

Pointed Sets

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0098 This chapter contains some foundational material on pointed sets.

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0099 6.1 Pointed Sets

009A 6.1.1 Foundations

009B **Definition 6.1.1.1.1.** A **pointed set**¹ is equivalently:

- An \mathbb{E}_0 -monoid in $(N_\bullet(\mathbf{Sets}), \text{pt})$.
- A pointed object in $(\mathbf{Sets}, \text{pt})$.

009C **Remark 6.1.1.1.2.** In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X , called the **underlying set of** (X, x_0) .
- *The Basepoint.* A morphism

$$[x_0]: \text{pt} \rightarrow X$$

in \mathbf{Sets} , determining an element $x_0 \in X$, called the **basepoint of** X .

009D **Example 6.1.1.1.3.** The **0-sphere**² is the pointed set $(S^0, 0)$ ³ consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0 .

¹*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, pointed sets are viewed as \mathbb{F}_1 -**modules**.

²*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

³*Further Notation:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

009E Example 6.1.1.1.4. The **trivial pointed set** is the pointed set (pt, \star) consisting of:

- *The Underlying Set.* The punctual set $\text{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element \star of pt .

01QB Example 6.1.1.1.5. The **standard pointed set with $n + 1$ elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{*\} \cup \{1, \dots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n \rangle$.

009H 6.1.2 Morphisms of Pointed Sets

009J Definition 6.1.2.1.1. A **morphism of pointed sets**^{4,5} is equivalently:

- A morphism of \mathbb{E}_0 -monoids in $(\mathbf{N}_\bullet(\mathbf{Sets}), \text{pt})$.
- A morphism of pointed objects in $(\mathbf{Sets}, \text{pt})$.

009K Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f: (X, x_0) \rightarrow (Y, y_0)$ is a morphism of sets $f: X \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} & \text{pt} & \\ [x_0] \swarrow & & \searrow [y_0] \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

⁴*Further Terminology:* Also called a **pointed function**.

⁵*Further Terminology:* In the context of monoids with zero as models for \mathbb{F}_1 -algebras, morphisms of pointed sets are also called **morphism of \mathbb{F}_1 -modules**.

009L 6.1.3 The Category of Pointed Sets

009M **Definition 6.1.3.1.1.** The **category of pointed sets** is the category \mathbf{Sets}_* defined equivalently as:

- The homotopy category of the ∞ -category $\mathbf{Mon}_{\mathbb{E}_0}(\mathbf{N}_\bullet(\mathbf{Sets}), \mathbf{pt})$ of ??, ??.
- The category \mathbf{Sets}_* of Constructions With Categories, ??.

009N **Remark 6.1.3.1.2.** In detail, the **category of pointed sets** is the category \mathbf{Sets}_* where:

- *Objects.* The objects of \mathbf{Sets}_* are pointed sets.
- *Morphisms.* The morphisms of \mathbf{Sets}_* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, the unit map

$$\mathbb{1}_{(X, x_0)}^{\mathbf{Sets}_*} : \mathbf{pt} \rightarrow \mathbf{Sets}_*((X, x_0), (X, x_0))$$

of \mathbf{Sets}_* at (X, x_0) is defined by⁶

$$\mathrm{id}_{(X, x_0)}^{\mathbf{Sets}_*} \stackrel{\mathrm{def}}{=} \mathrm{id}_X.$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Obj}(\mathbf{Sets}_*)$, the composition map

$$\circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} : \mathbf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathbf{Sets}_*((X, x_0), (Y, y_0)) \rightarrow \mathbf{Sets}_*((X, x_0), (Z, z_0))$$

of \mathbf{Sets}_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by⁷

$$g \circ_{(X, x_0), (Y, y_0), (Z, z_0)}^{\mathbf{Sets}_*} f \stackrel{\mathrm{def}}{=} g \circ f.$$

⁶Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

⁷Note that the composition of two morphisms of pointed sets is indeed a morphism of

009P 6.1.4 Elementary Properties of Pointed Sets

009Q **Proposition 6.1.4.1.1.** Let (X, x_0) be a pointed set.

009R 1. *Completeness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is complete, having in particular:

009S (a) Products, described as in [Definition 6.2.3.1.1](#).

009T (b) Pullbacks, described as in [Definition 6.2.4.1.1](#).

009U (c) Equalisers, described as in [Definition 6.2.5.1.1](#).

009V 2. *Cocompleteness.* The category \mathbf{Sets}_* of pointed sets and morphisms between them is cocomplete, having in particular:

009W (a) Coproducts, described as in [Definition 6.3.3.1.1](#).

009X (b) Pushouts, described as in [Definition 6.3.4.1.1](#);

009Y (c) Coequalisers, described as in [Definition 6.3.5.1.1](#).

009Z 3. *Failure To Be Cartesian Closed.* The category \mathbf{Sets}_* is not Cartesian closed.⁸

00A0 4. *Morphisms From the Monoidal Unit.* We have a bijection of sets⁹

$$\mathbf{Sets}_*(S^0, X) \cong X,$$

pointed sets, as we have

$$\begin{array}{c}
 \text{pt} \\
 \begin{array}{ccc}
 \swarrow [x_0] & \downarrow [y_0] & \searrow [z_0] \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
 \end{array}
 \end{array}$$

$g(f(x_0)) = g(y_0) = z_0,$

⁸The category \mathbf{Sets}_* does admit a natural monoidal closed structure, however; see [Tensor Products of Pointed Sets](#).

⁹In other words, the forgetful functor

$$\text{忘}: \mathbf{Sets}_* \rightarrow \mathbf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0 .

natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.

00A1 5. *Relation to Partial Functions.* We have an equivalence of categories¹⁰

$$\mathbf{Sets}_* \stackrel{\text{eq.}}{\cong} \mathbf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

024U (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi: \mathbf{Sets}_* \xrightarrow{\cong} \mathbf{Sets}^{\text{part.}}$$

sends:

024V i. A pointed set (X, x_0) to X .

024W ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \rightarrow Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

024X (b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}: \mathbf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathbf{Sets}_*$$

sends:



¹⁰ **Warning:** This is not an isomorphism of categories, only an equivalence.

- 024Y i. A set X is to the pointed set (X, \star) with \star an element that is not in X .
- 024Z ii. A partial function

$$f: X \rightarrow Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}: (X, x_0) \rightarrow (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. Item 1, Completeness: This follows from (the proofs) of [Definitions 6.2.3.1.1](#), [6.2.4.1.1](#) and [6.2.5.1.1](#) and ??.

Item 2, Cocompleteness: This follows from (the proofs) of [Definitions 6.3.3.1.1](#), [6.3.4.1.1](#) and [6.3.5.1.1](#) and ??.

Item 3, Failure To Be Cartesian Closed: See [[MSE 2855868](#)].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X , we obtain a bijection between pointed maps $S^0 \rightarrow X$ and the elements of X .

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}: S^0 \rightarrow X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \rightarrow X$ picking the element x_0 of X , and thus we see that the bijection between pointed maps $S^0 \rightarrow X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [[MSE 884460](#)]. □

01QC 6.1.5 Active and Inert Morphisms of Pointed Sets

01QD **Definition 6.1.5.1.1.** Let $f: (X, x_0) \rightarrow (Y, y_0)$ be a morphism of pointed sets.

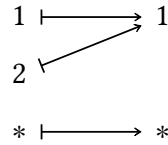
01QE 1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.

01QF 2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

01QG **Notation 6.1.5.1.2.** We write $\text{Sets}_*^{\text{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

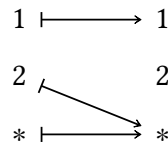
01QH **Example 6.1.5.1.3.** Here are some examples of active and inert maps of pointed sets.

01QJ 1. The map $\mu: \langle 2 \rangle \rightarrow \langle 1 \rangle$ given by



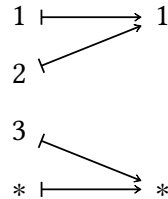
is active but not inert.

01QK 2. The map $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$ given by



is inert but not active.

01QL 3. The map $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$ given by



is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i: \langle 3 \rangle \rightarrow \langle 2 \rangle,$$

$$a: \langle 2 \rangle \rightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by

$$\begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \xrightarrow{\quad} & 2 \\ 3 & \searrow & \\ * & \xrightarrow{\quad} & * \end{array} \quad \begin{array}{ccc} 1 & \xrightarrow{\quad} & 1 \\ 2 & \searrow & 1 \\ * & \xrightarrow{\quad} & *, \end{array}$$

with i being inert and a being active.

01QM Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

01QN 1. *Active-Inert Factorisation.* Every morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

01QP (a) The map $i: (X, x_0) \rightarrow (K, k_0)$ is an inert morphism of pointed sets

01QQ (b) The map $a: (K, k_0) \rightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in \mathbf{Sets}_* .

Proof. Item 1, Active-Inert Factorisation: Let $f: X \rightarrow Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= (X \setminus f^{-1}(y_0)) \cup \{x_0\}; \end{aligned}$$

- $i: X \rightarrow K$ is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

- $a: K \rightarrow Y$ is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & & \downarrow g \\ A & \xrightarrow{a} & B \end{array}$$

be a commutative diagram in \mathbf{Sets}_* . Consider the morphism $\phi: Y \rightarrow A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ f \downarrow & \exists! \nearrow \phi & \downarrow g \\ A & \xrightarrow{a} & B. \end{array}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\stackrel{\text{def}}{=} \phi(i(x)) \\ &\stackrel{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned}
 [a \circ \phi](y) &\stackrel{\text{def}}{=} a(\phi(y)) \\
 &\stackrel{\text{def}}{=} a(f(i^{-1}(y))) \\
 &\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y)) \\
 &= [g \circ i](i^{-1}(y)) \\
 &\stackrel{\text{def}}{=} g(i(i^{-1}(y))) \\
 &\stackrel{\text{def}}{=} g(y)
 \end{aligned}$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{array}{ccc}
 X & \xrightarrow{i} & Y \\
 f \downarrow & \swarrow \psi & \downarrow g \\
 A & \xrightarrow{a} & B
 \end{array}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned}
 \psi(y) &= \psi(i(x)) \\
 &= f(x) \\
 &= f(i^{-1}(y)) \\
 &\stackrel{\text{def}}{=} \phi(y).
 \end{aligned}$$

This finishes the proof. □

00A2 6.2 Limits of Pointed Sets

00A3 6.2.1 The Terminal Pointed Set

00A4 **Definition 6.2.1.1.1.** The **terminal pointed set** is the terminal object of \mathbf{Sets}_* as in Limits and Colimits, ??.

0250 **Construction 6.2.1.1.2.** Concretely, the **terminal pointed set** is the pair $((\text{pt}, \star), \{!_X\}_{(X, x_0) \in \text{Obj}(\mathbf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set (pt, \star) .
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X: (X, x_0) \rightarrow (\text{pt}, \star)\}_{(X, x_0) \in \text{Obj}(\text{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\text{Sets})$.

Proof. We claim that (pt, \star) is the terminal object of Sets_* . Indeed, suppose we have a diagram of the form

$$(X, x_0) \quad (\text{pt}, \star)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (X, x_0) \rightarrow (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely $!_X$. □

00A5 6.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A6 **Definition 6.2.2.1.1.** The **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the product of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* as in Limits and Colimits, ??.

0251 **Construction 6.2.2.1.2.** Concretely, the **product** of $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\text{pr}_i\}_{i \in I})$ consisting of:

- *The Limit.* The pointed set $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$.
- *The Cone.* The collection

$$\left\{ \text{pr}_i: \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \rightarrow (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\text{pr}_i((x_j)_{j \in I}) \stackrel{\text{def}}{=} x_i$$

for each $(x_j)_{j \in I} \in \prod_{i \in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ is the categorical product of $\{(X_i, x_0^i)\}_{i \in I}$ in \mathbf{Sets}_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} (P, *) & & \\ & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$$

making the diagram

$$\begin{array}{ccc} (P, *) & & \\ \downarrow \phi \exists! & \searrow p_i & \\ (\prod_{i \in I} X_i, (x_0^i)_{i \in I}) & \xrightarrow{\text{pr}_i} & (X_i, x_0^i) \end{array}$$

commute, being uniquely determined by the condition $\text{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i \in I} \\ &= (x_0^i)_{i \in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. \square

00A7 Proposition 6.2.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00A8 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ defines a functor

$$\prod_{i \in I}: \mathbf{Fun}(I_{\text{disc}}, \mathbf{Sets}_*) \rightarrow \mathbf{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??. \square

00A9 6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA **Definition 6.2.3.1.1.** The **product of (X, x_0) and (Y, y_0)** is the product of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* as in Limits and Colimits, ??.

0252 **Construction 6.2.3.1.2.** Concretely, the **product of (X, x_0) and (Y, y_0)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{pr}_1 &: (X \times Y, (x_0, y_0)) \rightarrow (X, x_0), \\ \mathrm{pr}_2 &: (X \times Y, (x_0, y_0)) \rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\mathrm{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\mathrm{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (P, *) & & \\ & \swarrow p_1 & & \searrow p_2 & \\ (X, x_0) & \xleftarrow{\mathrm{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\mathrm{pr}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{array}{ccccc}
 & & (P, *) & & \\
 & \swarrow p_1 & \downarrow \phi \quad \exists! & \searrow p_2 & \\
 (X, x_0) & \xleftarrow{\text{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0)
 \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
 \text{pr}_1 \circ \phi &= p_1, \\
 \text{pr}_2 \circ \phi &= p_2
 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= (p_1(*), p_2(*)) \\
 &= (x_0, y_0),
 \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

00AB Proposition 6.2.3.1.3. Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets.

00AC 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned}
 A \times -: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\
 - \times B: \quad \text{Sets}_* &\rightarrow \text{Sets}_*, \\
 -_1 \times -_2: \text{Sets}_* \times \text{Sets}_* &\rightarrow \text{Sets}_*,
 \end{aligned}$$

defined in the same way as the functors of **Constructions With Sets, Item 1** of **Definition 4.1.3.1.3**.

- 01QR** 2. *Lack of Adjointness*. The functors $X \times -$ and $- \times Y$ do not admit right adjoints.
- 00AD** 3. *Associativity*. We have an isomorphism of pointed sets
- $$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$
- natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \text{Obj}(\text{Sets}_*)$.
- 00AE** 4. *Unitality*. We have isomorphisms of pointed sets
- $$\begin{aligned} (\text{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\text{pt}, \star) &\cong (X, x_0), \end{aligned}$$
- natural in $(X, x_0) \in \text{Obj}(\text{Sets}_*)$.
- 00AF** 5. *Commutativity*. We have an isomorphism of pointed sets
- $$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$
- natural in $(X, x_0), (Y, y_0) \in \text{Obj}(\text{Sets}_*)$.
- 00AG** 6. *Symmetric Monoidality*. The triple $(\text{Sets}_*, \times, (\text{pt}, \star))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality**: This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

Item 2, Lack of Adjointness: See [MSE 2855868].

Item 3, Associativity: This follows from **Constructions With Sets, Item 4** of **Definition 4.1.3.1.3**.

Item 4, Unitality: This follows from **Constructions With Sets, Item 5** of **Definition 4.1.3.1.3**.

Item 5, Commutativity: This follows from **Constructions With Sets, Item 6** of **Definition 4.1.3.1.3**.

Item 6, Symmetric Monoidality: This follows from **Constructions With Sets, Item 14** of **Definition 4.1.3.1.3**. \square

00AH 6.2.4 Pullbacks

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (X, x_0) \rightarrow (Z, z_0)$ and $g: (Y, y_0) \rightarrow (Z, z_0)$ be morphisms of pointed sets.

00AJ **Definition 6.2.4.1.1.** The **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in \mathbf{Sets}_* as in Limits and Colimits, ??.

0253 **Construction 6.2.4.1.2.** Concretely, the **pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \text{pr}_1: (X \times_Z Y, (x_0, y_0)) &\rightarrow (X, x_0), \\ \text{pr}_2: (X \times_Z Y, (x_0, y_0)) &\rightarrow (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \text{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \text{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$\begin{array}{ccc} (X \times_Z Y, (x_0, y_0)) & \xrightarrow{\text{pr}_2} & (Y, y_0) \\ \text{pr}_1 \downarrow & & \downarrow g \\ (X, x_0) & \xrightarrow{f} & (Z, z_0). \end{array}$$

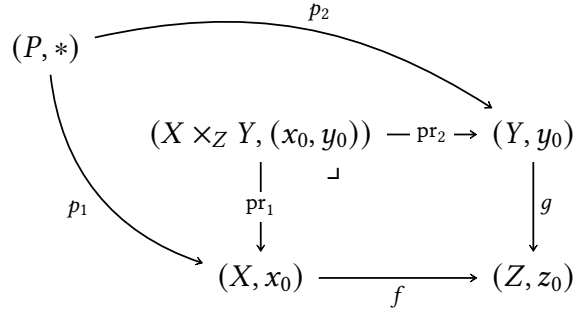
$f \circ \text{pr}_1 = g \circ \text{pr}_2$,

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \text{pr}_1](x, y) &= f(\text{pr}_1(x, y)) \\ &= f(x) \end{aligned}$$

$$\begin{aligned}
&= g(y) \\
&= g(\text{pr}_2(x, y)) \\
&= [g \circ \text{pr}_2](x, y),
\end{aligned}$$

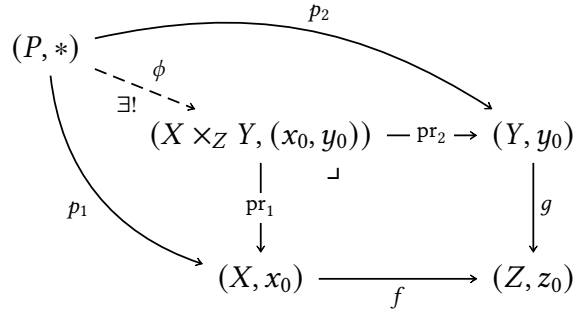
where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\begin{aligned}
\text{pr}_1 \circ \phi &= p_1, \\
\text{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. \square

00AK Proposition 6.2.4.1.3. Let (X, x_0) , (Y, y_0) , (Z, z_0) , and (A, a_0) be pointed sets.

00AL 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-3} -_1 : \text{Fun}(\mathcal{P}, \text{Sets}_*) \rightarrow \text{Sets}_*,$$

where \mathcal{P} is the category that looks like this:

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

In particular, the action on morphisms of $-_1 \times_{-3} -_1$ is given by sending a morphism

$$\begin{array}{ccccc} X \times_Z Y & \xrightarrow{\quad} & Y & & \\ \downarrow & \lrcorner & \downarrow g & \searrow \psi & \\ & & X' \times_{Z'} Y' & \xrightarrow{\quad} & Y' \\ \downarrow & \lrcorner & \downarrow & & \downarrow g' \\ X & \xrightarrow{f} & Z & \searrow \chi & \\ \downarrow \phi & & \downarrow & & \\ & & X' & \xrightarrow{f'} & Z' \end{array}$$

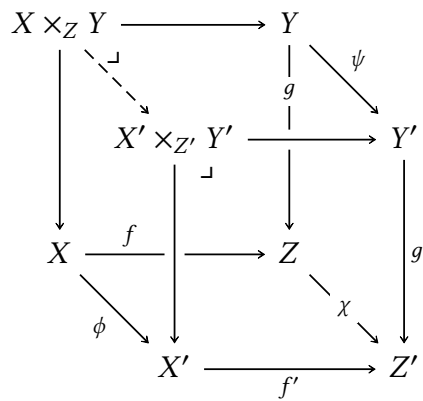
in $\text{Fun}(\mathcal{P}, \text{Sets}_*)$ to the morphism of pointed sets

$$\xi: (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

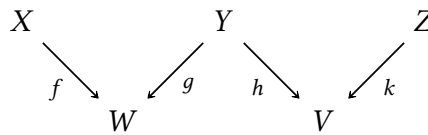
for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

00AM

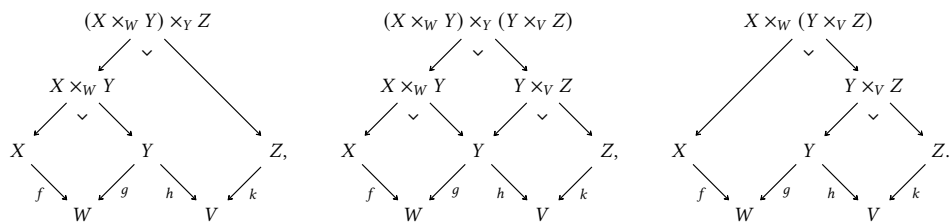
2. *Associativity.* Given a diagram



in Sets_* , we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams



00AN 3. *Unitality*. We have isomorphisms of pointed sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \downarrow f & \lrcorner & \downarrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 X \times_X A \cong A, \\
 A \times_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{f} & X \\
 \parallel \lrcorner \parallel & & \parallel \\
 X & \xrightarrow{f} & X.
 \end{array}$$

00AP 4. *Commutativity*. We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 A \times_X B & \longrightarrow & B \\
 \downarrow \lrcorner & & \downarrow g \\
 A & \xrightarrow{f} & X,
 \end{array}
 \quad
 A \times_X B \cong B \times_X A
 \quad
 \begin{array}{ccc}
 B \times_X A & \longrightarrow & A \\
 \downarrow \lrcorner & & \downarrow f \\
 B & \xrightarrow{g} & X.
 \end{array}$$

00AQ 5. *Interaction With Products*. We have an isomorphism of pointed sets

$$\begin{array}{ccc}
 X \times Y & \longrightarrow & Y \\
 \downarrow \lrcorner & & \downarrow !_Y \\
 X & \xrightarrow{!_X} & \text{pt.}
 \end{array}
 \quad
 X \times_{\text{pt}} Y \cong X \times Y,$$

00AR 6. *Symmetric Monoidality*. The triple $(\text{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality**: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from **Constructions With Sets, Item 4 of Definition 4.1.4.1.5**.

Item 3, Unitality: This follows from **Constructions With Sets, Item 6 of Definition 4.1.4.1.5**.

Item 4, Commutativity: This follows from **Constructions With Sets, Item 7 of Definition 4.1.4.1.5**.

Item 5, Interaction With Products: This follows from **Constructions With Sets**, **Item 10** of **Definition 4.1.4.1.5**.

Item 6, Symmetric Monoidality: This follows from **Constructions With Sets**, **Item 11** of **Definition 4.1.4.1.5**. \square

00AS 6.2.5 Equalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT Definition 6.2.5.1.1. The **equaliser of (f, g)** is the equaliser of f and g in \mathbf{Sets}_* as in Limits and Colimits, ??.

0254 Construction 6.2.5.1.2. Concretely, the **equaliser of (f, g)** is the pair consisting of:

- *The Limit.* The pointed set $(\text{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\text{eq}(f, g): (\text{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

given by the canonical inclusion $\text{eq}(f, g) \hookrightarrow \text{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\text{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in \mathbf{Sets}_* . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \text{eq}(f, g) = g \circ \text{eq}(f, g),$$

which indeed holds by the definition of the set $\text{Eq}(f, g)$. Next, we prove that $\text{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) \xrightleftharpoons[g]{f} (Y, y_0) \\ & \nearrow e & \\ (E, *) & & \end{array}$$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (E, *) \rightarrow (\text{Eq}(f, g), x_0)$$

making the diagram

$$\begin{array}{ccccc}
 (\text{Eq}(f, g), x_0) & \xrightarrow{\text{eq}(f, g)} & (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) \\
 \uparrow \phi \exists! & & \nearrow e & & \\
 (E, *) & & & &
 \end{array}$$

commute, being uniquely determined by the condition

$$\text{eq}(f, g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\text{Eq}(f, g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \text{Eq}(f, g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
 \phi(*) &= e(*) \\
 &= x_0,
 \end{aligned}$$

where we have used that e is a morphism of pointed sets. \square

00AU Proposition 6.2.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00AV 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{Eq}(f \circ \text{eq}(g, h), g \circ \text{eq}(g, h))}_{=\text{Eq}(f \circ \text{eq}(g, h), h \circ \text{eq}(g, h))} \cong \text{Eq}(f, g, h) \cong \underbrace{\text{Eq}(f \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}_{=\text{Eq}(g \circ \text{eq}(f, g), h \circ \text{eq}(f, g))}$$

where $\text{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \begin{array}{c} \xrightarrow{f} \\ \xrightarrow[g]{\quad} \\ \xrightarrow{h} \end{array} (Y, y_0)$$

in \mathbf{Sets}_* , being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW 2. *Unitality*. We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

00AX 3. *Commutativity*. We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from **Constructions With Sets, Item 1** of **Definition 4.1.5.1.3**.

Item 2, Unitality: This follows from **Constructions With Sets, Item 4** of **Definition 4.1.5.1.3**.

Item 3, Commutativity: This follows from **Constructions With Sets, Item 5** of **Definition 4.1.5.1.3**. \square

00AY 6.3 Colimits of Pointed Sets

00AZ 6.3.1 The Initial Pointed Set

00B0 **Definition 6.3.1.1.1**. The **initial pointed set** is the initial object of \mathbf{Sets}_* as in **Limits and Colimits, ??**.

0255 **Construction 6.3.1.1.2**. Concretely, the **initial pointed set** is the pair $((\mathrm{pt}, \star), \{\iota_X\}_{(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)})$ consisting of:

- *The Limit*. The pointed set (pt, \star) .
- *The Cone*. The collection of morphisms of pointed sets

$$\{\iota_X : (\mathrm{pt}, \star) \rightarrow (X, x_0)\}_{(X, x_0) \in \mathrm{Obj}(\mathbf{Sets}_*)}$$

defined by

$$\iota_X(\star) \stackrel{\mathrm{def}}{=} x_0.$$

Proof. We claim that (pt, \star) is the initial object of Sets_* . Indeed, suppose we have a diagram of the form

$$(\text{pt}, \star) \quad (X, x_0)$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: (\text{pt}, \star) \rightarrow (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow[\exists!]{\phi} (X, x_0)$$

commute, namely ι_X . □

00B1 6.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B2 **Definition 6.3.2.1.1.** The **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ ¹¹ is the coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* as in Limits and Colimits, ??.

0256 **Construction 6.3.2.1.2.** Concretely, the **coproduct of the family** $\{(X_i, x_0^i)\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \{\text{inj}_i\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:

- *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left(\bigsqcup_{i \in I} X_i \right) / \sim,$$

where \sim is the equivalence relation on $\bigsqcup_{i \in I} X_i$ given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each $i, j \in I$.

¹¹*Further Terminology:* Also called the **wedge sum of the family** $\{(X_i, x_0^i)\}_{i \in I}$.

– *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(i, x_0^i)] \\ &= [(j, x_0^j)] \end{aligned}$$

for any $i, j \in I$.

• *The Cocone.* The collection

$$\left\{ \text{inj}_i: (X_i, x_0^i) \rightarrow \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\text{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i \in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i \in I}$ in Sets_* . Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow l_i & \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

in Sets_* . Then there exists a unique morphism of pointed sets

$$\phi: \left(\bigvee_{i \in I} X_i, p_0 \right) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccc} & & (C, *) \\ & \nearrow l_i & \uparrow \phi \mid \exists! \\ (X_i, x_0^i) & \xrightarrow{\text{inj}_i} & \left(\bigvee_{i \in I} X_i, p_0 \right) \end{array}$$

commute, being uniquely determined by the condition $\phi \circ \text{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i, x)]) = \iota_i(x)$$

for each $[(i, x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i([(i, x_0^i)]) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. \square

00B3 Proposition 6.3.2.1.3. Let $\{(X_i, x_0^i)\}_{i \in I}$ be a family of pointed sets.

00B4 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \text{Fun}(I_{\text{disc}}, \text{Sets}_*) \rightarrow \text{Sets}_*.$$

Proof. **Item 1, Functoriality:** This follows from Limits and Colimits, ?? of ??. \square

00B5 6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

00B6 Definition 6.3.3.1.1. The **coproduct of (X, x_0) and (Y, y_0)** ¹² is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, ??.

0257 Construction 6.3.3.1.2. Concretely, the **coproduct of (X, x_0) and (Y, y_0)** , also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

– *The Underlying Set.* The set $X \vee Y$ defined by

$$\begin{aligned} (X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \amalg (Y, y_0) \\ &\cong (X \amalg_{\text{pt}} Y, p_0) \\ &\cong (X \amalg Y / \sim, p_0), \end{aligned} \quad \begin{array}{ccc} X \vee Y & \longleftarrow & Y \\ \uparrow \scriptstyle \Gamma & & \uparrow \scriptstyle [y_0] \\ X & \xleftarrow{\scriptstyle [x_0]} & \text{pt}, \end{array}$$

¹²*Further Terminology:* Also called the **wedge sum of (X, x_0) and (Y, y_0)** .

where \sim is the equivalence relation on $X \amalg Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

– *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$\begin{aligned} p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\ &= [(1, y_0)]. \end{aligned}$$

- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned} \text{inj}_1 &: (X, x_0) \rightarrow (X \vee Y, p_0), \\ \text{inj}_2 &: (Y, y_0) \rightarrow (X \vee Y, p_0), \end{aligned}$$

given by

$$\begin{aligned} \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\ \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)], \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in \mathbf{Sets}_* . Indeed, suppose we have a diagram of the form

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_1 & & \nwarrow \iota_2 & \\ (X, x_0) & \xrightarrow{\text{inj}_1} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \end{array}$$

in \mathbf{Sets} . Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \rightarrow (C, *)$$

making the diagram

$$\begin{array}{ccccc} & & (C, *) & & \\ & \nearrow \iota_1 & \uparrow \phi \mid \exists! & \nwarrow \iota_2 & \\ (X, x_0) & \xrightarrow{\text{inj}_1} & (X \vee Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \end{array}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \text{inj}_X &= \iota_X, \\ \phi \circ \text{inj}_Y &= \iota_Y\end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \iota_X([(0, x_0)]) \\ &= \iota_Y([(1, y_0)]) \\ &= *,\end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

00B7 Proposition 6.3.3.1.3. Let (X, x_0) and (Y, y_0) be pointed sets.

00B8 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned}X \vee - &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ - \vee Y &: \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*, \\ -_1 \vee -_2 &: \mathbf{Sets}_* \times \mathbf{Sets}_* \rightarrow \mathbf{Sets}_*.\end{aligned}$$

00B9 2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Sets}_*$.

00BA 3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}(\text{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\text{pt}, *) &\cong (X, x_0),\end{aligned}$$

natural in $(X, x_0) \in \mathbf{Sets}_*$.

00BB 4. *Commutativity*. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathbf{Sets}_*$.

00BC 5. *Symmetric Monoidality*. The triple $(\mathbf{Sets}_*, \vee, \text{pt})$ is a symmetric monoidal category.

00BD 6. *The Fold Map*. We have a natural transformation

$$\nabla: \vee \circ \Delta_{\mathbf{Sets}_*}^{\text{Cats}} \Rightarrow \text{id}_{\mathbf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X: X \vee X \rightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. **Item 1, Functoriality**: This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

Item 5, Symmetric Monoidality: Omitted.

Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f: (X, x_0) \rightarrow (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

Indeed, we have

$$\begin{aligned}
 [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\
 &= f(x) \\
 &= f(\nabla_X([(i, x)])) \\
 &= [f \circ \nabla_X]([(i, x)])
 \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. \square

00BE 6.3.4 Pushouts

Let (X, x_0) , (Y, y_0) , and (Z, z_0) be pointed sets and let $f: (Z, z_0) \rightarrow (X, x_0)$ and $g: (Z, z_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BF Definition 6.3.4.1.1. The **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in \mathbf{Sets}_* as in Limits and Colimits, ??.

0258 Construction 6.3.4.1.2. Concretely, the **pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g)** is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f, Z, g} Y, p_0)$, where:
 - The set $X \coprod_{f, Z, g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g ;
 - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}
 \text{inj}_1: (X, x_0) &\rightarrow (X \coprod_Z Y, p_0), \\
 \text{inj}_2: (Y, y_0) &\rightarrow (X \coprod_Z Y, p_0)
 \end{aligned}$$

given by

$$\begin{aligned}
 \text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)] \\
 \text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)]
 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned} x_0 &= f(z_0), \\ y_0 &= g(z_0) \end{aligned}$$

since f and g are morphisms of pointed sets, with the relation \sim on $X \amalg_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \amalg_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in \mathbf{Sets}_* . First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\begin{array}{ccc} (X \amalg_Z Y, p_0) & \xleftarrow{\text{inj}_2} & (Y, y_0) \\ \text{inj}_1 \uparrow & & \uparrow g \\ (X, x_0) & \xleftarrow{f} & (Z, z_0) \end{array}$$

$\text{inj}_1 \circ f = \text{inj}_2 \circ g$,

Indeed, given $z \in Z$, we have

$$\begin{aligned} [\text{inj}_1 \circ f](z) &= \text{inj}_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \text{inj}_2(g(z)) \\ &= [\text{inj}_2 \circ g](z), \end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation \sim on $X \amalg Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \amalg_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

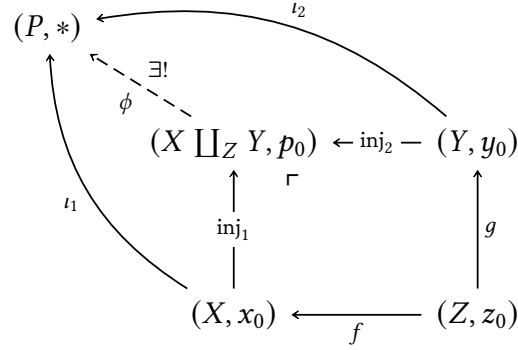
$$\begin{array}{ccc} (P, *) & \xleftarrow{\iota_2} & (X \amalg_Z Y, p_0) \xleftarrow{\text{inj}_2} (Y, y_0) \\ & \uparrow \iota_1 & \uparrow \text{inj}_1 \\ (X, x_0) & \xleftarrow{f} & (Z, z_0) \end{array}$$

$\uparrow g$

in \mathbf{Sets}_* . Then there exists a unique morphism of pointed sets

$$\phi: (X \amalg_Z Y, p_0) \rightarrow (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \text{inj}_1 = \iota_1,$$

$$\phi \circ \text{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of **Constructions With Sets, Definition 4.2.4.1.1**. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \phi([(0, x_0)]) \\ &= \iota_1(x_0) \\ &= *, \end{aligned}$$

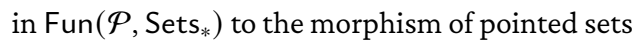
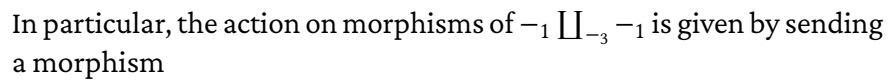
or alternatively

$$\begin{aligned} \phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *, \end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. \square

00BH 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \amalg_{f, Z, g} Y$ defines a functor

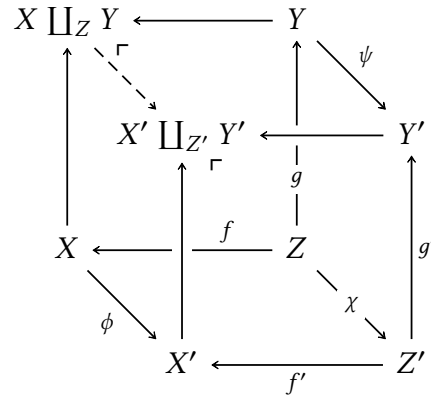
where \mathcal{P} is the category that looks like this:



given by

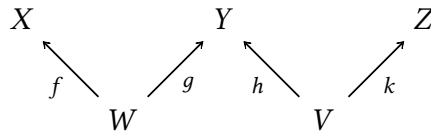
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \amalg_Z Y$, which is the unique morphism of pointed sets making the diagram



commute.

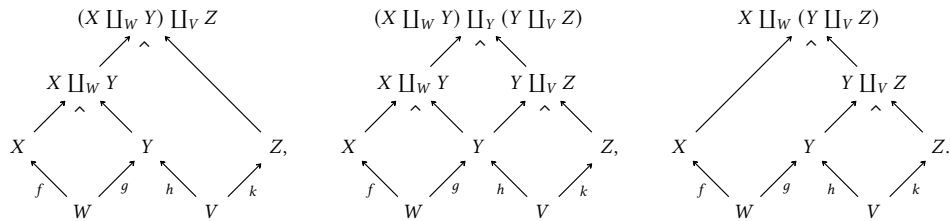
00BJ 2. *Associativity.* Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \amalg_W Y) \amalg_V Z \cong (X \amalg_W Y) \amalg_Y (Y \amalg_V Z) \cong X \amalg_W (Y \amalg_V Z),$$

where these pullbacks are built as in the diagrams



00BK 3. *Unitality*. We have isomorphisms of sets

$$\begin{array}{ccc}
 A & \xlongequal{\quad} & A \\
 \uparrow f & \ulcorner & \uparrow f \\
 X & \xlongequal{\quad} & X
 \end{array}
 \quad
 \begin{array}{l}
 X \coprod_X A \cong A, \\
 A \coprod_X X \cong A,
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xleftarrow{f} & X \\
 \uparrow \ulcorner & & \uparrow \llcorner \\
 X & \xleftarrow{f} & X.
 \end{array}$$

00BL 4. *Commutativity*. We have an isomorphism of sets

$$\begin{array}{ccc}
 X \coprod_Z Y & \xleftarrow{\quad} & Y \\
 \uparrow \ulcorner & & \uparrow g \\
 X & \xleftarrow{f} & Z,
 \end{array}
 \quad
 X \coprod_Z Y \cong Y \coprod_Z X
 \quad
 \begin{array}{ccc}
 Y \coprod_Z X & \xleftarrow{\quad} & X \\
 \uparrow \ulcorner & & \uparrow f \\
 Y & \xleftarrow{g} & Z.
 \end{array}$$

00BM 5. *Interaction With Coproducts*. We have

$$\begin{array}{ccc}
 X \vee Y & \xleftarrow{\quad} & Y \\
 \uparrow \ulcorner & & \uparrow [y_0] \\
 X & \xleftarrow{[x_0]} & \text{pt.}
 \end{array}
 \quad
 X \coprod_{\text{pt}} Y \cong X \vee Y,$$

00BN 6. *Symmetric Monoidality*. The triple $(\text{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. **Item 1, Functoriality:** This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: This follows from **Constructions With Sets, Item 3 of Definition 4.2.4.1.6**.

Item 3, Unitality: This follows from **Constructions With Sets, Item 5 of Definition 4.2.4.1.6**.

Item 4, Commutativity: This follows from **Constructions With Sets, Item 6 of Definition 4.2.4.1.6**.

Item 5, Interaction With Coproducts: Omitted.

Item 6, Symmetric Monoidality: Omitted. □

00BP 6.3.5 Coequalisers

Let $f, g: (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ **Definition 6.3.5.1.1.** The **coequaliser of** (f, g) is the pointed set $(\text{CoEq}(f, g), [y_0])$.

0259 **Construction 6.3.5.1.2.** The **coequaliser of** (f, g) is the pair $((\text{CoEq}(f, g), [y_0]), \text{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\text{CoEq}(f, g), [y_0])$, where $\text{CoEq}(f, g)$ is the coequaliser of f and g as in **Constructions With Sets, Definition 4.2.5.1.1.**
- *The Cocone.* The map

$$\text{coeq}(f, g): Y \twoheadrightarrow (\text{CoEq}(f, g), [y_0])$$

given by the quotient map, as in **Constructions With Sets, Item 2 of Definition 4.2.5.1.2.**

Proof. We claim that $(\text{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_* . First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\text{coeq}(f, g) \circ f = \text{coeq}(f, g) \circ g.$$

Indeed, we have

$$\begin{aligned} [\text{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](f(x)) \\ &\stackrel{\text{def}}{=} [f(x)] \\ &= [g(x)] \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g)](g(x)) \\ &\stackrel{\text{def}}{=} [\text{coeq}(f, g) \circ g](x) \end{aligned}$$

for each $x \in X$. Next, we prove that $\text{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{array}{ccc} (X, x_0) & \xrightleftharpoons[g]{f} & (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} & (\text{CoEq}(f, g), [y_0]) \\ & & & \searrow c & \\ & & & & (C, *) \end{array}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from **Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3** that there exists a unique map $\phi: \text{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{array}{ccc} (X, x_0) & \xrightarrow[g]{f} & (Y, y_0) & \xrightarrow{\text{coeq}(f, g)} & (\text{CoEq}(f, g), [y_0]) \\ & & \searrow c & & \downarrow \phi \mid \exists! \\ & & & & (C, *) \end{array}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \text{coeq}(f, g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that c is a morphism of pointed sets. \square

00BR Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h: (X, x_0) \rightarrow (Y, y_0)$ be morphisms of pointed sets.

00BS 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\text{CoEq}(\text{coeq}(f, g) \circ f, \text{coeq}(f, g) \circ h)}_{= \text{CoEq}(\text{coeq}(f, g) \circ g, \text{coeq}(f, g) \circ h)} \cong \text{CoEq}(f, g, h) \cong \underbrace{\text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ g)}_{= \text{CoEq}(\text{coeq}(g, h) \circ f, \text{coeq}(g, h) \circ h)},$$

where $\text{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \xrightarrow[g]{f} (Y, y_0) \xrightarrow[h]{g}$$

in Sets_* .

00BT 2. *Unitality.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, f) \cong B.$$

00BU 3. *Commutativity.* We have an isomorphism of pointed sets

$$\text{CoEq}(f, g) \cong \text{CoEq}(g, f).$$

Proof. **Item 1, Associativity:** This follows from **Constructions With Sets, Item 1** of **Definition 4.2.5.1.5**.

Item 2, Unitality: This follows from **Constructions With Sets, Item 4** of **Definition 4.2.5.1.5**.

Item 3, Commutativity: This follows from **Constructions With Sets, Item 5** of **Definition 4.2.5.1.5**. \square

00BV 6.4 Constructions With Pointed Sets

00BW 6.4.1 Free Pointed Sets

Let X be a set.

00BX **Definition 6.4.1.1.1.** The **free pointed set on X** is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by¹³

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \coprod \text{pt} \\ &\stackrel{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element \star of X^+ .

00BY **Proposition 6.4.1.1.2.** Let X be a set.

00BZ 1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ : \mathbf{Sets} \rightarrow \mathbf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \mathbf{Obj}(\mathbf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of **Definition 6.4.1.1.1**.

¹³*Further Notation:* We sometimes write \star_X for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

- *Action on Morphisms.* For each morphism $f: X \rightarrow Y$ of **Sets**, the image

$$f^+: X^+ \rightarrow Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

- 00C0 2. *Adjointness.* We have an adjunction

$$((-)^+ \dashv \text{forget}) : \text{Sets} \begin{array}{c} \xrightarrow{(-)^+} \\ \perp \\ \xleftarrow{\text{forget}} \end{array} \text{Sets}_*,$$

witnessed by a bijection of sets

$$\text{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \text{Sets}(X, Y),$$

natural in $X \in \text{Obj}(\text{Sets})$ and $(Y, y_0) \in \text{Obj}(\text{Sets}_*)$.

- 00C1 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^+, \amalg, (-)^+, \amalg) : (\text{Sets}, \amalg, \emptyset) \rightarrow (\text{Sets}_*, \vee, \text{pt}),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} \amalg : X^+ \vee Y^+ &\xrightarrow{\sim} (X \amalg Y)^+, \\ (-)^+_{\mathbb{1}} \amalg : \text{pt} &\xrightarrow{\sim} \emptyset^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

- 00C2 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^+, (-)^+, (-)^+_{\mathbb{1}}) : (\text{Sets}, \times, \text{pt}) \rightarrow (\text{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^+_{X,Y} : X^+ \wedge Y^+ &\xrightarrow{\sim} (X \times Y)^+, \\ (-)^+_{\mathbb{1}} : S^0 &\xrightarrow{\sim} \text{pt}^+, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\text{Sets})$.

Proof. **Item 1, Functoriality:** We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \text{Obj}(\text{Sets})$. We have

$$\text{id}_X^+(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\text{id}_X^+ = \text{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$\begin{aligned} f &: X \rightarrow Y, \\ g &: Y \rightarrow Z, \end{aligned}$$

we have

$$\begin{aligned} [g^+ \circ f^+](x) &\stackrel{\text{def}}{=} g^+(f^+(x)) \\ &\stackrel{\text{def}}{=} g^+(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^+(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned} [g^+ \circ f^+](\star_X) &\stackrel{\text{def}}{=} g^+(f^+(\star_X)) \\ &\stackrel{\text{def}}{=} g^+(\star_Y) \\ &\stackrel{\text{def}}{=} \star_Z \\ &\stackrel{\text{def}}{=} [g \circ f]^+(\star_X), \end{aligned}$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

- 025A** • *Map I.* We define a map

$$\Phi_{X,Y}: \text{Sets}_*(X^+, Y) \rightarrow \text{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi: (X^+, \star_X) \rightarrow (Y, y_0)$$

to the function

$$\xi^\dagger : X \rightarrow Y$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- 025B** • *Map II.* We define a map

$$\Psi_{X,Y} : \text{Sets}(X, Y) \rightarrow \text{Sets}_*(X^+, Y)$$

given by sending a function $\xi : X \rightarrow Y$ to the morphism of pointed sets

$$\xi^\dagger : (X^+, \star_X) \rightarrow (Y, y_0)$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- 025C** • *Invertibility I.* Given a morphism of pointed sets

$$\xi : (X^+, \star_X) \rightarrow (Y, y_0),$$

we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^\dagger) \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket \\ &= \xi \\ &\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}_*(X^+, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}_*(X^+, Y)}.$$

- 025D • *Invertibility II.* Given a map of sets $\xi: X \rightarrow Y$, we have

$$\begin{aligned}
 [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\
 &= \Phi_{X,Y}(\xi^\dagger) \\
 &= \Phi_{X,Y}(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket) \\
 &= \llbracket x \mapsto \xi(x) \rrbracket \\
 &= \xi \\
 &\stackrel{\text{def}}{=} [\text{id}_{\text{Sets}(X,Y)}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}(X,Y)}.$$

- 025E • *Naturality for Φ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc}
 \text{Sets}_*(X'^+, Y) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}(X', Y) \\
 f^* \downarrow & & \downarrow f^* \\
 \text{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}(X, Y)
 \end{array}$$

commutes. Indeed, given a morphism of pointed sets $\xi: X'^+ \rightarrow Y$, we have

$$\begin{aligned}
 [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
 &= \Phi_{X,Y}(\xi \circ f) \\
 &= \xi \circ f \\
 &= \Phi_{X',Y}(\xi) \circ f \\
 &= f^*(\Phi_{X',Y}(\xi)) \\
 &= f^*(\Phi_{X',Y}(\xi)) \\
 &= [f^* \circ \Phi_{X',Y}](\xi).
 \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

025F

- *Naturality for Φ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc} \mathbf{Sets}_*(X^+, Y) & \xrightarrow{\Phi_{X,Y}} & \mathbf{Sets}(X, Y) \\ g_* \downarrow & & \downarrow g_* \\ \mathbf{Sets}_*(X^+, Y') & \xrightarrow{\Phi_{X,Y'}} & \mathbf{Sets}(X, Y') \end{array}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger: X^+ \rightarrow Y,$$

we have

$$\begin{aligned} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

025G

- *Naturality for Ψ .* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from [Categories, Item 2](#) of [Definition 11.9.7.1.2](#) that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to \amalg and \vee as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{+, \amalg} : X^+ \vee Y^+ \xrightarrow{\sim} (X \amalg Y)^+$$

is given by

$$(-)_{X,Y}^{+, \amalg}(z) = \begin{cases} x & \text{if } z = [(0, x)] \text{ with } x \in X, \\ y & \text{if } z = [(1, y)] \text{ with } y \in Y, \\ \star_X \amalg_Y & \text{if } z = [(0, \star_X)], \\ \star_X \amalg_Y & \text{if } z = [(1, \star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)_{X,Y}^{+, \amalg, -1} : (X \amalg Y)^+ \xrightarrow{\sim} X^+ \vee Y^+$$

given by

$$(-)_{X,Y}^{+, \amalg, -1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0, x)] & \text{if } z = [(0, x)], \\ [(1, y)] & \text{if } z = [(1, y)], \\ p_0 & \text{if } z = \star_X \amalg_Y \end{cases}$$

for each $z \in (X \amalg Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+, \amalg, \mathbb{1}} : \text{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending \star_X to \star_\emptyset .

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^+$ with respect to \times and \wedge as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+ : X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^+(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)_{X,Y}^{+,-1} : (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\mathbb{1}} : S^0 \xrightarrow{\sim} \text{pt}^+$$

is given by sending 0 to \star_{pt} and 1 to \star , where $\text{pt}^+ = \{\star, \star_{\text{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

□

01QS 6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

01QT **Definition 6.4.2.1.1.** The **set with deleted basepoint associated to X** is the set X^- defined by

$$X^- \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

01QU **Proposition 6.4.2.1.2.** Let (X, x_0) be a pointed set.

01QV 1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- : \mathbf{Sets}_*^{\text{actv}} \rightarrow \mathbf{Sets},$$

where:

- *Action on Objects.* For each $X \in \text{Obj}(\mathbf{Sets}_*^{\text{actv}})$, we have

$$[(-)^-](X) \stackrel{\text{def}}{=} X^-,$$

where X^- is the set of **Definition 6.4.2.1.1**.

- *Action on Morphisms.* For each morphism $f : X \rightarrow Y$ of $\mathbf{Sets}_*^{\text{actv}}$, the image

$$f^- : X^- \rightarrow Y^-$$

of f by $(-)^-$ is the map defined by

$$f^-(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in X^-$.

01QW 2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$((-)^- \dashv (-)^+) : \mathbf{Sets}_*^{\text{actv}} \begin{array}{c} \xrightarrow{(-)^-} \\ \perp_{\text{eq}} \\ \xleftarrow{(-)^+} \end{array} \mathbf{Sets},$$

witnessed by a bijection of sets

$$\mathbf{Sets}(X^-, Y) \cong \mathbf{Sets}_*(X, Y^+),$$

natural in $X \in \text{Obj}(\mathbf{Sets}_*)$ and $Y \in \text{Obj}(\mathbf{Sets})$, and by isomorphisms

$$\begin{aligned} (X^-)^+ &\cong X, \\ (Y^+)^- &\cong Y, \end{aligned}$$

once again natural in $X \in \text{Obj}(\mathbf{Sets}_*)$ and $Y \in \text{Obj}(\mathbf{Sets})$.

01QX 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor

of **Item 1** has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\vee}, (-)^{-,\vee}_{\mathbb{1}}): (\mathbf{Sets}_*^{\text{actv}}, \vee, \text{pt}), \rightarrow (\mathbf{Sets}, \coprod, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^{-,\vee}_{X,Y}: X^- \coprod Y^- &\xrightarrow{\sim} (X \vee Y)^-, \\ (-)^{-,\vee}_{\mathbb{1}}: \emptyset &\xrightarrow{\sim} \text{pt}^-, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

01QY

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of **Item 1** has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)^{-,\times}_{\mathbb{1}}): (\mathbf{Sets}_*^{\text{actv}}, \wedge, S^0), \rightarrow (\mathbf{Sets}, \times, \text{pt})$$

being equipped with isomorphisms of pointed sets

$$\begin{aligned} (-)^{-,\times}_{X,Y}: X^- \times Y^- &\xrightarrow{\sim} (X \wedge Y)^-, \\ (-)^{-,\times}_{\mathbb{1}}: \text{pt} &\xrightarrow{\sim} (S^0)^-, \end{aligned}$$

natural in $X, Y \in \text{Obj}(\mathbf{Sets})$.

Proof. Item 1, Functoriality: We claim that $(-)^-$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \text{Obj}(\mathbf{Sets})$. We have

$$\text{id}_X^-(x) \stackrel{\text{def}}{=} x$$

for each $x \in X^-$, so $\text{id}_X^- = \text{id}_{X^-}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$\begin{aligned} f: (X, x_0) &\rightarrow (Y, y_0), \\ g: (Y, y_0) &\rightarrow (Z, z_0), \end{aligned}$$

we have

$$\begin{aligned} [g^- \circ f^-](x) &\stackrel{\text{def}}{=} g^-(f^-(x)) \\ &\stackrel{\text{def}}{=} g^-(f(x)) \\ &\stackrel{\text{def}}{=} g(f(x)) \\ &\stackrel{\text{def}}{=} [g \circ f]^-(x) \end{aligned}$$

for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2, Adjoint Equivalence: We proceed in a few steps:

025H 1. *Map I.* We define a map

$$\Phi_{X,Y}: \text{Sets}(X^-, Y) \rightarrow \text{Sets}_*^{\text{actv}}(X, Y^+)$$

by sending a map $\xi: X^- \rightarrow Y$ to the active morphism of pointed sets

$$\xi^\dagger: X \rightarrow Y^+$$

given by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

025J 2. *Map II.* We define a map

$$\Psi_{X,Y}: \text{Sets}_*^{\text{actv}}(X, Y^+) \rightarrow \text{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi: X \rightarrow Y^+$ to the map

$$\xi^\dagger: X^- \rightarrow Y$$

defined by

$$\xi^\dagger(x) \stackrel{\text{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

025K 3. *Invertibility I.* Given a map of sets $\xi: X^- \rightarrow Y$, we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket) \\ &= \llbracket x \mapsto \xi(x) \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}(X^-, Y)}](\xi). \end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \text{id}_{\text{Sets}(X^-, Y)}.$$

025L 4. *Invertibility II.* Given a morphism of pointed sets

$$\xi: (X, x_0) \rightarrow (Y^+, \star_Y),$$

we have

$$\begin{aligned} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\ &= \xi \\ &= [\text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \text{id}_{\text{Sets}_*^{\text{actv}}(X, Y^+)}.$$

025M 5. *Naturality for Φ , Part I.* We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \rightarrow (X', x'_0),$$

the diagram

$$\begin{array}{ccc} \text{Sets}(X', -, Y) & \xrightarrow{\Phi_{X',Y}} & \text{Sets}_*^{\text{actv}}(X', Y^+) \\ f^* \downarrow & & \downarrow f^* \\ \text{Sets}_*(X^-, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}_*^{\text{actv}}(X, Y^+) \end{array}$$

commutes. Indeed, given a map of sets $\xi: X' \rightarrow Y$, we have

$$\begin{aligned} [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= \llbracket x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^- \\ \star_Y & \text{if } f(x) = x'_0 \end{cases} \rrbracket \\ &= f^*(\llbracket x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^- \\ \star_Y & \text{if } x' = x'_0 \end{cases} \rrbracket) \end{aligned}$$

$$\begin{aligned}
&= f^*(\Phi_{X',Y}(\xi)) \\
&= [f^* \circ \Phi_{X',Y}](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

025N 6. *Naturality for Φ , Part II.* We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \rightarrow (Y', y'_0),$$

the diagram

$$\begin{array}{ccc}
\text{Sets}(X^-, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Sets}_*^{\text{actv}}(X, Y^+) \\
\downarrow g_* & & \downarrow g_* \\
\text{Sets}(X^-, Y') & \xrightarrow{\Phi_{X,Y'}} & \text{Sets}_*^{\text{actv}}(X, Y'^+)
\end{array}$$

commutes. Indeed, given a map of sets $\xi: X^- \rightarrow Y$, we have

$$\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= \llbracket x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases} \rrbracket \\
&= g_*\left(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket\right) \\
&= g_*(\Phi_{X,Y}(\xi)) \\
&= [g_* \circ \Phi_{X,Y}](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y},$$

and the naturality diagram for Φ above indeed commutes.

- 025P 7. *Naturality for Ψ* . Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from **Categories, Item 2** of **Definition 11.9.7.1.2** that Ψ is also natural in each argument.
- 025Q 8. *Fully Faithfulness of $(-)^-$* . We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)_{X,Y}^-: \mathbf{Sets}_*^{\text{actv}}(X, Y) \xrightarrow{\sim} \mathbf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)_{X,Y}^{-,-1}: \mathbf{Sets}(X^-, Y^-) \xrightarrow{\sim} \mathbf{Sets}_*^{\text{actv}}(X, Y)$$

is given by sending a map of sets $f: X^- \rightarrow Y^-$ to the active morphism of pointed sets $f^\dagger: X \rightarrow Y$ defined by

$$f^\dagger(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

- 025R 9. *Essential Surjectivity of $(-)^-$* . We need to show that, given an object $X \in \mathbf{Obj}(\mathbf{Sets})$, there exists some $X' \in \mathbf{Obj}(\mathbf{Sets}_*^{\text{actv}})$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$\begin{aligned} (X^+)^- &\stackrel{\text{def}}{=} (X \cup \{\star_X\})^- \\ &\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

- 025S 10. *The Functor $(-)^-$ Is an Equivalence*. Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by **Categories, Item 1** of **Definition 11.6.7.1.2**.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^-$ with respect to \vee and \coprod as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{-,\vee}: X^- \amalg Y^- \xrightarrow{\sim} (X \vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0, x)] & \text{if } z = (0, x) \text{ with } x \in X, \\ [(1, y)] & \text{if } z = (1, y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \amalg Y^-$, with inverse

$$(-)_{X,Y}^{-,\vee,-1}: (X \vee Y)^- \xrightarrow{\sim} X^- \amalg Y^-$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = [(0, x)], \\ (1, y) & \text{if } z = [(1, y)], \end{cases}$$

for each $z \in (X \vee Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\vee,\mathbb{1}}: \emptyset \xrightarrow{\sim} \text{pt}^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^-$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^+$ with respect to \wedge and \times as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^-: X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)_{X,Y}^-(x, y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1}: (X \wedge Y)^- \xrightarrow{\sim} X^- \times Y^-$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x, y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{-,1}_{X,Y} : \text{pt} \xrightarrow{\sim} (S^0)^{-}$$

is given by sending \star to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

□

Appendices

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Extra Part

15. [Notes](#)

References

- [MSE 2855868] **Qiaochu Yuan**. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/2855868> (cit. on pp. 7, 16).
- [MSE 884460] **Martin Brandenburg**. *Why are the category of pointed sets and the category of sets and partial functions “essentially the same”?* Mathematics Stack Exchange. URL: <https://math.stackexchange.com/q/884460> (cit. on p. 7).