

Constructions With Relations

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This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (??).
2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, converse relations, composition of relations, and collages ([Section 9.2](#)).

This chapter is under revision. TODO:

1. Rename range to image
2. Co/limits in **Rel**.

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9.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

9.2 More Constructions With Relations

9.2.1 The Domain and Range of a Relation

Let A and B be sets.

Definition 9.2.1.1.1. Let $R: A \multimap B$ be a relation.^{1,2}

¹Following ??, ??, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned}
 \chi_{\text{dom}(R)}(a) &\cong \text{colim}_{b \in B} (R_a^b) & (a \in A) \\
 &\cong \bigvee_{b \in B} R_a^b, \\
 \chi_{\text{range}(R)}(b) &\cong \text{colim}_{a \in A} (R_a^b) & (b \in B) \\
 &\cong \bigvee_{a \in A} R_a^b,
 \end{aligned}$$

where the join \bigvee is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of **Constructions With Sets**,

Definition 3.2.2.1.3.

²Viewing R as a function $R: A \rightarrow \mathcal{P}(B)$, we have

$$\begin{aligned}
 \text{dom}(R) &\cong \text{colim}_{y \in Y} (R(y)) \\
 &\cong \bigcup_{y \in Y} R(y), \\
 \text{range}(R) &\cong \text{colim}_{x \in X} (R(x)) \\
 &\cong \bigcup_{x \in X} R(x),
 \end{aligned}$$

1. The **domain of** R is the subset $\text{dom}(R)$ of A defined by

$$\text{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \left| \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right. \right\}.$$

2. The **range of** R is the subset $\text{range}(R)$ of B defined by

$$\text{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \left| \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right. \right\}.$$

9.2.2 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B .

Definition 9.2.2.1.1. The **union of R and S** ³ is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁴

$$R \cup S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cup S](a) \stackrel{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

Proposition 9.2.2.1.2. Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

1. *Interaction With Converses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \stackrel{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

³*Further Terminology:* Also called the **binary union of R and S** , for emphasis.

⁴This is the same as the union of R and S as subsets of $A \times B$.

Proof. **Item 1, Interaction With Converses:** Clear.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
 - There exists some $b \in B$ such that:
 - * $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
 - or
 - * $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
 - There exists some $b \in B$ such that:
 - * $a \sim_{R_1} b$ or $a \sim_{R_2} b$;
 - and
 - * $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. \square

9.2.3 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

Definition 9.2.3.1.1. The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁵

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \left| \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right. \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcup_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

for each $a \in A$.

⁵This is the same as the union of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

Proposition 9.2.3.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

1. *Interaction With Converses.* We have

$$\left(\bigcup_{i \in I} R_i\right)^\dagger = \bigcup_{i \in I} R_i^\dagger.$$

Proof. **Item 1, Interaction With Converses:** Clear. □

9.2.4 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B .

Definition 9.2.4.1.1. The **intersection of R and S** ⁶ is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁷

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

Proposition 9.2.4.1.2. Let R, S, R_1 , and R_2 be relations from A to B , and let S_1 and S_2 be relations from B to C .

1. *Interaction With Converses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

⁶*Further Terminology:* Also called the **binary intersection of R and S** , for emphasis.

⁷This is the same as the intersection of R and S as subsets of $A \times B$.

Proof. **Item 1, Interaction With Converses:** Clear.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
 - There exists some $b \in B$ such that:
 - * $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
 - and
 - * $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
 - There exists some $b \in B$ such that:
 - * $a \sim_{R_1} b$ and $a \sim_{R_2} b$;
 - and
 - * $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. \square

9.2.5 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

Definition 9.2.5.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define⁸

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \left| \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right. \right\}.$$

- Viewing relations from A to B as functions $A \rightarrow \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right] (a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

for each $a \in A$.

⁸This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

Proposition 9.2.5.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B .

1. *Interaction With Converses.* We have

$$\left(\bigcap_{i \in I} R_i\right)^\dagger = \bigcap_{i \in I} R_i^\dagger.$$

Proof. **Item 1, Interaction With Converses:** Clear. \square

9.2.6 Binary Products of Relations

Let A , B , X , and Y be sets, let $R: A \rightarrowtail B$ be a relation from A to B , and let $S: X \rightarrowtail Y$ be a relation from X to Y .

Definition 9.2.6.1.1. The **product of R and S** ⁹ is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.¹⁰
- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \rightarrow \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \xrightarrow{\mathcal{P}_{B,Y}^\otimes} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

Proposition 9.2.6.1.2. Let A , B , X , and Y be sets.

1. *Interaction With Converses.* Let

$$R: A \rightarrowtail B,$$

$$S: X \rightarrowtail Y$$

We have

$$(R \times S)^\dagger = R^\dagger \times S^\dagger.$$

⁹*Further Terminology:* Also called the **binary product of R and S** , for emphasis.

¹⁰That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &: A \rightarrowtail B, \\ S_1 &: B \rightarrowtail C, \\ R_2 &: X \rightarrowtail Y, \\ S_2 &: Y \rightarrowtail Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. **Item 1, Interaction With Converses:** Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(R \times S)^\dagger} (b, y)$ iff:
 - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
 - * We have $b \sim_R a$;
 - * We have $y \sim_S x$;
- We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
 - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
 - * We have $b \sim_R a$;
 - * We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2, Interaction With Composition: Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
 - We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
 - * There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
 - * There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
- We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:

- There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
 - * We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
 - * We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. \square

9.2.7 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i: A_i \rightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 9.2.7.1.1. The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \left| \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right. \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \stackrel{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} A_i$.

9.2.8 The Collage of a Relation

Let A and B be sets and let $R: A \rightarrow B$ be a relation from A to B .

Definition 9.2.8.1.1. The **collage of R** ¹¹ is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

¹¹*Further Terminology:* Also called the **cograph of R** .

- *The Underlying Set.* The set $\text{Coll}(R)$ defined by

$$\text{Coll}(R) \stackrel{\text{def}}{=} A \amalg B.$$

- *The Partial Order.* The partial order

$$\preceq_{\text{Coll}(R)} : \text{Coll}(R) \times \text{Coll}(R) \rightarrow \{\text{true}, \text{false}\}$$

on $\text{Coll}(R)$ defined by

$$\preceq(a, b) \stackrel{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

Notation 9.2.8.1.2. We write $\text{Pos}_{/\Delta^1}(A, B)$ for the category defined as the pull-back

$$\text{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \text{pt}_{[A], \text{Pos}, \text{ev}_0} \times_{\text{Pos}_{/\Delta^1}} \times_{\text{ev}_1, \text{Pos}, [B]} \text{pt},$$

as in the diagram

$$\begin{array}{ccccc}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) & & \\
 & \swarrow & \downarrow \vee & \searrow & \\
 \mathbf{Pos}_{/\Delta^1} \times_{\mathbf{Pos}} \text{pt} & & & & \text{pt} \times_{\mathbf{Pos}} \mathbf{Pos}_{/\Delta^1} \\
 \swarrow \vee & & \downarrow \vee & & \searrow \vee \\
 \text{pt} & & \mathbf{Pos}_{/\Delta^1} & & \text{pt.} \\
 \swarrow [A] & \swarrow \text{ev}_{[0]} & \downarrow \text{ev}_{[1]} & \searrow [B] & \\
 & \mathbf{Pos} & & \mathbf{Pos} &
 \end{array}$$

Remark 9.2.8.1.3. In detail, $\text{Pos}_{/\Delta^1}(A, B)$ is the category where:

- *Objects.* An object of $\text{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of
 - A poset X ;
 - A morphism $\phi_X : X \rightarrow \Delta^1$;

such that we have

$$\begin{aligned}
 \phi_X^{-1}(0) &= A, \\
 \phi_X^{-1}(1) &= B.
 \end{aligned}$$

- *Morphisms.* A morphism of $\text{Pos}_{/\Delta^1}(A, B)$ from (X, ϕ_X) to (Y, ϕ_Y) is a morphism of posets $f: X \rightarrow Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi_X \searrow & & \swarrow \phi_Y \\ & \Delta^1 & \end{array}$$

commute.

Proposition 9.2.8.1.4. Let A and B be sets and let $R: A \rightarrowtail B$ be a relation from A to B .

- I. *Functoriality.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor

$$\mathbf{Coll}: \mathbf{Rel}(A, B) \rightarrow \text{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \text{Obj}(\mathbf{Rel}(A, B))$, we have

$$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A, B)$, where

- The poset $\mathbf{Coll}(R)$ is the collage of R of [Definition 9.2.8.1.1](#).
- The morphism $\phi_R: \mathbf{Coll}(R) \rightarrow \Delta^1$ is given by

$$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \text{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}: \text{Hom}_{\mathbf{Rel}(A,B)}(R, S) \rightarrow \text{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of \mathbf{Coll} at (R, S) is given by sending an inclusion

$$\iota: R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota) : \mathbf{Coll}(R) \rightarrow \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.¹²

2. *Equivalence.* The functor of **Item 1** is an equivalence of categories.

Proof. **Item 1, Functoriality:** Clear.

Item 2, Equivalence: Omitted. □

Appendices

A Other Chapters

Preliminaries

1. **Introduction**
2. **A Guide to the Literature**

Sets

3. **Sets**
4. **Constructions With Sets**
5. **Monoidal Structures on the Category of Sets**
6. **Pointed Sets**
7. **Tensor Products of Pointed Sets**

Relations

8. **Relations**
9. **Constructions With Relations**
10. **Conditions on Relations**

Categories

11. **Categories**
12. **Presheaves and the Yoneda Lemma**

Monoidal Categories

13. **Constructions With Monoidal Categories**

¹²Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

Bicategories

14. Types of Morphisms in Bicat-
egories

Extra Part

15. Notes