# Pointed Sets

## The Clowder Project Authors

## July 29, 2025

70098 This chapter contains some foundational material on pointed sets.

## **Contents**

6.1	Point	ed Sets	2
	6.1.1	Foundations	2
		Morphisms of Pointed Sets	
		The Category of Pointed Sets	
		Elementary Properties of Pointed Sets	
	6.1.5	Active and Inert Morphisms of Pointed Sets	7
6.2	Limits of Pointed Sets		10
	6.2.1	The Terminal Pointed Set	10
	6.2.2	Products of Families of Pointed Sets	11
	6.2.3	Products	12
	6.2.4	Pullbacks	15
	6.2.5	Equalisers	19
6.3	Colin	nits of Pointed Sets	22
	6.3.1	The Initial Pointed Set.	22
	6.3.2	Coproducts of Families of Pointed Sets	22
	6.3.3	Coproducts	24
	6.3.4	Pushouts	28
	6.3.5	Coequalisers	33
6.4	Const	ructions With Pointed Sets	35
	6.4.1	Free Pointed Sets	35
	6.4.2	Deleting Basepoints	42

## 0099 6.1 Pointed Sets

#### 009A 6.1.1 Foundations

- **Definition 6.1.1.1.1.** A **pointed set**<sup>1</sup> is equivalently:
  - An  $\mathbb{E}_0$ -monoid in  $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$ .
  - A pointed object in (Sets, pt).
- **Remark 6.1.1.1.2.** In detail, a **pointed set** is a pair  $(X, x_0)$  consisting of:
  - The Underlying Set. A set X, called the **underlying set of**  $(X, x_0)$ .
  - The Basepoint. A morphism

$$[x_0]: pt \rightarrow X$$

in Sets, determining an element  $x_0 \in X$ , called the **basepoint of** X.

- **Example 6.1.1.1.3.** The 0-sphere<sup>2</sup> is the pointed set  $(S^0, 0)^3$  consisting of:
  - The Underlying Set. The set  $S^0$  defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint*. The element 0 of  $S^0$ .
- **Example 6.1.1.1.4.** The **trivial pointed set** is the pointed set  $(pt, \star)$  consisting of:
  - *The Underlying Set.* The punctual set  $pt \stackrel{\text{def}}{=} \{ \star \}$ .
  - *The Basepoint*. The element ★ of pt.
- **Example 6.1.1.1.5.** The **standard pointed set with** n + 1 **elements** is the pointed set  $\langle n \rangle$  consisting of
  - *The Underlying Set.* The set  $\langle n \rangle$  defined by

$$\langle n \rangle \stackrel{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

• *The Basepoint*. The element \* of  $\langle n \rangle$ .

<sup>&</sup>lt;sup>1</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, pointed sets are viewed as  $\mathbb{F}_1$ -modules.

<sup>&</sup>lt;sup>2</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element.** 

<sup>&</sup>lt;sup>3</sup>Further Notation: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras,  $S^0$  is also

## **009H 6.1.2** Morphisms of Pointed Sets

Definition 6.1.2.1.1. A morphism of pointed sets<sup>4,5</sup> is equivalently:

- A morphism of  $\mathbb{E}_0$ -monoids in  $(N_{\bullet}(\mathsf{Sets}), \mathsf{pt})$ .
- A morphism of pointed objects in (Sets, pt).

**Remark 6.1.2.1.2.** In detail, a **morphism of pointed sets**  $f: (X, x_0) \to (Y, y_0)$  is a morphism of sets  $f: X \to Y$  such that the diagram

$$\begin{array}{c|c}
pt \\
[x_0] & & \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

## **009L** 6.1.3 The Category of Pointed Sets

- **Definition 6.1.3.1.1.** The **category of pointed sets** is the category Sets<sub>\*</sub> defined equivalently as:
  - The homotopy category of the  $\infty$ -category  $\mathsf{Mon}_{\mathbb{E}_0}(\mathsf{N}_{\bullet}(\mathsf{Sets}),\mathsf{pt})$  of  $\ref{eq:sets}$ ??.
  - The category Sets\* of Constructions With Categories, ??.
- **Remark 6.1.3.1.2.** In detail, the **category of pointed sets** is the category Sets\* where:
  - *Objects*. The objects of Sets\* are pointed sets.
  - Morphisms. The morphisms of Sets\* are morphisms of pointed sets.
  - *Identities.* For each  $(X, x_0) \in Obj(Sets_*)$ , the unit map

$$\mathbb{1}_{(X,x_0)}^{\mathsf{Sets}_*} : \mathsf{pt} \to \mathsf{Sets}_*((X,x_0),(X,x_0))$$

of Sets<sub>\*</sub> at  $(X, x_0)$  is defined by<sup>6</sup>

$$id_{(X,x_0)}^{\mathsf{Sets}_*} \stackrel{\text{def}}{=} id_X$$
.

denoted  $(\mathbb{F}_1, 0)$ .

<sup>&</sup>lt;sup>4</sup>Further Terminology: Also called a **pointed function**.

<sup>&</sup>lt;sup>5</sup>Further Terminology: In the context of monoids with zero as models for  $\mathbb{F}_1$ -algebras, morphisms of pointed sets are also called **morphism of**  $\mathbb{F}_1$ -**modules**.

<sup>&</sup>lt;sup>6</sup>Note that id<sub>X</sub> is indeed a morphism of pointed sets, as we have id<sub>X</sub>( $x_0$ ) =  $x_0$ .

• *Composition*. For each  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0) \in Obj(Sets_*)$ , the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$
of  $\mathsf{Sets}_*$  at  $((X,x_0),(Y,y_0),(Z,z_0))$  is defined by<sup>7</sup>

$$g \circ_{(X,x_0),(Y,y_0),(Z,z_0)}^{\mathsf{Sets}_*} f \stackrel{\mathsf{def}}{=} g \circ f.$$

## **009P** 6.1.4 Elementary Properties of Pointed Sets

- **Proposition 6.1.4.1.1.** Let  $(X, x_0)$  be a pointed set.
- 009R 1. Completeness. The category Sets<sub>\*</sub> of pointed sets and morphisms between them is complete, having in particular:
- 009S (a) Products, described as in Definition 6.2.3.1.1.
- (b) Pullbacks, described as in Definition 6.2.4.1.1.
- (c) Equalisers, described as in Definition 6.2.5.1.1.
- 2. *Cocompleteness*. The category Sets<sub>\*</sub> of pointed sets and morphisms between them is cocomplete, having in particular:
- (a) Coproducts, described as in Definition 6.3.3.1.1.
- (b) Pushouts, described as in Definition 6.3.4.1.1;
- (c) Coequalisers, described as in Definition 6.3.5.1.1.
- 3. Failure To Be Cartesian Closed. The category Sets\* is not Cartesian closed.

$$g(f(x_0)) = g(y_0)$$

$$= z_0,$$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

<sup>&</sup>lt;sup>7</sup>Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

<sup>&</sup>lt;sup>8</sup>The category Sets<sub>\*</sub> does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

4. Morphisms From the Monoidal Unit. We have a bijection of sets<sup>9</sup>

$$\mathsf{Sets}_*(S^0, X) \cong X$$
,

natural in  $(X, x_0) \in \mathsf{Obj}(\mathsf{Sets}_*)$ , internalising also to an isomorphism of pointed sets

**Sets**<sub>\*</sub>(
$$S^0, X$$
)  $\cong (X, x_0),$ 

again natural in  $(X, x_0) \in Obj(Sets_*)$ .

5. Relation to Partial Functions. We have an equivalence of categories 10

$$\mathsf{Sets}_* \stackrel{\mathsf{eq.}}{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) From Pointed Sets to Sets With Partial Functions. The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\mathsf{part.}}$$

sends:

024V i. A pointe

i. A pointed set  $(X, x_0)$  to X.

024W ii. A pointed function

$$f: (X, x_0) \rightarrow (Y, y_0)$$

to the partial function

$$\xi_f: X \to Y$$

defined on  $f^{-1}(Y \setminus y_0)$  and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in f^{-1}(Y \setminus y_0)$ .

忘: 
$$\mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by  $S^0$ .

<sup>&</sup>lt;sup>9</sup>In other words, the forgetful functor

 $<sup>^{10}</sup>$  Warning: This is not an isomorphism of categories, only an equivalence.

024X (b) From Sets With Partial Functions to Pointed Sets. The equivalence

$$\xi^{-1}$$
: Sets<sup>part.</sup>  $\stackrel{\cong}{\to}$  Sets<sub>\*</sub>

sends:

024Y i. A set X is to the pointed set  $(X, \star)$  with  $\star$  an element that is not in X.

024Z ii. A partial function

$$f: X \to Y$$

defined on  $U \subset X$  to the pointed function

$$\xi_f^{-1} \colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each  $x \in X$ .

*Proof.* Item 1, Completeness: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and ??.

*Item 2, Cocompleteness:* This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and ??.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from  $S^0$  to a pointed set  $(X, x_0)$  sends  $0 \in S^0$  to  $x_0$  and then can send  $1 \in S^0$  to any element of X, we obtain a bijection between pointed maps  $S^0 \to X$  and the elements of X.

The isomorphism then

$$\mathsf{Sets}_*(S^0,X)\cong (X,x_0)$$

follows by noting that  $\Delta_{x_0}: S^0 \to X$ , the basepoint of **Sets**<sub>\*</sub>( $S^0, X$ ), corresponds to the pointed map  $S^0 \to X$  picking the element  $x_0$  of X, and thus we see that the bijection between pointed maps  $S^0 \to X$  and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

*Item 5*, *Relation to Partial Functions*: See [MSE 884460].

010C 6.1.5 Active and Inert Morphisms of Pointed Sets

**Oldow Definition 6.1.5.1.1.** Let  $f: (X, x_0) \to (Y, y_0)$  be a morphism of pointed sets.

01QE 1. The morphism f is active if  $f^{-1}(y_0) = x_0$ .

01QF 2. The morphism f is **inert** if, for each  $y \in Y$ , the set  $f^{-1}(y)$  has exactly one element.

**Notation 6.1.5.1.2.** We write Sets\*\* for the wide subcategory of Sets\* spanned by pointed sets and the active maps between them.

**Example 6.1.5.1.3.** Here are some examples of active and inert maps of pointed sets.

01QJ 1. The map  $\mu: \langle 2 \rangle \to \langle 1 \rangle$  given by



is active but not inert.

01QK 2. The map  $f: \langle 2 \rangle \rightarrow \langle 2 \rangle$  given by

$$1 \longmapsto 1$$

$$2 \longmapsto 2$$

$$* \longmapsto *$$

is inert but not active.

**01QL** 3. The map  $f: \langle 3 \rangle \rightarrow \langle 1 \rangle$  given by



3 \* : : : : :

is neither inert nor active. However, it factors as  $f = a \circ i$ , where

$$i: \langle 3 \rangle \rightarrow \langle 2 \rangle$$
,

$$a: \langle 2 \rangle \rightarrow \langle 1 \rangle$$

are the morphisms of pointed sets given by

with *i* being inert and *a* being active.

- **Proposition 6.1.5.1.4.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.
- 01QN 1. Active-Inert Factorisation. Every morphism of pointed sets  $f:(X,x_0) \to (Y,y_0)$  factors uniquely as

$$f = a \circ i$$

where:

- 01QP (a) The map  $i: (X, x_0) \to (K, k_0)$  is an inert morphism of pointed sets
- 01QQ (b) The map  $a: (K, k_0) \to (Y, y_0)$  is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets..

*Proof. Item* 1, *Active-Inert Factorisation*: Let  $f: X \to Y$  be a morphism of pointed sets. We can factor f as

$$X \stackrel{i}{\longrightarrow} K \stackrel{a}{\longrightarrow} Y$$

where:

• *K* is the pointed set given by

$$K = \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\}$$
  
=  $(X \setminus f^{-1}(y_0)) \cup \{x_0\};$ 

•  $i: X \to K$  is the inert morphism of pointed sets given by

$$i(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

for each  $x \in X$ ;

•  $a: K \to Y$  is the active morphism of pointed sets given by

$$a(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in K$ .

Next, let

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow f & & \downarrow g \\
A & \xrightarrow{g} & B
\end{array}$$

be a commutative diagram in Sets<sub>\*</sub>. Consider the morphism  $\phi: Y \to A$  given by

$$\phi(y) = f(i^{-1}(y))$$

for each  $y \in Y$  (which is well-defined since, as i is inert,  $i^{-1}(y)$  is a singleton for all  $y \in Y$ ). We claim that  $\phi$  is the unique diagonal filler in the diagram

$$X \xrightarrow{i} Y$$

$$f \downarrow \exists ! \text{ } / \downarrow g$$

$$A \xrightarrow{a} B.$$

Indeed, this diagram commutes, as we have

$$[\phi \circ i](x) \stackrel{\text{def}}{=} \phi(i(x))$$
$$\stackrel{\text{def}}{=} f(i^{-1}(i(x)))$$
$$= f(x)$$

for each  $x \in X$  and

$$[a \circ \phi](y) \stackrel{\text{def}}{=} a(\phi(y))$$

$$\stackrel{\text{def}}{=} a(f(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} [a \circ f](i^{-1}(y))$$

$$= [g \circ i](i^{-1}(y))$$

$$\stackrel{\text{def}}{=} g(i(i^{-1}(y)))$$

$$\stackrel{\text{def}}{=} g(y)$$

for each  $y \in Y$ . Moreover, given another morphism  $\psi$  such that the diagram

$$\begin{array}{c|c}
X & \xrightarrow{i} & Y \\
f & & \downarrow & \downarrow g \\
A & \xrightarrow{g} & B
\end{array}$$

commutes, it follows that we must have  $\psi = \phi$ , since, given  $y \in Y$ , there exists a unique  $x \in X$  such that i(x) = y, so we have

$$\psi(y) = \psi(i(x))$$

$$= f(x)$$

$$= f(i^{-1}(y))$$

$$\stackrel{\text{def}}{=} \phi(y).$$

This finishes the proof.

## **6.2** Limits of Pointed Sets

#### **00A3 6.2.1** The Terminal Pointed Set

- **Definition 6.2.1.1.1.** The **terminal pointed set** is the terminal object of Sets<sub>\*</sub> as in Limits and Colimits, **??**.
- **Construction 6.2.1.1.2.** Concretely, the **terminal pointed set** is the pair  $((pt, \star), \{!_X\}_{(X,x_0) \in Obj(Sets_*)})$  consisting of:
  - *The Limit.* The pointed set  $(pt, \star)$ .
  - *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathsf{pt}, \star)\}_{(X, x_0) \in \mathsf{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \stackrel{\text{def}}{=} \star$$

for each  $x \in X$  and each  $(X, x_0) \in Obj(Sets)$ .

*Proof.* We claim that  $(pt, \star)$  is the terminal object of Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

$$(X, x_0)$$
  $(pt, \star)$ 

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (X, x_0) \to (\mathsf{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow{-\frac{\phi}{\exists !}} (pt, \star)$$

commute, namely  $!_X$ .

#### 00A5 6.2.2 Products of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

- **Definition 6.2.2.1.1.** The **product of**  $\{(X_i, x_0^i)\}_{i \in I}$  is the product of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- **Construction 6.2.2.1.2.** Concretely, the **product of**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $((\prod_{i \in I} X_i, (x_0^i)_{i \in I}), \{\operatorname{pr}_i\}_{i \in I})$  consisting of:
  - *The Limit.* The pointed set  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$ .
  - The Cone. The collection

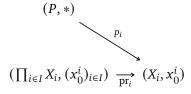
$$\left\{ \operatorname{pr}_i \colon \left( \prod_{i \in I} X_i, (x_0^i)_{i \in I} \right) \to (X_i, x_0^i) \right\}_{i \in I}$$

of maps given by

$$\operatorname{pr}_i((x_j)_{j\in I})\stackrel{\mathrm{def}}{=} x_i$$

for each  $(x_i)_{i \in I} \in \prod_{i \in I} X_i$  and each  $i \in I$ .

*Proof.* We claim that  $(\prod_{i \in I} X_i, (x_0^i)_{i \in I})$  is the categorical product of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub>. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

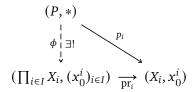


in Sets<sub>\*</sub>. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*)\to (\prod_{i\in I}X_i,(x_0^i)_{i\in I})$$

6.2.3 Products 12

making the diagram



commute, being uniquely determined by the condition  $pr_i \circ \phi = p_i$  for each  $i \in I$  via

$$\phi(x) = (p_i(x))_{i \in I}$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_i(*))_{i \in I} 
= (x_0^i)_{i \in I},$$

where we have used that  $p_i$  is a morphism of pointed sets for each  $i \in I$ .

- **Proposition 6.2.2.1.3.** Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.
- 00A8 1. Functoriality. The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\prod_{i \in I} X_i, (x_0^i)_{i \in I})$  defines a functor  $\prod_{i \in I} : \operatorname{Fun}(I_{\operatorname{disc}}, \operatorname{\mathsf{Sets}}_*) \to \operatorname{\mathsf{Sets}}_*.$

*Proof.* Item 1, Functoriality: This follows from Limits and Colimits, ?? of ??. □

**00A9 6.2.3 Products** 

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

- **Definition 6.2.3.1.1.** The **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the product of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- O252 **Construction 6.2.3.1.2.** Concretely, the **product of**  $(X, x_0)$  **and**  $(Y, y_0)$  is the pair consisting of:
  - *The Limit.* The pointed set  $(X \times Y, (x_0, y_0))$ .
  - The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 : (X \times Y, (x_0, y_0)) \to (X, x_0),$$
  
 $\operatorname{pr}_2 : (X \times Y, (x_0, y_0)) \to (Y, y_0)$ 

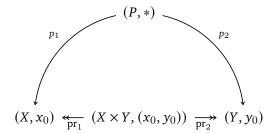
6.2.3 Products 13

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$
  
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$ 

for each  $(x, y) \in X \times Y$ .

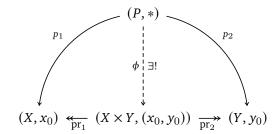
*Proof.* We claim that  $(X \times Y, (x_0, y_0))$  is the categorical product of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form



in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
  
 $\operatorname{pr}_2 \circ \phi = p_2$ 

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ . Note that this is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$
  
=  $(x_0, y_0),$ 

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets.

6.2.3 Products 14

- **Proposition 6.2.3.1.3.** Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets.
- 00AC 1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$A \times -:$$
 Sets<sub>\*</sub>  $\rightarrow$  Sets<sub>\*</sub>,  
 $- \times B:$  Sets<sub>\*</sub>  $\rightarrow$  Sets<sub>\*</sub>,  
 $-_1 \times -_2:$  Sets<sub>\*</sub>  $\times$  Sets<sub>\*</sub>  $\rightarrow$  Sets<sub>\*</sub>,

defined in the same way as the functors of Constructions With Sets, Item 1 of Definition 4.1.3.1.3.

- 01QR 2. Lack of Adjointness. The functors  $X \times -$  and  $\times Y$  do not admit right adjoints.
- 3. Associativity. We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0) \in Obj(Sets_*)$ .

**OOAE** 4. *Unitality*. We have isomorphisms of pointed sets

$$(pt, \star) \times (X, x_0) \cong (X, x_0),$$
  
 $(X, x_0) \times (pt, \star) \cong (X, x_0),$ 

natural in  $(X, x_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

00AF 5. Commutativity. We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in  $(X, x_0), (Y, y_0) \in \text{Obj}(\mathsf{Sets}_*).$ 

6. *Symmetric Monoidality*. The triple (Sets<sub>\*</sub>,  $\times$ , (pt,  $\star$ )) is a symmetric monoidal category.

*Proof. Item 1, Functoriality*: This is a special case of functoriality of limits, Limits and Colimits, ?? of ??.

*Item 2, Lack of Adjointness:* See [MSE 2855868].

*Item 3, Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.3.1.3.

*Item 4, Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.1.3.1.3.

*Item 5, Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.1.3.1.3.

*Item 6, Symmetric Monoidality*: This follows from Constructions With Sets, Item 14 of Definition 4.1.3.1.3. □

#### 00AH 6.2.4 Pullbacks

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (X, x_0) \to (Z, z_0)$  and  $g: (Y, y_0) \to (Z, z_0)$  be morphisms of pointed sets.

- **Definition 6.2.4.1.1.** The **pullback of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along** (f, g) is the pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- O253 Construction 6.2.4.1.2. Concretely, the pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) is the pair consisting of:
  - *The Limit.* The pointed set  $(X \times_Z Y, (x_0, y_0))$ .
  - The Cone. The morphisms of pointed sets

$$\operatorname{pr}_1 \colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0),$$
  
 $\operatorname{pr}_2 \colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0)$ 

defined by

$$\operatorname{pr}_{1}(x, y) \stackrel{\text{def}}{=} x,$$
  
 $\operatorname{pr}_{2}(x, y) \stackrel{\text{def}}{=} y$ 

for each  $(x, y) \in X \times_Z Y$ .

*Proof.* We claim that  $X \times_Z Y$  is the categorical pullback of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to (f, g) in Sets<sub>\*</sub>. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \operatorname{pr}_{1} = g \circ \operatorname{pr}_{2}, \qquad p_{1} \downarrow \qquad p_{1} \downarrow g$$

$$(X \times_{Z} Y, (x_{0}, y_{0})) \xrightarrow{\operatorname{pr}_{2}} (Y, y_{0}) \downarrow g$$

$$(X, x_{0}) \xrightarrow{f} (Z, z_{0}).$$

Indeed, given  $(x, y) \in X \times_Z Y$ , we have

$$[f \circ pr_1](x, y) = f(pr_1(x, y))$$

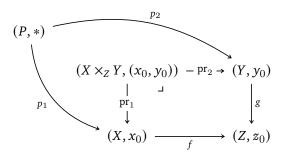
$$= f(x)$$

$$= g(y)$$

$$= g(pr_2(x, y))$$

$$= [g \circ pr_2](x, y),$$

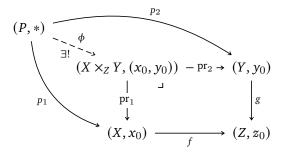
where f(x) = g(y) since  $(x, y) \in X \times_Z Y$ . Next, we prove that  $X \times_Z Y$  satisfies the universal property of the pullback. Suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi: (P, *) \rightarrow (X \times_Z Y, (x_0, y_0))$$

making the diagram



commute, being uniquely determined by the conditions

$$\operatorname{pr}_1 \circ \phi = p_1,$$
  
 $\operatorname{pr}_2 \circ \phi = p_2$ 

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each  $x \in P$ , where we note that  $(p_1(x), p_2(x)) \in X \times Y$  indeed lies in  $X \times_Z Y$  by the condition

$$f \circ p_1 = g \circ p_2$$
,

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each  $x \in P$ , so that  $(p_1(x), p_2(x)) \in X \times_Z Y$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(*) = (p_1(*), p_2(*))$$

$$=(x_0,y_0),$$

where we have used that  $p_1$  and  $p_2$  are morphisms of pointed sets.

**OOAK Proposition 6.2.4.1.3.** Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

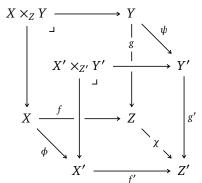
00AL 1. Functoriality. The assignment  $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$  defines a functor

$$-_1 \times_{-_3} -_1 : \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \times_{-3} -1$  is given by sending a morphism



in  $Fun(\mathcal{P}, \mathsf{Sets}_*)$  to the morphism of pointed sets

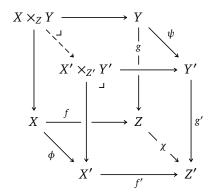
$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists !} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))$$

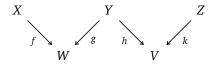
for each  $(x, y) \in X \times_Z Y$ , which is the unique morphism of pointed sets making the diagram

18



commute.

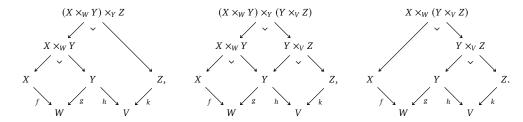
#### 00AM 2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

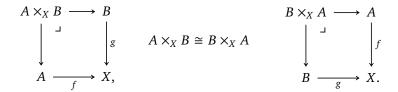
where these pullbacks are built as in the diagrams



#### 00AN 3. *Unitality*. We have isomorphisms of pointed sets



4. Commutativity. We have an isomorphism of pointed sets



00AQ 5. Interaction With Products. We have an isomorphism of pointed sets

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \times_{\text{pt}} Y \cong X \times Y,$$

$$X \xrightarrow{!_X} \text{pt.}$$

6. *Symmetric Monoidality.* The triple (Sets<sub>\*</sub>,  $\times_X$ , X) is a symmetric monoidal category.

*Proof. Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pullback diagram.

*Item 2, Associativity*: This follows from Constructions With Sets, Item 4 of Definition 4.1.4.1.5.

*Item 3, Unitality*: This follows from Constructions With Sets, Item 6 of Definition 4.1.4.1.5.

*Item 4, Commutativity*: This follows from Constructions With Sets, Item 7 of Definition 4.1.4.1.5.

*Item 5, Interaction With Products*: This follows from Constructions With Sets, Item 10 of Definition 4.1.4.1.5.

*Item 6, Symmetric Monoidality*: This follows from Constructions With Sets, Item 11 of Definition 4.1.4.1.5. □

## **00AS 6.2.5** Equalisers

Let  $f, g: (X, x_0) \Rightarrow (Y, y_0)$  be morphisms of pointed sets.

- **Definition 6.2.5.1.1.** The **equaliser of** (f, g) is the equaliser of f and g in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- **Construction 6.2.5.1.2.** Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set  $(Eq(f,g),x_0)$ .
- The Cone. The morphism of pointed sets

$$eq(f,g): (Eq(f,g),x_0) \hookrightarrow (X,x_0)$$

given by the canonical inclusion  $eq(f,g) \hookrightarrow Eq(f,g) \hookrightarrow X$ .

*Proof.* We claim that  $(\text{Eq}(f,g),x_0)$  is the categorical equaliser of f and g in  $\text{Sets}_*$ . First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ eq(f,g) = g \circ eq(f,g),$$

which indeed holds by the definition of the set Eq(f,g). Next, we prove that Eq(f,g) satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$(\text{Eq}(f,g),x_0) \xrightarrow{\text{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$(E,*)$$

in Sets\*. Then there exists a unique morphism of pointed sets

$$\phi \colon (E,*) \to (\text{Eq}(f,g),x_0)$$

making the diagram

$$(\operatorname{Eq}(f,g),x_0) \xrightarrow{\operatorname{eq}(f,g)} (X,x_0) \xrightarrow{f} (Y,y_0)$$

$$\downarrow f$$

$$\downarrow g$$

$$(E,*)$$

commute, being uniquely determined by the condition

$$eq(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each  $x \in E$ , where we note that  $e(x) \in A$  indeed lies in Eq(f,g) by the condition

$$f \circ e = g \circ e$$
,

which gives

$$f(e(x)) = g(e(x))$$

for each  $x \in E$ , so that  $e(x) \in \text{Eq}(f,g)$ . Lastly, we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(*) = e(*)$$
$$= x_0,$$

where we have used that e is a morphism of pointed sets.

- **Proposition 6.2.5.1.3.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h: (X, x_0) \rightarrow (Y, y_0)$  be morphisms of pointed sets.
- 00AV 1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}$$

where Eq(f, g, h) is the limit of the diagram

$$(X, x_0) \xrightarrow{f \atop -g \Rightarrow \atop h} (Y, y_0)$$

in Sets, being explicitly given by

$$Eq(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality*. We have an isomorphism of pointed sets

$$\operatorname{Eq}(f, f) \cong X$$
.

3. *Commutativity.* We have an isomorphism of pointed sets

$$\operatorname{Eq}(f,g) \cong \operatorname{Eq}(g,f).$$

*Proof. Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.1.5.1.3.

*Item 2, Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.1.5.1.3.

*Item 3, Commutativity*: This follows from Constructions With Sets, Item 5 of Definition 4.1.5.1.3.

## **6.3** Colimits of Pointed Sets

#### **00AZ** 6.3.1 The Initial Pointed Set

- **Definition 6.3.1.1.1.** The **initial pointed set** is the initial object of Sets<sub>\*</sub> as in Limits and Colimits, ??.
- **Construction 6.3.1.1.2.** Concretely, the **initial pointed set** is the pair  $((pt, \star), \{\iota_X\}_{(X,x_0) \in Obj(Sets_*)})$  consisting of:
  - *The Limit*. The pointed set  $(pt, \star)$ .
  - The Cone. The collection of morphisms of pointed sets

$$\{\iota_X \colon (\mathsf{pt}, \star) \to (X, x_0)\}_{(X, x_0) \in \mathsf{Obi}(\mathsf{Sets})}$$

defined by

$$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

*Proof.* We claim that  $(pt, \star)$  is the initial object of Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form

$$(pt, \star)$$
  $(X, x_0)$ 

in Sets. Then there exists a unique morphism of pointed sets

$$\phi \colon (\mathsf{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\text{pt}, \star) \xrightarrow{-\frac{\phi}{\exists !}} (X, x_0)$$

commute, namely  $\iota_X$ .

### **00B1** 6.3.2 Coproducts of Families of Pointed Sets

Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

- **Definition 6.3.2.1.1.** The **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets\* as in Limits and Colimits, ??.
- **Construction 6.3.2.1.2.** Concretely, the **coproduct of the family**  $\{(X_i, x_0^i)\}_{i \in I}$  is the pair  $((\bigvee_{i \in I} X_i, p_0), \{\inf_i\}_{i \in I})$  consisting of:

<sup>&</sup>lt;sup>11</sup>Further Terminology: Also called the wedge sum of the family  $\{(X_i, x_0^i)\}_{i \in I}$ .

- *The Colimit.* The pointed set  $(\bigvee_{i \in I} X_i, p_0)$  consisting of:
  - The Underlying Set. The set  $\bigvee_{i \in I} X_i$  defined by

$$\bigvee_{i \in I} X_i \stackrel{\text{def}}{=} \left( \prod_{i \in I} X_i \right) / \sim,$$

where  $\sim$  is the equivalence relation on  $\coprod_{i \in I} X_i$  given by declaring

$$(i, x_0^i) \sim (j, x_0^j)$$

for each  $i, j \in I$ .

**-** *The Basepoint*. The element  $p_0$  of  $\bigvee_{i ∈ I} X_i$  defined by

$$p_0 \stackrel{\text{def}}{=} [(i, x_0^i)]$$
$$= [(j, x_0^j)]$$

for any  $i, j \in I$ .

• The Cocone. The collection

$$\left\{ \operatorname{inj}_i \colon (X_i, x_0^i) \to (\bigvee_{i \in I} X_i, p_0) \right\}_{i \in I}$$

of morphism of pointed sets given by

$$\operatorname{inj}_{i}(x) \stackrel{\text{def}}{=} (i, x)$$

for each  $x \in X_i$  and each  $i \in I$ .

*Proof.* We claim that  $(\bigvee_{i \in I} X_i, p_0)$  is the categorical coproduct of  $\{(X_i, x_0^i)\}_{i \in I}$  in Sets<sub>\*</sub>. Indeed, suppose we have, for each  $i \in I$ , a diagram of the form

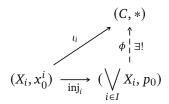
$$(C,*)$$

$$(X_i, x_0^i) \xrightarrow{\operatorname{inj}_i} (\bigvee_{i \in I} X_i, p_0)$$

in Sets<sub>\*</sub>. Then there exists a unique morphism of pointed sets

$$\phi\colon (\bigvee_{i\in I} X_i, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the condition  $\phi \circ \text{inj}_i = \iota_i$  for each  $i \in I$  via

$$\phi([(i,x)]) = \iota_i(x)$$

for each  $[(i, x)] \in \bigvee_{i \in I} X_i$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_i([(i, x_0^i)])$$
= \*

as  $\iota_i$  is a morphism of pointed sets.

**Proposition 6.3.2.1.3.** Let  $\{(X_i, x_0^i)\}_{i \in I}$  be a family of pointed sets.

00B4 1. Functoriality. The assignment  $\{(X_i, x_0^i)\}_{i \in I} \mapsto (\bigvee_{i \in I} X_i, p_0)$  defines a functor

$$\bigvee_{i \in I} : \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

*Proof. Item* 1, *Functoriality*: This follows from Limits and Colimits, ?? of ??. □

00B5 6.3.3 Coproducts

Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

- **Definition 6.3.3.1.1.** The **coproduct of**  $(X, x_0)$  **and**  $(Y, y_0)^{12}$  is the coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub> as in Limits and Colimits, ??.
- O257 Construction 6.3.3.1.2. Concretely, the coproduct of  $(X, x_0)$  and  $(Y, y_0)$ , also called their wedge sum, is the pair consisting of:
  - *The Colimit.* The pointed set  $(X \vee Y, p_0)$  consisting of:

<sup>12</sup> Further Terminology: Also called the wedge sum of  $(X, x_0)$  and  $(Y, y_0)$ .

- The Underlying Set. The set  $X \vee Y$  defined by

where  $\sim$  is the equivalence relation on  $X \coprod Y$  obtained by declaring  $(0, x_0) \sim (1, y_0)$ .

- The Basepoint. The element  $p_0$  of  $X \vee Y$  defined by

$$p_0 \stackrel{\text{def}}{=} [(0, x_0)]$$
  
=  $[(1, y_0)]$ .

• *The Cocone*. The morphisms of pointed sets

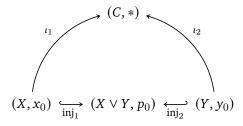
$$\operatorname{inj}_1 \colon (X, x_0) \to (X \vee Y, p_0),$$
  
 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \vee Y, p_0),$ 

given by

$$inj_1(x) \stackrel{\text{def}}{=} [(0, x)], 
inj_2(y) \stackrel{\text{def}}{=} [(1, y)],$$

for each  $x \in X$  and each  $y \in Y$ .

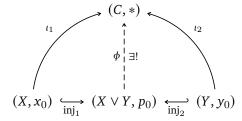
*Proof.* We claim that  $(X \lor Y, p_0)$  is the categorical coproduct of  $(X, x_0)$  and  $(Y, y_0)$  in Sets<sub>\*</sub>. Indeed, suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi: (X \vee Y, p_0) \to (C, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_X = \iota_X,$$
  
$$\phi \circ \operatorname{inj}_V = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0, x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1, y)] \text{ with } y \in Y \end{cases}$$

for each  $z \in X \vee Y$ , where we note that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \iota_X([(0, x_0)])$$
  
=  $\iota_Y([(1, y_0)])$   
= \*,

as  $\iota_X$  and  $\iota_Y$  are morphisms of pointed sets.

**Proposition 6.3.3.1.3.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets.

00B8 1. Functoriality. The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$X \lor -: \mathsf{Sets}_* \to \mathsf{Sets}_*,$$
  
 $- \lor Y : \mathsf{Sets}_* \to \mathsf{Sets}_*,$   
 $-_1 \lor -_2 : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.$ 

00B9 2. Associativity. We have an isomorphism of pointed sets

$$(X \lor Y) \lor Z \cong X \lor (Y \lor Z),$$

natural in  $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$ .

00BA 3. *Unitality*. We have isomorphisms of pointed sets

$$(pt, *) \lor (X, x_0) \cong (X, x_0),$$
  
 $(X, x_0) \lor (pt, *) \cong (X, x_0),$ 

natural in  $(X, x_0) \in \mathsf{Sets}_*$ .

00BB 4. Commutativity. We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X$$

natural in  $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$ .

**OOBC** 5. *Symmetric Monoidality*. The triple (Sets<sub>\*</sub>, ∨, pt) is a symmetric monoidal category.

6. The Fold Map. We have a natural transformation

$$\nabla\colon \vee\circ\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}\Longrightarrow id_{\mathsf{Sets}_*}, \qquad \begin{array}{c} \mathsf{Sets}_*\times\mathsf{Sets}_*\\ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}& & \\ & & \nabla\\ \mathsf{Sets}_*& & \\ & & & \\ & & & \\ \mathsf{Sets}_*& & \\ & & & \\ & & & \\ \mathsf{Sets}_*, \end{array}$$

called the **fold map**, whose component

$$\nabla_X : X \vee X \to X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each  $p \in X \vee X$ .

*Proof.* Item 1, Functoriality: This follows from Limits and Colimits, ?? of ??.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Omitted.

*Item 5*, *Symmetric Monoidality*: Omitted.

*Item 6, The Fold Map:* Naturality for the transformation  $\nabla$  is the statement that,

given a morphism of pointed sets  $f: (X, x_0) \to (Y, y_0)$ , we have

$$\nabla_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad X \vee X \xrightarrow{\nabla_{X}} X$$

$$V_{Y} \circ (f \vee f) = f \circ \nabla_{X}, \quad f \vee f \downarrow \qquad \downarrow f$$

$$Y \vee Y \xrightarrow{\nabla_{Y}} Y.$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{aligned}$$

for each  $[(i,x)] \in X \vee X$ , and thus  $\nabla$  is indeed a natural transformation.  $\square$ 

#### 00BE 6.3.4 Pushouts

Let  $(X, x_0)$ ,  $(Y, y_0)$ , and  $(Z, z_0)$  be pointed sets and let  $f: (Z, z_0) \to (X, x_0)$  and  $g: (Z, z_0) \to (Y, y_0)$  be morphisms of pointed sets.

- **Definition 6.3.4.1.1.** The **pushout of**  $(X, x_0)$  **and**  $(Y, y_0)$  **over**  $(Z, z_0)$  **along** (f, g) is the pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) in Sets<sub>\*</sub> as in Limits and Colimits, **??**.
- O258 Construction 6.3.4.1.2. Concretely, the pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  along (f, g) is the pair consisting of:
  - *The Colimit.* The pointed set  $(X \coprod_{f,Z,g} Y, p_0)$ , where:
    - The set  $X \coprod_{f,Z,g} Y$  is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
    - We have  $p_0 = [x_0] = [y_0]$ .
  - *The Cocone*. The morphisms of pointed sets

$$\operatorname{inj}_1 \colon (X, x_0) \to (X \coprod_Z Y, p_0),$$
  
 $\operatorname{inj}_2 \colon (Y, y_0) \to (X \coprod_Z Y, p_0)$ 

given by

$$\operatorname{inj}_{1}(x) \stackrel{\text{def}}{=} [(0, x)]$$
  
 $\operatorname{inj}_{2}(y) \stackrel{\text{def}}{=} [(1, y)]$ 

for each  $x \in X$  and each  $y \in Y$ .

*Proof.* Firstly, we note that indeed  $[x_0] = [y_0]$ , as we have

$$x_0 = f(z_0),$$
  
$$y_0 = g(z_0)$$

since f and g are morphisms of pointed sets, with the relation  $\sim$  on  $X \coprod_Z Y$  then identifying  $x_0 = f(z_0) \sim g(z_0) = y_0$ .

We now claim that  $(X \coprod_Z Y, p_0)$  is the categorical pushout of  $(X, x_0)$  and  $(Y, y_0)$  over  $(Z, z_0)$  with respect to (f, g) in Sets<sub>\*</sub>. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$(X \coprod_{Z} Y, p_{0}) \stackrel{\text{inj}_{2}}{\longleftarrow} (Y, y_{0})$$

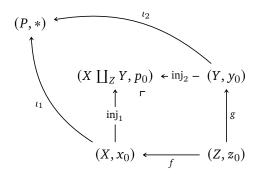
$$\text{inj}_{1} \circ f = \text{inj}_{2} \circ g, \qquad \text{inj}_{1} \qquad \qquad \uparrow g$$

$$(X, x_{0}) \stackrel{f}{\longleftarrow} (Z, z_{0}).$$

Indeed, given  $z \in Z$ , we have

$$\begin{aligned} [\inf_1 \circ f](z) &= \inf_1(f(z)) \\ &= [(0, f(z))] \\ &= [(1, g(z))] \\ &= \inf_2(g(z)) \\ &= [\inf_2 \circ g](z), \end{aligned}$$

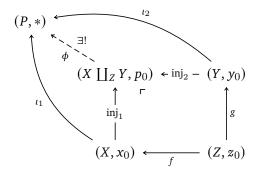
where [(0, f(z))] = [(1, g(z))] by the definition of the relation  $\sim$  on  $X \coprod Y$  (the coproduct of unpointed sets of X and Y). Next, we prove that  $X \coprod_Z Y$  satisfies the universal property of the pushout. Suppose we have a diagram of the form



in Sets. Then there exists a unique morphism of pointed sets

$$\phi: (X \mid I_Z Y, p_0) \rightarrow (P, *)$$

making the diagram



commute, being uniquely determined by the conditions

$$\phi \circ \operatorname{inj}_1 = \iota_1,$$
  
$$\phi \circ \operatorname{inj}_2 = \iota_2$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each  $p \in X \coprod_Z Y$ , where the well-definedness of  $\phi$  is proven in the same way as in the proof of Constructions With Sets, Definition 4.2.4.1.1. Finally, we show that  $\phi$  is indeed a morphism of pointed sets, as we have

$$\phi(p_0) = \phi([(0, x_0)])$$
  
=  $\iota_1(x_0)$   
= \*,

or alternatively

$$\phi(p_0) = \phi([(1, y_0)])$$
  
=  $\iota_2(y_0)$   
= \*,

where we use that  $\iota_1$  (resp.  $\iota_2$ ) is a morphism of pointed sets.

**Proposition 6.3.4.1.3.** Let  $(X, x_0)$ ,  $(Y, y_0)$ ,  $(Z, z_0)$ , and  $(A, a_0)$  be pointed sets.

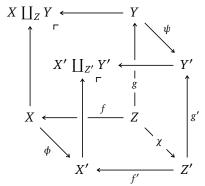
00BH 1. Functoriality. The assignment  $(X, Y, Z, f, g) \mapsto X \coprod_{f, Z, g} Y$  defines a functor

$$-_1 \coprod_{-_3} -_1 : \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where  $\mathcal{P}$  is the category that looks like this:



In particular, the action on morphisms of  $-1 \coprod_{-3} -1$  is given by sending a morphism



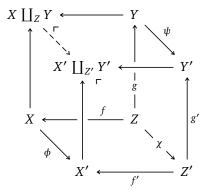
in  $\operatorname{Fun}(\mathcal{P},\operatorname{\mathsf{Sets}}_*)$  to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists !} (X' \coprod_{Z'} Y', p'_0)$$

given by

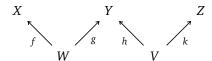
$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each  $p \in X \coprod_Z Y$ , which is the unique morphism of pointed sets making the diagram



commute.

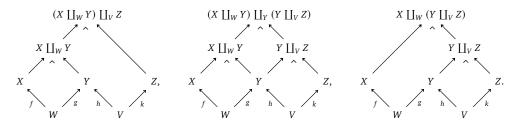
#### 00BJ 2. Associativity. Given a diagram



in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams



## **00BK** 3. *Unitality*. We have isomorphisms of sets



### 4. *Commutativity.* We have an isomorphism of sets

#### **OOBM** 5. *Interaction With Coproducts*. We have

6. *Symmetric Monoidality*. The triple (Sets<sub>\*</sub>,  $\coprod_X$ ,  $(X, x_0)$ ) is a symmetric monoidal category.

*Proof. Item 1, Functoriality*: This is a special case of functoriality of co/limits, Limits and Colimits, ?? of ??, with the explicit expression for  $\xi$  following from the commutativity of the cube pushout diagram.

*Item 2, Associativity*: This follows from Constructions With Sets, Item 3 of Definition 4.2.4.1.6.

*Item 3, Unitality*: This follows from Constructions With Sets, Item 5 of Definition 4.2.4.1.6.

*Item 4, Commutativity*: This follows from Constructions With Sets, Item 6 of Definition 4.2.4.1.6.

*Item 5*, *Interaction With Coproducts*: Omitted.

Item 6, Symmetric Monoidality: Omitted.

## **00BP** 6.3.5 Coequalisers

Let  $f, g: (X, x_0) \rightrightarrows (Y, y_0)$  be morphisms of pointed sets.

- **Definition 6.3.5.1.1.** The **coequaliser of** (f,g) is the pointed set  $(CoEq(f,g), [y_0])$ .
- **Construction 6.3.5.1.2.** The **coequaliser of** (f, g) is the pair  $((CoEq(f, g), [y_0]), coeq(f, g))$  consisting of:
  - *The Colimit*. The pointed set  $(CoEq(f,g), [y_0])$ , where CoEq(f,g) is the coequaliser of f and g as in Constructions With Sets, Definition 4.2.5.1.1.
  - The Cocone. The map

$$coeq(f,g): Y \rightarrow (CoEq(f,g), [y_0])$$

given by the quotient map, as in Constructions With Sets, Item 2 of Definition 4.2.5.1.2.

*Proof.* We claim that  $(CoEq(f, g), [y_0])$  is the categorical coequaliser of f and g in Sets<sub>\*</sub>. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$coeq(f, g) \circ f = coeq(f, g) \circ g$$
.

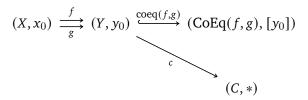
Indeed, we have

$$[\operatorname{coeq}(f,g) \circ f](x) \stackrel{\text{def}}{=} [\operatorname{coeq}(f,g)](f(x))$$
$$\stackrel{\text{def}}{=} [f(x)]$$
$$= [g(x)]$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f, g)](g(x))$$

$$\stackrel{\text{def}}{=} [\operatorname{coeq}(f, g) \circ g](x)$$

for each  $x \in X$ . Next, we prove that CoEq(f, g) satisfies the universal property of the coequaliser. Suppose we have a diagram of the form



in Sets. Then, since c(f(a)) = c(g(a)) for each  $a \in A$ , it follows from Conditions on Relations, Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map  $\phi \colon CoEq(f,g) \xrightarrow{\exists !} C$  making the diagram

commute, where we note that  $\phi$  is indeed a morphism of pointed sets since

$$\phi([y_0]) = [\phi \circ \operatorname{coeq}(f, g)]([y_0])$$

$$= c([y_0])$$

$$= *,$$

where we have used that c is a morphism of pointed sets.

**Proposition 6.3.5.1.3.** Let  $(X, x_0)$  and  $(Y, y_0)$  be pointed sets and let  $f, g, h \colon (X, x_0) \to (Y, y_0)$  be morphisms of pointed sets.

00BS 1. Associativity. We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)}\cong \mathrm{CoEq}(f,g,h)\cong \underbrace{\underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)}}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)}$$

where CoEq(f, g, h) is the colimit of the diagram

$$(X, x_0) \xrightarrow{f} (Y, y_0)$$

in Sets\*.

00BT 2. *Unitality*. We have an isomorphism of pointed sets

$$CoEq(f, f) \cong B$$
.

**00BU** 3. Commutativity. We have an isomorphism of pointed sets

$$CoEq(f, g) \cong CoEq(g, f)$$
.

*Proof. Item 1, Associativity*: This follows from Constructions With Sets, Item 1 of Definition 4.2.5.1.5.

*Item 2, Unitality*: This follows from Constructions With Sets, Item 4 of Definition 4.2.5.1.5.

*Item 3, Commutativity*: This follows from Constructions With Sets, Item 5 of Definition 4.2.5.1.5.

## **6.4 Constructions With Pointed Sets**

### **00BW** 6.4.1 Free Pointed Sets

Let *X* be a set.

**Definition 6.4.1.1.1.** The **free pointed set on** X is the pointed set  $X^+$  consisting of:

• The Underlying Set. The set  $X^+$  defined by <sup>13</sup>

$$X^{+} \stackrel{\text{def}}{=} X \coprod \text{pt}$$

$$\stackrel{\text{def}}{=} X \coprod \{ \star \}.$$

• *The Basepoint*. The element  $\star$  of  $X^+$ .

**OOBY** Proposition 6.4.1.1.2. Let X be a set.

1. Functoriality. The assignment  $X \mapsto X^+$  defines a functor

$$(-)^+$$
: Sets  $\rightarrow$  Sets<sub>\*</sub>,

where:

00BZ

<sup>&</sup>lt;sup>13</sup>Further Notation: We sometimes write  $\star_X$  for the basepoint of  $X^+$  for clarity, specially when there are multiple free pointed sets involved in the current discussion.

• *Action on Objects.* For each  $X \in Obj(Sets)$ , we have

$$[(-)^+](X) \stackrel{\text{def}}{=} X^+,$$

where  $X^+$  is the pointed set of Definition 6.4.1.1.1.

• *Action on Morphisms*. For each morphism  $f: X \to Y$  of Sets, the image

$$f^+: X^+ \to Y^+$$

of f by  $(-)^+$  is the map of pointed sets defined by

$$f^{+}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_{Y} & \text{if } x = \star_{X}. \end{cases}$$

00C0 2. Adjointness. We have an adjunction

$$((-)^+ \dashv \stackrel{\leftarrow}{\sim}): \operatorname{Sets}_{\stackrel{\leftarrow}{\sim}}^{(-)^+} \operatorname{Sets}_*,$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*((X^+, \star_X), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in  $X \in \text{Obj}(\mathsf{Sets})$  and  $(Y, y_0) \in \text{Obj}(\mathsf{Sets}_*)$ .

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+, (-)^{+, \coprod}, (-)^{+, \coprod}_{\mathbb{1}}) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathsf{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{+,\coprod}: X^{+} \vee Y^{+} \xrightarrow{\sim} (X \coprod Y)^{+},$$
$$(-)_{1}^{+,\coprod}: \operatorname{pt} \xrightarrow{\sim} \emptyset^{+},$$

natural in  $X, Y \in Obj(Sets)$ .

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^+,(-)^+,(-)^+_{1})\colon (\mathsf{Sets},\times,\mathsf{pt})\to (\mathsf{Sets}_*,\wedge,S^0),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+,$$
$$(-)_{\sharp}^+ \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+,$$

natural in  $X, Y \in Obj(Sets)$ .

*Proof. Item* 1, *Functoriality*: We claim that  $(-)^+$  is indeed a functor:

• Preservation of Identities. Let  $X \in Obj(Sets)$ . We have

$$\operatorname{id}_{X}^{+}(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_{X} & \text{if } x = \star_{X}, \end{cases}$$

for each  $x \in X^+$ , so  $id_X^+ = id_{X^+}$ .

• Preservation of Composition. Given morphisms of sets

$$f: X \to Y$$
,  $g: Y \to Z$ ,

we have

$$[g^+ \circ f^+](x) \stackrel{\text{def}}{=} g^+(f^+(x))$$

$$\stackrel{\text{def}}{=} g^+(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^+(x)$$

for each  $x \in X$  and

$$[g^{+} \circ f^{+}](\star_{X}) \stackrel{\text{def}}{=} g^{+}(f^{+}(\star_{X}))$$

$$\stackrel{\text{def}}{=} g^{+}(\star_{Y})$$

$$\stackrel{\text{def}}{=} \star_{Z}$$

$$\stackrel{\text{def}}{=} [g \circ f]^{+}(\star_{X}),$$

so 
$$(g \circ f)^+ = g^+ \circ f^+$$
.

This finishes the proof.

Item 2, Adjointness: We proceed in a few steps:

• *Map I*. We define a map

$$\Phi_{X,Y} : \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

by sending a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

to the function

$$\xi^{\dagger}: X \to Y$$

given by

$$\xi^{\dagger}(x) \stackrel{\mathrm{def}}{=} \xi(x)$$

for each  $x \in X$ .

• Map II. We define a map

$$\Psi_{XY} : \mathsf{Sets}(X,Y) \to \mathsf{Sets}_*(X^+,Y)$$

given by sending a function  $\xi: X \to Y$  to the morphism of pointed sets

$$\xi^{\dagger} \colon (X^+, \star_X) \to (Y, \gamma_0)$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each  $x \in X^+$ .

• *Invertibility I.* Given a morphism of pointed sets

$$\xi \colon (X^+, \star_X) \to (Y, y_0),$$

we have

$$\begin{split} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &= \Psi_{X,Y}(\xi^{\dagger}) \\ &\stackrel{\text{def}}{=} \llbracket x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \end{bmatrix} \\ &= \xi \\ \stackrel{\text{def}}{=} [\text{id}_{\mathsf{Sets}_*(X^+,Y)}](\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^+,Y)}$$
.

• *Invertibility II.* Given a map of sets  $\xi: X \to Y$ , we have

$$\begin{split} \big[ \Phi_{X,Y} \circ \Psi_{X,Y} \big] (\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y} (\Psi_{X,Y} (\xi)) \\ &= \Phi_{X,Y} (\xi^{\dagger}) \\ &= \Phi_{X,Y} (\big[ x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases} \big] \big) \\ &= \big[ x \mapsto \xi(x) \big] \end{split}$$

$$= \xi$$

$$\stackrel{\text{def}}{=} [id_{Sets(X,Y)}](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}$$
.

• Naturality for  $\Phi$ , Part I. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \to (X', x_0'),$$

the diagram

$$\operatorname{\mathsf{Sets}}_*(X'^{,+},Y) \xrightarrow{\Phi_{X',Y}} \operatorname{\mathsf{Sets}}(X',Y)$$

$$f^* \qquad \qquad \downarrow f^*$$

$$\operatorname{\mathsf{Sets}}_*(X^+,Y) \xrightarrow{\Phi_{X,Y}} \operatorname{\mathsf{Sets}}(X,Y)$$

commutes. Indeed, given a morphism of pointed sets  $\xi: X'^{+} \to Y$ , we have

$$[\Phi_{X,Y} \circ f^*](\xi) = \Phi_{X,Y}(f^*(\xi))$$

$$= \Phi_{X,Y}(\xi \circ f)$$

$$= \xi \circ f$$

$$= \Phi_{X',Y}(\xi) \circ f$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= f^*(\Phi_{X',Y}(\xi))$$

$$= [f^* \circ \Phi_{X',Y}(\xi)](\xi).$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

• Naturality for  $\Phi$ , Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \to (Y', y_0'),$$

the diagram

$$\mathsf{Sets}_*(X^+,Y) \xrightarrow{\Phi_{X,Y}} \mathsf{Sets}(X,Y)$$

$$\downarrow^{g_*} \qquad \qquad \downarrow^{g_*}$$

$$\mathsf{Sets}_*(X^+,Y'), \xrightarrow{\Phi_{X,Y'}} \mathsf{Sets}(X,Y')$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^{\dagger}: X^+ \to Y,$$

we have

$$\begin{split} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= g \circ \xi \\ &= g \circ \Phi_{X,Y'}(\xi) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y'}\circ g_*=g_*\circ \Phi_{X,Y'}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

• Naturality for  $\Psi$ . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\Psi$  is also natural in each argument.

This finishes the proof.

*Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums*: We construct the strong monoidal structure on  $(-)^+$  with respect to [] and  $\lor$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{XY}^{+,\coprod}: X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)_{X,Y}^{+,\coprod}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each  $z \in X^+ \vee Y^+$ , with inverse

$$(-)_{X,Y}^{+,\coprod,-1}\colon (X\coprod Y)^+\stackrel{\sim}{\longrightarrow} X^+\vee Y^+$$

given by

$$(-)_{X,Y}^{+,\coprod,-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(1,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each  $z \in (X \mid Y)^+$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,\coprod,\mathbb{1}} \colon \operatorname{pt} \xrightarrow{\sim} \emptyset^+$$

is given by sending  $\star_X$  to  $\star_\emptyset$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on  $(-)^+$  with respect to  $\times$  and  $\wedge$  as follows:

• *The Strong Monoidality Constraints.* The isomorphism

$$(-)_{X,Y}^+ \colon X^+ \wedge Y^+ \xrightarrow{\sim} (X \times Y)^+$$

is given by

$$(-)_{X,Y}^{+}(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_{X} \text{ and } y \neq \star_{Y} \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each  $x \land y \in X^+ \land Y^+$ , with inverse

$$(-)^{+,-1}_{X,Y} \colon (X \times Y)^+ \xrightarrow{\sim} X^+ \wedge Y^+$$

given by

$$(-)_{X,Y}^{+,-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each  $z \in (X \times Y)^+$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{+,1} \colon S^0 \xrightarrow{\sim} \mathsf{pt}^+$$

is given by sending 0 to  $\star_{pt}$  and 1 to  $\star$ , where  $pt^+ = {\star, \star_{pt}}$ .

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

### 01QS 6.4.2 Deleting Basepoints

Let  $(X, x_0)$  be a pointed set.

O1QT Definition 6.4.2.1.1. The set with deleted basepoint associated to X is the set  $X^-$  defined by

$$X^{-} \stackrel{\text{def}}{=} X \setminus \{x_0\}.$$

- **O1QU** Proposition 6.4.2.1.2. Let  $(X, x_0)$  be a pointed set.
- 01QV 1. Functoriality. The assignment  $(X, x_0) \mapsto X^-$  defines a functor

$$X^-: \mathsf{Sets}^{\mathsf{actv}}_* \to \mathsf{Sets},$$

where:

• Action on Objects. For each  $X \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}}_*)$ , we have

$$[(-)^{-}](X) \stackrel{\text{def}}{=} X^{-},$$

where  $X^-$  is the set of Definition 6.4.2.1.1.

• *Action on Morphisms*. For each morphism  $f: X \to Y$  of  $\mathsf{Sets}^\mathsf{actv}_*$ , the image

$$f^-: X^- \to Y^-$$

of f by  $(-)^-$  is the map defined by

$$f^-(x) \stackrel{\text{def}}{=} f(x)$$

for each  $x \in X^-$ .

2. Adjoint Equivalence. We have an adjoint equivalence of categories

$$((-)^- + (-)^+)$$
: Sets\*\*  $(-)^-$  Sets,

witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*(X, Y^+),$$

natural in  $X \in Obj(Sets_*)$  and  $Y \in Obj(Sets)$ , and by isomorphisms

$$(X^-)^+ \cong X,$$

$$(Y^+)^- \cong Y$$
.

once again natural in  $X \in Obj(Sets_*)$  and  $Y \in Obj(Sets)$ .

3. Symmetric Strong Monoidality With Respect to Wedge Sums. The functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-,(-)^{-,\vee},(-)^{-,\vee}_{\mathbb{1}})\colon (\mathsf{Sets}^{\mathsf{actv}}_*,\vee,\mathsf{pt}),\to (\mathsf{Sets}, {\textstyle\coprod}, \emptyset),$$

being equipped with isomorphisms of pointed sets

$$(-)_{X,Y}^{-,\vee} \colon X^{-} \coprod Y^{-} \xrightarrow{\sim} (X \vee Y)^{-},$$
$$(-)_{1}^{-,\vee} \colon \varnothing \xrightarrow{\sim} \mathsf{pt}^{-},$$

natural in  $X, Y \in Obj(Sets)$ .

4. Symmetric Strong Monoidality With Respect to Smash Products. The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$((-)^-, (-)^{-,\times}, (-)^{-,\times}_1) \colon (\mathsf{Sets}^{\mathsf{actv}}_*, \wedge, S^0), \to (\mathsf{Sets}, \times, \mathsf{pt})$$

being equipped with isomorphisms of pointed sets

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \xrightarrow{\sim} (X \wedge Y)^{-},$$
$$(-)^{-}_{1} \colon \operatorname{pt} \xrightarrow{\sim} (S^{0})^{-},$$

natural in  $X, Y \in Obj(Sets)$ .

*Proof. Item* 1, *Functoriality*: We claim that  $(-)^-$  is indeed a functor:

• Preservation of Identities. Let  $X \in Obj(Sets)$ . We have

$$id_X^-(x) \stackrel{\text{def}}{=} x$$

for each  $x \in X^-$ , so  $id_X^- = id_{X^-}$ .

• Preservation of Composition. Given morphisms of pointed sets

$$f: (X, x_0) \to (Y, y_0),$$
  
 $g: (Y, y_0) \to (Z, z_0),$ 

we have

$$[g^{-} \circ f^{-}](x) \stackrel{\text{def}}{=} g^{-}(f^{-}(x))$$

$$\stackrel{\text{def}}{=} g^{-}(f(x))$$

$$\stackrel{\text{def}}{=} g(f(x))$$

$$\stackrel{\text{def}}{=} [g \circ f]^{-}(x)$$

for each  $x \in X$ , so  $(g \circ f)^- = g^- \circ f^-$ .

This finishes the proof.

*Item 2*, *Adjoint Equivalence*: We proceed in a few steps:

025H 1. Map I. We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}(X^-,Y) \to \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)$$

by sending a map  $\xi \colon X^- \to Y$  to the active morphism of pointed sets

$$\xi^{\dagger} \colon X \to Y^{+}$$

given by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^{-}, \\ \star_{Y} & \text{if } x = x_{0}, \end{cases}$$

for each  $x \in X$ , where this morphism is indeed active since  $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$  for all  $x \in X^-$ .

025J 2. Map II. We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \to \mathsf{Sets}(X^-,Y)$$

given by sending an active morphism of pointed sets  $\xi: X \to Y^+$  to the map

$$\xi^{\dagger}: X^{-} \to Y$$

defined by

$$\xi^{\dagger}(x) \stackrel{\text{def}}{=} \xi(x)$$

for each  $x \in X^-$ , which is indeed well-defined (in that  $\xi(x) \in Y$  for all  $x \in X^-$ ) since  $\xi$  is active.

**3.** *Invertibility I.* Given a map of sets  $\xi: X^- \to Y$ , we have

$$\begin{split} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}([x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}]) \\ &= [x \mapsto \xi(x)] \\ &= \xi \\ &= [\text{id}_{\mathsf{Sets}(X^-,Y)}](\xi). \end{split}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-Y)}$$
.

**4.** *Invertibility II.* Given a morphism of pointed sets

$$\xi \colon (X, x_0) \to (Y^+, \star_Y),$$

we have

$$\begin{split} [\Phi_{X,Y} \circ \Psi_{X,Y}](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(\xi)) \\ &= \Phi_{X,Y}(\llbracket x \mapsto \xi(x) \rrbracket) \\ &= \llbracket x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases} \end{bmatrix} \\ &= \xi \\ &= [id_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)}](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+)} .$$

025M 5. *Naturality for* Φ, *Part I*. We need to show that, given a morphism of pointed sets

$$f: (X, x_0) \to (X', x_0'),$$

the diagram

$$\begin{array}{ccc} \mathsf{Sets}(X^{',-},Y) & \xrightarrow{\Phi_{X',Y}} & \mathsf{Sets}^{\mathsf{actv}}_*(X',Y^+) \\ & & & \downarrow^{f^*} & & \downarrow^{f^*} \\ \mathsf{Sets}_*(X^-,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \end{array}$$

commutes. Indeed, given a map of sets  $\xi: X' \to Y$ , we have

$$\begin{split} [\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\ &= \Phi_{X,Y}(\xi \circ f) \\ &= [\![x \mapsto \begin{cases} \xi(f(x)) & \text{if } f(x) \in X'^{,-} \\ \star_Y & \text{if } f(x) = x'_0 \end{cases} ]\!] \\ &= f^*([\![x' \mapsto \begin{cases} \xi(x') & \text{if } x' \in X'^{,-} \\ \star_Y & \text{if } x' = x'_0 \end{cases} ]\!]) \\ &= f^*(\Phi_{X',Y}(\xi)) \\ &= [f^* \circ \Phi_{X',Y}](\xi). \end{split}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for  $\Phi$  above indeed commutes.

025N 6. Naturality for Φ, Part II. We need to show that, given a morphism of pointed sets

$$g: (Y, y_0) \to (Y', y_0'),$$

the diagram

$$\begin{array}{ccc} \mathsf{Sets}(X^-,Y) & \xrightarrow{\Phi_{X,Y}} & \mathsf{Sets}^{\mathsf{actv}}_*(X,Y^+) \\ & & & \downarrow^{g_*} & & \downarrow^{g_*} \\ \mathsf{Sets}(X^-,Y') & \xrightarrow{\Phi_{X,Y'}} & \mathsf{Sets}^{\mathsf{actv}}_*(X,Y'^{,+}) \end{array}$$

commutes. Indeed, given a map of sets  $\xi: X^- \to Y$ , we have

$$\begin{split} [\Phi_{X,Y'} \circ g_*](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= [\![x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}]\!] \\ &= g_*([\![x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases}]\!]) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= [g_* \circ \Phi_{X,Y'}](\xi). \end{split}$$

Therefore we have

$$\Phi_{XY'} \circ g_* = g_* \circ \Phi_{XY'},$$

and the naturality diagram for  $\Phi$  above indeed commutes.

- 7. Naturality for  $\Psi$ . Since  $\Phi$  is natural in each argument and  $\Phi$  is a componentwise inverse to  $\Psi$  in each argument, it follows from Categories, Item 2 of Definition 11.9.7.1.2 that  $\Psi$  is also natural in each argument.
- 8. Fully Faithfulness of  $(-)^-$ . We aim to show that the assignment  $f \mapsto f^-$  sets up a bijection

$$(-)_{X,Y}^- \colon \mathsf{Sets}^{\mathsf{actv}}_*(X,Y) \overset{\sim}{\dashrightarrow} \mathsf{Sets}(X^-,Y^-).$$

Indeed, the inverse map

$$(-)_{XY}^{-,-1} : \mathsf{Sets}(X^-, Y^-) \xrightarrow{\sim} \mathsf{Sets}^{\mathsf{actv}}_*(X, Y)$$

is given by sending a map of sets  $f: X^- \to Y^-$  to the active morphism of pointed sets  $f^{\dagger}: X \to Y$  defined by

$$f^{\dagger}(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X^{-}, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each  $x \in X$ .

9. Essential Surjectivity of  $(-)^-$ . We need to show that, given an object  $X \in \text{Obj}(\mathsf{Sets})$ , there exists some  $X' \in \text{Obj}(\mathsf{Sets}^{\mathsf{actv}})$  such that  $(X')^- \cong X$ . Indeed, taking  $X' = X^+$ , we have

$$(X^+)^- \stackrel{\text{def}}{=} (X \cup \{\star_X\})^-$$
$$\stackrel{\text{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\}$$
$$= X.$$

and thus we have in fact an *equality*  $(X^+)^- = X$ , showing  $(-)^-$  to be essentially surjective.

025S 10. *The Functor* (–)<sup>–</sup> *Is an Equivalence*. Since (–)<sup>–</sup> is fully faithful and essentially surjective, it is an equivalence by Categories, Item 1 of Definition 11.6.7.1.2.

This finishes the proof.

*Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums*: We construct the strong monoidal structure on (-)<sup>-</sup> with respect to  $\lor$  and  $\coprod$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)^{-,\vee}_{X,Y}\colon X^-\coprod Y^-\stackrel{\sim}{\dashrightarrow} (X\vee Y)^-$$

is given by

$$(-)_{X,Y}^{-,\vee}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each  $z \in X^- \coprod Y^-$ , with inverse

$$(-)_{X,Y}^{-,\vee,-1} \colon (X \vee Y)^{-} \xrightarrow{\sim} X^{-} \coprod Y^{-}$$

given by

$$(-)_{X,Y}^{-,\vee,-1}(z) \stackrel{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each  $z \in (X \vee Y)^-$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{XY}^{+,\vee,1}: \not O \xrightarrow{\sim} pt^-$$

is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^-$  into a symmetric strong monoidal functor is omitted. *Item 4, Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on  $(-)^+$  with respect to  $\wedge$  and  $\times$  as follows:

• The Strong Monoidality Constraints. The isomorphism

$$(-)_{X,Y}^- \colon X^- \times Y^- \xrightarrow{\sim} (X \wedge Y)^-$$

is given by

$$(-)^-_{X,Y}(x,y) = x \wedge y$$

for each  $(x, y) \in X^- \times Y^-$ , with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \xrightarrow{\sim} X^{-} \times Y^{-}$$

given by

$$(-)_{X,Y}^{-,-1}(x \wedge y) \stackrel{\text{def}}{=} (x,y)$$

for each  $x \wedge y \in (X \wedge Y)^-$ .

• The Strong Monoidal Unity Constraint. The isomorphism

$$(-)_{X,Y}^{-,\mathbb{1}} : \operatorname{pt} \xrightarrow{\sim} (S^0)^-$$

is given by sending  $\star$  to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor  $(-)^+$  into a symmetric strong monoidal functor is omitted.

# **Appendices**

## **A** Other Chapters

_			
Dre	lim	ina	rinc

- 1. Introduction
- 2. A Guide to the Literature

#### Sets

- 3. Sets
- 4. Constructions With Sets
- Monoidal Structures on the Category of Sets
- 6. Pointed Sets
- 7. Tensor Products of Pointed Sets

## Relations

- 8. Relations
- 9. Constructions With Relations

#### 10. Conditions on Relations

#### Categories

- 11. Categories
- 12. Presheaves and the Yoneda Lemma

#### **Monoidal Categories**

13. Constructions With Monoidal Categories

#### **Bicategories**

14. Types of Morphisms in Bicategories

#### Extra Part

15. Notes

# References

[MSE 2855868] Qiaochu Yuan. Is the category of pointed sets Cartesian closed? Mathematics Stack Exchange. URL: https://math.stackexchange.

com/q/2855868 (cit. on pp. 6, 14).

[MSE 884460] Martin Brandenburg. Why are the category of pointed sets and

the category of sets and partial functions "essentially the same"? Mathematics Stack Exchange. URL: https://math.stackexchange.

com/q/884460 (cit. on p. 6).