Conditions on Relations

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00TJ   This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

Contents

02D0 ## 10.1   Functional and Total Relations

00JC ### 10.1.1   Functional Relations

Let A and B be sets.

00JD **DEFINITION 10.1.1.1.1 ▶ FUNCTIONAL RELATIONS**

A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

00JE **PROPOSITION 10.1.1.1.2 ▶ PROPERTIES OF FUNCTIONAL RELATIONS**

Let $R\colon A \nrightarrow B$ be a relation.

00JF   1. *Characterisations.* The following conditions are equivalent:

00JG   (a) The relation R is functional.

00JH   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

PROOF 10.1.1.1.3 ▶ PROOF OF PROPOSITION 10.1.1.1.2

Item 1: Characterisations

We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* Let $(b, b') \in B \times B$. We need to show that

$$[R \diamond R^{\dagger}](b, b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both

$a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b $\implies$ Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  - Since $a \sim_R b$, we have $b \sim_{R^\dagger} a$.
  - Since $R \diamond R^\dagger \subset \chi_B$, we have

    $$[R \diamond R^\dagger](b, b') \preceq_{\{t, f\}} \chi_B(b, b'),$$

    and since $b \sim_{R^\dagger} a$ and $a \sim_R b'$, it follows that $[R \diamond R^\dagger](b, b') =$ true, and thus $\chi_B(b, b') =$ true as well, i.e. $b = b'$.

This finishes the proof. 　　　　　　　　　　　　　　　　　　　　　　　　　　　◳

00JJ　10.1.2   Total Relations

Let A and B be sets.

00JK　**DEFINITION 10.1.2.1.1 ▶ TOTAL RELATIONS**

A relation $R : A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \varnothing$.

00JL　**PROPOSITION 10.1.2.1.2 ▶ PROPERTIES OF TOTAL RELATIONS**

Let $R : A \nrightarrow B$ be a relation.

00JM　　　1. *Characterisations.* The following conditions are equivalent:

00JN　　　　　(a) The relation R is total.

00JP　　　　　(b) We have $\chi_A \subset R^\dagger \diamond R$.

> **PROOF 10.1.2.1.3 ▶ PROOF OF PROPOSITION 10.1.2.1.2**
>
> ### Item 1: Characterisations
>
> We claim that Items 1a and 1b are indeed equivalent:
>
> - *Item 1a $\implies$ Item 1b:* We have to show that, for each $(a, a') \in A$, we have
> $$\chi_A(a, a') \preceq_{\{t,f\}} [R^\dagger \diamond R](a, a'),$$
>   i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.
>
> - *Item 1b $\implies$ Item 1a:* Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have
> $$\{a\} \subset [R^\dagger \diamond R](a),$$
>   implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \varnothing$, as $b \in R(a)$.
>
> This finishes the proof.

00TK 10.2 Reflexive Relations

00TL 10.2.1 Foundations

Let A be a set.

> **00TM** **DEFINITION 10.2.1.1.1 ▶ REFLEXIVE RELATIONS**
>
> A **reflexive relation** is equivalently:[1]
>
> - An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.
>
> - A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.
>
> ---
>
> [1]Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

00TN   **REMARK 10.2.1.1.2 ▶ UNWINDING DEFINITION 10.2.1.1.1**

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R : \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

00TP   **DEFINITION 10.2.1.1.3 ▶ THE PO/SET OF REFLEXIVE RELATIONS ON A SET**

Let A be a set.

00TQ   1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.

00TR   2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

00TS   **PROPOSITION 10.2.1.1.4 ▶ PROPERTIES OF REFLEXIVE RELATIONS**

Let R and S be relations on A.

00TT   1. *Interaction With Inverses.* If R is reflexive, then so is $R^{\dagger}$.

00TU   2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

PROOF 10.2.1.1.5 ▶ PROOF OF PROPOSITION 10.2.1.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.

00TV   ## 10.2.2   The Reflexive Closure of a Relation

Let R be a relation on A.

00TW

DEFINITION 10.2.2.1.1 ▶ THE REFLEXIVE CLOSURE OF A RELATION

The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\text{refl}}$[1] satisfying the following universal property:[2]

($\star$) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{refl}} \subset \sim_S$.

[1]*Further Notation:* Also written R^{refl}.
[2]*Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

00TX

CONSTRUCTION 10.2.2.1.2 ▶ THE REFLEXIVE CLOSURE OF A RELATION

Concretely, $\sim_R^{\text{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[1], being given by

$$
\begin{aligned}
R^{\text{refl}} &\overset{\text{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\
&= R \cup \Delta_A \\
&= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}.
\end{aligned}
$$

[1]Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

PROOF 10.2.2.1.3 ▶ PROOF OF CONSTRUCTION 10.2.2.1.2

Clear.

00TY

PROPOSITION 10.2.2.1.4 ▶ PROPERTIES OF THE REFLEXIVE CLOSURE OF A RELATION

Let R be a relation on A.

00TZ

1. *Adjointness.* We have an adjunction

$$
\left((-)^{\text{refl}} \dashv \text{忘}\right): \quad \mathbf{Rel}(A, A) \underset{\text{忘}}{\overset{(-)^{\text{refl}}}{\rightleftarrows}} \mathbf{Rel}^{\text{refl}}(A, A),
$$

witnessed by a bijection of sets

$$
\mathbf{Rel}^{\text{refl}}(R^{\text{refl}}, S) \cong \mathbf{Rel}(R, S),
$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\text{refl}}(A, A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, A))$.

00U0

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\text{refl}} = R$.

00U1

3. *Idempotency.* We have

$$(R^{\text{refl}})^{\text{refl}} = R^{\text{refl}}.$$

00U2

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\text{refl}} = \left(R^{\text{refl}}\right)^{\dagger},$$

$$
\begin{array}{ccc}
\text{Rel}(A, A) & \xrightarrow{(-)^{\text{refl}}} & \text{Rel}(A, A) \\
{\scriptstyle (-)^{\dagger}}\downarrow & & \downarrow{\scriptstyle (-)^{\dagger}} \\
\text{Rel}(A, A) & \xrightarrow[(-)^{\text{refl}}]{} & \text{Rel}(A, A).
\end{array}
$$

00U3

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{refl}} = S^{\text{refl}} \diamond R^{\text{refl}},$$

$$
\begin{array}{ccc}
\text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow{\diamond} & \text{Rel}(A, A) \\
{\scriptstyle (-)^{\text{refl}}\times(-)^{\text{refl}}}\downarrow & & \downarrow{\scriptstyle (-)^{\text{refl}}} \\
\text{Rel}(A, A) \times \text{Rel}(A, A) & \xrightarrow[\diamond]{} & \text{Rel}(A, A).
\end{array}
$$

PROOF 10.2.2.1.5 ▶ PROOF OF PROPOSITION 10.2.2.1.4

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 10.2.2.1.1.

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

> **Item 4: Interaction With Inverses**

Clear.

> **Item 5: Interaction With Composition**

This follows from Item 2 of Proposition 10.2.1.1.4.

00U4   10.3   Symmetric Relations

00U5   10.3.1   Foundations

Let A be a set.

> **00U6**   **DEFINITION 10.3.1.1.1 ▶ SYMMETRIC RELATIONS**
>
> A relation R on A is **symmetric** if we have $R^\dagger = R$.

> **00U7**   **REMARK 10.3.1.1.2 ▶ UNWINDING DEFINITION 10.3.1.1.1**
>
> In detail, a relation R is symmetric if it satisfies the following condition:
>
> ($\star$)   For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

> **00U8**   **DEFINITION 10.3.1.1.3 ▶ THE PO/SET OF SYMMETRIC RELATIONS ON A SET**
>
> Let A be a set.
>
> **00U9**   1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathsf{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
>
> **00UA**   2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UB | **PROPOSITION 10.3.1.1.4 ▶ PROPERTIES OF SYMMETRIC RELATIONS**

Let R and S be relations on A.

00UC | 1. *Interaction With Inverses.* If R is symmetric, then so is $R^\dagger$.

00UD | 2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

PROOF 10.3.1.1.5 ▶ PROOF OF PROPOSITION 10.3.1.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear.

00UE ## 10.3.2   The Symmetric Closure of a Relation

Let R be a relation on A.

00UF | **DEFINITION 10.3.2.1.1 ▶ THE SYMMETRIC CLOSURE OF A RELATION**

The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[1] satisfying the following universal property:[2]

  (★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

     [1]*Further Notation:* Also written R^{symm}.
     [2]*Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

00UG | **CONSTRUCTION 10.3.2.1.2 ▶ THE SYMMETRIC CLOSURE OF A RELATION**

Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$R^{\mathrm{symm}} \overset{\mathrm{def}}{=} R \cup R^\dagger$$
$$= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}.$$

PROOF 10.3.2.1.3 ▶ PROOF OF CONSTRUCTION 10.3.2.1.2

Clear.   ▬

00UH | **PROPOSITION 10.3.2.1.4 ▶ PROPERTIES OF THE SYMMETRIC CLOSURE OF A RELATION**

Let R be a relation on A.

00UJ 1. *Adjointness.* We have an adjunction

$$\left((-)^{\text{symm}} \dashv \text{忘}\right): \quad \mathbf{Rel}(A,A) \underset{\text{忘}}{\overset{(-)^{\text{symm}}}{\rightleftarrows}} \mathbf{Rel}^{\text{symm}}(A,A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\text{symm}}(R^{\text{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \text{Obj}(\mathbf{Rel}^{\text{symm}}(A,A))$ and $S \in \text{Obj}(\mathbf{Rel}(A,A))$.

00UK 2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\text{symm}} = R$.

00UL 3. *Idempotency.* We have

$$(R^{\text{symm}})^{\text{symm}} = R^{\text{symm}}.$$

00UM 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\text{symm}} = \left(R^{\text{symm}}\right)^{\dagger},$$

$$
\begin{array}{ccc}
\text{Rel}(A,A) & \xrightarrow{(-)^{\text{symm}}} & \text{Rel}(A,A) \\
{\scriptstyle (-)^{\dagger}}\downarrow & & \downarrow{\scriptstyle (-)^{\dagger}} \\
\text{Rel}(A,A) & \xrightarrow[(-)^{\text{symm}}]{} & \text{Rel}(A,A).
\end{array}
$$

00UN 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\text{symm}} = S^{\text{symm}} \diamond R^{\text{symm}},$$

$$
\begin{array}{ccc}
\text{Rel}(A,A) \times \text{Rel}(A,A) & \xrightarrow{\diamond} & \text{Rel}(A,A) \\
{\scriptstyle (-)^{\text{symm}} \times (-)^{\text{symm}}}\downarrow & & \downarrow{\scriptstyle (-)^{\text{symm}}} \\
\text{Rel}(A,A) \times \text{Rel}(A,A) & \xrightarrow[\diamond]{} & \text{Rel}(A,A).
\end{array}
$$

> **Item 1: Adjointness**
>
> This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 10.3.2.1.1.
>
> **Item 2: The Symmetric Closure of a Symmetric Relation**
>
> Clear.
>
> **Item 3: Idempotency**
>
> This follows from Item 2.
>
> **Item 4: Interaction With Inverses**
>
> Clear.
>
> **Item 5: Interaction With Composition**
>
> This follows from Item 2 of Proposition 10.3.1.1.4.

00UP 10.4 Transitive Relations

00UQ 10.4.1 Foundations

Let A be a set.

00UR **DEFINITION 10.4.1.1.1 ▶ TRANSITIVE RELATIONS**

A **transitive relation** is equivalently:[1]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)), \diamond)$.

- A non-unital monoid in $(\mathbf{Rel}(A,A), \diamond)$.

---
[1]Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

00US **REMARK 10.4.1.1.2 ▶ UNWINDING DEFINITION 10.4.1.1.1**

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00UT **DEFINITION 10.4.1.1.3 ▶ THE PO/SET OF TRANSITIVE RELATIONS ON A SET**

Let A be a set.

00UU   1. The **set of transitive relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.

00UV   2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UW **PROPOSITION 10.4.1.1.4 ▶ PROPERTIES OF TRANSITIVE RELATIONS**

Let R and S be relations on A.

00UX   1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.

00UY   2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

PROOF 10.4.1.1.5 ▶ PROOF OF PROPOSITION 10.4.1.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [MSE 2096272].[1]

---
[1]*Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
    - There is some $b \in A$ such that:

> * $a \sim_R b$;
> * $b \sim_S c$;
> – There is some $d \in A$ such that:
>   * $c \sim_R d$;
>   * $d \sim_S e$.

10.4.2   The Transitive Closure of a Relation

Let R be a relation on A.

DEFINITION 10.4.2.1.1 ▶ THE TRANSITIVE CLOSURE OF A RELATION

The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[1] satisfying the following universal property:[2]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

---

[1]*Further Notation:* Also written R^{trans}.
[2]*Slogan:* The transitive closure of R is the smallest transitive relation containing R.

CONSTRUCTION 10.4.2.1.2 ▶ THE TRANSITIVE CLOSURE OF A RELATION

Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[1], being given by

$$R^{\mathrm{trans}} \overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n}$$

$$\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n}$$

$$\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}.$$

---

[1]Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_{\bullet}(\mathbf{Rel}(A, A)), \diamond)$.

> **PROOF 10.4.2.1.3 ▶ PROOF OF CONSTRUCTION 10.4.2.1.2**
>
> Clear.   ▬

00V2 **PROPOSITION 10.4.2.1.4 ▶ PROPERTIES OF THE TRANSITIVE CLOSURE OF A RELATION**

Let R be a relation on A.

00V3 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv \text{忘}\right): \quad \mathbf{Rel}(A,A) \underset{\text{忘}}{\overset{(-)^{\mathrm{trans}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Rel}^{\mathrm{trans}}(A,A),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}(R^{\mathrm{trans}}, S) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{trans}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

00V4 2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

00V5 3. *Idempotency.* We have

$$(R^{\mathrm{trans}})^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

00V6 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger},$$

$$\begin{array}{ccc}
\mathrm{Rel}(A,A) & \xrightarrow{(-)^{\mathrm{trans}}} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^{\dagger}}\downarrow & & \downarrow{\scriptstyle (-)^{\dagger}} \\
\mathrm{Rel}(A,A) & \xrightarrow[(-)^{\mathrm{trans}}]{} & \mathrm{Rel}(A,A).
\end{array}$$

00V7 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\mathrm{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}},$$

$$\begin{array}{ccc}
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) & \xrightarrow{\diamond} & \mathrm{Rel}(A,A) \\
{\scriptstyle (-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}}\downarrow & \textcolor{red}{\times} & \downarrow{\scriptstyle (-)^{\mathrm{trans}}} \\
\mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) & \xrightarrow[\diamond]{} & \mathrm{Rel}(A,A).
\end{array}$$

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 10.4.2.1.1.

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

We have

$$
\begin{aligned}
(R^\dagger)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty} (R^\dagger)^{\diamond n} \\
&= \bigcup_{n=1}^{\infty} (R^{\diamond n})^\dagger \\
&= \left(\bigcup_{n=1}^{\infty} R^{\diamond n} \right)^\dagger \\
&= (R^{\mathrm{trans}})^\dagger,
\end{aligned}
$$

where we have used, respectively:

- Construction 10.4.2.1.2.

- Constructions With Relations, **??** of **??**.

- Constructions With Relations, **??** of Proposition 9.2.3.1.2.

- Construction 10.4.2.1.2.

This finishes the proof.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 10.4.1.1.4.

00V8 10.5 Equivalence Relations

00V9 10.5.1 Foundations

Let A be a set.

00VA

DEFINITION 10.5.1.1.1 ▶ EQUIVALENCE RELATIONS

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[1]

---

[1]*Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

00VB

EXAMPLE 10.5.1.1.2 ▶ THE KERNEL OF A FUNCTION

The **kernel of a function** $f : A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[1]

---

[1]The kernel $\mathrm{Ker}(f) \colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1} \colon A \rightleftarrows B$ in **Rel** of Constructions With Relations, **??** of **??**.

00VC

DEFINITION 10.5.1.1.3 ▶ THE PO/SET OF EQUIVALENCE RELATIONS ON A SET

Let A and B be sets.

00VD

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

00VE

2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

00VF 10.5.2 The Equivalence Closure of a Relation

Let R be a relation on A.

00VG

DEFINITION 10.5.2.1.1 ▶ THE EQUIVALENCE CLOSURE OF A RELATION

The **equivalence closure**[1] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[2] satisfying the following universal property:[3]

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

---

[1]*Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.
[2]*Further Notation:* Also written R^{eq}.
[3]*Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

00VH   **CONSTRUCTION 10.5.2.1.2 ▶ THE EQUIVALENCE CLOSURE OF A RELATION**

Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$R^{\mathrm{eq}} \overset{\mathrm{def}}{=} ((R^{\mathrm{refl}})^{\mathrm{symm}})^{\mathrm{trans}}$$

$$= ((R^{\mathrm{symm}})^{\mathrm{trans}})^{\mathrm{refl}}$$

$$= \left\{ (a,b) \in A \times B \,\middle|\, \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\[6pt] \quad 1. \quad \text{The following conditions are satisfied:} \\[4pt] \qquad \text{(a)} \quad \text{We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b)} \quad \text{We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\qquad\quad \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c)} \quad \text{We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\[4pt] \quad 2. \quad \text{We have } a = b. \end{array} \right\}$$

PROOF 10.5.2.1.3 ▶ PROOF OF CONSTRUCTION 10.5.2.1.2

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

00VJ   1. The symmetric closure of a reflexive relation is still reflexive.

00VK   2. The transitive closure of a symmetric relation is still symmetric.

which are both clear.

00VL **PROPOSITION 10.5.2.1.4** ▶ **PROPERTIES OF EQUIVALENCE RELATIONS**

Let R be a relation on A.

00VM 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv \text{忘}\right): \quad \mathbf{Rel}(A, B) \underset{\text{忘}}{\overset{(-)^{\mathrm{eq}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Rel}^{\mathrm{eq}}(A, B),$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A, B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$.

00VN 2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

00VP 3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

PROOF 10.5.2.1.5 ▶ **PROOF OF PROPOSITION 10.5.2.1.4**

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 10.5.2.1.1.

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

`00VQ` # 10.6   Quotients by Equivalence Relations

`00VR` ## 10.6.1   Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

`02B1`
> **DEFINITION 10.6.1.1.1 ▶ EQUIVALENCE CLASSES**
>
> The **equivalence class associated to** a is the set $[a]$ defined by
>
> $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\}$$
> $$= \{x \in X \mid a \sim_R x\}. \qquad \text{(since } R \text{ is symmetric)}$$

`02B2` ## 10.6.2   Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

`02B3`
> **DEFINITION 10.6.2.1.1 ▶ QUOTIENTS OF SETS BY EQUIVALENCE RELATIONS**
>
> The **quotient of** X **by** R is the set $X/\!\sim_R$ defined by
>
> $$X/\!\sim_R \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

`00VV`
> **REMARK 10.6.2.1.2 ▶ WHY USE "EQUIVALENCE" RELATIONS FOR QUOTIENT SETS**
>
> The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:
>
> - *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
>
> - *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by
> $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$
>   but we could equally well define
> $$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[1]

- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \not\sim_R b$, and equal otherwise.

---

[1]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, **??**.

00VW **PROPOSITION 10.6.2.1.3 ▶ PROPERTIES OF QUOTIENT SETS**

Let $f: X \to Y$ be a function and let R be a relation on X.

02B4 1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\text{eq}} \cong \text{CoEq}(R \to X \times X \overset{\text{pr}_1}{\underset{\text{pr}_2}{\rightrightarrows}} X),$$

where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.

02B5 2. *As a Pushout.* We have an isomorphism of sets[1]

$$X/\sim_R^{\text{eq}} \cong X \coprod_{\text{Eq}(\text{pr}_1,\text{pr}_2)} X,$$

$$\begin{array}{ccc} X/\sim_R^{\text{eq}} & \longleftarrow & X \\ \uparrow & \ulcorner & \uparrow \\ X & \longleftarrow & \text{Eq}(\text{pr}_1, \text{pr}_2). \end{array}$$

where $\sim_R^{\text{eq}}$ is the equivalence relation generated by $\sim_R$.

02B6 3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[2,3]

$$X/\sim_{\text{Ker}(f)} \cong \text{Im}(f).$$

00W0 4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

02B7 (a) There exists a map

$$\bar{f}: X/\sim_R \to Y$$

making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle q}\downarrow & {\scriptstyle \exists}\;\nearrow & \\
 & \;\;\overline{f} & \\
X/\!\sim_R & &
\end{array}
$$

commute.

02B8   (b) We have $R \subset \mathrm{Ker}(f)$.

02B9   (c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

00W1   5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle q}\downarrow & {\scriptstyle \exists!}\;\nearrow & \\
 & \;\;\overline{f} & \\
X/\!\sim_R & &
\end{array}
$$

commute.

00W2   6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/\!\sim_R, Y) \cong \mathrm{Hom}_{\mathsf{Sets}}^{R}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}_{\mathsf{Sets}}^{R}(X, Y)$ is the set defined by

$$\mathrm{Hom}_{\mathsf{Sets}}^{R}(X, Y) \overset{\mathrm{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

02BA   7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BB   (a) The map $\overline{f}$ is an injection.

02BC   (b) We have $R = \mathrm{Ker}(f)$.

02BD   (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

02BE   8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

02BF   (a) The map $f \colon X \to Y$ is surjective.

02BG   (b) The map $\overline{f} \colon X/\sim_R \to Y$ is surjective.

02BH   9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

02BJ   (a) The map f satisfies the equivalent conditions of Item 4:

            • There exists a map

            $$\overline{f} \colon X/\sim_R^{\mathrm{eq}} \to Y$$

            making the diagram

            $$
            \begin{array}{ccc}
            X & \xrightarrow{\ f\ } & Y \\
            {\scriptstyle q}\downarrow & {\scriptstyle \exists}\ \diagdown\ {\scriptstyle \overline{f}} & \\
            X/\sim_R^{\mathrm{eq}} & &
            \end{array}
            $$

            commute.
            • For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

02BK   (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

[1]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) \cong X \times_{X/\sim_R^{\mathrm{eq}}} X,$$

$$
\begin{array}{ccc}
\mathrm{Eq}(\mathrm{pr}_1, \mathrm{pr}_2) & \longrightarrow & X \\
\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & X/\sim_R^{\mathrm{eq}}.
\end{array}
$$

[2]*Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{CoIm}(f)$.

[3]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f)\colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$
\bigl(\mathrm{Gr}(f) \dashv f^{-1}\bigr)\colon \quad A \underset{f^{-1}}{\overset{\mathrm{Gr}(f)}{\rightleftarrows}} B
$$

of Constructions With Relations, **??** of **??**.

PROOF 10.6.2.1.4 ▶ PROOF OF PROPOSITION 10.6.2.1.3

Item 1: As a Coequaliser

Omitted.

Item 2: As a Pushout

Omitted.

Item 3: The First Isomorphism Theorem for Sets

Clear.

Item 4: Descending Functions to Quotient Sets, I

See [Pro25c].

Item 5: Descending Functions to Quotient Sets, II

See [Pro25d].

> **Item 6: Descending Functions to Quotient Sets, III**

This follows from Items 5 and 6.

> **Item 7: Descending Functions to Quotient Sets, IV**

See [Pro25b].

> **Item 8: Descending Functions to Quotient Sets, V**

See [Pro25a].

> **Item 9: Descending Functions to Quotient Sets, VI**

The implication Item 8a $\implies$ Item 8b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.
Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\text{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

- The following conditions are satisfied:
  * We have $x \sim_R x_1$ or $x_1 \sim_R x$;
  * We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
  * We have $y \sim_R x_n$ or $x_n \sim_R y$;
- We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$f(x) = f(x_1),$$
$$f(x_1) = f(x_2),$$
$$\vdots$$
$$f(x_{n-1}) = f(x_n),$$
$$f(x_n) = f(y),$$

and $f(x) = f(y)$, as we wanted to show.

Appendices

A   Other Chapters

References

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