Pointed Sets

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This chapter contains some foundational material on pointed sets.

Contents

0099 6.1 Pointed Sets

009A 6.1.1 Foundations

009B **Definition 6.1.1.1.1.** A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(N_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

009C **Remark 6.1.1.1.2.** In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).

- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

009D **Example 6.1.1.1.3.** The 0-**sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\mathrm{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

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[1]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$-**modules**.

[2]*Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3]*Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

009E   **Example 6.1.1.1.4.** The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.

- *The Basepoint.* The element $\star$ of pt.

01QB   **Example 6.1.1.1.5.** The **standard pointed set with** $n+1$ **elements** is the pointed set $\langle n \rangle$ consisting of

- *The Underlying Set.* The set $\langle n \rangle$ defined by

$$\langle n \rangle \overset{\mathrm{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n \rangle$.

009H   ## 6.1.2   Morphisms of Pointed Sets

009J   **Definition 6.1.2.1.1.** A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.

- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

009K   **Remark 6.1.2.1.2.** In detail, a **morphism of pointed sets** $f \colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f \colon X \to Y$ such that the diagram

$$
\begin{array}{ccc}
 & \mathrm{pt} & \\
{\scriptstyle [x_0]} \swarrow & & \searrow {\scriptstyle [y_0]} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

----

[4] *Further Terminology:* Also called a **pointed function**.
[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

009L   6.1.3   **The Category of Pointed Sets**

009M   **Definition 6.1.3.1.1.** The **category of pointed sets** is the category Sets_* defined equivalently as:

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**, **??**.

- The category Sets_* of Constructions With Categories, **??**.

009N   **Remark 6.1.3.1.2.** In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.

- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.

- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

of Sets_* at (X, x_0) is defined by[6]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\mathrm{def}}{=} \mathrm{id}_X.$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \overset{\mathrm{def}}{=} g \circ f.$$

---

[6]Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[7]Note that the composition of two morphisms of pointed sets is indeed a morphism of

009P **6.1.4   Elementary Properties of Pointed Sets**

009Q **Proposition 6.1.4.1.1.** Let (X, x_0) be a pointed set.

009R   1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

009S   (a) Products, described as in Definition 6.2.3.1.1.

009T   (b) Pullbacks, described as in Definition 6.2.4.1.1.

009U   (c) Equalisers, described as in Definition 6.2.5.1.1.

009V   2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

009W   (a) Coproducts, described as in Definition 6.3.3.1.1.

009X   (b) Pushouts, described as in Definition 6.3.4.1.1;

009Y   (c) Coequalisers, described as in Definition 6.3.5.1.1.

009Z   3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

00A0   4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*(S^0, X) \cong X,$$

pointed sets, as we have

$$g(f(x_0)) = g(y_0)$$
$$= z_0,$$

---

[8]The category Sets_* does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

[9]In other words, the forgetful functor

$$忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathsf{Sets}_*(S^0, X) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00A1   5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\mathrm{eq.}}{\cong} \mathsf{Sets}^{\mathrm{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

024U   (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \overset{\cong}{\to} \mathsf{Sets}^{\mathrm{part.}}$$

sends:

024V   i. A pointed set (X, x_0) to X.

024W   ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\mathrm{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

024X   (b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} \colon \mathsf{Sets}^{\mathrm{part.}} \overset{\cong}{\to} \mathsf{Sets}_*$$

sends:

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[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

024Y   i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is
                            not in X.

024Z   ii. A partial function

$$f : X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} : (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. *Item 1, Completeness*: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.
Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.
Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].
Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathsf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} : S^0 \to X$, the basepoint of $\mathsf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.
Item 5, Relation to Partial Functions: See [MSE 884460].   □

01QC   6.1.5   Active and Inert Morphisms of Pointed Sets

01QD   **Definition 6.1.5.1.1.** Let $f : (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

01QE   1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.

01QF   2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

01QG **Notation 6.1.5.1.2.** We write $\mathsf{Sets}_*^{\mathrm{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.
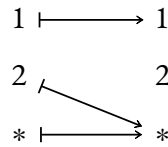
01QH **Example 6.1.5.1.3.** Here are some examples of active and inert maps of pointed sets.

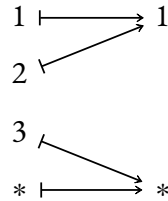01QJ   1. The map $\mu\colon \langle 2 \rangle \to \langle 1 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & \nearrow & \\
* & \longmapsto & *
\end{array}
$$
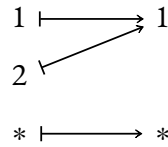
is active but not inert.

01QK   2. The map $f\colon \langle 2 \rangle \to \langle 2 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & & 2 \\
* & \longmapsto & *
\end{array}
$$

is inert but not active.

01QL   3. The map $f\colon \langle 3 \rangle \to \langle 1 \rangle$ given by

$$
\begin{array}{ccc}
1 & \longmapsto & 1 \\
2 & \nearrow & \\
3 & & \\
* & \longmapsto & *
\end{array}
$$

is neither inert nor active. However, it factors as $f = a \circ i$, where

$$i\colon \langle 3 \rangle \to \langle 2 \rangle,$$

$$a \colon \langle 2 \rangle \longrightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by

with i being inert and a being active.

01QM   **Proposition 6.1.5.1.4.** Let (X, x_0) and (Y, y_0) be pointed sets.

01QN   1. *Active-Inert Factorisation.* Every morphism of pointed sets $f \colon (X, x_0) \longrightarrow (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

01QP   (a) The map $i \colon (X, x_0) \longrightarrow (K, k_0)$ is an inert morphism of pointed sets

01QQ   (b) The map $a \colon (K, k_0) \longrightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_*.

Proof. *Item 1, Active-Inert Factorisation*: Let $f \colon X \longrightarrow Y$ be a morphism of pointed sets. We can factor f as

$$X \overset{i}{\longrightarrow} K \overset{a}{\longrightarrow} Y,$$

where:

- K is the pointed set given by

$$
\begin{aligned}
K &= \left\{ x \in X \,\middle|\, f(x) \neq y_0 \right\} \cup \{x_0\} \\
&= (X \setminus f^{-1}(y_0)) \cup \{x_0\};
\end{aligned}
$$

- $i\colon X \to K$ is the inert morphism of pointed sets given by

$$i(x) \overset{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each $x \in X$;

- $a\colon K \to Y$ is the active morphism of pointed sets given by

$$a(x) \overset{\text{def}}{=} f(x)$$

for each $x \in K$.

Next, let

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A & \underset{a}{\longrightarrow} & B \end{array}$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y \to A$ given by

$$\phi(y) = f(i^{-1}(y))$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & Y \\ {\scriptstyle f}\downarrow & {\scriptstyle \exists!}\diagup{\scriptstyle \phi} & \downarrow{\scriptstyle g} \\ A & \underset{a}{\longrightarrow} & B. \end{array}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\overset{\text{def}}{=} \phi(i(x)) \\ &\overset{\text{def}}{=} f(i^{-1}(i(x))) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$
\begin{aligned}
[a \circ \phi](y) &\overset{\text{def}}{=} a(\phi(y)) \\
&\overset{\text{def}}{=} a(f(i^{-1}(y))) \\
&\overset{\text{def}}{=} [a \circ f](i^{-1}(y)) \\
&= [g \circ i](i^{-1}(y)) \\
&\overset{\text{def}}{=} g(i(i^{-1}(y))) \\
&\overset{\text{def}}{=} g(y)
\end{aligned}
$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & Y \\
{\scriptstyle f}\downarrow & {\scriptstyle \psi} & \downarrow{\scriptstyle g} \\
A & \xrightarrow[\ a\ ]{} & B
\end{array}
$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$
\begin{aligned}
\psi(y) &= \psi(i(x)) \\
&= f(x) \\
&= f(i^{-1}(y)) \\
&\overset{\text{def}}{=} \phi(y).
\end{aligned}
$$

This finishes the proof. $\qquad\square$

00A2 6.2   Limits of Pointed Sets

00A3 6.2.1   The Terminal Pointed Set

00A4 **Definition 6.2.1.1.1.** The **terminal pointed set** is the terminal object of Sets_* as in Limits and Colimits, **??**.

0250 **Construction 6.2.1.1.2.** Concretely, the **terminal pointed set** is the pair $\big((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\big)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\left\{ !_X \colon (X, x_0) \to (\mathrm{pt}, \star) \right\}_{(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \overset{\mathrm{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \underset{\exists!}{-\overset{\phi}{-}\!\!\to} (\mathrm{pt}, \star)$$

commute, namely $!_X$.   □

00A5   6.2.2   Products of Families of Pointed Sets

Let $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ be a family of pointed sets.

00A6   **Definition 6.2.2.1.1.** The **product of** $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ is the product of $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ in Sets_* as in Limits and Colimits, **??**.

0251   **Construction 6.2.2.1.2.** Concretely, the **product of** $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ is the pair $\left(\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right), \left\{ \mathrm{pr}_i \right\}_{i \in I} \right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right)$.

- *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \to (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i((x_j)_{j\in I}) \overset{\mathrm{def}}{=} x_i$$

for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $(\prod_{i\in I} X_i, (x_0^i)_{i\in I})$ is the categorical product of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$(P, *)$$
$$\searrow^{p_i}$$
$$(\textstyle\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \xrightarrow[\mathrm{pr}_i]{} (X_i, x_0^i)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (\prod_{i\in I} X_i, (x_0^i)_{i\in I})$$

making the diagram

$$(P, *)$$
$$\phi \downarrow \exists! \qquad \searrow^{p_i}$$
$$(\textstyle\prod_{i\in I} X_i, (x_0^i)_{i\in I}) \xrightarrow[\mathrm{pr}_i]{} (X_i, x_0^i)$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i\in I}$$
$$= (x_0^i)_{i\in I},$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. $\qquad \square$

00A7 **Proposition 6.2.2.1.3.** Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

00A8 1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i \in I} \mapsto \left(\prod_{i \in I} X_i, (x_0^i)_{i \in I}\right)$ defines a functor

$$\prod_{i \in I} \colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**. □

00A9 6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA **Definition 6.2.3.1.1.** The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

0252 **Construction 6.2.3.1.2.** Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$\mathrm{pr}_1 \colon (X \times Y, (x_0, y_0)) \to (X, x_0),$$
$$\mathrm{pr}_2 \colon (X \times Y, (x_0, y_0)) \to (Y, y_0)$$

defined by

$$\mathrm{pr}_1(x, y) \overset{\mathrm{def}}{=} x,$$
$$\mathrm{pr}_2(x, y) \overset{\mathrm{def}}{=} y$$

for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$
\begin{array}{ccc}
& (P, *) & \\
{\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\
(X, x_0) \xleftarrow{\mathrm{pr}_1} & (X \times Y, (x_0, y_0)) & \xrightarrow{\mathrm{pr}_2} (Y, y_0)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \longrightarrow (X \times Y, (x_0, y_0))$$

making the diagram

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
$$= (x_0, y_0),$$

where we have used that p_1 and p_2 are morphisms of pointed sets.   □

00AB   **Proposition 6.2.3.1.3.** Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00AC   1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned}
A \times -&\colon & \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\
- \times B&\colon & \mathsf{Sets}_* & \to \mathsf{Sets}_*, \\
-_1 \times -_2&\colon & \mathsf{Sets}_* \times \mathsf{Sets}_* & \to \mathsf{Sets}_*,
\end{aligned}$$

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

01QR   2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

00AD   3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AE   4. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, \star) \times (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \times (\mathrm{pt}, \star) \cong (X, x_0),$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AF   5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AG   6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of limits, Limits and Colimits, **??** of **??**.
Item 2, Lack of Adjointness: See [MSE 2855868].
Item 3, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.
Item 4, Unitality: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.
Item 5, Commutativity: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.
Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3.   □

00AH **6.2.4   Pullbacks**

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

00AJ   **Definition 6.2.4.1.1.** The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

0253   **Construction 6.2.4.1.2.** Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.

- *The Cone.* The morphisms of pointed sets

$$\mathrm{pr}_1\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$
$$\mathrm{pr}_2\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$$

defined by

$$\mathrm{pr}_1(x, y) \overset{\mathrm{def}}{=} x,$$
$$\mathrm{pr}_2(x, y) \overset{\mathrm{def}}{=} y$$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2,$$

$$
\begin{array}{ccc}
(X \times_Z Y, (x_0, y_0)) & \xrightarrow{\ \mathrm{pr}_2\ } & (Y, y_0) \\
{\scriptstyle \mathrm{pr}_1} \downarrow & & \downarrow {\scriptstyle g} \\
(X, x_0) & \xrightarrow[\ f\ ]{} & (Z, z_0).
\end{array}
$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$[f \circ \mathrm{pr}_1](x, y) = f(\mathrm{pr}_1(x, y))$$
$$= f(x)$$

$$= g(y)$$
$$= g(\mathrm{pr}_2(x, y))$$
$$= [g \circ \mathrm{pr}_2](x, y),$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

commute, being uniquely determined by the conditions

$$\mathrm{pr}_1 \circ \phi = p_1,$$
$$\mathrm{pr}_2 \circ \phi = p_2$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. $\square$

00AK   **Proposition 6.2.4.1.3.** Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

00AL   1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \longrightarrow \bullet.
\end{array}
$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi\colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x_0', y_0'))$$

given by

$$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram
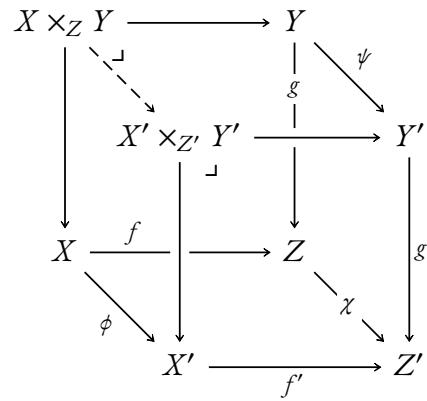
commute.

00AM 2. *Associativity.* Given a diagram

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

00AN   3. *Unitality.* We have isomorphisms of pointed sets
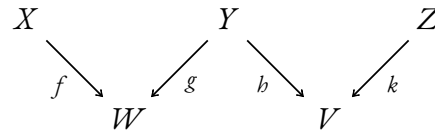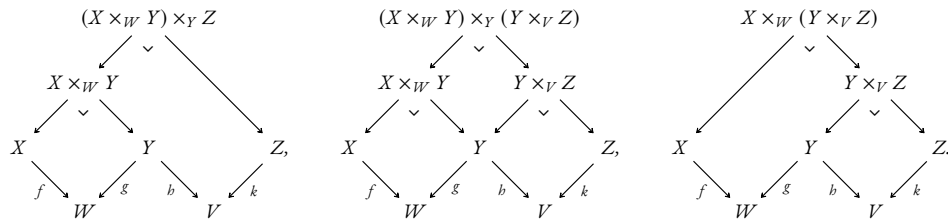
$$
\begin{array}{ccc}
A & =\!=\!= & A \\
{\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
X & =\!=\!= & X
\end{array}
\qquad
\begin{array}{l}
X \times_X A \cong A, \\
A \times_X X \cong A,
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & X \\
\| & \lrcorner & \| \\
X & \xrightarrow[\ f\ ]{} & X.
\end{array}
$$

00AP   4. *Commutativity.* We have an isomorphism of pointed sets

$$
\begin{array}{ccc}
A \times_X B & \longrightarrow & B \\
\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
A & \xrightarrow[\ f\ ]{} & X,
\end{array}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{array}{ccc}
B \times_X A & \longrightarrow & A \\
\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
B & \xrightarrow[\ g\ ]{} & X.
\end{array}
$$

00AQ   5. *Interaction With Products.* We have an isomorphism of pointed sets

$$
X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{array}{ccc}
X \times Y & \longrightarrow & Y \\
\downarrow & \lrcorner & \downarrow{\scriptstyle !_Y} \\
X & \xrightarrow[\ !_X\ ]{} & \mathrm{pt.}
\end{array}
$$

00AR   6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.

Item 3, Unitality: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.

Item 4, Commutativity: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.

Item 5, Interaction With Products: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.

Item 6, Symmetric Monoidality: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5. $\qquad\square$

00AS ## 6.2.5 Equalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT **Definition 6.2.5.1.1.** The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in Limits and Colimits, **??**.

0254 **Construction 6.2.5.1.2.** Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.

- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g) \colon (\mathrm{Eq}(f, g), x_0) \to (X, x_0)$$

given by the canonical inclusion $\mathrm{eq}(f, g) \to \mathrm{Eq}(f, g) \to X$.

Proof. We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\mathrm{Eq}(f, g), x_0) \xhookrightarrow{\mathrm{eq}(f,g)} (X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$$
$$\nearrow{\scriptstyle e}$$
$$(E, *)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$(\mathrm{Eq}(f,g), x_0) \xleftarrow{\;\;\mathrm{eq}(f,g)\;\;} (X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0)$$

$$\phi \mid \exists! \qquad e$$

$$(E, *)$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.   □

00AU **Proposition 6.2.5.1.3.** Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00AV   1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f, g, h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$(X, x_0) \xrightarrow[\substack{-g \to \\ h}]{f} (Y, y_0)$$

in Sets$_*$, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \big\{ a \in A \,\big|\, f(a) = g(a) = h(a) \big\}.$$

00AW   2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

00AX   3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.
Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.
Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3.   □

00AY   6.3   Colimits of Pointed Sets

00AZ   6.3.1   The Initial Pointed Set

00B0   **Definition 6.3.1.1.1.** The **initial pointed set** is the initial object of Sets$_*$ as in Limits and Colimits, **??**.

0255   **Construction 6.3.1.1.2.** Concretely, the **initial pointed set** is the pair $((\mathrm{pt}, \star), \{\iota_X\}_{(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)})$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.

- *The Cone.* The collection of morphisms of pointed sets

$$\big\{ \iota_X \colon (\mathrm{pt}, \star) \to (X, x_0) \big\}_{(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \overset{\mathrm{def}}{=} x_0.$$

Proof. We claim that $(\text{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\text{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (\text{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \; \underset{\exists!}{-\overset{\phi}{-}\!\!\to} \; (X, x_0)$$

commute, namely ι_X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

00B1   6.3.2   Coproducts of Families of Pointed Sets

Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

00B2   **Definition 6.3.2.1.1.** The **coproduct of the family** $\left\{(X_i, x_0^i)\right\}_{i \in I}$[II] is the coproduct of $\left\{(X_i, x_0^i)\right\}_{i \in I}$ in Sets_* as in Limits and Colimits, **??**.

0256   **Construction 6.3.2.1.2.** Concretely, the **coproduct of the family** $\left\{(X_i, x_0^i)\right\}_{i \in I}$ is the pair $((\bigvee_{i \in I} X_i, p_0), \left\{\text{inj}_i\right\}_{i \in I})$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i \in I} X_i, p_0)$ consisting of:

    - *The Underlying Set.* The set $\bigvee_{i \in I} X_i$ defined by

    $$\bigvee_{i \in I} X_i \overset{\text{def}}{=} (\coprod_{i \in I} X_i)/\!\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i \in I} X_i$ given by declaring

    $$(i, x_0^i) \sim (j, x_0^j)$$

    for each $i, j \in I$.

[II]*Further Terminology:* Also called the **wedge sum of the family** $\left\{(X_i, x_0^i)\right\}_{i \in I}$.

– *The Basepoint.* The element p_0 of $\bigvee_{i \in I} X_i$ defined by

$$p_0 \overset{\text{def}}{=} [(i, x_0^i)]$$
$$= [(j, x_0^j)]$$

for any $i, j \in I$.

- *The Cocone.* The collection

$$\left\{ \mathrm{inj}_i \colon (X_i, x_0^i) \to \left(\bigvee_{i \in I} X_i, p_0 \right) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $\left(\bigvee_{i \in I} X_i, p_0 \right)$ is the categorical coproduct of $\left\{ (X_i, x_0^i) \right\}_{i \in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$
\begin{array}{ccc}
 & & (C, *) \\
 & \nearrow^{\iota_i} & \\
(X_i, x_0^i) & \xrightarrow[\mathrm{inj}_i]{} & \left(\bigvee_{i \in I} X_i, p_0 \right)
\end{array}
$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon \left(\bigvee_{i \in I} X_i, p_0 \right) \to (C, *)$$

making the diagram

$$
\begin{array}{ccc}
 & & (C, *) \\
 & \nearrow^{\iota_i} & \uparrow{\scriptstyle \phi} \;{\scriptstyle \exists!} \\
(X_i, x_0^i) & \xrightarrow[\mathrm{inj}_i]{} & \left(\bigvee_{i \in I} X_i, p_0 \right)
\end{array}
$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i \in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i,x_0^i)])$$
$$= *,$$

as ι_i is a morphism of pointed sets. $\square$

`00B3` **Proposition 6.3.2.1.3.** Let $\left\{(X_i, x_0^i)\right\}_{i \in I}$ be a family of pointed sets.

`00B4`   1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$ defines a functor

$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**. $\square$

`00B5` ## 6.3.3   Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

`00B6` **Definition 6.3.3.1.1.** The **coproduct of** (X, x_0) **and** (Y, y_0)[12] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in Limits and Colimits, **??**.

`0257` **Construction 6.3.3.1.2.** Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

   • *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

      − *The Underlying Set.* The set $X \vee Y$ defined by

$$(X \vee Y, p_0) \overset{\mathrm{def}}{=} (X, x_0) \coprod (Y, y_0)$$
$$\cong (X \coprod_{\mathrm{pt}} Y, p_0)$$
$$\cong (X \coprod Y/\sim, p_0),$$

$$
\begin{array}{ccc}
X \vee Y & \longleftarrow & Y \\
\uparrow & \ulcorner & \big\downarrow {\scriptstyle [y_0]} \\
\big\uparrow & & \\
X & \underset{[x_0]}{\longleftarrow} & \mathrm{pt},
\end{array}
$$

---

[12]*Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

– *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$p_0 \overset{\text{def}}{=} [(0, x_0)]$$
$$= [(1, y_0)].$$

- *The Cocone.* The morphisms of pointed sets

$$\mathrm{inj}_1 \colon (X, x_0) \longrightarrow (X \vee Y, p_0),$$
$$\mathrm{inj}_2 \colon (Y, y_0) \longrightarrow (X \vee Y, p_0),$$

given by

$$\mathrm{inj}_1(x) \overset{\text{def}}{=} [(0, x)],$$
$$\mathrm{inj}_2(y) \overset{\text{def}}{=} [(1, y)],$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \vee Y, p_0) \longrightarrow (C, *)$$

making the diagram

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_X = \iota_X,$$
$$\phi \circ \mathrm{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$
$$= \iota_Y([(1, y_0)])$$
$$= *,$$

as ι_X and ι_Y are morphisms of pointed sets. $\qquad\square$

00B7 **Proposition 6.3.3.1.3.** Let (X, x_0) and (Y, y_0) be pointed sets.

00B8   1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

      define functors

$$X \vee - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$- \vee Y \colon \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
$$-_1 \vee -_2 \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$$

00B9   2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

      natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

00BA   3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, *) \vee (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \vee (\mathrm{pt}, *) \cong (X, x_0),$$

      natural in $(X, x_0) \in \mathsf{Sets}_*$.

00BB 4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

00BC 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

00BD 6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*},$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1, Functoriality*: This follows from Limits and Colimits, **??** of **??**.
Item 2, Associativity: Omitted.
Item 3, Unitality: Omitted.
Item 4, Commutativity: Omitted.
Item 5, Symmetric Monoidality: Omitted.
Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f \colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X,$$

Indeed, we have

$$[\nabla_Y \circ (f \vee f)]([(i,x)]) = \nabla_Y([(i,f(x))])$$
$$= f(x)$$
$$= f(\nabla_X([(i,x)]))$$
$$= [f \circ \nabla_X]([(i,x)])$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. $\square$

00BE ## 6.3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f \colon (Z, z_0) \to (X, x_0)$ and $g \colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BF **Definition 6.3.4.1.1.** The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pushout of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in Limits and Colimits, **??**.

0258 **Construction 6.3.4.1.2.** Concretely, the **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $(X \coprod_{f,Z,g} Y, p_0)$, where:

  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;

  - We have $p_0 = [x_0] = [y_0]$.

- *The Cocone.* The morphisms of pointed sets

$$\mathrm{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$
$$\mathrm{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$$

given by

$$\mathrm{inj}_1(x) \overset{\mathrm{def}}{=} [(0, x)]$$
$$\mathrm{inj}_2(y) \overset{\mathrm{def}}{=} [(1, y)]$$

for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$x_0 = f(z_0),$$
$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g,$$

$$
\begin{array}{ccc}
(X \coprod_Z Y, p_0) & \xleftarrow{\mathrm{inj}_2} & (Y, y_0) \\
\uparrow{\scriptstyle\mathrm{inj}_1} & & \uparrow{\scriptstyle g} \\
(X, x_0) & \xleftarrow{\ f\ } & (Z, z_0).
\end{array}
$$

Indeed, given $z \in Z$, we have

$$
\begin{aligned}
[\mathrm{inj}_1 \circ f](z) &= \mathrm{inj}_1(f(z)) \\
&= [(0, f(z))] \\
&= [(1, g(z))] \\
&= \mathrm{inj}_2(g(z)) \\
&= [\mathrm{inj}_2 \circ g](z),
\end{aligned}
$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$
\begin{array}{ccc}
(P, *) & & \\
& & \\
& (X \coprod_Z Y, p_0) & \xleftarrow{\ \mathrm{inj}_2\ } (Y, y_0) \\
& \uparrow{\scriptstyle\mathrm{inj}_1} & \uparrow{\scriptstyle g} \\
& (X, x_0) & \xleftarrow{\ f\ } (Z, z_0)
\end{array}
$$

with arrows ι_2 from (Y, y_0) to $(P, *)$ and ι_1 from (X, x_0) to $(P, *)$.

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_1 = \iota_1,$$
$$\phi \circ \mathrm{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$
$$= \iota_1(x_0)$$
$$= *,$$

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$
$$= \iota_2(y_0)$$
$$= *,$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. $\square$

00BG   **Proposition 6.3.4.1.3.** Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

00BH   1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f, Z, g} Y$ defines a functor
$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$
where $\mathcal{P}$ is the category that looks like this:
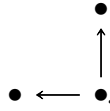
In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism
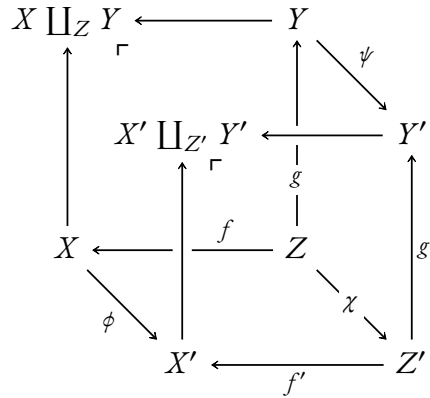
in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets
$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p_0')$$
given by
$$\xi(p) \overset{\mathrm{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram
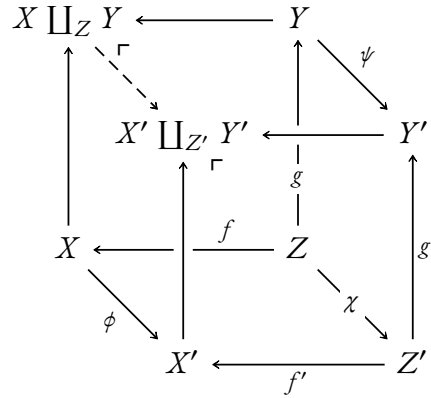
$$
\begin{array}{ccc}
X \coprod_Z Y & \longleftarrow & Y \\
\uparrow & \searrow & \downarrow \psi \\
& X' \coprod_{Z'} Y' \longleftarrow Y' \\
\uparrow & & \uparrow g' \\
X \longleftarrow & Z & \\
\phi \searrow & \searrow \chi & \uparrow g' \\
X' & \longleftarrow & Z'
\end{array}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{array}{ccccc}
X & & Y & & Z \\
& \nwarrow \ f \quad g \nearrow & & \nwarrow h \quad k \nearrow & \\
& W & & V &
\end{array}
$$
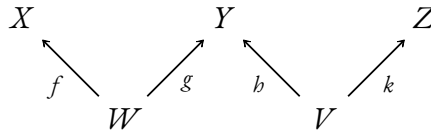
in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams
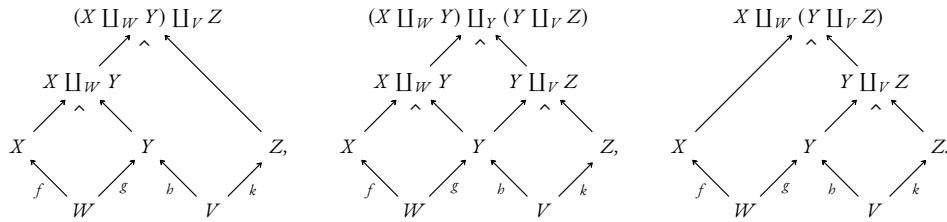
00BK   3. *Unitality.* We have isomorphisms of sets

$$
\begin{array}{ccc}
A = A \\
f\uparrow \quad \ulcorner \quad \uparrow f \\
X = X
\end{array}
\qquad
\begin{array}{l}
X \coprod_X A \cong A, \\
A \coprod_X X \cong A,
\end{array}
\qquad
\begin{array}{ccc}
A \xleftarrow{f} X \\
\| \quad \ulcorner \quad \| \\
X \xleftarrow{f} X.
\end{array}
$$

00BL   4. *Commutativity.* We have an isomorphism of sets

$$
\begin{array}{ccc}
X \coprod_Z Y \longleftarrow Y \\
\uparrow \quad \ulcorner \quad \uparrow g \\
X \xleftarrow{f} Z,
\end{array}
\qquad
X \coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{array}{ccc}
Y \coprod_Z X \longleftarrow X \\
\uparrow \quad \ulcorner \quad \uparrow f \\
Y \xleftarrow{g} Z.
\end{array}
$$

00BM   5. *Interaction With Coproducts.* We have

$$
X \coprod_{\mathrm{pt}} Y \cong X \vee Y,
\qquad
\begin{array}{ccc}
X \vee Y \longleftarrow Y \\
\uparrow \quad \ulcorner \quad \uparrow [y_0] \\
X \xleftarrow{[x_0]} \mathrm{pt.}
\end{array}
$$

00BN   6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric
            monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits,
Limits and Colimits, **??** of **??**, with the explicit expression for ξ following from
the commutativity of the cube pushout diagram.
Item 2, Associativity: This follows from Constructions With Sets, Item 3 of
Definition 4.2.4.1.6.
Item 3, Unitality: This follows from Constructions With Sets, Item 5 of Defini-
tion 4.2.4.1.6.
Item 4, Commutativity: This follows from Constructions With Sets, Item 6 of
Definition 4.2.4.1.6.
Item 5, Interaction With Coproducts: Omitted.
Item 6, Symmetric Monoidality: Omitted.   □

6.3.5   Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f, g), [y_0]), \mathrm{coeq}(f, g))$ consisting of:

- *The Colimit.* The pointed set $(\mathrm{CoEq}(f, g), [y_0])$, where $\mathrm{CoEq}(f, g)$ is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.

- *The Cocone.* The map

$$\mathrm{coeq}(f, g)\colon Y \twoheadrightarrow (\mathrm{CoEq}(f, g), [y_0])$$

  given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$
\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](x) &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](f(x)) \\
&\overset{\mathrm{def}}{=} [f(x)] \\
&= [g(x)] \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g)](g(x)) \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f, g) \circ g](x)
\end{aligned}
$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$(X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0) \xrightarrow{\mathrm{coeq}(f,g)} (\mathrm{CoEq}(f, g), [y_0])$$

$$\searrow^{c}$$

$$(C, *)$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\phi\colon \mathrm{CoEq}(f, g) \xrightarrow{\exists!} C$ making the diagram

$$(X, x_0) \underset{g}{\overset{f}{\rightrightarrows}} (Y, y_0) \xleftarrow{\mathrm{coeq}(f,g)} (\mathrm{CoEq}(f, g), [y_0])$$

$$c \qquad\qquad \phi \mid \exists!$$

$$(C, *)$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \mathrm{coeq}(f, g)]([y_0])$$
$$= c([y_0])$$
$$= *,$$

where we have used that c is a morphism of pointed sets.   □

00BR   **Proposition 6.3.5.1.3.** Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BS   1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f, g) \circ f, \mathrm{coeq}(f, g) \circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g) \circ g, \mathrm{coeq}(f,g) \circ h)} \cong \mathrm{CoEq}(f, g, h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g, h) \circ f, \mathrm{coeq}(g, h) \circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h) \circ f, \mathrm{coeq}(g,h) \circ h)},$$

where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$(X, x_0) \overset{f}{\underset{h}{\overset{}{=}\!\!\underset{}{=}\!\!\overset{g}{\rightrightarrows}}} (Y, y_0)$$

in Sets_*.

00BT   2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, f) \cong B.$$

00BU   3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

Proof. Item 1, Associativity: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.

Item 2, Unitality: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.

Item 3, Commutativity: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5. □

00BV 6.4 Constructions With Pointed Sets

00BW 6.4.1 Free Pointed Sets

Let X be a set.

00BX **Definition 6.4.1.1.1.** The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[13]

$$X^+ \stackrel{\text{def}}{=} X \coprod \text{pt}$$
$$\stackrel{\text{def}}{=} X \coprod \{\star\}.$$

- *The Basepoint.* The element $\star$ of X^+.

00BY **Proposition 6.4.1.1.2.** Let X be a set.

00BZ   1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where:

- *Action on Objects.* For each $X \in \text{Obj}(\mathsf{Sets})$, we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 6.4.1.1.1.

---

[13]*Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially when there are multiple free pointed sets involved in the current discussion.

- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^+ \colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

00C0 2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv \text{忘}\right) \colon \quad \mathsf{Sets} \overset{(-)^+}{\underset{\text{忘}}{\rightleftarrows}} \perp \mathsf{Sets}_*,$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00C1 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)_{\mathbb{1}}^{+,\coprod}\right) \colon (\mathsf{Sets}, \coprod, \varnothing) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod} \colon X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \coprod Y)^+,$$

$$(-)_{\mathbb{1}}^{+,\coprod} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \varnothing^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

00C2 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, (-)_{\mathbb{1}}^+\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}_*, \wedge, S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+,$$

$$(-)_{\mathbb{1}}^+ \colon S^0 \overset{\sim}{\dashrightarrow} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}_X^+(x) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\mathrm{id}_X^+ = \mathrm{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$f : X \to Y,$$
$$g : Y \to Z,$$

we have

$$\begin{aligned}[g^+ \circ f^+](x) &\overset{\mathrm{def}}{=} g^+(f^+(x)) \\ &\overset{\mathrm{def}}{=} g^+(f(x)) \\ &\overset{\mathrm{def}}{=} g(f(x)) \\ &\overset{\mathrm{def}}{=} [g \circ f]^+(x)\end{aligned}$$

for each $x \in X$ and

$$\begin{aligned}[g^+ \circ f^+](\star_X) &\overset{\mathrm{def}}{=} g^+(f^+(\star_X)) \\ &\overset{\mathrm{def}}{=} g^+(\star_Y) \\ &\overset{\mathrm{def}}{=} \star_Z \\ &\overset{\mathrm{def}}{=} [g \circ f]^+(\star_X),\end{aligned}$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

025A

- *Map I.* We define a map

$$\Phi_{X,Y} : \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi : (X^+, \star_X) \to (Y, y_0)$$

to the function
$$\xi^\dagger \colon X \to Y$$

given by
$$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X$.

025B   • *Map II.* We define a map
$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*(X^+, Y)$$

given by sending a function $\xi \colon X \to Y$ to the morphism of pointed sets
$$\xi^\dagger \colon (X^+, \star_X) \to (Y, y_0)$$

defined by
$$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

025C   • *Invertibility I.* Given a morphism of pointed sets
$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have
$$
\begin{aligned}
[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\
&= \Psi_{X,Y}(\xi^\dagger) \\
&\overset{\text{def}}{=} \left[\!\!\left[x \mapsto \begin{cases} \xi^\dagger(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\!\right] \\
&= \left[\!\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \right]\!\!\right] \\
&= \xi \\
&\overset{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}_*(X^+, Y)}\right](\xi).
\end{aligned}
$$

Therefore we have
$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+, Y)}.$$

025D   • *Invertibility II.* Given a map of sets $\xi\colon X \to Y$, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\
&= \Phi_{X,Y}(\xi^{\dagger}) \\
&= \Phi_{X,Y}(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \rrbracket) \\
&= \llbracket x \mapsto \xi(x) \rrbracket \\
&= \xi \\
&\overset{\text{def}}{=} [\mathrm{id}_{\mathsf{Sets}(X,Y)}](\xi).
\end{aligned}
$$

Therefore we have

$$
\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}.
$$

025E   • *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$
f\colon (X, x_0) \to (X', x_0'),
$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*(X'^{,+}, Y) & \overset{\Phi_{X',Y}}{\longrightarrow} & \mathsf{Sets}(X', Y) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*(X^+, Y) & \underset{\Phi_{X,Y}}{\longrightarrow} & \mathsf{Sets}(X, Y)
\end{array}
$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+} \to Y$, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= [f^* \circ \Phi_{X',Y}](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

025F • *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g \colon (Y, y_0) \to (Y', y_0'),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}_*(X^+, Y) & \xrightarrow{\ \Phi_{X,Y}\ } & \mathsf{Sets}(X, Y) \\
\Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle g_*} \\
\mathsf{Sets}_*(X^+, Y'), & \xrightarrow[\ \Phi_{X,Y'}\ ]{} & \mathsf{Sets}(X, Y')
\end{array}
$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger \colon X^+ \to Y,$$

we have

$$
\begin{aligned}
[\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*(\Phi_{X,Y'}(\xi)) \\
&= [g_* \circ \Phi_{X,Y'}](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

025G • *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct
the strong monoidal structure on $(-)^+$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+,\coprod}_{X,Y} : X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \coprod Y)^+$$

is given by

$$(-)^{+,\coprod}_{X,Y}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)^{+,\coprod,-1}_{X,Y} : (X \coprod Y)^+ \overset{\sim}{\dashrightarrow} X^+ \vee Y^+$$

given by

$$(-)^{+,\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\coprod,1}_{X,Y} : \mathrm{pt} \overset{\sim}{\dashrightarrow} \varnothing^+$$

is given by sending $\star_X$ to $\star_\varnothing$.

The verification that these isomorphisms satisfy the coherence conditions making
the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We
construct the strong monoidal structure on $(-)^+$ with respect to $\times$ and $\wedge$ as
follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^+_{X,Y}\colon X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+$$

is given by

$$(-)^+_{X,Y}(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X\times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$(-)^{+,-1}_{X,Y}\colon (X \times Y)^+ \overset{\sim}{\dashrightarrow} X^+ \wedge Y^+$$

given by

$$(-)^{+,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X\times Y}, \end{cases}$$

for each $z \in (X \times Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\mathbb{1}}_{X,Y}\colon S^0 \overset{\sim}{\dashrightarrow} \mathrm{pt}^+$$

is given by sending 0 to $\star_{\mathrm{pt}}$ and 1 to $\star$, where $\mathrm{pt}^+ = \{\star, \star_{\mathrm{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted.   □

01QS   6.4.2   Deleting Basepoints

Let (X, x_0) be a pointed set.

01QT   **Definition 6.4.2.1.1.** The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\text{def}}{=} X \setminus \{x_0\}.$$

01QU   **Proposition 6.4.2.1.2.** Let (X, x_0) be a pointed set.

01QV   1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- : \mathsf{Sets}_*^{\mathrm{actv}} \to \mathsf{Sets},$$

where:

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*^{\mathrm{actv}})$, we have

$$[(-)^-](X) \overset{\mathrm{def}}{=} X^-,$$

where X^- is the set of Definition 6.4.2.1.1.

- *Action on Morphisms.* For each morphism $f : X \to Y$ of $\mathsf{Sets}_*^{\mathrm{actv}}$, the image

$$f^- : X^- \to Y^-$$

of f by $(-)^-$ is the map defined by

$$f^-(x) \overset{\mathrm{def}}{=} f(x)$$

for each $x \in X^-$.

01QW   2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$\left((-)^- \dashv (-)^+\right): \quad \mathsf{Sets}_*^{\mathrm{actv}} \underset{(-)^+}{\overset{(-)^-}{\rightleftarrows}}_{\perp_{\mathrm{eq}}} \mathsf{Sets},$$

witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*(X, Y^+),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms

$$(X^-)^+ \cong X,$$
$$(Y^+)^- \cong Y,$$

once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

01QX   3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor

of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\vee}, (-)_1^{-,\vee}) \colon (\mathsf{Sets}_*^{\mathrm{actv}}, \vee, \mathrm{pt}), \to (\mathsf{Sets}, \coprod, \varnothing),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^- \coprod Y^- \xdashrightarrow{\sim} (X \vee Y)^-,$$
$$(-)_1^{-,\vee} \colon \varnothing \xdashrightarrow{\sim} \mathrm{pt}^-,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

01QY 4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free
       pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)_1^{-,\times}) \colon (\mathsf{Sets}_*^{\mathrm{actv}}, \wedge, S^0), \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^- \colon X^- \times Y^- \xdashrightarrow{\sim} (X \wedge Y)^-,$$
$$(-)_1^- \colon \mathrm{pt} \xdashrightarrow{\sim} (S^0)^-,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: We claim that $(-)^-$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}_X^-(x) \stackrel{\mathrm{def}}{=} x$$

  for each $x \in X^-$, so $\mathrm{id}_X^- = \mathrm{id}_{X^-}$.

- *Preservation of Composition.* Given morphisms of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$
$$g \colon (Y, y_0) \to (Z, z_0),$$

  we have

$$\begin{aligned}
[g^- \circ f^-](x) &\stackrel{\mathrm{def}}{=} g^-(f^-(x)) \\
&\stackrel{\mathrm{def}}{=} g^-(f(x)) \\
&\stackrel{\mathrm{def}}{=} g(f(x)) \\
&\stackrel{\mathrm{def}}{=} [g \circ f]^-(x)
\end{aligned}$$

  for each $x \in X$, so $(g \circ f)^- = g^- \circ f^-$.

This finishes the proof.

Item 2, Adjoint Equivalence: We proceed in a few steps:

025H   1. *Map I.* We define a map

$$\Phi_{X,Y}\colon \mathsf{Sets}(X^-, Y) \to \mathsf{Sets}^{\mathrm{actv}}_*(X, Y^+)$$

by sending a map $\xi\colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^\dagger \colon X \to Y^+$$

given by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

025J   2. *Map II.* We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}^{\mathrm{actv}}_*(X, Y^+) \to \mathsf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi\colon X \to Y^+$ to the map

$$\xi^\dagger \colon X^- \to Y$$

defined by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

025K   3. *Invertibility I.* Given a map of sets $\xi\colon X^- \to Y$, we have

$$[\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) \overset{\mathrm{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi))$$

$$\overset{\mathrm{def}}{=} \Psi_{X,Y}(\llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket)$$

$$= \llbracket x \mapsto \xi(x) \rrbracket$$

$$= \xi$$

$$= [\mathrm{id}_{\mathsf{Sets}(X^-,Y)}](\xi).$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}.$$

025L 4. *Invertibility II.* Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\
&= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\
&= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \rrbracket \\
&= \xi \\
&= [\mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)}](\xi).
\end{aligned}
$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)}.$$

025M 5. *Naturality for Φ, Part I.* We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (X', x_0'),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(X'^{,-}, Y) & \xrightarrow{\Phi_{X',Y}} & \mathsf{Sets}_*^{\mathrm{actv}}(X', Y^+) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathsf{Sets}_*(X^-, Y) & \xrightarrow[\Phi_{X,Y}]{} & \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)
\end{array}
$$

commutes. Indeed, given a map of sets $\xi \colon X' \to Y$, we have

$$
\begin{aligned}
[\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \llbracket x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x_0' \end{cases} \rrbracket \\
&= f^*(\llbracket x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x_0' \end{cases} \rrbracket)
\end{aligned}
$$

$$= f^*(\Phi_{X',Y}(\xi))$$
$$= [f^* \circ \Phi_{X',Y}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

025N   6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g \colon (Y, y_0) \longrightarrow (Y', y_0'),$$

the diagram

$$
\begin{array}{ccc}
\mathsf{Sets}(X^-, Y) & \xrightarrow{\;\Phi_{X,Y}\;} & \mathsf{Sets}^{\mathrm{actv}}_*(X, Y^+) \\
\downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\
\mathsf{Sets}(X^-, Y') & \xrightarrow[\;\Phi_{X,Y'}\;]{} & \mathsf{Sets}^{\mathrm{actv}}_*(X, Y'^{,+})
\end{array}
$$

commutes. Indeed, given a map of sets $\xi \colon X^- \to Y$, we have

$$[\Phi_{X,Y'} \circ g_*](\xi) = \Phi_{X,Y'}(g_*(\xi))$$
$$= \Phi_{X,Y'}(g \circ \xi)$$
$$= [\![x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}]\!]$$
$$= g_*([\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}]\!])$$
$$= g_*(\Phi_{X,Y'}(\xi))$$
$$= [g_* \circ \Phi_{X,Y'}](\xi).$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

025P 7. *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a compo-
nentwise inverse to Ψ in each argument, it follows from Categories, Item 2
of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

025Q 8. *Fully Faithfulness of* $(-)^-$. We aim to show that the assignment $f \mapsto f^-$
sets up a bijection

$$(-)^-_{X,Y} : \mathsf{Sets}^{\mathrm{actv}}_*(X, Y) \dashrightarrow^{\sim} \mathsf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)^{-,-1}_{X,Y} : \mathsf{Sets}(X^-, Y^-) \dashrightarrow^{\sim} \mathsf{Sets}^{\mathrm{actv}}_*(X, Y)$$

is given by sending a map of sets $f : X^- \to Y^-$ to the active morphism of
pointed sets $f^\dagger : X \to Y$ defined by

$$f^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

025R 9. *Essential Surjectivity of* $(-)^-$. We need to show that, given an object $X \in$
$\mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}(\mathsf{Sets}^{\mathrm{actv}}_*)$ such that $(X')^- \cong X$.
Indeed, taking $X' = X^+$, we have

$$\begin{aligned} (X^+)^- &\overset{\mathrm{def}}{=} (X \cup \{\star_X\})^- \\ &\overset{\mathrm{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be
essentially surjective.

025S 10. *The Functor* $(-)^-$ *Is an Equivalence*. Since $(-)^-$ is fully faithful and es-
sentially surjective, it is an equivalence by Categories, Item 1 of Defini-
tion 11.6.7.1.2.

This finishes the proof.
Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct
the strong monoidal structure on $(-)^-$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{-,\vee}\colon X^{-} \amalg Y^{-} \xdashrightarrow{\sim} (X \vee Y)^{-}$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^{-} \amalg Y^{-}$, with inverse

$$(-)_{X,Y}^{-,\vee,-1}\colon (X \vee Y)^{-} \xdashrightarrow{\sim} X^{-} \amalg Y^{-}$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{+,\vee,\mathbb{1}}\colon \varnothing \xdashrightarrow{\sim} \mathrm{pt}^{-}$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{-}$ into a symmetric strong monoidal functor is omitted.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^{-}\colon X^{-} \times Y^{-} \xdashrightarrow{\sim} (X \wedge Y)^{-}$$

is given by

$$(-)_{X,Y}^{-}(x,y) = x \wedge y$$

for each $(x, y) \in X^- \times Y^-$, with inverse

$$(-)_{X,Y}^{-,-1} \colon (X \wedge Y)^- \overset{\sim}{\dashrightarrow} X^- \times Y^-$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \overset{\text{def}}{=} (x, y)$$

for each $x \wedge y \in (X \wedge Y)^-$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)_{X,Y}^{-,1} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} (S^0)^-$$

is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ into a symmetric strong monoidal functor is omitted. □

Appendices

A Other Chapters

References

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