# Monoidal Structures on the Category of Sets

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**O1NK** This chapter contains some material on monoidal structures on Sets.

# **Contents**

5.1	The M	Ionoidal Category of Sets and Products	2
	5.1.1	Products of Sets	2
	5.1.2	The Internal Hom of Sets	2
	5.1.3	The Monoidal Unit	2
	5.1.4	The Associator	3
	5.1.5	The Left Unitor	6
	5.1.6	The Right Unitor	8
	5.1.7	The Symmetry	11
	5.1.8	The Diagonal	13
	5.1.9	The Monoidal Category of Sets and Products	17
	5.1.10	The Universal Property of (Sets, ×, pt)	22
5.2	The M	Ionoidal Category of Sets and Coproducts	42
	5.2.1	Coproducts of Sets	42
	5.2.2	The Monoidal Unit	42
	5.2.3	The Associator	42
	5.2.4	The Left Unitor	46
	5.2.5	The Right Unitor	49
	5.2.6	The Symmetry	52
	5.2.7	The Monoidal Category of Sets and Coproducts	56

	5.3	The Bimonoidal Category of Sets, Products, and Coproducts645.3.1The Left Distributor645.3.2The Right Distributor665.3.3The Left Annihilator685.3.4The Right Annihilator68			
	A	5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts 69  Other Chapters			
)1NL	5.1	The Monoidal Category of Sets and Products			
)1NM	5.1.	r Products of Sets			
	See	Constructions With Sets, Section 4.1.3.			
)1NN	5.1.	2 The Internal Hom of Sets			
	See	Constructions With Sets, Section 4.3.5.			
)1NP	5.1.	3 The Monoidal Unit			
1NQ	D	EFINITION 5.1.3.1.1 ► THE MONOIDAL UNIT OF ×			
	The monoidal unit of the product of sets is the functor				
		$1^{Sets} \colon pt \to Sets$			
	d	efined by $\mathbb{1}_{Sets} \stackrel{\scriptscriptstyle \mathrm{def}}{=} pt,$			

where pt is the terminal set of Constructions With Sets, Definition 4.1.1.1.

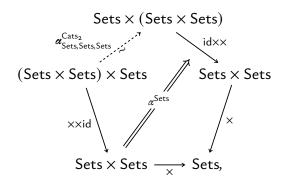
# 01NR 5.1.4 The Associator

### **01NS DEFINITION 5.1.4.1.1** ► THE ASSOCIATOR OF ×

The associator of the product of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \mathsf{X} \circ (\mathsf{X} \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\widetilde{}}{\Longrightarrow} \mathsf{X} \circ (\mathsf{id}_{\mathsf{Sets}} \, \mathsf{X} \mathsf{X}) \circ \pmb{\alpha}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}'}^{\mathsf{Cats_2}}$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets}} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\text{Sets}}((x,y),z) \stackrel{\text{def}}{=} (x,(y,z))$$

for each  $((x, y), z) \in (X \times Y) \times Z$ .

### PROOF 5.1.4.1.2 ▶ Proof of the Claims Made in Definition 5.1.4.1.1

Invertibility

The inverse of  $\alpha_{X,Y,Z}^{Sets}$  is the morphism

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha_{X,Y,Z}^{\text{Sets},-1}(x,(y,z)) \stackrel{\text{def}}{=} ((x,y),z)$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ . Indeed:

• *Invertibility I.* We have

$$\begin{bmatrix} \alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} \end{bmatrix} ((x,y),z) \stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1} \Big( \alpha_{X,Y,Z}^{\mathsf{Sets}} ((x,y),z) \Big)$$

$$\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets},-1} (x,(y,z))$$

$$\stackrel{\text{def}}{=} ((x,y),z)$$

$$\stackrel{\text{def}}{=} [\mathrm{id}_{(X\times Y)\times Z}] ((x,y),z)$$

for each  $((x, y), z) \in (X \times Y) \times Z$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{(X \times Y) \times Z}.$$

• Invertibility II. We have

$$\begin{bmatrix} \alpha_{X,Y,Z}^{\mathsf{Sets}} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},-1} \end{bmatrix} (x, (y, z)) \stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} (\alpha_{X,Y,Z}^{\mathsf{Sets},-1} (x, (y, z)))$$

$$\stackrel{\text{def}}{=} \alpha_{X,Y,Z}^{\mathsf{Sets}} ((x, y), z)$$

$$\stackrel{\text{def}}{=} (x, (y, z))$$

$$\stackrel{\text{def}}{=} [\mathrm{id}_{(X \times Y) \times Z}] (x, (y, z))$$

for each  $(x, (y, z)) \in X \times (Y \times Z)$ , and therefore we have

$$\alpha_{X,Y,Z}^{\mathsf{Sets},-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets}} = \mathrm{id}_{X \times (Y \times Z)}$$
.

Therefore  $\alpha_{X,Y,Z}^{\mathsf{Sets}}$  is indeed an isomorphism.

5.1.4 The Associator

### **Naturality**

We need to show that, given functions

$$f: X \to X',$$

$$g: Y \to Y',$$

$$h: Z \to Z'$$

the diagram

$$\begin{array}{c|c} (X\times Y)\times Z & \xrightarrow{(f\times g)\times h} & (X'\times Y')\times Z' \\ & \alpha^{\mathsf{Sets}}_{X,Y,Z} & & & \downarrow \alpha^{\mathsf{Sets}}_{X',Y',Z'} \\ X\times (Y\times Z) & \xrightarrow{f\times (g\times h)} & X'\times (Y'\times Z') \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$((x, y), z) \qquad ((x, y), z) \longmapsto ((f(x), g(y)), h(z))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes, showing  $\alpha^{Sets}$  to be a natural transformation.

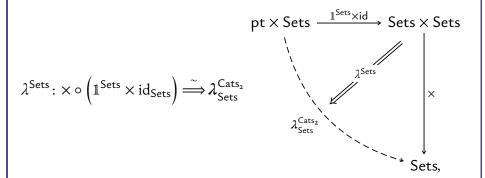
### Being a Natural Isomorphism

Since  $\alpha^{\text{Sets}}$  is natural and  $\alpha^{\text{Sets},-1}$  is a componentwise inverse to  $\alpha^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\alpha^{\text{Sets},-1}$  is also natural. Thus  $\alpha^{\text{Sets}}$  is a natural isomorphism.

# 01NT 5.1.5 The Left Unitor

### **O1NU DEFINITION 5.1.5.1.1** ► THE LEFT UNITOR OF ×

The **left unitor of the product of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets}} \colon \mathsf{pt} \times X \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets}}(\star, x) \stackrel{\text{def}}{=} x$$

for each  $(\star, x) \in pt \times X$ .

### PROOF 5.1.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.5.1.1

# Invertibility

The inverse of  $\lambda_X^{\rm Sets}$  is the morphism

$$\lambda_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} \mathsf{pt} \times X$$

defined by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (\star, x)$$

for each  $x \in X$ . Indeed:

• Invertibility I. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}}\right] \big(\mathsf{pt},x\big) &= \lambda_X^{\mathsf{Sets},-1} \Big(\lambda_X^{\mathsf{Sets}} \big(\mathsf{pt},x\big)\Big) \\ &= \lambda_X^{\mathsf{Sets},-1} \big(x\big) \\ &= \big(\mathsf{pt},x\big) \\ &= \big[\mathsf{id}_{\mathsf{pt} \times X}\big] \big(\mathsf{pt},x\big) \end{split}$$

for each  $(pt, x) \in pt \times X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},-1} \circ \lambda_X^{\mathsf{Sets}} = \mathrm{id}_{\mathsf{pt} \times X}.$$

• Invertibility II. We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1}\right](x) &= \lambda_X^{\mathsf{Sets}} \left(\lambda_X^{\mathsf{Sets},-1}(x)\right) \\ &= \lambda_X^{\mathsf{Sets},-1} \big(\mathsf{pt},x\big) \\ &= x \\ &= [\mathsf{id}_X](x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}} \circ \lambda_X^{\mathsf{Sets},-1} = \mathrm{id}_X.$$

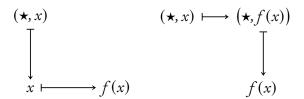
Therefore  $\lambda_X^{\mathsf{Sets}}$  is indeed an isomorphism.

### Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc} \operatorname{pt} \times X & \xrightarrow{\operatorname{id}_{\operatorname{pt}} \times f} & \operatorname{pt} \times Y \\ \lambda_X^{\operatorname{Sets}} & & & \downarrow \lambda_Y^{\operatorname{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\lambda^{\mathsf{Sets}}$  is a natural transformation.

# Being a Natural Isomorphism

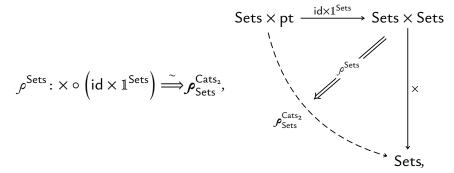
Since  $\lambda^{\text{Sets}}$  is natural and  $\lambda^{\text{Sets},-1}$  is a componentwise inverse to  $\lambda^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\lambda^{\text{Sets},-1}$  is also natural. Thus  $\lambda^{\text{Sets}}$  is a natural isomorphism.

# 01NV 5.1.6 The Right Unitor

### **01NW**

# **DEFINITION 5.1.6.1.1** ► THE RIGHT UNITOR OF ×

The **right unitor of the product of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets}} \colon X \times \mathsf{pt} \xrightarrow{\sim} X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\rho_X^{\mathsf{Sets}}(x, \star) \stackrel{\mathrm{def}}{=} x$$

for each  $(x, \star) \in X \times pt$ .

### PROOF 5.1.6.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.6.1.1

# Invertibility

The inverse of  $\rho_X^{\rm Sets}$  is the morphism

$$\rho_X^{\mathsf{Sets},-1} \colon X \xrightarrow{\sim} X \times \mathsf{pt}$$

defined by

$$\rho_X^{\mathsf{Sets},-1}(x) \stackrel{\mathrm{def}}{=} (x, \star)$$

for each  $x \in X$ . Indeed:

• *Invertibility I.* We have

$$\left[\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}}\right](x, \star) = \rho_X^{\mathsf{Sets},-1} \left(\rho_X^{\mathsf{Sets}}(x, \star)\right)$$

$$= \rho_X^{\mathsf{Sets},-1}(x)$$

$$= (x, \star)$$

$$= [\mathrm{id}_{X \times \mathsf{pt}}](x, \star)$$

for each  $(x, \star) \in X \times pt$ , and therefore we have

$$\rho_X^{\mathsf{Sets},-1} \circ \rho_X^{\mathsf{Sets}} = \mathrm{id}_{X \times \mathsf{pt}}.$$

• Invertibility II. We have

$$\begin{aligned} \left[ \rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} \right](x) &= \rho_X^{\mathsf{Sets}} \left( \rho_X^{\mathsf{Sets},-1}(x) \right) \\ &= \rho_X^{\mathsf{Sets},-1}(x, \bigstar) \\ &= x \\ &= \left[ \mathrm{id}_X \right](x) \end{aligned}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X.$$

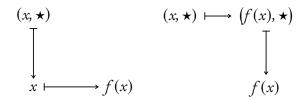
Therefore  $\rho_X^{\rm Sets}$  is indeed an isomorphism.

### Naturality

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{c|c} X \times \operatorname{pt} & \xrightarrow{f \times \operatorname{id}_{\operatorname{pt}}} & Y \times \operatorname{pt} \\ \rho_X^{\operatorname{Sets}} & & & \downarrow \rho_Y^{\operatorname{Sets}} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes. Therefore  $\rho^{\mathsf{Sets}}$  is a natural transformation.

# Being a Natural Isomorphism

Since  $\rho^{\text{Sets}}$  is natural and  $\rho^{\text{Sets},-1}$  is a componentwise inverse to  $\rho^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\rho^{\text{Sets},-1}$  is also natural. Thus  $\rho^{\text{Sets}}$  is a natural isomorphism.

# 01NX 5.1.7 The Symmetry

### **01NY DEFINITION 5.1.7.1.1** ► THE SYMMETRY OF ×

The **symmetry of the product of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{array}{c} \mathsf{Sets} \times \mathsf{Sets} & \xrightarrow{\times} \mathsf{Sets}, \\ & \downarrow & \\ & & \downarrow & \\ & & \mathsf{Sets} \times \mathsf{Sets} \end{array}$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets}} \colon X \times Y \xrightarrow{\sim} Y \times X$$

at  $X, Y \in \text{Obj}(\mathsf{Sets})$  is defined by

$$\sigma_{XY}^{\text{Sets}}(x,y) \stackrel{\text{def}}{=} (y,x)$$

for each  $(x, y) \in X \times Y$ .

### PROOF 5.1.7.1.2 ▶ Proof of the Claims Made in Definition 5.1.7.1.1

### Invertibility

The inverse of  $\sigma_{X,Y}^{\mathsf{Sets}}$  is the morphism

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \colon Y \times X \xrightarrow{\sim} X \times Y$$

defined by

$$\sigma_{X,Y}^{\text{Sets},-1}(y,x) \stackrel{\text{def}}{=} (x,y)$$

for each  $(y, x) \in Y \times X$ . Indeed:

• *Invertibility I.* We have

$$\begin{bmatrix}
\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}}
\end{bmatrix} (x,y) \stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \left(\sigma_{X,Y}^{\mathsf{Sets}}(x,y)\right) \\
\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1}(y,x) \\
\stackrel{\text{def}}{=} (x,y) \\
\stackrel{\text{def}}{=} [\mathrm{id}_{X\times Y}](x,y)$$

for each  $(x, y) \in X \times Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets},-1} \circ \sigma_{X,Y}^{\mathsf{Sets}} = \mathrm{id}_{X \times Y}.$$

• Invertibility II. We have

$$\begin{bmatrix}
\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} \\
\end{bmatrix} (y, x) \stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} \left( \sigma_{X,Y}^{\mathsf{Sets}} (y, x) \right) \\
\stackrel{\text{def}}{=} \sigma_{X,Y}^{\mathsf{Sets},-1} (x, y) \\
\stackrel{\text{def}}{=} (y, x) \\
\stackrel{\text{def}}{=} [\mathrm{id}_{Y \times X}] (y, x)$$

for each  $(y, x) \in Y \times X$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}} \circ \sigma_{X,Y}^{\mathsf{Sets},-1} = \mathrm{id}_{Y \times X}$$
.

Therefore  $\sigma_{X,Y}^{\mathsf{Sets}}$  is indeed an isomorphism.

### **Naturality**

We need to show that, given functions

$$f: X \to A$$

$$g: Y \to B$$

the diagram

$$\begin{array}{c|c} X \times Y & \xrightarrow{f \times g} & A \times B \\ \sigma_{X,Y}^{\mathsf{Sets}} & & & & & \\ \sigma_{A,B}^{\mathsf{Sets}} & & & & \\ Y \times X & \xrightarrow{\sigma \times f} & B \times A \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$(x, y) \qquad (x, y) \longmapsto (f(x), g(y))$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$(y, x) \longmapsto (g(y), f(x)) \qquad (g(y), f(x))$$

and hence indeed commutes, showing  $\sigma^{Sets}$  to be a natural transformation.

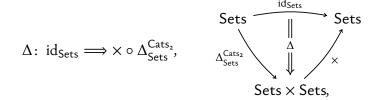
# Being a Natural Isomorphism

Since  $\sigma^{\text{Sets}}$  is natural and  $\sigma^{\text{Sets},-1}$  is a componentwise inverse to  $\sigma^{\text{Sets}}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\sigma^{\text{Sets},-1}$  is also natural. Thus  $\sigma^{\text{Sets}}$  is a natural isomorphism.

# 01NZ 5.1.8 The Diagonal

### 01P0 DEFINITION 5.1.8.1.1 ► THE DIAGONAL OF ×

The diagonal of the product of sets is the natural transformation



whose component

$$\Delta_X \colon X \to X \times X$$

at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\Delta_X(x) \stackrel{\text{def}}{=} (x, x)$$

for each  $x \in X$ .

### PROOF 5.1.8.1.2 ▶ PROOF OF THE CLAIMS MADE IN DEFINITION 5.1.8.1.1

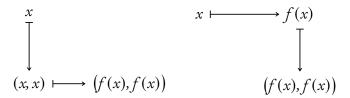
We need to show that, given a function  $f: X \to Y$ , the diagram

$$X \xrightarrow{f} Y$$

$$\Delta_X \downarrow \qquad \qquad \downarrow \Delta_Y$$

$$X \times X \xrightarrow{f \times f} Y \times Y$$

commutes. Indeed, this diagram acts on elements as



and hence indeed commutes, showing  $\Delta$  to be natural.

### **PROPOSITION 5.1.8.1.3** ▶ PROPERTIES OF THE DIAGONAL MAP

Let *X* be a set.

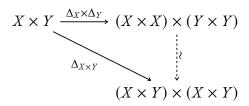
01P2 I. Monoidality. The diagonal map

$$\Delta \colon \operatorname{id}_{\mathsf{Sets}} \Longrightarrow \mathsf{X} \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

is a monoidal natural transformation:

01P3

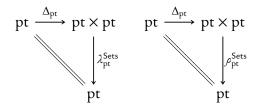
(a) Compatibility With Strong Monoidality Constraints. For each  $X, Y \in \mathsf{Obj}(\mathsf{Sets})$ , the diagram



commutes.

01P4

(b) Compatibility With Strong Unitality Constraints. The diagrams



commute, i.e. we have

$$\begin{split} \Delta_{\text{pt}} &= \lambda_{\text{pt}}^{\text{Sets,-1}} \\ &= \rho_{\text{pt}}^{\text{Sets,-1}}, \end{split}$$

where we recall that the equalities

$$\begin{split} \lambda_{\text{pt}}^{\text{Sets}} &= \rho_{\text{pt}}^{\text{Sets}}, \\ \lambda_{\text{pt}}^{\text{Sets},-1} &= \rho_{\text{pt}}^{\text{Sets},-1} \end{split}$$

are always true in any monoidal category by Monoidal Categories, ?? of ??.

01P5

2. The Diagonal of the Unit. The component

$$\Delta_{pt} \colon pt \xrightarrow{\sim} pt \times pt$$

of  $\Delta$  at pt is an isomorphism.

### PROOF 5.1.8.1.4 ► PROOF OF PROPOSITION 5.1.8.1.3

### Item 1: Monoidality

We claim that  $\Delta$  is indeed monoidal:

o24S

I. Item 1a: Compatibility With Strong Monoidality Constraints: We need to show that the diagram

$$X \times Y \xrightarrow{\Delta_X \times \Delta_Y} (X \times X) \times (Y \times Y)$$

$$\downarrow (X \times Y) \times (X \times Y)$$

commutes. Indeed, this diagram acts on elements as

$$(x, y) \longmapsto ((x, x), (y, y)) \qquad (x, y)$$

$$((x, y), (x, y)) \qquad ((x, y), (x, y))$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at  $X \in \text{Obj}(\mathsf{Sets})$  is given by

$$\lambda_X^{\mathsf{Sets},-1}(x) \stackrel{\mathsf{def}}{=} (\star, x)$$

for each  $x \in X$ , so when X = pt, we have

$$\lambda_{\mathrm{pt}}^{\mathrm{Sets},-1}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

and also

024T

$$\Delta_{\mathrm{pt}}^{\mathrm{Sets}}(\star)\stackrel{\mathrm{def}}{=}(\star,\star),$$

so we have  $\Delta_{pt} = \lambda_{pt}^{Sets,-1}$ .

This finishes the proof.

### Item 2: The Diagonal of the Unit

This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to ×, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor.

### The Monoidal Category of Sets and Products 01P6 **5.I.9**

PROPOSITION 5.1.9.1.1 ➤ THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO THE 01P7 **PRODUCT** 

> The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category*. The category Sets of pointed sets.
- *The Monoidal Product.* The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

• The Internal Hom. The internal Hom functor

Sets: Sets
$$^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.3.5.1.2.

• *The Monoidal Unit*. The functor

$$1^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\text{Sets}} : \times \circ (\times \times \text{id}_{\text{Sets}}) \stackrel{\sim}{\Longrightarrow} \times \circ (\text{id}_{\text{Sets}} \times \times) \circ \alpha^{\text{Cats}}_{\text{Sets},\text{Sets},\text{Sets}}$$
of Definition 5.1.4.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \mathsf{X} \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}}\right) \stackrel{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.6.1.1.

• The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Diagonals. The monoidal natural transformation

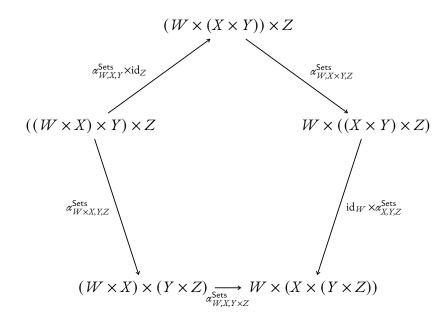
$$\Delta \colon id_{\mathsf{Sets}} \Longrightarrow \mathsf{X} \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

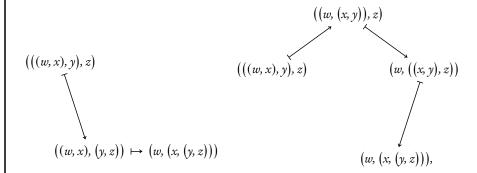
### PROOF 5.1.9.1.2 ► PROOF OF PROPOSITION 5.1.9.1.1

The Pentagon Identity

Let W, X, Y and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the pentagon identity is satisfied.

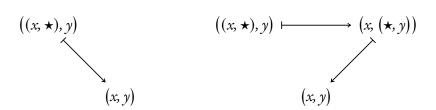
The Triangle Identity

Let X and Y be sets. We have to show that the diagram

$$(X \times \operatorname{pt}) \times Y \xrightarrow{\alpha_{X,\operatorname{pt},Y}^{\operatorname{Sets}}} X \times (\operatorname{pt} \times Y)$$

$$\rho_X^{\operatorname{Sets}} \times \operatorname{id}_Y \xrightarrow{\operatorname{id}_X \times \lambda_Y^{\operatorname{Sets}}} X \times Y$$

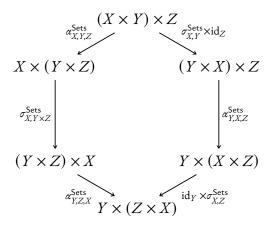
commutes. Indeed, this diagram acts on elements as



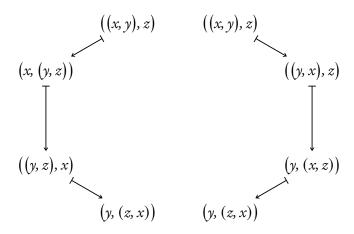
and thus the triangle identity is satisfied.

### The Left Hexagon Identity

Let *X*, *Y*, and *Z* be sets. We have to show that the diagram



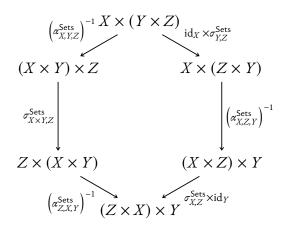
commutes. Indeed, this diagram acts on elements as



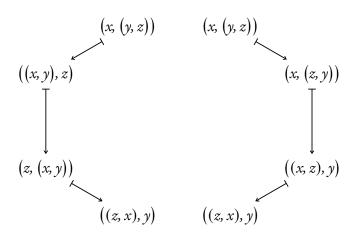
and thus the left hexagon identity is satisfied.

### The Right Hexagon Identity

Let X, Y, and Z be sets. We have to show that the diagram



commutes. Indeed, this diagram acts on elements as



and thus the right hexagon identity is satisfied.

### Monoidal Closedness

01PA

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This follows from Constructions With Sets, Item 2 of Proposition 4.3.5.1.2

### Existence of Monoidal Diagonals

This follows from Items 1 and 2 of Proposition 5.1.8.1.3.

# **1018** 5.1.10 The Universal Property of (Sets, ×, pt)

### 01P9 THEOREM 5.1.10.1.1 ► THE UNIVERSAL PROPERTY OF (Sets, ×, pt)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

I. Existence of an Internal Hom. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets admits an internal Hom  $[-1, -2]_{Sets}$ .

2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

More precisely, the full subcategory of the category  $\mathcal{M}^{cld}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\mathsf{P}^{\mathsf{Sets}}$  spanned by the closed symmetric monoidal categories ( $\mathsf{Sets}$ ,  $\mathsf{P}^{\mathsf{Sets}}$ ),

 $[-1, -2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda^{Sets}$ ,  $\rho^{Sets}$ ,  $\sigma^{Sets}$ ) satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

### PROOF 5.1.10.1.2 ▶ PROOF OF THEOREM 5.1.10.1.1

### Unwinding the Statement

Let (Sets,  $\otimes_{Sets}$ ,  $[-1, -2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda', \rho', \sigma'$ ) be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$id_{Sets} : Sets \rightarrow Sets$$

admits a unique closed symmetric monoidal functor structure

making it into a symmetric monoidal strongly closed isomorphism of categories from (Sets,  $\otimes_{Sets}$ ,  $[-_1, -_2]_{Sets}$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda'$ ,  $\rho'$ ,  $\sigma'$ ) to the closed symmetric monoidal category (Sets,  $\times$ , Sets $(-_1, -_2)$ ,  $\mathbb{1}_{Sets}$ ,  $\lambda^{Sets}$ ,  $\rho^{Sets}$ ,  $\sigma^{Sets}$ ) of Proposition 5.1.9.1.1.

## Constructing an Isomorphism $[-1, -2]_{Sets} \cong Sets(-1, -2)$

By ??, we have a natural isomorphism

$$Sets(pt, [-1, -2]_{Sets}) \cong Sets(-1, -2).$$

By Constructions With Sets, Item 3 of Proposition 4.3.5.1.2, we also have a natural isomorphism

Sets(pt, 
$$[-1, -2]_{Sets}$$
)  $\cong [-1, -2]_{Sets}$ .

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}(-_1,-_2) \cong [-_1,-_2]_{\mathsf{Sets}}.$$

Given  $A, B \in \text{Obj}(\mathsf{Sets})$ , we will write

$$id_{A,B}^{Hom}$$
:  $Sets(A, B) \xrightarrow{\sim} [A, B]_{Sets}$ 

for the component of this isomorphism at (A, B).

# Constructing an Isomorphism ⊗<sub>Sets</sub> ≅ ×

Since  $\otimes_{Sets}$  is adjoint in each variable to  $[-1, -2]_{Sets}$  by assumption and  $\times$  is adjoint in each variable to Sets(-1, -2) by Constructions With Sets, Item 2 of Proposition 4.3.5.1.2, uniqueness of adjoints (??) gives us natural isomorphisms

$$A \otimes_{\mathsf{Sets}} - \cong A \times -,$$
  
 $- \otimes_{\mathsf{Sets}} B \cong B \times -.$ 

By ??, we then have  $\otimes_{\mathsf{Sets}} \cong \times$ . We will write

$$\operatorname{id}_{\operatorname{\mathsf{Sets}}|A,B}^{\otimes} \colon A \otimes_{\operatorname{\mathsf{Sets}}} B \xrightarrow{\sim} A \times B$$

for the component of this isomorphism at (A, B).

### Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Alternatively, we may construct a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \mathsf{x}$  as follows:

I. Let  $A \in Obj(Sets)$ .

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- 2. Since  $\otimes_{Sets}$  is part of a closed monoidal structure, it preserves colimits in each variable by ??.
- 3. Since  $A \cong \coprod_{a \in A} \text{pt}$  and  $\otimes_{\text{Sets}}$  preserves colimits in each variable, we have

$$A \otimes_{\mathsf{Sets}} B \cong \left( \coprod_{a \in A} \mathsf{pt} \right) \otimes_{\mathsf{Sets}} B$$
$$\cong \coprod_{a \in A} \left( \mathsf{pt} \otimes_{\mathsf{Sets}} B \right)$$

$$\cong \coprod_{a \in A} B$$
$$\cong A \times B$$

naturally in  $B \in \text{Obj}(Sets)$ , where we have used that pt is the monoidal unit for  $\otimes_{Sets}$ . Thus  $A \otimes_{Sets} - \cong A \times -$  for each  $A \in \text{Obj}(\mathsf{Sets}).$ 

4. Similarly,  $- \otimes_{\mathsf{Sets}} B \cong - \times B$  for each  $B \in \mathsf{Obj}(\mathsf{Sets})$ .

Below, we'll show that if a natural isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  exists, then it

must be unique. This will show that the isomorphism constructed above is equal to the isomorphism  $\operatorname{id}_{\operatorname{Sets}|A,B}^{\otimes} \colon A \otimes_{\operatorname{Sets}} B \xrightarrow{1} A \times B$  from before.

# Constructing an Isomorphism $id_1^{\otimes}: \mathbb{1}_{Sets} \longrightarrow pt$

5. By ??, we then have  $\otimes_{\mathsf{Sets}} \cong \times$ .

We define an isomorphism  $id_1^{\otimes} : \mathbb{1}_{\mathsf{Sets}} \to \mathsf{pt}$  as the composition

$$\mathbb{1}_{\mathsf{Sets}} \overset{\rho^{\mathsf{Sets},-1}_{\mathsf{1}_{\mathsf{Sets}}}}{\overset{\circ}{\underset{\sim}{\longrightarrow}}} \mathbb{1}_{\mathsf{Sets}} \times \mathsf{pt} \overset{\mathrm{id}_{\mathsf{Sets}|\mathsf{1}_{\mathsf{Sets}}}^{\otimes}}{\overset{\circ}{\underset{\sim}{\longrightarrow}}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathsf{pt} \overset{\lambda'_{\mathsf{pt}}}{\overset{\circ}{\underset{\sim}{\longrightarrow}}} \mathsf{pt}$$

in Sets.

# Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$\operatorname{pt} \otimes_{\mathsf{Sets}} A \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes}} \operatorname{pt} \times A$$

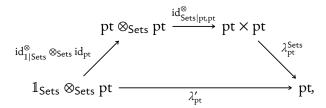
$$\operatorname{id}_{\mathsf{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \operatorname{id}_{A} \xrightarrow{\lambda_{A}'} A$$

$$\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \xrightarrow{\lambda_{A}'} A$$

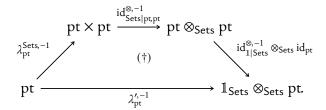
01PF

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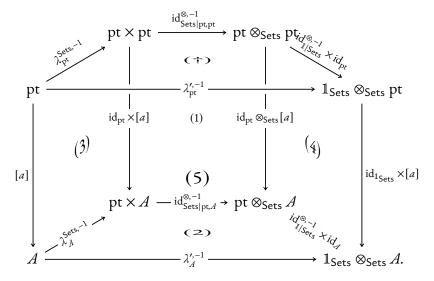
commutes. First, note that the diagram



corresponding to the case A = pt, commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes, so does the diagram



Now, let  $A \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , and consider the diagram



Since:

- Subdiagram (5) commutes by the naturality of  $\lambda'^{-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{1|Sets}^{\otimes,-1}$ .
- Subdiagram (1) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\lambda^{\text{Sets},-1}$ .

it follows that the diagram

$$\operatorname{pt} \times A \xrightarrow{\operatorname{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1}} \operatorname{pt} \otimes_{\mathsf{Sets}} A$$

$$\operatorname{pt} \xrightarrow{\lambda_A^{\mathsf{Sets},-1}} A \xrightarrow{\operatorname{id}_{\mathsf{1}|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_A}$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{split} \lambda_A^{\prime,-1}(a) &= \left[\lambda_A^{\prime,-1} \circ [a]\right](\bigstar) \\ &= \left[\left(\mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1} \circ [a]\right](\bigstar) \\ &= \left[\left(\mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}\right](a) \end{split}$$

for each  $a \in A$ , and thus we have

$$\lambda_A^{\prime,-1} = \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \times \mathrm{id}_A \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \lambda_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\lambda_A' = \lambda_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ \left(\mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \times \mathrm{id}_A\right),$$

showing that the diagram

$$\operatorname{pt} \otimes_{\mathsf{Sets}} A \xrightarrow{\operatorname{id}_{\mathsf{Sets}}^{\otimes} | \operatorname{pt} A} \operatorname{pt} \times A$$

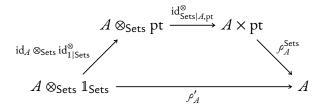
$$\operatorname{id}_{\mathsf{1}|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \operatorname{id}_{A} \xrightarrow{\lambda_{A}'} A$$

$$1_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \xrightarrow{\lambda_{A}'} A$$

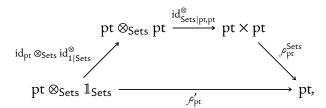
indeed commutes.

### Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We can use the same argument we used to prove the monoidal left unity of the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  above. For completeness, we repeat it below. We have to show that the diagram

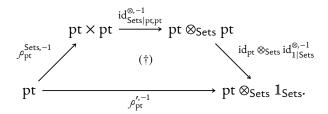


commutes. First, note that the diagram

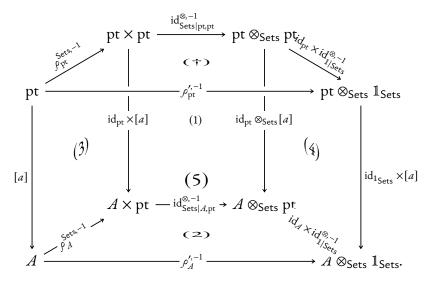


corresponding to the case A = pt, commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since this diagram commutes,

so does the diagram



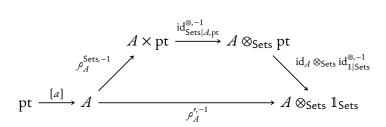
Now, let  $A \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , and consider the diagram



### Since:

- Subdiagram (5) commutes by the naturality of  $\rho'^{-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $id_{1\mid Sets}^{\otimes,-1}$
- Subdiagram (1) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $\rho^{\text{Sets},-1}$ .

it follows that the diagram



Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{split} \rho_A^{\prime,-1}(a) &= \left[\rho_A^{\prime,-1} \circ [a]\right](\bigstar) \\ &= \left[\left(\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1} \circ [a]\right](\bigstar) \\ &= \left[\left(\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1}\right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}\right](a) \end{split}$$

for each  $a \in A$ , and thus we have

$$\rho_A^{\prime,-1} = \left( \mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \right) \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes,-1} \circ \rho_A^{\mathsf{Sets},-1}.$$

Taking inverses then gives

$$\rho_A' = \rho_A^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|\mathsf{pt},A}^{\otimes} \circ \left(\mathrm{id}_A \times \mathrm{id}_{1|\mathsf{Sets}}^{\otimes}\right),$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} \mathsf{pt} \xrightarrow{\mathrm{id}_{\mathsf{Sets}}^{\otimes} |A,\mathsf{pt}|} A \times \mathsf{pt}$$

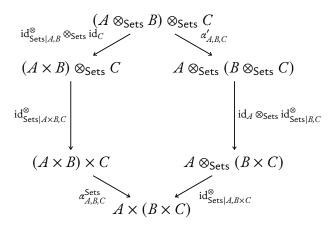
$$\mathrm{id}_{A} \otimes_{\mathsf{Sets}} \mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \longrightarrow A$$

$$A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \longrightarrow A$$

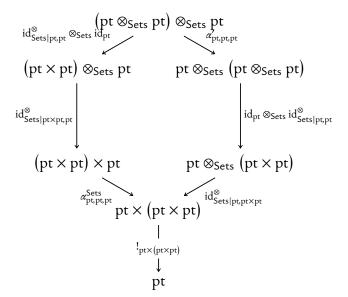
indeed commutes.

Monoidality of the Isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$ 

We have to show that the diagram

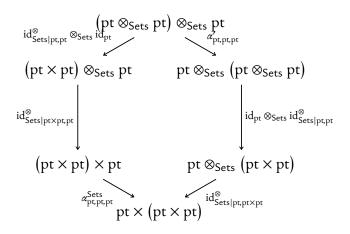


commutes. First, note that the diagram

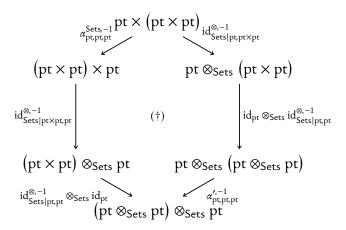


commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since the map  $!_{pt\times(pt\times pt)}: pt\times(pt\times pt) \to pt$  is an isomor-

phism (e.g. having inverse  $\lambda_{pt}^{Sets,-1}\circ\lambda_{pt}^{Sets,-1}),$  it follows that the diagram

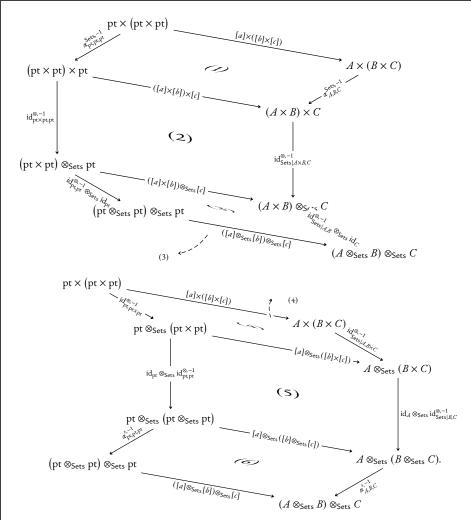


also commutes. Taking inverses, we see that the diagram



commutes as well. Now, let A, B,  $C \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , let

# $c \in C, \text{ and consider the diagram}$ $pt \times (pt \times pt)$ $pt \otimes pt$ $pt \otimes sets (pt \times pt)$ $(pt \times pt) \otimes sets pt$ $(pt \times pt) \otimes sets pt$ $(pt \times pt) \otimes sets pt$ $(pt \otimes sets pt) \otimes sets pt$ $(pt \otimes sets pt) \otimes sets pt$

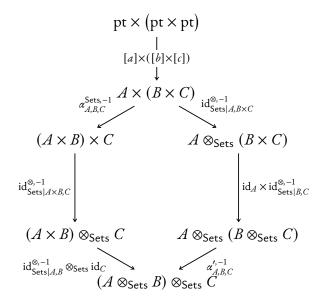


### Since:

- Subdiagram (1) commutes by the naturality of  $\alpha^{\text{Sets},-1}$ .
- Subdiagram (2) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (3) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (†) commutes, as proved above.

- Subdiagram (4) commutes by the naturality of  $id_{\mathsf{Sets}}^{\otimes,-1}$ .
- Subdiagram (5) commutes by the naturality of  $id_{Sets}^{\otimes,-1}$ .
- Subdiagram (6) commutes by the naturality of  $\alpha'^{-1}$ .

it follows that the diagram



also commutes. We then have

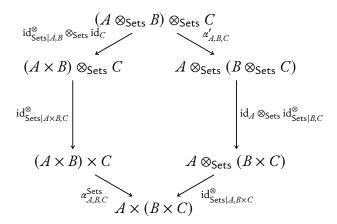
$$\begin{split} \left[ \left( \operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_{C} \right) \circ \operatorname{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \right] (a,(b,c)) &= \left[ \left( \operatorname{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \otimes_{\mathsf{Sets}} \operatorname{id}_{C} \right) \circ \operatorname{id}_{\mathsf{Sets}|A \times B,C}^{\otimes,-1} \\ \circ \alpha_{A,B,C}^{\mathsf{Sets},-1} \circ \left( [a] \times \left( [b] \times [c] \right) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \\ \circ \operatorname{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \circ \left( [a] \times \left( [b] \times [c] \right) \right) \right] (\star,(\star,\star)) \\ &= \left[ \alpha_{A,B,C}^{\prime,-1} \circ \left( \operatorname{id}_{A} \times \operatorname{id}_{\mathsf{Sets}|B,C}^{\otimes,-1} \right) \circ \operatorname{id}_{\mathsf{Sets}|A,B \times C}^{\otimes,-1} \right] (a,(b,c)) \end{split}$$

for each  $(a, (b, c)) \in A \times (B \times C)$ , and thus we have

$$\left(\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1}\otimes_{\mathsf{Sets}}\mathrm{id}_{C}\right)\circ\mathrm{id}_{\mathsf{Sets}|A\times B,C}^{\otimes,-1}\circ\alpha_{A,B,C}^{\mathsf{Sets},-1}=\alpha_{A,B,C}^{\prime,-1}\circ\left(\mathrm{id}_{A}\times\mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes,-1}\right)\circ\mathrm{id}_{\mathsf{Sets}|A,B\times C}^{\otimes,-1}.$$

Taking inverses then gives

$$\alpha_{A,B,C}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A \times B,C}^{\otimes} \circ \left( \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_{C} \right) = \mathrm{id}_{\mathsf{Sets}|A,B \times C}^{\otimes} \circ \left( \mathrm{id}_{A} \times \mathrm{id}_{\mathsf{Sets}|B,C}^{\otimes} \right) \circ \alpha_{A,B,C}',$$
 showing that the diagram



indeed commutes.

# Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

We have to show that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\sigma'_{A,B} \downarrow \qquad \qquad \qquad \downarrow \sigma_{A,B}^{\mathsf{Sets}}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

commutes. First, note that the diagram

commutes by the terminality of pt (Constructions With Sets, Construction 4.1.1.1.2). Since the map  $!_{pt \times pt}$ : pt  $\times$  pt is invertible (e.g. with inverse  $\lambda_{pt}^{Sets,-1}$ ), the diagram

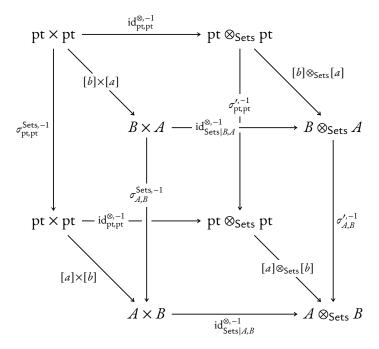
$$\begin{array}{c|c} pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{id^{\otimes}_{\mathsf{Sets}|pt,pt}} & pt \times pt \\ \\ \sigma'_{\mathsf{pt,pt}} & & & & & \\ \phi'_{\mathsf{pt,pt}} & & & & \\ pt \otimes_{\mathsf{Sets}} pt & \xrightarrow{id^{\otimes}_{\mathsf{Sets}|pt,pt}} & pt \times pt \end{array}$$

also commutes. Taking inverses, we see that the diagram

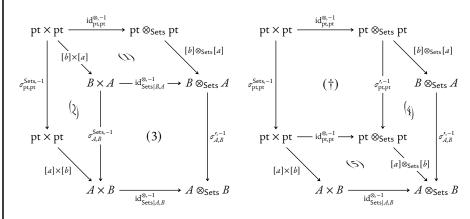
$$\begin{array}{c|c} pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}^{\otimes,-1}} pt \otimes_{\mathsf{Sets}} pt \\ \hline \sigma_{\mathsf{pt,pt}}^{\mathsf{Sets,-1}} & & (\dagger) & & \sigma_{\mathsf{pt,pt}}^{\prime,-1} \\ pt \times pt & \xrightarrow{id_{\mathsf{Sets}|pt,pt}} pt \otimes_{\mathsf{Sets}} pt \end{array}$$

commutes as well. Now, let  $A, B \in \text{Obj}(\mathsf{Sets})$ , let  $a \in A$ , let  $b \in B$ , and

#### consider the diagram



which we partition into subdiagrams as follows:

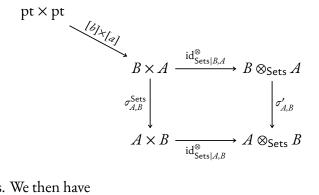


Since:

• Subdiagram (2) commutes by the naturality of  $\sigma^{\rm Sets,-1}.$ 

- Subdiagram (5) commutes by the naturality of  $id^{\otimes,-1}$ .
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of  $\sigma'^{-1}$ .
- Subdiagram (1) commutes by the naturality of  $id^{\otimes,-1}$ .

it follows that the diagram



commutes. We then have

$$\begin{split} \left[ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \right] (b,a) &= \left[ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} \circ ([b] \times [a]) \right] (\star, \star) \\ &= \left[ \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \circ ([b] \times [a]) \right] (\star, \star) \\ &= \left[ \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1} \right] (b,a) \end{split}$$

for each  $(b, a) \in B \times A$ , and thus we have

$$\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes,-1} \circ \sigma_{A,B}^{\mathsf{Sets},-1} = \sigma_{A,B}'^{,-1} \circ \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes,-1}.$$

Taking inverses then gives

$$\sigma_{A,B}^{\mathsf{Sets}} \circ \mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes} = \mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes} \circ \sigma_{A,B}',$$

showing that the diagram

$$A \otimes_{\mathsf{Sets}} B \xrightarrow{\mathrm{id}_{\mathsf{Sets}|A,B}^{\otimes}} A \times B$$

$$\downarrow^{\sigma'_{A,B}} \qquad \qquad \downarrow^{\sigma_{\mathsf{A},B}^{\mathsf{Sets}}}$$

$$B \otimes_{\mathsf{Sets}} A \xrightarrow{\mathrm{id}_{\mathsf{Sets}|B,A}^{\otimes}} B \times A$$

indeed commutes.

#### Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$

Let  $\phi$ ,  $\psi$ :  $-_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$  be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to  $\times$  and  $\otimes$  (as shown above), we have

$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \phi_{\mathsf{pt},B} \circ \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right),$$
$$\lambda_B' = \lambda_B^{\mathsf{Sets}} \circ \psi_{\mathsf{pt},B} \circ \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_Y \right).$$

Postcomposing both sides with  $\lambda_B^{\text{Sets},-1}$  gives

$$\lambda_{B}^{\mathsf{Sets},-1} \circ \lambda_{B}' \circ \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes,-1} \otimes_{\mathsf{Sets}} \mathrm{id}_{Y} \right) = \phi_{\mathsf{pt},B},$$

$$\lambda_{B}^{\mathsf{Sets},-1} \circ \lambda_{B}' \circ \left( \mathrm{id}_{1|\mathsf{Sets}}^{\otimes} \otimes_{\mathsf{Sets}} \mathrm{id}_{Y} \right) = \psi_{\mathsf{pt},B},$$

and thus we have

$$\phi_{\text{pt},B} = \psi_{\text{pt},B}$$

for each  $B \in \text{Obj}(\mathsf{Sets})$ . Now, let  $a \in A$  and consider the naturality diagrams

for  $\phi$  and  $\psi$  with respect to the morphisms [a] and id<sub>B</sub>. Having shown that  $\phi_{\text{pt},B} = \psi_{\text{pt},B}$ , we have

$$\phi_{A,B}(a,b) = [\phi_{A,B} \circ ([a] \times id_B)](\star,b)$$

$$= [([a] \otimes_{\mathsf{Sets}} id_B) \circ \phi_{\mathsf{pt},B}](\star,b)$$

$$= [([a] \otimes_{\mathsf{Sets}} id_B) \circ \psi_{\mathsf{pt},B}](\star,b)$$

$$= [\psi_{A,B} \circ ([a] \times id_B)](\star,b)$$

$$= \psi_{A,B}(a,b)$$

for each  $(a, b) \in A \times B$ . Therefore we have

01PJ

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$$\phi_{A,B} = \psi_{A,B}$$

for each  $A, B \in \text{Obj}(\mathsf{Sets})$  and thus  $\phi = \psi$ , showing the isomorphism  $\otimes_{\mathsf{Sets}} \cong \times$  to be unique.

#### **O1PH** COROLLARY 5.1.10.1.3 ► A SECOND UNIVERSAL PROPERTY FOR (Sets, ×, pt)

The symmetric monoidal structure on the category Sets of Proposition 5.1.9.1.1 is uniquely determined by the following requirements:

1. Two-Sided Preservation of Colimits. The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

2. The Unit Object Is pt. We have  $\mathbb{1}_{Sets} \cong pt$ .

More precisely, the full subcategory of the category  $\mathcal{M}_{\mathbb{E}_{\infty}}(\mathsf{Sets})$  of  $\ref{eq:Sets}$  spanned by the symmetric monoidal categories (Sets,  $\otimes_{\mathsf{Sets}}$ ,  $1_{\mathsf{Sets}}$ ,  $\rho^{\mathsf{Sets}}$ ,  $\rho^{\mathsf{Sets}}$ ,  $\sigma^{\mathsf{Sets}}$ ) satisfying Items 1 and 2 is contractible.

#### PROOF 5.1.10.1.4 ► PROOF OF COROLLARY 5.1.10.1.3

Since Sets is locally presentable (??), it follows from ?? that Item I is equivalent to the existence of an internal Hom as in Item I of Theorem 5.I.IO.I.I.

The result then follows from Theorem 5.I.IO.I.I.

# other 5.2 The Monoidal Category of Sets and Coproducts

# 01PM 5.2.1 Coproducts of Sets

See Constructions With Sets, Section 4.2.3.

#### 01PN 5.2.2 The Monoidal Unit

#### 01PP DEFINITION 5.2.2.1.1 ► THE MONOIDAL UNIT OF [

The monoidal unit of the coproduct of sets is the functor

$$0^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

defined by

$$0_{\mathsf{Sets}} \stackrel{\mathrm{def}}{=} \emptyset$$

where  $\emptyset$  is the empty set of Constructions With Sets, Definition 4.3.1.1.1.

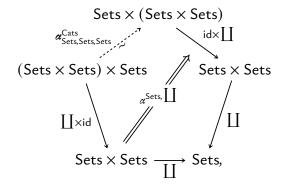
#### 01PQ 5.2.3 The Associator

#### **01PR DEFINITION 5.2.3.1.1** ► THE ASSOCIATOR OF ∐

The associator of the coproduct of sets is the natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\scriptstyle \sim}{\Longrightarrow} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \pmb{\alpha}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}'}^{\mathsf{Cats}}$$

as in the diagram



whose component

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \colon (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z)$$

at (X, Y, Z) is given by

$$\alpha_{X,Y,Z}^{\text{Sets},\coprod}(a) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } a = (0,(0,x)), \\ (1,(0,y)) & \text{if } a = (0,(1,y)), \\ (1,(1,a)) & \text{if } a = (1,z) \end{cases}$$

for each  $a \in (X \coprod Y) \coprod Z$ .

#### PROOF 5.2.3.1.2 ▶ Proof of the Claims Made in Definition 5.2.3.1.1

#### Unwinding the Definitions of $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$

Firstly, we unwind the expressions for  $(X \coprod Y) \coprod Z$  and  $X \coprod (Y \coprod Z)$ . We have

$$(X \coprod Y) \coprod Z \stackrel{\text{def}}{=} \{ (0, a) \in S \mid a \in X \coprod Y \} \cup \{ (1, z) \in S \mid z \in Z \}$$
$$= \{ (0, (0, x)) \in S \mid x \in X \} \cup \{ (0, (1, y)) \in S \mid y \in Y \}$$
$$\cup \{ (1, z) \in S \mid z \in Z \},$$

where 
$$S = \{0, 1\} \times ((X \coprod Y) \cup Z)$$
 and

$$X \coprod (Y \coprod Z) \stackrel{\text{def}}{=} \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, a) \in S' \mid a \in Y \coprod Z \}$$
$$= \{ (0, x) \in S' \mid x \in X \} \cup \{ (1, (0, y)) \in S' \mid y \in Y \}$$
$$\cup \{ (1, (1, z)) \in S' \mid z \in Z \},$$

where  $S' = \{0, 1\} \times (X \cup (Y \coprod Z))$ .

#### Invertibility

The inverse of  $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$  is the map

$$\alpha_{X,Y,Z}^{\mathsf{Sets}, \coprod, -1} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}(a) \stackrel{\text{def}}{=} \begin{cases} (0,(0,x)) & \text{if } a = (0,x), \\ (0,(1,y)) & \text{if } a = (1,(0,y)), \\ (1,z) & \text{if } a = (1,(1,z)) \end{cases}$$

for each  $a \in X \coprod Y(\coprod Z)$ . Indeed:

• *Invertibility I.* The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod}$  acts on elements as

$$(0, (0, x)) \mapsto (0, x) \mapsto (0, (0, x)),$$
  

$$(0, (0, y)) \mapsto (1, (0, y)) \mapsto (0, (0, y)),$$
  

$$(1, z) \mapsto (1, (1, z)) \mapsto (1, z)$$

and hence is equal to the identity map of  $(X \coprod Y) \coprod Z$ .

• *Invertibility II.* The map  $\alpha_{X,Y,Z}^{\mathsf{Sets},\coprod} \circ \alpha_{X,Y,Z}^{\mathsf{Sets},\coprod,-1}$  acts on elements as

$$(0,x) \mapsto (0,(0,x)) \mapsto (0,x), (1,(0,y)) \mapsto (0,(0,y)) \mapsto (1,(0,y)), (1,(1,z)) \mapsto (1,z) \mapsto (1,(1,z))$$

and hence is equal to the identity map of  $X \coprod (Y \coprod Z)$ .

5.2.3 The Associator

Therefore  $\alpha_{X,Y,Z}^{\text{Sets},\coprod}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions

$$f: X \to X',$$
  
 $g: Y \to Y',$   
 $h: Z \to Z'$ 

the diagram

$$(X \coprod Y) \coprod Z \xrightarrow{\left(f \coprod g\right) \coprod b} (X' \coprod Y') \coprod Z'$$

$$\downarrow^{\text{Sets,} \coprod}_{\alpha_{X,Y,Z}} \downarrow$$

$$X \coprod (Y \coprod Z) \xrightarrow{f \coprod \left(g \coprod b\right)} X' \coprod (Y' \coprod Z')$$

commutes. Indeed, this diagram acts on elements as

$$(0, (0, x)) \qquad (0, (0, x)) \longmapsto (0, (0, f(x)))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad$$

and hence indeed commutes, showing  $\alpha^{\text{Sets},\coprod}$  to be a natural transformation.

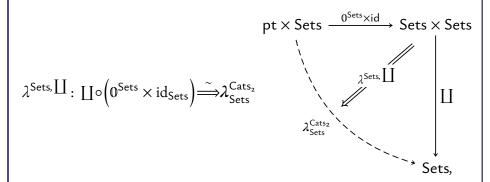
## Being a Natural Isomorphism

Since  $\alpha^{\text{Sets}, \coprod}$  is natural and  $\alpha^{\text{Sets}, \coprod, -1}$  is a componentwise inverse to  $\alpha^{\text{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\alpha^{\text{Sets}, \coprod}$  is a natural isomorphism.

#### 01PS 5.2.4 The Left Unitor

#### 01PT DEFINITION 5.2.4.1.1 ► THE LEFT UNITOR OF \[ \]

The **left unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\lambda_X^{\mathsf{Sets},\coprod} : \varnothing \coprod X \xrightarrow{\sim} X$$

at X is given by

$$\lambda_X^{\text{Sets},\coprod}((1,x))\stackrel{\text{def}}{=} x$$

for each  $(1, x) \in \emptyset \mid \mid X$ .

#### PROOF 5.2.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.4.1.1

#### Unwinding the Definition of $\emptyset \coprod X$

Firstly, we unwind the expressions for  $\emptyset \coprod X$ . We have

where  $S = \{0, 1\} \times (\emptyset \cup X)$ .

# Invertibility

The inverse of  $\lambda_X^{\mathsf{Sets},\coprod}$  is the map

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \colon X \to \emptyset \coprod X$$

given by

$$\lambda_X^{\mathsf{Sets}, \coprod, -1}(x) \stackrel{\mathrm{def}}{=} (1, x)$$

for each  $x \in X$ . Indeed:

• *Invertibility I.* We have

$$\begin{split} \left[\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod}\right] (1, x) &= \lambda_X^{\mathsf{Sets}, \coprod, -1} \left(\lambda_X^{\mathsf{Sets}, \coprod} (1, x)\right) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (1, x) \\ &= \left[\mathrm{id}_{\varnothing \coprod X}\right] (1, x) \end{split}$$

for each  $(1, x) \in \emptyset \coprod X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets}, \coprod, -1} \circ \lambda_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\varnothing \coprod X}.$$

• Invertibility II. We have

$$\begin{bmatrix} \lambda_X^{\mathsf{Sets}, \coprod} \circ \lambda_X^{\mathsf{Sets}, \coprod, -1} \end{bmatrix} (x) = \lambda_X^{\mathsf{Sets}, \coprod} \left( \lambda_X^{\mathsf{Sets}, \coprod, -1} (x) \right)$$

$$= \lambda_X^{\mathsf{Sets}, \coprod, -1} (1, x)$$

$$= x$$

$$= [\mathrm{id}_X](x)$$

for each  $x \in X$ , and therefore we have

$$\lambda_X^{\mathsf{Sets},\coprod} \circ \lambda_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore  $\lambda_X^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

**Naturality** 

We need to show that, given a function  $f: X \to Y$ , the diagram

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(1,x) & (1,x) & \longmapsto (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\lambda^{\text{Sets},\coprod}$  is a natural transformation.

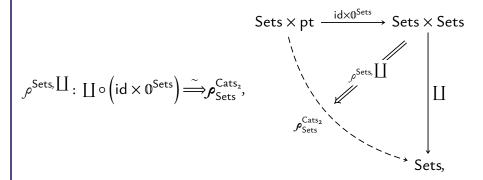
# Being a Natural Isomorphism

Since  $\lambda^{\text{Sets}, \coprod}$  is natural and  $\lambda^{\text{Sets}, -1}$  is a componentwise inverse to  $\lambda^{\text{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\lambda^{\text{Sets}, -1}$  is also natural. Thus  $\lambda^{\text{Sets}, \coprod}$  is a natural isomorphism.

# 01PU 5.2.5 The Right Unitor

01PV DEFINITION 5.2.5.1.1 ► THE RIGHT UNITOR OF [

The **right unitor of the coproduct of sets** is the natural isomorphism



whose component

$$\rho_X^{\mathsf{Sets},\coprod}: X \coprod \varnothing \xrightarrow{\sim} X$$

at X is given by

$$\rho_X^{\mathsf{Sets},\coprod}((0,x))\stackrel{\mathrm{def}}{=} x$$

for each  $(0, x) \in X \coprod \emptyset$ .

#### PROOF 5.2.5.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.5.1.1

#### Unwinding the Definition of $X \coprod \emptyset$

Firstly, we unwind the expression for  $X \coprod \emptyset$ . We have

$$X \coprod \emptyset \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, z) \in S \mid z \in \emptyset\}$$
$$= \{(0, x) \in S \mid x \in X\} \cup \emptyset$$
$$= \{(0, x) \in S \mid x \in X\},$$

where  $S = \{0, 1\} \times (X \cup \emptyset) = \{0, 1\} \times (\emptyset \cup X) = S$ .

# Invertibility

The inverse of  $\rho_X^{\mathrm{Sets},\coprod}$  is the map

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \colon X \to X \coprod \varnothing$$

given by

$$\rho_X^{\mathsf{Sets}, \coprod, -1}(x) \stackrel{\mathrm{def}}{=} (0, x)$$

for each  $x \in X$ . Indeed:

• *Invertibility I.* We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} \right] (0, x) &= \rho_X^{\mathsf{Sets}, \coprod, -1} \left( \rho_X^{\mathsf{Sets}, \coprod} (0, x) \right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \\ &= (0, x) \\ &= \left[ \mathrm{id}_{X \coprod \varnothing} \right] (0, x) \end{split}$$

for each  $(0, x) \in \emptyset \coprod X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod, -1} \circ \rho_X^{\mathsf{Sets}, \coprod} = \mathrm{id}_{\emptyset \coprod X}.$$

• Invertibility II. We have

$$\begin{split} \left[ \rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1} \right] (x) &= \rho_X^{\mathsf{Sets}, \coprod} \left( \rho_X^{\mathsf{Sets}, \coprod, -1} (x) \right) \\ &= \rho_X^{\mathsf{Sets}, \coprod, -1} (0, x) \\ &= x \\ &= \left[ \mathrm{id}_X \right] (x) \end{split}$$

for each  $x \in X$ , and therefore we have

$$\rho_X^{\mathsf{Sets}, \coprod} \circ \rho_X^{\mathsf{Sets}, \coprod, -1} = \mathrm{id}_X.$$

Therefore  $\rho_X^{\text{Sets},\coprod}$  is indeed an isomorphism.

**Naturality** 

We need to show that, given a function  $f: X \to Y$ , the diagram

$$\begin{array}{ccc} X \coprod \varnothing & f \coprod \mathrm{id}_{\varnothing} & Y \coprod \varnothing \\ & & \downarrow \rho_X^{\mathsf{Sets}, \coprod} & & \downarrow \rho_Y^{\mathsf{Sets}, \coprod} \\ & & X & \longrightarrow & Y \end{array}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{array}{ccc}
(0,x) & (0,x) & \longmapsto (1,f(x)) \\
\downarrow & & \downarrow \\
x & \longmapsto f(x) & f(x)
\end{array}$$

and hence indeed commutes. Therefore  $\rho^{\text{Sets},\coprod}$  is a natural transformation.

# Being a Natural Isomorphism

Since  $\rho^{\text{Sets}, \coprod}$  is natural and  $\rho^{\text{Sets}, -1}$  is a componentwise inverse to  $\rho^{\text{Sets}, \coprod}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\rho^{\text{Sets}, -1}$  is also natural. Thus  $\rho^{\text{Sets}, \coprod}$  is a natural isomorphism.

## 01PW 5.2.6 The Symmetry

#### 01PX DEFINITION 5.2.6.1.1 ► THE SYMMETRY OF [

The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}, \coprod} : \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}}, \qquad \begin{array}{c} \mathsf{Sets} \times \mathsf{Sets} & \overset{\coprod}{\longleftrightarrow} \mathsf{Sets}, \\ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}, \mathsf{Sets}} & & \downarrow & \downarrow \\ \mathsf{Sets} \times \mathsf{Sets} & & \mathsf{Sets} & \\ \mathsf{Sets} \times \mathsf{Sets} & \\ \mathsf{Sets} \times \mathsf{Sets} & \\ \mathsf{Sets} \times \mathsf{Sets} & & \\$$

whose component

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod}: X \coprod Y \xrightarrow{\sim} Y \coprod X$$

at  $X, Y \in \text{Obj}(\mathsf{Sets})$  is defined by

$$\sigma_{X,Y}^{\text{Sets,}\coprod}(x,y)\stackrel{\text{def}}{=}(y,x)$$

for each  $(x, y) \in X \times Y$ .

#### PROOF 5.2.6.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.2.6.1.1

#### Unwinding the Definitions of $X \coprod Y$ and $Y \coprod X$

Firstly, we unwind the expressions for  $X \coprod Y$  and  $Y \coprod X$ . We have

$$X \coprod Y \stackrel{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},\$$

where  $S = \{0, 1\} \times (X \cup Y)$  and

$$Y \coprod X \stackrel{\text{def}}{=} \left\{ \left(0, y\right) \in S' \mid y \in Y \right\} \cup \left\{ (1, x) \in S' \mid x \in X \right\},$$

where 
$$S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$$
.

#### Invertibility

The inverse of  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is the map

$$\sigma^{\mathsf{Sets}, \coprod, -1}_{X,Y} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets},\coprod,-1} \stackrel{\scriptscriptstyle\mathsf{def}}{=} \sigma_{Y,X}^{\mathsf{Sets},\coprod}$$

and hence given by

$$\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}(z) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = (1, x), \\ (1, y) & \text{if } z = (0, y) \end{cases}$$

for each  $z \in Y \coprod X$ . Indeed:

• *Invertibility I.* We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod}\right](0, x) &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} \left(\sigma_{X}^{\mathsf{Sets}, \coprod}(0, x)\right) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1}(1, x) \\ &= (0, x) \\ &= \left[\mathrm{id}_{X \coprod Y}\right](0, x) \end{split}$$

for each  $(0, x) \in X \coprod Y$  and

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets,}\coprod,-1} \circ \sigma_{X,Y}^{\mathsf{Sets,}\coprod}\right] (1,y) &= \sigma_{X}^{\mathsf{Sets,}\coprod,-1} \left(\sigma_{X}^{\mathsf{Sets,}\coprod} (1,y)\right) \\ &= \sigma_{X}^{\mathsf{Sets,}\coprod,-1} (0,y) \\ &= (1,y) \\ &= \left[\mathrm{id}_{X\coprod Y}\right] (1,y) \end{split}$$

for each  $(1, y) \in X \coprod Y$ , and therefore we have

$$\sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod} = \mathrm{id}_{X \coprod Y}.$$

• Invertibility II. We have

$$\begin{split} \left[\sigma_{X,Y}^{\mathsf{Sets}, \coprod} \circ \sigma_{X,Y}^{\mathsf{Sets}, \coprod, -1}\right] (0, y) &= \sigma_{X}^{\mathsf{Sets}, \coprod} \left(\sigma_{X}^{\mathsf{Sets}, \coprod, -1}(0, y)\right) \\ &= \sigma_{X}^{\mathsf{Sets}, \coprod, -1} (1, y) \\ &= (0, y) \\ &= \left[\mathrm{id}_{Y \coprod X}\right] (0, y) \end{split}$$

for each  $(0, y) \in Y \coprod X$  and

$$\left[\sigma_{X,Y}^{\mathsf{Sets},\coprod} \circ \sigma_{X,Y}^{\mathsf{Sets},\coprod,-1}\right](1,x) = \sigma_{X}^{\mathsf{Sets},\coprod} \left(\sigma_{X}^{\mathsf{Sets},\coprod,-1}(1,x)\right)$$

$$= \sigma_X^{\mathsf{Sets}, \coprod, -1}(0, x)$$
$$= (1, x)$$
$$= \left[ \mathsf{id}_{Y \coprod X} \right] (1, x)$$

for each  $(1, x) \in Y \coprod X$ , and therefore we have

$$\sigma_X^{\mathsf{Sets},\coprod} \circ \sigma_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_{Y\coprod X}.$$

Therefore  $\sigma_{X,Y}^{\mathsf{Sets},\coprod}$  is indeed an isomorphism.

#### Naturality

We need to show that, given functions  $f:A\to X$  and  $g:B\to Y$ , the diagram

$$A \coprod B \xrightarrow{f \coprod g} X \coprod Y$$

$$\downarrow_{\sigma_{A,B}^{\mathsf{Sets}}, \coprod} \qquad \qquad \downarrow_{\sigma_{X,Y}^{\mathsf{Sets}}, \coprod}$$

$$B \coprod A \xrightarrow{g \coprod f} Y \coprod X$$

$$(0,a) \qquad (0,a) \longmapsto (0,f(a))$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

and hence indeed commutes. Therefore  $\sigma^{\text{Sets},\coprod}$  is a natural transformation.

#### Being a Natural Isomorphism

Since  $\sigma^{Sets, \coprod}$  is natural and  $\sigma^{Sets, -1}$  is a componentwise inverse to  $\sigma^{Sets, \coprod}$ , it follows from Categories, Item 2 of Proposition II.9.7.I.2 that  $\sigma^{Sets, -1}$  is also natural. Thus  $\sigma^{Sets, \coprod}$  is a natural isomorphism.

## 01PY 5.2.7 The Monoidal Category of Sets and Coproducts

#### 01PZ PROPOSITION 5.2.7.1.1 ► THE MONOIDAL STRUCTURE ON SETS ASSOCIATED TO

The category Sets admits a closed symmetric monoidal category structure consisting of:

- The Underlying Category. The category Sets of pointed sets.
- The Monoidal Product. The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

• The Monoidal Unit. The functor

$$\mathbb{O}^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.2.2.1.1.

• The Associators. The natural isomorphism

$$\alpha^{\text{Sets}, \coprod} : \coprod \circ (\coprod \times \text{id}_{\text{Sets}}) \xrightarrow{\sim} \coprod \circ (\text{id}_{\text{Sets}} \times \coprod) \circ \alpha^{\text{Cats}}_{\text{Sets}, \text{Sets}, \text{Sets}}$$
of Definition 5.2.3.1.1.

• The Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Right Unitors. The natural isomorphism

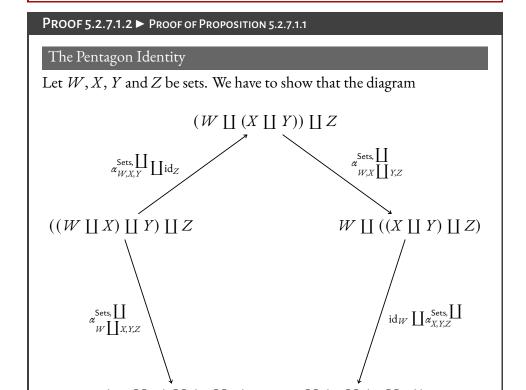
$$\rho^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathsf{id} \times \mathbb{0}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

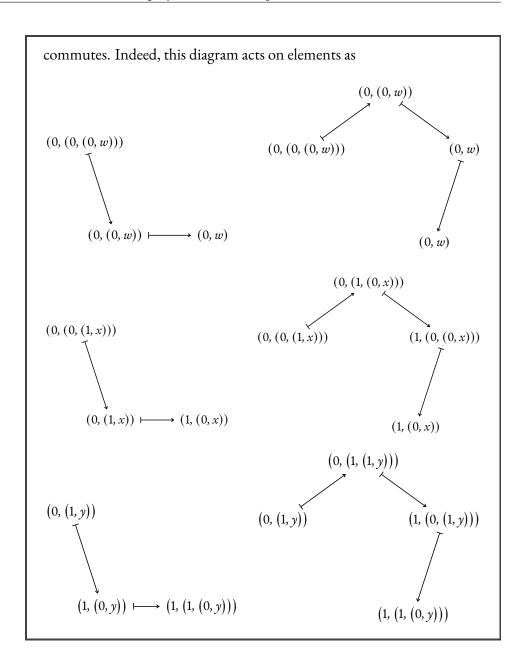
of Definition 5.2.5.1.1.

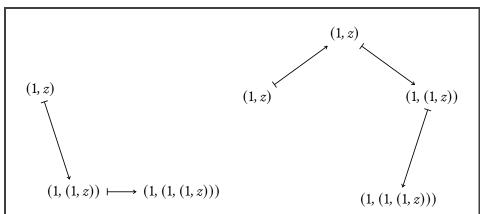
• The Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod}: imes \stackrel{\sim}{\Longrightarrow} imes \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.



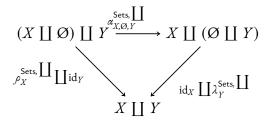


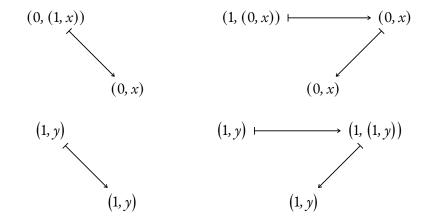


and therefore the pentagon identity is satisfied.

#### The Triangle Identity

Let X and Y be sets. We have to show that the diagram

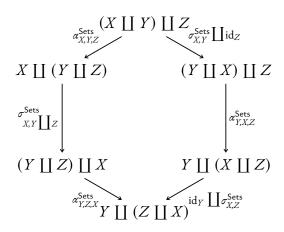


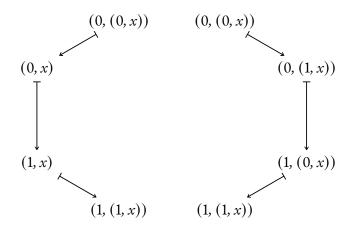


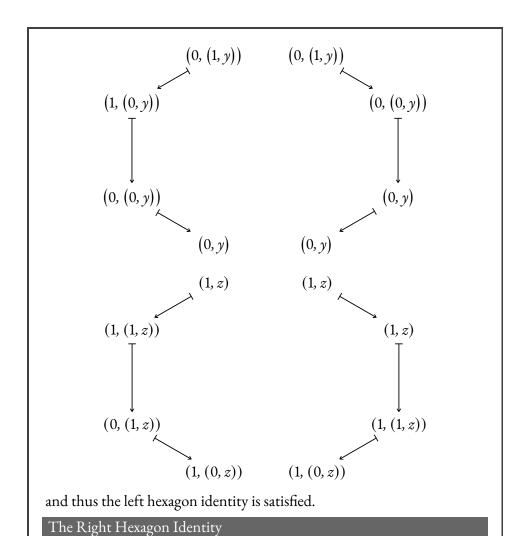
and therefore the triangle identity is satisfied.

#### The Left Hexagon Identity

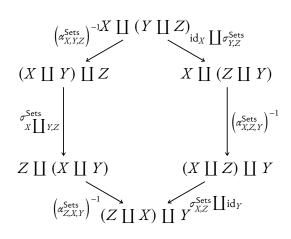
Let X, Y, and Z be sets. We have to show that the diagram

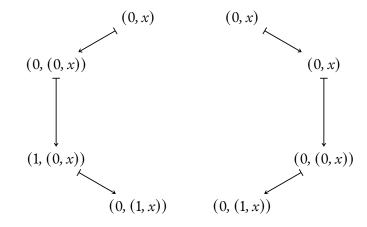


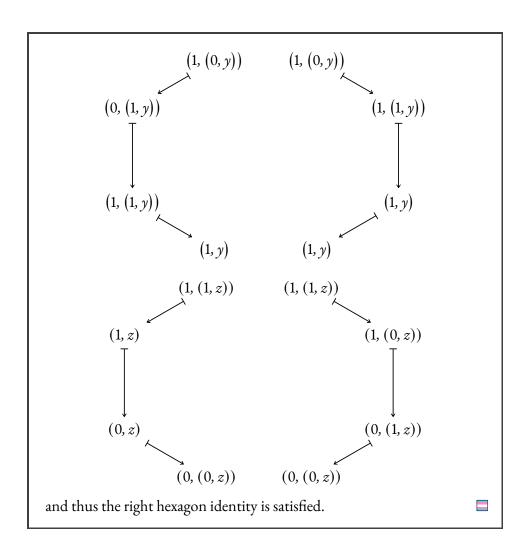




Let X, Y, and Z be sets. We have to show that the diagram







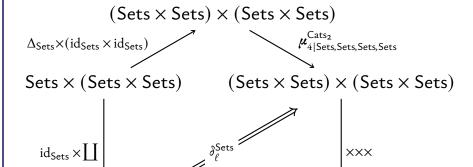
# o100 5.3 The Bimonoidal Category of Sets, Products, and Coproducts

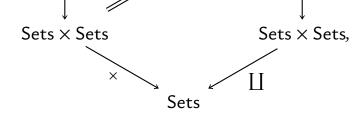
01Q1 5.3.1 The Left Distributor

#### 01Q2 DEFINITION 5.3.1.1.1 ► THE LEFT DISTRIBUTOR OF × OVER ∐

The left distributor of the product of sets over the coproduct of sets is the natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} : \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ (\Delta_{\mathsf{Sets}} \times (\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}))$$
 as in the diagram





whose component

$$\delta^{\mathsf{Sets}}_{\ell|X,Y,Z} \colon X \times (Y \coprod Z) \stackrel{\sim}{\dashrightarrow} (X \times Y) \coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets}}(x,a) \stackrel{\text{def}}{=} \begin{cases} (0,(x,y)) & \text{if } a = (0,y), \\ (1,(x,z)) & \text{if } a = (1,z) \end{cases}$$

for each  $(x, a) \in X \times (Y \coprod Z)$ .

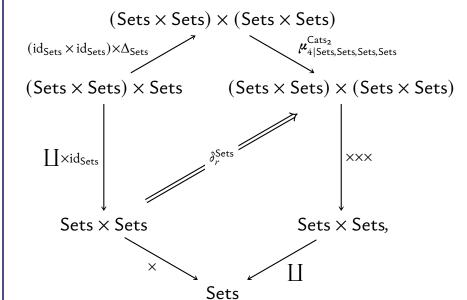


# 01Q3 5.3.2 The Right Distributor

#### 01Q4 DEFINITION 5.3.2.1.1 ► THE RIGHT DISTRIBUTOR OF × OVER \[ \]

The right distributor of the product of sets over the coproduct of sets is the natural isomorphism

 $\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$  as in the diagram



whose component

$$\delta^{\mathsf{Sets}}_{r|X,Y\!,Z} \colon (X \coprod Y) \times Z \xrightarrow{\sim} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{r|X,Y,Z}^{\mathsf{Sets}}(a,z) \stackrel{\text{def}}{=} \begin{cases} (0,(x,z)) & \text{if } a = (0,x), \\ (1,(y,z)) & \text{if } a = (1,y) \end{cases}$$

for each  $(a, z) \in (X \coprod Y) \times Z$ .

#### PROOF 5.3.2.1.2 ▶ Proof of the Claims Made in Definition 5.3.2.1.1

Omitted.

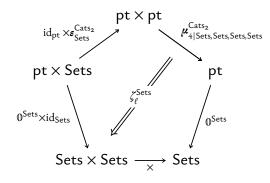
#### 01Q5 5.3.3 The Left Annihilator

#### 01Q6 DEFINITION 5.3.3.1.1 ► THE LEFT ANNIHILATOR OF ×

The left annihilator of the product of sets is the natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\varepsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$

as in the diagram



with components

$$\zeta_{\ell|A}^{\mathsf{Sets}} \colon \mathscr{O} \times A \xrightarrow{\sim} \mathscr{O}.$$

#### PROOF 5.3.3.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.3.1.1

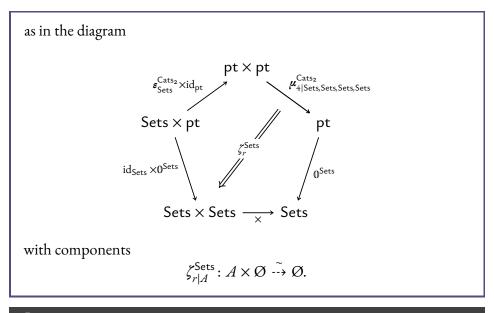
Omitted. For a partial proof, see [Pro25].

# 01Q7 5.3.4 The Right Annihilator

#### 01Q8 DEFINITION 5.3.4.1.1 ► THE RIGHT ANNIHILATOR OF ×

The right annihilator of the product of sets is the natural isomorphism

$$\zeta_r^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \pmb{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left( \pmb{\varepsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}} \right) \xrightarrow{\sim} \times \circ \left( \mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}} \right)$$



#### PROOF 5.3.4.1.2 ► PROOF OF THE CLAIMS MADE IN DEFINITION 5.3.4.1.1

Omitted. For a partial proof, see [Pro25].

01QA

# 0109 5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

PROPOSITION 5.3.5.1.1 ► THE BIMONOIDAL STRUCTURE ON SETS ASSOCIATED TO × AND

The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Additive Monoidal Product.* The coproduct functor

$$II: Sets \times Sets \rightarrow Sets$$

of Constructions With Sets, Item 1 of Proposition 4.2.3.1.4.

• The Multiplicative Monoidal Product. The product functor

$$\times$$
: Sets  $\times$  Sets  $\rightarrow$  Sets

#### of Constructions With Sets, Item 1 of Proposition 4.1.3.1.4.

• The Monoidal Unit. The functor

$$1^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Monoidal Zero. The functor

$$0^{\mathsf{Sets}} \colon \mathsf{pt} \to \mathsf{Sets}$$

of Definition 5.1.3.1.1.

• The Internal Hom. The internal Hom functor

Sets: 
$$Sets^{op} \times Sets \rightarrow Sets$$

of Constructions With Sets, ?? of ??.

• The Additive Associators. The natural isomorphism

$$\alpha^{\mathsf{Sets}, \coprod} : \coprod \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \coprod \circ (\mathsf{id}_{\mathsf{Sets}} \times \coprod) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets}, \mathsf{Sets}, \mathsf{Sets}}$$
of Definition 5.2.3.1.1.

• The Additive Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathbb{O}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

• The Additive Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}, \coprod} : \coprod \circ \left( \mathsf{id} \times 0^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.2.5.1.1.

• The Additive Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} : \coprod \stackrel{\widetilde{}}{\Longrightarrow} \coprod \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

- The Multiplicative Associators. The natural isomorphism  $\alpha^{\mathsf{Sets}} : \times \circ (\times \times \mathsf{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \times \circ (\mathsf{id}_{\mathsf{Sets}} \times \times) \circ \alpha^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$ of Definition 5.1.4.1.1.
- The Multiplicative Left Unitors. The natural isomorphism

$$\lambda^{\mathsf{Sets}} : \times \circ \left( \mathbb{1}^{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

• The Multiplicative Right Unitors. The natural isomorphism

$$\rho^{\mathsf{Sets}} : \times \circ \left( \mathsf{id} \times \mathbb{1}^{\mathsf{Sets}} \right) \stackrel{\sim}{\Longrightarrow} \rho_{\mathsf{Sets}}^{\mathsf{Cats}_2}$$

of Definition 5.1.6.1.1.

• The Multiplicative Symmetry. The natural isomorphism

$$\sigma^{\mathsf{Sets}} : \times \stackrel{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

• The Left Distributor. The natural isomorphism

$$\delta_{\ell}^{Sets} : \times \circ (id_{Sets} \times \coprod) \stackrel{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{4|Sets, Sets, Sets, Sets}^{Cats_2} \circ (\Delta_{Sets} \times (id_{Sets} \times id_{Sets}))$$
of Definition 5.3.1.1.1.

• The Right Distributor. The natural isomorphism

$$\delta_r^{\mathsf{Sets}} : \times \circ (\coprod \times \mathsf{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \mu_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ ((\mathsf{id}_{\mathsf{Sets}} \times \mathsf{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$
of Definition 5.3.2.1.1.

• The Left Annihilator. The natural isomorphism

$$\zeta_{\ell}^{\mathsf{Sets}} \colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{\mathsf{4}|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left(\mathrm{id}_{\mathsf{pt}} \times \boldsymbol{\varepsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2}\right) \stackrel{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$
of Definition 5.3.3.1.1.

• The Right Annihilator. The natural isomorphism

$$\mathcal{Z}_r^{\mathsf{Sets}} : 0^{\mathsf{Sets}} \circ \boldsymbol{\mu}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}^{\mathsf{Cats}_2} \circ \left( \boldsymbol{\varepsilon}_{\mathsf{Sets}}^{\mathsf{Cats}_2} \times \mathrm{id}_{\mathsf{pt}} \right) \xrightarrow{\sim} \times \circ \left( \mathrm{id}_{\mathsf{Sets}} \times 0^{\mathsf{Sets}} \right)$$
of Definition 5.3.4.1.1.

#### PROOF 5.3.5.1.2 ► PROOF OF PROPOSITION 5.3.5.1.1

Omitted.

# **Appendices**

# A Other Chapters

#### **Preliminaries**

- I. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- 5. Monoidal Structures on the Category of Sets
- 6. Pointed Sets

7. Tensor Products of Pointed Sets

#### Relations

- 8. Relations
- 9. Constructions With Relations
- 10. Conditions on Relations

#### Categories

- II. Categories
- 12. Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

References 73

13. Constructions With Monoidal gories
Categories

#### **Bicategories**

#### Extra Part

14. Types of Morphisms in Bicate- 15. Notes

# References

[Pro25] Proof Wiki Contributors. *Cartesian Product Is Empty Iff Factor Is Empty — Proof Wiki*. 2025. URL: https://proofwiki.org/wiki/Cartesian\_Product\_is\_Empty\_iff\_Factor\_is\_Empty (cit. on pp. 68, 69).