R^{-1}(U). \end{aligned}$$

This finishes the proof. □

Proposition 8.7.4.1.3. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[R_*](U) \overset{\mathrm{def}}{=} R_*(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(X)$:
     - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$(R^{-1} \dashv R_*) \colon \quad \begin{tikzcd} \mathcal{P}(Y) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "R_*"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R^{-1}(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

(★) The following conditions are equivalent:
   - We have $R^{-1}(U) \subset V$.
   - We have $U \subset R_*(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_*(U_i) \subset R_*(\bigcup_{i \in I} U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$R_*(U) \cup R_*(V) \subset R_*(U \cup V),$$
$$\emptyset \subset R_*(\emptyset),$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_*(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$R_*(U \cap V) = R_*(U) \cap R_*(V),$$
$$R_*(X) = Y,$$

natural in $U, V \in \mathcal{P}(X)$.

---

see Item 7 of Definition 8.7.4.1.3.

5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$(R_*, R_*^{\otimes}, R_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} \colon R_*(U) \cup R_*(V) \subset R_*(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_*(\emptyset),$$

natural in $U, V \in \mathcal{P}(X)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$(R_*, R_*^{\otimes}, R_{!|\mathbb{1}}^{\otimes}) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} \colon R_*(U \cap V) \xrightarrow{=} R_*(U) \cap R_*(V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon R_*(X) \xrightarrow{=} Y,$$

natural in $U, V \in \mathcal{P}(X)$.

7. *Relation to Direct Images.* We have

$$R_*(U) = Y \setminus R_!(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from Kan Extensions, **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??**, **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: This follows from Item 7 of Definition 8.7.1.1.4. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Constructions With Sets, Item 16 of Definition 4.6.3.1.7).

We claim that $R_*(U) = Y \setminus R_!(X \setminus U)$:

- *The First Implication.* We claim that

$$R_*(U) \subset Y \setminus R_!(X \setminus U).$$

  Let $b \in R_*(U)$. We need to show that $b \notin R_!(X \setminus U)$, i.e. that there is no $a \in X \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_*(U)$).

  Thus $b \in Y \setminus R_!(X \setminus U)$.

- *The Second Implication.* We claim that

$$Y \setminus R_!(X \setminus U) \subset R_*(U).$$

  Let $b \in Y \setminus R_!(X \setminus U)$. We need to show that $b \in R_*(U)$, i.e. that $R^{-1}(b) \subset U$.

  Since $b \notin R_!(X \setminus U)$, there exists no $a \in X \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

  Thus $b \in R_*(U)$.

This finishes the proof. $\square$

Proposition 8.7.4.1.4. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Sets}(X, Y) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(Y) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functoriality I*: Clear.
Item 2, Functoriality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_X)_*(U) &\stackrel{\text{def}}{=} \left\{a \in X \,\middle|\, \chi_X^{-1}(a) \subset U\right\} \\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\} \\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_* = \mathrm{id}_{\mathcal{P}(X)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_*(U) &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\} \\ &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}(R^{-1}(c)) \subset U\right\} \\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_*(U)\right\} \\ &\stackrel{\text{def}}{=} R_*(S_*(U)) \\ &\stackrel{\text{def}}{=} [R_* \circ S_*](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Definition 8.7.4.1.3, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_*(U)$.

are equivalent. Thus $(S \diamond R)_* = S_* \circ R_*$. □

8.7.5 Functoriality of Powersets

Proposition 8.7.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors[44]

$$\begin{aligned}\mathcal{P}_! &\colon \mathsf{Rel} \longrightarrow \mathsf{Sets}, \\ \mathcal{P}_{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Sets}, \\ \mathcal{P}^{-1} &\colon \mathsf{Rel}^{\mathsf{op}} \longrightarrow \mathsf{Sets}, \\ \mathcal{P}_* &\colon \mathsf{Rel} \longrightarrow \mathsf{Sets}\end{aligned}$$

where

[44] The functor $\mathcal{P}_! \colon \mathsf{Rel} \longrightarrow \mathsf{Sets}$ admits a left adjoint; see Item 2 of Definition 8.2.2.1.2.

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}\mathcal{P}_!(X) &\overset{\mathrm{def}}{=} \mathcal{P}(X),\\ \mathcal{P}_{-1}(X) &\overset{\mathrm{def}}{=} \mathcal{P}(X),\\ \mathcal{P}^{-1}(X) &\overset{\mathrm{def}}{=} \mathcal{P}(X),\\ \mathcal{P}_*(X) &\overset{\mathrm{def}}{=} \mathcal{P}(X).\end{aligned}$$

- *Action on Morphisms.* For each morphism $R\colon X \nrightarrow Y$ of Rel, the images

$$\begin{aligned}\mathcal{P}_!(R)&\colon \mathcal{P}(X) \to \mathcal{P}(Y),\\ \mathcal{P}_{-1}(R)&\colon \mathcal{P}(Y) \to \mathcal{P}(X),\\ \mathcal{P}^{-1}(R)&\colon \mathcal{P}(Y) \to \mathcal{P}(X),\\ \mathcal{P}_*(R)&\colon \mathcal{P}(X) \to \mathcal{P}(Y)\end{aligned}$$

  of R by $\mathcal{P}_!$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ are defined by

$$\begin{aligned}\mathcal{P}_!(R) &\overset{\mathrm{def}}{=} R_!,\\ \mathcal{P}_{-1}(R) &\overset{\mathrm{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R) &\overset{\mathrm{def}}{=} R^{-1},\\ \mathcal{P}_*(R) &\overset{\mathrm{def}}{=} R_*,\end{aligned}$$

  as in Definitions 8.7.1.1.1, 8.7.2.1.1, 8.7.3.1.1 and 8.7.4.1.1.

Proof. This follows from Items 3 and 4 of Definition 8.7.1.1.5, Items 3 and 4 of Definition 8.7.2.1.4, Items 3 and 4 of Definition 8.7.3.1.4, and Items 3 and 4 of Definition 8.7.4.1.4. □

8.7.6 Functoriality of Powersets: Relations on Powersets

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.6.1.1. The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$$

defined by[45]

$$\mathcal{P}(R)^{V}_{U} \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\mathrm{pt}}, V \diamond R \diamond U)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

Remark 8.7.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathsf{true}$, i.e. we have

$$\int^{a \in X} \int^{b \in Y} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathsf{true}.$$

- There exists some $a \in X$ and some $b \in Y$ such that:
  - We have $U^{a}_{\star} = \mathsf{true}$.
  - We have $R^{b}_{a} = \mathsf{true}$.
  - We have $V^{\star}_{b} = \mathsf{true}$.
- There exists some $a \in X$ and some $b \in Y$ such that:
  - We have $a \in U$.
  - We have $a \sim_{R} b$.
  - We have $b \in V$.

Proposition 8.7.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathrm{Rel} \to \mathrm{Rel}.$$

Proof. Omitted. □

[45]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\shortmid"{marking}, "U"'] \arrow[rrr, bend left, "\shortmid"{marking}, "\chi_{\mathrm{pt}}"] & X \arrow[r, "\shortmid"{marking}, "R"'] & Y \arrow[r, "\shortmid"{marking}, "V"'] & \mathrm{pt}. \end{tikzcd}$$

8.8 The Left Skew Monoidal Structure on $\mathrm{Rel}(A, B)$

8.8.1 The Left Skew Monoidal Product

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

Definition 8.8.1.1.1. The **left J-skew monoidal product of** $\mathrm{Rel}(A, B)$ is the functor

$$\lhd_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\text{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\mapstochar"'] \arrow[d, "J", "\mapstochar"'] \arrow[dl, phantom, "\Swarrow"] & B. \\ A \arrow[ur, dashed, "\mathrm{Rift}_J(R)"] \arrow[r, "R"', "\mapstochar"] & B & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \lhd_J R, S' \lhd_J R')$$

  of $\lhd_J$ at $((R, S), (R', S'))$ is defined by[46]

$$\beta \lhd_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[d, "J"] \arrow[dl, phantom, "\Swarrow"] & B \\ A \arrow[ur, dashed, bend left=40, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[ur, phantom, bend left=20, "\mathrm{Rift}_J(\alpha)\Downarrow"] \arrow[r, "R"] \arrow[r, bend right, "R'"'] \arrow[r, phantom, shift right=2, "\Downarrow\alpha"] & B & \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[46] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset S' \lhd_J R'$.

8.8.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.2.1.1. The **left J-skew monoidal unit of** $\mathbf{Rel}(A,B)$ is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\colon \mathsf{pt} \to \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\triangleleft_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

8.8.3 The Left Skew Associators

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.3.1.1. The **left J-skew associator of** $\mathbf{Rel}(A,B)$ is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ (\triangleleft_J \times \mathsf{id}) \Longrightarrow \triangleleft_J \circ (\mathsf{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & \mathbf{Rel}(A,B) \times (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \arrow[dr, "\mathsf{id}\times\triangleleft_J"] & \\
(\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \times \mathbf{Rel}(A,B) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\triangleleft_J\times\mathsf{id}"'] & & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \\
\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[rr, "\triangleleft_J"'] \arrow[rr, phantom, bend left, "\Nearrow\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\colon \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\overset{\text{def}}{=} T\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\overset{\text{def}}{=} T\diamond\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} \overset{\text{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon\ \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}\colon\ \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\overset{\text{def}}{=} J_!(\mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_! \dashv \mathrm{Rift}_J$, where $\epsilon\colon\ J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon\ A \nrightarrow B$ be a relation.

Definition 8.8.4.1.1. The **left J-skew left unitor of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon\ \triangleleft_J \circ (\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathbf{Rel}(A, B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\Swarrow \lambda^{\mathbf{Rel}(A,B),\triangleleft_J}"] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleleft_J"] \\
& \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R\colon\ \underbrace{J \triangleleft_J R}_{\overset{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon\ J_! \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.5 The Left Skew Right Unitors

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

Definition 8.8.5.1.1. The **left J-skew right unitor of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\lhd_J} : \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \lhd_J \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J})$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B) \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\lhd_J"] \arrow[dl, phantom, "\Nearrow \rho^{\mathbf{Rel}(A,B),\lhd_J}"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\lhd_J}_R : R \hookrightarrow \underbrace{R \lhd_J J}_{\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} R \diamond \chi_A \\
&\overset{\mathrm{id}_R \diamond \eta_{\chi_A}}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J_!(\chi_A)) \\
&\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J \diamond \chi_A) \\
&\overset{\sim}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J) \\
&\overset{\text{def}}{=} R \lhd_J J,
\end{aligned}$$

where $\eta : \mathrm{id}_{\mathbf{Rel}(A,A)} \Longrightarrow \mathrm{Rift}_J \circ J_!$ is the unit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Proposition 8.8.6.1.1. The category $\mathbf{Rel}(A, B)$ admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A,B)$ of relations from A to B of **??** of **??**.
- *The Left Skew Monoidal Product.* The left J-skew monoidal product

$$\lhd_J \colon \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,B)$$

  of Definition 8.8.1.1.1.
- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J} \colon \mathrm{pt} \to \mathbf{Rel}(A,B)$$

  of Definition 8.8.2.1.1.
- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\lhd_J} \colon \lhd_J \circ (\lhd_J \times \mathrm{id}) \Longrightarrow \lhd_J \circ (\mathrm{id} \times \lhd_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 8.8.3.1.1.
- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\lhd_J} \colon \lhd_J \circ (\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \times \mathrm{id}) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 8.8.4.1.1.
- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\lhd_J} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \lhd_J \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J})$$

  of Definition 8.8.5.1.1.

Proof. Since $\mathbf{Rel}(A,B)$ is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and thus $\mathbf{Rel}(A,B)$ together with the data in the statement forms a left skew monoidal category. □

8.9 The Right Skew Monoidal Structure on $\mathrm{Rel}(A,B)$

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

8.9.1 The Right Skew Monoidal Product

Definition 8.9.1.1.1. The **right J-skew monoidal product of** $\mathbf{Rel}(A, B)$ is the functor

$$\triangleright_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \stackrel{\text{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mapstochar"'] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\mapstochar"'] & B. \\ & A \arrow[u, "J", "\mapstochar"'] \arrow[ur, "S"', "\mapstochar"] \arrow[u, phantom, shift left=4, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[47]

$$\beta \triangleright_J \alpha \stackrel{\text{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'"] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[r, dashed, bend left, "\mathrm{Ran}_J(S')"] \arrow[r, dashed, bend right, "\mathrm{Ran}_J(S)"'] \arrow[r, phantom, "\mathrm{Ran}_J(\beta)\Uparrow"] & B \\ & A \arrow[u, "J"] \arrow[ur, "S'"] \arrow[ur, bend right=60, "S"'] \arrow[ur, phantom, bend right=30, "\beta\Uparrow"] & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

8.9.2 The Right Skew Monoidal Unit

Definition 8.9.2.1.1. The **right J-skew monoidal unit of** $\mathbf{Rel}(A, B)$ is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \colon \mathrm{pt} \to \mathbf{Rel}(A, B)$$

[47] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset S' \triangleright_J R'$.

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

8.9.3 The Right Skew Associators

Definition 8.9.3.1.1. The **right J-skew associator of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \times \mathbf{Rel}(A,B) \arrow[dr, "\triangleright_J \times \mathsf{id}"] & \\
\mathbf{Rel}(A,B) \times (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\mathsf{id} \times \triangleright_J"'] \arrow[rr, phantom, "\Uparrow \alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[rr, "\triangleright_J"'] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\overset{\text{def}}{=} \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond R} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.4 The Right Skew Left Unitors

Definition 8.9.4.1.1. The **right J-skew left unitor of** $\mathbf{Rel}(A,B)$ is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ (\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id}),$$

as in the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[r, phantom, shift right=6ex, "\lambda^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
& \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} \chi_B \diamond R \\
&\xRightarrow{\eta_{\chi_B} \diamond \mathrm{id}_R} \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \\
&\overset{\text{def}}{=} \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R \\
&\overset{\sim}{\Longrightarrow} \mathrm{Ran}_J(J) \diamond R \\
&\overset{\text{def}}{=} R \triangleright_J J,
\end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.5 The Right Skew Right Unitors

Definition 8.9.5.1.1. The **right J-skew right unitor of** $\mathbf{Rel}(A,B)$ is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B)\times\mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B), \arrow[d, "\triangleright_J"] \arrow[dl, phantom, "\Swarrow\rho^{\mathbf{Rel}(A,B),\triangleright_J}"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S \colon \underbrace{S \triangleright_J J}_{\overset{\mathrm{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon\colon J^* \circ \mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Proposition 8.9.6.1.1. The category $\mathbf{Rel}(A, B)$ admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of **??** of **??**.
- *The Right Skew Monoidal Product.* The right J-skew monoidal product

  $$\triangleleft_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$

  of Definition 8.9.1.1.1.
- *The Right Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A, B)$$

  of Definition 8.9.2.1.1.

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} : \triangleright_J \circ (\mathrm{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathrm{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 8.9.3.1.1.

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J} : \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ (\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id})$$

  of Definition 8.9.4.1.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J} : \triangleright_J \circ (\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 8.9.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic (Categories, Item 4 of Definition 11.2.7.1.2), and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a right skew monoidal category. □

Appendices

A Other Chapters

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