Constructions With Monoidal Categories

The Clowder Project Authors

July 29, 2025

01UF This chapter contains some material on constructions with monoidal categories.

Contents

01UG 13.1 Moduli Categories of Monoidal Structures

01UH 13.1.1 The Moduli Category of Monoidal Structures on a Category

Let $\mathcal{C}$ be a category.

01UJ **Definition 13.1.1.1.1 ▶ The Moduli Category of Monoidal Structures on a Category**

The **moduli category of monoidal structures on** C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(C) \overset{\text{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, \qquad \begin{tikzcd} \mathcal{M}_{\mathbb{E}_1}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{MonCats} \arrow[d, "忘"] \\ \mathsf{pt} \arrow[r, "{[C]}"'] & \mathsf{Cats}. \end{tikzcd}$$

01UK **Remark 13.1.1.1.2 ▶ Unwinding Definition 13.1.1.1.1, I**

In detail, **the moduli category of monoidal structures on** C is the category $\mathcal{M}_{\mathbb{E}_1}(C)$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(C)$ are monoidal categories $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ whose underlying category is C.
- *Morphisms.* A morphism from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ is a strong monoidal functor structure

$$\begin{aligned} \mathrm{id}^{\otimes}_{C} &\colon A \boxtimes_C B \overset{\sim}{\dashrightarrow} A \otimes_C B, \\ \mathrm{id}^{\otimes}_{\mathbb{1}|C} &\colon \mathbb{1}'_C \overset{\sim}{\dashrightarrow} \mathbb{1}_C \end{aligned}$$

  on the identity functor $\mathrm{id}_C \colon C \to C$ of C.
- *Identities.* For each $M \overset{\text{def}}{=} (C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C) \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(C))$, the unit map

$$\mathbb{1}^{\mathcal{M}_{\mathbb{E}_1}(C)}_{M,M} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(C)}(M, M)$$

  of $\mathcal{M}_{\mathbb{E}_1}(C)$ at M is defined by

$$\mathrm{id}^{\mathcal{M}_{\mathbb{E}_1}(C)}_{M} \overset{\text{def}}{=} (\mathrm{id}^{\otimes}_{C}, \mathrm{id}^{\otimes}_{\mathbb{1}|C}),$$

  where $\left(\mathrm{id}^{\otimes}_{C}, \mathrm{id}^{\otimes}_{\mathbb{1}|C}\right)$ is the identity monoidal functor of C of **??**.

- *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the composition map

$$\circ^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,N,P} \colon \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(N,P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M,N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M,P)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at (M,N,P) is defined by

$$\left(\mathrm{id}^{\otimes,\prime}_{\mathcal{C}}, \mathrm{id}^{\otimes,\prime}_{\mathbb{1}|\mathcal{C}}\right) \circ^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,N,P} \left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right) \overset{\text{def}}{=} \left(\mathrm{id}^{\otimes,\prime}_{\mathcal{C}} \circ \mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes,\prime}_{\mathbb{1}|\mathcal{C}} \circ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right).$$

01UL **REMARK 13.1.1.1.3 ▶ UNWINDING DEFINITION 13.1.1.1.1, II**

In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ satisfies the following conditions:

01UM 1. *Naturality.* For each pair $f\colon A \to X$ and $g\colon B \to Y$ of morphisms of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
A \boxtimes_{\mathcal{C}} B \arrow[r, "f \boxtimes_{\mathcal{C}} g"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B}"'] & X \boxtimes_{\mathcal{C}} Y \arrow[d, "\mathrm{id}^{\otimes}_{X,Y}"] \\
A \otimes_{\mathcal{C}} B \arrow[r, "f \otimes_{\mathcal{C}} g"'] & X \otimes_{\mathcal{C}} Y
\end{tikzcd}$$

commutes.

01UN 2. *Monoidality.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] \arrow[dr, "\alpha^{\mathcal{C},\prime}_{A,B,C}"] & \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"'] & & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[dr, "\alpha^{\mathcal{C}}_{A,B,C}"'] & & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[dl, "\mathrm{id}^{\otimes}_{A, B \otimes_{\mathcal{C}} C}"] \\
 & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) &
\end{tikzcd}$$

commutes.

01UP 3. *Left Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}'_{\mathcal{C}},A}"] & \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A \arrow[dr, "\lambda^{\mathcal{C}}_{A}"] & \\ \mathbb{1}'_{\mathcal{C}} \boxtimes_{\mathcal{C}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_{\mathcal{C}} \mathrm{id}_{A}"] \arrow[rrr, "\lambda^{\mathcal{C},\prime}_{A}"'] & & & A \end{tikzcd}$$

commutes.

01UQ 4. *Right Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd} & A \boxtimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes}_{A,\mathbb{1}'_{\mathcal{C}}}"] & A \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[dr, "\rho^{\mathcal{C}}_{A}"] & \\ A \boxtimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[ur, "\mathrm{id}_{A} \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[rrr, "\rho^{\mathcal{C},\prime}_{A}"'] & & & A \end{tikzcd}$$

commutes.

01UR **PROPOSITION 13.1.1.1.4 ► PROPERTIES OF THE MODULI CATEGORY OF MONOIDAL STRUCTURES ON A CATEGORY**

Let $\mathcal{C}$ be a category.

01US 1. *Extra Monoidality Conditions.* Let $\left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$.

01UT (a) The diagram

$$\begin{tikzcd} (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_{C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A \boxtimes_{\mathcal{C}} B, C}"'] & (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"] \\ (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_{C}"'] & (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \end{tikzcd}$$

commutes.

01UU (b) The diagram

$$\begin{tikzcd}
A \boxtimes_C (B \boxtimes_C C) \arrow[r, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B\boxtimes_C C}"'] & A \boxtimes_C (B \otimes_C C) \arrow[d, "\mathrm{id}^{\otimes}_{A,B\otimes_C C}"] \\
A \otimes_C (B \boxtimes_C C) \arrow[r, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"'] & A \otimes_C (B \otimes_C C)
\end{tikzcd}$$

commutes.

01WB 2. *Extra Monoidal Unity Constraints.* Let $\left(\mathrm{id}^{\otimes}_{C}, \mathrm{id}^{\otimes}_{\mathbb{1}|C}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ to $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$.

01WC (a) The diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^C_{\mathbb{1}'_C}"'] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

commutes.

01WD (b) The diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^C_{\mathbb{1}'_C}"'] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

commutes.

01WE (c) The diagram

$$\begin{tikzcd}
\mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"'] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"] \\
\mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes.

01WF (d) The diagram

$$\begin{tikzcd}
\mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^{C}_{\mathbb{1}'_C}"] \\
\mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes.

01UV 3. *Mixed Associators.* Let $(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C)$ and $(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime})$ be monoidal structures on C and let

$$\mathrm{id}^{\otimes}_{-_1,-_2} \colon -_1 \boxtimes_C -_2 \to -_1 \otimes_C -_2$$

be a natural transformation.

01UW (a) If there exists a natural transformation

$$\alpha^{\otimes}_{A,B,C} \colon (A \otimes_C B) \boxtimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{tikzcd}
(A \otimes_C B) \boxtimes_C C \arrow[r, "\alpha^{\otimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A\otimes_C B,C}"'] & A \otimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[r, "\alpha^{C}_{A,B,C}"'] & A \otimes_C (B \otimes_C C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_C B) \boxtimes_C C \arrow[r, "\alpha^{C,\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_{A,B \boxtimes_C C}"] \\
(A \otimes_C B) \boxtimes_C C \arrow[r, "\alpha^{\otimes}_{A,B,C}"'] & A \otimes_C (B \boxtimes_C C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UX (b) If there exists a natural transformation

$$\alpha^{\boxtimes}_{A,B,C} \colon (A \boxtimes_C B) \otimes_C C \to A \boxtimes_C (B \otimes_C C)$$

making the diagrams

$$\begin{tikzcd}
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C"'] & A \boxtimes_C (B \otimes_C C) \arrow[d, "\mathrm{id}^{\otimes}_{A,B \otimes_C C}"] \\
(A \otimes_C B) \otimes_C C \arrow[r, "\alpha^{C}_{A,B,C}"'] & A \otimes_C (B \otimes_C C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_C B) \boxtimes_C C \arrow[r, "\alpha^{C,\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A \boxtimes_C B,C}"'] & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes}_{A,B,C}"'] & A \boxtimes_C (B \otimes_C C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01UY (c) If there exists a natural transformation

$$\alpha^{\boxtimes,\otimes}_{A,B,C} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$
\begin{tikzcd}
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes,\otimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C"'] & A \otimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[r, "\alpha^{C}_{A,B,C}"'] & A \otimes_C (B \otimes_C C)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(A \boxtimes_C B) \boxtimes_C C \arrow[r, "\alpha^{C,\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A \boxtimes_C B, C}"'] & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}^{\otimes}_{A, B \boxtimes_C C}"] \\
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes,\otimes}_{A,B,C}"'] & A \otimes_C (B \boxtimes_C C)
\end{tikzcd}
$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

Proof 13.1.1.1.5 ▶ Proof of Proposition 13.1.1.1.4

Item 1: Extra Monoidality Conditions

We claim that Items 1a and 1b are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}^{\otimes}_{A,B}$ and id_C.

2. *Proof of Item 1b:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms id_A and $\mathrm{id}^{\otimes}_{B,C}$.

This finishes the proof.

Item 2: Extra Monoidal Unity Constraints

We claim that Items 2a and 2b are indeed true:

1. *Proof of Item 1a*: Indeed, consider the diagram

$$
\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[dr, "\mathrm{id}_{\mathbb{1}_C} \otimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[dd, "\lambda^C_{\mathbb{1}'_C}"'] & & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[d, "\mathrm{id}_{\mathbb{1}_C} \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[ddd, bend left=60, "\rho^{C,\prime}_{\mathbb{1}_C}"] \arrow[ddd, phantom, bend left=30, "(4)"] \\
& \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[ur, phantom, "(1)"] \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dr, equal] \arrow[dr, phantom, "(2)", shift left=3ex] & \mathbb{1}_C \boxtimes_C \mathbb{1}_C \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}"] \\
\mathbb{1}'_C \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] \arrow[rr, phantom, "(3)"] & & \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[d, "\lambda^C_{\mathbb{1}_C} = \rho^C_{\mathbb{1}_C}"] \\
& & \mathbb{1}_C
\end{tikzcd}
$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}_{C}^{\otimes,-1}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^{C}, where the equality $\rho^{C}_{\mathbb{1}_C} = \lambda^{C}_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $(\mathrm{id}_C, \mathrm{id}^{\otimes}_C, \mathrm{id}^{\otimes}_{C|\mathbb{1}})$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b*: Indeed, consider the diagram

$$
\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}'_C,\mathbb{1}_C}"] \arrow[dr, "\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_C \mathrm{id}_{\mathbb{1}_C}"] \arrow[dd, "\rho^C_{\mathbb{1}'_C}"'] & \arrow[d, phantom, "(1)"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_C \mathrm{id}_{\mathbb{1}_C}"] \arrow[ddd, rounded corners, to path={-- ([xshift=18ex]\tikztostart.east) |- (\tikztotarget.east) \tikztonodes}, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \\
 & \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dr, equal] \arrow[d, phantom, "(3)"] & \mathbb{1}_C \boxtimes_C \mathbb{1}_C \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dl, phantom, "(2)"] \\
\mathbb{1}'_C \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}}"] & {} & \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}"] \arrow[r, phantom, "(4)"] & {} \\
 & & \mathbb{1}_C
\end{tikzcd}
$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{C}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^{C}, where the equality $\rho^{C}_{\mathbb{1}_C} = \lambda^{C}_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $(\mathrm{id}_C, \mathrm{id}^{\otimes}_{C}, \mathrm{id}^{\otimes}_{C|\mathbb{1}})$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c*: Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"'] \arrow[dr, phantom, "(1)"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C.
\end{tikzcd}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"] \\
& \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}$ is an isomorphism, it follows that the diagram $(\dagger)$ also commutes, and we are done.

4. *Proof of Item 2d*: Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^C_{\mathbb{1}'_C}"'] \arrow[dr, phantom, "(1)"] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^C_{\mathbb{1}'_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^C_{\mathbb{1}'_C}"] \\
 & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}}$ is an isomorphism, it follows that the diagram (†) also commutes, and we are done.

This finishes the proof.

Item 3: Mixed Associators

We claim that Items 3a to 3c are indeed true:

01UZ 1. *Proof of Item 3a*: We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:

$$
\begin{tikzcd}
 & (A \boxtimes_C B) \boxtimes_C C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[dr, "\alpha^{C}_{A,B,C}"] \arrow[dd, "\mathrm{id}^{\otimes}_{A \boxtimes_C B, C}"] & \\
(A \otimes_C B) \boxtimes_C C \arrow[dd, "\mathrm{id}^{\otimes}_{A \otimes_C B, C}"'] \arrow[r, phantom, "(1)"] & & A \boxtimes_C (B \boxtimes_C C) \arrow[dd, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \arrow[l, phantom, "(2)"] \\
 & (A \boxtimes_C B) \otimes_C C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C"'] \arrow[dr, "\alpha^{\otimes}_{A,B,C}"] & \\
(A \otimes_C B) \otimes_C C \arrow[dr, "\alpha^{C,\prime}_{A,B,C}"'] \arrow[rr, phantom, "(3)"] & & A \boxtimes_C (B \otimes_C C) \arrow[dl, "\mathrm{id}^{\otimes}_{A, B \otimes_C C}"] \\
 & A \otimes_C (B \otimes_C C) &
\end{tikzcd}
$$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01V0 2. *Proof of Item 3b*: We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:

$$
\begin{tikzcd}
 & (A \boxtimes_C B) \boxtimes_C C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[dr, "\alpha^{C}_{A,B,C}"] & \\
(A \otimes_C B) \boxtimes_C C \arrow[dr, dashed, "\alpha^{\boxtimes}_{A,B,C}"] \arrow[dd, "\mathrm{id}^{\otimes}_{A \otimes_C B,C}"'] & (1) & A \boxtimes_C (B \boxtimes_C C) \arrow[dl, dashed, "\mathrm{id}^{\otimes}_{A,B \boxtimes_C C}"'] \arrow[dd, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
 & A \otimes_C (B \boxtimes_C C) \arrow[dd, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(3)"] & \\
(A \otimes_C B) \otimes_C C \arrow[dr, "\alpha^{C,\prime}_{A,B,C}"'] & & A \boxtimes_C (B \otimes_C C) \arrow[dl, "\mathrm{id}^{\otimes}_{A,B \otimes_C C}"] \\
 & A \otimes_C (B \otimes_C C) &
\end{tikzcd}
$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

01V1 3. *Proof of Item 3c*: We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Remark 13.1.1.1.3 as follows:

$$
\begin{tikzcd}
 & (A \boxtimes_C B) \boxtimes_C C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[dr, "\alpha^C_{A,B,C}"] & \\
(A \otimes_C B) \boxtimes_C C \arrow[d, "\mathrm{id}^{\otimes}_{A\otimes_C B,C}"'] \arrow[drr, "\alpha^{\boxtimes,\otimes}_{A,B,C}"] & (1) & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[dr, "\alpha^{C,\prime}_{A,B,C}"'] & (2) & A \boxtimes_C (B \otimes_C C) \arrow[dl, "\mathrm{id}^{\otimes}_{A,B\otimes_C C}"] \\
 & A \otimes_C (B \otimes_C C) &
\end{tikzcd}
$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Remark 13.1.1.1.3.

This finishes the proof.

01V2 13.1.2 The Moduli Category of Braided Monoidal Structures on a Category

01V3 13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category

01V4 13.2 Moduli Categories of Closed Monoidal Structures

01V5 13.3 Moduli Categories of Refinements of Monoidal Structures

01V6 13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

Appendices

A Other Chapters