The Clowder Project

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Part I

Preliminaries

Chapter 1

Introduction

This chapter contains some general information about the Clowder Project.

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1.1 Introduction

1.1.1 Project Description and Goals

In short, the Clowder Project is an online reference work and wiki for category theory and mathematics that aims to essentially become a Stacks Project for category theory.

The project arose from a desire to improve upon a number of issues with the existing category theory literature, as well as fill several gaps in it.

In this section, we list and discuss the goals of the Clowder Project.

1.1.1.1 Provide a Unified and Complete Reference for Category Theory

The category theory literature is at times rather fragmented, and often it takes a long while for book-long treatments on a given subject to appear.

For example, although the theory of bicategories dates back to the late 1960s, it was not until 2020 that the subject would receive its first textbook in the topic, namely [JY21].

The Clowder Project aims to bridge this gap, providing a complete overview of the foundational material on category theory (see also Section 1.2.1).

1.1.1.2 Gather Hard to Find Results

As an extension of the previous goal, the Clowder Project also aims to gather in a single place results that are hard to find in the literature. These tend to be recorded only on original sources, which often means papers, notes or theses from the 1970s.

Since the Clowder Project is organized as a wiki, it becomes rather easy to search and find such results, as one merely needs to go to the page for a given concept and then look at the properties listed there.

1.1.1.3 Elaborate on Details That Are Often Left Out

Another goal of the Clowder Project is to include all kinds of details and intuitions that often don't make their way into textbooks, papers, monographs, etc.

For instance, one sometimes finds claims that a given diagram commutes and that it is "easy" to fill in the details. This also tends to happen particularly when the details are rather unwieldy.

One of the goals of the Clowder Project is to provide such proofs in great detail, including discussions of technical results, even when these are indeed “obvious”.

1.1.1.4 Homogenize Conventions, Notation, and Terminology

Another issue with practice in the field is that there are often a number of conflicting conventions, notations, and terminology.

Being organized as a comprehensive and encyclopedic wiki, the Clowder Project tries to homogenize these conventions, notations, and terminology.

1.1.1.5 Fill Gaps in the Category Theory Literature

There are quite a few significant gaps in the category theory literature, some of which we hope to fill with the Clowder Project. For a list of (some of) these gaps, see Section 1.3.4.

1.1.1.6 Provide a Citable Reference for All Kinds of Results

It is a common situation to require a well-known result for a paper. Although proving it might be straightforward, it is often more convenient to cite a reference instead. Finding such a reference, however, may be hard and/or time-consuming.

With its encyclopedic nature, the Clowder Project hopes to serve as that convenient reference.

1.1.2 Navigating the Clowder Project

Hopefully, it should be intuitive to navigate through the web version of project. Nevertheless, here we mention a couple things that might be useful to know.

1.1.2.1 Preferences

You can change the font of the site, the style of the PDFs, as well as turn on dark mode by clicking the gear button located at the top right corner of the page.

1.1.2.2 Large Diagrams and the Zoom in Feature

This work features many diagrams that are unfortunately a bit too large to be comfortably legible in their native size.

To compensate for this, it's possible to click on them to expand their size by 200%.

In addition, you may also right-click on diagrams and then select "Open image in new tab" to allow for even higher amounts of zoom.

1.1.2.3 PDF Styles

The PDFs for each chapter as well as for the whole book are generated using twelve different styles, as summarised in the following table:

Typeface	Theorem Environments
Alegreya Sans	`tcbthm`
Alegreya	`tcbthm`
EB Garamond	`tcbthm`
Crimson Pro	`tcbthm`
XCharter	`tcbthm`
Computer Modern	`tcbthm`
Alegreya Sans	`amsthm`
Alegreya	`amsthm`
EB Garamond	`amsthm`
Crimson Pro	`amsthm`
XCharter	`amsthm`
Computer Modern	`amsthm`

The default style uses Alegreya Sans and `tcbthm`.

1.1.3 Prerequisites/Assumed Background

The Clowder Project assumes at least a background on basic category theory corresponding to e.g. [Rie16], as well as some comfort in working with category-theoretic notions.

In particular, it should be viewed as a reference work/wiki, and *not* as a textbook. This, however, doesn't mean it shouldn't be pedagogical. Indeed, a number of stylistic choices are made aiming to make the material as easily digestible as possible.

For an outline of several introductory references for different topics in category theory, see A Guide to the Literature.

1.1.4 Community Engagement, Contributions and Collaboration

All kinds of feedback and contributions to the Clowder Project are extremely welcome: pointing out typos, errors, historical remarks, references, layout of webpages, spelling errors, improvements to the overall structure, missing lemmas, etc.

The Clowder Project has an official Discord server in which people can ask questions, carry out discussions and give feedback. Please join it if you'd like to contribute to the Clowder Project. Alternatively, you may also reach out to the project maintainer at emily.de.oliveira.santos.tmf@gmail.com.

1.1.4.1 How to Contribute

There's a number of ways to contribute to the Clowder Project, some of which will be detailed a bit below. However, please keep in mind that they are not just examples, and are most definitely not meant to be exhaustive.

If there's another way in which you'd like to contribute, by all means feel free to drop by the project's Discord (or, alternatively, reach out to the project maintainer).

1.1.4.2 Ways to Contribute: Missing Proofs

There is a large number of missing proofs in the project, ranging from trivial proofs to simple lemmas to more involved results.

Missing proofs are listed in Section 1.3.1.

Note: The following chapters are undergoing revision. If you're interested in contributing, please disregard them for now:

- Relations
- Constructions With Relations
- Conditions on Relations
- Categories
- Constructions With Monoidal Categories
- Types of Morphisms in Bicategories

1.1.4.3 Ways to Contribute: Missing Examples

New examples to the Clowder Project are always welcome. These could be examples illustrating a new concept, examples showing why certain conditions are necessary in a given proof, counterexamples to be aware of, etc.

Some examples which would be particularly nice to have in Clowder are listed in Section 1.3.2. Please do keep in mind however that ***all examples are welcome***, even if they fall outside the examples listed in Section 1.3.2.

1.1.4.4 Ways to Contribute: Questions

A number of questions appear throughout the Clowder Project; tackling these would be an amazing way to contribute to the project.

The questions appearing throughout the Clowder Project are listed in Section 1.3.3.

1.1.5 Frequently Asked Questions

1.1.5.1 How does Clowder differ from the nLab?

Clowder is meant to be much more comprehensive than the nLab, which includes even filling a number of gaps in the category theory literature. Additionally, it also has a different set of goals and stylistic choices. For a more in-depth explanation, see Section 1.1.1.

1.1.5.2 Why not just use the nLab instead?

There are a number of reasons why Clowder was built as a separate project, instead of e.g. just editing the nLab:

1. *Curation.* All content on Clowder is personally curated by the project maintainer. This ensures an even quality to everything in the project.

2. *Cohesion.* As a consequence of Item 1, the Clowder Project ends up being much more cohesive than the nLab, having a clear and coherent organization, consistent notation and conventions, as well as a consistent style.

3. *Referenceability.* Clowder employs Gerby's Tag system, meaning that every citable statement in Clowder (e.g. definitions, examples, constructions, propositions, remarks, even individual items in lists,

etc.) carries a corresponding tag.

This makes the project easy to cite and reference, since although the numbering of e.g. a given definition may change, its associated tag will forever be the same. See also Clowder — The Tag System.

4. *Crowdsourcing and Crowdfunding.* Clowder is meant to be a crowdfunded project in which the community can help directly finance its development. As a result, the project has a dedicated project maintainer whose role is to continuously take care of the project, coordinating contributions, developing infrastructure, and expanding the content of the project.

5. *Infrastructure.* The Clowder Project makes use of several very specific features which simply wouldn't be possible to implement in the nLab. This includes:

   (a) An elaborate fork of `gerby-website`, implementing a variety of new features and quality-of-life additions.

   (b) Another elaborate fork, this time of Gerby (which is itself a fork of plasTeX), implement a number of similarly needed features for the website to work as intended.

   See Section 1.2.3.2 for a (slightly) more in-depth description of the features and additions that have been created specifically for Clowder.

1.1.6 Goodies

In this section we list a few sample nice results and things from the Clowder Project.

1.1.6.1 General Utility

- Section 15.1 contains several `tikz-cd` snippets producing somewhat-hard-to-draw diagrams. Examples include cube, pentagon, and hexagon diagrams, as well as e.g. co/product diagrams with perfectly circular arrows.

1.1.6.2 Set Theory Through a Categorical Lens

Sets:

- Section 4.4.7 contains a discussion of internal Homs in powersets viewed as categories.
- More generally, Section 4.4 discusses several properties of powersets that are analogous to those of presheaf categories.
- **??** discusses the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ induced by a function $f\colon X \to Y$.
- Section 4.6.4 constructs a kind of "six functor formalism for (power)sets".
- Monoidal Structures on the Category of Sets contains explicit proofs that product/coproduct of sets form a monoidal structure.
- Section 5.1.10 gives a completely 1-categorical proof of the universal property of $(\mathsf{Sets}, \times, \mathrm{pt})$.

Pointed Sets:

- Tensor Products of Pointed Sets constructs several tensor products of pointed sets, including a few unusual ones giving rise to skew monoidal structures on Sets_*.
- Section 7.5.10 gives a completely 1-categorical proof of the universal property of $\left(\mathsf{Sets}_*, \wedge, S^0\right)$.
- Definition 7.5.12.1.1 contains a description of comonoids in Sets_* with respect to $\wedge$.

Relations:

- Section 8.5 contains a discussion of several properties of the 2-category of relations like descriptions of internal adjunctions and internal monads.
- Sections 8.8 and 8.9 contains a discussion of two skew monoidal structures on the category $\mathbf{Rel}(A, B)$ of relations from a set A to a set B.
- Definition 15.2.1.1.8 contains a description of left/right Kan extensions and lifts internal to the 2-category of relations.

1.1.6.3 Category Theory

- Categories contains a description of several properties of functors, including somewhat lesser known ones such as dominant functors or pseudoepic functors.

1.2 Project Overview

1.2.1 Content and Scope

In this section, we outline what content is expected to be covered in the Clowder Project.

1.2.1.1 Elementary Category Theory

First and foremost, the Clowder Project aims to cover the foundations of category theory. This comprises all the usual topics treated in basic textbooks in category theory, such as [Mac98] or [Rie16], like adjunctions, co/limits, Kan extensions, co/ends, monoidal categories, etc.

1.2.1.2 Variants of Category Theory

Second, the Clowder Project aims to cover variants of category theory such as internal, fibred, or enriched category theory. The literature on these topics is often quite scattered and scarce, and so having a comprehensive discussion of them in Clowder aims to fill a large gap in the literature. See also Definition 1.3.4.1.14.

1.2.1.3 Higher Category Theory

Third, a detailed presentation of the theories of bicategories and double categories is planned, along with *some* material on tricategories.

Bicategories are another topic for which the literature is rather scattered, and, for some topics, scarce. As mentioned in the introduction, only recently has a proper textbook on bicategories appeared, [JY21]. Moreover, one finds several gaps in the literature, with a number of important results missing. As one particular example, one could look at the theory of 2-dimensional co/ends, in which case a comprehensive treatment based upon lax/oplax/pseudo dinatural transformations seems to be missing.

All of the elementary and not-so-elementary topics in the theory of bicategories are planned to appear in Clowder, and the same holds true for the theory of double categories.

1.2.1.4 ∞-Categories

Lastly, some material on ∞-categories is planned, although the precise scope of this remains to be defined. Ideally, this would include both model categories as well as synthetic and concrete models for ∞-categories (e.g.

quasicategories, complete Segal spaces, cubical quasicategories, etc.).

In this way, we view Clowder as a good *complement* to [Lur25].

1.2.1.5 Other Topics

Occasionally, material on topics not a-priori related to category theory will be included. This may be done for a variety of reasons, including:

- Illustrating general theory.
- Comparison with classical concepts, such as e.g. ionads vs.topological spaces.
- Providing a more consistent and unified treatment of a particular topic, with hyperlinks to relevant concepts or examples.

1.2.2 Style

The Clowder Project makes several unusual stylistic choices, aligned with its goals.

1.2.2.1 Presentation of Topics

The presentation of topics is encyclopedic, non-linear, and sometimes idiosyncratic.

In particular, there's some amount of repetition throughout the project. This is a result of simultaneously wanting to cover as much material as possible while still allowing Clowder to be used as an online reference work/wiki.

1.2.2.2 Provable Items Come With Proofs

Every proposition, theorem, lemma, etc. needs to come with a proof. In case a proof has not been written yet, it shall read as "Omitted". This is to ensure results without proof are clearly labelled as such.

1.2.2.3 Proper Justification of Proofs

Every proof must read either "Omitted" or be properly justified, no matter how trivial the details are.

Expressions like "it is clear that", "it is straightforward to show that", "it is obvious", etc. inside proofs should not be used.

1.2.3 Infrastructure and Technical Implementation

1.2.3.1 Removed Features (in Comparison With the Stacks Project)

A few features present in the general infrastructure of the Stacks Project were removed in Clowder, including:

1. The python back-end, in favour of static pages.
2. The comment system, as a result of the static nature of the website.

1.2.3.2 Gerby and the Tags System

Clowder is built using Gerby, similarly to the Stacks Project. However, a number of additional features and quality-of-life additions not implemented in plasTeX or Gerby were required by Clowder, including:

1. Clowder uses `tcbtheorem`-like environments, which affects the placement of footnotes (which are often used).
2. Clowder implements a dangerous bend symbol to help visually highlight warnings (example).
3. There are a few aesthetic changes in Clowder's HTML/CSS structure, including font selection as well as a dark mode.
4. `tikz-cd` diagrams are very frequently used, and they need to be separately compiled and converted to `svg` files.
5. Code in Clowder can be copied easily using a "Copy" button, with code for bibliography entries also having proper syntax highlighting (example).
6. Clowder is automatically built using GitHub actions.
7. Non-sectioning tags are rendered differently and shown in context (example).

These have been implemented using a fork of Gerby along with a few build scripts.

1.2.3.3 Placeholder Symbols and Future Style

Currently, a number of macros have been defined using placeholder symbols, and look very ugly as a result.

They will eventually be replaced with proper symbols coming from the math fonts of Hundar, a free and open-source typeface project currently being worked on.

Hundar

A free, open-source typeface optimized
for academic, scientific, and book writing.

You can find more details about Hundar at its GitHub repository or website.

1.3 Lists

1.3.1 List of Omitted Proofs

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>
> ---
>
> □□□□□□□ □□□□□□□□□□□
> Truth does not do as much good in the world as the appearance of truth does evil.
>
> ---
>
> *Daniil Dankovsky*

There's a very large number of omitted proofs throughout these notes. In this section we list them in order of decreasing importance.

- If a proof relies on material that has yet to be developed on Clowder, we mark it by a ⚠ sign. If you're interested in contributing, please disregard those for now.
- The following chapters are undergoing revision. If you're interested in contributing, please disregard them for now:

  - Relations
  - Constructions With Relations
  - Conditions on Relations
  - Categories
  - Constructions With Monoidal Categories
  - Types of Morphisms in Bicategories

- This list is under construction.

Remark 1.3.1.1.1. Proofs that *need* to be added at some point:

- Extra proof of Definition 7.5.10.1.1 using the machinery of presentable categories, following Maxime Ranzi's answer to MO 466593 ⚠.
- Horizontal composition of natural transformations is associative: Item 2 of Definition 11.9.5.1.3.
- Fully faithful functors are essentially injective: Item 4 of Definition 11.6.3.1.2.

Proofs that *would be very nice* to be added at some point:

- Properties of pseudomonic functors: Definition 11.7.4.1.2 ⚠.
- Characterisation of fully faithful functors: Item 1 of Definition 11.6.3.1.2.
- The quadruple adjunction between categories and sets: Definition 11.3.1.1.1.
- F_* faithful iff F faithful: Item 2 of Definition 11.6.1.1.2.
- Properties of groupoid completions: Definition 11.4.3.1.3.
- Properties of cores: Definition 11.4.4.1.4.
- Rel is isomorphic to the category of free algebras of the powerset monad: Definition 8.5.20.1.1 ⚠.
- Non/existence of left Kan extensions in **Rel**:
  - **??** of **??**.
  - **??** of **??**.
- Non/existence of left Kan lifts in **Rel**:

- **??** of **??**.
- **??** of **??**.

Proofs that *would be nice* to be added at some point:

- Properties of posetal categories: Definition 11.2.7.1.2.
- Injective on objects functors are precisely the isocofibrations in Cats_2: Item 1 of Definition 11.8.1.1.2 ⚠.
- Characterisations of monomorphisms of categories: Item 1 of Definition 11.7.2.1.2.
- Epimorphisms of categories are surjective on objects: Item 2 of Definition 11.7.3.1.2.
- Properties of pseudoepic functors: Definition 11.7.5.1.2 ⚠.

Proofs that *would be nice but not essential* to be added at some point:

- Proof that $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category: Item 15 of Definition 4.1.3.1.3 ⚠.
- Proof that $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category: Definition 5.3.5.1.1 ⚠.
- Proof that $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category: Item 11 of Definition 4.1.4.1.5 ⚠.
- Proof that $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category: Item 6 of Definition 4.2.3.1.3 ⚠.
- Proof that $(\mathsf{Sets}, \coprod_X, X)$ is a symmetric monoidal category: Item 8 of Definition 4.2.4.1.6 ⚠.

Proofs that have been (temporarily) omitted because they are “clear”, “straightforward”, or “tedious”:

- Properties of pushouts of sets:
  - Associativity: Item 3 of Definition 4.2.4.1.6.
  - Unitality: Item 5 of Definition 4.2.4.1.6.
  - Commutativity: Item 6 of Definition 4.2.4.1.6.
  - Pushout of sets over the empty set recovers the coproduct of sets: Item 7 of Definition 4.2.4.1.6.

- Properties of coequalisers of sets:
  - Associativity: Item 1 of Definition 4.2.5.1.5.
  - Unitality: Item 4 of Definition 4.2.5.1.5.
  - Commutativity: Item 5 of Definition 4.2.5.1.5.
  - Interaction with composition: Item 6 of Definition 4.2.5.1.5.
- Properties of set differences:
  - Item 4 of Definition 4.3.10.1.2.
  - Item 11 of Definition 4.3.10.1.2.
  - Item 13 of Definition 4.3.10.1.2.
  - Item 15 of Definition 4.3.10.1.2.
- Complements and characteristic functions: Item 4 of Definition 4.3.11.1.2.
- Properties of symmetric differences:
  - Item 1 of Definition 4.3.12.1.2.
  - Item 16 of Definition 4.3.12.1.2.
- Properties of direct images:
  - Functoriality: Item 1 of Definition 4.6.1.1.5.
  - Interaction with coproducts: Item 15 of Definition 4.6.1.1.5.
  - Interaction with products: Item 16 of Definition 4.6.1.1.5.
- Properties of inverse images:
  - Functoriality: Item 1 of Definition 4.6.2.1.3.
  - Interaction with coproducts: Item 15 of Definition 4.6.2.1.3.
  - Interaction with products; Item 16 of Definition 4.6.2.1.3.
- Properties of codirect images:
  - Functoriality: Item 1 of Definition 4.6.3.1.7.
  - Lax preservation of colimits: Item 10 of Definition 4.6.3.1.7.
  - Interaction with coproducts: Item 14 of Definition 4.6.3.1.7.
  - Interaction with products: Item 15 of Definition 4.6.3.1.7.

- Left distributor of $\times$ over $\coprod$ is a natural isomorphism: Definition 5.3.1.1.1.
- Right distributor of $\times$ over $\coprod$ is a natural isomorphism: Definition 5.3.2.1.1.
- Left annihilator of $\times$ is a natural isomorphism: Definition 5.3.3.1.1.
- Right annihilator of $\times$ is a natural isomorphism: Definition 5.3.4.1.1.
- Properties of wedge products of pointed sets:
  - Associativity: Item 2 of Definition 6.3.3.1.3.
  - Unitality: Item 3 of Definition 6.3.3.1.3.
  - Commutativity: Item 4 of Definition 6.3.3.1.3.
  - Symmetric monoidality: Item 5 of Definition 6.3.3.1.3.
- Properties of pushouts of pointed sets:
  - Interaction with coproducts: Item 5 of Definition 6.3.4.1.3.
  - Symmetric monoidality: Item 6 of Definition 6.3.4.1.3.

1.3.2 List of Missing Examples

Adding new examples is always welcome! In this section, we list some subjects and sections which could do with more examples:

Remark 1.3.2.1.1. Potentially interesting examples to add include, but are definitely not limited to:

- Examples of 2-categorical monomorphisms in Rel, following Section 8.5.11.
- Examples of 2-categorical epimorphisms in Rel, following Section 8.5.13.
- Examples of left Kan extensions and left Kan lifts in Rel.
- Examples of functors satisfying the conditions described in Categories.

1.3.3 List of Questions

There's a number of questions listed throughout this project. Here we collect them in a single place.

Remark 1.3.3.1.1. On relations:

- Definition 8.5.11.1.2, on better characterisations of representably full morphisms in **Rel**. This question also appears as [MO 467527].
- Definition 8.5.13.1.2, on better characterisations of corepresentably full morphisms in **Rel**. This question also appears as [MO 467527].
- **??**, seeking a characterisation of which left Kan extensions exist in **Rel**. This question also appears as [MO 461592].
- **??**, seeking an explicit descriptions of left Kan extensions along relations of the form f^{-1} (which always exist in **Rel**). This question also appears as [MO 461592].
- **??**, seeking a characterisation of which left Kan lifts exist in **Rel**. This question also appears as [MO 461592].
- **??**, seeking an explicit descriptions of left Kan lifts along relations of the form $\mathrm{Gr}(f)$ (which always exist in **Rel**). This question also appears as [MO 461592].

On categories:

- Definition 11.6.2.1.3, seeking a better characterisation of necessary and sufficient conditions on F for F^* to always be full. This question also appears as [MO 468121b].
- Definition 11.6.4.1.3, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be conservative. This question also appears as [MO 468121a].
- Definition 11.6.5.1.2, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be essentially injective. This question also appears as [MO 468121a].
- Definition 11.6.6.1.2, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be essentially surjective. This question also appears as [MO 468121a].

- Definition 11.7.1.1.3, , seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be dominant. This question also appears as [MO 468121a].
- Definition 11.7.2.1.3, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be monic. This question also appears as [MO 468121a].
- Definition 11.7.3.1.3, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be epic. This question also appears as [MO 468121a].
- Definition 11.7.5.1.5, seeking a characterisation of necessary and sufficient conditions on F for F^* or F_* to be pseudoepic. This question also appears as [MO 468121a].
- Definition 11.7.5.1.3, seeking a characterisation of pseudoepic functors. This question also appears as [MO 321971].
- Definition 11.7.5.1.4, which asks whether a pseudomonic and pseudoepic functor must necessarily be an equivalence of categories. This question also appears as [MO 468334].
- Definition 11.8.4.1.3, seeking a characterisation of functors representably faithful on cores.
- Definition 11.8.5.1.3, seeking a characterisation of functors representably full on cores.
- Definition 11.8.6.1.3, seeking a characterisation of functors representably fully faithful on cores.
- Definition 11.8.7.1.3, seeking a characterisation of functors corepresentably faithful on cores.
- Definition 11.8.8.1.3, seeking a characterisation of functors corepresentably full on cores.
- Definition 11.8.9.1.3, seeking a characterisation of functors corepresentably fully faithful on cores.

1.3.4 List of Gaps in the Category Theory Literature

The Clowder Project aims to address several significant gaps in the existing literature on category theory, as detailed below. See also [MO 494959].

Gap 1.3.4.1.1. Even though its analogue for ∞-categories has for years been a widely used tool[1], a comprehensive treatment of the tensor product of presentable categories seems to be currently missing.

Gap 1.3.4.1.2. An exhaustive concrete description of the various limits and colimits of categories, including 2-dimensional ones, is missing.

Gap 1.3.4.1.3. There seems to be no unified presentation of dinatural transformation co/classifiers in the literature. These are characterised by isomorphisms of the form

$$\mathrm{Nat}(F, G) \cong \mathrm{DiNat}(\Gamma(F), G),$$

and were originally studied in Dubuc–Street's paper introducing dinatural transformations, [DS06].

Even though these arguably form a fundamental piece of the framework of co/end calculus, it seems that all foundational treatments that followed after ended up not covering this concept.

Gap 1.3.4.1.4. The tensor product of symmetric monoidal categories had been a missing concept from the literature for years. Recently, [GJO24] covered the case of permutative categories. It would be nice, however, to also have a treatment of the non-strict case available.

Gap 1.3.4.1.5. A comprehensive and exhaustive treatment of the theory of promonoidal categories is currently missing. There are several important notions undefined, like:

- Promonoidal profunctors.
- Dualisability internal to a promonoidal category.
- Invertibility internal to a promonoidal category.

Moreover, it would be nice to record how promonoidal categories may be viewed as categorifications of "hypermonoids" (i.e. monoids in Rel).

Gap 1.3.4.1.6. A comprehensive and exhaustive treatment of the theory of multicategories is currently missing. There are several important notions undefined, like:

[1] See [MO 490557].

- Co/limits internal to multicategories.

See [MO 484647].

Gap 1.3.4.1.7. It would be nice to have an extensive collection of examples of what a given 2-categorical notion looks like in a 2-category. For instance, it would be nice to explicitly list what internal adjunctions look like in **Rel**, Span, Prof, etc.

See Section 8.5 for a concrete example of what is meant by this gap.

Gap 1.3.4.1.8. The literature on centres and traces of categories is really small. There are lots of results missing[2] and very few worked examples[3].

Gap 1.3.4.1.9. Natural transformations satisfy an isomorphism of the form

$$\mathrm{Nat}(F, G) \cong \int_{A \in C} \mathrm{Hom}_{\mathcal{D}}(F_A, G_A).$$

It is then exceedingly natural to define *natural cotransformations* via an isomorphism of the form

$$\mathrm{CoNat}(F, G) \cong \int^{A \in C} \mathrm{Hom}_{\mathcal{D}}(F_A, G_A)$$

and study their properties. This generalises traces of categories, since we have

$$\mathrm{Tr}(C) = \mathrm{CoNat}(\mathrm{id}_C, \mathrm{id}_C),$$

much like $\mathrm{Z}(C) = \mathrm{Nat}(\mathrm{id}_C, \mathrm{id}_C)$.

Gap 1.3.4.1.10. There are several results, notions, and examples in the theory of Isbell duality missing from the literature, and a truly comprehensive treatment is still lacking.[4]

Gap 1.3.4.1.11. The currently available treatments of 2-dimensional co/ends

[2] E.g. There's a certain interaction between traces of categories and Leinster's eventual image.

[3] E.g. what is the trace of Connes's cycle category? Such a computation doesn't seem to be available.

[4] For instance, there appears to be no mention of the duality pairings

$$\mathrm{Spec}(F) \boxtimes F \to \mathrm{Tr}(C),$$
$$\mathcal{F} \boxtimes \mathsf{O}(\mathcal{F}) \to \mathrm{Tr}(C)$$

in the currently available literature.

are unsatisfactory.[5]

Gap 1.3.4.1.12. A comprehensive treatment of factorisation systems is currently missing; see [MO 495003].

Gap 1.3.4.1.13. Several proofs of coherence theorems for string diagrams currently have gaps; see [MO 497309].

Gap 1.3.4.1.14. The currently available treatments of variants of category theory such as fibred category theory, enriched category theory, or internal category theory are unsatisfactory for a number of reasons.

Ideally, there should be a comprehensive and (simultaneously) approachable treatment for these topics. See also [MO 497419].

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[5]For instance, none of them define 2-dimensional co/ends via 2-dimensional dinatu-

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ral transformations and then go on to develop a general theory from there.

Chapter 2

A Guide to the Literature

This chapter contains some material about category theory literature.

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Part II

Sets

Chapter 3

Sets

This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

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3.1 Sets and Functions

3.1.1 Functions

Definition 3.1.1.1.1. A **function** is a functional and total relation.

Notation 3.1.1.1.2. Throughout this work, we will sometimes denote a function $f: X \to Y$ by

$$f \stackrel{\text{def}}{=} [\![x \mapsto f(x)]\!].$$

1. For example, given a function

$$\Phi\colon \mathrm{Hom}_{\mathsf{Sets}}(X,Y) \to K$$

taking values on a set of functions such as $\mathrm{Hom}_{\mathsf{Sets}}(X,Y)$, we will sometimes also write

$$\Phi(f) \stackrel{\text{def}}{=} \Phi(\llbracket x \mapsto f(x) \rrbracket).$$

2. This notational choice is based on the lambda notation

$$f \stackrel{\text{def}}{=} (\lambda x.\, f(x)),$$

but uses a "$\mapsto$" symbol for better spacing and double brackets instead of either:

   (a) Square brackets $[x \mapsto f(x)]$;
   (b) Parentheses $(x \mapsto f(x))$;

   hoping to improve readability when dealing with e.g.:

   (a) Equivalence classes, cf.:
      i. $\llbracket [x] \mapsto f([x]) \rrbracket$
      ii. $[[x] \mapsto f([x])]$
      iii. $(\lambda[x].\, f([x]))$
   (b) Function evaluations, cf.:
      i. $\Phi(\llbracket x \mapsto f(x) \rrbracket)$
      ii. $\Phi((x \mapsto f(x)))$
      iii. $\Phi((\lambda x.\, f(x)))$

3. We will also sometimes write $-$, $-_1$, $-_2$, etc. for the arguments of a function. Some examples include:

   (a) Writing $f(-_1)$ for a function $f\colon A \to B$.
   (b) Writing $f(-_1, -_2)$ for a function $f\colon A \times B \to C$.
   (c) Given a function $f\colon A \times B \to C$, writing

   $$f(a,-)\colon B \to C$$

   for the function $\llbracket b \mapsto f(a,b) \rrbracket$.

(d) Denoting a composition of the form

$$A \times B \xrightarrow{\phi \times \mathrm{id}_B} A' \times B \xrightarrow{f} C$$

by $f(\phi(-_1), -_2)$.

4. Finally, given a function $f : A \to B$, we will sometimes write

$$\mathrm{ev}_a(f) \overset{\text{def}}{=} f(a)$$

for the value of f at some $a \in A$.

For an example of the above notations being used in practice, see the proof of the adjunction

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

stated in Item 2 of Definition 4.1.3.1.3.

3.2 The Enrichment of Sets in Classical Truth Values

3.2.1 (-2)-Categories

Definition 3.2.1.1.1. A (-2)**-category** is the “necessarily true” truth value.[1,2,3]

3.2.2 (-1)-Categories

Definition 3.2.2.1.1. A (-1)**-category** is a classical truth value.

Remark 3.2.2.1.2. [4](-1)-categories should be thought of as being “categories enriched in (-2)-categories”, having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).

As a result, a (-1)-category C is either:[5]

[1]Thus, there is only one (-2)-category.

[2]A $(-n)$-category for $n = 3, 4, \ldots$ is also the “necessarily true” truth value, coinciding with a (-2)-category.

[3]For motivation, see [BS10, p. 13].

[4]For more motivation, see [BS10, p. 13].

[5]See [BS10, pp. 33–34].

1. *Empty*, having no objects.
2. *Contractible*, having a collection of objects $\{a, b, c, \dots\}$, but with $\mathrm{Hom}_{C}(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(C)$, forcing all objects of C to be uniquely isomorphic to each other.

Thus there are only two (-1)-categories up to equivalence:

1. The (-1)-category false (the empty one);
2. The (-1)-category true (the contractible one).

Definition 3.2.2.1.3. The **poset of truth values**[6] is the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ consisting of:

- *The Underlying Set.* The set $\{\mathsf{true}, \mathsf{false}\}$ whose elements are the truth values true and false.
- *The Partial Order.* The partial order

  $$\preceq\colon \{\mathsf{true}, \mathsf{false}\} \times \{\mathsf{true}, \mathsf{false}\} \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

  on $\{\mathsf{true}, \mathsf{false}\}$ defined by[7]

  $$\begin{aligned} \mathsf{false} \preceq \mathsf{false} &\stackrel{\text{def}}{=} \mathsf{true}, \\ \mathsf{true} \preceq \mathsf{false} &\stackrel{\text{def}}{=} \mathsf{false}, \\ \mathsf{false} \preceq \mathsf{true} &\stackrel{\text{def}}{=} \mathsf{true}, \\ \mathsf{true} \preceq \mathsf{true} &\stackrel{\text{def}}{=} \mathsf{true}. \end{aligned}$$

Notation 3.2.2.1.4. We also write $\{\mathsf{t}, \mathsf{f}\}$ for the poset $\{\mathsf{true}, \mathsf{false}\}$.

Proposition 3.2.2.1.5. The poset of truth values $\{\mathsf{t}, \mathsf{f}\}$ is Cartesian closed with product given by[8]

$$\begin{aligned} \mathsf{t} \times \mathsf{t} &= \mathsf{t}, & \mathsf{f} \times \mathsf{t} &= \mathsf{f}, \\ \mathsf{t} \times \mathsf{f} &= \mathsf{f}, & \mathsf{f} \times \mathsf{f} &= \mathsf{f}, \end{aligned}$$

×	t	f
t	t	f
f	f	f

and internal Hom $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ given by the partial order of $\{\mathsf{t}, \mathsf{f}\}$, i.e. by

$$\begin{aligned} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) &= \mathsf{t}, & \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) &= \mathsf{f}, & \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) &= \mathsf{t}, \end{aligned}$$

$\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$	t	f
t	t	t
f	t	f

.

[6] *Further Terminology:* Also called the **poset of (-1)-categories**.

[7] This partial order coincides with logical implication.

[8] Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ coincides with the

Proof. Existence of Products: We claim that the products $\mathrm{t} \times \mathrm{t}$, $\mathrm{t} \times \mathrm{f}$, $\mathrm{f} \times \mathrm{t}$, and $\mathrm{f} \times \mathrm{f}$ satisfy the universal property of the product in $\{\mathrm{t}, \mathrm{f}\}$. Indeed, suppose we have diagrams of the form

$$\begin{tikzcd} & P_1 \arrow[dl, bend right, "p_1^1"'] \arrow[dr, bend left, "p_2^1"] & \\ \mathrm{t} & \mathrm{t} \times \mathrm{t} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t} \end{tikzcd} \quad \begin{tikzcd} & P_2 \arrow[dl, bend right, "p_1^2"'] \arrow[dr, bend left, "p_2^2"] & \\ \mathrm{t} & \mathrm{t} \times \mathrm{f} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f} \end{tikzcd} \quad \begin{tikzcd} & P_3 \arrow[dl, bend right, "p_1^3"'] \arrow[dr, bend left, "p_2^3"] & \\ \mathrm{f} & \mathrm{f} \times \mathrm{t} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t} \end{tikzcd} \quad \begin{tikzcd} & P_4 \arrow[dl, bend right, "p_1^4"'] \arrow[dr, bend left, "p_2^4"] & \\ \mathrm{f} & \mathrm{f} \times \mathrm{f} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f} \end{tikzcd}$$

where the pr_1 and pr_2 morphisms are the only possible ones (since $\{\mathrm{t}, \mathrm{f}\}$ is posetal). We claim that there are unique morphisms making the diagrams

$$\begin{tikzcd} & P_1 \arrow[dl, bend right, "p_1^1"'] \arrow[dr, bend left, "p_2^1"] \arrow[d, dashed, "\exists!"] & \\ \mathrm{t} & \mathrm{t} \times \mathrm{t} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t} \end{tikzcd} \quad \begin{tikzcd} & P_2 \arrow[dl, bend right, "p_1^2"'] \arrow[dr, bend left, "p_2^2"] \arrow[d, dashed, "\exists!"] & \\ \mathrm{t} & \mathrm{t} \times \mathrm{f} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f} \end{tikzcd} \quad \begin{tikzcd} & P_3 \arrow[dl, bend right, "p_1^3"'] \arrow[dr, bend left, "p_2^3"] \arrow[d, dashed, "\exists!"] & \\ \mathrm{f} & \mathrm{f} \times \mathrm{t} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{t} \end{tikzcd} \quad \begin{tikzcd} & P_4 \arrow[dl, bend right, "p_1^4"'] \arrow[dr, bend left, "p_2^4"] \arrow[d, dashed, "\exists!"] & \\ \mathrm{f} & \mathrm{f} \times \mathrm{f} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathrm{f} \end{tikzcd}$$

commute. Indeed:

1. If $P_1 = \mathrm{t}$, then $p_1^1 = p_2^1 = \mathrm{id}_{\mathrm{t}}$, so there's a unique morphism from P_1 to t making the diagram commute, namely id_{t}.

2. If $P_1 = \mathrm{f}$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, so there's a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t.

3. If $P_2 = \mathrm{t}$, then there is no morphism p_2^2.

4. If $P_2 = \mathrm{f}$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \mathrm{id}_{\mathrm{f}}$, so there's a unique morphism from P_2 to f making the diagram commute, namely id_{f}.

5. The proof for P_3 is similar to the one for P_2.

6. If $P_4 = \mathrm{t}$, then there is no morphism p_1^4 or p_2^4.

7. If $P_4 = \mathrm{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathrm{f}}$, so there's a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

This finishes the existence of products part of the proof.
Cartesian Closedness: We claim there's a bijection

$$\mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(A, \mathbf{Hom}_{\{\mathrm{t},\mathrm{f}\}}(B, C)\right),$$

natural in $A, B, C \in \{\mathrm{t}, \mathrm{f}\}$. Indeed:

- For $(A,B,C) = (\mathsf{t},\mathsf{t},\mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}\times\mathsf{t},\mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t})\\ &= \{\mathrm{id}_{\mathsf{true}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t})\big).\end{aligned}$$

- For $(A,B,C) = (\mathsf{t},\mathsf{t},\mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}\times\mathsf{t},\mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{f})\\ &= \emptyset\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{f})\big).\end{aligned}$$

- For $(A,B,C) = (\mathsf{t},\mathsf{f},\mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}\times\mathsf{f},\mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\big).\end{aligned}$$

- For $(A,B,C) = (\mathsf{t},\mathsf{f},\mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}\times\mathsf{f},\mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})\\ &\cong \{\mathrm{id}_{\mathsf{false}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{t},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{f})\big).\end{aligned}$$

- For $(A,B,C) = (\mathsf{f},\mathsf{t},\mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}\times\mathsf{t},\mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f},\mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f},\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t},\mathsf{t})\big).\end{aligned}$$

logical implication operator.

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\big). \end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &= \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\big). \end{aligned}$$

Since $\{\mathsf{t}, \mathsf{f}\}$ is posetal, naturality is automatic (**??** of **??**). □

3.2.3 0-Categories

Definition 3.2.3.1.1. A 0-**category** is a poset.[9]

Definition 3.2.3.1.2. A 0-**groupoid** is a 0-category in which every morphism is invertible.[10]

3.2.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. The analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.

[9] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.

[10] That is, a *set*.

Remark 3.2.4.1.1. The basic analogies between set theory and category theory are summarised in the following table:

Set Theory	Category Theory
Enrichment in $\{\mathsf{true}, \mathsf{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathrm{Obj}(\mathcal{C})$
Function $f\colon X \to Y$	Functor $F\colon \mathcal{C} \to \mathcal{D}$
Function $X \to \{\mathsf{true}, \mathsf{false}\}$	Copresheaf $\mathcal{C} \to \mathsf{Sets}$
Function $X \to \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$

Remark 3.2.4.1.2. The category of presheaves $\mathsf{PSh}(\mathcal{C})$ and the category of copresheaves $\mathsf{CoPSh}(\mathcal{C})$ on a category $\mathcal{C}$ are the 1-categorical counterparts to the powerset $\mathcal{P}(X)$ of subsets of a set X. The further analogies built upon this are summarised in the following table:

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Characteristic function $\chi_{\{x\}}\colon X \to \{\mathsf{t}, \mathsf{f}\}$	Representable presheaf $h_X\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{Sets}$
Characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $\text{よ}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)\colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$	Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$
The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathcal{P}(U)} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$

Remark 3.2.4.1.3. We summarise the analogies between un/straightening in set theory and category theory in the following table:

Set Theory	**Category Theory**
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_C \mathcal{F}$
Un/straightening isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Un/straightening equivalence $\mathsf{DFib}(C) \overset{\text{eq.}}{\cong} \mathsf{PSh}(C)$

Remark 3.2.4.1.4. We summarise the analogies between functions $\mathcal{P}(X) \to \mathcal{P}(Y)$ and functors $\mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$ in the following table:

Set Theory	**Category Theory**
Direct image function $f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Left Kan extension functor $F_! \colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Precomposition functor $F^* \colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(C)$
Codirect image function $f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Right Kan extension functor $F_* \colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

Remark 3.2.4.1.5. We summarise the analogies between functions, relations and profunctors in the following table:

Set Theory	**Category Theory**
Relation $R \colon X \times Y \to \{\mathsf{t}, \mathsf{f}\}$	Profunctor $\mathfrak{p} \colon \mathcal{D}^{\mathsf{op}} \times C \to \mathsf{Sets}$
Relation $R \colon X \to \mathcal{P}(Y)$	Profunctor $\mathfrak{p} \colon C \to \mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p} \colon \mathsf{PSh}(C) \to \mathsf{PSh}(\mathcal{D})$

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Chapter 4

Constructions With Sets

This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. Of particular interest are perhaps the following:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 4.2.4.1.1, 4.2.4.1.3, 4.2.5.1.1 and 4.2.5.1.3).

2. A discussion of powersets as decategorifications of categories of presheaves, including in particular results such as:

    (a) A discussion of the internal Hom of a powerset (Section 4.4.7).

    (b) A 0-categorical version of the Yoneda lemma (Definition 12.1.5.1.1), which we term the *Yoneda lemma for sets* (Definition 4.5.5.1.1).

    (c) A characterisation of powersets as free cocompletions (Section 4.4.5), mimicking the corresponding statement for categories of presheaves (**??**).

    (d) A characterisation of powersets as free completions (Section 4.4.6), mimicking the corresponding statement for categories of copresheaves (**??**).

    (e) A (-1)-categorical version of un/straightening (Item 2 of Definition 4.5.1.1.4 and Definition 4.5.1.1.5).

    (f) A 0-categorical form of Isbell duality internal to powersets (Section 4.4.8).

3. A lengthy discussion of the adjoint triple

$$f_! \dashv f^{-1} \dashv f_* \colon \mathcal{P}(A) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(B)$$

of functors (i.e. morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f\colon A \to B$, including in particular:

(a) How f^{-1} can be described as a precomposition while $f_!$ and f_* can be described as Kan extensions (Definitions 4.6.1.1.4, 4.6.2.1.2 and 4.6.3.1.4).

(b) An extensive list of the properties of $f_!$, f^{-1}, and f_* (Definitions 4.6.1.1.5, 4.6.1.1.6, 4.6.2.1.3, 4.6.2.1.4, 4.6.3.1.7 and 4.6.3.1.8).

(c) How the functors $f_!$, f^{-1}, f_*, along with the functors

$$-_1 \cap -_2 \colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X),$$
$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

may be viewed as a six-functor formalism with the empty set $\emptyset$ as the dualising object (Section 4.6.4).

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4.1 Limits of Sets

4.1.1 The Terminal Set

Definition 4.1.1.1.1. The **terminal set** is the terminal object of Sets as in **??**.

Construction 4.1.1.1.2. Concretely, the terminal set is the pair $\left(\mathrm{pt}, \{!_A\}_{A\in\mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

1. *The Limit.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
2. *The Cone.* The collection of maps

$$\{!_A\colon A \to \mathrm{pt}\}_{A\in\mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_A(a) \overset{\mathrm{def}}{=} \star$$

for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi\colon A \to \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. □

4.1.2 Products of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 4.1.2.1.1. The **product**[1] **of** $\{A_i\}_{i\in I}$ is the product of $\{A_i\}_{i\in I}$ in Sets as in **??**.

Construction 4.1.2.1.2. Concretely, the product of $\{A_i\}_{i\in I}$ is the pair $\left(\prod_{i\in I} A_i, \left\{\mathrm{pr}_i\right\}_{i\in I}\right)$ consisting of:

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i\in I}$.

1. *The Limit.* The set $\prod_{i\in I} A_i$ defined by
$$\prod_{i\in I} A_i \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i\in I} A_i\right) \,\middle|\, \begin{array}{l}\text{for each } i \in I, \text{we}\\ \text{have } f(i) \in A_i\end{array} \right\}.$$

2. *The Cone.* The collection
$$\left\{ \mathrm{pr}_i \colon \prod_{i\in I} A_i \to A_i \right\}_{i\in I}$$
of maps given by
$$\mathrm{pr}_i(f) \overset{\text{def}}{=} f(i)$$
for each $f \in \prod_{i\in I} A_i$ and each $i \in I$.

Proof. We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form
$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$
in Sets. Then there exists a unique map $\phi \colon P \to \prod_{i\in I} A_i$ making the diagram
$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$
commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via
$$\phi(x) = (p_i(x))_{i\in I}$$
for each $x \in P$. □

Remark 4.1.2.1.3. Less formally, we may think of Cartesian products and projection maps as follows:

1. We think of $\prod_{i\in I} A_i$ as the set whose elements are I-indexed collections $(a_i)_{i\in I}$ with $a_i \in A_i$ for each $i \in I$.

2. We view the projection maps

$$\left\{\mathrm{pr}_i\colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

as being given by

$$\mathrm{pr}_i\Big(\big(a_j\big)_{j\in I}\Big) \stackrel{\text{def}}{=} a_i$$

for each $\big(a_j\big)_{j\in I} \in \prod_{i\in I} A_i$ and each $i \in I$.

Proposition 4.1.2.1.4. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathrm{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}}\colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i\colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\prod_{i\in I} f_i\colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

4.1.3 Binary Products of Sets

Let A and B be sets.

Definition 4.1.3.1.1. The **product of** A **and** B[2] is the product of A and B in Sets as in **??**.

Construction 4.1.3.1.2. Concretely, the product of A and B is the pair $\left(A \times B, \left\{\mathrm{pr}_1, \mathrm{pr}_2\right\}\right)$ consisting of:

1. *The Limit.* The set $A \times B$ defined by

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z \in \{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\} \\ &\cong \left\{ \begin{matrix} \text{ordered pairs } (a,b) \text{ with} \\ a \in A \text{ and } b \in B \end{matrix} \right\}. \end{aligned}$$

2. *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(a,b) &\stackrel{\text{def}}{=} a, \\ \mathrm{pr}_2(a,b) &\stackrel{\text{def}}{=} b \end{aligned}$$

for each $(a,b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in the category of sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

[2] *Further Terminology:* Also called the **Cartesian product of** A **and** B.

in Sets. Then there exists a unique map $\phi\colon P \to A\times B$ making the diagram

$$\begin{tikzcd}
 & P \arrow[dl, bend right, "p_1"'] \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, bend left, "p_2"] & \\
A & A\times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1,\\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

Proposition 4.1.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignments $A, B, (A, B) \mapsto A\times B$ define functors

$$\begin{aligned}
A\times -\colon &\quad \mathsf{Sets} \to \mathsf{Sets},\\
-\times B\colon &\quad \mathsf{Sets} \to \mathsf{Sets},\\
-_1\times -_2\colon &\quad \mathsf{Sets}\times\mathsf{Sets} \to \mathsf{Sets},
\end{aligned}$$

   where $-_1\times -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets}\times\mathsf{Sets})$, we have

   $$[-_1\times -_2](A, B) \overset{\text{def}}{=} A\times B.$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

   $$\times_{(A,B),(X,Y)}\colon \mathsf{Sets}(A, X)\times\mathsf{Sets}(B, Y) \to \mathsf{Sets}(A\times B, X\times Y)$$

   of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

   $$f\times g\colon A\times B \to X\times Y$$

   defined by

   $$[f\times g](a, b) \overset{\text{def}}{=} (f(a), g(b))$$

   for each $(a, b) \in A\times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness I.* We have adjunctions

$$(A \times - \dashv \mathsf{Sets}(A,-))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathsf{Sets}(B,-))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Adjointness II.* We have an adjunction

$$(\Delta_{\mathsf{Sets}} \dashv -_1 \times -_2)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\Delta_{\mathsf{Sets}}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \times \mathsf{Sets}, \arrow[l, bend left, "-_1\times-_2"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}\times\mathsf{Sets}}((A, A), (B, C)) \cong \mathsf{Sets}(A, B \times C),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and in $(B, C) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$.

4. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets}}_{A,B,C}\colon (A \times B) \times C \overset{\sim}{\dashrightarrow} A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \lambda^{\mathsf{Sets}}_A &\colon \mathrm{pt} \times A \overset{\sim}{\dashrightarrow} A, \\ \rho^{\mathsf{Sets}}_A &\colon A \times \mathrm{pt} \overset{\sim}{\dashrightarrow} A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Commutativity.* We have an isomorphism of sets

$$\sigma^{\mathsf{Sets}}_{A,B} \colon A \times B \overset{\sim}{\dashrightarrow} B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

7. *Distributivity Over Coproducts.* We have isomorphisms of sets

$$\begin{aligned}\delta^{\mathsf{Sets}}_{\ell} &\colon A \times \left(B \textstyle\coprod C\right) \overset{\sim}{\dashrightarrow} (A \times B) \textstyle\coprod (A \times C),\\ \delta^{\mathsf{Sets}}_{r} &\colon \left(A \textstyle\coprod B\right) \times C \overset{\sim}{\dashrightarrow} (A \times C) \textstyle\coprod (B \times C),\end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

8. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned}\zeta^{\mathsf{Sets}}_{\ell} &\colon \emptyset \times A \overset{\sim}{\dashrightarrow} \emptyset,\\ \zeta^{\mathsf{Sets}}_{r} &\colon A \times \emptyset \overset{\sim}{\dashrightarrow} \emptyset,\end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

9. *Distributivity Over Unions.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \cup W) &= (U \times V) \cup (U \times W),\\ (U \cup V) \times W &= (U \times W) \cup (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

10. *Distributivity Over Intersections.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \cap W) &= (U \times V) \cap (U \times W),\\ (U \cap V) \times W &= (U \times W) \cap (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

11. *Distributivity Over Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned}U \times (V \setminus W) &= (U \times V) \setminus (U \times W),\\ (U \setminus V) \times W &= (U \times W) \setminus (V \times W)\end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

12. *Distributivity Over Symmetric Differences.* Let X be a set. For each $U, V, W \in \mathcal{P}(X)$, we have equalities

$$\begin{aligned} U \times (V \mathbin{\triangle} W) &= (U \times V) \mathbin{\triangle} (U \times W), \\ (U \mathbin{\triangle} V) \times W &= (U \times W) \mathbin{\triangle} (V \times W) \end{aligned}$$

of subsets of $\mathcal{P}(X \times X)$.

13. *Middle-Four Exchange with Respect to Intersections.* The diagram

$$\begin{tikzcd} (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[r, "\cap \times \cap"] \arrow[d, "\mathcal{P}^{\times}_{X,X} \times \mathcal{P}^{\times}_{X,X}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\mathcal{P}^{\times}_{X,X}"] \\ \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \arrow[r, "\cap"'] & \mathcal{P}(X \times X) \end{tikzcd}$$

commutes, i.e. we have

$$(U \times V) \cap (W \times T) = (U \cap V) \times (W \cap T).$$

for each $U, V, W, T \in \mathcal{P}(X)$.

14. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}, \times, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}\right)$ is a closed symmetric monoidal category.

15. *Symmetric Bimonoidality.* The 18-tuple

$$\begin{gathered}\Big(\mathsf{Sets}, \coprod, \times, \emptyset, \mathrm{pt}, \mathsf{Sets}(-_1, -_2), \alpha^{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}}, \\ \alpha^{\mathsf{Sets}, \coprod}, \lambda^{\mathsf{Sets}, \coprod}, \rho^{\mathsf{Sets}, \coprod}, \sigma^{\mathsf{Sets}, \coprod}, \delta^{\mathsf{Sets}}_{\ell}, \delta^{\mathsf{Sets}}_{r}, \zeta^{\mathsf{Sets}}_{\ell}, \zeta^{\mathsf{Sets}}_{r}\Big),\end{gathered}$$

is a symmetric closed bimonoidal category, where $\alpha^{\mathsf{Sets}, \coprod}$, $\lambda^{\mathsf{Sets}, \coprod}$, $\rho^{\mathsf{Sets}, \coprod}$, and $\sigma^{\mathsf{Sets}, \coprod}$ are the natural transformations from Items 3 to 5 of Definition 4.2.3.1.3.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Adjointness: We prove only that there's an adjunction $- \times B \dashv \mathsf{Sets}(B, -)$, witnessed by a bijection

$$\mathsf{Sets}(A \times B, C) \cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathsf{Sets}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

  $$\Phi_{B,C} \colon \mathsf{Sets}(A \times B, C) \to \mathsf{Sets}(A, \mathsf{Sets}(B, C)),$$

  by sending a function

  $$\xi \colon A \times B \to C$$

  to the function

  $$\begin{aligned} \xi^{\dagger} \colon A &\longrightarrow \mathsf{Sets}(B, C), \\ a &\longmapsto \left(\xi^{\dagger}_a \colon B \to C\right), \end{aligned}$$

  where we define

  $$\xi^{\dagger}_a(b) \stackrel{\text{def}}{=} \xi(a, b)$$

  for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Definition 3.1.1.1.2, we have

  $$\xi^{\dagger} \stackrel{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

- *Map II.* We define a map

  $$\Psi_{B,C} \colon \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \to \mathsf{Sets}(A \times B, C)$$

  given by sending a function

  $$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}(B, C), \\ a &\longmapsto (\xi_a \colon B \to C), \end{aligned}$$

  to the function

  $$\xi^{\dagger} \colon A \times B \to C$$

  defined by

  $$\begin{aligned} \xi^{\dagger}(a, b) &\stackrel{\text{def}}{=} \mathrm{ev}_b(\mathrm{ev}_a(\xi)) \\ &\stackrel{\text{def}}{=} \mathrm{ev}_b(\xi_a) \\ &\stackrel{\text{def}}{=} \xi_a(b) \end{aligned}$$

  for each $(a, b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A\times B,C)}\,.$$

Indeed, given a function $\xi\colon A\times B\to C$, we have

$$\begin{aligned}
[\Psi_{A,B}\circ\Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi))\\
&= \Psi_{A,B}(\Phi_{A,B}([\![(a,b)\mapsto \xi(a,b)]\!]))\\
&= \Psi_{A,B}([\![a\mapsto [\![b\mapsto \xi(a,b)]\!]]\!])\\
&= \Psi_{A,B}([\![a'\mapsto [\![b'\mapsto \xi(a',b')]\!]]\!])\\
&= [\![(a,b)\mapsto \mathrm{ev}_b(\mathrm{ev}_a([\![a'\mapsto [\![b'\mapsto \xi(a',b')]\!]]\!]))]\!]\\
&= [\![(a,b)\mapsto \mathrm{ev}_b([\![b'\mapsto \xi(a,b')]\!])]\!]\\
&= [\![(a,b)\mapsto \xi(a,b)]\!]\\
&= \xi.
\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}(B,C))}\,.$$

Indeed, given a function

$$\begin{aligned}
\xi\colon A &\longrightarrow \mathsf{Sets}(B,C),\\
a &\longmapsto (\xi_a\colon B\to C),
\end{aligned}$$

we have

$$\begin{aligned}
[\Phi_{A,B}\circ\Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi))\\
&\overset{\text{def}}{=} \Phi_{A,B}([\![(a,b)\mapsto \xi_a(b)]\!])\\
&\overset{\text{def}}{=} \Phi_{A,B}([\![(a',b')\mapsto \xi_{a'}(b')]\!])\\
&\overset{\text{def}}{=} [\![a\mapsto [\![b\mapsto \mathrm{ev}_{(a,b)}([\![(a',b')\mapsto \xi_{a'}(b')]\!])]\!]]\!]\\
&\overset{\text{def}}{=} [\![a\mapsto [\![b\mapsto \xi_a(b)]\!]]\!]\\
&\overset{\text{def}}{=} [\![a\mapsto \xi_a]\!]\\
&\overset{\text{def}}{=} \xi.
\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g\colon B\to B'$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A\times B',C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A\times g^*"'] & \mathsf{Sets}(A,\mathsf{Sets}(B',C)), \arrow[d, "(g^*)_!"]\\
\mathsf{Sets}(A\times B,C) \arrow[r, "\Phi_{B,C}"'] & \mathsf{Sets}(A,\mathsf{Sets}(B,C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A\times B'\to C,$$

we have

$$\begin{aligned}\left[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)\right](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\ &= \Phi_{B,C}(\xi(-_1,g(-_2)))\\ &= [\xi(-_1,g(-_2))]^\dagger\\ &= \xi^\dagger_{-_1}(g(-_2))\\ &= (g^*)_!\left(\xi^\dagger\right)\\ &= (g^*)_!\left(\Phi_{B',C}(\xi)\right)\\ &= \left[(g^*)_!\circ\Phi_{B',C}\right](\xi).\end{aligned}$$

Alternatively, using the $[\![a\mapsto f(a)]\!]$ notation of Definition 3.1.1.1.2, we have

$$\begin{aligned}\left[\Phi_{B,C}\circ(\mathrm{id}_A\times g^*)\right](\xi) &= \Phi_{B,C}([\mathrm{id}_A\times g^*](\xi))\\ &= \Phi_{B,C}([\mathrm{id}_A\times g^*]([\![(a,b')\mapsto\xi(a,b')]\!]))\\ &= \Phi_{B,C}([\![(a,b)\mapsto\xi(a,g(b))]\!])\\ &= [\![a\mapsto[\![b\mapsto\xi(a,g(b))]\!]]\!]\\ &= [\![a\mapsto g^*([\![b'\mapsto\xi(a,b')]\!])]\!]\\ &= (g^*)_!([\![a\mapsto[\![b'\mapsto\xi(a,b')]\!]]\!])\\ &= (g^*)_!\left(\Phi_{B',C}([\![(a,b')\mapsto\xi(a,b')]\!])\right)\\ &= (g^*)_!\left(\Phi_{B',C}(\xi)\right)\\ &= \left[(g^*)_!\circ\Phi_{B',C}\right](\xi).\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function $h\colon C\to C'$, the diagram

$$\begin{tikzcd}\mathsf{Sets}(A\times B,C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_!"'] & \mathsf{Sets}(A,\mathsf{Sets}(B,C)), \arrow[d, "(h_!)_!"]\\ \mathsf{Sets}(A\times B,C') \arrow[r, "\Phi_{B,C'}"'] & \mathsf{Sets}(A,\mathsf{Sets}(B,C'))\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A\times B\to C,$$

we have

$$\begin{aligned}
\left[\Phi_{B,C}\circ h_!\right](\xi) &= \Phi_{B,C}(h_!(\xi))\\
&= \Phi_{B,C}(h_!(\llbracket (a,b)\mapsto \xi(a,b)\rrbracket))\\
&= \Phi_{B,C}(\llbracket (a,b)\mapsto h(\xi(a,b))\rrbracket)\\
&= \llbracket a\mapsto \llbracket b\mapsto h(\xi(a,b))\rrbracket\rrbracket\\
&= \llbracket a\mapsto h_!(\llbracket b\mapsto \xi(a,b)\rrbracket\rrbracket)\\
&= (h_!)_!(\llbracket a\mapsto \llbracket b\mapsto \xi(a,b)\rrbracket\rrbracket)\\
&= (h_!)_!\left(\Phi_{B,C}(\llbracket (a,b)\mapsto \xi(a,b)\rrbracket)\right)\\
&= (h_!)_!\left(\Phi_{B,C}(\xi)\right)\\
&= \left[(h_!)_!\circ\Phi_{B,C}\right](\xi).
\end{aligned}$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Adjointness II: This follows from the universal property of the product.

Item 4, Associativity: This is proved in the proof of Definition 5.1.4.1.1.

Item 5, Unitality: This is proved in the proof of Definitions 5.1.5.1.1 and 5.1.6.1.1.

Item 6, Commutativity: This is proved in the proof of Definition 5.1.7.1.1.

Item 7, Distributivity Over Coproducts: This is proved in the proof of Definitions 5.3.1.1.1 and 5.3.2.1.1.

Item 8, Annihilation With the Empty Set: This is proved in the proof of Definitions 5.3.3.1.1 and 5.3.4.1.1.

Item 9, Distributivity Over Unions: See [Pro25c].

Item 10, Distributivity Over Intersections: See [Pro25e, Corollary 1].

Item 11, Distributivity Over Differences: See [Pro25a].

Item 12, Distributivity Over Symmetric Differences: See [Pro25b].

Item 13, Middle-Four Exchange With Respect to Intersections: See [Pro25e, Corollary 1].

Item 14, Symmetric Monoidality: This is a repetition of Definition 5.1.9.1.1, and is proved there.

Item 15, Symmetric Bimonoidality: This is a repetition of Definition 5.3.5.1.1, and is proved there. □

Remark 4.1.3.1.4. As shown in Item 1 of Definition 4.1.3.1.3, the Cartesian product of sets defines a functor

$$-_1\times -_2\colon \mathsf{Sets}\times\mathsf{Sets}\longrightarrow\mathsf{Sets}.$$

This functor is the $(k, \ell) = (-1, -1)$ case of a family of functors

$$\otimes_{k,\ell} \colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets})$$

of tensor products of $\mathbb{E}_k$-monoid objects on Sets with $\mathbb{E}_\ell$-monoid objects on Sets; see **??**.

Remark 4.1.3.1.5. We may state the equalities in Items 9 to 12 of Definition 4.1.3.1.3 as the commutativity of the following diagrams:

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cup"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cup"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cup"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cap"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\cap"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\setminus"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\setminus"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\setminus\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\setminus"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\triangle"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\triangle"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu_4^{\mathsf{Sets}}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)}\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\triangle\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\mathcal{P}^{\times}_{X,X}\times\mathcal{P}^{\times}_{X,X}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\mathcal{P}^{\times}_{X,X}"'] & & \mathcal{P}(X\times X)\times\mathcal{P}(X\times X) \arrow[dl, "\triangle"] \\
 & \mathcal{P}(X\times X) &
\end{tikzcd}$$

4.1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

Definition 4.1.4.1.1. The **pullback of** A **and** B **over** C **along** f **and** g[3] is the pullback of A and B over C along f and g in Sets as in **??**.

Construction 4.1.4.1.2. Concretely, the pullback of A and B over C along f and g is the pair $\left(A \times_C B, \left\{\mathrm{pr}_1, \mathrm{pr}_2\right\}\right)$ consisting of:

1. *The Limit.* The set $A \times_C B$ defined by

$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$

2. *The Cone.* The maps[4]

$$\begin{aligned}\mathrm{pr}_1 &\colon A \times_C B \to A,\\ \mathrm{pr}_2 &\colon A \times_C B \to B\end{aligned}$$

defined by

$$\begin{aligned}\mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b\end{aligned}$$

for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned}\left[f \circ \mathrm{pr}_1\right](a, b) &= f\left(\mathrm{pr}_1(a, b)\right)\\ &= f(a)\end{aligned}$$

[3] *Further Terminology:* Also called the **fibre product of** A **and** B **over** C **along** f **and** g.

[4] *Further Notation:* Also written $\mathrm{pr}_1^{A \times_C B}$ and $\mathrm{pr}_2^{A \times_C B}$.

$$
\begin{aligned}
&= g(b) \\
&= g\big(\mathrm{pr}_2(a,b)\big) \\
&= \big[g \circ \mathrm{pr}_2\big](a,b),
\end{aligned}
$$

where $f(a) = g(b)$ since $(a,b) \in A \times_C B$. Next, we prove that $A \times_C B$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$
\begin{tikzcd}
P \arrow[rrd, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times_C B$ making the diagram

$$
\begin{tikzcd}
P \arrow[rrd, bend left, "p_2"] \arrow[ddr, bend right, "p_1"'] \arrow[dr, dashed, "\phi", "\exists!"'] & & \\
& A \times_C B \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[d, two heads, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}
$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

Remark 4.1.4.1.3. It is common practice to write $A \times_C B$ for the pullback of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \times_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \times_{f,C,g} B$ or $A \times_C^{f,g} B$ for $A \times_C B$.

Example 4.1.4.1.4. Here are some examples of pullbacks of sets.

1. *Unions via Intersections.* Let X be a set. We have

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook, "\iota_B"] \\ A \arrow[r, hook, "\iota_A"'] & A \cup B \end{tikzcd}$$

for each $A, B \in \mathcal{P}(X)$.

Proof. *Item 1*, *Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

Proposition 4.1.4.1.5. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \overset{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a, b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& A' \times_{C'} B' \arrow[rr, crossing over] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & C'
\end{tikzcd}$$

commute.

2. *Adjointness I.* We have adjunctions

$$\left(A \times_X - \dashv \mathbf{Sets}_{/X}(A, -)\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "A \times_X -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X}, \arrow[l, bend left, "\mathbf{Sets}_{/X}(A,-)"] \end{tikzcd}$$

$$\left(- \times_X B \dashv \mathbf{Sets}_{/X}(B, -)\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "- \times_X B"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X}, \arrow[l, bend left, "\mathbf{Sets}_{/X}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}\big(A, \mathbf{Sets}_{/X}(B, C)\big),$$
$$\mathsf{Sets}_{/X}(A \times_X B, C) \cong \mathsf{Sets}_{/X}\big(B, \mathbf{Sets}_{/X}(A, C)\big),$$

natural in $(A, \phi_A), (B, \phi_B), (C, \phi_C) \in \mathrm{Obj}\big(\mathsf{Sets}_{/X}\big)$, where $\mathbf{Sets}_{/X}(A, B)$ is the object of $\mathsf{Sets}_{/X}$ consisting of (see **??**):

- *The Set.* The set $\mathbf{Sets}_{/X}(A, B)$ defined by

  $$\mathbf{Sets}_{/X}(A, B) \overset{\text{def}}{=} \coprod_{x \in X} \mathsf{Sets}\big(\phi_A^{-1}(x), \phi_Y^{-1}(x)\big)$$

- *The Map to X.* The map

  $$\phi_{\mathbf{Sets}_{/X}(A,B)} \colon \mathbf{Sets}_{/X}(A, B) \to X$$

  defined by

  $$\phi_{\mathbf{Sets}_{/X}(A,B)}(x, f) \overset{\text{def}}{=} x$$

  for each $(x, f) \in \mathbf{Sets}_{/X}(A, B)$.

3. *Adjointness II.* We have an adjunction

$$\big(\Delta_{\mathsf{Sets}_{/X}} \dashv -_1 \times -_2\big)\colon \quad \begin{tikzcd} \mathsf{Sets}_{/X} \arrow[r, bend left, "\Delta_{\mathsf{Sets}_{/X}}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}, \arrow[l, bend left, "-_1 \times -_2"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}}((A, A), (B, C)) \cong \mathsf{Sets}_{/X}(A, B \times_X C),$$

natural in $A \in \mathrm{Obj}\big(\mathsf{Sets}_{/X}\big)$ and in $(B, C) \in \mathrm{Obj}\big(\mathsf{Sets}_{/X} \times \mathsf{Sets}_{/X}\big)$.

4. *Associativity.* Given a diagram

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\ & X & & Y & \end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}
 & & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}
 & & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & A \times_X B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
\quad
\begin{tikzcd}
 & & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & & \\
 & & & B \times_Y C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner", very near start] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C. \arrow[dl, "k"] \\
 & X & & Y &
\end{tikzcd}
$$

5. *Interaction With Composition.* Given a diagram

$$
\begin{tikzcd}
X \arrow[dr, "\phi"'] & & & & Y \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

in Sets, we have isomorphisms of sets

$$
\begin{aligned}
X \times_K^{f\circ\phi, g\circ\psi} Y &\cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f,g} B\right)\right) \times^{p_2, p_1}_{A \times_K^{f,g} B} \left(\left(A \times_K^{f,g} B\right) \times_B^{q_2, \psi} Y\right) \\
&\cong X \times_A^{\phi, p} \left(\left(A \times_K^{f,g} B\right) \times_B^{q_2, \psi} Y\right) \\
&\cong \left(X \times_A^{\phi, q_1} \left(A \times_K^{f,g} B\right)\right) \times_B^{q, \psi} Y
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &= \mathrm{pr}_1^{A \times_K^{f,g} B}, & q_2 &= \mathrm{pr}_2^{A \times_K^{f,g} B}, \\
p_1 &= \mathrm{pr}_1^{\left(A \times_K^{f,g} B\right) \times_Y^{q_2, \psi}}, & p_2 &= \mathrm{pr}_2^{X \times^{\phi, q_1}_{A \times_K^{f,g} B} \left(A \times_K^{f,g} B\right)}, \\
p &= q_1 \circ \mathrm{pr}_1^{\left(A \times_K^{f,g} B\right) \times_B^{q_2, \psi} Y}, & q &= q_2 \circ \mathrm{pr}_2^{X \times_A^{\phi, q_1} \left(A \times_K^{f,g} B\right)},
\end{aligned}
$$

and where these pullbacks are built as in the following diagrams:

$$
\begin{tikzcd}[column sep=tiny]
 & & (X \times_A (A \times_K B)) \times_{A \times_K B} ((A \times_K B) \times_B Y) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & \\
 & X \times_A (A \times_K B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & (A \times_K B) \times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & \\
X \arrow[dr, "\phi"'] & & A \times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & X \times_A ((A \times_K B) \times_B Y) \arrow[ddll] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & \\
 & & & (A \times_K B) \times_B Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & \\
X \arrow[dr, "\phi"'] & & A \times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=tiny]
 & & (X \times_A (A \times_K B)) \times_B Y \arrow[dl] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & \\
 & X \times_A (A \times_K B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & & \\
X \arrow[dr, "\phi"'] & & A \times_K B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & Y, \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & & X \times_K Y \arrow[ddll] \arrow[ddrr] \arrow[dd, phantom, "\lrcorner" {rotate=-45}, very near start] & & \\
 & & & & \\
X \arrow[dr, "\phi"'] & & & & Y. \arrow[dl, "\psi"] \\
 & A \arrow[dr, "f"'] & & B \arrow[dl, "g"] & \\
 & & K & &
\end{tikzcd}
$$

6. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
\lambda^{\mathsf{Sets}_{/X}}_{A} &\colon X \times_X A \overset{\sim}{\dashrightarrow} A,\\
\rho^{\mathsf{Sets}_{/X}}_{A} &\colon A \times_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X,
\end{tikzcd}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

7. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
\sigma^{\mathsf{Sets}_{/X}}_{A,B} \colon A \times_X B \overset{\sim}{\dashrightarrow} B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X,
\end{tikzcd}
$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{/X})$.

8. *Distributivity Over Coproducts.* Let A, B, and C be sets and let $\phi_A\colon A \to X$, $\phi_B\colon B \to X$, and $\phi_C\colon C \to X$ be morphisms of sets. We have isomorphisms of sets

$$\delta_\ell^{\mathsf{Sets}_{/X}}\colon A \times_X (B \textstyle\coprod C) \overset{\sim}{\dashrightarrow} (A \times_X B) \coprod (A \times_X C),$$
$$\delta_r^{\mathsf{Sets}_{/X}}\colon (A \textstyle\coprod B) \times_X C \overset{\sim}{\dashrightarrow} (A \times_X C) \coprod (B \times_X C),$$

as in the diagrams

$$\begin{tikzcd} (A \times_X B) \coprod (A \times_X C) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \coprod C \arrow[d, "\phi_B \coprod \phi_C"] \\ A \arrow[r, "\phi_A"'] & X \end{tikzcd} \qquad \begin{tikzcd} (A \times_X C) \coprod (B \times_X C) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\phi_C"] \\ A \coprod B \arrow[r, "\phi_A \coprod \phi_B"'] & X \end{tikzcd}$$

natural in $A, B, C \in \operatorname{Obj}(\mathsf{Sets}_{/X})$.

9. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \begin{aligned} \zeta_\ell^{\mathsf{Sets}_{/X}}&\colon A \times_X \emptyset \overset{\sim}{\dashrightarrow} \emptyset, \\ \zeta_r^{\mathsf{Sets}_{/X}}&\colon \emptyset \times_X A \overset{\sim}{\dashrightarrow} \emptyset, \end{aligned} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X, \end{tikzcd}$$

natural in $(A, f) \in \operatorname{Obj}(\mathsf{Sets}_{/X})$.

10. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

11. *Symmetric Monoidality.* The 8-tuple $\left(\mathsf{Sets}_{/X}, \times_X, X, \mathbf{Sets}_{/X}, \alpha^{\mathsf{Sets}_{/X}}, \lambda^{\mathsf{Sets}_{/X}}, \rho^{\mathsf{Sets}_{/X}}, \sigma^{\mathsf{Sets}_{/X}}\right)$ is a symmetric closed monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Adjointness I: This is a repetition of **??** of **??**, and is proved there.

Item 3, *Adjointness II*: This follows from the universal property of the product (pullbacks are products in $\mathsf{Sets}_{/X}$).
Item 4, *Associativity*: We have

$$\begin{aligned}(A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\}\\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\}\\ &\cong A \times_X (B \times_Y C)\end{aligned}$$

and

$$\begin{aligned}(A \times_X B) \times_B (B \times_Y C) &\cong \{((a,b),(b',c)) \in (A \times_X B) \times (B \times_Y C) \mid b = b'\}\\ &\cong \left\{((a,b),(b',c)) \in (A \times B) \times (B \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\}\\ &\cong \left\{(a,(b,(b',c))) \in A \times (B \times (B \times C)) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\}\\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b), b = b', \\ \text{and } h(b') = k(c) \end{array}\right\}\\ &\cong \left\{(a,((b,b'),c)) \in A \times ((B \times_B B) \times C) \,\middle|\, \begin{array}{l} f(a) = g(b) \text{ and} \\ h(b') = k(c) \end{array}\right\}\\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\}\\ &\cong A \times_X (B \times_Y C),\end{aligned}$$

where we have used Item 6 for the isomorphism $B \times_B B \cong B$.
Item 5, *Interaction With Composition*: By Item 4, it suffices to construct only the isomorphism

$$X \times_K^{f\circ\phi, g\circ\psi} Y \cong \left(X \times_A^{\phi,q_1} \left(A \times_K^{f,g} B\right)\right) \times^{p_2,p_1}_{A \times_K^{f,g} B} \left(\left(A \times_K^{f,g} B\right) \times_B^{q_2,\psi} Y\right).$$

We have

$$\begin{aligned}\left(X \times_A^{\phi,q_1} \left(A \times_K^{f,g} B\right)\right) &\stackrel{\text{def}}{=} \left\{(x,(a,b)) \in X \times \left(A \times_K^{f,g} B\right) \,\middle|\, \phi(x) = q_1(a,b)\right\}\\ &\stackrel{\text{def}}{=} \left\{(x,(a,b)) \in X \times \left(A \times_K^{f,g} B\right) \,\middle|\, \phi(x) = a\right\}\\ &\cong \{(x,(a,b)) \in X \times (A \times B) \mid \phi(x) = a \text{ and } f(a) = g(b)\},\\ \left(\left(A \times_K^{f,g} B\right) \times_B^{q_2,\psi} Y\right) &\stackrel{\text{def}}{=} \left\{((a,b),y) \in \left(A \times_K^{f,g} B\right) \times Y \,\middle|\, q_2(a,b) = \psi(y)\right\}\\ &\stackrel{\text{def}}{=} \left\{((a,b),y) \in \left(A \times_K^{f,g} B\right) \times Y \,\middle|\, b = \psi(y)\right\}\\ &\cong \{((a,b),y) \in (A \times B) \times Y \mid b = \psi(y) \text{ and } f(a) = g(b)\},\end{aligned}$$

so writing

$$S = \left(X \times_A^{\phi,q_1} \left(A \times_K^{f,g} B\right)\right)$$

$$S' = \left(\left(A \times_K^{f,g} B\right) \times_B^{q_2,\psi} Y\right),$$

we have

$$\begin{aligned}
S \times^{p_2,p_1}_{A\times_K^{f,g}B} S' &\overset{\text{def}}{=} \{((x,(a,b)),((a',b'),y)) \in S \times S' \mid p_1(x,(a,b)) = p_2((a',b'),y)\} \\
&\overset{\text{def}}{=} \{((x,(a,b)),((a',b'),y)) \in S \times S' \mid (a,b) = (a',b')\} \\
&\cong \{((x,a,b,y)) \in X \times A \times B \times Y \mid \phi(x) = a, \psi(y) = b, \text{and } f(a) = g(b)\} \\
&\cong \{((x,a,b,y)) \in X \times A \times B \times Y \mid f(\phi(x)) = g(\psi(y))\} \\
&\overset{\text{def}}{=} X \times_K Y.
\end{aligned}$$

This finishes the proof.

Item 6, Unitality: We have

$$\begin{aligned}
X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\}, \\
A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\},
\end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$. The proof of the naturality of $\lambda^{\mathsf{Sets}/X}$ and $\rho^{\mathsf{Sets}/X}$ is omitted.

Item 7, Commutativity: We have

$$\begin{aligned}
A \times_C B &\overset{\text{def}}{=} \{(a,b) \in A \times B \mid f(a) = g(b)\} \\
&= \{(a,b) \in A \times B \mid g(b) = f(a)\} \\
&\cong \{(b,a) \in B \times A \mid g(b) = f(a)\} \\
&\overset{\text{def}}{=} B \times_C A.
\end{aligned}$$

The proof of the naturality of $\sigma^{\mathsf{Sets}/X}$ is omitted.

Item 8, Distributivity Over Coproducts: We have

$$\begin{aligned}
A \times_X (B \textstyle\coprod C) &\overset{\text{def}}{=} \left\{(a,z) \in A \times (B \textstyle\coprod C) \,\middle|\, \phi_A(a) = \phi_{B\coprod C}(z)\right\} \\
&= \left\{(a,z) \in A \times (B \textstyle\coprod C) \,\middle|\, z = (0,b) \text{ and } \phi_A(a) = \phi_{B\coprod C}(z)\right\} \\
&\quad \cup \left\{(a,z) \in A \times (B \textstyle\coprod C) \,\middle|\, z = (1,c) \text{ and } \phi_A(a) = \phi_{B\coprod C}(z)\right\} \\
&= \{(a,z) \in A \times (B \textstyle\coprod C) \mid z = (0,b) \text{ and } \phi_A(a) = \phi_B(b)\} \\
&\quad \cup \{(a,z) \in A \times (B \textstyle\coprod C) \mid z = (1,c) \text{ and } \phi_A(a) = \phi_C(c)\} \\
&\cong \{(a,b) \in A \times B \mid \phi_A(a) = \phi_B(b)\} \\
&\quad \cup \{(a,c) \in A \times C \mid \phi_A(a) = \phi_C(c)\} \\
&\overset{\text{def}}{=} (A \times_X B) \cup (A \times_X C) \\
&\cong (A \times_X B) \textstyle\coprod (A \times_X C),
\end{aligned}$$

with the construction of the isomorphism

$$\delta_r^{\mathsf{Sets}/X} \colon (A \textstyle\coprod B) \times_X C \overset{\sim}{\dashrightarrow} (A \times_X C) \textstyle\coprod (B \times_X C)$$

being similar. The proof of the naturality of $\delta_\ell^{\mathsf{Sets}/X}$ and $\delta_r^{\mathsf{Sets}/X}$ is omitted.
Item 9, Annihilation With the Empty Set: We have

$$\begin{aligned} A \times_X \emptyset &\stackrel{\text{def}}{=} \{(a,b) \in A \times \emptyset \mid f(a) = g(b)\} \\ &= \{k \in \emptyset \mid f(a) = g(b)\} \\ &= \emptyset, \end{aligned}$$

and similarly for $\emptyset \times_X A$, where we have used Item 8 of Definition 4.1.3.1.3. The proof of the naturality of $\zeta_\ell^{\mathsf{Sets}/X}$ and $\zeta_r^{\mathsf{Sets}/X}$ is omitted.
Item 10, Interaction With Products: We have

$$\begin{aligned} A \times_{\mathrm{pt}} B &\stackrel{\text{def}}{=} \{(a,b) \in A \times B \mid !_A(a) = !_B(b)\} \\ &\stackrel{\text{def}}{=} \{(a,b) \in A \times B \mid \star = \star\} \\ &= \{(a,b) \in A \times B\} \\ &= A \times B. \end{aligned}$$

Item 11, Symmetric Monoidality: Omitted. □

4.1.5 Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 4.1.5.1.1. The **equaliser of** f **and** g is the equaliser of f and g in Sets as in **??**.

Construction 4.1.5.1.2. Concretely, the equaliser of f and g is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

1. *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by

$$\mathrm{Eq}(f,g) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$

2. *The Cone.* The inclusion map

$$\mathrm{eq}(f,g) \colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \longrightarrow \mathrm{Eq}(f,g)$ making the diagram

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & & \end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. □

Proposition 4.1.5.1.3. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[5]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f \circ \mathrm{eq}(g,h), h \circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))},$$

[5]That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\xrightarrow{\;-g\to\;}}} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

---

1. Take the equaliser of (f, g, h), i.e. the limit of the diagram

$$A \overset{f}{\underset{h}{\xrightarrow{\;-g\to\;}}} B$$

in Sets.

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))$$

of $\mathrm{Eq}(f, g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g, h), h \circ \mathrm{eq}(g, h))$$

of $\mathrm{Eq}(g, h)$.

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f, g), k \circ g \circ \mathrm{eq}(f, g))$ is the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1*, *Associativity*: We first prove that $\mathrm{Eq}(f, g, h)$ is indeed given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \longrightarrow \mathrm{Eq}(f, g, h)$, uniquely determined by the condition

$$\mathrm{eq}(f, g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f, g, h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives
$$f(e(x)) = g(e(x)) = h(e(x))$$
for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f, g, h)$.

We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) \cong \mathrm{Eq}(f,g,h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) &\cong \{x \in \mathrm{Eq}(g,h) \mid [f \circ \mathrm{eq}(g,h)](a) = [g \circ \mathrm{eq}(g,h)](a)\} \\
&\cong \{x \in \mathrm{Eq}(g,h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)) &\cong \{x \in \mathrm{Eq}(f,g) \mid [f \circ \mathrm{eq}(f,g)](a) = [h \circ \mathrm{eq}(f,g)](a)\} \\
&\cong \{x \in \mathrm{Eq}(f,g) \mid f(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Item 4, *Unitality*: Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f,f) &\overset{\text{def}}{=} \{a \in A \mid f(a) = f(a)\} \\
&= A.
\end{aligned}$$

Item 5, *Commutativity*: Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f,g) &\overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\} \\
&= \{a \in A \mid g(a) = f(a)\} \\
&\overset{\text{def}}{=} \mathrm{Eq}(g,f).
\end{aligned}$$

Item 6, *Interaction With Composition*: Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) &\cong \{a \in \mathrm{Eq}(f,g) \mid h(f(a)) = k(g(a))\} \\
&\cong \{a \in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.
\end{aligned}$$

and
$$\mathrm{Eq}(h \circ f, k \circ g) \cong \{a \in A \mid h(f(a)) = k(g(a))\},$$
and thus there's an inclusion from $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h \circ f, k \circ g)$. □

4.1.6 Inverse Limits

Let $\left(X_\alpha, f_{\alpha\beta}\right)_{\alpha,\beta\in I}\colon (I, \preceq) \to \mathsf{Sets}$ be an inverse system of sets.

Definition 4.1.6.1.1. The **inverse limit of** $\left(X_\alpha, f_{\alpha\beta}\right)_{\alpha,\beta\in I}$ is the inverse limit of $\left(X_\alpha, f_{\alpha\beta}\right)_{\alpha,\beta\in I}$ in Sets as in **??**.

Construction 4.1.6.1.2. Concretely, the inverse limit of $\left(X_\alpha, f_{\alpha\beta}\right)_{\alpha,\beta\in I}$ is the pair $\left(\varprojlim_{\alpha\in I}(X_\alpha), \{\mathrm{pr}_\alpha\}_{\alpha\in I}\right)$ consisting of:

1. *The Limit.* The set $\varprojlim_{\alpha\in I}(X_\alpha)$ defined by

$$\varprojlim_{\alpha\in I}(X_\alpha) \stackrel{\text{def}}{=} \left\{ (x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha \;\middle|\; \begin{array}{l} \text{for each } \alpha,\beta\in I, \text{ if } \alpha\preceq\beta, \\ \text{then we have } x_\alpha = f_{\alpha\beta}(x_\beta) \end{array} \right\}.$$

2. *The Cone.*The collection

$$\left\{ \mathrm{pr}_\gamma \colon \varprojlim_{\alpha\in I}(X_\alpha) \to X_\gamma \right\}_{\gamma\in I}$$

of maps of sets defined as the restriction of the maps

$$\left\{ \mathrm{pr}_\gamma \colon \prod_{\alpha\in I} X_\alpha \to X_\gamma \right\}_{\gamma\in I}$$

of Item 2 of Definition 4.1.2.1.2 to $\varprojlim_{\alpha\in I}(X_\alpha)$ and hence given by

$$\mathrm{pr}_\gamma((x_\alpha)_{\alpha\in I}) \stackrel{\text{def}}{=} x_\gamma$$

for each $\gamma\in I$ and each $(x_\alpha)_{\alpha\in I} \in \varprojlim_{\alpha\in I}(X_\alpha)$.

Proof. We claim that $\varprojlim_{\alpha\in I}(X_\alpha)$ is the limit of the inverse system of sets $\left(X_\alpha, f_{\alpha\beta}\right)_{\alpha,\beta\in I}$. First we need to check that the limit diagram defined by it commutes, i.e. that we have

$$f_{\alpha\beta}\circ \mathrm{pr}_\alpha = \mathrm{pr}_\beta, \qquad \begin{tikzcd} & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"'] \arrow[dr, "\mathrm{pr}_\beta"] & \\ X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \end{tikzcd}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $(x_\gamma)_{\gamma\in I} \in \varprojlim_{\gamma\in I}(X_\gamma)$, we have

$$\begin{aligned}
[f_{\alpha\beta} \circ \mathrm{pr}_\alpha]\Big((x_\gamma)_{\gamma\in I}\Big) &\stackrel{\text{def}}{=} f_{\alpha\beta}\Big(\mathrm{pr}_\alpha\Big((x_\gamma)_{\gamma\in I}\Big)\Big) \\
&\stackrel{\text{def}}{=} f_{\alpha\beta}(x_\alpha) \\
&= x_\beta \\
&\stackrel{\text{def}}{=} \mathrm{pr}_\beta\Big((x_\gamma)_{\gamma\in I}\Big),
\end{aligned}$$

where the third equality comes from the definition of $\varprojlim_{\alpha\in I}(X_\alpha)$. Next, we prove that $\varprojlim_{\alpha\in I}(X_\alpha)$ satisfies the universal property of an inverse limit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{tikzcd}
 & L \arrow[ddl, bend right, "p_\alpha"'] \arrow[ddr, bend left, "p_\beta"] & \\
 & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"] \arrow[dr, "\mathrm{pr}_\beta"'] & \\
X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta
\end{tikzcd}$$

in Sets. Then there indeed exists a unique map $\phi\colon L \xrightarrow{\exists!} \varprojlim_{\alpha\in I}(X_\alpha)$ making the diagram

$$\begin{tikzcd}
 & L \arrow[d, dashed, "\phi"', "\exists!"] \arrow[ddl, bend right, "p_\alpha"'] \arrow[ddr, bend left, "p_\beta"] & \\
 & \varprojlim_{\alpha\in I}(X_\alpha) \arrow[dl, "\mathrm{pr}_\alpha"] \arrow[dr, "\mathrm{pr}_\beta"'] & \\
X_\alpha \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta
\end{tikzcd}$$

commute, being uniquely determined by the family of conditions

$$\{p_\alpha = \mathrm{pr}_\alpha \circ \phi\}_{\alpha\in I}$$

via

$$\phi(\ell) = (p_\alpha(\ell))_{\alpha\in I}$$

for each $\ell \in L$, where we note that $(p_\alpha(\ell))_{\alpha\in I} \in \prod_{\alpha\in I} X_\alpha$ indeed lies in $\lim_{\overleftarrow{\alpha\in I}}(X_\alpha)$, as we have

$$\begin{aligned} f_{\alpha\beta}(p_\alpha(\ell)) &\overset{\text{def}}{=} \left[f_{\alpha\beta} \circ p_\alpha\right](\ell) \\ &\overset{\text{def}}{=} p_\beta(\ell) \end{aligned}$$

for each $\beta \in I$ with $\alpha \preceq \beta$ by the commutativity of the diagram for $\left(L, \{p_\alpha\}_{\alpha\in I}\right)$. □

Example 4.1.6.1.3. Here are some examples of inverse limits of sets.

1. *The p-Adic Integers.* The ring of p-adic integers $\mathbb{Z}_p$ of **??** is the inverse limit

$$\mathbb{Z}_p \cong \lim_{\overleftarrow{n\in\mathbb{N}}}\left(\mathbb{Z}_{/p^n}\right);$$

   see **??**.

2. *Rings of Formal Power Series.* The ring $R[\![t]\!]$ of formal power series in a variable t is the inverse limit

$$R[\![t]\!] \cong \lim_{\overleftarrow{n\in\mathbb{N}}}(R[t]/t^n R[t]);$$

   see **??**.

3. *Profinite Groups.* Profinite groups are inverse limits of finite groups; see **??**.

4.2 Colimits of Sets

4.2.1 The Initial Set

Definition 4.2.1.1.1. The **initial set** is the initial object of Sets as in **??**.

Construction 4.2.1.1.2. Concretely, the initial set is the pair $\left(\emptyset, \{\iota_A\}_{A\in\mathrm{Obj(Sets)}}\right)$ consisting of:

1. *The Colimit.* The empty set $\emptyset$ of Definition 4.3.1.1.1.

2. *The Cocone.* The collection of maps

$$\{\iota_A \colon \emptyset \longrightarrow A\}_{A\in\mathrm{Obj(Sets)}}$$

   given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi\colon \emptyset \to A$ making the diagram

$$\emptyset \xrightarrow[\exists!]{\phi} A$$

commute, namely the inclusion map ι_A. □

4.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

Definition 4.2.2.1.1. The **coproduct of** $\{A_i\}_{i\in I}$[6] is the coproduct of $\{A_i\}_{i\in I}$ in Sets as in **??**.

Construction 4.2.2.1.2. Concretely, the disjoint union of $\{A_i\}_{i\in I}$ is the pair $\left(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I}\right)$ consisting of:

1. *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \stackrel{\text{def}}{=} \left\{ (i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \;\middle|\; x \in A_i \right\}.$$

2. *The Cocone.* The collection
$$\left\{ \mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i \right\}_{i\in I}$$
of maps given by
$$\mathrm{inj}_i(x) \stackrel{\text{def}}{=} (i,x)$$
for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets.

[6] *Further Terminology:* Also called the **disjoint union of the family** $\{A_i\}_{i\in I}$.

Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[r, "\mathrm{inj}_i"'] \arrow[ur, "\iota_i"] & \coprod_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[r, "\mathrm{inj}_i"'] \arrow[ur, "\iota_i"] & \coprod_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi((i,x)) = \iota_i(x)$$

for each $(i,x) \in \coprod_{i\in I} A_i$. □

Proposition 4.2.2.1.3. Let $\{A_i\}_{i\in I}$ be a family of sets.

1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}}\colon \mathrm{Nat}((A_i)_{i\in I}, (B_i)_{i\in I}) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

of $\coprod_{i\in I}$ at $((A_i)_{i\in I}, (B_i)_{i\in I})$ is defined by sending a map

$$\{f_i\colon A_i \longrightarrow B_i\}_{i\in I}$$

in $\mathsf{Nat}((A_i)_{i\in I}, (B_i)_{i\in I})$ to the map of sets

$$\coprod_{i\in I} f_i\colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

defined by

$$\left[\coprod_{i\in I} f_i\right](i,a) \stackrel{\text{def}}{=} f_i(a)$$

for each $(i,a) \in \coprod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

4.2.3 Binary Coproducts

Let A and B be sets.

Definition 4.2.3.1.1. The **coproduct of** A **and** B[7] is the coproduct of A and B in Sets as in **??**.

Construction 4.2.3.1.2. Concretely, the coproduct of A and B is the pair $\left(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\}\right)$ consisting of:

1. *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\stackrel{\text{def}}{=} \{(0,a) \in S \mid a \in A\} \cup \{(1,b) \in S \mid b \in B\}, \end{aligned}$$

   where $S = \{0,1\} \times (A \cup B)$.

2. *The Cocone.* The maps

$$\begin{aligned} \mathrm{inj}_1 &\colon A \longrightarrow A \coprod B, \\ \mathrm{inj}_2 &\colon B \longrightarrow A \coprod B, \end{aligned}$$

   given by

$$\begin{aligned} \mathrm{inj}_1(a) &\stackrel{\text{def}}{=} (0,a), \\ \mathrm{inj}_2(b) &\stackrel{\text{def}}{=} (1,b), \end{aligned}$$

   for each $a \in A$ and each $b \in B$.

[7] *Further Terminology:* Also called the **disjoint union of** A **and** B.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & A \coprod B & B \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"]
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"]
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_A &= \iota_A, \\
\phi \circ \mathrm{inj}_B &= \iota_B
\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a), \\ \iota_B(b) & \text{if } x = (1, b) \end{cases}$$

for each $x \in A \coprod B$. □

Proposition 4.2.3.1.3. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned}
A \coprod - &\colon \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\
- \coprod B &\colon \quad \mathsf{Sets} \quad \to \mathsf{Sets}, \\
-_1 \coprod -_2 &\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets},
\end{aligned}$$

   where $-_1 \coprod -_2$ is the functor where

   - *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$

   - *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)}\colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A,B),(X,Y))$ is defined by sending (f,g) to the function

$$f \coprod g \colon A \coprod B \to X \coprod Y$$

defined by

$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0,a), \\ (1, g(b)) & \text{if } x = (1,b), \end{cases}$$

for each $x \in A \coprod B$.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

2. *Adjointness.* We have an adjunction

$$(-_1 \coprod -_2 \dashv \Delta_{\mathsf{Sets}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \times \mathsf{Sets} \arrow[r, bend left, "-_1 \coprod -_2"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\Delta_{\mathsf{Sets}}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}(A \coprod B, C), \cong \mathrm{Hom}_{\mathsf{Sets} \times \mathsf{Sets}}((A,B),(C,C))$$

natural in $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$ and in $C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Associativity.* We have an isomorphism of sets

$$\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z} \colon (X \coprod Y) \coprod Z \xrightarrow{\sim} X \coprod (Y \coprod Z),$$

natural in $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Unitality.* We have isomorphisms of sets

$$\lambda^{\mathsf{Sets},\coprod}_{X} \colon \emptyset \coprod X \xrightarrow{\sim} X,$$
$$\rho^{\mathsf{Sets},\coprod}_{X} \colon X \coprod \emptyset \xrightarrow{\sim} X,$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

5. *Commutativity.* We have an isomorphism of sets

$$\sigma^{\mathsf{Sets},\coprod}_{X,Y} \colon X \coprod Y \xrightarrow{\sim} Y \coprod X,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Symmetric Monoidality.* The 7-tuple $\left(\mathsf{Sets}, \coprod, \emptyset, \alpha^{\mathsf{Sets}}_{\coprod}, \lambda^{\mathsf{Sets}}_{\coprod}, \rho^{\mathsf{Sets}}_{\coprod}, \sigma^{\mathsf{Sets}}\right)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Adjointness: This follows from the universal property of the coproduct.
Item 3, Associativity: This is proved in the proof of Definition 5.2.3.1.1.
Item 4, Unitality: This is proved in the proof of Definitions 5.2.4.1.1 and 5.2.5.1.1.
Item 5, Commutativity: This is proved in the proof of Definition 5.2.6.1.1.
Item 6, Symmetric Monoidality: This is a repetition of Definition 5.2.7.1.1, and is proved there. □

4.2.4 Pushouts

Let A, B, and C be sets and let $f: C \to A$ and $g: C \to B$ be functions.

Definition 4.2.4.1.1. The **pushout of A and B over C along f and g**[8] is the pushout of A and B over C along f and g in Sets as in **??**.

Construction 4.2.4.1.2. Concretely, the pushout of A and B over C along f and g is the pair $\left(A \coprod_C B, \left\{\mathrm{inj}_1, \mathrm{inj}_2\right\}\right)$ consisting of:

1. *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

2. *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 &: A \to A \textstyle\coprod_C B,\\ \mathrm{inj}_2 &: B \to A \textstyle\coprod_C B\end{aligned}$$

given by

$$\begin{aligned}\mathrm{inj}_1(a) &\overset{\text{def}}{=} [(0, a)]\\ \mathrm{inj}_2(b) &\overset{\text{def}}{=} [(1, b)]\end{aligned}$$

for each $a \in A$ and each $b \in B$.

[8] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and**

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned} \left[\mathrm{inj}_1 \circ f\right](c) &= \mathrm{inj}_1(f(c)) \\ &= [(0, f(c))] \\ &= [(1, g(c))] \\ &= \mathrm{inj}_2(g(c)) \\ &= \left[\mathrm{inj}_2 \circ g\right](c), \end{aligned}$$

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd} P & & \\ & A \coprod_C B \arrow[ul, dashed] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\ & A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"'] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon A \coprod_C B \to P$ making the

---

g.

diagram

$$\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\phi", "\exists!"'] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] \arrow[ur, phantom, "\ulcorner" very near end] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').$$

2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').$$

3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by Definition 4.2.4.1.3, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\phi([x]) = \phi([(0, f(c))])$$

$$
\begin{aligned}
&\stackrel{\text{def}}{=} \iota_1(f(c)) \\
&= \iota_2(g(c)) \\
&\stackrel{\text{def}}{=} \phi([(1, g(c))]) \\
&= \phi([y]),
\end{aligned}
$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$
\begin{aligned}
(0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\
(1, b) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\
(0, a) &\sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)
\end{aligned}
$$

gives

$$
\begin{aligned}
\phi([(0, a)]) &= \phi([(0, a')]), \\
\phi([(1, b)]) &= \phi([(1, b')]), \\
\phi([(0, a)]) &= \phi([(1, b)]),
\end{aligned}
$$

showing ϕ to be well-defined. □

Remark 4.2.4.1.3. In detail, by Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a, b \in A$ and $a = b$.
2. We have $a, b \in B$ and $a = b$.
3. There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
   (a) There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
   (b) There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

   In other words, there exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

   (c) There exists $c_0 \in C$ satisfying one of the following conditions:
      i. We have $a = f(c_0)$ and $x_1 = g(c_0)$.
      ii. We have $a = g(c_0)$ and $x_1 = f(c_0)$.

(d) For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:

i. We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.

ii. We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.

(e) There exists $c_n \in C$ satisfying one of the following conditions:

i. We have $x_n = f(c_n)$ and $b = g(c_n)$.

ii. We have $x_n = g(c_n)$ and $b = f(c_n)$.

Remark 4.2.4.1.4. It is common practice to write $A \coprod_C B$ for the pushout of A and B over C along f and g, omitting the maps f and g from the notation and instead leaving them implicit, to be understood from the context.

However, the set $A \coprod_C B$ depends very much on the maps f and g, and sometimes it is necessary or useful to note this dependence explicitly. In such situations, we will write $A \coprod_{f,C,g} B$ or $A \coprod_C^{f,g} B$ for $A \coprod_C B$.

Example 4.2.4.1.5. Here are some examples of pushouts of sets.

1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Definition 6.3.3.1.1 is an example of a pushout of sets.

2. *Intersections via Unions.* Let X be a set. We have

$$A \cup B \cong A \textstyle\coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u] & A \cap B \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

for each $A, B \in \mathcal{P}(X)$.

Proof. *Item 1, Wedge Sums of Pointed Sets*: This follows by definition, as the wedge sum of two pointed sets is defined as a pushout.

Item 2, Intersections via Unions: Indeed, $A \coprod_{A\cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. □

Proposition 4.2.4.1.6. Let A, B, C, and X be sets.

1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \textstyle\coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd}
& \bullet \\
\bullet & \bullet. \arrow[l] \arrow[u]
\end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Adjointness.* We have an adjunction

$$\left(-_1 \coprod_X -_2 \dashv \Delta_{\mathsf{Sets}_{X/}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/} \arrow[r, bend left, "-_1 \coprod_X -_2"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{X/}, \arrow[l, bend left, "\Delta_{\mathsf{Sets}_{X/}}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_{X/}(A \coprod{}_X B, C), \cong \mathrm{Hom}_{\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/}}((A, B), (C, C))$$

natural in $(A, B) \in \mathrm{Obj}(\mathsf{Sets}_{X/} \times \mathsf{Sets}_{X/})$ and in $C \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

3. *Associativity.* Given a diagram

$$\begin{tikzcd} A & & B & & C \\ & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \textstyle\coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C)$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small] & & (A \coprod_X B) \coprod_Y C & & \\ & A \coprod_X B \arrow[ur] & & & \\ A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\ & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\ & A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\ A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\ & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & & A \coprod_X (B \coprod_Y C) & & \\ & & & B \coprod_Y C \arrow[ul] & \\ A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\ & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] & \end{tikzcd}$$

4. *Interaction With Composition.* Given a diagram

$$\begin{tikzcd} X & & & & Y \\ & A \arrow[ul, "\phi"] & & B \arrow[ur, "\psi"'] & \\ & & K \arrow[ul, "f"] \arrow[ur, "g"'] & & \end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$X \coprod{}^{\phi\circ f, \psi\circ g}_K Y \cong \left(X \coprod{}^{\phi, j_1}_A \left(A \coprod{}^{f,g}_K B\right)\right) \coprod{}^{i_2, i_1}_{A \coprod^{f,g}_K B} \left(\left(A \coprod{}^{f,g}_K B\right) \coprod{}^{j_2, \psi}_B Y\right)$$

$$
\begin{aligned}
&\cong X \coprod\nolimits_A^{\phi,i}\left(\left(A \coprod\nolimits_K^{f,g} B\right) \coprod\nolimits_B^{j_2,\psi} Y\right)\\
&\cong \left(X \coprod\nolimits_A^{\phi,i_1}\left(A \coprod\nolimits_K^{f,g} B\right)\right) \coprod\nolimits_B^{j,\psi} Y
\end{aligned}
$$

where

$$
\begin{aligned}
j_1 &= \mathrm{inj}_1^{A\times_K^{f,g}B}, & j_2 &= \mathrm{inj}_2^{A\times_K^{f,g}B},\\
i_1 &= \mathrm{inj}_1^{\left(A\times_K^{f,g}B\right)\times_Y^{q_2,\psi}}, & i_2 &= \mathrm{inj}_2^{X\times^{\phi,q_1}_{A\times_K^{f,g}B}\left(A\times_K^{f,g}B\right)},\\
i &= j_1 \circ \mathrm{inj}_1^{\left(A\times_K^{f,g}B\right)\times_B^{q_2,\psi}Y}, & j &= j_2 \circ \mathrm{inj}_2^{X\times_A^{\phi,q_1}\left(A\times_K^{f,g}B\right)},
\end{aligned}
$$

and where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}
 & & (X \coprod_A (A \coprod_K B)) \coprod_{A \coprod_K B} ((A \coprod_K B) \coprod_B Y) \arrow[d, phantom, "\wedge"] & & \\
 & X \coprod_A (A \coprod_K B) \arrow[ur, hook] & {} & (A \coprod_K B) \coprod_B Y \arrow[ul, hook'] & \\
X \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & A \coprod_K B \arrow[ul, hook'] \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & Y, \arrow[ul, hook'] \\
 & A \arrow[ul, "\phi"] \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & B \arrow[ul, hook'] \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X \coprod_A ((A \coprod_K B) \coprod_B Y) \arrow[d, phantom, "\wedge"] & & \\
 & & {} & (A \coprod_K B) \coprod_B Y \arrow[ul, hook'] & \\
X \arrow[uurr, hook] & & A \coprod_K B \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & Y, \arrow[ul, hook'] \\
 & A \arrow[ul, "\phi"] \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & B \arrow[ul, hook'] \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & (X \coprod_A (A \coprod_K B)) \coprod_B Y \arrow[d, phantom, "\wedge"] & & \\
 & X \coprod_A (A \coprod_K B) \arrow[ur, hook] & {} & & \\
X \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & A \coprod_K B \arrow[ul, hook'] & & Y, \arrow[uull, hook'] \\
 & A \arrow[ul, "\phi"] \arrow[ur, hook] \arrow[rr, phantom, "\wedge"] & & B \arrow[ul, hook'] \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X \coprod_K Y \arrow[d, phantom, "\wedge"] & & \\
 & & {} & & \\
X \arrow[uurr, hook] & & & & Y. \arrow[uull, hook'] \\
 & A \arrow[ul, "\phi"] & & B \arrow[ur, "\psi"'] & \\
 & & K \arrow[ul, "f"] \arrow[ur, "g"'] & &
\end{tikzcd}
$$

5. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[dr, phantom, very near start, "\ulcorner"] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
\lambda_A^{\mathsf{Sets}_{X/}} &: X \coprod\nolimits_X A \overset{\sim}{\dashrightarrow} A,\\
\rho_A^{\mathsf{Sets}_{X/}} &: A \coprod\nolimits_X X \overset{\sim}{\dashrightarrow} A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X, \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}
$$

natural in $(A, f) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

6. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
A \coprod_X B & B \arrow[l] \\
A \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\quad
\sigma^{\mathsf{Sets}_{X/}}_{A} \colon A \coprod_X B \xrightarrow{\sim} B \coprod_X A
\quad
\begin{tikzcd}
B \coprod_X A & A \arrow[l] \\
B \arrow[u] \arrow[ur, phantom, "\ulcorner"] & X. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}
$$

natural in $(A, f), (B, g) \in \mathrm{Obj}(\mathsf{Sets}_{X/})$.

7. *Interaction With Coproducts.* We have

$$
A \coprod_{\emptyset} B \cong A \coprod B,
\quad
\begin{tikzcd}
A \coprod B & B \arrow[l] \\
A \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \emptyset. \arrow[l, hook, "\iota_A"] \arrow[u, hook, "\iota_B"']
\end{tikzcd}
$$

8. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_{X/}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, *: Adjointness*: This follows from the universal property of the coproduct (pushouts are coproducts in $\mathsf{Sets}_{X/}$).
Item 3, *Associativity*: Omitted.
Item 4, *Interaction With Composition*: Omitted.
Item 5, *Unitality*: Omitted.
Item 6, *Commutativity*: Omitted.
Item 7, *Interaction With Coproducts*: Omitted.
Item 8, *Symmetric Monoidality*: Omitted. □

4.2.5 Coequalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

Definition 4.2.5.1.1. The **coequaliser of** f **and** g is the coequaliser of f and g in Sets as in **??**.

Construction 4.2.5.1.2. Concretely, the coequaliser of f and g is the pair $(\mathrm{CoEq}(f, g), \mathrm{coeq}(f, g))$ consisting of:

1. *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by
$$\mathrm{CoEq}(f,g) \overset{\mathrm{def}}{=} B/{\sim},$$
where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

2. *The Cocone.* The map
$$\mathrm{coeq}(f,g)\colon B \twoheadrightarrow \mathrm{CoEq}(f,g)$$
given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have
$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$
Indeed, we have
$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\overset{\mathrm{def}}{=} [f(a)] \\
&= [g(a)] \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\overset{\mathrm{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$
for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form
$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \\
& & C
\end{tikzcd}$$
in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram
$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\
& & C
\end{tikzcd}$$
commute. □

Remark 4.2.5.1.3. In detail, by Definition 10.5.2.1.2, the relation $\sim$ of Definition 4.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

1. We have $a = b$;
2. There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
   (a) There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
   (b) There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

   In other words, there exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:

   (a) There exists $z_0 \in A$ satisfying one of the following conditions:
      i. We have $a = f(z_0)$ and $x_1 = g(z_0)$.
      ii. We have $a = g(z_0)$ and $x_1 = f(z_0)$.
   (b) For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
      i. We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
      ii. We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
   (c) There exists $z_n \in A$ satisfying one of the following conditions:
      i. We have $x_n = f(z_n)$ and $b = g(z_n)$.
      ii. We have $x_n = g(z_n)$ and $b = f(z_n)$.

Example 4.2.5.1.4. Here are some examples of coequalisers of sets.

1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/{\sim_R} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof. *Item 1, Quotients by Equivalence Relations*: See [Pro25ae]. □

Proposition 4.2.5.1.5. Let A, B, and C be sets.

1. *Associativity.* We have isomorphisms of sets[9]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

---

[9]That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

of $\mathrm{CoEq}(g,h)$.

4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,f) \cong B.$$

5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h,k) \circ h \circ f, \mathrm{coeq}(h,k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Omitted.
Item 5, *Commutativity*: Omitted.
Item 6, *Interaction With Composition*: Omitted. □

4.2.6 Direct Colimits

Let $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I} \colon (I, \preceq) \to \mathbb{T}$ be a direct system of sets.

Definition 4.2.6.1.1. The **direct colimit of** $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ in Sets as in **??**.

Construction 4.2.6.1.2. Concretely, the direct colimit of $(X_\alpha, f_{\alpha\beta})_{\alpha,\beta \in I}$ is the pair $\left(\underset{\overrightarrow{\alpha \in I}}{\mathrm{colim}}(X_\alpha), \{\mathrm{inj}_\alpha\}_{\alpha \in I}\right)$ consisting of:

1. *The Colimit.* The set $\underset{\overrightarrow{\alpha \in I}}{\mathrm{colim}}(X_\alpha)$ defined by

$$\underset{\overrightarrow{\alpha \in I}}{\mathrm{colim}}(X_\alpha) \overset{\mathrm{def}}{=} \left(\coprod_{\alpha \in I} X_\alpha\right) \Big/ \sim,$$

where $\sim$ is the equivalence relation on $\coprod_{\alpha \in I} X_\alpha$ generated by declaring $(\alpha, x) \sim (\beta, y)$ iff there exists some $\gamma \in I$ satisfying the following conditions:

(a) We have $\alpha \preceq \gamma$.

(b) We have $\beta \preceq \gamma$.

(c) We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

2. *The Cocone.*The collection

$$\left\{ \mathrm{inj}_\gamma \colon X_\gamma \to \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) \right\}_{\gamma\in I}$$

of maps of sets defined by

$$\mathrm{inj}_\gamma(x) \overset{\text{def}}{=} [(\gamma, x)]$$

for each $\gamma \in I$ and each $x \in X_\gamma$.

Proof. We will prove Definition 4.2.6.1.2 below in a bit, but first we need a lemma (which is interesting in its own right). □

Lemma 4.2.6.1.3. For each $\alpha, \beta \in I$ and each $x \in X_\alpha$, if $\alpha \preceq \beta$, then we have

$$(\alpha, x) \sim \big(\beta, f_{\alpha\beta}(x)\big)$$

in $\operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha)$.

Proof. Taking $\gamma = \beta$, we have $f_{\alpha\gamma} = f_{\alpha\beta}$, we have $f_{\beta\gamma} = f_{\beta\beta} \overset{\text{def}}{=} \mathrm{id}_{X_\beta}$, and we have

$$\begin{aligned} f_{\alpha\beta}(x) &= f_{\beta\beta}\big(f_{\alpha\beta}(x)\big) \\ &\overset{\text{def}}{=} \mathrm{id}_{X_\beta}\big(f_{\alpha\beta}(x)\big), \\ &= f_{\alpha\beta}(x). \end{aligned}$$

As a result, since $\alpha \preceq \beta$ and $\beta \preceq \beta$ as well, Items 1a to 1c of Definition 4.2.6.1.2 are met. Thus we have $(\alpha, x) \sim \big(\beta, f_{\alpha\beta}(x)\big)$. □

We can now prove Definition 4.2.6.1.2:

Proof. We claim that $\operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha)$ is the colimit of the direct system of sets $\big(X_\alpha, f_{\alpha\beta}\big)_{\alpha,\beta\in I}$.

Commutativity of the Colimit Diagram: First, we need to check that the colimit diagram defined by $\operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha)$ commutes, i.e. that we have

$$\mathrm{inj}_\alpha = \mathrm{inj}_\beta \circ f_{\alpha\beta}, \qquad \begin{tikzcd} & \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) & \\ X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[ul, "\mathrm{inj}_\beta"'] \end{tikzcd}$$

for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$. Indeed, given $x \in X_\alpha$, we have

$$\begin{aligned} \left[\mathrm{inj}_\beta \circ f_{\alpha\beta}\right](x) &\stackrel{\text{def}}{=} \mathrm{inj}_\beta(f_{\alpha\beta}(x)) \\ &\stackrel{\text{def}}{=} [(\beta, f_{\alpha\beta}(x))] \\ &= [(\alpha, x)] \\ &\stackrel{\text{def}}{=} \mathrm{inj}_\alpha(x), \end{aligned}$$

where we have used Definition 4.2.6.1.3 for the third equality.

Proof of the Universal Property of the Colimit: Next, we prove that $\operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha)$ as constructed in Definition 4.2.6.1.2 satisfies the universal property of a direct colimit. Suppose that we have, for each $\alpha, \beta \in I$ with $\alpha \preceq \beta$, a diagram of the form

$$\begin{tikzcd} & C & \\ & \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) & \\ X_\alpha \arrow[uur, bend left, "i_\alpha"] \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[uul, bend right, "i_\beta"'] \arrow[ul, "\mathrm{inj}_\beta"'] \end{tikzcd}$$

in Sets. We claim that there exists a unique map $\phi\colon \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) \xrightarrow{\exists!} C$

making the diagram

$$\begin{tikzcd} & C & \\ & \underset{\alpha\in I}{\underrightarrow{\mathrm{colim}}}(X_\alpha) \arrow[u, dashed, "\phi", "\exists!"'] & \\ X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[uur, bend left, "i_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta \arrow[ul, "\mathrm{inj}_\beta"'] \arrow[uul, bend right, "i_\beta"'] \end{tikzcd}$$

commute. To this end, first consider the diagram

$$\begin{tikzcd} \coprod\limits_{\alpha\in I} X_\alpha \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, "\coprod\limits_{\alpha\in I} i_\alpha"'] & \underset{\alpha\in I}{\underrightarrow{\mathrm{colim}}}(X_\alpha) \\ & C. \end{tikzcd}$$

Lemma. If $(\alpha, x) \sim (\beta, y)$, then we have

$$\left[\coprod_{\alpha\in I} i_\alpha\right](x) = \left[\coprod_{\alpha\in I} i_\alpha\right](y).$$

Proof. Indeed, if $(\alpha, x) \sim (\beta, y)$, then there exists some $\gamma \in I$ satisfying the following conditions:

1. We have $\alpha \preceq \gamma$.

2. We have $\beta \preceq \gamma$.

3. We have $f_{\alpha\gamma}(x) = f_{\beta\gamma}(y)$.

We then have

$$\begin{aligned} \left[\coprod_{\alpha\in I} i_\alpha\right](x) &\stackrel{\text{def}}{=} i_\alpha(x) \\ &\stackrel{\text{def}}{=} \left[i_\gamma \circ f_{\alpha\gamma}\right](x) \\ &\stackrel{\text{def}}{=} i_\gamma\big(f_{\alpha\gamma}(x)\big) \end{aligned}$$

$$
\begin{aligned}
&= i_\gamma\big(f_{\beta\gamma}(x)\big)\\
&\overset{\text{def}}{=} \big[i_\gamma \circ f_{\beta\gamma}\big](x)\\
&= i_\beta(y)\\
&\overset{\text{def}}{=} \left[\coprod_{\alpha\in I} i_\alpha\right](y).
\end{aligned}
$$

This finishes the proof of the lemma. Continuing, by **??** of Definition 10.6.2.1.3, there then exists a map $\phi\colon \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) \xrightarrow{\exists!} C$ making the diagram

$$
\begin{tikzcd}
\coprod_{\alpha\in I} X_\alpha \arrow[r, two heads, "\mathrm{pr}"] \arrow[dr, "\coprod_{\alpha\in I} i_\alpha"'] & \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) \arrow[d, "\phi"] \\
& C
\end{tikzcd}
$$

commute. In particular, this implies that the diagram

$$
\begin{tikzcd}
X_\alpha \arrow[r, two heads, "\mathrm{inj}_\alpha"] \arrow[dr, "i_\alpha"'] & \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) \arrow[d, "\phi"] \\
& C
\end{tikzcd}
$$

also commutes, and thus so does the diagram

$$
\begin{tikzcd}
& C & \\
& \operatorname*{colim}_{\overrightarrow{\alpha\in I}}(X_\alpha) \arrow[u, dashed, "\phi"', "\exists!"] & \\
X_\alpha \arrow[ur, "\mathrm{inj}_\alpha"] \arrow[uur, bend left, "i_\alpha"] \arrow[rr, "f_{\alpha\beta}"'] & & X_\beta. \arrow[ul, "\mathrm{inj}_\beta"'] \arrow[uul, bend right, "i_\beta"']
\end{tikzcd}
$$

This finishes the proof.[10] □

Example 4.2.6.1.4. Here are some examples of direct colimits of sets.

1. *The Prüfer Group.* The Prüfer group $\mathbb{Z}(p^\infty)$ is defined as the direct colimit
$$\mathbb{Z}(p^\infty) \stackrel{\text{def}}{=} \underset{\substack{\longrightarrow \\ n\in\mathbb{N}}}{\operatorname{colim}}\left(\mathbb{Z}_{/p^n}\right);$$
see **??**.

4.3 Operations With Sets

4.3.1 The Empty Set

Definition 4.3.1.1.1. The **empty set** is the set $\emptyset$ defined by

$$\emptyset \stackrel{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where X is the set in the set existence axiom, **??** of **??**.

4.3.2 Singleton Sets

Let X be a set.

Definition 4.3.2.1.1. The **singleton set containing** X is the set $\{X\}$ defined by
$$\{X\} \stackrel{\text{def}}{=} \{X, X\},$$
where $\{X, X\}$ is the pairing of X with itself of Definition 4.3.3.1.1.

[10] Incidentally, the conditions

$$\left\{i_\alpha = \phi \circ \operatorname{inj}_\alpha\right\}_{\alpha\in I}$$

show that ϕ must be given by

$$\phi([(\alpha, x)]) = (i_\alpha(x))_{\alpha\in I}$$

for each $[(\alpha, x)] \in \underset{\substack{\longrightarrow \\ \alpha\in I}}{\operatorname{colim}}(X_\alpha)$, although we would need to show that this assignment is well-defined were we to prove Definition 4.2.6.1.2 in this way. Instead, invoking **??** of Definition 10.6.2.1.3 gave us a way to avoid having to prove this, leading to a cleaner alternative proof.

4.3.3 Pairings of Sets

Let X and Y be sets.

Definition 4.3.3.1.1. The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

4.3.4 Ordered Pairs

Let A and B be sets.

Definition 4.3.4.1.1. The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

Proposition 4.3.4.1.2. Let A and B be sets.

1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

   (a) We have $(A, B) = (C, D)$.

   (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3]. □

4.3.5 Sets of Maps

Let A and B be sets.

Definition 4.3.5.1.1. The **set of maps from** A **to** B[11] is the set $\mathsf{Sets}(A, B)$[12] whose elements are the functions from A to B.

Proposition 4.3.5.1.2. Let A and B be sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathsf{Sets}(X, -) &\colon & \mathsf{Sets} &\to \mathsf{Sets}, \\ \mathsf{Sets}(-, Y) &\colon & \mathsf{Sets}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathsf{Sets}(-_1, -_2) &\colon & \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} &\to \mathsf{Sets}. \end{aligned}$$

[11] *Further Terminology:* Also called the **Hom set from** A **to** B.

[12] *Further Notation:* Also written $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$.

2. *Adjointness.* We have adjunctions

$$\begin{tikzcd}[ampersand replacement=\&] (A \times - \dashv \mathsf{Sets}(A,-))\colon \quad \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] \& \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(A,-)"] \end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&] (- \times B \dashv \mathsf{Sets}(B,-))\colon \quad \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] \& \mathsf{Sets}, \arrow[l, bend left, "\mathsf{Sets}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, C)), \\ \mathsf{Sets}(A \times B, C) &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Maps From the Punctual Set.* We have a bijection

$$\mathsf{Sets}(\mathrm{pt}, A) \cong A,$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Maps to the Punctual Set.* We have a bijection

$$\mathsf{Sets}(A, \mathrm{pt}) \cong \mathrm{pt},$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: This follows from Items 2 and 5 of Definition 11.1.4.1.2.
Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.1.3.1.3 and is proved there.
Item 3, Maps From the Punctual Set: The bijection

$$\Phi_A \colon \mathsf{Sets}(\mathrm{pt}, A) \xrightarrow{\sim} A$$

is given by

$$\Phi_A(f) \overset{\mathrm{def}}{=} f(\star)$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$, admitting an inverse

$$\Phi_A^{-1} \colon A \xrightarrow{\sim} \mathsf{Sets}(\mathrm{pt}, A)$$

given by

$$\Phi_A^{-1}(a) \overset{\mathrm{def}}{=} [\![\star \mapsto a]\!]$$

for each $a \in A$. Indeed, we have

$$\begin{aligned}
\left[\Phi_A^{-1} \circ \Phi_A\right](f) &\stackrel{\text{def}}{=} \Phi_A^{-1}(\Phi_A(f)) \\
&\stackrel{\text{def}}{=} \Phi_A^{-1}(f(\star)) \\
&\stackrel{\text{def}}{=} [\![\star \mapsto f(\star)]\!] \\
&\stackrel{\text{def}}{=} f \\
&\stackrel{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(\mathrm{pt},A)}\right](f)
\end{aligned}$$

for each $f \in \mathsf{Sets}(\mathrm{pt}, A)$ and

$$\begin{aligned}
\left[\Phi_A \circ \Phi_A^{-1}\right](a) &\stackrel{\text{def}}{=} \Phi_A\left(\Phi_A^{-1}(a)\right) \\
&\stackrel{\text{def}}{=} \Phi_A([\![\star \mapsto a]\!]) \\
&\stackrel{\text{def}}{=} \mathrm{ev}_\star([\![\star \mapsto a]\!]) \\
&\stackrel{\text{def}}{=} a \\
&\stackrel{\text{def}}{=} [\mathrm{id}_A](a)
\end{aligned}$$

for each $a \in A$, and thus we have

$$\begin{aligned}
\Phi_A^{-1} \circ \Phi_A &= \mathrm{id}_{\mathsf{Sets}(\mathrm{pt},A)} \\
\Phi_A \circ \Phi_A^{-1} &= \mathrm{id}_A .
\end{aligned}$$

To prove naturality, we need to show that the diagram

$$\begin{tikzcd}
\mathsf{Sets}(\mathrm{pt}, A) \arrow[r, "f_!"] \arrow[d, dashed, "\Phi_A"', "\wr"] & \mathsf{Sets}(\mathrm{pt}, B) \arrow[d, dashed, "\Phi_B", "\wr"'] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
[f \circ \Phi_A](\phi) &\stackrel{\text{def}}{=} f(\Phi_A(\phi)) \\
&\stackrel{\text{def}}{=} f(\phi(\star)) \\
&\stackrel{\text{def}}{=} [f \circ \phi](\star) \\
&\stackrel{\text{def}}{=} \Phi_B(f \circ \phi) \\
&\stackrel{\text{def}}{=} \Phi_B(f_!(\phi)) \\
&\stackrel{\text{def}}{=} [\Phi_B \circ f_!](\phi)
\end{aligned}$$

for each $\phi \in \mathsf{Sets}(\mathrm{pt}, A)$. This finishes the proof.

Item 4, Maps to the Punctual Set: This follows from the universal property of pt as the terminal set, Definition 4.1.1.1.1. □

4.3.6 Unions of Families of Subsets

Let X be a set and let $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.6.1.1. The **union of** $\mathcal{U}$ is the set $\bigcup_{U\in\mathcal{U}} U$ defined by

$$\bigcup_{U\in\mathcal{U}} U \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } U \in \mathcal{U}\\ \text{such that we have } x \in U\end{array} \right\}.$$

Proposition 4.3.6.1.2. Let X be a set.

1. *Functoriality.* The assignment $\mathcal{U} \mapsto \bigcup_{U\in\mathcal{U}} U$ defines a functor

$$\bigcup\colon (\mathcal{P}(\mathcal{P}(X)), \subset) \to (\mathcal{P}(X), \subset).$$

   In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

   ($\star$) If $\mathcal{U} \subset \mathcal{V}$, then $\displaystyle\bigcup_{U\in\mathcal{U}} U \subset \bigcup_{V\in\mathcal{V}} V$.

2. *Associativity.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\ \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X) \end{tikzcd}$$

   commutes, i.e. we have

$$\bigcup_{U\in\bigcup_{A\in\mathcal{A}} A} U = \bigcup_{A\in\mathcal{A}}\left(\bigcup_{U\in A} U\right)$$

   for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \arrow[d, "\chi_{\mathcal{P}(X)}"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\ \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V \in \{U\}} V = U$$

for each $U \in \mathcal{P}(X)$.

4. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\mathcal{P}(\chi_X)"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{\{u\} \in \chi_X(U)} \{u\} = U$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Unions I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cup"] \arrow[d, "\bigcup \times \bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W = \left(\bigcup_{U \in \mathcal{U}} U\right) \cup \left(\bigcup_{V \in \mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cup -)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "U \cup -"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cup V)"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \arrow[r, "- \cup V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U \cup \left(\bigcup_{V \in \mathcal{V}} V\right) = \bigcup_{V \in \mathcal{V}} (U \cup V),$$
$$\left(\bigcup_{U \in \mathcal{U}} U\right) \cup V = \bigcup_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cap"] \arrow[d, "\cup \times \cup"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X),
\end{tikzcd}$$

with components

$$\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W \subset \left(\bigcup_{U \in \mathcal{U}} U\right) \cap \left(\bigcup_{V \in \mathcal{V}} V\right)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)"] \arrow[d, "\cup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "U \cap -"'] & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)"] \arrow[d, "\cup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "- \cap V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U \cup \left(\bigcup_{V \in \mathcal{V}} V\right) = \bigcup_{V \in \mathcal{V}} (U \cup V),$$
$$\left(\bigcup_{U \in \mathcal{U}} U\right) \cup V = \bigcup_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\setminus"] \arrow[d, "\bigcup \times \bigcup"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\setminus"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcup_{U \in \mathcal{U}} U\right) \setminus \left(\bigcup_{V \in \mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[d, "\bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{U \in \mathcal{U}^{\mathsf{c}}} U \neq \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[dr, dashed, "\sim"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[d, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[d, "\bigcap^{\mathsf{op}}"] \\
\mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}},
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$\begin{tikzcd}
 & & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[drr, dashed, "\sim"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[dr, "\bigcap"'] & & & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dl, "\bigcup^{\mathsf{op}}"] \\
 & \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, &
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcap_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\triangle"] \arrow[d, "\bigcup\times\bigcup"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathrm{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcup^{\mathrm{op}} \times \bigcup^{\mathrm{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathrm{op}} \arrow[dr, "\bigcup^{\mathrm{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathrm{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & \mathcal{P}(X)^{\mathrm{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X}"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V\right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} \left(f^{-1}\right)^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U\in\mathcal{U}} f_*(U) = \bigcup_{V\in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Intersections of Families I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(x)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "\bigcap"'] & X
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U\in\bigcup_{A\in\mathcal{A}} A} U = \bigcap_{A\in\mathcal{A}}\left(\bigcap_{U\in A} U\right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Intersections of Families II.* Let X be a set and consider the compositions

$$\begin{tikzcd}
 & & & \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[dlll, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dl, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"'] \arrow[drrr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] & & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[drrr, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dr, "\bigcap"] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dl, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dlll, "\bigcap"] \\
 & & & \mathcal{P}(X) & & &
\end{tikzcd}$$

given by

$$\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$\begin{tikzcd}
 & \bigcup\limits_{U \in \bigcap_{A \in \mathcal{A}} A} U \arrow[dr, hook] & \\
\bigcup\limits_{A \in \mathcal{A}} \left(\bigcap\limits_{U \in A} U \right) & & \bigcap\limits_{A \in \mathcal{A}} \left(\bigcup\limits_{U \in A} U \right) \\
 & \bigcap\limits_{U \in \bigcup_{A \in \mathcal{A}} A} U \arrow[ul, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

All other possible inclusions fail to hold in general.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition $(\star)$. So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcup_{U \in \mathcal{U}} U \subset \bigcup_{V \in \mathcal{V}} V.$$

Indeed, given $x \in \bigcup_{U \in \mathcal{U}} U$, there exists some $U \in \mathcal{U}$ such that $x \in U$, but since $\mathcal{U} \subset \mathcal{V}$, we have $U \in \mathcal{V}$ as well, and thus $x \in \bigcup_{V \in \mathcal{V}} V$, which gives our desired inclusion.

Item 2, Associativity: We have

$$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U \stackrel{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{there exists some } U \in \bigcup\limits_{A \in \mathcal{A}} A \\ \text{such that we have } x \in U \end{array} \right\}$$

$$
\begin{aligned}
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{and some } U \in A \text{ such that} \\ \text{we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } A \in \mathcal{A} \\ \text{such that we have } x \in \bigcup_{U \in A} U \end{array} \right\} \\
&\overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right).
\end{aligned}
$$

This finishes the proof.
Item 3, Left Unitality: We have

$$
\begin{aligned}
\bigcup_{V \in \{U\}} V &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } V \in \{U\} \\ \text{such that we have } x \in U \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.
Item 4, Right Unitality: We have

$$
\begin{aligned}
\bigcup_{\{u\} \in \chi_X(U)} \{u\} &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x \in \{u\} \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } \{u\} \in \chi_X(U) \\ \text{such that we have } x = u \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } u \in U \\ \text{such that we have } x = u \end{array} \right\} \\
&= \{x \in X \mid x \in U\} \\
&= U.
\end{aligned}
$$

This finishes the proof.
Item 5, Interaction With Unions I: We have

$$
\begin{aligned}
\bigcup_{W \in \mathcal{U} \cup \mathcal{V}} W &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \cup \mathcal{V} \\ \text{such that we have } x \in W \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } W \in \mathcal{U} \text{ or some} \\ W \in \mathcal{V} \text{ such that we have } x \in W \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{U}\\ \text{such that we have } x \in W\end{array} \right\}\\
&\quad \cup \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{V}\\ \text{such that we have } x \in W\end{array} \right\}\\
&\overset{\text{def}}{=} \left(\bigcup_{W \in \mathcal{U}} W \right) \cup \left(\bigcup_{W \in \mathcal{V}} W \right)\\
&= \left(\bigcup_{U \in \mathcal{U}} U \right) \cup \left(\bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}
$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$
\begin{aligned}
\bigcup_{W \in \mathcal{U} \cap \mathcal{V}} W &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } W \in \mathcal{U} \cap \mathcal{V}\\ \text{such that we have } x \in W\end{array} \right\}\\
&\subset \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } U \in \mathcal{U} \text{ and some } V \in \mathcal{V}\\ \text{such that we have } x \in U \text{ and } x \in V\end{array} \right\}\\
&= \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } U \in \mathcal{U}\\ \text{such that we have } x \in U\end{array} \right\}\\
&\quad \cup \left\{ x \in X \;\middle|\; \begin{array}{l}\text{there exists some } V \in \mathcal{V}\\ \text{such that we have } x \in V\end{array} \right\}\\
&\overset{\text{def}}{=} \left(\bigcup_{U \in \mathcal{U}} U \right) \cap \left(\bigcup_{V \in \mathcal{V}} V \right).
\end{aligned}
$$

This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$
\begin{aligned}
\bigcup_{W \in \mathcal{U} \setminus \mathcal{V}} U &= \bigcup_{W \in \{\{0,1\}\}} W\\
&= \{0,1\},
\end{aligned}
$$

whereas

$$
\left(\bigcup_{U \in \mathcal{U}} U \right) \setminus \left(\bigcup_{V \in \mathcal{V}} V \right) = \{0,1\} \setminus \{0\}
$$

$$= \{1\}.$$

Thus we have

$$\bigcup_{W\in\mathcal{U}\setminus\mathcal{V}} W = \{0,1\} \neq \{1\} = \left(\bigcup_{U\in\mathcal{U}} U\right) \setminus \left(\bigcup_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0,1\}$ and let $\mathcal{U} = \{0\}$. We have

$$\begin{aligned}\bigcup_{U\in\mathcal{U}^{\mathsf{c}}} U &= \bigcup_{U\in\{\emptyset,\{1\},\{0,1\}\}} U\\ &= \{0,1\},\end{aligned}$$

whereas

$$\begin{aligned}\bigcup_{U\in\mathcal{U}} U^{\mathsf{c}} &= \{0\}^{\mathsf{c}}\\ &= \{1\}.\end{aligned}$$

Thus we have

$$\bigcup_{U\in\mathcal{U}^{\mathsf{c}}} U = \{0,1\} \neq \{1\} = \bigcup_{U\in\mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.

Item 11, Interaction With Complements II: Omitted.

Item 12, Interaction With Complements III: Omitted.

Item 13, Interaction With Symmetric Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$\begin{aligned}\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W &= \bigcup_{W\in\{\{0\}\}} W\\ &= \{0\},\end{aligned}$$

whereas

$$\begin{aligned}\left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right) &= \{0,1\} \triangle \{0,1\}\\ &= \emptyset,\end{aligned}$$

Thus we have

$$\bigcup_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcup_{U\in\mathcal{U}} U\right) \triangle \left(\bigcup_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 14, Interaction With Internal Homs I: This is a repetition of Item 7 of Definition 4.4.7.1.3 and is proved there.

Item 15, Interaction With Internal Homs II: This is a repetition of Item 8 of Definition 4.4.7.1.3 and is proved there.

Item 16, Interaction With Internal Homs III: This is a repetition of Item 9 of Definition 4.4.7.1.3 and is proved there.

Item 17, Interaction With Direct Images: This is a repetition of Item 3 of Definition 4.6.1.1.5 and is proved there.

Item 18, Interaction With Inverse Images: This is a repetition of Item 3 of Definition 4.6.2.1.3 and is proved there.

Item 19, Interaction With Codirect Images: This is a repetition of Item 3 of Definition 4.6.3.1.7 and is proved there.

Item 20, Interaction With Intersections of Families I: We have

$$\begin{aligned}\bigcap_{U\in\bigcup_{A\in\mathcal{A}} A} U &\stackrel{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } U\in\bigcup_{A\in\mathcal{A}} A,\\ \text{we have } x\in U\end{array}\right\}\\ &= \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } A\in\mathcal{A} \text{ and each}\\ U\in A, \text{ we have } x\in U\end{array}\right\}\\ &\stackrel{\text{def}}{=} \bigcap_{A\in\mathcal{A}}\left(\bigcap_{U\in A} U\right).\end{aligned}$$

This finishes the proof.

Item 21, Interaction With Intersections of Families II: Omitted. □

4.3.7 Intersections of Families of Subsets

Let X be a set and let $\mathcal{U}\in\mathcal{P}(\mathcal{P}(X))$.

Definition 4.3.7.1.1. The **intersection of** $\mathcal{U}$ is the set $\bigcap_{U\in\mathcal{U}} U$ defined by

$$\bigcap_{U\in\mathcal{U}} U \stackrel{\text{def}}{=} \left\{x\in X \;\middle|\; \begin{array}{l}\text{for each } U\in\mathcal{U},\\ \text{we have } x\in U\end{array}\right\}.$$

Proposition 4.3.7.1.2. Let X be a set.

1. *Functoriality*. The assignment $\mathcal{U}\mapsto\bigcap_{U\in\mathcal{U}} U$ defines a functor

$$\bigcap\colon (\mathcal{P}(\mathcal{P}(X)),\supset)\to(\mathcal{P}(X),\subset).$$

In particular, for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$, the following condition is satisfied:

($\star$) If $\mathcal{U} \subset \mathcal{V}$, then $\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U$.

2. *Oplax Associativity.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

with components

$$\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) \subset \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$.

3. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\chi_{\mathcal{P}(X)}"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V \in \{U\}} V = U.$$

for each $U \in \mathcal{P}(X)$.

4. *Oplax Right Unitality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\mathcal{P}(\chi_X)"'] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"] & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] \arrow[ur, phantom, "\times"] & \mathcal{P}(X)
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{\{x\}\in\chi_X(U)}\{x\}\neq U$$

in general, where $U\in\mathcal{P}(X)$. However, when U is nonempty, we have

$$\bigcap_{\{x\}\in\chi_X(U)}\{x\}\subset U.$$

5. *Interaction With Unions I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))\times\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cup"] \arrow[d, "\cap\times\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{W\in\mathcal{U}\cup\mathcal{V}}W=\left(\bigcap_{U\in\mathcal{U}}U\right)\cap\left(\bigcap_{V\in\mathcal{V}}V\right)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$.

6. *Interaction With Unions II.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(U\cup-)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] & \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))}\times(-\cup V)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "U\cup-"'] & \mathcal{P}(X) & \mathcal{P}(X) \arrow[r, "-\cup V"'] & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$U\cup\left(\bigcap_{V\in\mathcal{V}}V\right)=\bigcap_{V\in\mathcal{V}}(U\cup V),$$
$$\left(\bigcap_{U\in\mathcal{U}}U\right)\cup V=\bigcap_{U\in\mathcal{U}}(U\cup V)$$

for each $\mathcal{U},\mathcal{V}\in\mathcal{P}(\mathcal{P}(X))$ and each $U,V\in\mathcal{P}(X)$.

7. *Interaction With Intersections I.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\cap"] \arrow[d, "\cap \times \cap"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X), \end{tikzcd}$$

with components

$$\left(\bigcap_{U \in \mathcal{U}} U\right) \cap \left(\bigcap_{V \in \mathcal{V}} V\right) \subset \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Intersections II.* The diagrams

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (U \cap -)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "U \cap -"'] & \mathcal{P}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(\mathcal{P}(X))} \times (- \cap V)"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "- \cap V"'] & \mathcal{P}(X) \end{tikzcd}$$

commute, i.e. we have

$$U \cup \left(\bigcap_{V \in \mathcal{V}} V\right) = \bigcap_{V \in \mathcal{V}} (U \cup V),$$

$$\left(\bigcap_{U \in \mathcal{U}} U\right) \cup V = \bigcap_{U \in \mathcal{U}} (U \cup V)$$

for each $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Differences.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\setminus"] \arrow[d, "\cap \times \cap"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\setminus"'] & \mathcal{P}(X), \end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} W \neq \left(\bigcap_{U \in \mathcal{U}} U\right) \setminus \left(\bigcap_{V \in \mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Complements I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[d, "\bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in \mathcal{U}^{\mathsf{c}}} W \neq \bigcap_{U \in \mathcal{U}} U^{\mathsf{c}}$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Complements II.* The diagram

$$\begin{tikzcd}
& & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[drr, dashed, "\sim"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[dr, "\bigcap"'] & & & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dl, "\bigcup^{\mathsf{op}}"] \\
& \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, &
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcap_{U \in \mathcal{U}} U\right)^{\mathsf{c}} = \bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

12. *Interaction With Complements III.* The diagram

$$\begin{tikzcd}
& & \mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[drr, dashed, "\sim"] & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \star (-)^{\mathsf{c}}"] \arrow[dr, "\bigcup"'] & & & & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dl, "\bigcap^{\mathsf{op}}"] \\
& \mathcal{P}(X) \arrow[rr, "(-)^{\mathsf{c}}"'] & & \mathcal{P}(X)^{\mathsf{op}}, &
\end{tikzcd}$$

commutes, i.e. we have

$$\left(\bigcup_{U\in\mathcal{U}} U\right)^{\mathsf{c}} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "\triangle"] \arrow[d, "\bigcap\times\bigcap"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W \neq \left(\bigcap_{U\in\mathcal{U}} U\right) \triangle \left(\bigcap_{V\in\mathcal{V}} V\right)$$

in general, where $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

14. *Interaction With Internal Homs I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcap^{\mathsf{op}}\times\bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W\in[\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U\in\mathcal{U}} U, \bigcap_{V\in\mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

15. *Interaction With Internal Homs II.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcap^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcup"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcap_{U\in\mathcal{U}} U, V\right]_X = \bigcup_{U\in\mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

16. *Interaction With Internal Homs III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V\in\mathcal{V}} V\right]_X = \bigcap_{V\in\mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

17. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U\in\mathcal{U}} f_!(U) = \bigcap_{V\in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \overset{\text{def}}{=} (f_!)_!(\mathcal{U})$.

18. *Interaction With Inverse Images.* Let $f\colon X \longrightarrow Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V\in\mathcal{V}} f^{-1}(V) = \bigcap_{U\in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} \left(f^{-1}\right)^{-1}(\mathcal{V})$.

19. *Interaction With Codirect Images.* Let $f\colon X \longrightarrow Y$ be a map of sets. The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U\in\mathcal{U}} f_*(U) = \bigcap_{V\in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

20. *Interaction With Unions of Families I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"] \arrow[d, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(\mathcal{P}(x)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X) \arrow[r, "\bigcap"'] & X
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(X))$.

21. *Interaction With Unions of Families II.* Let X be a set and consider the compositions

$$\begin{tikzcd}
 & & & \mathcal{P}(\mathcal{P}(\mathcal{P}(X))) \arrow[dlll, "\bigcap \star \mathrm{id}_{\mathcal{P}(X)}"'] \arrow[dl, "\bigcup \star \mathrm{id}_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcap"'] \arrow[drrr, "\mathrm{id}_{\mathcal{P}(X)} \star \bigcup"] & & & \\
\mathcal{P}(\mathcal{P}(X)) \arrow[drrr, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dr, "\bigcap"] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dl, "\bigcup"'] & & \mathcal{P}(\mathcal{P}(X)) \arrow[dlll, "\bigcap"] \\
 & & & \mathcal{P}(X) & & &
\end{tikzcd}$$

given by

$$\mathcal{A} \mapsto \bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U, \qquad \mathcal{A} \mapsto \bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U,$$

$$\mathcal{A} \mapsto \bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right), \quad \mathcal{A} \mapsto \bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right)$$

for each $\mathcal{A} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$. We have the following inclusions:

$$
\begin{tikzcd}
& \displaystyle\bigcup_{U \in \bigcap_{A \in \mathcal{A}} A} U \arrow[dr, hook] & \\
\displaystyle\bigcup_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) & & \displaystyle\bigcap_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) \\
& \displaystyle\bigcap_{U \in \bigcup_{A \in \mathcal{A}} A} U \arrow[ul, hook] \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

All other possible inclusions fail to hold in general.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove the condition $(\star)$. So let $\mathcal{U}, \mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$ with $\mathcal{U} \subset \mathcal{V}$. We claim that

$$\bigcap_{V \in \mathcal{V}} V \subset \bigcap_{U \in \mathcal{U}} U.$$

Indeed, if $x \in \bigcap_{V \in \mathcal{V}} V$, then $x \in V$ for all $V \in \mathcal{V}$. But since $\mathcal{U} \subset \mathcal{V}$, it follows that $x \in U$ for all $U \in \mathcal{U}$ as well. Thus $x \in \bigcap_{U \in \mathcal{U}} U$, which gives our desired inclusion.

Item 2, Oplax Associativity: We have

$$
\begin{aligned}
\bigcap_{A \in \mathcal{A}} \left(\bigcap_{U \in A} U \right) &\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A}, \\ \text{we have } x \in \displaystyle\bigcap_{U \in A} U \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } A \in \mathcal{A} \text{ and each} \\ U \in A, \text{ we have } x \in U \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \displaystyle\bigcup_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\} \\
&\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \displaystyle\bigcap_{A \in \mathcal{A}} A, \\ \text{we have } x \in U \end{array} \right\}
\end{aligned}
$$

$$\stackrel{\text{def}}{=} \bigcap_{U \in \bigcap_{A \in \mathcal{A}} A} U.$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**??** of **??**). This finishes the proof.
Item 3, Left Unitality: We have

$$\begin{aligned} \bigcap_{V \in \{U\}} V &\stackrel{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{for each } V \in \{U\}, \\ \text{we have } x \in U \end{array} \right\} \\ &= \{x \in X \mid x \in U\} \\ &= U. \end{aligned}$$

This finishes the proof.
Item 4, Oplax Right Unitality: If $U = \emptyset$, then we have

$$\begin{aligned} \bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \bigcap_{\{u\} \in \emptyset} \{u\} \\ &= X, \end{aligned}$$

so $\bigcap_{\{u\} \in \chi_X(U)} \{u\} = X \neq \emptyset = U$. When U is nonempty, we have two cases:

1. If U is a singleton, say $U = \{u\}$, we have

$$\begin{aligned} \bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \{u\} \\ &\stackrel{\text{def}}{=} U. \end{aligned}$$

2. If U contains at least two elements, we have

$$\begin{aligned} \bigcap_{\{u\} \in \chi_X(U)} \{u\} &= \emptyset \\ &\subset U. \end{aligned}$$

This finishes the proof.
Item 5, Interaction With Unions I: We have

$$\bigcap_{W \in \mathcal{U} \cup \mathcal{V}} W \stackrel{\text{def}}{=} \left\{ x \in X \,\middle|\, \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\}$$

$$
\begin{aligned}
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \text{ and each} \\ W \in \mathcal{V}, \text{ we have } x \in W \end{array} \right\} \\
&\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U}, \\ \text{we have } x \in W \end{array} \right\} \\
&\quad \cap \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
&\overset{\text{def}}{=} \left(\bigcap_{W \in \mathcal{U}} W \right) \cap \left(\bigcap_{W \in \mathcal{V}} W \right) \\
&= \left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right).
\end{aligned}
$$

This finishes the proof.

Item 6, Interaction With Unions II: Omitted.

Item 7, Interaction With Intersections I: We have

$$
\begin{aligned}
\left(\bigcap_{U \in \mathcal{U}} U \right) \cap \left(\bigcap_{V \in \mathcal{V}} V \right) &\overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } U \in \mathcal{U}, \\ \text{we have } x \in U \end{array} \right\} \\
&\quad \cup \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } V \in \mathcal{V}, \\ \text{we have } x \in V \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \cap \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
&\subset \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for each } W \in \mathcal{U} \cup \mathcal{V}, \\ \text{we have } x \in W \end{array} \right\} \\
&\overset{\text{def}}{=} \bigcap_{W \in \mathcal{U} \cap \mathcal{V}} W.
\end{aligned}
$$

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**??** of **??**). This finishes the proof.

Item 8, Interaction With Intersections II: Omitted.

Item 9, Interaction With Differences: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0\}, \{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}\}$. We have

$$
\begin{aligned}
\bigcap_{W \in \mathcal{U} \setminus \mathcal{V}} U &= \bigcap_{W \in \{\{0,1\}\}} W \\
&= \{0, 1\},
\end{aligned}
$$

whereas

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \setminus \left(\bigcap_{V\in\mathcal{V}} V\right) = \{0\} \setminus \{0\}$$
$$= \emptyset.$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\setminus\mathcal{V}} W = \{0,1\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right) \setminus \left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.

Item 10, Interaction With Complements I: Let $X = \{0,1\}$ and let $\mathcal{U} = \{\{0\}\}$. We have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U = \bigcap_{W\in\{\emptyset,\{1\},\{0,1\}\}} W$$
$$= \emptyset,$$

whereas

$$\bigcap_{U\in\mathcal{U}} U^{\mathsf{c}} = \{0\}^{\mathsf{c}}$$
$$= \{1\}.$$

Thus we have

$$\bigcap_{W\in\mathcal{U}^{\mathsf{c}}} U = \emptyset \neq \{1\} = \bigcap_{U\in\mathcal{U}} U^{\mathsf{c}}.$$

This finishes the proof.

Item 11, Interaction With Complements II: This is a repetition of Item 12 of Definition 4.3.6.1.2 and is proved there.

Item 12, Interaction With Complements III: This is a repetition of Item 11 of Definition 4.3.6.1.2 and is proved there.

Item 13, Interaction With Symmetric Differences: Let $X = \{0,1\}$, let $\mathcal{U} = \{\{0,1\}\}$, and let $\mathcal{V} = \{\{0\},\{0,1\}\}$. We have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W = \bigcap_{W\in\{\{0\}\}} W$$
$$= \{0\},$$

whereas

$$\left(\bigcap_{U\in\mathcal{U}} U\right) \triangle \left(\bigcap_{V\in\mathcal{V}} V\right) = \{0,1\} \triangle \{0\}$$

$$= \emptyset,$$

Thus we have

$$\bigcap_{W\in\mathcal{U}\triangle\mathcal{V}} W = \{0\} \neq \emptyset = \left(\bigcap_{U\in\mathcal{U}} U\right) \triangle \left(\bigcap_{V\in\mathcal{V}} V\right).$$

This finishes the proof.
Item 14, Interaction With Internal Homs I: This is a repetition of Item 10 of Definition 4.4.7.1.3 and is proved there.
Item 15, Interaction With Internal Homs II: This is a repetition of Item 11 of Definition 4.4.7.1.3 and is proved there.
Item 16, Interaction With Internal Homs III: This is a repetition of Item 12 of Definition 4.4.7.1.3 and is proved there.
Item 17, Interaction With Direct Images: This is a repetition of Item 4 of Definition 4.6.1.1.5 and is proved there.
Item 18, Interaction With Inverse Images: This is a repetition of Item 4 of Definition 4.6.2.1.3 and is proved there.
Item 19, Interaction With Codirect Images: This is a repetition of Item 4 of Definition 4.6.3.1.7 and is proved there.
Item 20, Interaction With Unions of Families I: This is a repetition of Item 20 of Definition 4.3.6.1.2 and is proved there.
Item 21, Interaction With Unions of Families II: This is a repetition of Item 21 of Definition 4.3.6.1.2 and is proved there. □

4.3.8 Binary Unions

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.8.1.1. The **union of** U **and** V is the set $U \cup V$ defined by

$$\begin{aligned} U \cup V &\stackrel{\text{def}}{=} \bigcup_{z\in\{U,V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

Proposition 4.3.8.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$\begin{aligned} U \cup -\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ - \cup V\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ -_1 \cup -_2\colon & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

(a) If $U \subset A$, then $U \cup V \subset A \cup V$.

(b) If $V \subset B$, then $U \cup V \subset U \cup B$.

(c) If $U \subset A$ and $V \subset B$, then $U \cup V \subset A \cup B$.

2. *Associativity.* The diagram

$$\begin{tikzcd}
 & & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[drr, "\mathrm{id}_{\mathcal{P}(X)} \times \cup"] & & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[urr, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}", "\sim"'] \arrow[dr, "\cup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X), &
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $U, V, W \in \mathcal{P}(X)$.

3. *Unitality.* The diagrams

$$\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[\emptyset] \times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\sim", "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
 & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [\emptyset]}"] \arrow[dr, dashed, "\sim", "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
\emptyset \cup U &= U, \\
U \cup \emptyset &= U
\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

4. *Commutativity.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup V = V \cup U$$

for each $U, V \in \mathcal{P}(X)$.

5. *Annihilation With X.* The diagrams

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathrm{pt}\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}"] \arrow[d, "{[X]\times\mathrm{id}_{\mathcal{P}(X)}}"'] & & \mathrm{pt} \arrow[d, "{[X]}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times\mathrm{pt} \arrow[ur, "\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)}\times[X]}"'] & & \mathrm{pt} \arrow[d, "{[X]}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[rr, "\cup"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
U \cup X &= X, \\
X \cup V &= X
\end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

6. *Distributivity of Unions Over Intersections.* The diagrams

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned}
U \cup (V \cap W) &= (U \cup V) \cap (U \cup W), \\
(U \cap V) \cup W &= (U \cup W) \cap (V \cup W)
\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Distributivity of Intersections Over Unions.* The diagrams

$$\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cup"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Idempotency.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\Delta_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup U = U$$

for each $U \in \mathcal{P}(X)$.

9. *Via Intersections and Symmetric Differences.* The diagram

$$\begin{tikzcd} (\mathcal{P}(X) \times \mathcal{P}(X)) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[r, "\triangle \times \cap"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[dr, "\triangle"] & \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[u, "\Delta_{\mathcal{P}(X) \times \mathcal{P}(X)}"] \arrow[rr, "\cup"'] & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cup V = (U \triangle V) \triangle (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd} \mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "f_!\times f_!"] \arrow[d, "\cup"'] & \mathcal{P}(Y)\times\mathcal{P}(Y) \arrow[d, "\cup"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(U\cup V) = f_!(U)\cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd} \mathcal{P}(Y)\times\mathcal{P}(Y) \arrow[r, "f^{-1}\times f^{-1}"] \arrow[d, "\cup"'] & \mathcal{P}(X)\times\mathcal{P}(X) \arrow[d, "\cup"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U\cup V) = f^{-1}(U)\cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X)\times\mathcal{P}(X) \arrow[r, "f_*\times f_*"] \arrow[d, "\cup"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y)\times\mathcal{P}(Y) \arrow[d, "\cup"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_*(U)\cup f_*(V) \subset f_*(U\cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro25as].
Item 2, Associativity: See [Pro25bf].
Item 3, Unitality: This follows from [Pro25bi] and Item 4.
Item 4, Commutativity: See [Pro25bg].
Item 5, Annihilation With X: We have

$$\begin{aligned} U \cup X &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in X\} \\ &= \{x \in X \mid x \in X\}, \\ &= X \end{aligned}$$

and

$$\begin{aligned} X \cup V &\stackrel{\text{def}}{=} \{x \in X \mid x \in X \text{ or } x \in V\} \\ &= \{x \in X \mid x \in X\} \\ &= X. \end{aligned}$$

This finishes the proof.
Item 6, Distributivity of Unions Over Intersections: See [Pro25be].
Item 7, Distributivity of Intersections Over Unions: See [Pro25ao].
Item 8, Idempotency: See [Pro25ar].
Item 9, Via Intersections and Symmetric Differences: See [Pro25bd].
Item 10, Interaction With Characteristic Functions I: See [Pro25i].
Item 11, Interaction With Characteristic Functions II: See [Pro25i].
Item 12, Interaction With Direct Images: See [Pro25t].
Item 13, Interaction With Inverse Images: See [Pro25ad].
Item 14, Interaction With Codirect Images: This is a repetition of Item 5 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 of this propostition and Items 3 to 6 and 8 of Definition 4.3.9.1.2.
□

4.3.9 Binary Intersections

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.9.1.1. The **intersection of** U **and** V is the set $U \cap V$ defined by

$$\begin{aligned} U \cap V &\stackrel{\text{def}}{=} \bigcap_{z \in \{U,V\}} z \\ &\stackrel{\text{def}}{=} \{x \in X \mid x \in U \text{ or } x \in V\}. \end{aligned}$$

Proposition 4.3.9.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \cap -\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ - \cap V\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ -_1 \cap -_2\colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \cap V \subset A \cap V$.

   (b) If $V \subset B$, then $U \cap V \subset U \cap B$.

   (c) If $U \subset A$ and $V \subset B$, then $U \cap V \subset A \cap B$.

2. *Adjointness.* We have adjunctions

$$(U \cap - \dashv [U, -]_X)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[U,-]_X}"] \end{tikzcd}$$

$$(- \cap V \dashv [V, -]_X)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[V,-]_X}"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X), \end{aligned}$$

   natural in $U, V, W \in \mathcal{P}(X)$, where

$$[-_1, -_2]_X \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

   is the bifunctor of Section 4.4.7. In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

   (a) The following conditions are equivalent:

      i. We have $U \cap V \subset W$.

      ii. We have $U \subset [V, W]_X$.

   (b) The following conditions are equivalent:

      i. We have $U \cap V \subset W$.

ii. We have $V \subset [U, W]_X$.

3. *Associativity.* The diagram

$$\begin{tikzcd}
 & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \cap"] & \\
(\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}", "\sim"'] \arrow[d, "\cap \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X),
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Unitality.* The diagrams

$$\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[X] \times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\sim", "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [X]}"] \arrow[dr, dashed, "\sim", "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have equalities of sets

$$\begin{aligned} X \cap U &= U, \\ U \cap X &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

5. *Commutativity.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
 & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cap V = V \cap U$$

for each $U, V \in \mathcal{P}(X)$.

6. *Annihilation With the Empty Set.* The diagrams

$$
\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathrm{pt}\times\mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}"] \arrow[d, "{[\emptyset]\times\mathrm{id}_{\mathcal{P}(X)}}"'] & & \mathrm{pt} \arrow[d, "{[\emptyset]}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times\mathrm{pt} \arrow[ur, "\epsilon^{\mathsf{Sets}}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)}\times[\emptyset]}"'] & & \mathrm{pt} \arrow[d, "{[\emptyset]}"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[rr, "\cap"'] & & \mathcal{P}(X)
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
\emptyset\cap X &= \emptyset,\\
X\cap\emptyset &= \emptyset
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$.

7. *Distributivity of Unions Over Intersections.* The diagrams

$$
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cap"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cap\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cup\times\cup"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cup"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cap"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$
\begin{aligned}
U\cup(V\cap W) &= (U\cup V)\cap(U\cup W),\\
(U\cap V)\cup W &= (U\cup W)\cap(V\cup W)
\end{aligned}
$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Distributivity of Intersections Over Unions.* The diagrams

$$
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
\mathcal{P}(X)\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[ur, "\Delta_{\mathcal{P}(X)}\times(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)}\times\cup"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[dr, "\mu^{\mathsf{Sets}}_{4|\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}"] & \\
(\mathcal{P}(X)\times\mathcal{P}(X))\times\mathcal{P}(X) \arrow[ur, "(\mathrm{id}_{\mathcal{P}(X)}\times\mathrm{id}_{\mathcal{P}(X)})\times\Delta_{\mathcal{P}(X)}"] \arrow[d, "\cup\times\mathrm{id}_{\mathcal{P}(X)}"'] & & (\mathcal{P}(X)\times\mathcal{P}(X))\times(\mathcal{P}(X)\times\mathcal{P}(X)) \arrow[d, "\cap\times\cap"] \\
\mathcal{P}(X)\times\mathcal{P}(X) \arrow[dr, "\cap"'] & & \mathcal{P}(X)\times\mathcal{P}(X), \arrow[dl, "\cup"] \\
 & \mathcal{P}(X) &
\end{tikzcd}
$$

commute, i.e. we have equalities of sets

$$\begin{aligned} U \cap (V \cup W) &= (U \cap V) \cup (U \cap W), \\ (U \cup V) \cap W &= (U \cap W) \cup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Idempotency.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\Delta_{\mathcal{P}(X)}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\ & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$U \cap U = U$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

12. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cap"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

13. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

14. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cap"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

15. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

16. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1*, *Functoriality*: See [Pro25aq].
Item 2, *Adjointness*: See [MSE 267469].
Item 3, *Associativity*: See [Pro25v].
Item 4, *Unitality*: This follows from [Pro25z] and Item 5.
Item 5, *Commutativity*: See [Pro25w].
Item 6, *Annihilation With the Empty Set*: This follows from [Pro25x] and Item 5.

Item 7, Distributivity of Unions Over Intersections: See [Pro25be].
Item 8, Distributivity of Intersections Over Unions: See [Pro25ao].
Item 9, Idempotency: See [Pro25ap].
Item 10, Interaction With Characteristic Functions I: See [Pro25f].
Item 11, Interaction With Characteristic Functions II: See [Pro25f].
Item 12, Interaction With Direct Images: See [Pro25r].
Item 13, Interaction With Inverse Images: See [Pro25ab].
Item 14, Interaction With Codirect Images: This is a repetition of Item 6 of Definition 4.6.3.1.7 and is proved there.
Item 15, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 6.
Item 16, Interaction With Powersets and Semirings: This follows from Items 2 to 4 and 8 and Items 3 to 6 and 8 of Definition 4.3.9.1.2. □

4.3.10 Differences

Let X and Y be sets.

Definition 4.3.10.1.1. The **difference of** X **and** Y is the set $X \setminus Y$ defined by

$$X \setminus Y \stackrel{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

Proposition 4.3.10.1.2. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -\colon & \quad (\mathcal{P}(X), \supset) & \to (\mathcal{P}(X), \subset), \\ - \setminus V\colon & \quad (\mathcal{P}(X), \subset) & \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2\colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) & \to (\mathcal{P}(X), \subset). \end{aligned}$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

   (a) If $U \subset A$, then $U \setminus V \subset A \setminus V$.

   (b) If $V \subset B$, then $U \setminus B \subset U \setminus V$.

   (c) If $U \subset A$ and $V \subset B$, then $U \setminus B \subset A \setminus V$.

2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

   for each $U, V \in \mathcal{P}(X)$.

3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $U, V, W \in \mathcal{P}(X)$.

7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \bigtriangleup (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $U, V, W \in \mathcal{P}(X)$.

11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $U \in \mathcal{P}(X)$.

13. *Right Annihilation.* We have

$$U \setminus X = U$$

for each $U \in \mathcal{P}(X)$.

14. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

15. *Interaction With Containment.* The following conditions are equivalent:

    (a) We have $V \setminus U \subset W$.

    (b) We have $V \setminus W \subset U$.

16. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Direct Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

18. *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathsf{op},-1} \times f^{-1}"] \arrow[d, "\setminus"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\setminus"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

19. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: See [Pro25ai] and [Pro25am].
Item 2, *De Morgan's Laws*: See [Pro25o].
Item 3, *Interaction With Unions I*: See [Pro25p].
Item 4, *Interaction With Unions II*: Omitted.
Item 5, *Interaction With Unions III*: See [Pro25an].
Item 6, *Interaction With Unions IV*: See [Pro25ah].
Item 7, *Interaction With Intersections*: See [Pro25y].
Item 8, *Interaction With Complements*: See [Pro25af].
Item 9, *Interaction With Symmetric Differences*: See [Pro25ag].
Item 10, *Triple Differences*: See [Pro25al].
Item 11, *Left Annihilation*: Omitted.
Item 12, *Right Unitality*: See [Pro25aj].
Item 13, *Right Annihilation*: Omitted.
Item 14, *Invertibility*: See [Pro25ak].
Item 15, *Interaction With Containment*: Omitted.
Item 16, *Interaction With Characteristic Functions*: See [Pro25g].
Item 17, *Interaction With Direct Images*: See [Pro25s].
Item 18, *Interaction With Inverse Images*: See [Pro25ac]. □

4.3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.3.11.1.1. The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

Proposition 4.3.11.1.2. Let X be a set.

1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X).$$

   In particular, the following statements hold for each $U, V \in \mathcal{P}(X)$:

   (★) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

2. *De Morgan's Laws.* The diagrams

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cup^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cap"'] & \mathcal{P}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\cup"'] & \mathcal{P}(X) \end{tikzcd}$$

commute, i.e. we have equalities of sets

$$(U \cup V)^{\mathsf{c}} = U^{\mathsf{c}} \cap V^{\mathsf{c}},$$
$$(U \cap V)^{\mathsf{c}} = U^{\mathsf{c}} \cup V^{\mathsf{c}}$$

for each $U, V \in \mathcal{P}(X)$.

3. *Involutority.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "(-)^{\mathsf{c}}"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}"'] & \mathcal{P}(X) \arrow[d, "(-)^{\mathsf{c},\mathsf{op}}"] \\
 & \mathcal{P}(X)^{\mathsf{op}}
\end{tikzcd}$$

commutes, i.e. we have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $U \in \mathcal{P}(X)$.

4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} \equiv 1 - \chi_U \pmod 2$$

for each $U \in \mathcal{P}(X)$.

5. *Interaction With Direct Images.* Let $f \colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

6. *Interaction With Inverse Images.* Let $f \colon X \to Y$ be a function. The

diagram

$$\begin{tikzcd} \mathcal{P}(Y)^{\mathrm{op}} \arrow[r, "f^{-1,\mathrm{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathrm{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Codirect Images.* Let $f : X \longrightarrow Y$ be a function. The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathrm{op}} \arrow[r, "f_!^{\mathrm{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathrm{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: This follows from Item 1 of Definition 4.3.10.1.2.
Item 2, De Morgan's Laws: See [Pro25o].
Item 3, Involutority: See [Pro25j].
Item 4, Interaction With Characteristic Functions: Omitted.
Item 5, Interaction With Direct Images: This is a repetition of Item 8 of Definition 4.6.1.1.5 and is proved there.
Item 6, Interaction With Inverse Images: This is a repetition of Item 8 of Definition 4.6.2.1.3 and is proved there.
Item 7, Interaction With Codirect Images: This is a repetition of Item 7 of Definition 4.6.3.1.7 and is proved there. □

4.3.12 Symmetric Differences

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Definition 4.3.12.1.1. The **symmetric difference of** U **and** V is the set $U \bigtriangleup V$ defined by[13]

$$U \bigtriangleup V \overset{\text{def}}{=} (U \setminus V) \cup (V \setminus U).$$

Proposition 4.3.12.1.2. Let X be a set.

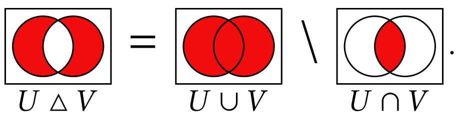
1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \bigtriangleup V$ ***does not*** in general define functors

$$\begin{aligned} U \bigtriangleup -\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ - \bigtriangleup V\colon & \quad (\mathcal{P}(X), \subset) && \to (\mathcal{P}(X), \subset), \\ -_1 \bigtriangleup -_2\colon & \quad (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) && \to (\mathcal{P}(X), \subset). \end{aligned}$$

2. *Via Unions and Intersections.* We have

$$U \bigtriangleup V = (U \cup V) \setminus (U \cap V)$$

for each $U, V \in \mathcal{P}(X)$, as in the Venn diagram



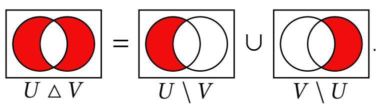
3. *Symmetric Differences of Disjoint Sets.* If U and V are disjoint, then we have

$$U \bigtriangleup V = U \cup V.$$

4. *Associativity.* The diagram

$$\begin{tikzcd} & \mathcal{P}(X) \times (\mathcal{P}(X) \times \mathcal{P}(X)) \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)} \times \bigtriangleup"] & \\ (\mathcal{P}(X) \times \mathcal{P}(X)) \times \mathcal{P}(X) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X),\mathcal{P}(X)}", "\sim"'] \arrow[d, "\bigtriangleup \times \mathrm{id}_{\mathcal{P}(X)}"'] & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\bigtriangleup"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[rr, "\bigtriangleup"'] & & \mathcal{P}(X), \end{tikzcd}$$

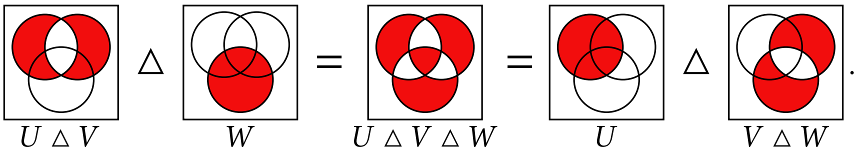
[13] *Illustration:*



commutes, i.e. we have

$$(U \mathbin{\triangle} V) \mathbin{\triangle} W = U \mathbin{\triangle} (V \mathbin{\triangle} W)$$

for each $U, V, W \in \mathcal{P}(X)$, as in the Venn diagram



5. *Unitality*. The diagrams

$$\begin{tikzcd}
\mathrm{pt} \times \mathcal{P}(X) \arrow[r, "{[\emptyset] \times \mathrm{id}_{\mathcal{P}(X)}}"] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
& \mathcal{P}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \times \mathrm{pt} \arrow[r, "{\mathrm{id}_{\mathcal{P}(X)} \times [\emptyset]}"] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathcal{P}(X)}"', "\sim"] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
& \mathcal{P}(X)
\end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned} U \mathbin{\triangle} \emptyset &= U, \\ \emptyset \mathbin{\triangle} U &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

6. *Commutativity*. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\sigma^{\mathsf{Sets}}_{\mathcal{P}(X),\mathcal{P}(X)}"] \arrow[dr, "\triangle"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
& \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$U \mathbin{\triangle} V = V \mathbin{\triangle} U$$

for each $U, V \in \mathcal{P}(X)$.

7. *Invertibility*. We have

$$U \mathbin{\triangle} U = \emptyset$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Unions.* We have

$$(U \mathbin{\triangle} V) \cup (V \mathbin{\triangle} T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $U, V, W \in \mathcal{P}(X)$.

9. *Interaction With Complements I.* We have

$$U \mathbin{\triangle} U^{\mathsf{c}} = X$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Complements II.* We have

$$\begin{aligned} U \mathbin{\triangle} X &= U^{\mathsf{c}}, \\ X \mathbin{\triangle} U &= U^{\mathsf{c}} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

11. *Interaction With Complements III.* The diagram

$$\begin{tikzcd} \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"] \arrow[d, "(-)^{\mathsf{c}} \times (-)^{\mathsf{c}}"'] & \mathcal{P}(X) \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\triangle"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$U^{\mathsf{c}} \mathbin{\triangle} V^{\mathsf{c}} = U \mathbin{\triangle} V$$

for each $U, V \in \mathcal{P}(X)$.

12. *"Transitivity".* We have

$$(U \mathbin{\triangle} V) \mathbin{\triangle} (V \mathbin{\triangle} W) = U \mathbin{\triangle} W$$

for each $U, V, W \in \mathcal{P}(X)$.

13. *The Triangle Inequality for Symmetric Differences.* We have

$$U \mathbin{\triangle} W \subset U \mathbin{\triangle} V \cup V \mathbin{\triangle} W$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \bigtriangleup W) &= (U \cap V) \bigtriangleup (U \cap W), \\ (U \bigtriangleup V) \cap W &= (U \cap W) \bigtriangleup (V \cap W) \end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \bigtriangleup V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \bigtriangleup V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

16. *Bijectivity.* Given $U, V \in \mathcal{P}(X)$, the maps

$$\begin{aligned} U \bigtriangleup - &\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \bigtriangleup V &\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (U \bigtriangleup -)^{-1} &= - \cup (U \cap -), \\ (- \bigtriangleup V)^{-1} &= - \cup (V \cap -). \end{aligned}$$

Moreover, the map

$$\begin{aligned} \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ C &\longmapsto C \bigtriangleup (U \bigtriangleup V) \end{aligned}$$

is a bijection of $\mathcal{P}(X)$ onto itself sending U to V and V to U.

17. *Interaction With Powersets and Groups.* Let X be a set.

    (a) The quadruple $\left(\mathcal{P}(X), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group.[14]

---

[14] Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \bigtriangleup, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

4. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

   - The group $\mathcal{P}(X)$ of Item 17;
   - The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$0 \cdot U \overset{\text{def}}{=} \emptyset,$$
$$1 \cdot U \overset{\text{def}}{=} U;$$

   is an $\mathbb{F}_2$-vector space.

5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

   (a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 4.

   (b) We have
$$\dim(\mathcal{P}(X)) = \#X.$$

6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[15]

7. *Interaction With Direct Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\triangle"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$

3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

15 *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro25bb] for a proof.

with components

$$f_!(U) \mathbin{\triangle} f_!(V) \subset f_!(U \mathbin{\triangle} V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathrm{op},-1} \times f^{-1}"] \arrow[d, "\triangle"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

i.e. we have

$$f^{-1}(U) \mathbin{\triangle} f^{-1}(V) = f^{-1}(U \mathbin{\triangle} V)$$

for each $U, V \in \mathcal{P}(Y)$.

9. *Interaction With Codirect Images.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathrm{op}} \times f_*"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U \mathbin{\triangle} V) \subset f_*(U) \mathbin{\triangle} f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Lack of Functoriality*: Omitted.
Item 2, Via Unions and Intersections: See [Pro25q].
Item 3, Symmetric Differences of Disjoint Sets: Since U and V are disjoint, we have $U \cap V = \emptyset$, and therefore we have

$$\begin{aligned}
U \mathbin{\triangle} V &= (U \cup V) \setminus (U \cap V) \\
&= (U \cup V) \setminus \emptyset \\
&= U \cup V,
\end{aligned}$$

where we've used Item 2 and Item 12 of Definition 4.3.10.1.2.
Item 4, Associativity: See [Pro25at].
Item 5, Unitality: This follows from Item 6 and [Pro25ay].
Item 6, Commutativity: See [Pro25au].
Item 7, Invertibility: See [Pro25ba].
Item 8, Interaction With Unions: See [Pro25bh].
Item 9, Interaction With Complements I: See [Pro25ax].
Item 10, Interaction With Complements II: This follows from Item 6 and [Pro25bc].
Item 11, Interaction With Complements III: See [Pro25av].
Item 12, "Transitivity": We have

$$\begin{aligned}
(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 4)}\\
&= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 4)}\\
&= U \triangle (\emptyset \triangle W) && \text{(by Item 7)}\\
&= U \triangle W. && \text{(by Item 5)}
\end{aligned}$$

This finishes the proof.
Item 13, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 12.
Item 14, Distributivity Over Intersections: See [Pro25u].
Item 15, Interaction With Characteristic Functions: See [Pro25h].
Item 16, Bijectivity: Omitted.
Item 17, Interaction With Powersets and Groups: Item 17a follows from Items 4 to 7, while Item 3b follows from Item 7.[16]
Item 4, Interaction With Powersets and Vector Spaces I: See [MSE 2719059].
Item 5, Interaction With Powersets and Vector Spaces II: See [MSE 2719059].
Item 6, Interaction With Powersets and Rings: This follows from Items 6 and 15 of Definition 4.3.9.1.2 and Items 14 and 17.[17]
Item 7, Interaction With Direct Images: This is a repetition of Item 9 of Definition 4.6.1.1.5 and is proved there.
Item 8, Interaction With Inverse Images: This is a repetition of Item 9 of Definition 4.6.2.1.3 and is proved there.
Item 9, Interaction With Codirect Images: This is a repetition of Item 8 of Definition 4.6.3.1.7 and is proved there. □

[16] *Reference:* [Pro25aw].
[17] *Reference:* [Pro25az].

4.4 Powersets

4.4.1 Foundations

Let X be a set.

Definition 4.4.1.1.1. The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \overset{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

Remark 4.4.1.1.2. Under the analogy that $\{\mathsf{t}, \mathsf{f}\}$ should be the (-1)-categorical analogue of Sets, we may view the powerset of a set as a decategorification of the category of presheaves of a category (or of the category of copresheaves):

- The powerset of a set X is equivalently (Item 2 of Definition 4.5.1.1.4) the set
$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$
of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category C is the category
$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Sets})$$
of functors from C^{op} to the category Sets of sets.

Notation 4.4.1.1.3. Let X be a set.

1. We write $\mathcal{P}_0(X)$ for the set of nonempty subsets of X.

2. We write $\mathcal{P}_{\text{fin}}(X)$ for the set of finite subsets of X.

Proposition 4.4.1.1.4. Let X be a set.

1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

   (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

(c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at most one morphisms between any two objects, so co/equalisers are trivial.

2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed.

3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned} \mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X), \\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}), \end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Interaction With Products I.* The map

$$\begin{aligned} \mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \textstyle\coprod Y) \\ (U, V) &\longmapsto U \cup V \end{aligned}$$

is an isomorphism of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

5. *Interaction With Products II.* The map

$$\begin{aligned} \mathcal{P}(X) \times \mathcal{P}(Y) &\longrightarrow \mathcal{P}(X \textstyle\coprod Y) \\ (U, V) &\longmapsto U \boxtimes_{X \times Y} V, \end{aligned}$$

where[18]

$$U \boxtimes_{X \times Y} V \overset{\mathrm{def}}{=} \{(u, v) \in X \times Y \mid u \in U \text{ and } v \in V\}$$

is an inclusion of sets, natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

[18] The set $U \boxtimes_{X \times Y} V$ is usually denoted simply $U \times V$. Here we denote it in this somewhat weird way to highlight the similarity to external tensor products in six-functor formalisms (see also Section 4.6.4).

6. *Interaction With Products III.* We have an isomorphism

$$\mathcal{P}(X) \otimes \mathcal{P}(Y) \cong \mathcal{P}(X \times Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ with respect to each of the functor structures $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ on $\mathcal{P}$ of Definition 4.4.2.1.1, where $\otimes$ denotes the tensor product of suplattices of **??**. Moreover, this makes each of $\mathcal{P}_!$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ into a symmetric monoidal functor.

Proof. *Item 1, Co/Completeness*: Omitted.
Item 2, Cartesian Closedness: See Section 4.4.7.
Item 3, Powersets as Sets of Relations: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(\mathrm{pt}, X) &\stackrel{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X)\\ &\cong \mathcal{P}(X)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Rel}(X, \mathrm{pt}) &\stackrel{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt})\\ &\cong \mathcal{P}(X),\end{aligned}$$

where we have used Item 5 of Definition 4.1.3.1.3.
Item 4, Interaction With Products I: The inverse of the map in the statement is the map

$$\Phi\colon \mathcal{P}(X \textstyle\coprod Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$$

defined by

$$\Phi(S) \stackrel{\text{def}}{=} (S_X, S_Y)$$

for each $S \in \mathcal{P}(X \coprod Y)$, where

$$\begin{aligned}S_X &\stackrel{\text{def}}{=} \{x \in X \mid (0, x) \in S\}\\ S_Y &\stackrel{\text{def}}{=} \{y \in Y \mid (1, y) \in S\}.\end{aligned}$$

The rest of the proof is omitted.
Item 5, Interaction With Products II: Omitted.
Item 6, Interaction With Products III: Omitted. □

4.4.2 Functoriality of Powersets

Proposition 4.4.2.1.1. Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_!\colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_!(f) \stackrel{\text{def}}{=} f_!,$$

as in Definition 4.6.1.1.1.

2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$\mathcal{P}^{-1}(f) \stackrel{\text{def}}{=} f^{-1},$$

as in Definition 4.6.2.1.1.

3. *Functoriality III*. The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects*. For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms*. For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in Definition 4.6.3.1.1.

Proof. Item 1, Functoriality I: This follows from Items 3 and 4 of Definition 4.6.1.1.6.
Item 2, Functoriality II: This follows from Items 3 and 4 of Definition 4.6.2.1.4.
Item 3, Functoriality III: This follows from Items 3 and 4 of Definition 4.6.3.1.8. □

4.4.3 Adjointness of Powersets I

Proposition 4.4.3.1.1. We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(X), Y)}_{\stackrel{\text{def}}{=} \mathsf{Sets}(Y, \mathcal{P}(X))} \cong \mathsf{Sets}(X, \mathcal{P}(Y)),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $Y \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

Proof. We have

$$
\begin{aligned}
\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\overset{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\
&\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{aligned}
$$

where all bijections are natural in A and B.[19] □

4.4.4 Adjointness of Powersets II

Proposition 4.4.4.1.1. We have an adjunction

$$
(\mathrm{Gr} \dashv \mathcal{P}_!): \quad
\begin{tikzcd}
\mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_!"]
\end{tikzcd}
$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(X), Y) \cong \mathsf{Sets}(X, \mathcal{P}(Y))$$

natural in $X \in \mathsf{Obj}(\mathsf{Sets})$ and $Y \in \mathsf{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Item 1 of Definition 8.2.2.1.2 and $\mathcal{P}_!$ is the functor of Definition 8.7.5.1.1.

Proof. We have

$$
\begin{aligned}
\mathrm{Rel}(\mathrm{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) && \text{(by Item 2 of Definition 4.5.1.1.4)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) && \text{(by Item 2 of Definition 4.1.3.1.3)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)), && \text{(by Item 2 of Definition 4.5.1.1.4)}
\end{aligned}
$$

where all bijections are natural in A, (where we are using Item 3 of Definition 4.5.1.1.4). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \nrightarrow B$ to the map $R^{\dagger}: A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Definition 8.1.1.1.1.

Naturality in B is then the statement that given a relation $R\colon B \nrightarrow B'$,

[19] Here we are using Item 3 of Definition 4.5.1.1.4.

the diagram

$$\begin{tikzcd}
\mathsf{Rel}(\mathrm{Gr}(A), B) \arrow[r, "R\diamond-"] \arrow[d, dashed, "\wr"'] & \mathsf{Rel}(\mathrm{Gr}(A), B') \arrow[d, dashed, "\wr"] \\
\mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_!"'] & \mathsf{Sets}(A, \mathcal{P}(B'))
\end{tikzcd}$$

commutes, which follows from Definition 8.7.1.1.3. □

4.4.5 Powersets as Free Cocompletions

Let X be a set.

Proposition 4.4.5.1.1. The pair $(\mathcal{P}(X), \chi_{(-)})$ consisting of

- The powerset $(\mathcal{P}(X), \subset)$ of X of Definition 4.4.1.1.1;
- The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

   – A suplattice $(Y, \preceq)$;
   – A function $f \colon X \to Y$;

   there exists a unique morphism of suplattices

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

   making the diagram

$$\begin{tikzcd}
 & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\
X \arrow[ru, "\chi_X"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

   commute.

Proof. This is a rephrasing of Definition 4.4.5.1.2, which we prove below.[20] □

Proposition 4.4.5.1.2. We have an adjunction

$$(\mathcal{P} \dashv 忘)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathcal{P}"] \arrow[r, phantom, "\perp"] & \mathsf{SupLat} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection

$$\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{SupLat})$, where:

- The category SupLat is the category of suplattices of **??**.
- The map

  $$\chi_X^* \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

  witnessing the above bijection is defined by

  $$\chi_X^*(f) \overset{\mathrm{def}}{=} f \circ \chi_X,$$

  i.e. by sending a morphism of suplattices $f\colon \mathcal{P}(X) \to Y$ to the composition

  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

  witnessing the above bijection is given by sending a function $f\colon X \to Y$ to its left Kan extension along χ_X,

  $$\mathrm{Lan}_{\chi_X}(f)\colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, "\Uparrow"] & Y. \end{tikzcd}$$

[20]Here we only remark that the unique morphism of suplattices in the statement is

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned}
\left[\mathrm{Lan}_{\chi_X}(f)\right](U) &= \int^{x\in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\
&= \int^{x\in X} \chi_U(x) \odot f(x) \\
&= \bigvee_{x\in X} (\chi_U(x) \odot f(x)) \\
&= \left(\bigvee_{x\in U} (\chi_U(x) \odot f(x))\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} (\chi_U(x) \odot f(x))\right) \\
&= \left(\bigvee_{x\in U} f(x)\right) \vee \left(\bigvee_{x\in U^{\mathsf{c}}} \varnothing_Y\right) \\
&= \bigvee_{x\in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

- We have used **??** for the first equality.
- We have used Definition 4.5.5.1.1 for the second equality.
- We have used **??** for the third equality.
- The symbol $\bigvee$ denotes the join in $(Y, \preceq)$.
- The symbol $\odot$ denotes the tensor of an element of Y by a truth value as in **??**. In particular, we have

$$\begin{aligned}
\mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\
\mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y,
\end{aligned}$$

  where $\varnothing_Y$ is the bottom element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathrm{Lan}_{\chi_X}(f)$ is given by

$$\begin{aligned}
\left[\mathrm{Lan}_{\chi_X}(f)\right](U) &= \bigvee_{x\in U} f(x) \\
&= \bigcup_{x\in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Map I*: We define a map

$$\Phi_{X,Y} \colon \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.
Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, ""{name=A}] & Y, \arrow[u, Rightarrow, shorten=8pt] \end{tikzcd}$$

for each $f \in \mathsf{Sets}(X, Y)$.
Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))} .$$

We have

$$\begin{aligned} \left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](f) &\stackrel{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(f)) \\ &\stackrel{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X) \\ &\stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f \circ \chi_X) \end{aligned}$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$. Indeed, we have

$$\left[\mathrm{Lan}_{\chi_X}(f \circ \chi_X)\right](U) = \bigvee_{x \in U} f(\chi_X(x))$$

$$
\begin{aligned}
&= f\left(\bigvee_{x\in U} \chi_X(x)\right)\\
&= f\left(\bigcup_{x\in U}\{x\}\right)\\
&= f(U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of suplattices and hence preserves joins for the second equality. This proves our claim. Since we have shown that

$$\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](f) = f$$

for each $f \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq))}$ of $\mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.

Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)} .$$

We have

$$
\begin{aligned}
\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](f) &\stackrel{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(f)\right)\\
&\stackrel{\text{def}}{=} \Phi_{X,Y}\left(\mathrm{Lan}_{\chi_X}(f)\right)\\
&\stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f) \circ \chi_X
\end{aligned}
$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Lan}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$
\begin{aligned}
\left[\mathrm{Lan}_{\chi_X}(f) \circ \chi_X\right](x) &= \bigvee_{y\in\{x\}} f(y)\\
&= f(x)
\end{aligned}
$$

for each $x \in X$. This proves our claim. Since we have shown that

$$\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, Part I: We need to show that, given a function $f\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{SupLat}((\mathcal{P}(X'),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_!(f)^*"'] & \mathsf{Sets}(X',Y) \arrow[d, "f^*"] \\
\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X,Y)
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
\left[\Phi_{X,Y}\circ\mathcal{P}_!(f)^*\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\left(\mathcal{P}_!(f)^*(\xi)\right)\\
&\overset{\text{def}}{=} \Phi_{X,Y}(\xi\circ f_!)\\
&\overset{\text{def}}{=} (\xi\circ f_!)\circ\chi_X\\
&= \xi\circ(f_!\circ\chi_X)\\
&\overset{(\dagger)}{=} \xi\circ(\chi_{X'}\circ f)\\
&= (\xi\circ\chi_{X'})\circ f\\
&\overset{\text{def}}{=} \Phi_{X',Y}(\xi)\circ f\\
&\overset{\text{def}}{=} f^*\left(\Phi_{X',Y}(\xi)\right)\\
&\overset{\text{def}}{=} \left[f^*\circ\Phi_{X',Y}\right](\xi),
\end{aligned}$$

for each $\xi\in\mathsf{SupLat}((\mathcal{P}(X'),\subset),(Y,\preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, Part II: We need to show that, given a morphism of suplattices

$$g\colon (Y,\preceq_Y)\to(Y',\preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_!"'] & \mathsf{Sets}(X,Y) \arrow[d, "g_!"] \\
\mathsf{SupLat}((\mathcal{P}(X),\subset),(Y',\preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X,Y')
\end{tikzcd}$$

commutes. Indeed, we have

$$\left[\Phi_{X,Y'}\circ g_!\right](\xi) \overset{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi))$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\
&\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X\\
&= g \circ (\xi \circ \chi_X)\\
&\overset{\text{def}}{=} g \circ \left(\Phi_{X,Y}(\xi)\right)\\
&\overset{\text{def}}{=} g_!\left(\Phi_{X,Y}(\xi)\right)\\
&\overset{\text{def}}{=} \left[g_! \circ \Phi_{X,Y}\right](\xi).
\end{aligned}
$$

for each $\xi \in \mathsf{SupLat}((\mathcal{P}(X), \subset), (Y, \preceq))$.
Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. □

Warning 4.4.5.1.3. Although the assignment $X \mapsto \mathcal{P}(X)$ is called the *free cocompletion of* X, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)) \neq \mathcal{P}(X)$.

4.4.6 Powersets as Free Completions

Let X be a set.

Proposition 4.4.6.1.1. The pair $\left(\mathcal{P}(X), \chi_{(-)}\right)$ consisting of

- The powerset of X together with reverse inclusion $\mathcal{P}(X)^{\mathsf{op}} = (\mathcal{P}(X), \supset)$ of Definition 4.4.1.1.1;
- The characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1;

satisfies the following universal property:

(★) Given another pair (Y, f) consisting of

  - An inflattice $(Y, \preceq)$;
  - A function $f \colon X \to Y$;

  there exists a unique morphism of inflattices

$$(\mathcal{P}(X), \supset) \xrightarrow{\exists!} (Y, \preceq)$$

---

given by the left Kan extension $\mathrm{Lan}_{\chi_X}(f)$ of f along χ_X.

making the diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commute.

Proof. This is a rephrasing of Definition 4.4.6.1.2, which we prove below.[21] □

Proposition 4.4.6.1.2. We have an adjunction

$$(\mathcal{P} \dashv 忘): \quad \mathsf{Sets} \underset{忘}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{InfLat},$$

witnessed by a bijection

$$\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{InfLat})$, where:

- The category InfLat is the category of inflattices of **??**.
- The map
  $$\chi_X^* \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$
  witnessing the above bijection is defined by
  $$\chi_X^*(f) \overset{\text{def}}{=} f \circ \chi_X,$$
  i.e. by sending a morphism of inflattices $f \colon \mathcal{P}(X)^{\mathsf{op}} \to Y$ to the composition
  $$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X)^{\mathsf{op}} \xrightarrow{f} Y.$$

[21] Here we only remark that the unique morphism of inflattices in the statement is given by the right Kan extension $\mathrm{Ran}_{\chi_X}(f)$ of f along χ_X.

- The map

$$\mathrm{Ran}_{\chi_X}\colon \mathsf{Sets}(X,Y) \longrightarrow \mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))$$

witnessing the above bijection is given by sending a function $f\colon X \to Y$ to its right Kan extension along χ_X,

$$\mathrm{Ran}_{\chi_X}(f)\colon \mathcal{P}(X)^{\mathsf{op}} \longrightarrow Y, \qquad \begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\mathrm{Ran}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[r, phantom, bend left, "\Downarrow"] & Y. \end{tikzcd}$$

Moreover, invoking the bijection $\mathcal{P}(X) \cong \mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})$ of Item 2 of Definition 4.5.1.1.4, $\mathrm{Ran}_{\chi_X}(f)$ can be explicitly computed by

$$\begin{aligned}
\left[\mathrm{Ran}_{\chi_X}(f)\right](U) &= \int_{x\in X} \chi_{\mathcal{P}(X)^{\mathsf{op}}}(\chi_x, U) \pitchfork f(x)\\
&= \int_{x\in X} \chi_{\mathcal{P}(X)}(U, \chi_x) \pitchfork f(x)\\
&= \int_{x\in X} \chi_U(x) \pitchfork f(x)\\
&= \bigwedge_{x\in X} \chi_U(x) \pitchfork f(x)\\
&= \left(\bigwedge_{x\in U} \chi_U(x) \pitchfork f(x)\right) \wedge \left(\bigwedge_{x\in U^{\mathsf{c}}} \chi_U(x) \pitchfork f(x)\right)\\
&= \left(\bigwedge_{x\in U} f(x)\right) \wedge \left(\bigwedge_{x\in U^{\mathsf{c}}} \infty_Y\right)\\
&= \left(\bigwedge_{x\in U} f(x)\right) \wedge \infty_Y\\
&= \bigwedge_{x\in U} f(x)
\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where:

  - We have used **??** for the first equality.
  - We have used Definition 4.5.5.1.1 for the second equality.
  - We have used **??** for the third equality.

– The symbol $\bigwedge$ denotes the meet in $(Y, \preceq)$.
– The symbol $\pitchfork$ denotes the cotensor of an element of Y by a truth value as in **??**. In particular, we have

$$\begin{aligned} \mathsf{true} \pitchfork f(x) &\stackrel{\text{def}}{=} f(x), \\ \mathsf{false} \pitchfork f(x) &\stackrel{\text{def}}{=} \infty_Y, \end{aligned}$$

where ∞_Y is the top element of $(Y, \preceq)$.

In particular, when $(Y, \preceq_Y) = (\mathcal{P}(B), \subset)$ for some set B, the Kan extension $\mathrm{Ran}_{\chi_X}(f)$ is given by

$$\begin{aligned} \left[\mathrm{Ran}_{\chi_X}(f)\right](U) &= \bigwedge_{x \in U} f(x) \\ &= \bigcap_{x \in U} f(x) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Map I*: We define a map

$$\Phi_{X,Y} \colon \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, i.e. by

$$\Phi_{X,Y}(f) \stackrel{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
Map II: We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$$

as in the statement, i.e. by

$$\Psi_{X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, dashed, "\mathrm{Ran}_{\chi_X}(f)"] \arrow[dl, phantom, "\Swarrow"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y, \end{tikzcd}$$

for each $f \in \mathsf{Sets}(X, Y)$.
Invertibility I: We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))}.$$

We have

$$\begin{aligned}\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](f) &\overset{\text{def}}{=} \Psi_{X,Y}\left(\Phi_{X,Y}(f)\right)\\ &\overset{\text{def}}{=} \Psi_{X,Y}(f \circ \chi_X)\\ &\overset{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f \circ \chi_X)\end{aligned}$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f \circ \chi_X) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$. Indeed, we have

$$\begin{aligned}\left[\mathrm{Ran}_{\chi_X}(f \circ \chi_X)\right](U) &= \bigwedge_{x \in U} f(\chi_X(x))\\ &= f\left(\bigwedge_{x \in U} \chi_X(x)\right)\\ &= f\left(\bigcup_{x \in U} \{x\}\right)\\ &= f(U)\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we have used that f is a morphism of inflattices and hence preserves meets in $(\mathcal{P}(X), \supset)$ (i.e. joins in $(\mathcal{P}(X), \subset)$) for the second equality. This proves our claim. Since we have shown that

$$\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](f) = f$$

for each $f \in \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$, it follows that $\Psi_{X,Y} \circ \Phi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))}$ of $\mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq))$.
Invertibility II: We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}\,.$$

We have

$$\begin{aligned}\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](f) &\overset{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(f)\right)\\ &\overset{\text{def}}{=} \Phi_{X,Y}\left(\mathrm{Ran}_{\chi_X}(f)\right)\\ &\overset{\text{def}}{=} \mathrm{Ran}_{\chi_X}(f) \circ \chi_X\end{aligned}$$

for each $f \in \mathsf{Sets}(X, Y)$. We now claim that

$$\mathrm{Ran}_{\chi_X}(f) \circ \chi_X = f$$

for each $f \in \mathsf{Sets}(X, Y)$. Indeed, we have

$$\begin{aligned}\left[\mathrm{Ran}_{\chi_X}(f) \circ \chi_X\right](x) &= \bigwedge_{y \in \{x\}} f(y)\\ &= f(x)\end{aligned}$$

for each $x \in X$. This proves our claim. Since we have shown that

$$\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](f) = f$$

for each $f \in \mathsf{Sets}(X, Y)$, it follows that $\Phi_{X,Y} \circ \Psi_{X,Y}$ must be equal to the identity map $\mathrm{id}_{\mathsf{Sets}(X,Y)}$ of $\mathsf{Sets}(X, Y)$.

Naturality for Φ, Part I: We need to show that, given a function $f : X \to X'$, the diagram

$$\begin{tikzcd} \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_!(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\ \mathsf{InfLat}((\mathcal{P}(X), \supset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y) \end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}\left[\Phi_{X,Y} \circ \mathcal{P}_!(f)^*\right](\xi) &\overset{\text{def}}{=} \Phi_{X,Y}\left(\mathcal{P}_!(f)^*(\xi)\right)\\ &\overset{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_!)\\ &\overset{\text{def}}{=} (\xi \circ f_!) \circ \chi_X\\ &= \xi \circ (f_! \circ \chi_X)\\ &\overset{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f)\\ &= (\xi \circ \chi_{X'}) \circ f\\ &\overset{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f\\ &\overset{\text{def}}{=} f^*\left(\Phi_{X',Y}(\xi)\right)\\ &\overset{\text{def}}{=} \left[f^* \circ \Phi_{X',Y}\right](\xi),\end{aligned}$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X'), \supset), (Y, \preceq))$, where we have used Item 1 of Definition 4.5.4.1.3 for the fifth equality above.

Naturality for Φ, Part II: We need to show that, given a cocontinuous morphism of posets

$$g : (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

the diagram

$$\begin{tikzcd}
\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_!"'] & \mathsf{Sets}(X,Y) \arrow[d, "g_!"] \\
\mathsf{InfLat}((\mathcal{P}(X),\supset),(Y',\preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X,Y')
\end{tikzcd}$$

commutes. Indeed, we have

$$\begin{aligned}
\left[\Phi_{X,Y'} \circ g_!\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y'}(g_!(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi) \\
&\stackrel{\text{def}}{=} (g \circ \xi) \circ \chi_X \\
&= g \circ (\xi \circ \chi_X) \\
&\stackrel{\text{def}}{=} g \circ \left(\Phi_{X,Y}(\xi)\right) \\
&\stackrel{\text{def}}{=} g_!\left(\Phi_{X,Y}(\xi)\right) \\
&\stackrel{\text{def}}{=} \left[g_! \circ \Phi_{X,Y}\right](\xi).
\end{aligned}$$

for each $\xi \in \mathsf{InfLat}((\mathcal{P}(X),\supset),(Y,\preceq))$.
Naturality for Ψ: Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument. □

Warning 4.4.6.1.3. Although the assignment $X \mapsto \mathcal{P}(X)^{\mathsf{op}}$ is called the *free completion of* X, it is not an idempotent operation, i.e. we have $\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}})^{\mathsf{op}} \neq \mathcal{P}(X)^{\mathsf{op}}$.

4.4.7 The Internal Hom of a Powerset

Let X be a set and let $U, V \in \mathcal{P}(X)$.

Proposition 4.4.7.1.1. The **internal Hom of** $\mathcal{P}(X)$ **from** U **to** V is the subset $[U,V]_X$[22] of X given by

$$\begin{aligned}
[U,V]_X &= U^{\textsf{c}} \cup V \\
&= (U \setminus V)^{\textsf{c}}
\end{aligned}$$

where $U^{\textsf{c}}$ is the complement of U of Definition 4.3.11.1.1.

[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathcal{P}(X)}(U,V)$.

Proof. *Proof of the Equality $U^{\textsf{c}} \cup V = (U \setminus V)^{\textsf{c}}$*: We have

$$\begin{aligned}(U \setminus V)^{\textsf{c}} &\overset{\text{def}}{=} X \setminus (U \setminus V)\\ &= (X \cap V) \cup (X \setminus U)\\ &= V \cup (X \setminus U)\\ &\overset{\text{def}}{=} V \cup U^{\textsf{c}}\\ &= U^{\textsf{c}} \cup V,\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the second equality.
2. Item 4 of Definition 4.3.9.1.2 for the third equality.
3. Item 4 of Definition 4.3.8.1.2 for the last equality.

This finishes the proof.
Proof that $U^{\textsf{c}} \cup V$ Is Indeed the Internal Hom: This follows from Item 2 of Definition 4.3.9.1.2. □

Remark 4.4.7.1.2. Henning Makholm suggests the following heuristic intuition for the internal Hom of $\mathcal{P}(X)$ from U to V ([MSE 267365]):

1. Since products in $\mathcal{P}(X)$ are given by binary intersections (Item 1 of Definition 4.4.1.1.4), the right adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U, -)$ of $U \cap -$ may be thought of as a function type $[U, V]$.
2. Under the Curry–Howard correspondence (**??**), the function type $[U, V]$ corresponds to implication $U \Rightarrow V$.
3. Implication $U \Rightarrow V$ is logically equivalent to $\neg U \vee V$.
4. The expression $\neg U \vee V$ then corresponds to the set $U^{\textsf{c}} \cup V$ in $\mathcal{P}(X)$.
5. The set $U^{\textsf{c}} \vee V$ turns out to indeed be the internal Hom of $\mathcal{P}(X)$.

Proposition 4.4.7.1.3. Let X be a set.

1. *Functoriality.* The assignments $U, V, (U, V) \mapsto \mathbf{Hom}_{\mathcal{P}(X)}$ define functors

$$\begin{aligned}[U, -]_X &\colon & (\mathcal{P}(X), \supset) &\to (\mathcal{P}(X), \subset),\\ [-, V]_X &\colon & (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset),\\ [-_1, -_2]_X &\colon & (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) &\to (\mathcal{P}(X), \subset).\end{aligned}$$

   In particular, the following statements hold for each $U, V, A, B \in \mathcal{P}(X)$:

(a) If $U \subset A$, then $[A, V]_X \subset [U, V]_X$.

(b) If $V \subset B$, then $[U, V]_X \subset [U, B]_X$.

(c) If $U \subset A$ and $V \subset B$, then $[A, V]_X \subset [U, B]_X$.

2. *Adjointness.* We have adjunctions

$$
\begin{tikzcd}
(U \cap - \dashv [U, -]_X)\colon \quad \mathcal{P}(X) \arrow[r, bend left, "U\cap-"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[U,-]_X}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(- \cap V \dashv [V, -]_X)\colon \quad \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "{[V,-]_X}"]
\end{tikzcd}
$$

witnessed by bijections

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, [V, W]_X), \\
\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(V, [U, W]_X).
\end{aligned}
$$

In particular, the following statements hold for each $U, V, W \in \mathcal{P}(X)$:

(a) The following conditions are equivalent:

i. We have $U \cap V \subset W$.

ii. We have $U \subset [V, W]_X$.

(b) The following conditions are equivalent:

i. We have $U \cap V \subset W$.

ii. We have $V \subset [U, W]_X$.

3. *Interaction With the Empty Set I.* We have

$$
\begin{aligned}
[U, \emptyset]_X &= U^{\mathsf{c}}, \\
[\emptyset, V]_X &= X,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Interaction With X.* We have

$$
\begin{aligned}
[U, X]_X &= X, \\
[X, V]_X &= V,
\end{aligned}
$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Interaction With the Empty Set II.* The functor

$$D_X\colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X)$$

defined by

$$\begin{aligned} D_X &\overset{\text{def}}{=} [-,\emptyset]_X \\ &= (-)^{\mathsf{c}} \end{aligned}$$

is an involutory isomorphism of categories, making $\emptyset$ into a dualising object for $(\mathcal{P}(X),\cap,X,[-,-]_X)$ in the sense of **??**. In particular:

(a) The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "D_X"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}}"'] & \mathcal{P}(X) \arrow[d, "D_X"] \\ & \mathcal{P}(X)^{\mathsf{op}} \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(D_X(U))}_{\overset{\text{def}}{=}[[U,\emptyset]_X,\emptyset]_X} = U$$

for each $U \in \mathcal{P}(X)$.

(b) The diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "\cap^{\mathsf{op}}"] & \mathcal{P}(X)^{\mathsf{op}} \arrow[dr, "D_X"] & \\ \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)^{\mathsf{op}}} \times D_X"] \arrow[rrr, "{[-_1,-_2]_X}"'] & & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\text{def}}{=}[U\cap[V,\emptyset]_X,\emptyset]_X} = [U,V]_X$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With the Empty Set III.* Let $f\colon X \to Y$ be a function.

(a) *Interaction With Direct Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(b) *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \arrow[r, "f^{-1,\mathsf{op}}"] \arrow[d, "D_Y"'] & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "D_X"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X\big(f^{-1}(U)\big)$$

for each $U \in \mathcal{P}(X)$.

(c) *Interaction With Codirect Images.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "D_Y"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

7. *Interaction With Unions of Families of Subsets I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcup^{\mathsf{op}} \times \bigcup^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcup_{W \in [\mathcal{U},\mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcup_{U \in \mathcal{U}} U, \bigcup_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

8. *Interaction With Unions of Families of Subsets II.* The diagram

$$\begin{tikzcd}
& \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcup^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcap"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcup_{U \in \mathcal{U}} U, V\right]_X = \bigcap_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

9. *Interaction With Unions of Families of Subsets III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X}"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcup"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcup_{V \in \mathcal{V}} V\right]_X = \bigcup_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

10. *Interaction With Intersections of Families of Subsets I.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \times \mathcal{P}(\mathcal{P}(X)) \arrow[r, "{[-_1,-_2]_{\mathcal{P}(X)}}"] \arrow[d, "\bigcap^{\mathsf{op}} \times \bigcap^{\mathsf{op}}"'] \arrow[dr, phantom, "\times"] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcap"] \\
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "{[-_1,-_2]_X}"'] & \mathcal{P}(X),
\end{tikzcd}$$

does not commute in general, i.e. we may have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W \neq \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X$$

in general, where $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$.

11. *Interaction With Intersections of Families of Subsets II.* The diagram

$$\begin{tikzcd}
& \mathcal{P}(\mathcal{P}(X))^{\mathsf{op}} \arrow[dr, "\bigcap^{\mathsf{op}}"] & \\
\mathcal{P}(\mathcal{P}(X)^{\mathsf{op}}) \arrow[ur, dashed, "\sim"] \arrow[d, "{\mathrm{id}_{\mathcal{P}(X)} \star [-,V]_X}"'] & & \mathcal{P}(X)^{\mathsf{op}} \arrow[d, "{[-,V]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[rr, "\bigcup"'] & & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[\bigcap_{U \in \mathcal{U}} U, V\right]_X = \bigcup_{U \in \mathcal{U}} [U, V]_X$$

for each $\mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$ and each $V \in \mathcal{P}(X)$.

12. *Interaction With Intersections of Families of Subsets III.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"] \arrow[d, "\mathrm{id}_{\mathcal{P}(X)} \star [U,-]_X"'] & \mathcal{P}(X) \arrow[d, "{[U,-]_X}"] \\
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "\bigcap"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[U, \bigcap_{V \in \mathcal{V}} V\right]_X = \bigcap_{V \in \mathcal{V}} [U, V]_X$$

for each $U \in \mathcal{P}(X)$ and each $\mathcal{V} \in \mathcal{P}(\mathcal{P}(X))$.

13. *Interaction With Binary Unions.* We have equalities of sets

$$\begin{aligned}
[U \cap V, W]_X &= [U, W]_X \cup [V, W]_X, \\
[U, V \cap W]_X &= [U, V]_X \cap [U, W]_X
\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

14. *Interaction With Binary Intersections.* We have equalities of sets

$$\begin{aligned}
[U \cup V, W]_X &= [U, W]_X \cap [V, W]_X, \\
[U, V \cup W]_X &= [U, V]_X \cup [U, W]_X
\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

15. *Interaction With Differences.* We have equalities of sets

$$\begin{aligned}
[U \setminus V, W]_X &= [U, W]_X \cup [V^{\mathsf{c}}, W]_X \\
&= [U, W]_X \cup [U, V]_X, \\
[U, V \setminus W]_X &= [U, V]_X \setminus (U \cap W)
\end{aligned}$$

for each $U, V, W \in \mathcal{P}(X)$.

16. *Interaction With Complements.* We have equalities of sets

$$\begin{aligned}
[U^{\mathsf{c}}, V]_X &= U \cup V, \\
[U, V^{\mathsf{c}}]_X &= U \cap V, \\
[U, V]^{\mathsf{c}}_X &= U \setminus V
\end{aligned}$$

for each $U, V \in \mathcal{P}(X)$.

17. *Interaction With Characteristic Functions.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}}(x) = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

18. *Interaction With Direct Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_!"] \arrow[d, "{[-_1,-_2]_X}"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f_!([U,V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

19. *Interaction With Inverse Images.* Let $f\colon X \to Y$ be a function. The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = \left[f^{-1}(U), f^{-1}(V)\right]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

20. *Interaction With Codirect Images.* Let $f\colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_*"] \arrow[d, "{[-_1,-_2]_X}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U,V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Since $\mathcal{P}(X)$ is posetal, it suffices to prove Items 1a to 1c.

1. *Proof of Item 1a:* We have

$$\begin{aligned} [A,V]_X &\stackrel{\text{def}}{=} A^{\mathsf{c}} \cup V \\ &\subset U^{\mathsf{c}} \cup V \\ &\stackrel{\text{def}}{=} [U,V]_X, \end{aligned}$$

   where we have used:

   (a) Item 1 of Definition 4.3.11.1.2, which states that if $U \subset A$, then $A^{\mathsf{c}} \subset U^{\mathsf{c}}$.

   (b) Item 1a of Item 1 of Definition 4.3.11.1.2, which states that if $A^{\mathsf{c}} \subset U^{\mathsf{c}}$, then $A^{\mathsf{c}} \cup K \subset U^{\mathsf{c}} \cup K$ for any $K \in \mathcal{P}(X)$.

2. *Proof of Item 1b:* We have

$$\begin{aligned} [U,V]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup V \\ &\subset U^{\mathsf{c}} \cup B \\ &\stackrel{\text{def}}{=} [U,B]_X, \end{aligned}$$

   where we have used Item 1b of Item 1 of Definition 4.3.11.1.2, which states that if $V \subset B$, then $K \cup V \subset K \cup B$ for any $K \in \mathcal{P}(X)$.

3. *Proof of Item 1c:* We have

$$\begin{aligned} [A,V]_X &\subset [U,V]_X \\ &\subset [U,B]_X, \end{aligned}$$

   where we have used Items 1a and 1b.

This finishes the proof.
Item 2, Adjointness: This is a repetition of Item 2 of Definition 4.3.9.1.2 and is proved there.
Item 3, Interaction With the Empty Set I: We have

$$[U, \emptyset]_X \stackrel{\text{def}}{=} U^{\mathsf{c}} \cup \emptyset$$

$$= U^{\mathsf{c}},$$

where we have used Item 3 of Definition 4.3.8.1.2, and we have

$$\begin{aligned}
[\emptyset, V]_X &\stackrel{\text{def}}{=} \emptyset^{\mathsf{c}} \cup V \\
&\stackrel{\text{def}}{=} (X \setminus \emptyset) \cup V \\
&= X \cup V \\
&= X,
\end{aligned}$$

where we have used:

1. Item 12 of Definition 4.3.10.1.2 for the first equality.

2. Item 5 of Definition 4.3.8.1.2 for the last equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**??** of **??**).
Item 4, Interaction With X: We have

$$\begin{aligned}
[U, X]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup X \\
&= X,
\end{aligned}$$

where we have used Item 5 of Definition 4.3.8.1.2, and we have

$$\begin{aligned}
[X, V]_X &\stackrel{\text{def}}{=} X^{\mathsf{c}} \cup V \\
&\stackrel{\text{def}}{=} (X \setminus X) \cup V \\
&= \emptyset \cup V \\
&= V,
\end{aligned}$$

where we have used Item 3 of Definition 4.3.8.1.2 for the last equality. Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**??** of **??**).
Item 5, Interaction With the Empty Set II: We have

$$\begin{aligned}
D_X(D_X(U)) &\stackrel{\text{def}}{=} [[U, \emptyset]_X, \emptyset]_X \\
&= [U^{\mathsf{c}}, \emptyset]_X \\
&= (U^{\mathsf{c}})^{\mathsf{c}} \\
&= U,
\end{aligned}$$

where we have used:

1. Item 3 for the second and third equalities.

2. Item 3 of Definition 4.3.11.1.2 for the fourth equality.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**??** of **??**), and thus we have

$$[[-,\emptyset]_X,\emptyset]_X \cong \mathrm{id}_{\mathcal{P}(X)}$$

This finishes the proof.
Item 6, Interaction With the Empty Set III: Since $D_X = (-)^{\textsf{c}}$, this is essentially a repetition of the corresponding results for $(-)^{\textsf{c}}$, namely Items 5 to 7 of Definition 4.3.11.1.2.
Item 7, Interaction With Unions of Families of Subsets I: By Item 3 of Definition 4.4.7.1.3, we have

$$\begin{aligned}[\mathcal{U},\emptyset]_{\mathcal{P}(X)} &= \mathcal{U}^{\textsf{c}},\\ [U,\emptyset]_X &= U^{\textsf{c}}.\end{aligned}$$

With this, the counterexample given in the proof of Item 10 of Definition 4.3.6.1.2 then applies.
Item 8, Interaction With Unions of Families of Subsets II: We have

$$\begin{aligned}\left[\bigcup_{U\in\mathcal{U}} U, V\right]_X &\stackrel{\text{def}}{=} \left(\bigcup_{U\in\mathcal{U}} U\right)^{\textsf{c}} \cup V\\ &= \left(\bigcap_{U\in\mathcal{U}} U^{\textsf{c}}\right) \cup V\\ &= \bigcap_{U\in\mathcal{U}} (U^{\textsf{c}} \cup V)\\ &\stackrel{\text{def}}{=} \bigcap_{U\in\mathcal{U}} [U,V]_X,\end{aligned}$$

where we have used:

1. Item 11 of Definition 4.3.6.1.2 for the second equality.

2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.
Item 9, Interaction With Unions of Families of Subsets III: We have

$$\begin{aligned}\bigcup_{V\in\mathcal{V}} [U,V]_X &\stackrel{\text{def}}{=} \bigcup_{V\in\mathcal{V}} (U^{\textsf{c}} \cup V)\\ &= U^{\textsf{c}} \cup \left(\bigcup_{V\in\mathcal{V}} V\right)\end{aligned}$$

$$\stackrel{\text{def}}{=} \left[U, \bigcup_{V \in \mathcal{V}} V\right]_X.$$

where we have used Item 6. This finishes the proof.

Item 10, Interaction With Intersections of Families of Subsets I: Let $X = \{0, 1\}$, let $\mathcal{U} = \{\{0, 1\}\}$, and let $\mathcal{V} = \{\{0\}, \{0, 1\}\}$. We have

$$\begin{aligned} \bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W &= \bigcap_{W \in \mathcal{P}(X)} W \\ &= \{0, 1\}, \end{aligned}$$

whereas

$$\begin{aligned} \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X &= [\{0, 1\}, \{0\}] \\ &= \{0\}, \end{aligned}$$

Thus we have

$$\bigcap_{W \in [\mathcal{U}, \mathcal{V}]_{\mathcal{P}(X)}} W = \{0, 1\} \neq \{0\} = \left[\bigcap_{U \in \mathcal{U}} U, \bigcap_{V \in \mathcal{V}} V\right]_X.$$

This finishes the proof.

Item 11, Interaction With Intersections of Families of Subsets II: We have

$$\begin{aligned} \left[\bigcap_{U \in \mathcal{U}} U, V\right]_X &\stackrel{\text{def}}{=} \left(\bigcap_{U \in \mathcal{U}} U\right)^{\mathsf{c}} \cup V \\ &= \left(\bigcup_{U \in \mathcal{U}} U^{\mathsf{c}}\right) \cup V \\ &= \bigcup_{U \in \mathcal{U}} (U^{\mathsf{c}} \cup V) \\ &\stackrel{\text{def}}{=} \bigcup_{U \in \mathcal{U}} [U, V]_X, \end{aligned}$$

where we have used:

1. Item 12 of Definition 4.3.6.1.2 for the second equality.
2. Item 6 of Definition 4.3.7.1.2 for the third equality.

This finishes the proof.

Item 12, Interaction With Intersections of Families of Subsets III: We have

$$\begin{aligned}\bigcap_{V\in\mathcal{V}}[U,V]_X &\stackrel{\text{def}}{=} \bigcap_{V\in\mathcal{V}}(U^{\mathsf{c}}\cup V)\\ &= U^{\mathsf{c}}\cup\left(\bigcap_{V\in\mathcal{V}}V\right)\\ &\stackrel{\text{def}}{=}\left[U,\bigcap_{V\in\mathcal{V}}V\right]_X.\end{aligned}$$

where we have used Item 6. This finishes the proof.

Item 13, Interaction With Binary Unions: We have

$$\begin{aligned}[U\cap V,W]_X &\stackrel{\text{def}}{=} (U\cap V)^{\mathsf{c}}\cup W\\ &= (U^{\mathsf{c}}\cup V^{\mathsf{c}})\cup W\\ &= (U^{\mathsf{c}}\cup V^{\mathsf{c}})\cup(W\cup W)\\ &= (U^{\mathsf{c}}\cup W)\cup(V^{\mathsf{c}}\cup W)\\ &\stackrel{\text{def}}{=} [U,W]_X\cup[V,W]_X,\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.
2. Item 8 of Definition 4.3.8.1.2 for the third equality.
3. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the fourth equality.

For the second equality in the statement, we have

$$\begin{aligned}[U,V\cap W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}}\cup(V\cap W)\\ &= (U^{\mathsf{c}}\cup V)\cap(U^{\mathsf{c}}\cap W)\\ &\stackrel{\text{def}}{=} [U,V]_X\cap[U,W]_X,\end{aligned}$$

where we have used Item 6 of Definition 4.3.8.1.2 for the second equality.

Item 14, Interaction With Binary Intersections: We have

$$\begin{aligned}[U\cup V,W]_X &\stackrel{\text{def}}{=} (U\cup V)^{\mathsf{c}}\cup W\\ &= (U^{\mathsf{c}}\cap V^{\mathsf{c}})\cup W\\ &= (U^{\mathsf{c}}\cup W)\cap(V^{\mathsf{c}}\cup W)\\ &\stackrel{\text{def}}{=} [U,W]_X\cap[V,W]_X,\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.11.1.2 for the second equality.
2. Item 6 of Definition 4.3.8.1.2 for the third equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}[U, V \cup W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup (V \cup W)\\ &= (U^{\mathsf{c}} \cup U^{\mathsf{c}}) \cup (V \cup W)\\ &= (U^{\mathsf{c}} \cup V) \cup (U^{\mathsf{c}} \cup W)\\ &\stackrel{\text{def}}{=} [U, V]_X \cup [U, W]_X,\end{aligned}$$

where we have used:

1. Item 8 of Definition 4.3.8.1.2 for the second equality.
2. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the third equality.

This finishes the proof.
Item 15, Interaction With Differences: We have

$$\begin{aligned}[U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W\\ &\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W\\ &= ((X \cap V) \cup (X \setminus U)) \cup W\\ &= (V \cup (X \setminus U)) \cup W\\ &\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W\\ &= (V \cup (U^{\mathsf{c}} \cup U^{\mathsf{c}})) \cup W\\ &= (U^{\mathsf{c}} \cup W) \cup (U^{\mathsf{c}} \cup V)\\ &\stackrel{\text{def}}{=} [U, W]_X \cup [U, V]_X,\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.
2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.
3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.
4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.

We also have

$$\begin{aligned}
[U \setminus V, W]_X &\stackrel{\text{def}}{=} (U \setminus V)^{\mathsf{c}} \cup W \\
&\stackrel{\text{def}}{=} (X \setminus (U \setminus V)) \cup W \\
&= ((X \cap V) \cup (X \setminus U)) \cup W \\
&= (V \cup (X \setminus U)) \cup W \\
&\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \cup W \\
&= (V \cup U^{\mathsf{c}}) \cup (W \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup (V \cup W) \\
&= (U^{\mathsf{c}} \cup W) \cup \left((V^{\mathsf{c}})^{\mathsf{c}} \cup W\right) \\
&\stackrel{\text{def}}{=} [U, W]_X \cup [V^{\mathsf{c}}, W]_X,
\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.10.1.2 for the third equality.
2. Item 4 of Definition 4.3.9.1.2 for the fourth equality.
3. Item 8 of Definition 4.3.8.1.2 for the sixth equality.
4. Several applications of Items 2 and 4 of Definition 4.3.8.1.2 and for the seventh equality.
5. Item 3 of Definition 4.3.11.1.2 for the eighth equality.

Now, for the second equality in the statement, we have

$$\begin{aligned}
[U, V \setminus W]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup (V \setminus W) \\
&= (V \setminus W) \cup U^{\mathsf{c}} \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \setminus U^{\mathsf{c}}) \\
&\stackrel{\text{def}}{=} (V \cup U^{\mathsf{c}}) \setminus (W \setminus (X \setminus U)) \\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup (W \setminus X)) \\
&= (V \cup U^{\mathsf{c}}) \setminus ((W \cap U) \cup \emptyset) \\
&= (V \cup U^{\mathsf{c}}) \setminus (W \cap U) \\
&= (V \cup U^{\mathsf{c}}) \setminus (U \cap W) \\
&\stackrel{\text{def}}{=} [U, V]_X \setminus (U \cap W)
\end{aligned}$$

where we have used:

1. Item 4 of Definition 4.3.8.1.2 for the second equality.

2. Item 4 of Definition 4.3.10.1.2 for the third equality.

3. Item 10 of Definition 4.3.10.1.2 for the fifth equality.

4. Item 13 of Definition 4.3.10.1.2 for the sixth equality.

5. Item 3 of Definition 4.3.8.1.2 for the seventh equality.

6. Item 5 of Definition 4.3.9.1.2 for the eighth equality.

This finishes the proof.
Item 16, Interaction With Complements: We have

$$\begin{aligned}[U^{\mathsf{c}}, V]_X &\stackrel{\text{def}}{=} (U^{\mathsf{c}})^{\mathsf{c}} \cup V,\\ &= U \cup V,\end{aligned}$$

where we have used Item 3 of Definition 4.3.11.1.2. We also have

$$\begin{aligned}[U, V^{\mathsf{c}}]_X &\stackrel{\text{def}}{=} U^{\mathsf{c}} \cup V^{\mathsf{c}}\\ &= U \cap V\end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2. Finally, we have

$$\begin{aligned}[U, V]_X^{\mathsf{c}} &= \left((U \setminus V)^{\mathsf{c}}\right)^{\mathsf{c}}\\ &= U \setminus V,\end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2.
Item 17, Interaction With Characteristic Functions: We have

$$\begin{aligned}\chi_{[U,V]_{\mathcal{P}(X)}}(x) &\stackrel{\text{def}}{=} \chi_{U^{\mathsf{c}} \cup V}(x)\\ &= \max(\chi_{U^{\mathsf{c}}}, \chi_V)\\ &= \max(1 - \chi_U \pmod 2, \chi_V),\end{aligned}$$

where we have used:

1. Item 10 of Definition 4.3.8.1.2 for the second equality.

2. Item 4 of Definition 4.3.11.1.2 for the third equality.

This finishes the proof.
Item 18, Interaction With Direct Images: This is a repetition of Item 10 of Definition 4.6.1.1.5 and is proved there.
Item 19, Interaction With Inverse Images: This is a repetition of Item 10 of Definition 4.6.2.1.3 and is proved there.
Item 20, Interaction With Codirect Images: This is a repetition of Item 9 of Definition 4.6.3.1.7 and is proved there. □

4.4.8 Isbell Duality for Sets

Let X be a set.

Definition 4.4.8.1.1. The **Isbell function** of X is the map

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

defined by

$$\mathsf{I}(U) \stackrel{\text{def}}{=} [\![x \mapsto [U, \{x\}]_X]\!]$$

for each $U \in \mathcal{P}(X)$.

Remark 4.4.8.1.2. Recall from Definition 4.4.1.1.2 that we may view the powerset $\mathcal{P}(X)$ of a set X as the decategorification of the category of presheaves $\mathsf{PSh}(\mathcal{C})$ of a category $\mathcal{C}$. Building upon this analogy, we want to mimic the definition of the Isbell Spec functor, which is given on objects by

$$\mathsf{Spec}(\mathcal{F}) \stackrel{\text{def}}{=} \mathsf{Nat}\big(\mathcal{F}, h_{(-)}\big)$$

for each $\mathcal{F} \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$. To this end, we could define

$$\mathsf{I}(U) \stackrel{\text{def}}{=} \big[U, \chi_{(-)}\big]_X,$$

replacing:

- The Yoneda embedding $X \mapsto h_X$ of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ with the characteristic embedding $x \mapsto \chi_x$ of X into $\mathcal{P}(X)$ of Definition 4.5.4.1.1.
- The internal Hom Nat of $\mathsf{PSh}(\mathcal{C})$ with the internal Hom $[-,-]_X$ of $\mathcal{P}(X)$ of Definition 4.4.7.1.1.

However, since $[U, \chi_x]_X$ is a subset of U instead of a truth value, we get a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathsf{Sets}(X, \mathcal{P}(X))$$

instead of a function

$$\mathsf{I}\colon \mathcal{P}(X) \to \mathcal{P}(X).$$

This makes some of the properties involving I a bit more cumbersome to state, although we still have an analogue of Isbell duality in that $\mathsf{I}_! \circ \mathsf{I}$ evaluates to $\mathrm{id}_{\mathcal{P}(X)}$ in the sense of Definition 4.4.8.1.3.

Proposition 4.4.8.1.3. The diagram

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, "\mathsf{I}"] \arrow[dr, "\Delta_{\Delta_{\mathrm{id}_{\mathcal{P}(X)}}}"'] & \mathsf{Sets}(X, \mathcal{P}(X)) \arrow[d, "\mathsf{I}_!"] \\ & \mathsf{Sets}(X, \mathsf{Sets}(X, \mathcal{P}(X))) \end{tikzcd}$$

commutes, i.e. we have

$$\mathsf{I}_!(\mathsf{I}(U)) = [\![x \mapsto [\![y \mapsto U]\!]]\!]$$

for each $U \in \mathcal{P}(X)$.

Proof. We have

$$\begin{aligned} \mathsf{I}_!(\mathsf{I}(U)) &\stackrel{\text{def}}{=} \mathsf{I}_!([\![x \mapsto U^{\mathsf{c}} \cup \{x\}]\!]) \\ &\stackrel{\text{def}}{=} [\![x \mapsto \mathsf{I}(U^{\mathsf{c}} \cup \{x\})]\!] \\ &\stackrel{\text{def}}{=} [\![x \mapsto [\![y \mapsto (U^{\mathsf{c}} \cup \{x\})^{\mathsf{c}} \cup \{x\}]\!]]\!] \\ &= [\![x \mapsto [\![y \mapsto (U \cap (X \setminus \{x\})) \cup \{x\}]\!]]\!] \\ &= [\![x \mapsto [\![y \mapsto (U \setminus \{x\}) \cup \{x\}]\!]]\!] \\ &= [\![x \mapsto [\![y \mapsto U]\!]]\!], \end{aligned}$$

where we have used Item 2 of Definition 4.3.11.1.2 for the fourth equality above. □

4.5 Characteristic Functions

4.5.1 The Characteristic Function of a Subset

Let X be a set and let $U \in \mathcal{P}(X)$.

Definition 4.5.1.1.1. The **characteristic function of** U[23] is the function $\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$[24] defined by

$$\chi_U(x) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

[23] *Further Terminology:* Also called the **indicator function of** U.

[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

Remark 4.5.1.1.2. Under the analogy that $\{\mathrm{t}, \mathrm{f}\}$ should be the (-1)-categorical analogue of Sets, we may view a function

$$f : X \to \{\mathrm{t}, \mathrm{f}\}$$

as a decategorification of presheaves and copresheaves

$$\mathcal{F} : C^{\mathrm{op}} \to \mathsf{Sets},$$
$$F : C \to \mathsf{Sets}.$$

The characteristic functions χ_U of the subsets of X are then the primordial examples of such functions (and, in fact, all of them).

Notation 4.5.1.1.3. We will often employ the bijection $\{\mathrm{t}, \mathrm{f}\} \cong \{0, 1\}$ to make use of the arithmetical operations defined on $\{0, 1\}$ when disucssing characteristic functions.

Examples of this include Items 4 to 11 of Definition 4.5.1.1.4 below.

Proposition 4.5.1.1.4. Let X be a set.

1. *Functionality.* The assignment $U \mapsto \chi_U$ defines a function

$$\chi_{(-)} : \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathrm{t}, \mathrm{f}\}).$$

2. *Bijectivity.* The function $\chi_{(-)}$ from Item 1 is bijective.

3. *Naturality.* The collection

$$\left\{\chi_{(-)} : \mathcal{P}(X) \to \mathsf{Sets}(X, \{\mathrm{t}, \mathrm{f}\})\right\}_{X \in \mathrm{Obj}(\mathsf{Sets})}$$

defines a natural isomorphism between $\mathcal{P}^{-1}$ and $\mathsf{Sets}(-, \{\mathrm{t}, \mathrm{f}\})$. In particular, given a function $f : X \to Y$, the diagram

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\ \mathsf{Sets}(Y, \{\mathrm{t}, \mathrm{f}\}) \arrow[r, "f^*"'] & \mathsf{Sets}(X, \{\mathrm{t}, \mathrm{f}\}) \end{tikzcd}$$

commutes, i.e. we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}$$

for each $V \in \mathcal{P}(Y)$.

4. *Interaction With Unions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

5. *Interaction With Unions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Intersections I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Intersections II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Differences.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $U, V \in \mathcal{P}(X)$.

9. *Interaction With Complements.* We have

$$\chi_{U^c} \equiv 1 - \chi_U \pmod 2$$

for each $U \in \mathcal{P}(X)$.

10. *Interaction With Symmetric Differences.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \pmod 2$$

for each $U, V \in \mathcal{P}(X)$.

11. *Interaction With Internal Homs.* We have

$$\chi_{[U,V]_{\mathcal{P}(X)}} = \max(1 - \chi_U \pmod 2, \chi_V)$$

for each $U, V \in \mathcal{P}(X)$.

Proof. *Item 1, Functionality*: There is nothing to prove.
Item 2, Bijectivity: We proceed in three steps:

1. *The Inverse of $\chi_{(-)}$.* The inverse of $\chi_{(-)}$ is the map

$$\Phi\colon \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}) \overset{\sim}{\dashrightarrow} \mathcal{P}(X),$$

defined by

$$\begin{aligned}\Phi(f) &\overset{\text{def}}{=} U_f\\ &\overset{\text{def}}{=} f^{-1}(\mathsf{true})\\ &\overset{\text{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\}\end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$.

2. *Invertibility I.* We have

$$\begin{aligned}\left[\Phi \circ \chi_{(-)}\right](U) &\overset{\text{def}}{=} \Phi(\chi_U)\\ &\overset{\text{def}}{=} \chi_U^{-1}(\mathsf{true})\\ &\overset{\text{def}}{=} \{x \in X \mid \chi_U(x) = \mathsf{true}\}\\ &\overset{\text{def}}{=} \{x \in X \mid x \in U\}\\ &= U\\ &\overset{\text{def}}{=} \left[\mathrm{id}_{\mathcal{P}(X)}\right](U)\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus, we have

$$\Phi \circ \chi_{(-)} = \mathrm{id}_{\mathcal{P}(X)}.$$

3. *Invertibility II.* We have

$$\begin{aligned}\left[\chi_{(-)} \circ \Phi\right](U) &\overset{\text{def}}{=} \chi_{\Phi(f)}\\ &\overset{\text{def}}{=} \chi_{f^{-1}(\mathsf{true})}\\ &\overset{\text{def}}{=} \left[\!\left[x \mapsto \begin{cases}\mathsf{true} & \text{if } x \in f^{-1}(\mathsf{true})\\ \mathsf{false} & \text{otherwise}\end{cases}\right]\!\right]\\ &= [\![x \mapsto f(x)]\!]\end{aligned}$$

$$\begin{aligned} &= f \\ &\stackrel{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})}\right](f) \end{aligned}$$

for each $f \in \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$. Thus, we have

$$\chi_{(-)} \circ \Phi = \mathrm{id}_{\mathsf{Sets}(X,\{\mathsf{t},\mathsf{f}\})} \,.$$

This finishes the proof.

Item 3, Naturality: We proceed in two steps:

1. *Naturality of $\chi_{(-)}$*. We have

$$\begin{aligned} [\chi_V \circ f](v) &\stackrel{\text{def}}{=} \chi_V(f(v)) \\ &= \begin{cases} \mathsf{true} & \text{if } f(v) \in V, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_{f^{-1}(V)}(v) \end{aligned}$$

   for each $v \in V$.

2. *Naturality of Φ*. Since $\chi_{(-)}$ is natural and a componentwise inverse to Φ, it follows from Item 2 of Definition 11.9.7.1.2 that Φ is also natural in each argument.

This finishes the proof.

Item 4, Interaction With Unions I: This is a repetition of Item 10 of Definition 4.3.8.1.2 and is proved there.

Item 5, Interaction With Unions II: This is a repetition of Item 11 of Definition 4.3.8.1.2 and is proved there.

Item 6, Interaction With Intersections I: This is a repetition of Item 10 of Definition 4.3.9.1.2 and is proved there.

Item 7, Interaction With Intersections II: This is a repetition of Item 11 of Definition 4.3.9.1.2 and is proved there.

Item 8, Interaction With Differences: This is a repetition of Item 16 of Definition 4.3.10.1.2 and is proved there.

Item 9, Interaction With Complements: This is a repetition of Item 4 of Definition 4.3.11.1.2 and is proved there.

Item 10, Interaction With Symmetric Differences: This is a repetition of Item 15 of Definition 4.3.12.1.2 and is proved there.

Item 11, Interaction With Internal Homs: This is a repetition of Item 17 of Definition 4.4.7.1.3 and is proved there. □

Remark 4.5.1.1.5. The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 2 of Definition 4.5.1.1.4, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U : X \to \{\mathsf{true}, \mathsf{false}\}$;
- Takes a function $f : X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the 0-categorical un/straightening isomorphism between indexed and fibred sets

$$\underbrace{\mathsf{FibSets}_X}_{\stackrel{\text{def}}{=}\mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}_X}_{\stackrel{\text{def}}{=}\mathsf{Fun}(X_{\mathsf{disc}},\mathsf{Sets})}$$

of **??**. Here we view:

- Subsets $U \hookrightarrow X$ as being analogous to X-fibred sets $\phi_X : A \to X$.
- Functions $f : X \to \{\mathsf{t}, \mathsf{f}\}$ as being analogous to X-indexed sets $A : X_{\mathsf{disc}} \to \mathsf{Sets}$.

4.5.2 The Characteristic Function of a Point

Let X be a set and let $x \in X$.

Definition 4.5.2.1.1. The **characteristic function of** x is the function[25]

$$\chi_x : X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \stackrel{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

[25] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.

Remark 4.5.2.1.2. Expanding upon Definition 4.5.1.1.2, we may think of the characteristic function

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

of an *element* x of X as a decategorification of the representable presheaf and of the representable copresheaf

$$h_X \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
$$h^X \colon \mathcal{C} \to \mathsf{Sets}$$

associated of an *object* X of a category $\mathcal{C}$.

4.5.3 The Characteristic Relation of a Set

Let X be a set.

Definition 4.5.3.1.1. The **characteristic relation on** X[26] is the relation[27]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

Remark 4.5.3.1.2. Expanding upon Definitions 4.5.1.1.2 and 4.5.2.1.2, we may view the characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

of X as a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2) \colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

[26] *Further Terminology:* Also called the **identity relation on** X.

[27] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\mathrm{id}}$ in the context of relations.

[28] Under the bijection $\mathsf{Sets}(X \times X, \{\mathsf{t}, \mathsf{f}\}) \cong \mathcal{P}(X \times X)$ of Item 2 of Definition 4.5.1.1.4, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.

Proposition 4.5.3.1.3. Let $f: X \to Y$ be a function.

1. *The Inclusion of Characteristic Relations Associated to a Function.* Let $f: A \to B$ be a function. We have an inclusion[29]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f \times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t}, \mathsf{f}\}. & \end{tikzcd}$$

Proof. Item 1, *The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the statement "if $a = b$, then $f(a) = f(b)$", which is true. □

4.5.4 The Characteristic Embedding of a Set

Let X be a set.

Definition 4.5.4.1.1. The **characteristic embedding**[30] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$$

defined by[31]

$$\begin{aligned} \chi_{(-)}(x) &\stackrel{\text{def}}{=} \chi_x \\ &= \{x\} \end{aligned}$$

for each $x \in X$.

Remark 4.5.4.1.2. Expanding upon Definitions 4.5.1.1.2, 4.5.2.1.2 and 4.5.3.1.2, we may view the characteristic embedding

$$\chi_{(-)}: X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ as a decategorification of the Yoneda embedding

$$よ: \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

[29] *Note:* This is the 0-categorical version of Definition 11.5.4.1.1.

[30] The name "characteristic *embedding*" is justified by Definition 4.5.5.1.2, which gives an analogue of fully faithfulness for $\chi_{(-)}$.

[31] Here we are identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ as per Item 2 of Definition 4.5.1.1.4.

Proposition 4.5.4.1.3. Let $f: X \to Y$ be a map of sets.

1. *Interaction With Functions.* We have

$$f_! \circ \chi_X = \chi_Y \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & Y \arrow[d, "\chi_Y"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y). \end{tikzcd}$$

Proof. *Item 1, Interaction With Functions*: Indeed, we have

$$\begin{aligned} [f_! \circ \chi_X](x) &\stackrel{\text{def}}{=} f_!(\chi_X(x)) \\ &\stackrel{\text{def}}{=} f_!(\{x\}) \\ &= \{f(x)\} \\ &\stackrel{\text{def}}{=} \chi_{X'}(f(x)) \\ &\stackrel{\text{def}}{=} [\chi_{X'} \circ f](x), \end{aligned}$$

for each $x \in X$, showing the desired equality. □

4.5.5 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

Proposition 4.5.5.1.1. We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}\big(\chi_{(-)}, \chi_U\big) = \chi_U,$$

where

$$\chi_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } U \subset V, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

Proof. We have

$$\begin{aligned} \chi_{\mathcal{P}(X)}(\chi_x, \chi_U) &\stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } \{x\} \subset U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if } x \in U \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &\stackrel{\text{def}}{=} \chi_U(x). \end{aligned}$$

This finishes the proof. □

Corollary 4.5.5.1.2. The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) \cong \chi_X(x, y)$$

for each $x, y \in X$.

Proof. We have

$$\begin{aligned} \chi_{\mathcal{P}(X)}\big(\chi_x, \chi_y\big) &= \chi_y(x) \\ &\overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x \in \{y\} \\ \text{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \text{true} & \text{if } x = y \\ \text{false} & \text{otherwise} \end{cases} \\ &\overset{\text{def}}{=} \chi_X(x, y). \end{aligned}$$

where we have used Definition 4.5.5.1.1 for the first equality. □

4.6 The Adjoint Triple $f_! \dashv f^{-1} \dashv f_*$

4.6.1 Direct Images

Let $f : X \to Y$ be a function.

Definition 4.6.1.1.1. The **direct image function associated to** f is the function[32]

$$f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[33]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } y = f(x) \end{array} \right\} \\ &= \{f(x) \in Y \mid x \in U\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.1.1.2. Sometimes one finds the notation

$$\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

[32] *Further Notation:* Also written simply $f : \mathcal{P}(X) \to \mathcal{P}(Y)$.
[33] *Further Terminology:* The set $f(U)$ is called the **direct image of** U **by** f.

- We have $y \in \exists_f(U)$.
- There exists some $x \in U$ such that $f(x) = y$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.1.1.3. Notation for direct images between powersets is tricky:

1. Direct images for powersets and presheaves are both adjoint to their corresponding inverse image functors. However, the direct image functor for powersets is a *left* adjoint, while the direct image functor for presheaves is a *right* adjoint:
   (a) *Powersets.* Given a function $f\colon X \longrightarrow Y$, we have an inverse image functor
   $$f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X).$$
   The *left* adjoint of this functor is the usual direct image, defined above in Definition 4.6.1.1.1.
   (b) *Presheaves.* Given a morphism of topological spaces $f\colon X \longrightarrow Y$, we have an inverse image functor
   $$f^{-1}\colon \mathsf{PSh}(Y) \to \mathsf{PSh}(X).$$
   The *right* adjoint of this functor is the direct image functor of presheaves, defined in **??**.
2. The presheaf direct image functor is denoted f_*, but the direct image functor for powersets is denoted $f_!$ (as it's a left adjoint).
3. Adding to the confusion, it's somewhat common for $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ to be denoted f_*.

We chose to write $f_!$ for the direct image to keep the notation aligned with the following similar adjoint situations:

Situation	Adjoint String
Functoriality of Powersets	$(f_! \dashv f^{-1} \dashv f_*)\colon \mathcal{P}(X) \overset{\rightleftarrows}{\rightarrow} \mathcal{P}(Y)$
Functoriality of Presheaf Categories	$(f_! \dashv f^{-1} \dashv f_*)\colon \mathsf{PSh}(X) \overset{\rightleftarrows}{\rightarrow} \mathsf{PSh}(Y)$
Base Change	$(f_! \dashv f^* \dashv f_*)\colon \mathcal{C}_{/X} \overset{\rightleftarrows}{\rightarrow} \mathcal{C}_{/Y}$
Kan Extensions	$(F_! \dashv F^* \dashv F_*)\colon \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \overset{\rightleftarrows}{\rightarrow} \mathsf{Fun}(\mathcal{D}, \mathcal{E})$

Remark 4.6.1.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the direct image function associated to f is equivalently the function

$$f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by

$$\begin{aligned} f_!(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Lan}_f(\chi_U) \\ &= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\ &= \operatorname*{colim}_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)) \\ &= \bigvee_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)), \end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned} [f_!(\chi_U)](y) &= \bigvee_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } x \in X \text{ such} \\ & \text{that } f(x) = y \text{ and } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\ &= \begin{cases} \mathsf{true} & \text{if there exists some } x \in U \\ & \text{such that } f(x) = y, \\ \mathsf{false} & \text{otherwise} \end{cases} \end{aligned}$$

for each $y \in Y$.

Proposition 4.6.1.1.5. Let $f \colon X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

   In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

   ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\big(f_! \dashv f^{-1} \dashv f_*\big)\colon\quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "f_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, bend right, "f_*"'] & \mathcal{P}(Y) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by:

(a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \mathrm{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \mathrm{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*\big(f^{-1}(V)\big), \\ f_!\big(f^{-1}(V)\big) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, f^{-1}(V)\big), \\ \mathrm{Hom}_{\mathcal{P}(X)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

i. The following conditions are equivalent:
   A. We have $f_!(U) \subset V$.
   B. We have $U \subset f^{-1}(V)$.

ii. The following conditions are equivalent:
   A. We have $f^{-1}(U) \subset V$.
   B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_!(U) = \bigcup_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_!)_!"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_!(U) = \bigcap_{V \in f_!(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_!(\mathcal{U}) \stackrel{\text{def}}{=} (f_!)_!(\mathcal{U})$.

5. *Interaction With Binary Unions.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cup"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_!(U \cup V) = f_!(U) \cup f_!(V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_! \times f_!"] \arrow[d, "\cap"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_!(U \cap V) \subset f_!(U) \cap f_!(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

7. *Interaction With Differences.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\setminus"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\setminus"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_!(U) \setminus f_!(V) \subset f_!(U \setminus V)$$

indexed by $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_*^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(U^{\mathsf{c}}) = f_*(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{tikzcd} \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_!"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

with components

$$f_!(U) \triangle f_!(V) \subset f_!(U \triangle V)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_!"] \arrow[d, "{[-_1,-_2]_X}"'] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f_!([U, V]_X) = [f_*(U), f_!(V)]_Y,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f_!\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f_!(U) \cup f_!(V) &= f_!(U \cup V), \\ f_!(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_!\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} f_!(U \cap V) &\subset f_!(U) \cap f_!(V), \\ f_!(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$f^{\otimes}_{!|U,V} \colon f_!(U) \cup f_!(V) \xrightarrow{=} f_!(U \cup V),$$
$$f^{\otimes}_{!|\mathbb{1}} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_!, f^{\otimes}_!, f^{\otimes}_{!|\mathbb{1}}\right) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$f^{\otimes}_{!|U,V} \colon f_!(U \cap V) \hookrightarrow f_!(U) \cap f_!(V),$$
$$f^{\otimes}_{!|\mathbb{1}} \colon f_!(X) \hookrightarrow Y,$$

natural in $U, V \in \mathcal{P}(X)$.

15. *Interaction With Coproducts.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \textstyle\coprod g)_!(U \coprod V) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Interaction With Products.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X \times Y} g)_!(U \boxtimes_{X \times Y} V) = f_!(U) \boxtimes_{X' \times Y'} g_!(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

17. *Relation to Codirect Images.* We have

$$\begin{aligned} f_!(U) &= f_*(U^{\mathsf{c}})^{\mathsf{c}} \\ &\overset{\text{def}}{=} Y \setminus f_*(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and **??** of **??**.

Item 3, Interaction With Unions of Families of Subsets: We have

$$\begin{aligned}\bigcup_{V \in f_!(\mathcal{U})} V &= \bigcup_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcup_{U \in \mathcal{U}} f_!(U).\end{aligned}$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned}\bigcap_{V \in f_!(\mathcal{U})} V &= \bigcap_{V \in \{f_!(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_!(U).\end{aligned}$$

This finishes the proof.
Item 5, Interaction With Binary Unions: See [Pro25t].
Item 6, Interaction With Binary Intersections: See [Pro25r].
Item 7, Interaction With Differences: See [Pro25s].
Item 8, Interaction With Complements: Applying Item 17 to $X \setminus U$, we have

$$\begin{aligned}f_!(U^{\mathsf{c}}) &= f_!(X \setminus U) \\ &= Y \setminus f_*(X \setminus (X \setminus U)) \\ &= Y \setminus f_*(U) \\ &= f_*(U)^{\mathsf{c}}.\end{aligned}$$

This finishes the proof.
Item 9, Interaction With Symmetric Differences: We have

$$\begin{aligned}f_!(U) \triangle f_!(V) &= (f_!(U) \cup f_!(V)) \setminus (f_!(U) \cap f_!(V)) \\ &\subset (f_!(U) \cup f_!(V)) \setminus (f_!(U \cap V)) \\ &= (f_!(U \cup V)) \setminus (f_!(U \cap V)) \\ &\subset f_!((U \cup V) \setminus (U \cap V)) \\ &= f_!(U \triangle V),\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.
2. Item 6 of this proposition together with Item 1 of Definition 4.3.10.1.2 for the first inclusion.

3. Item 5 for the second equality.
4. Item 7 for the second inclusion.
5. Item 2 of Definition 4.3.12.1.2 for the tchird equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**??** of **??**). This finishes the proof.
Item 10, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned} f_!([U,V]_X) &\stackrel{\text{def}}{=} f_!(U^{\mathsf{c}} \cup V) \\ &= f_!(U^{\mathsf{c}}) \cup f_!(V) \\ &= f_*(U)^{\mathsf{c}} \cup f_!(V) \\ &\stackrel{\text{def}}{=} [f_*(U), f_!(V)]_Y, \end{aligned}$$

where we have used:

1. Item 5 for the second equality.
2. Item 17 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**??** of **??**). This finishes the proof.
Item 11, Preservation of Colimits: This follows from Item 2 and **??** of **??**.[34]
Item 12, Oplax Preservation of Limits: The inclusion $f_!(X) \subset Y$ is automatic. See [Pro25r] for the other inclusions.
Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.
Item 14, Symmetric Oplax Monoidality With Respect to Intersections: The inclusions in the statement follow from Item 12. Since $\mathcal{P}(Y)$ is posetal, the commutativity of the diagrams in the definition of a symmetric oplax monoidal functor is automatic (**??** of **??**).
Item 15, Interaction With Coproducts: Omitted.
Item 16, Interaction With Products: Omitted.
Item 17, Relation to Codirect Images: Applying Item 16 of Definition 4.6.3.1.7 to $X \setminus U$, we have

$$\begin{aligned} f_*(X \setminus U) &= B \setminus f_!(X \setminus (X \setminus U)) \\ &= B \setminus f_!(U). \end{aligned}$$

[34] *Reference:* [Pro25t].

Taking complements, we then obtain

$$\begin{aligned} f_!(U) &= B \setminus (B \setminus f_!(U)), \\ &= B \setminus f_*(X \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 4.6.1.1.6. Let $f \colon X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function

$$(-)_{*|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f \colon X \to Y$ and $g \colon Y \to Z$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(Y) \arrow[d, "g_!"] \\ & \mathcal{P}(Z). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: There is nothing to prove.
Item 2, Functionality II: This follows from Item 1 of Definition 4.6.1.1.5.
Item 3, Interaction With Identities: This follows from Definition 4.6.1.1.4 and **??** of **??**.
Item 4, Interaction With Composition: This follows from Definition 4.6.1.1.4 and **??** of **??**. □

4.6.2 Inverse Images

Let $f \colon X \to Y$ be a function.

Definition 4.6.2.1.1. The **inverse image function associated to** f is the function[35]

$$f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[36]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{x \in X \mid \text{we have } f(x) \in V\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 4.6.2.1.2. Identifying $\mathcal{P}(Y)$ with $\mathrm{Sets}(Y, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the inverse image function associated to f is equivalently the function

$$f^* \colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(Y)$, where $\chi_V \circ f$ is the composition

$$X \xrightarrow{f} Y \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

Proposition 4.6.2.1.3. Let $f \colon X \to Y$ be a function.

1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1} \colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset).$$

In particular, for each $U, V \in \mathcal{P}(Y)$, the following condition is satisfied:

($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$(f_! \dashv f^{-1} \dashv f_*)\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left=50, "f_!"] \arrow[r, bend right=50, "f_*"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathcal{P}(Y), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by:

[35] *Further Notation:* Also written $f^* \colon \mathcal{P}(Y) \to \mathcal{P}(X)$.

[36] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

(a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \mathrm{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \mathrm{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*\bigl(f^{-1}(V)\bigr), \\ f_!\bigl(f^{-1}(V)\bigr) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\bigl(U, f^{-1}(V)\bigr), \\ \mathrm{Hom}_{\mathcal{P}(X)}\bigl(f^{-1}(U), V\bigr) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

i. The following conditions are equivalent:
   A. We have $f_!(U) \subset V$.
   B. We have $U \subset f^{-1}(V)$.

ii. The following conditions are equivalent:
   A. We have $f^{-1}(U) \subset V$.
   B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\bigcup"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V) = \bigcup_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\mathrm{def}}{=} \bigl(f^{-1}\bigr)^{-1}(\mathcal{V})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(\mathcal{P}(Y)) \arrow[r, "(f^{-1})^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(\mathcal{P}(X)) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{V \in \mathcal{V}} f^{-1}(V) = \bigcap_{U \in f^{-1}(\mathcal{U})} U$$

for each $\mathcal{V} \in \mathcal{P}(Y)$, where $f^{-1}(\mathcal{V}) \overset{\text{def}}{=} \left(f^{-1}\right)^{-1}(\mathcal{V})$.

5. *Interaction With Binary Unions.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cup"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cup"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

6. *Interaction With Binary Intersections.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[r, "f^{-1} \times f^{-1}"] \arrow[d, "\cap"'] & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[d, "\cap"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$$

for each $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathrm{op},-1} \times f^{-1}"] \arrow[d, "\setminus"'] & \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[d, "\setminus"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$$

for each $U, V \in \mathcal{P}(X)$.

8. *Interaction With Complements.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathrm{op}} \arrow[r, "f^{-1,\mathrm{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(X)^{\mathrm{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(U^{\mathsf{c}}) = f^{-1}(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

9. *Interaction With Symmetric Differences.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathrm{op}} \times \mathcal{P}(Y) \arrow[r, "f^{\mathrm{op},-1} \times f^{-1}"] \arrow[d, "\triangle"'] & \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[d, "\triangle"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

i.e. we have

$$f^{-1}(U) \triangle f^{-1}(V) = f^{-1}(U \triangle V)$$

for each $U, V \in \mathcal{P}(Y)$.

10. *Interaction With Internal Homs of Powersets.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U, V]_Y) = \left[f^{-1}(U), f^{-1}(V)\right]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cup f^{-1}(V) &= f^{-1}(U \cup V), \\ f^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

12. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(Y) &= X, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

13. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f_{\mathbb{1}}^{-1,\otimes}\right) \colon (\mathcal{P}(Y), \cup, \emptyset) \to (\mathcal{P}(X), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} &: f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f^{-1,\otimes}_{\mathbb{1}} &: \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

14. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) : (\mathcal{P}(Y), \cap, Y) \to (\mathcal{P}(X), \cap, X),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} &: f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V), \\ f^{-1,\otimes}_{\mathbb{1}} &: X \xrightarrow{=} f^{-1}(Y), \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

15. *Interaction With Coproducts.* Let $f : X \to X'$ and $g : Y \to Y'$ be maps of sets. We have

$$(f \textstyle\coprod g)^{-1}(U' \textstyle\coprod V') = f^{-1}(U') \textstyle\coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

16. *Interaction With Products.* Let $f : X \to X'$ and $g : Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X' \times Y'} g)^{-1}(U' \boxtimes_{X' \times Y'} V') = f^{-1}(U') \boxtimes_{X \times Y} g^{-1}(V')$$

for each $U' \in \mathcal{P}(X')$ and each $V' \in \mathcal{P}(Y')$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and **??** of **??**.
Item 3, *Interaction With Unions of Families of Subsets*: We have

$$\begin{aligned} \bigcup_{U \in f^{-1}(\mathcal{V})} U &= \bigcup_{U \in \{f^{-1}(V) \in \mathcal{P}(X) \mid V \in \mathcal{V}\}} U \\ &= \bigcup_{V \in \mathcal{V}} f^{-1}(V). \end{aligned}$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned}\bigcap_{U\in f^{-1}(\mathcal{V})} U &= \bigcap_{U\in\{f^{-1}(V)\in\mathcal{P}(X) \mid V\in\mathcal{V}\}} U\\ &= \bigcap_{V\in\mathcal{V}} f^{-1}(V).\end{aligned}$$

This finishes the proof.
Item 5, Interaction With Binary Unions: See [Pro25ad].
Item 6, Interaction With Binary Intersections: See [Pro25ab].
Item 7, Interaction With Differences: See [Pro25ac].
Item 8, Interaction With Complements: See [Pro25k].
Item 9, Interaction With Symmetric Differences: We have

$$\begin{aligned}f^{-1}(U \mathbin{\triangle} V) &= f^{-1}((U\cup V)\setminus(U\cap V))\\ &= f^{-1}(U\cup V)\setminus f^{-1}(U\cap V)\\ &= f^{-1}(U)\cup f^{-1}(V)\setminus f^{-1}(U\cap V)\\ &= f^{-1}(U)\cup f^{-1}(V)\setminus f^{-1}(U)\cap f^{-1}(V)\\ &= f^{-1}(U)\mathbin{\triangle} f^{-1}(V),\end{aligned}$$

where we have used:

1. Item 2 of Definition 4.3.12.1.2 for the first equality.
2. Item 7 for the second equality.
3. Item 5 for the third equality.
4. Item 6 for the fourth equality.
5. Item 2 of Definition 4.3.12.1.2 for the fifth equality.

This finishes the proof.
Item 10, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned}f^{-1}([U,V]_Y) &\stackrel{\text{def}}{=} f^{-1}(U^{\textsf{c}}\cup V)\\ &= f^{-1}(U^{\textsf{c}})\cup f^{-1}(V)\\ &= f^{-1}(U)^{\textsf{c}}\cup f^{-1}(V)\\ &\stackrel{\text{def}}{=} \left[f^{-1}(U), f^{-1}(V)\right]_X,\end{aligned}$$

where we have used:

1. Item 8 for the second equality.

2. Item 5 for the third equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**??** of **??**). This finishes the proof.

Item 11, Preservation of Colimits: This follows from Item 2 and **??** of **??**.[37]

Item 12, Preservation of Limits: This follows from Item 2 and **??** of **??**.[38]

Item 13, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 11.

Item 14, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 12.

Item 15, Interaction With Coproducts: Omitted.

Item 16, Interaction With Products: Omitted. □

Proposition 4.6.2.1.4. Let $f : X \to Y$ be a function.

1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(Y), \mathcal{P}(X)).$$

2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function
$$(-)^{-1}_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(Y), \subset), (\mathcal{P}(X), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathrm{id}_X^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable functions $f : X \to Y$ and $g : Y \to Z$, we have
$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(Z) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(Y) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: There is nothing to prove.

Item 2, Functionality II: This follows from Item 1 of Definition 4.6.2.1.3.

Item 3, Interaction With Identities: This follows from Definition 4.6.2.1.2 and Item 5 of Definition 11.1.4.1.2.

Item 4, Interaction With Composition: This follows from Definition 4.6.2.1.2 and Item 2 of Definition 11.1.4.1.2. □

[37] *Reference:* [Pro25ad].

[38] *Reference:* [Pro25ab].

4.6.3 Codirect Images

Let $f\colon X \to Y$ be a function.

Definition 4.6.3.1.1. The **codirect image function associated to** f is the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[39,40]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{for each } x \in X \text{, if we have} \\ f(x) = y \text{, then } x \in U \end{array} \right\} \\ &= \left\{ y \in Y \;\middle|\; \text{we have } f^{-1}(y) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Notation 4.6.3.1.2. Sometimes one finds the notation

$$\forall_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

for $f_!$. This notation comes from the fact that the following statements are equivalent, where $y \in Y$ and $U \in \mathcal{P}(X)$:

- We have $y \in \forall_f(U)$.
- For each $x \in X$, if $y = f(x)$, then $x \in U$.

We will not make use of this notation elsewhere in Clowder.

Warning 4.6.3.1.3. See Definition 4.6.1.1.3.

Remark 4.6.3.1.4. Identifying $\mathcal{P}(X)$ with $\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$ via Item 2 of Definition 4.5.1.1.4, we see that the codirect image function associated to f is equivalently the function

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

---

[39] *Further Terminology:* The set $f_*(U)$ is called the **codirect image of** U **by** f.

[40] We also have

$$\begin{aligned} f_*(U) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \\ &\overset{\text{def}}{=} Y \setminus f_!(X \setminus U); \end{aligned}$$

see Item 16 of Definition 4.6.3.1.7.

defined by

$$\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\
&= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} X \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\
&= \lim_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)) \\
&= \bigwedge_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)).
\end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned}
[f_*(\chi_U)](y) &= \bigwedge_{\substack{x \in X \\ f(x) = y}} (\chi_U(x)) \\
&= \begin{cases} \mathsf{true} & \text{if, for each } x \in X \text{ such that} \\ & f(x) = y, \text{ we have } x \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if } f^{-1}(y) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $y \in Y$.

Definition 4.6.3.1.5. Let U be a subset of X.[41,42]

---

[41]Note that we have

$$f_*(U) = f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U),$$

as

$$\begin{aligned}
f_*(U) &= f_*(U) \cap Y \\
&= f_*(U) \cap (\mathrm{Im}(f) \cup (Y \setminus \mathrm{Im}(f))) \\
&= (f_*(U) \cap \mathrm{Im}(f)) \cup (f_*(U) \cap (Y \setminus \mathrm{Im}(f))) \\
&\overset{\text{def}}{=} f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U).
\end{aligned}$$

[42]In terms of the meet computation of $f_*(U)$ of Definition 4.6.3.1.4, namely

$$f_*(\chi_U) = \bigwedge_{\substack{x \in X \\ f(x) = -_1}} (\chi_U(x)),$$

we see that $f_{*,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{*,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

1. The **image part of the codirect image $f_*(U)$ of U** is the set $f_{*,\mathrm{im}}(U)$ defined by

$$
\begin{aligned}
f_{*,\mathrm{im}}(U) &\overset{\text{def}}{=} f_*(U) \cap \mathrm{Im}(f) \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) \neq \emptyset. \end{array} \right\}.
\end{aligned}
$$

2. The **complement part of the codirect image $f_*(U)$ of U** is the set $f_{*,\mathrm{cp}}(U)$ defined by

$$
\begin{aligned}
f_{*,\mathrm{cp}}(U) &\overset{\text{def}}{=} f_*(U) \cap (Y \setminus \mathrm{Im}(f)) \\
&= Y \setminus \mathrm{Im}(f) \\
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(y) \subset U \\ \text{and } f^{-1}(y) = \emptyset. \end{array} \right\} \\
&= \left\{ y \in Y \;\middle|\; f^{-1}(y) = \emptyset \right\}.
\end{aligned}
$$

Example 4.6.3.1.6. Here are some examples of codirect images.

1. *Multiplication by Two.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$
\begin{aligned}
f_{*,\mathrm{im}}(U) &= f_!(U) \\
f_{*,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\}
\end{aligned}
$$

for any $U \subset \mathbb{N}$. In particular, we have

$$f_*(\{\text{even natural numbers}\}) = \mathbb{N}.$$

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \left\{-\sqrt{x}, \sqrt{x}\right\}$, we have e.g.:

$$
\begin{aligned}
f_{*,\mathrm{im}}([0,1]) &= \{0\}, \\
f_{*,\mathrm{im}}([-1,1]) &= [0,1], \\
f_{*,\mathrm{im}}([1,2]) &= \emptyset, \\
f_{*,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4].
\end{aligned}
$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{*,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{*,\mathrm{im}}([-1,1]\times[-1,1]) &= [0,1], \\ f_{*,\mathrm{im}}(([-1,1]\times[-1,1]) \setminus [-1,1]\times\{0\}) &= \emptyset. \end{aligned}$$

Proposition 4.6.3.1.7. Let $f\colon X \to Y$ be a function.

1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_*\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset).$$

In particular, for each $U, V \in \mathcal{P}(X)$, the following condition is satisfied:

(★) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_! \dashv f^{-1} \dashv f_*\right)\colon\quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left=50, "f_!"] \arrow[r, bend right=50, "f_*"'] & \mathcal{P}(Y), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by:

(a) Units and counits of the form

$$\begin{aligned} \mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow f^{-1} \circ f_!, & \mathrm{id}_{\mathcal{P}(Y)} &\hookrightarrow f_* \circ f^{-1}, \\ f_! \circ f^{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)}, & f^{-1} \circ f_* &\hookrightarrow \mathrm{id}_{\mathcal{P}(X)}, \end{aligned}$$

having components of the form

$$\begin{aligned} U &\subset f^{-1}(f_!(U)), & V &\subset f_*\big(f^{-1}(V)\big),\\ f_!\big(f^{-1}(V)\big) &\subset V, & f^{-1}(f_*(U)) &\subset U \end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

(b) Bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(Y)}(f_!(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\big(U, f^{-1}(V)\big),\\ \mathrm{Hom}_{\mathcal{P}(X)}\big(f^{-1}(U), V\big) &\cong \mathrm{Hom}_{\mathcal{P}(X)}(U, f_*(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$ and (respectively) $V \in \mathcal{P}(X)$ and $U \in \mathcal{P}(Y)$. In particular:

i. The following conditions are equivalent:
   A. We have $f_!(U) \subset V$.
   B. We have $U \subset f^{-1}(V)$.

ii. The following conditions are equivalent:
   A. We have $f^{-1}(U) \subset V$.
   B. We have $U \subset f_*(V)$.

3. *Interaction With Unions of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcup"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcup"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcup_{U \in \mathcal{U}} f_*(U) = \bigcup_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \overset{\text{def}}{=} (f_*)_*(\mathcal{U})$.

4. *Interaction With Intersections of Families of Subsets.* The diagram

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}(X)) \arrow[r, "(f_*)_*"] \arrow[d, "\bigcap"'] & \mathcal{P}(\mathcal{P}(Y)) \arrow[d, "\bigcap"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$\bigcap_{U \in \mathcal{U}} f_*(U) = \bigcap_{V \in f_*(\mathcal{U})} V$$

for each $\mathcal{U} \in \mathcal{P}(X)$, where $f_*(\mathcal{U}) \stackrel{\text{def}}{=} (f_*)_*(\mathcal{U})$.

5. *Interaction With Binary Unions.* Let $f \colon X \to Y$ be a function. We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cup"'] \arrow[dr, phantom, "\subset"] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cup"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U) \cup f_*(V) \subset f_*(U \cup V)$$

indexed by $U, V \in \mathcal{P}(X)$.

6. *Interaction With Binary Intersections.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "f_* \times f_*"] \arrow[d, "\cap"'] & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[d, "\cap"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U) \cap f_*(V) = f_*(U \cap V)$$

for each $U, V \in \mathcal{P}(X)$.

7. *Interaction With Complements.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \arrow[r, "f_!^{\mathsf{op}}"] \arrow[d, "(-)^{\mathsf{c}}"'] & \mathcal{P}(Y)^{\mathsf{op}} \arrow[d, "(-)^{\mathsf{c}}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

commutes, i.e. we have

$$f_*(U^{\mathsf{c}}) = f_!(U)^{\mathsf{c}}$$

for each $U \in \mathcal{P}(X)$.

8. *Interaction With Symmetric Differences.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_*^{\mathsf{op}} \times f_*"] \arrow[d, "\triangle"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "\triangle"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$f_*(U \triangle V) \subset f_*(U) \triangle f_*(V)$$

indexed by $U, V \in \mathcal{P}(X)$.

9. *Interaction With Internal Homs of Powersets.* We have a natural transformation

$$\begin{tikzcd}
\mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[r, "f_!^{\mathsf{op}} \times f_*"] \arrow[d, "{[-_1,-_2]_X}"'] \arrow[dr, phantom, "\supset"] & \mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[d, "{[-_1,-_2]_Y}"] \\
\mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y)
\end{tikzcd}$$

with components

$$[f_!(U), f_*(V)]_Y \subset f_*([U, V]_X)$$

indexed by $U, V \in \mathcal{P}(X)$.

10. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} f_*(U_i) \subset f_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V),$$
$$\emptyset \hookrightarrow f_*(\emptyset),$$

natural in $U, V \in \mathcal{P}(X)$.

11. *Preservation of Limits.* We have an equality of sets

$$f_*\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U \cap V) &= f_*(U) \cap f^{-1}(V), \\ f_*(X) &= Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

12. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{aligned} f_{*|U,V}^{\otimes} &: f_*(U) \cup f_*(V) \hookrightarrow f_*(U \cup V), \\ f_{*|\mathbb{1}}^{\otimes} &: \emptyset \hookrightarrow f_*(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

13. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$\begin{aligned} f_{*|U,V}^{\otimes} &: f_*(U \cap V) \xrightarrow{=} f_*(U) \cap f_*(V), \\ f_{*|\mathbb{1}}^{\otimes} &: f_*(X) \xrightarrow{=} Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

14. *Interaction With Coproducts.* Let $f : X \to X'$ and $g : Y \to Y'$ be maps of sets. We have

$$(f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

15. *Interaction With Products.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be maps of sets. We have

$$(f \boxtimes_{X\times Y} g)_*(U \boxtimes_{X\times Y} V) = f_*(U) \boxtimes_{X'\times Y'} g_*(V)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

16. *Relation to Direct Images.* We have

$$\begin{aligned} f_*(U) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \\ &= Y \setminus f_!(X \setminus U) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

17. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &= f_!(U), \\ f_{*,\mathrm{cp}}(U) &= Y \setminus \mathrm{Im}(f), \end{aligned}$$

and so

$$\begin{aligned} f_*(U) &= f_{*,\mathrm{im}}(U) \cup f_{*,\mathrm{cp}}(U) \\ &= f_!(U) \cup (Y \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

18. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{*,\mathrm{im}}(U) &\subset f_!(U), \\ f_{*,\mathrm{cp}}(U) &= \emptyset, \end{aligned}$$

and so

$$f_*(U) \subset f_!(U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Triple Adjointness*: This follows from Definition 4.6.1.1.4, Definition 4.6.2.1.2, Definition 4.6.3.1.4, and **??** of **??**.
Item 3, *Interaction With Unions of Families of Subsets*: We have

$$\bigcup_{V \in f_*(\mathcal{U})} V = \bigcup_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V$$

$$= \bigcup_{U \in \mathcal{U}} f_*(U).$$

This finishes the proof.
Item 4, Interaction With Intersections of Families of Subsets: We have

$$\begin{aligned} \bigcap_{V \in f_*(\mathcal{U})} V &= \bigcap_{V \in \{f_*(U) \in \mathcal{P}(X) \mid U \in \mathcal{U}\}} V \\ &= \bigcap_{U \in \mathcal{U}} f_*(U). \end{aligned}$$

This finishes the proof.
Item 5, Interaction With Binary Unions: We have

$$\begin{aligned} f_*(U) \cup f_*(V) &= f_!(U^{\mathsf{c}})^{\mathsf{c}} \cup f_!(V^{\mathsf{c}})^{\mathsf{c}} \\ &= (f_!(U^{\mathsf{c}}) \cap f_!(V^{\mathsf{c}}))^{\mathsf{c}} \\ &\subset (f_!(U^{\mathsf{c}} \cap V^{\mathsf{c}}))^{\mathsf{c}} \\ &= f_!\big((U \cup V)^{\mathsf{c}}\big)^{\mathsf{c}} \\ &= f_*(U \cup V), \end{aligned}$$

where:

1. We have used Item 16 for the first equality.
2. We have used Item 2 of Definition 4.3.11.1.2 for the second equality.
3. We have used Item 6 of Definition 4.6.1.1.5 for the third equality.
4. We have used Item 2 of Definition 4.3.11.1.2 for the fourth equality.
5. We have used Item 16 for the last equality.

This finishes the proof.
Item 6, Interaction With Binary Intersections: This follows from Item 11.
Item 7, Interaction With Complements: Omitted.
Item 8, Interaction With Symmetric Differences: Omitted.
Item 9, Interaction With Internal Homs of Powersets: We have

$$\begin{aligned} \big[f_!(U), f^!(V)\big]_X &\stackrel{\text{def}}{=} f_!(U)^{\mathsf{c}} \cup f_*(V) \\ &= f_*(U^{\mathsf{c}}) \cup f_*(V) \\ &\subset f_*(U^{\mathsf{c}} \cup V) \\ &\stackrel{\text{def}}{=} f_*([U, V]_X), \end{aligned}$$

where we have used:

1. Item 7 of Definition 4.6.3.1.7 for the second equality.
2. Item 5 of Definition 4.6.3.1.7 for the inclusion.

Since $\mathcal{P}(X)$ is posetal, naturality is automatic (**??** of **??**). This finishes the proof.
Item 10, Lax Preservation of Colimits: Omitted.
Item 11, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 12, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 10.
Item 13, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 11.
Item 14, Interaction With Coproducts: Omitted.
Item 15, Interaction With Products: Omitted.
Item 16, Relation to Direct Images: We claim that $f_*(U) = Y \setminus f_!(X \setminus U)$.

- *The First Implication.* We claim that
$$f_*(U) \subset Y \setminus f_!(X \setminus U).$$
Let $y \in f_*(U)$. We need to show that $y \notin f_!(X \setminus U)$, i.e. that there is no $x \in X \setminus U$ such that $f(x) = y$.
This is indeed the case, as otherwise we would have $x \in f^{-1}(y)$ and $x \notin U$, contradicting $f^{-1}(y) \subset U$ (which holds since $y \in f_*(U)$).
Thus $y \in Y \setminus f_!(X \setminus U)$.
- *The Second Implication.* We claim that
$$Y \setminus f_!(X \setminus U) \subset f_*(U).$$
Let $y \in Y \setminus f_!(X \setminus U)$. We need to show that $y \in f_*(U)$, i.e. that $f^{-1}(y) \subset U$.
Since $y \notin f_!(X \setminus U)$, there exists no $x \in X \setminus U$ such that $y = f(x)$, and hence $f^{-1}(y) \subset U$.
Thus $y \in f_*(U)$.

This finishes the proof of Item 16.
Item 17, Interaction With Injections: Omitted.
Item 18, Interaction With Surjections: Omitted. □

Proposition 4.6.3.1.8. Let $f : X \to B$ be a function.

1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{!|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$
2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function
$$(-)_{!|X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$
3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)} .$$
4. *Interaction With Composition.* For each pair of composable functions $f \colon X \to Y$ and $g \colon Y \to Z$, we have
$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(Y) \arrow[d, "g_*"] \\ & \mathcal{P}(Z). \end{tikzcd}$$

Proof. Item 1, *Functionality I*: There is nothing to prove.
Item 2, *Functionality II*: This follows from Item 1 of Definition 4.6.3.1.7.
Item 3, *Interaction With Identities*: This follows from Definition 4.6.3.1.4 and **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Definition 4.6.3.1.4 and **??** of **??**. □

4.6.4 A Six-Functor Formalism for Sets

Remark 4.6.4.1.1. The assignment $X \mapsto \mathcal{P}(X)$ together with the functors f_*, f^{-1}, and $f_!$ of Item 1 of Definition 4.6.1.1.5, Item 1 of Definition 4.6.2.1.3, and Item 1 of Definition 4.6.3.1.7, and the functors
$$\begin{aligned} -_1 \cap -_2 &\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X), \\ [-_1, -_2]_X &\colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$
of Item 1 of Definition 4.3.9.1.2 and Item 1 of Definition 4.4.7.1.3 satisfy several properties reminiscent of a six functor formalism in the sense of **??**.

We collect these properties in Definition 4.6.4.1.2 below.[43]

[43] See also [nLa25b].

Proposition 4.6.4.1.2. Let X be a set.

1. *The Beck–Chevalley Condition.* Let

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

be a pullback diagram in Sets. We have

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\mathrm{pr}_1^{-1}"] \arrow[d, "f_!"'] & \mathcal{P}(X \times_Z Y) \arrow[d, "(\mathrm{pr}_2)_!"] \\
\mathcal{P}(Z) \arrow[r, "g^{-1}"'] & \mathcal{P}(Y),
\end{tikzcd}
\qquad
\begin{aligned}
g^{-1} \circ f_! &= (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}, \\
f^{-1} \circ g_! &= (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1},
\end{aligned}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \arrow[r, "\mathrm{pr}_2^{-1}"] \arrow[d, "g_!"'] & \mathcal{P}(X \times_Z Y) \arrow[d, "(\mathrm{pr}_1)_!"] \\
\mathcal{P}(Z) \arrow[r, "f^{-1}"'] & \mathcal{P}(Y).
\end{tikzcd}$$

2. *The Projection Formula I.* The diagram

$$\begin{tikzcd}
 & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[drr, "\cap"] & & \\
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}"] \arrow[dr, "f_! \times \mathrm{id}_{\mathcal{P}(Y)}"'] & & & & \mathcal{P}(X) \arrow[dl, "f_!"] \\
 & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[rr, "\cap"'] & & \mathcal{P}(Y), &
\end{tikzcd}$$

commutes, i.e. we have

$$f_!\big(U \cap f^{-1}(V)\big) = f_!(U) \cap V$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

3. *The Projection Formula II.* We have a natural transformation

$$\begin{tikzcd}
 & & \mathcal{P}(X) \times \mathcal{P}(X) \arrow[drr, "\cap"] & & \\
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[urr, "\mathrm{id}_{\mathcal{P}(X)} \times f^{-1}"] \arrow[dr, "f_* \times \mathrm{id}_{\mathcal{P}(Y)}"'] \arrow[rrrr, phantom, "\cup", shift right=4ex] & & & & \mathcal{P}(X) \arrow[dl, "f_*"] \\
 & \mathcal{P}(Y) \times \mathcal{P}(Y) \arrow[rr, "\cap"'] & & \mathcal{P}(Y), &
\end{tikzcd}$$

with components

$$f_*(U) \cap V \subset f_*\big(U \cap f^{-1}(V)\big)$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$.

4. *Strong Closed Monoidality.* The diagram

$$\begin{tikzcd}
\mathcal{P}(Y)^{\mathsf{op}} \times \mathcal{P}(Y) \arrow[r, "f^{-1,\mathsf{op}} \times f^{-1}"] \arrow[d, "{[-_1,-_2]_Y}"'] & \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \arrow[d, "{[-_1,-_2]_X}"] \\
\mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X)
\end{tikzcd}$$

commutes, i.e. we have an equality of sets

$$f^{-1}([U,V]_Y) = \big[f^{-1}(U), f^{-1}(V)\big]_X,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *The External Tensor Product.* We have an external tensor product

$$-_1 \boxtimes_{X\times Y} -_2 \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$$

given by

$$\begin{aligned} U \boxtimes_{X\times Y} V &\overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\ &= \{(u,v) \in X \times Y \mid u \in U \text{ and } v \in V\}. \end{aligned}$$

This is the same map as the one in Item 5 of Definition 4.4.1.1.4. Moreover, the following conditions are satisfied:

   (a) *Interaction With Direct Images.* Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_! \times g_!"] \arrow[d, "\boxtimes_{X\times Y}"'] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, "\boxtimes_{X'\times Y'}"] \\
\mathcal{P}(X \times Y) \arrow[r, "f_! \times g_!"'] & \mathcal{P}(X' \times Y')
\end{tikzcd}$$

   commutes, i.e. we have

$$[f_! \times g_!](U \boxtimes_{X\times Y} V) = f_!(U) \boxtimes_{X'\times Y'} g_!(V)$$

   for each $(U,V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(b) *Interaction With Inverse Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd}
\mathcal{P}(X') \times \mathcal{P}(Y') \arrow[r, "f^{-1}\times g^{-1}"] \arrow[d, "\boxtimes_{X'\times Y'}"'] & \mathcal{P}(X) \times \mathcal{P}(Y) \arrow[d, "\boxtimes_{X\times Y}"] \\
\mathcal{P}(X' \times Y') \arrow[r, "f^{-1}\times g^{-1}"'] & \mathcal{P}(X \times Y)
\end{tikzcd}$$

commutes, i.e. we have

$$\left[f^{-1} \times g^{-1}\right](U \boxtimes_{X'\times Y'} V) = f^{-1}(U) \boxtimes_{X\times Y} g^{-1}(V)$$

for each $(U, V) \in \mathcal{P}(X') \times \mathcal{P}(Y')$.

(c) *Interaction With Codirect Images.* Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be functions. The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_*\times g_*"] \arrow[d, "\boxtimes_{X\times Y}"'] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, "\boxtimes_{X'\times Y'}"] \\
\mathcal{P}(X \times Y) \arrow[r, "f_*\times g_*"'] & \mathcal{P}(X' \times Y')
\end{tikzcd}$$

commutes, i.e. we have

$$[f_* \times g_*](U \boxtimes_{X\times Y} V) = f_*(U) \boxtimes_{X'\times Y'} g_*(V)$$

for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$.

(d) *Interaction With Diagonals.* The diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(X) \arrow[r, "\boxtimes_{X\times X}"] \arrow[dr, "\cap"'] & \mathcal{P}(X \times X) \arrow[d, "\Delta_X^{-1}"] \\
 & \mathcal{P}(X),
\end{tikzcd}$$

i.e. we have

$$U \cap V = \Delta_X^{-1}(U \boxtimes_{X\times X} V)$$

for each $U, V \in \mathcal{P}(X)$.

6. *The Dualisation Functor.* We have a functor

$$D_X \colon \mathcal{P}(X)^{\mathrm{op}} \to \mathcal{P}(X)$$

given by

$$\begin{aligned} D_X(U) &\overset{\text{def}}{=} [U, \emptyset]_X \\ &\overset{\text{def}}{=} U^{\mathsf{c}} \end{aligned}$$

for each $U \in \mathcal{P}(X)$, as in Item 5 of Definition 4.4.7.1.3, satisfying the following conditions:

(a) *Duality.* We have

$$D_X(D_X(U)) = U, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "D_X"] \arrow[dr, "\mathrm{id}_{\mathcal{P}(X)}"'] & \mathcal{P}(X) \arrow[d, "D_X"] \\ & \mathcal{P}(X). \end{tikzcd}$$

(b) *Duality.* The diagram

$$\begin{tikzcd} & \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X)^{\mathrm{op}} \arrow[r, "\cap^{\mathrm{op}}"] & \mathcal{P}(X)^{\mathrm{op}} \arrow[dr, "D_X"] & \\ \mathcal{P}(X)^{\mathrm{op}} \times \mathcal{P}(X) \arrow[ur, "\mathrm{id}_{\mathcal{P}(X)^{\mathrm{op}}} \times D_X"] \arrow[rrr, "{[-_1, -_2]_X}"'] & & & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$\underbrace{D_X(U \cap D_X(V))}_{\overset{\text{def}}{=} [U \cap [V, \emptyset]_X, \emptyset]_X} = [U, V]_X$$

for each $U, V \in \mathcal{P}(X)$.

(c) *Interaction With Direct Images.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathrm{op}} \arrow[r, "f_*^{\mathrm{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathrm{op}} \arrow[d, "D_Y"] \\ \mathcal{P}(X) \arrow[r, "f_!"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_!(D_X(U)) = D_Y(f_*(U))$$

for each $U \in \mathcal{P}(X)$.

(d) *Interaction With Inverse Images.* The diagram

$$\begin{tikzcd} \mathcal{P}(Y)^{\mathrm{op}} \arrow[r, "f^{-1,\mathrm{op}}"] \arrow[d, "D_Y"'] & \mathcal{P}(X)^{\mathrm{op}} \arrow[d, "D_X"] \\ \mathcal{P}(Y) \arrow[r, "f^{-1}"'] & \mathcal{P}(X) \end{tikzcd}$$

commutes, i.e. we have

$$f^{-1}(D_Y(U)) = D_X\left(f^{-1}(U)\right)$$

for each $U \in \mathcal{P}(X)$.

(e) *Interaction With Codirect Images.* The diagram

$$\begin{tikzcd} \mathcal{P}(X)^{\mathrm{op}} \arrow[r, "f_!^{\mathrm{op}}"] \arrow[d, "D_X"'] & \mathcal{P}(Y)^{\mathrm{op}} \arrow[d, "D_Y"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(Y) \end{tikzcd}$$

commutes, i.e. we have

$$f_*(D_X(U)) = D_Y(f_!(U))$$

for each $U \in \mathcal{P}(X)$.

Proof. Item 1, The Beck–Chevalley Condition: We have

$$\begin{aligned}
\left[g^{-1} \circ f_!\right](U) &\overset{\text{def}}{=} g^{-1}(f_!(U)) \\
&\overset{\text{def}}{=} \{y \in Y \mid g(y) \in f_!(U)\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some } x \in U \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid x \in U\} \end{array} \right\} \\
&= \left\{ y \in Y \,\middle|\, \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } y = y \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid x \in U\} \\ \text{such that } \mathrm{pr}_2(x,y) = y \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\{(x,y) \in X \times_Z Y \mid x \in U\}) \\
&= (\mathrm{pr}_2)_!\left(\left\{(x,y) \in X \times_Z Y \;\middle|\; \mathrm{pr}_1(x,y) \in U\right\}\right) \\
&\overset{\text{def}}{=} (\mathrm{pr}_2)_!(\mathrm{pr}_1^{-1}(U)) \\
&\overset{\text{def}}{=} \left[(\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}\right](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(X)$. Therefore, we have

$$
g^{-1} \circ f_! = (\mathrm{pr}_2)_! \circ \mathrm{pr}_1^{-1}.
$$

For the second equality, we have

$$
\begin{aligned}
\left[f^{-1} \circ g_!\right](U) &\overset{\text{def}}{=} f^{-1}(g_!(U)) \\
&\overset{\text{def}}{=} \{x \in X \mid f(x) \in g_!(V)\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some } y \in V \\ \text{such that } f(x) = g(y) \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid y \in V\} \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } x = x \end{array} \right\} \\
&= \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there exists some} \\ (x,y) \in \{(x,y) \in X \times_Z Y \mid y \in V\} \\ \text{such that } \mathrm{pr}_1(x,y) = x \end{array} \right\} \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\{(x,y) \in X \times_Z Y \mid y \in V\}) \\
&= (\mathrm{pr}_1)_!\left(\left\{(x,y) \in X \times_Z Y \;\middle|\; \mathrm{pr}_2(x,y) \in V\right\}\right) \\
&\overset{\text{def}}{=} (\mathrm{pr}_1)_!(\mathrm{pr}_2^{-1}(V)) \\
&\overset{\text{def}}{=} \left[(\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}\right](V)
\end{aligned}
$$

for each $V \in \mathcal{P}(Y)$. Therefore, we have

$$
f^{-1} \circ g_! = (\mathrm{pr}_1)_! \circ \mathrm{pr}_2^{-1}.
$$

This finishes the proof.

Item 2, The Projection Formula I: We claim that

$$f_!(U) \cap V \subset f_!\big(U \cap f^{-1}(V)\big).$$

Indeed, we have

$$\begin{aligned} f_!(U) \cap V &\subset f_!(U) \cap f_!\big(f^{-1}(V)\big) \\ &= f_!\big(U \cap f^{-1}(V)\big), \end{aligned}$$

where we have used:

1. Item 2 of Definition 4.6.1.1.5 for the inclusion.
2. Item 6 of Definition 4.6.1.1.5 for the equality.

Conversely, we claim that

$$f_!\big(U \cap f^{-1}(V)\big) \subset f_!(U) \cap V.$$

Indeed:

1. Let $y \in f_!\big(U \cap f^{-1}(V)\big)$.
2. Since $y \in f_!\big(U \cap f^{-1}(V)\big)$, there exists some $x \in U \cap f^{-1}(V)$ such that $f(x) = y$.
3. Since $x \in U \cap f^{-1}(V)$, we have $x \in U$, and thus $f(x) \in f_!(U)$.
4. Since $x \in U \cap f^{-1}(V)$, we have $x \in f^{-1}(V)$, and thus $f(x) \in V$.
5. Since $f(x) \in f_!(U)$ and $f(x) \in V$, we have $f(x) \in f_!(U) \cap V$.
6. But $y = f(x)$, so $y \in f_!(U) \cap V$.
7. Thus $f_!\big(U \cap f^{-1}(V)\big) \subset f_!(U) \cap V$.

This finishes the proof.
Item 3, The Projection Formula II: We have

$$\begin{aligned} f_*(U) \cap V &\subset f_*(U) \cap f_*\big(f^{-1}(V)\big) \\ &= f_*\big(U \cap f^{-1}(V)\big), \end{aligned}$$

where we have used:

1. Item 2 of Definition 4.6.3.1.7 for the inclusion.

2. Item 6 of Definition 4.6.3.1.7 for the equality.

Since $\mathcal{P}(Y)$ is posetal, naturality is automatic (**??** of **??**).
Item 4, Strong Closed Monoidality: This is a repetition of Item 19 of Definition 4.4.7.1.3 and is proved there.
Item 5, The External Tensor Product: We have

$$\begin{aligned}
U \boxtimes_{X\times Y} V &\overset{\text{def}}{=} \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V) \\
&\overset{\text{def}}{=} \left\{(x,y) \in X \times Y \,\middle|\, \mathrm{pr}_1(x,y) \in U\right\} \\
&\quad \cup \left\{(x,y) \in X \times Y \,\middle|\, \mathrm{pr}_2(x,y) \in V\right\} \\
&= \{(x,y) \in X \times Y \mid x \in U\} \\
&\quad \cup \{(x,y) \in X \times Y \mid y \in V\} \\
&= \{(x,y) \in X \times Y \mid x \in U \text{ and } y \in V\} \\
&\overset{\text{def}}{=} U \times V.
\end{aligned}$$

Next, we claim that Items 5a to 5d are indeed true:

1. *Proof of Item 5a:* This is a repetition of Item 16 of Definition 4.6.1.1.5 and is proved there.

2. *Proof of Item 5b:* This is a repetition of Item 16 of Definition 4.6.2.1.3 and is proved there.

3. *Proof of Item 5c:* This is a repetition of Item 15 of Definition 4.6.3.1.7 and is proved there.

4. *Proof of Item 5d:* We have

$$\begin{aligned}
\Delta_X^{-1}(U \boxtimes_{X\times X} V) &\overset{\text{def}}{=} \{x \in X \mid (x,x) \in U \boxtimes_{X\times X} V\} \\
&= \{x \in X \mid (x,x) \in \{(u,v) \in X \times X \mid u \in U \text{ and } v \in V\}\} \\
&= U \cap V.
\end{aligned}$$

This finishes the proof.
Item 6, The Dualisation Functor: This is a repetition of Items 5 and 6 of Definition 4.4.7.1.3 and is proved there. □

Appendices

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2. A Guide to the Literature

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4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

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8. Relations
9. Constructions With Relations
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Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

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Chapter 5

Monoidal Structures on the Category of Sets

This chapter contains some material on monoidal structures on Sets.

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5.1 The Monoidal Category of Sets and Products

5.1.1 Products of Sets

See Section 4.1.3.

5.1.2 The Internal Hom of Sets

See Section 4.3.5.

5.1.3 The Monoidal Unit

Definition 5.1.3.1.1. The **monoidal unit of the product of sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathrm{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{1}_{\mathsf{Sets}} \overset{\text{def}}{=} \mathrm{pt},$$

where pt is the terminal set of Definition 4.1.1.1.1.

5.1.4 The Associator

Definition 5.1.4.1.1. The **associator of the product of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}},$$

as in the diagram

$$
\begin{tikzcd}
 & & \mathsf{Sets} \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[drr, "\mathrm{id} \times \times"] & & \\
(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets} \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}", "\sim"'] \arrow[dr, "\times \times \mathrm{id}"'] & & & & \mathsf{Sets} \times \mathsf{Sets} \arrow[dl, "\times"] \\
 & \mathsf{Sets} \times \mathsf{Sets} \arrow[rr, "\times"'] \arrow[uur, phantom, "\Uparrow \alpha^{\mathsf{Sets}}"] & & \mathsf{Sets}, &
\end{tikzcd}
$$

whose component

$$\alpha^{\mathsf{Sets}}_{X,Y,Z} \colon (X \times Y) \times Z \xrightarrow{\sim} X \times (Y \times Z)$$

at (X, Y, Z) is given by

$$\alpha^{\mathsf{Sets}}_{X,Y,Z}((x, y), z) \stackrel{\text{def}}{=} (x, (y, z))$$

for each $((x, y), z) \in (X \times Y) \times Z$.

Proof. *Invertibility*: The inverse of $\alpha^{\mathsf{Sets}}_{X,Y,Z}$ is the morphism

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z} \colon X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$$

defined by

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z}(x, (y, z)) \stackrel{\text{def}}{=} ((x, y), z)$$

for each $(x, (y, z)) \in X \times (Y \times Z)$. Indeed:

- *Invertibility I*. We have

$$
\begin{aligned}
\left[\alpha^{\mathsf{Sets},-1}_{X,Y,Z} \circ \alpha^{\mathsf{Sets}}_{X,Y,Z}\right]((x, y), z) &\stackrel{\text{def}}{=} \alpha^{\mathsf{Sets},-1}_{X,Y,Z}\left(\alpha^{\mathsf{Sets}}_{X,Y,Z}((x, y), z)\right) \\
&\stackrel{\text{def}}{=} \alpha^{\mathsf{Sets},-1}_{X,Y,Z}(x, (y, z)) \\
&\stackrel{\text{def}}{=} ((x, y), z) \\
&\stackrel{\text{def}}{=} \left[\mathrm{id}_{(X \times Y) \times Z}\right]((x, y), z)
\end{aligned}
$$

  for each $((x, y), z) \in (X \times Y) \times Z$, and therefore we have

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z} \circ \alpha^{\mathsf{Sets}}_{X,Y,Z} = \mathrm{id}_{(X \times Y) \times Z}\,.$$

- *Invertibility II*. We have

$$\begin{aligned}\left[\alpha^{\mathsf{Sets}}_{X,Y,Z} \circ \alpha^{\mathsf{Sets},-1}_{X,Y,Z}\right](x,(y,z)) &\overset{\text{def}}{=} \alpha^{\mathsf{Sets}}_{X,Y,Z}\left(\alpha^{\mathsf{Sets},-1}_{X,Y,Z}(x,(y,z))\right)\\ &\overset{\text{def}}{=} \alpha^{\mathsf{Sets}}_{X,Y,Z}((x,y),z)\\ &\overset{\text{def}}{=} (x,(y,z))\\ &\overset{\text{def}}{=} \left[\mathrm{id}_{(X\times Y)\times Z}\right](x,(y,z))\end{aligned}$$

for each $(x,(y,z)) \in X \times (Y \times Z)$, and therefore we have

$$\alpha^{\mathsf{Sets},-1}_{X,Y,Z} \circ \alpha^{\mathsf{Sets}}_{X,Y,Z} = \mathrm{id}_{X\times(Y\times Z)}.$$

Therefore $\alpha^{\mathsf{Sets}}_{X,Y,Z}$ is indeed an isomorphism.
Naturality: We need to show that, given functions

$$\begin{aligned}f &: X \to X',\\ g &: Y \to Y',\\ h &: Z \to Z'\end{aligned}$$

the diagram

$$\begin{tikzcd}(X\times Y)\times Z \arrow[r, "(f\times g)\times h"] \arrow[d, "\alpha^{\mathsf{Sets}}_{X,Y,Z}"'] & (X'\times Y')\times Z' \arrow[d, "\alpha^{\mathsf{Sets}}_{X',Y',Z'}"] \\ X\times(Y\times Z) \arrow[r, "f\times(g\times h)"'] & X'\times(Y'\times Z')\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}((x,y),z) \arrow[d, mapsto] & \\ (x,(y,z)) \arrow[r, mapsto] & (f(x),(g(y),h(z)))\end{tikzcd} \qquad \begin{tikzcd}((x,y),z) \arrow[r, mapsto] & ((f(x),g(y)),h(z)) \arrow[d, mapsto] \\ & (f(x),(g(y),h(z)))\end{tikzcd}$$

and hence indeed commutes, showing α^{Sets} to be a natural transformation.
Being a Natural Isomorphism: Since α^{Sets} is natural and $\alpha^{\mathsf{Sets},-1}$ is a componentwise inverse to α^{Sets}, it follows from Item 2 of Definition 11.9.7.1.2 that $\alpha^{\mathsf{Sets},-1}$ is also natural. Thus α^{Sets} is a natural isomorphism. □

5.1.5 The Left Unitor

Definition 5.1.5.1.1. The **left unitor of the product of sets** is the natural isomorphism

$$
\lambda^{\mathsf{Sets}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}
\qquad
\begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets} \arrow[r, "\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Sets}}"] \\
& \mathsf{Sets},
\end{tikzcd}
$$

whose component

$$\lambda^{\mathsf{Sets}}_X \colon \mathrm{pt} \times X \overset{\sim}{\dashrightarrow} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\lambda^{\mathsf{Sets}}_X(\star, x) \overset{\mathrm{def}}{=} x$$

for each $(\star, x) \in \mathrm{pt} \times X$.

Proof. *Invertibility*: The inverse of $\lambda^{\mathsf{Sets}}_X$ is the morphism

$$\lambda^{\mathsf{Sets},-1}_X \colon X \overset{\sim}{\dashrightarrow} \mathrm{pt} \times X$$

defined by

$$\lambda^{\mathsf{Sets},-1}_X(x) \overset{\mathrm{def}}{=} (\star, x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$
\begin{aligned}
\left[\lambda^{\mathsf{Sets},-1}_X \circ \lambda^{\mathsf{Sets}}_X\right](\mathrm{pt}, x) &= \lambda^{\mathsf{Sets},-1}_X\left(\lambda^{\mathsf{Sets}}_X(\mathrm{pt}, x)\right) \\
&= \lambda^{\mathsf{Sets},-1}_X(x) \\
&= (\mathrm{pt}, x) \\
&= \left[\mathrm{id}_{\mathrm{pt}\times X}\right](\mathrm{pt}, x)
\end{aligned}
$$

  for each $(\mathrm{pt}, x) \in \mathrm{pt} \times X$, and therefore we have

$$\lambda^{\mathsf{Sets},-1}_X \circ \lambda^{\mathsf{Sets}}_X = \mathrm{id}_{\mathrm{pt}\times X}.$$

- *Invertibility II.* We have
$$
\begin{aligned}
\left[\lambda^{\mathsf{Sets}}_X \circ \lambda^{\mathsf{Sets},-1}_X\right](x) &= \lambda^{\mathsf{Sets}}_X\left(\lambda^{\mathsf{Sets},-1}_X(x)\right)\\
&= \lambda^{\mathsf{Sets},-1}_X(\mathrm{pt}, x)\\
&= x\\
&= [\mathrm{id}_X](x)
\end{aligned}
$$
for each $x \in X$, and therefore we have
$$\lambda^{\mathsf{Sets}}_X \circ \lambda^{\mathsf{Sets},-1}_X = \mathrm{id}_X .$$

Therefore $\lambda^{\mathsf{Sets}}_X$ is indeed an isomorphism.
Naturality: We need to show that, given a function $f\colon X \to Y$, the diagram
$$
\begin{tikzcd}
\mathrm{pt} \times X \arrow[r, "\mathrm{id}_{\mathrm{pt}} \times f"] \arrow[d, "\lambda^{\mathsf{Sets}}_X"'] & \mathrm{pt} \times Y \arrow[d, "\lambda^{\mathsf{Sets}}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$
commutes. Indeed, this diagram acts on elements as
$$
\begin{tikzcd}
(\star, x) \arrow[d, mapsto] & & & (\star, x) \arrow[r, mapsto] & (\star, f(x)) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x) & & & f(x)
\end{tikzcd}
$$
and hence indeed commutes. Therefore λ^{Sets} is a natural transformation.
Being a Natural Isomorphism: Since λ^{Sets} is natural and $\lambda^{\mathsf{Sets},-1}$ is a componentwise inverse to λ^{Sets}, it follows from Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus λ^{Sets} is a natural isomorphism. □

5.1.6 The Right Unitor

Definition 5.1.6.1.1. The **right unitor of the product of sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets} \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}}"] \\ & \mathsf{Sets}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}}_X \colon X \times \mathrm{pt} \overset{\sim}{\dashrightarrow} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\rho^{\mathsf{Sets}}_X(x, \star) \overset{\mathrm{def}}{=} x$$

for each $(x, \star) \in X \times \mathrm{pt}$.

Proof. *Invertibility*: The inverse of ρ^{Sets}_X is the morphism

$$\rho^{\mathsf{Sets},-1}_X \colon X \overset{\sim}{\dashrightarrow} X \times \mathrm{pt}$$

defined by

$$\rho^{\mathsf{Sets},-1}_X(x) \overset{\mathrm{def}}{=} (x, \star)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned} \left[\rho^{\mathsf{Sets},-1}_X \circ \rho^{\mathsf{Sets}}_X\right](x, \star) &= \rho^{\mathsf{Sets},-1}_X\left(\rho^{\mathsf{Sets}}_X(x, \star)\right) \\ &= \rho^{\mathsf{Sets},-1}_X(x) \\ &= (x, \star) \\ &= \left[\mathrm{id}_{X\times\mathrm{pt}}\right](x, \star) \end{aligned}$$

  for each $(x, \star) \in X \times \mathrm{pt}$, and therefore we have

$$\rho^{\mathsf{Sets},-1}_X \circ \rho^{\mathsf{Sets}}_X = \mathrm{id}_{X\times\mathrm{pt}}.$$

- *Invertibility II.* We have
$$\begin{aligned}\left[\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1}\right](x) &= \rho_X^{\mathsf{Sets}}\left(\rho_X^{\mathsf{Sets},-1}(x)\right)\\ &= \rho_X^{\mathsf{Sets},-1}(x,\star)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$
for each $x \in X$, and therefore we have
$$\rho_X^{\mathsf{Sets}} \circ \rho_X^{\mathsf{Sets},-1} = \mathrm{id}_X\,.$$

Therefore ρ_X^{Sets} is indeed an isomorphism.
Naturality: We need to show that, given a function $f\colon X \to Y$, the diagram
$$\begin{tikzcd} X \times \mathrm{pt} \arrow[r, "f\times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\rho_X^{\mathsf{Sets}}"'] & Y \times \mathrm{pt} \arrow[d, "\rho_Y^{\mathsf{Sets}}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
commutes. Indeed, this diagram acts on elements as
$$\begin{tikzcd} (x,\star) \arrow[d, mapsto] & \\ x \arrow[r, mapsto] & f(x) \end{tikzcd} \qquad \begin{tikzcd} (x,\star) \arrow[r, mapsto] & (f(x),\star) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$
and hence indeed commutes. Therefore ρ^{Sets} is a natural transformation.
Being a Natural Isomorphism: Since ρ^{Sets} is natural and $\rho^{\mathsf{Sets},-1}$ is a componentwise inverse to ρ^{Sets}, it follows from Item 2 of Definition 11.9.7.1.2 that $\rho^{\mathsf{Sets},-1}$ is also natural. Thus ρ^{Sets} is a natural isomorphism. □

5.1.7 The Symmetry

Definition 5.1.7.1.1. The **symmetry of the product of sets** is the natural isomorphism
$$\sigma^{\mathsf{Sets}}\colon \times \overset{\sim}{\Longrightarrow} \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets}\times\mathsf{Sets} \arrow[r, "\times"] \arrow[dr, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}"'] \arrow[dr, phantom, shift left=3, "\Downarrow\sigma^{\mathsf{Sets}}"] & \mathsf{Sets}, \\ & \mathsf{Sets}\times\mathsf{Sets} \arrow[u, "\times"'] \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}}_{X,Y} \colon X \times Y \overset{\sim}{\dashrightarrow} Y \times X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ is defined by

$$\sigma^{\mathsf{Sets}}_{X,Y}(x, y) \overset{\text{def}}{=} (y, x)$$

for each $(x, y) \in X \times Y$.

Proof. *Invertibility*: The inverse of $\sigma^{\mathsf{Sets}}_{X,Y}$ is the morphism

$$\sigma^{\mathsf{Sets},-1}_{X,Y} \colon Y \times X \overset{\sim}{\dashrightarrow} X \times Y$$

defined by

$$\sigma^{\mathsf{Sets},-1}_{X,Y}(y, x) \overset{\text{def}}{=} (x, y)$$

for each $(y, x) \in Y \times X$. Indeed:

- *Invertibility I*. We have

$$\begin{aligned}
\left[\sigma^{\mathsf{Sets},-1}_{X,Y} \circ \sigma^{\mathsf{Sets}}_{X,Y}\right](x, y) &\overset{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}\left(\sigma^{\mathsf{Sets}}_{X,Y}(x, y)\right) \\
&\overset{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}(y, x) \\
&\overset{\text{def}}{=} (x, y) \\
&\overset{\text{def}}{=} [\mathrm{id}_{X \times Y}](x, y)
\end{aligned}$$

  for each $(x, y) \in X \times Y$, and therefore we have

$$\sigma^{\mathsf{Sets},-1}_{X,Y} \circ \sigma^{\mathsf{Sets}}_{X,Y} = \mathrm{id}_{X \times Y}.$$

- *Invertibility II*. We have

$$\begin{aligned}
\left[\sigma^{\mathsf{Sets}}_{X,Y} \circ \sigma^{\mathsf{Sets},-1}_{X,Y}\right](y, x) &\overset{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}\left(\sigma^{\mathsf{Sets}}_{X,Y}(y, x)\right) \\
&\overset{\text{def}}{=} \sigma^{\mathsf{Sets},-1}_{X,Y}(x, y) \\
&\overset{\text{def}}{=} (y, x) \\
&\overset{\text{def}}{=} [\mathrm{id}_{Y \times X}](y, x)
\end{aligned}$$

  for each $(y, x) \in Y \times X$, and therefore we have

$$\sigma^{\mathsf{Sets}}_{X,Y} \circ \sigma^{\mathsf{Sets},-1}_{X,Y} = \mathrm{id}_{Y \times X}.$$

Therefore $\sigma^{\mathsf{Sets}}_{X,Y}$ is indeed an isomorphism.
Naturality: We need to show that, given functions

$$f\colon X \to A,$$
$$g\colon Y \to B$$

the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f\times g"] \arrow[d, "\sigma^{\mathsf{Sets}}_{X,Y}"'] & A \times B \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"] \\ Y \times X \arrow[r, "g\times f"'] & B \times A \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x,y) \arrow[d, mapsto] \\ (y,x) \arrow[r, mapsto] & (g(y), f(x)) \end{tikzcd} \qquad \begin{tikzcd} (x,y) \arrow[r, mapsto] & (f(x), g(y)) \arrow[d, mapsto] \\ & (g(y), f(x)) \end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets} to be a natural transformation.
Being a Natural Isomorphism: Since σ^{Sets} is natural and $\sigma^{\mathsf{Sets},-1}$ is a componentwise inverse to σ^{Sets}, it follows from Item 2 of Definition 11.9.7.1.2 that $\sigma^{\mathsf{Sets},-1}$ is also natural. Thus σ^{Sets} is a natural isomorphism. □

5.1.8 The Diagonal

Definition 5.1.8.1.1. The **diagonal of the product of sets** is the natural transformation

$$\Delta\colon \mathrm{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets} \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \arrow[d, phantom, "\Downarrow\Delta"] & \mathsf{Sets} \\ & \mathsf{Sets} \times \mathsf{Sets}, \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\Delta_X\colon X \to X \times X$$

at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\Delta_X(x) \overset{\mathrm{def}}{=} (x,x)$$

for each $x \in X$.

Proof. We need to show that, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\
X \times X \arrow[r, "f\times f"'] & Y \times Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[d, mapsto] & \\
(x, x) \arrow[r, mapsto] & (f(x), f(x))
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, mapsto] & f(x) \arrow[d, mapsto] \\
& (f(x), f(x))
\end{tikzcd}$$

and hence indeed commutes, showing Δ to be natural. □

Proposition 5.1.8.1.2. Let X be a set.

1. *Monoidality.* The diagonal map

$$\Delta\colon \mathrm{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}},$$

is a monoidal natural transformation:

(a) *Compatibility With Strong Monoidality Constraints.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, the diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "\Delta_X \times \Delta_Y"] \arrow[dr, "\Delta_{X\times Y}"'] & (X \times X) \times (Y \times Y) \arrow[d, dashed, "\wr"] \\
& (X \times Y) \times (X \times Y)
\end{tikzcd}$$

commutes.

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{tikzcd}
\mathrm{pt} \arrow[r, "\Delta_{\mathrm{pt}}"] \arrow[dr, equal] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\lambda^{\mathsf{Sets}}_{\mathrm{pt}}"] \\
& \mathrm{pt}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pt} \arrow[r, "\Delta_{\mathrm{pt}}"] \arrow[dr, equal] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\rho^{\mathsf{Sets}}_{\mathrm{pt}}"] \\
& \mathrm{pt}
\end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned}\Delta_{\mathrm{pt}} &= \lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}\\ &= \rho^{\mathsf{Sets},-1}_{\mathrm{pt}},\end{aligned}$$

where we recall that the equalities

$$\begin{aligned}\lambda^{\mathsf{Sets}}_{\mathrm{pt}} &= \rho^{\mathsf{Sets}}_{\mathrm{pt}},\\ \lambda^{\mathsf{Sets},-1}_{\mathrm{pt}} &= \rho^{\mathsf{Sets},-1}_{\mathrm{pt}}\end{aligned}$$

are always true in any monoidal category by **??** of **??**.

2. *The Diagonal of the Unit.* The component

$$\Delta_{\mathrm{pt}}\colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \mathrm{pt}\times\mathrm{pt}$$

of Δ at pt is an isomorphism.

Proof. *Item 1, Monoidality*: We claim that Δ is indeed monoidal:

1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{tikzcd} X\times Y \arrow[r, "\Delta_X\times\Delta_Y"] \arrow[dr, "\Delta_{X\times Y}"'] & (X\times X)\times(Y\times Y) \arrow[d, dashed, "\wr"] \\ & (X\times Y)\times(X\times Y) \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x,y) \arrow[r, mapsto] & ((x,x),(y,y)) \arrow[d, mapsto] & (x,y) \arrow[dr, mapsto] & \\ & ((x,y),(x,y)) & & ((x,y),(x,y)) \end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 5.1.5.1.1, the inverse of the left unitor of Sets with respect to to the product at $X \in \mathrm{Obj}(\mathsf{Sets})$ is given by

$$\lambda^{\mathsf{Sets},-1}_{X}(x) \overset{\mathrm{def}}{=} (\star, x)$$

for each $x \in X$, so when $X = \mathrm{pt}$, we have

$$\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}(\star) \stackrel{\text{def}}{=} (\star, \star),$$

and also

$$\Delta^{\mathsf{Sets}}_{\mathrm{pt}}(\star) \stackrel{\text{def}}{=} (\star, \star),$$

so we have $\Delta_{\mathrm{pt}} = \lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}$.

This finishes the proof.
Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets with respect to $\times$, proved in the proof of Definition 5.1.5.1.1 for the left unitor or the proof of Definition 5.1.6.1.1 for the right unitor. □

5.1.9 The Monoidal Category of Sets and Products

Proposition 5.1.9.1.1. The category Sets admits a closed symmetric monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Monoidal Product.* The product functor

  $$\times \colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Item 1 of Definition 4.1.3.1.3.
- *The Internal Hom.* The internal Hom functor

  $$\mathsf{Sets} \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Item 1 of Definition 4.3.5.1.2.
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}} \colon \mathrm{pt} \to \mathsf{Sets}$$

  of Definition 5.1.3.1.1.
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \xrightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.1.4.1.1.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.6.1.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}} \colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

- *The Diagonals.* The monoidal natural transformation

$$\Delta \colon \mathrm{id}_{\mathsf{Sets}} \Longrightarrow \times \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.8.1.1.

Proof. The Pentagon Identity: Let W, X, Y and Z be sets. We have to show that the diagram

$$\begin{tikzcd}[column sep=small]
 & (W \times (X \times Y)) \times Z \arrow[dr, "\alpha^{\mathsf{Sets}}_{W,X\times Y,Z}"] & \\
((W \times X) \times Y) \times Z \arrow[ur, "\alpha^{\mathsf{Sets}}_{W,X,Y} \times \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}}_{W\times X,Y,Z}"'] & & W \times ((X \times Y) \times Z) \arrow[d, "\mathrm{id}_W \times \alpha^{\mathsf{Sets}}_{X,Y,Z}"] \\
(W \times X) \times (Y \times Z) \arrow[rr, "\alpha^{\mathsf{Sets}}_{W,X,Y\times Z}"'] & & W \times (X \times (Y \times Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(((w,x),y),z) \arrow[d, mapsto] & & & ((w,(x,y)),z) \arrow[dr, mapsto] & \\
 & & (((w,x),y),z) \arrow[ur, mapsto] & & (w,((x,y),z)) \arrow[d, mapsto] \\
((w,x),(y,z)) \arrow[r, mapsto] & (w,(x,(y,z))) & & & (w,(x,(y,z))),
\end{tikzcd}
$$

and thus the pentagon identity is satisfied.

The Triangle Identity: Let X and Y be sets. We have to show that the diagram

$$
\begin{tikzcd}
(X \times \mathrm{pt}) \times Y \arrow[rr, "\alpha^{\mathsf{Sets}}_{X,\mathrm{pt},Y}"] \arrow[dr, "\rho^{\mathsf{Sets}}_{X}\times\mathrm{id}_Y"'] & & X \times (\mathrm{pt} \times Y) \arrow[dl, "\mathrm{id}_X \times\lambda^{\mathsf{Sets}}_{Y}"] \\
 & X \times Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
((x,\star),y) \arrow[dr, mapsto] & & & ((x,\star),y) \arrow[rr, mapsto] & & (x,(\star,y)) \arrow[dl, mapsto] \\
 & (x,y) & & & (x,y) &
\end{tikzcd}
$$

and thus the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$
\begin{tikzcd}
 & (X \times Y) \times Z \arrow[dl, "\alpha^{\mathsf{Sets}}_{X,Y,Z}"'] \arrow[dr, "\sigma^{\mathsf{Sets}}_{X,Y}\times\mathrm{id}_Z"] & \\
X \times (Y \times Z) \arrow[d, "\sigma^{\mathsf{Sets}}_{X,Y\times Z}"'] & & (Y \times X) \times Z \arrow[d, "\alpha^{\mathsf{Sets}}_{Y,X,Z}"] \\
(Y \times Z) \times X \arrow[dr, "\alpha^{\mathsf{Sets}}_{Y,Z,X}"'] & & Y \times (X \times Z) \arrow[dl, "\mathrm{id}_Y \times\sigma^{\mathsf{Sets}}_{X,Z}"] \\
 & Y \times (Z \times X) &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
& ((x,y),z) \arrow[dl, mapsto] & ((x,y),z) \arrow[dr, mapsto] & \\
(x,(y,z)) \arrow[d, mapsto] & & & ((y,x),z) \arrow[d, mapsto] \\
((y,z),x) \arrow[dr, mapsto] & & & (y,(x,z)) \arrow[dl, mapsto] \\
& (y,(z,x)) & (y,(z,x)) &
\end{tikzcd}$$

and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
& X \times (Y \times Z) \arrow[dl, "\left(\alpha^{\mathsf{Sets}}_{X,Y,Z}\right)^{-1}"'] \arrow[dr, "\mathrm{id}_X \times \sigma^{\mathsf{Sets}}_{Y,Z}"] & \\
(X \times Y) \times Z \arrow[d, "\sigma^{\mathsf{Sets}}_{X \times Y, Z}"'] & & X \times (Z \times Y) \arrow[d, "\left(\alpha^{\mathsf{Sets}}_{X,Z,Y}\right)^{-1}"] \\
Z \times (X \times Y) \arrow[dr, "\left(\alpha^{\mathsf{Sets}}_{Z,X,Y}\right)^{-1}"'] & & (X \times Z) \times Y \arrow[dl, "\sigma^{\mathsf{Sets}}_{X,Z} \times \mathrm{id}_Y"] \\
& (Z \times X) \times Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
& (x,(y,z)) \arrow[dl, mapsto] & (x,(y,z)) \arrow[dr, mapsto] & \\
((x,y),z) \arrow[d, mapsto] & & & (x,(z,y)) \arrow[d, mapsto] \\
(z,(x,y)) \arrow[dr, mapsto] & & & ((x,z),y) \arrow[dl, mapsto] \\
& ((z,x),y) & ((z,x),y) &
\end{tikzcd}$$

and thus the right hexagon identity is satisfied.
Monoidal Closedness: This follows from Item 2 of Definition 4.3.5.1.2
Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 5.1.8.1.2. ☐

5.1.10 The Universal Property of $(\mathsf{Sets}, \times, \mathrm{pt})$

Theorem 5.1.10.1.1. The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathsf{Sets}}\colon \mathsf{Sets}\times\mathsf{Sets}\to\mathsf{Sets}$$

of Sets admits an internal Hom $[-_1,-_2]_{\mathsf{Sets}}$.

2. *The Unit Object Is pt.* We have $\mathbb{1}_{\mathsf{Sets}}\cong\mathrm{pt}$.

More precisely, the full subcategory of the category $\mathcal{M}^{\mathrm{cld}}_{\mathbb{E}_\infty}(\mathsf{Sets})$ of **??** spanned by the closed symmetric monoidal categories $\left(\mathsf{Sets},\otimes_{\mathsf{Sets}},[-_1,-_2]_{\mathsf{Sets}},\mathbb{1}_{\mathsf{Sets}},\lambda^{\mathsf{Sets}},\rho^{\mathsf{Sets}},\sigma^{\mathsf{Sets}}\right)$ satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

Proof. Unwinding the Statement: Let $(\mathsf{Sets},\otimes_{\mathsf{Sets}},[-_1,-_2]_{\mathsf{Sets}},\mathbb{1}_{\mathsf{Sets}},\lambda',\rho',\sigma')$ be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$\mathrm{id}_{\mathsf{Sets}}\colon \mathsf{Sets}\to\mathsf{Sets}$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned}
\mathrm{id}^{\otimes}_{\mathsf{Sets}}\colon\ & A\otimes_{\mathsf{Sets}}B \overset{\sim}{\dashrightarrow} A\times B,\\
\mathrm{id}^{\mathrm{Hom}}_{\mathsf{Sets}}\colon\ & [A,B]_{\mathsf{Sets}} \overset{\sim}{\dashrightarrow} \mathsf{Sets}(A,B),\\
\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}\colon\ & \mathbb{1}_{\mathsf{Sets}} \overset{\sim}{\dashrightarrow} \mathrm{pt},
\end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from $(\mathsf{Sets},\otimes_{\mathsf{Sets}},[-_1,-_2]_{\mathsf{Sets}},\mathbb{1}_{\mathsf{Sets}},\lambda',\rho',\sigma')$ to the closed symmetric monoidal category $(\mathsf{Sets},\times,\mathsf{Sets}(-_1,-_2),\mathbb{1}_{\mathsf{Sets}},\lambda^{\mathsf{Sets}},\rho^{\mathsf{Sets}},\sigma^{\mathsf{Sets}})$ of Definition 5.1.9.1.1.
Constructing an Isomorphism $[-_1,-_2]_{\mathsf{Sets}}\cong\mathsf{Sets}(-_1,-_2)$: By **??**, we have a natural isomorphism

$$\mathsf{Sets}(\mathrm{pt},[-_1,-_2]_{\mathsf{Sets}})\cong\mathsf{Sets}(-_1,-_2).$$

By Item 3 of Definition 4.3.5.1.2, we also have a natural isomorphism

$$\mathsf{Sets}(\mathrm{pt}, [-_1, -_2]_{\mathsf{Sets}}) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}(-_1, -_2) \cong [-_1, -_2]_{\mathsf{Sets}}.$$

Given $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we will write

$$\mathrm{id}^{\mathrm{Hom}}_{A,B} \colon \mathsf{Sets}(A, B) \overset{\sim}{\dashrightarrow} [A, B]_{\mathsf{Sets}}$$

for the component of this isomorphism at (A, B).
Constructing an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Since $\otimes_{\mathsf{Sets}}$ is adjoint in each variable to $[-_1, -_2]_{\mathsf{Sets}}$ by assumption and $\times$ is adjoint in each variable to $\mathsf{Sets}(-_1, -_2)$ by Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (**??**) gives us natural isomorphisms

$$\begin{aligned} A \otimes_{\mathsf{Sets}} - &\cong A \times -, \\ - \otimes_{\mathsf{Sets}} B &\cong B \times -. \end{aligned}$$

By **??**, we then have $\otimes_{\mathsf{Sets}} \cong \times$. We will write

$$\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \colon A \otimes_{\mathsf{Sets}} B \overset{\sim}{\dashrightarrow} A \times B$$

for the component of this isomorphism at (A, B).
Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Alternatively, we may construct a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ as follows:

1. Let $A \in \mathrm{Obj}(\mathsf{Sets})$.

2. Since $\otimes_{\mathsf{Sets}}$ is part of a closed monoidal structure, it preserves colimits in each variable by **??**.

3. Since $A \cong \coprod_{a \in A} \mathrm{pt}$ and $\otimes_{\mathsf{Sets}}$ preserves colimits in each variable, we have

$$\begin{aligned} A \otimes_{\mathsf{Sets}} B &\cong \left(\coprod_{a \in A} \mathrm{pt} \right) \otimes_{\mathsf{Sets}} B \\ &\cong \coprod_{a \in A} (\mathrm{pt} \otimes_{\mathsf{Sets}} B) \\ &\cong \coprod_{a \in A} B \end{aligned}$$

$$\cong A \times B,$$

naturally in $B \in \mathrm{Obj}(\mathsf{Sets})$, where we have used that pt is the monoidal unit for $\otimes_{\mathsf{Sets}}$. Thus $A \otimes_{\mathsf{Sets}} - \cong A \times -$ for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

4. Similarly, $- \otimes_{\mathsf{Sets}} B \cong - \times B$ for each $B \in \mathrm{Obj}(\mathsf{Sets})$.

5. By **??**, we then have $\otimes_{\mathsf{Sets}} \cong \times$.

Below, we'll show that if a natural isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \colon A \otimes_{\mathsf{Sets}} B \to A \times B$ from before.

Constructing an Isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}} \to \mathrm{pt}$: We define an isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}} \to \mathrm{pt}$ as the composition

$$\mathbb{1}_{\mathsf{Sets}} \underset{\sim}{\overset{\rho^{\mathsf{Sets},-1}_{\mathbb{1}_{\mathsf{Sets}}}}{\dashrightarrow}} \mathbb{1}_{\mathsf{Sets}} \times \mathrm{pt} \underset{\sim}{\overset{\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathbb{1}_{\mathsf{Sets}}}}{\dashrightarrow}} \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt} \underset{\sim}{\overset{\lambda'_{\mathrm{pt}}}{\dashrightarrow}} \mathrm{pt}$$

in Sets.

Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A}"] & \mathrm{pt} \times A \arrow[dr, "\lambda^{\mathsf{Sets}}_A"] & \\
\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_A"] \arrow[rrr, "\lambda'_A"'] & & & A
\end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\lambda^{\mathsf{Sets}}_{\mathrm{pt}}"] & \\
\mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"] \arrow[rrr, "\lambda'_{\mathrm{pt}}"'] & & & \mathrm{pt},
\end{tikzcd}$$

corresponding to the case $A = \mathrm{pt}$, commutes by the terminality of pt (Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_{\mathrm{pt}}"] & \\
\mathrm{pt} \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrr, "\lambda'^{,-1}_{\mathrm{pt}}"'] & {} & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt}.
\end{tikzcd}$$

Now, let $A \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, and consider the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \times [a]"] \arrow[rr, phantom, "(\dagger)"', shift right=3] \arrow[ddrr, phantom, "(5)"] & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_{\mathrm{pt}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} [a]"] & \\
\mathrm{pt} \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrrr, "\lambda'^{,-1}_{\mathrm{pt}}"] \arrow[dd, "{[a]}"'] \arrow[ddr, phantom, "(3)"] \arrow[rrrr, phantom, "(1)"', shift right=4] & & & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}}} \times [a]"] \arrow[ddl, phantom, "(4)"] \\
 & \mathrm{pt} \times A \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A}"] \arrow[rr, phantom, "(2)"', shift right=3] & & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A"] & \\
A \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{A}"] \arrow[rrrr, "\lambda'^{,-1}_{A}"] & & & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A.
\end{tikzcd}$$

Since:

- Subdiagram (5) commutes by the naturality of $\lambda'^{,-1}$.
- Subdiagram ($\dagger$) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathsf{Sets},-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & & \mathrm{pt} \times A \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A}"] & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_A"] & \\
\mathrm{pt} \arrow[r, "{[a]}"] & A \arrow[ur, "\lambda^{\mathsf{Sets},-1}_{A}"] \arrow[rrr, "\lambda'^{,-1}_{A}"'] & & & \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A
\end{tikzcd}$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{aligned}
\lambda'^{,-1}_{A}(a) &= \left[\lambda'^{,-1}_{A} \circ [a]\right](\star) \\
&= \left[\left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \lambda^{\mathsf{Sets},-1}_{A} \circ [a]\right](\star) \\
&= \left[\left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \lambda^{\mathsf{Sets},-1}_{A}\right](a)
\end{aligned}$$

for each $a \in A$, and thus we have

$$\lambda'^{,-1}_A = \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \lambda^{\mathsf{Sets},-1}_A.$$

Taking inverses then gives

$$\lambda'_A = \lambda^{\mathsf{Sets}}_A \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A} \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \times \mathrm{id}_A\right),$$

showing that the diagram

$$\begin{tikzcd} & \mathrm{pt} \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A}"] & \mathrm{pt} \times A \arrow[dr, "\lambda^{\mathsf{Sets}}_A"] & \\ \mathbb{1}_{\mathsf{Sets}} \otimes_{\mathsf{Sets}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_A"] \arrow[rrr, "\lambda'_A"'] & & & A \end{tikzcd}$$

indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We can use the same argument we used to prove the monoidal left unity of the isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ above. For completeness, we repeat it below.

We have to show that the diagram

$$\begin{tikzcd} & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,\mathrm{pt}}"] & A \times \mathrm{pt} \arrow[dr, "\rho^{\mathsf{Sets}}_A"] & \\ A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[ur, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[rrr, "\rho'_A"'] & & & A \end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd} & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "\rho^{\mathsf{Sets}}_{\mathrm{pt}}"] & \\ \mathrm{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[rrr, "\rho'_{\mathrm{pt}}"'] & & & \mathrm{pt}, \end{tikzcd}$$

corresponding to the case $A = \mathrm{pt}$, commutes by the terminality of pt (Definition 4.1.1.1.2). Since this diagram commutes, so does the diagram

$$\begin{tikzcd} & \mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] & \\ \mathrm{pt} \arrow[ur, "\rho^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrr, "\rho'^{,-1}_{\mathrm{pt}}"'] & & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}}. \end{tikzcd}$$

Now, let $A \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, and consider the diagram

$$
\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \times [a]"] \arrow[dr, phantom, "(\dagger)"] & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_{\mathrm{pt}} \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[dd, "\mathrm{id}_{\mathrm{pt}} \otimes_{\mathsf{Sets}} [a]"] & \\
\mathrm{pt} \arrow[ur, "\rho^{\mathsf{Sets},-1}_{\mathrm{pt}}"] \arrow[rrr, "\rho'^{,-1}_{\mathrm{pt}}"] \arrow[dd, "{[a]}"'] \arrow[dr, phantom, "(3)"] & \arrow[r, phantom, "(1)"] & & \mathrm{pt} \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}}} \times [a]"] \arrow[dl, phantom, "(4)"] \\
 & A \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,\mathrm{pt}}"] \arrow[ur, phantom, "(5)"] \arrow[d, phantom, "(2)"] & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] & \\
A \arrow[ur, "\rho^{\mathsf{Sets},-1}_A"] \arrow[rrr, "\rho'^{,-1}_A"] & & & A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}}.
\end{tikzcd}
$$

Since:

- Subdiagram (5) commutes by the naturality of $\rho'^{,-1}$.
- Subdiagram ($\dagger$) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\mathsf{Sets},-1}$.

it follows that the diagram

$$
\begin{tikzcd}
 & & A \times \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,\mathrm{pt}}"] & A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[dr, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}"] & \\
\mathrm{pt} \arrow[r, "{[a]}"] & A \arrow[ur, "\rho^{\mathsf{Sets},-1}_A"] \arrow[rrr, "\rho'^{,-1}_A"'] & & & A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}}
\end{tikzcd}
$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$
\begin{aligned}
\rho'^{,-1}_A(a) &= \left[\rho'^{,-1}_A \circ [a]\right](\star) \\
&= \left[\left(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \rho^{\mathsf{Sets},-1}_A \circ [a]\right](\star) \\
&= \left[\left(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \rho^{\mathsf{Sets},-1}_A\right](a)
\end{aligned}
$$

for each $a \in A$, and thus we have

$$\rho'^{,-1}_A = \left(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},A} \circ \rho^{\mathsf{Sets},-1}_A.$$

Taking inverses then gives

$$\rho'_A = \rho^{\mathsf{Sets}}_A \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},A} \circ \left(\mathrm{id}_A \times \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}\right),$$

showing that the diagram

$$\begin{tikzcd}
& A \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,\mathrm{pt}}"] & A \times \mathrm{pt} \arrow[dr, "\rho^{\mathsf{Sets}}_A"] & \\
A \otimes_{\mathsf{Sets}} \mathbb{1}_{\mathsf{Sets}} \arrow[ur, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}}"] \arrow[rrr, "\rho'_A"'] & & & A
\end{tikzcd}$$

indeed commutes.

Monoidality of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$\begin{tikzcd}
& (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] \arrow[dr, "\alpha'_{A,B,C}"] & \\
(A \times B) \otimes_{\mathsf{Sets}} C \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A\times B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[d, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,C}"] \\
(A \times B) \times C \arrow[dr, "\alpha^{\mathsf{Sets}}_{A,B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B\times C}"] \\
& A \times (B \times C) &
\end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd}
 & (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] \arrow[dr, "\alpha'_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] & \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[dr, "\alpha^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] \\
 & \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[d, "!_{\mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt})}"] & \\
 & \mathrm{pt} &
\end{tikzcd}$$

commutes by the terminality of pt (Definition 4.1.1.1.2). Since the map $!_{\mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt})}\colon \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \longrightarrow \mathrm{pt}$ is an isomorphism (e.g. having inverse $\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}\circ\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}$), it follows that the diagram

$$\begin{tikzcd}
 & (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] \arrow[dr, "\alpha'_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] & \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[dr, "\alpha^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] \\
 & \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) &
\end{tikzcd}$$

also commutes. Taking inverses, we see that the diagram

$$
\begin{tikzcd}
 & \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] & \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & (\dagger) & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[dl, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] \\
 & (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} &
\end{tikzcd}
$$

commutes as well. Now, let $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, let $c \in C$, and consider the diagram

$$
\begin{tikzcd}
 & \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] \arrow[rrr, "{[a]\times([b]\times[c])}"] & & & A\times(B\times C) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{A,B,C}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}"] & \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] \arrow[rrr, "{([a]\times[b])\times[c]}"] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[rrr, "{[a]\otimes_{\mathsf{Sets}}([b]\times[c])}"] & (A\times B)\times C \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C}"] & & A\otimes_{\mathsf{Sets}}(B\times C) \arrow[d, "\mathrm{id}_A\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}"] \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] \arrow[rrr, "{([a]\times[b])\otimes_{\mathsf{Sets}}[c]}"] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[dl, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] \arrow[rrr, "{[a]\otimes_{\mathsf{Sets}}([b]\otimes_{\mathsf{Sets}}[c])}"] & (A\times B)\otimes_{\mathsf{Sets}} C \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}\otimes_{\mathsf{Sets}}\mathrm{id}_C"'] & & A\otimes_{\mathsf{Sets}}(B\otimes_{\mathsf{Sets}} C) \arrow[dl, "\alpha'^{,-1}_{A,B,C}"] \\
 & (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rrr, "{([a]\otimes_{\mathsf{Sets}}[b])\otimes_{\mathsf{Sets}}[c]}"'] & & & (A\otimes_{\mathsf{Sets}} B)\otimes_{\mathsf{Sets}} C &
\end{tikzcd}
$$

which we partition into subdiagrams as follows:

$$\begin{tikzcd}
 & \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] & \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & (\dagger) & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[dl, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"] \\
 & (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[rrr, "{[a]\times([b]\times[c])}"] & & & A\times(B\times C) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{A,B,C}"] \\
(\mathrm{pt}\times\mathrm{pt})\times\mathrm{pt} \arrow[rrr, "{([a]\times[b])\times[c]}"] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt}\times\mathrm{pt},\mathrm{pt}}"'] & & (1) & (A\times B)\times C \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C}"] & \\
(\mathrm{pt}\times\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rrr, "{([a]\times[b])\otimes_{\mathsf{Sets}}[c]}"] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}_{\mathrm{pt}}"'] & (2) & & (A\times B)\otimes_{\mathsf{Sets}} C \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}\otimes_{\mathsf{Sets}}\mathrm{id}_C"] & \\
 & (\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rrr, "{([a]\otimes_{\mathsf{Sets}}[b])\otimes_{\mathsf{Sets}}[c]}"] & & & (A\otimes_{\mathsf{Sets}} B)\otimes_{\mathsf{Sets}} C
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
\mathrm{pt}\times(\mathrm{pt}\times\mathrm{pt}) \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}\times\mathrm{pt}}"] \arrow[rrr, "{[a]\times([b]\times[c])}"] & & (4) & A\times(B\times C) \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}"] & \\
 & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\times\mathrm{pt}) \arrow[rrr, "{[a]\otimes_{\mathsf{Sets}}([b]\times[c])}"] \arrow[d, "\mathrm{id}_{\mathrm{pt}}\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"'] & & (5) & A\otimes_{\mathsf{Sets}}(B\times C) \arrow[d, "\mathrm{id}_A\otimes_{\mathsf{Sets}}\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}"] \\
 & \mathrm{pt}\otimes_{\mathsf{Sets}}(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}) \arrow[dl, "\alpha'^{,-1}_{\mathrm{pt},\mathrm{pt},\mathrm{pt}}"'] \arrow[rrr, "{[a]\otimes_{\mathsf{Sets}}([b]\otimes_{\mathsf{Sets}}[c])}"] & (6) & & A\otimes_{\mathsf{Sets}}(B\otimes_{\mathsf{Sets}} C). \arrow[dl, "\alpha'^{,-1}_{A,B,C}"] \\
(\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt})\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[rrr, "{([a]\otimes_{\mathsf{Sets}}[b])\otimes_{\mathsf{Sets}}[c]}"] & & & (A\otimes_{\mathsf{Sets}} B)\otimes_{\mathsf{Sets}} C &
\end{tikzcd}$$

Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{\mathsf{Sets},-1}$.
- Subdiagram (2) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (3) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}}$.
- Subdiagram (6) commutes by the naturality of $\alpha'^{,-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times (\mathrm{pt} \times \mathrm{pt}) \arrow[d, "{[a]\times([b]\times[c])}"] & \\
 & A \times (B \times C) \arrow[dl, "\alpha^{\mathsf{Sets},-1}_{A,B,C}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}"] & \\
(A \times B) \times C \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[d, "\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}"] \\
(A \times B) \otimes_{\mathsf{Sets}} C \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] & & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[dl, "\alpha'^{,-1}_{A,B,C}"] \\
 & (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C &
\end{tikzcd}$$

also commutes. We then have

$$\begin{aligned}
\Big[\Big(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C\Big) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C} & \\
\circ\, \alpha^{\mathsf{Sets},-1}_{A,B,C}\Big](a,(b,c)) &= \Big[\Big(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C\Big) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C} \\
&\quad \circ\, \alpha^{\mathsf{Sets},-1}_{A,B,C} \circ ([a] \times ([b] \times [c]))\Big](\star,(\star,\star)) \\
&= \Big[\alpha'^{,-1}_{A,B,C} \circ \Big(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}\Big) \\
&\quad \circ\, \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C} \circ ([a] \times ([b] \times [c]))\Big](\star,(\star,\star)) \\
&= \Big[\alpha'^{,-1}_{A,B,C} \circ \Big(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}\Big) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}\Big](a,(b,c))
\end{aligned}$$

for each $(a, (b, c)) \in A \times (B \times C)$, and thus we have

$$\left(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A\times B,C} \circ \alpha^{\mathsf{Sets},-1}_{A,B,C} = \alpha'^{,-1}_{A,B,C} \circ \left(\mathrm{id}_A \times \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,C}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B\times C}.$$

Taking inverses then gives

$$\alpha^{\mathsf{Sets}}_{A,B,C} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|A\times B,C} \circ \left(\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C\right) = \mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B\times C} \circ \left(\mathrm{id}_A \times \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,C}\right) \circ \alpha'_{A,B,C},$$

showing that the diagram

$$\begin{tikzcd}
& (A \otimes_{\mathsf{Sets}} B) \otimes_{\mathsf{Sets}} C \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} \otimes_{\mathsf{Sets}} \mathrm{id}_C"'] \arrow[dr, "\alpha'_{A,B,C}"] & \\
(A \times B) \otimes_{\mathsf{Sets}} C \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A\times B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \otimes_{\mathsf{Sets}} C) \arrow[d, "\mathrm{id}_A \otimes_{\mathsf{Sets}} \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,C}"] \\
(A \times B) \times C \arrow[dr, "\alpha^{\mathsf{Sets}}_{A,B,C}"'] & & A \otimes_{\mathsf{Sets}} (B \times C) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B\times C}"] \\
& A \times (B \times C) &
\end{tikzcd}$$

indeed commutes.

Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: We have to show that the diagram

$$\begin{tikzcd}
A \otimes_{\mathsf{Sets}} B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B}"] \arrow[d, "\sigma'_{A,B}"'] & A \times B \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"] \\
B \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A}"'] & B \times A
\end{tikzcd}$$

commutes. First, note that the diagram

$$\begin{tikzcd}
\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, "\sigma'_{\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \times \mathrm{pt} \arrow[d, "\sigma^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt}}"] & \\
\mathrm{pt} \otimes_{\mathsf{Sets}} \mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt} \times \mathrm{pt} \arrow[dr, "!_{pt\times pt}"] & \\
& & \mathrm{pt}
\end{tikzcd}$$

commutes by the terminality of pt (Definition 4.1.1.1.2). Since the map $!_{\mathrm{pt}\times\mathrm{pt}}\colon \mathrm{pt}\times\mathrm{pt}\longrightarrow\mathrm{pt}$ is invertible (e.g. with inverse $\lambda^{\mathsf{Sets},-1}_{\mathrm{pt}}$), the diagram

$$\begin{tikzcd}
\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, "\sigma'_{\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt}\times\mathrm{pt} \arrow[d, "\sigma^{\mathsf{Sets}}_{\mathrm{pt},\mathrm{pt}}"] \\
\mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt}\times\mathrm{pt}
\end{tikzcd}$$

also commutes. Taking inverses, we see that the diagram

$$\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"] \arrow[d, "\sigma^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] \arrow[dr, phantom, "(\dagger)"] & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[d, "\sigma'^{,-1}_{\mathrm{pt},\mathrm{pt}}"] \\
\mathrm{pt}\times\mathrm{pt} \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|\mathrm{pt},\mathrm{pt}}"'] & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt}
\end{tikzcd}$$

commutes as well. Now, let $A, B \in \mathrm{Obj}(\mathsf{Sets})$, let $a \in A$, let $b \in B$, and consider the diagram

$$\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dr, "{[b]\times[a]}"] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "{[b]\otimes_{\mathsf{Sets}}[a]}"] \arrow[dd, "\sigma'^{,-1}_{\mathrm{pt},\mathrm{pt}}" near start] & \\
& B\times A \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A}" near start] \arrow[dd, "\sigma^{\mathsf{Sets},-1}_{A,B}" near start] & & B\otimes_{\mathsf{Sets}} A \arrow[dd, "\sigma'^{,-1}_{A,B}"] \\
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}" near start] \arrow[dr, "{[a]\times[b]}"'] & & \mathrm{pt}\otimes_{\mathsf{Sets}}\mathrm{pt} \arrow[dr, "{[a]\otimes_{\mathsf{Sets}}[b]}"'] & \\
& A\times B \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B}"'] & & A\otimes_{\mathsf{Sets}} B
\end{tikzcd}$$

which we partition into subdiagrams as follows:

$$
\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dr, "{[b]\times[a]}"'] \arrow[dd, "\sigma^{\mathrm{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] & & \mathrm{pt}\otimes_{\mathrm{Sets}}\mathrm{pt} \arrow[d, "{[b]\otimes_{\mathrm{Sets}}[a]}"] \\
& B\times A \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|B,A}"] \arrow[dd, "\sigma^{\mathrm{Sets},-1}_{A,B}"'] \arrow[ur, phantom, "(1)"] \arrow[dl, phantom, "(2)"] \arrow[ddr, phantom, "(3)"] & B\otimes_{\mathrm{Sets}} A \arrow[dd, "\sigma'^{,-1}_{A,B}"] \\
\mathrm{pt}\times\mathrm{pt} \arrow[dr, "{[a]\times[b]}"'] & & \\
& A\times B \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|A,B}"'] & A\otimes_{\mathrm{Sets}} B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dd, "\sigma^{\mathrm{Sets},-1}_{\mathrm{pt},\mathrm{pt}}"'] \arrow[ddrr, phantom, "(\dagger)"] & & \mathrm{pt}\otimes_{\mathrm{Sets}}\mathrm{pt} \arrow[dr, "{[b]\otimes_{\mathrm{Sets}}[a]}"] \arrow[dd, "\sigma'^{,-1}_{\mathrm{pt},\mathrm{pt}}"] & \\
& & & B\otimes_{\mathrm{Sets}} A \arrow[dd, "\sigma'^{,-1}_{A,B}"] \arrow[dl, phantom, "(4)"] \\
\mathrm{pt}\times\mathrm{pt} \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{pt},\mathrm{pt}}"] \arrow[dr, "{[a]\times[b]}"'] & & \mathrm{pt}\otimes_{\mathrm{Sets}}\mathrm{pt} \arrow[dr, "{[a]\otimes_{\mathrm{Sets}}[b]}"] \arrow[d, phantom, "(5)"] & \\
& A\times B \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|A,B}"'] & & A\otimes_{\mathrm{Sets}} B
\end{tikzcd}
$$

Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\mathrm{Sets},-1}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\sigma'^{,-1}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.

it follows that the diagram

$$
\begin{tikzcd}
\mathrm{pt}\times\mathrm{pt} \arrow[dr, "{[b]\times[a]}"] & & \\
& B\times A \arrow[r, "\mathrm{id}^{\otimes}_{\mathrm{Sets}|B,A}"] \arrow[d, "\sigma^{\mathrm{Sets}}_{A,B}"'] & B\otimes_{\mathrm{Sets}} A \arrow[d, "\sigma'_{A,B}"] \\
& A\times B \arrow[r, "\mathrm{id}^{\otimes}_{\mathrm{Sets}|A,B}"'] & A\otimes_{\mathrm{Sets}} B
\end{tikzcd}
$$

commutes. We then have

$$
\begin{aligned}
\left[\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|A,B}\circ\sigma^{\mathrm{Sets},-1}_{A,B}\right](b,a) &= \left[\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|A,B}\circ\sigma^{\mathrm{Sets},-1}_{A,B}\circ([b]\times[a])\right](\star,\star)\\
&= \left[\sigma'^{,-1}_{A,B}\circ\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|B,A}\circ([b]\times[a])\right](\star,\star)\\
&= \left[\sigma'^{,-1}_{A,B}\circ\mathrm{id}^{\otimes,-1}_{\mathrm{Sets}|B,A}\right](b,a)
\end{aligned}
$$

for each $(b,a) \in B \times A$, and thus we have

$$\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|A,B} \circ \sigma^{\mathsf{Sets},-1}_{A,B} = \sigma'^{,-1}_{A,B} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}|B,A}.$$

Taking inverses then gives

$$\sigma^{\mathsf{Sets}}_{A,B} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B} = \mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A} \circ \sigma'_{A,B},$$

showing that the diagram

$$\begin{tikzcd}
A \otimes_{\mathsf{Sets}} B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|A,B}"] \arrow[d, "\sigma'_{A,B}"'] & A \times B \arrow[d, "\sigma^{\mathsf{Sets}}_{A,B}"] \\
B \otimes_{\mathsf{Sets}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}|B,A}"'] & B \times A
\end{tikzcd}$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}} \cong \times$: Let $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}} -_2 \Rightarrow -_1 \times -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets with respect to $\times$ and $\otimes$ (as shown above), we have

$$\lambda'_B = \lambda^{\mathsf{Sets}}_B \circ \phi_{\mathrm{pt},B} \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right),$$
$$\lambda'_B = \lambda^{\mathsf{Sets}}_B \circ \psi_{\mathrm{pt},B} \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right).$$

Postcomposing both sides with $\lambda^{\mathsf{Sets},-1}_B$ gives

$$\lambda^{\mathsf{Sets},-1}_B \circ \lambda'_B \circ \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right) = \phi_{\mathrm{pt},B},$$
$$\lambda^{\mathsf{Sets},-1}_B \circ \lambda'_B \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right) = \psi_{\mathrm{pt},B},$$

and thus we have

$$\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$$

for each $B \in \mathrm{Obj}(\mathsf{Sets})$. Now, let $a \in A$ and consider the naturality diagrams

$$\begin{tikzcd}
\mathrm{pt} \times B \arrow[r, "{[a]\times\mathrm{id}_B}"] \arrow[d, "\phi_{\mathrm{pt},B}"'] & A \times B \arrow[d, "\phi_{A,B}"] \\
\mathrm{pt} \otimes_{\mathsf{Sets}} B \arrow[r, "{[a]\otimes_{\mathsf{Sets}}\mathrm{id}_B}"'] & A \otimes_{\mathsf{Sets}} B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{pt} \times B \arrow[r, "{[a]\times\mathrm{id}_B}"] \arrow[d, "\psi_{\mathrm{pt},B}"'] & A \times B \arrow[d, "\psi_{A,B}"] \\
\mathrm{pt} \otimes_{\mathsf{Sets}} B \arrow[r, "{[a]\otimes_{\mathsf{Sets}}\mathrm{id}_B}"'] & A \otimes_{\mathsf{Sets}} B
\end{tikzcd}$$

for ϕ and ψ with respect to the morphisms $[a]$ and id_B. Having shown that $\phi_{\mathrm{pt},B} = \psi_{\mathrm{pt},B}$, we have

$$\begin{aligned}\phi_{A,B}(a,b) &= \big[\phi_{A,B} \circ ([a] \times \mathrm{id}_B)\big](\star, b)\\ &= \big[([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \phi_{\mathrm{pt},B}\big](\star, b)\\ &= \big[([a] \otimes_{\mathsf{Sets}} \mathrm{id}_B) \circ \psi_{\mathrm{pt},B}\big](\star, b)\\ &= \big[\psi_{A,B} \circ ([a] \times \mathrm{id}_B)\big](\star, b)\\ &= \psi_{A,B}(a,b)\end{aligned}$$

for each $(a,b) \in A \times B$. Therefore we have

$$\phi_{A,B} = \psi_{A,B}$$

for each $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathsf{Sets}} \cong \times$ to be unique. □

Corollary 5.1.10.1.2. The symmetric monoidal structure on the category Sets of Definition 5.1.9.1.1 is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The tensor product

$$\otimes_{\mathsf{Sets}} \colon \mathsf{Sets} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

of Sets preserves colimits separately in each variable.

2. *The Unit Object Is pt.* We have $\mathbb{1}_{\mathsf{Sets}} \cong \mathrm{pt}$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathsf{Sets})$ of **??** spanned by the symmetric monoidal categories $(\mathsf{Sets}, \otimes_{\mathsf{Sets}}, \mathbb{1}_{\mathsf{Sets}}, \lambda^{\mathsf{Sets}}, \rho^{\mathsf{Sets}}, \sigma^{\mathsf{Sets}})$ satisfying Items 1 and 2 is contractible.

Proof. Since Sets is locally presentable (**??**), it follows from **??** that Item 1 is equivalent to the existence of an internal Hom as in Item 1 of Definition 5.1.10.1.1. The result then follows from Definition 5.1.10.1.1. □

5.2 The Monoidal Category of Sets and Coproducts

5.2.1 Coproducts of Sets

See Section 4.2.3.

5.2.2 The Monoidal Unit

Definition 5.2.2.1.1. The **monoidal unit of the coproduct of sets** is the functor

$$\mathbb{0}^{\mathsf{Sets}} \colon \mathrm{pt} \to \mathsf{Sets}$$

defined by

$$\mathbb{0}_{\mathsf{Sets}} \stackrel{\text{def}}{=} \emptyset,$$

where $\emptyset$ is the empty set of Definition 4.3.1.1.1.

5.2.3 The Associator

Definition 5.2.3.1.1. The **associator of the coproduct of sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\coprod \times \mathrm{id}_{\mathsf{Sets}}\right) \xRightarrow{\sim} \coprod \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \coprod\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets} \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[dr, "\mathrm{id} \times \coprod"] & \\
(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}", "\sim"'] \arrow[d, "\coprod \times \mathrm{id}"'] \arrow[dr, phantom, "\Uparrow \alpha^{\mathsf{Sets},\coprod}"] & & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\coprod"] \\
\mathsf{Sets} \times \mathsf{Sets} \arrow[rr, "\coprod"'] & & \mathsf{Sets},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z} \colon \left(X \coprod Y\right) \coprod Z \xrightarrow{\sim} X \coprod \left(Y \coprod Z\right)$$

at (X, Y, Z) is given by

$$\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}(a) \stackrel{\text{def}}{=} \begin{cases} (0, x) & \text{if } a = (0, (0, x)), \\ (1, (0, y)) & \text{if } a = (0, (1, y)), \\ (1, (1, a)) & \text{if } a = (1, z) \end{cases}$$

for each $a \in \left(X \coprod Y\right) \coprod Z$.

Proof. Unwinding the Definitions of $\left(X \coprod Y\right) \coprod Z$ *and* $X \coprod \left(Y \coprod Z\right)$: Firstly, we unwind the expressions for $(X \coprod Y) \coprod Z$ and $X \coprod (Y \coprod Z)$. We have

$$\begin{aligned}(X \coprod Y) \coprod Z &\stackrel{\text{def}}{=} \{(0,a) \in S \mid a \in X \coprod Y\} \cup \{(1,z) \in S \mid z \in Z\}\\ &= \{(0,(0,x)) \in S \mid x \in X\} \cup \{(0,(1,y)) \in S \mid y \in Y\}\\ &\quad \cup \{(1,z) \in S \mid z \in Z\},\end{aligned}$$

where $S = \{0,1\} \times ((X \coprod Y) \cup Z)$ and

$$\begin{aligned}X \coprod (Y \coprod Z) &\stackrel{\text{def}}{=} \{(0,x) \in S' \mid x \in X\} \cup \{(1,a) \in S' \mid a \in Y \coprod Z\}\\ &= \{(0,x) \in S' \mid x \in X\} \cup \{(1,(0,y)) \in S' \mid y \in Y\}\\ &\quad \cup \{(1,(1,z)) \in S' \mid z \in Z\},\end{aligned}$$

where $S' = \{0,1\} \times (X \cup (Y \coprod Z))$.

Invertibility: The inverse of $\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}$ is the map

$$\alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z} \colon X \coprod (Y \coprod Z) \to (X \coprod Y) \coprod Z$$

given by

$$\alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z}(a) \stackrel{\text{def}}{=} \begin{cases}(0,(0,x)) & \text{if } a = (0,x),\\ (0,(1,y)) & \text{if } a = (1,(0,y)),\\ (1,z) & \text{if } a = (1,(1,z))\end{cases}$$

for each $a \in X \coprod Y(\coprod Z)$. Indeed:

- *Invertibility I*. The map $\alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z} \circ \alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}$ acts on elements as

$$\begin{array}{ccccc}(0,(0,x)) & \mapsto & (0,x) & \mapsto & (0,(0,x)),\\ (0,(0,y)) & \mapsto & (1,(0,y)) & \mapsto & (0,(0,y)),\\ (1,z) & \mapsto & (1,(1,z)) & \mapsto & (1,z)\end{array}$$

  and hence is equal to the identity map of $(X \coprod Y) \coprod Z$.

- *Invertibility II*. The map $\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z} \circ \alpha^{\mathsf{Sets},\coprod,-1}_{X,Y,Z}$ acts on elements as

$$\begin{array}{ccccc}(0,x) & \mapsto & (0,(0,x)) & \mapsto & (0,x),\\ (1,(0,y)) & \mapsto & (0,(0,y)) & \mapsto & (1,(0,y)),\\ (1,(1,z)) & \mapsto & (1,z) & \mapsto & (1,(1,z))\end{array}$$

  and hence is equal to the identity map of $X \coprod (Y \coprod Z)$.

Therefore $\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}$ is indeed an isomorphism.
Naturality: We need to show that, given functions

$$
\begin{aligned}
f &\colon X \to X',\\
g &\colon Y \to Y',\\
h &\colon Z \to Z'
\end{aligned}
$$

the diagram

$$
\begin{tikzcd}
(X \coprod Y) \coprod Z \arrow[r, "(f\coprod g)\coprod h"] \arrow[d, "\alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}"'] & (X' \coprod Y') \coprod Z' \arrow[d, "\alpha^{\mathsf{Sets},\coprod}_{X',Y',Z'}"] \\
X \coprod (Y \coprod Z) \arrow[r, "f\coprod(g\coprod h)"'] & X' \coprod (Y' \coprod Z')
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(0,(0,x)) \arrow[d, mapsto] & & (0,(0,x)) \arrow[r, mapsto] & (0,(0,f(x))) \arrow[d, mapsto] \\
(0,x) \arrow[r, mapsto] & (0,f(x)) & & (0,f(x))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,(1,y)) \arrow[d, mapsto] & & (0,(1,y)) \arrow[r, mapsto] & (0,(1,g(y))) \arrow[d, mapsto] \\
(1,(0,y)) \arrow[r, mapsto] & (1,(0,g(y))) & & (1,(0,g(y)))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(1,z) \arrow[d, mapsto] & & (1,z) \arrow[r, mapsto] & (1,h(z)) \arrow[d, mapsto] \\
(1,(1,z)) \arrow[r, mapsto] & (1,(1,h(z))) & & (1,(1,h(z)))
\end{tikzcd}
$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets},\coprod}$ to be a natural transformation.
Being a Natural Isomorphism: Since $\alpha^{\mathsf{Sets},\coprod}$ is natural and $\alpha^{\mathsf{Sets},\coprod,-1}$ is a componentwise inverse to $\alpha^{\mathsf{Sets},\coprod}$, it follows from Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus $\alpha^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

5.2.4 The Left Unitor

Definition 5.2.4.1.1. The **left unitor of the coproduct of sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets},\coprod}\colon \coprod\circ\left(\mathbb{0}^{\mathsf{Sets}}\times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}} \qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Sets} \arrow[r, "\mathbb{0}^{\mathsf{Sets}}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets}\times\mathsf{Sets} \arrow[d, "\coprod"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Sets},\coprod}"] \\ & \mathsf{Sets}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets},\coprod}_X\colon \emptyset \coprod X \overset{\sim}{\dashrightarrow} X$$

at X is given by

$$\lambda^{\mathsf{Sets},\coprod}_X((1,x)) \overset{\text{def}}{=} x$$

for each $(1,x) \in \emptyset \coprod X$.

Proof. *Unwinding the Definition of* $\emptyset \coprod X$: Firstly, we unwind the expressions for $\emptyset \coprod X$. We have

$$\begin{aligned} \emptyset \coprod X &\overset{\text{def}}{=} \{(0,z) \in S \mid z \in \emptyset\} \cup \{(1,x) \in S \mid x \in X\} \\ &= \emptyset \cup \{(1,x) \in S \mid x \in X\} \\ &= \{(1,x) \in S \mid x \in X\}, \end{aligned}$$

where $S = \{0,1\} \times (\emptyset \cup X)$.

Invertibility: The inverse of $\lambda^{\mathsf{Sets},\coprod}_X$ is the map

$$\lambda^{\mathsf{Sets},\coprod,-1}_X\colon X \to \emptyset \coprod X$$

given by

$$\lambda^{\mathsf{Sets},\coprod,-1}_X(x) \overset{\text{def}}{=} (1,x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\left[\lambda^{\mathsf{Sets},\coprod,-1}_X \circ \lambda^{\mathsf{Sets},\coprod}_X\right](1,x) = \lambda^{\mathsf{Sets},\coprod,-1}_X\left(\lambda^{\mathsf{Sets},\coprod}_X(1,x)\right)$$

$$
\begin{aligned}
&= \lambda^{\mathsf{Sets},\coprod,-1}_{X}(x)\\
&= (1,x)\\
&= \left[\mathrm{id}_{\emptyset\coprod X}\right](1,x)
\end{aligned}
$$

for each $(1,x) \in \emptyset \coprod X$, and therefore we have

$$\lambda^{\mathsf{Sets},\coprod,-1}_{X} \circ \lambda^{\mathsf{Sets},\coprod}_{X} = \mathrm{id}_{\emptyset\coprod X}.$$

- *Invertibility II*. We have

$$
\begin{aligned}
\left[\lambda^{\mathsf{Sets},\coprod}_{X} \circ \lambda^{\mathsf{Sets},\coprod,-1}_{X}\right](x) &= \lambda^{\mathsf{Sets},\coprod}_{X}\left(\lambda^{\mathsf{Sets},\coprod,-1}_{X}(x)\right)\\
&= \lambda^{\mathsf{Sets},\coprod,-1}_{X}(1,x)\\
&= x\\
&= [\mathrm{id}_X](x)
\end{aligned}
$$

for each $x \in X$, and therefore we have

$$\lambda^{\mathsf{Sets},\coprod}_{X} \circ \lambda^{\mathsf{Sets},\coprod,-1}_{X} = \mathrm{id}_X.$$

Therefore $\lambda^{\mathsf{Sets},\coprod}_{X}$ is indeed an isomorphism.
Naturality: We need to show that, given a function $f \colon X \to Y$, the diagram

$$
\begin{tikzcd}
\emptyset \coprod X \arrow[r, "\mathrm{id}_{\emptyset} \coprod f"] \arrow[d, "\lambda^{\mathsf{Sets},\coprod}_{X}"'] & \emptyset \coprod Y \arrow[d, "\lambda^{\mathsf{Sets},\coprod}_{Y}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(1,x) \arrow[d, mapsto] & & (1,x) \arrow[r, mapsto] & (1,f(x)) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x) & & f(x)
\end{tikzcd}
$$

and hence indeed commutes. Therefore $\lambda^{\mathsf{Sets},\coprod}$ is a natural transformation.
Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets},\coprod}$ is natural and $\lambda^{\mathsf{Sets},-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets},\coprod}$, it follows from Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets},-1}$ is also natural. Thus $\lambda^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

5.2.5 The Right Unitor

Definition 5.2.5.1.1. The **right unitor of the coproduct of sets** is the natural isomorphism

$$\rho^{\mathsf{Sets},\coprod}\colon \coprod \circ \left(\mathsf{id} \times \mathbb{0}^{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets} \times \mathsf{pt} \arrow[r, "\mathsf{id}\times\mathbb{0}^{\mathsf{Sets}}"] \arrow[rd, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}"'] & \mathsf{Sets} \times \mathsf{Sets} \arrow[d, "\coprod"] \arrow[ld, phantom, "\Swarrow \rho^{\mathsf{Sets},\coprod}"] \\ & \mathsf{Sets}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets},\coprod}_X \colon X \coprod \emptyset \overset{\sim}{\dashrightarrow} X$$

at X is given by

$$\rho^{\mathsf{Sets},\coprod}_X((0,x)) \overset{\text{def}}{=} x$$

for each $(0,x) \in X \coprod \emptyset$.

Proof. Unwinding the Definition of $X \coprod \emptyset$: Firstly, we unwind the expression for $X \coprod \emptyset$. We have

$$\begin{aligned} X \coprod \emptyset &\overset{\text{def}}{=} \{(0,x) \in S \mid x \in X\} \cup \{(1,z) \in S \mid z \in \emptyset\} \\ &= \{(0,x) \in S \mid x \in X\} \cup \emptyset \\ &= \{(0,x) \in S \mid x \in X\}, \end{aligned}$$

where $S = \{0,1\} \times (X \cup \emptyset) = \{0,1\} \times (\emptyset \cup X) = S$.

Invertibility: The inverse of $\rho^{\mathsf{Sets},\coprod}_X$ is the map

$$\rho^{\mathsf{Sets},\coprod,-1}_X \colon X \to X \coprod \emptyset$$

given by

$$\rho^{\mathsf{Sets},\coprod,-1}_X(x) \overset{\text{def}}{=} (0,x)$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

$$\left[\rho^{\mathsf{Sets},\coprod,-1}_X \circ \rho^{\mathsf{Sets},\coprod}_X\right](0,x) = \rho^{\mathsf{Sets},\coprod,-1}_X\left(\rho^{\mathsf{Sets},\coprod}_X(0,x)\right)$$

$$
\begin{aligned}
&= \rho_X^{\mathsf{Sets},\coprod,-1}(x) \\
&= (0,x) \\
&= \left[\mathrm{id}_{X\coprod\emptyset}\right](0,x)
\end{aligned}
$$

for each $(0,x) \in \emptyset \coprod X$, and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod,-1} \circ \rho_X^{\mathsf{Sets},\coprod} = \mathrm{id}_{\emptyset\coprod X}.$$

- *Invertibility II*. We have

$$
\begin{aligned}
\left[\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1}\right](x) &= \rho_X^{\mathsf{Sets},\coprod}\left(\rho_X^{\mathsf{Sets},\coprod,-1}(x)\right) \\
&= \rho_X^{\mathsf{Sets},\coprod,-1}(0,x) \\
&= x \\
&= [\mathrm{id}_X](x)
\end{aligned}
$$

for each $x \in X$, and therefore we have

$$\rho_X^{\mathsf{Sets},\coprod} \circ \rho_X^{\mathsf{Sets},\coprod,-1} = \mathrm{id}_X.$$

Therefore $\rho_X^{\mathsf{Sets},\coprod}$ is indeed an isomorphism.

Naturality: We need to show that, given a function $f \colon X \to Y$, the diagram

$$
\begin{tikzcd}
X \coprod \emptyset \arrow[r, "f\coprod \mathrm{id}_\emptyset"] \arrow[d, "\rho_X^{\mathsf{Sets},\coprod}"'] & Y \coprod \emptyset \arrow[d, "\rho_Y^{\mathsf{Sets},\coprod}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(0,x) \arrow[d, mapsto] & & (0,x) \arrow[r, mapsto] & (1,f(x)) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x) & & f(x)
\end{tikzcd}
$$

and hence indeed commutes. Therefore $\rho^{\mathsf{Sets},\coprod}$ is a natural transformation.

Being a Natural Isomorphism: Since $\rho^{\mathsf{Sets},\coprod}$ is natural and $\rho^{\mathsf{Sets},-1}$ is a componentwise inverse to $\rho^{\mathsf{Sets},\coprod}$, it follows from Item 2 of Definition 11.9.7.1.2 that $\rho^{\mathsf{Sets},-1}$ is also natural. Thus $\rho^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

5.2.6 The Symmetry

Definition 5.2.6.1.1. The **symmetry of the coproduct of sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} \colon \coprod \overset{\sim}{\Longrightarrow} \coprod \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}, \qquad \begin{tikzcd} \mathsf{Sets} \times \mathsf{Sets} \arrow[rr, "\coprod"] \arrow[dr, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}"'] & \arrow[d, phantom, "\overset{\sigma^{\mathsf{Sets},\coprod}}{\Downarrow}"] & \mathsf{Sets}, \\ & \mathsf{Sets} \times \mathsf{Sets} \arrow[ur, "\coprod"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets},\coprod}_{X,Y} \colon X \coprod Y \overset{\sim}{\dashrightarrow} Y \coprod X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets})$ is defined by

$$\sigma^{\mathsf{Sets},\coprod}_{X,Y}(x, y) \overset{\text{def}}{=} (y, x)$$

for each $(x, y) \in X \times Y$.

Proof. Unwinding the Definitions of $X \coprod Y$ *and* $Y \coprod X$: Firstly, we unwind the expressions for $X \coprod Y$ and $Y \coprod X$. We have

$$X \coprod Y \overset{\text{def}}{=} \{(0, x) \in S \mid x \in X\} \cup \{(1, y) \in S \mid y \in Y\},$$

where $S = \{0, 1\} \times (X \cup Y)$ and

$$Y \coprod X \overset{\text{def}}{=} \{(0, y) \in S' \mid y \in Y\} \cup \{(1, x) \in S' \mid x \in X\},$$

where $S' = \{0, 1\} \times (Y \cup X) = \{0, 1\} \times (X \cup Y) = S$.

Invertibility: The inverse of $\sigma^{\mathsf{Sets},\coprod}_{X,Y}$ is the map

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y} \colon Y \coprod X \to X \coprod Y$$

defined by

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y} \overset{\text{def}}{=} \sigma^{\mathsf{Sets},\coprod}_{Y,X}$$

and hence given by

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} (0, x) & \text{if } z = (1, x), \\ (1, y) & \text{if } z = (0, y) \end{cases}$$

for each $z \in Y \coprod X$. Indeed:

- *Invertibility I.* We have

$$\begin{aligned}\left[\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}\circ\sigma^{\mathsf{Sets},\coprod}_{X,Y}\right](0,x) &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}\left(\sigma^{\mathsf{Sets},\coprod}_{X}(0,x)\right)\\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(1,x)\\ &= (0,x)\\ &= \left[\mathrm{id}_{X\coprod Y}\right](0,x)\end{aligned}$$

  for each $(0,x)\in X\coprod Y$ and

$$\begin{aligned}\left[\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}\circ\sigma^{\mathsf{Sets},\coprod}_{X,Y}\right](1,y) &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}\left(\sigma^{\mathsf{Sets},\coprod}_{X}(1,y)\right)\\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(0,y)\\ &= (1,y)\\ &= \left[\mathrm{id}_{X\coprod Y}\right](1,y)\end{aligned}$$

  for each $(1,y)\in X\coprod Y$, and therefore we have

$$\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}\circ\sigma^{\mathsf{Sets},\coprod}_{X,Y}=\mathrm{id}_{X\coprod Y}.$$

- *Invertibility II.* We have

$$\begin{aligned}\left[\sigma^{\mathsf{Sets},\coprod}_{X,Y}\circ\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}\right](0,y) &= \sigma^{\mathsf{Sets},\coprod}_{X}\left(\sigma^{\mathsf{Sets},\coprod,-1}_{X}(0,y)\right)\\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(1,y)\\ &= (0,y)\\ &= \left[\mathrm{id}_{Y\coprod X}\right](0,y)\end{aligned}$$

  for each $(0,y)\in Y\coprod X$ and

$$\begin{aligned}\left[\sigma^{\mathsf{Sets},\coprod}_{X,Y}\circ\sigma^{\mathsf{Sets},\coprod,-1}_{X,Y}\right](1,x) &= \sigma^{\mathsf{Sets},\coprod}_{X}\left(\sigma^{\mathsf{Sets},\coprod,-1}_{X}(1,x)\right)\\ &= \sigma^{\mathsf{Sets},\coprod,-1}_{X}(0,x)\\ &= (1,x)\\ &= \left[\mathrm{id}_{Y\coprod X}\right](1,x)\end{aligned}$$

  for each $(1,x)\in Y\coprod X$, and therefore we have

$$\sigma^{\mathsf{Sets},\coprod}_{X}\circ\sigma^{\mathsf{Sets},\coprod,-1}_{X}=\mathrm{id}_{Y\coprod X}.$$

Therefore $\sigma^{\mathsf{Sets},\coprod}_{X,Y}$ is indeed an isomorphism.
Naturality: We need to show that, given functions $f\colon A \to X$ and $g\colon B \to Y$, the diagram

$$\begin{tikzcd}
A \coprod B \arrow[r, "f \coprod g"] \arrow[d, "\sigma^{\mathsf{Sets},\coprod}_{A,B}"'] & X \coprod Y \arrow[d, "\sigma^{\mathsf{Sets},\coprod}_{X,Y}"] \\
B \coprod A \arrow[r, "g \coprod f"'] & Y \coprod X
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0,a) \arrow[d, mapsto] & & (0,a) \arrow[r, mapsto] & (0,f(a)) \arrow[d, mapsto] \\
(1,a) \arrow[r, mapsto] & (1,f(a)) & & (1,f(a))
\end{tikzcd}$$

$$\begin{tikzcd}
(1,b) \arrow[d, mapsto] & & (1,b) \arrow[r, mapsto] & (1,g(b)) \arrow[d, mapsto] \\
(0,b) \arrow[r, mapsto] & (0,g(b)) & & (0,g(b))
\end{tikzcd}$$

and hence indeed commutes. Therefore $\sigma^{\mathsf{Sets},\coprod}$ is a natural transformation.
Being a Natural Isomorphism: Since $\sigma^{\mathsf{Sets},\coprod}$ is natural and $\sigma^{\mathsf{Sets},-1}$ is a componentwise inverse to $\sigma^{\mathsf{Sets},\coprod}$, it follows from Item 2 of Definition 11.9.7.1.2 that $\sigma^{\mathsf{Sets},-1}$ is also natural. Thus $\sigma^{\mathsf{Sets},\coprod}$ is a natural isomorphism. □

5.2.7 The Monoidal Category of Sets and Coproducts

Proposition 5.2.7.1.1. The category Sets admits a closed symmetric monoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Monoidal Product.* The coproduct functor

$$\coprod\colon \mathsf{Sets} \times \mathsf{Sets} \to \mathsf{Sets}$$

  of Item 1 of Definition 4.2.3.1.3.

- *The Monoidal Unit.* The functor

  $$\mathbb{0}^{\mathsf{Sets}} \colon \mathrm{pt} \to \mathsf{Sets}$$

  of Definition 5.2.2.1.1.

- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\coprod \times \mathrm{id}_{\mathsf{Sets}}\right) \xRightarrow{\sim} \coprod \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \coprod\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.2.3.1.1.

- *The Left Unitors.* The natural isomorphism

  $$\lambda^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \xRightarrow{\sim} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

  of Definition 5.2.4.1.1.

- *The Right Unitors.* The natural isomorphism

  $$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathrm{id} \times \mathbb{0}^{\mathsf{Sets}}\right) \xRightarrow{\sim} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

  of Definition 5.2.5.1.1.

- *The Symmetry.* The natural isomorphism

  $$\sigma^{\mathsf{Sets},\coprod} \colon \times \xRightarrow{\sim} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.2.6.1.1.

Proof. *The Pentagon Identity*: Let W, X, Y and Z be sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \coprod (X \coprod Y)) \coprod Z \arrow[dr, "\alpha^{\mathsf{Sets},\coprod}_{W,X \coprod Y,Z}"] & \\
((W \coprod X) \coprod Y) \coprod Z \arrow[ur, "\alpha^{\mathsf{Sets},\coprod}_{W,X,Y} \coprod \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets},\coprod}_{W \coprod X,Y,Z}"'] & & W \coprod ((X \coprod Y) \coprod Z) \arrow[d, "\mathrm{id}_W \coprod \alpha^{\mathsf{Sets},\coprod}_{X,Y,Z}"] \\
(W \coprod X) \coprod (Y \coprod Z) \arrow[rr, "\alpha^{\mathsf{Sets},\coprod}_{W,X,Y \coprod Z}"'] & & W \coprod (X \coprod (Y \coprod Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0,(0,(0,w))) \arrow[d, mapsto] & \\
(0,(0,w)) \arrow[r, mapsto] & (0,w)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (0,(0,w)) \arrow[dr, mapsto] & \\
(0,(0,(0,w))) \arrow[ur, mapsto] & & (0,w) \arrow[d, mapsto] \\
 & & (0,w)
\end{tikzcd}$$

$$\begin{tikzcd}
(0,(0,(1,x))) \arrow[d, mapsto] & \\
(0,(1,x)) \arrow[r, mapsto] & (1,(0,x))
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (0,(1,(0,x))) \arrow[dr, mapsto] & \\
(0,(0,(1,x))) \arrow[ur, mapsto] & & (1,(0,(0,x))) \arrow[d, mapsto] \\
 & & (1,(0,x))
\end{tikzcd}$$

$$
\begin{tikzcd}
(0,(1,y)) \arrow[d, mapsto] & & & (0,(1,(1,y))) \arrow[dr, mapsto] & \\
(1,(0,y)) \arrow[r, mapsto] & (1,(1,(0,y))) & (0,(1,y)) \arrow[ur, mapsto] & & (1,(0,(1,y))) \arrow[d, mapsto] \\
 & & & & (1,(1,(0,y)))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(1,z) \arrow[d, mapsto] & & & (1,z) \arrow[dr, mapsto] & \\
(1,(1,z)) \arrow[r, mapsto] & (1,(1,(1,z))) & (1,z) \arrow[ur, mapsto] & & (1,(1,z)) \arrow[d, mapsto] \\
 & & & & (1,(1,(1,z)))
\end{tikzcd}
$$

and therefore the pentagon identity is satisfied.

The Triangle Identity: Let X and Y be sets. We have to show that the diagram

$$
\begin{tikzcd}
(X \coprod \emptyset) \coprod Y \arrow[rr, "\alpha^{\mathsf{Sets},\coprod}_{X,\emptyset,Y}"] \arrow[dr, "\rho^{\mathsf{Sets},\coprod}_X \coprod \mathrm{id}_Y"'] & & X \coprod (\emptyset \coprod Y) \arrow[dl, "\mathrm{id}_X \coprod \lambda^{\mathsf{Sets},\coprod}_Y"] \\
 & X \coprod Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(0,(1,x)) \arrow[dr, mapsto] & & (1,(0,x)) \arrow[rr, mapsto] & & (0,x) \arrow[dl, mapsto] \\
 & (0,x) & & (0,x) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(1, y) \arrow[dr, mapsto] & & (1, y) \arrow[r, mapsto] & (1, (1, y)) \arrow[dl, mapsto] \\
& (1, y) & (1, y) &
\end{tikzcd}
$$

and therefore the triangle identity is satisfied.

The Left Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$
\begin{tikzcd}
& (X \coprod Y) \coprod Z \arrow[dl, "\alpha^{\mathrm{Sets}}_{X,Y,Z}"'] \arrow[dr, "\sigma^{\mathrm{Sets}}_{X,Y} \coprod \mathrm{id}_Z"] & \\
X \coprod (Y \coprod Z) \arrow[d, "\sigma^{\mathrm{Sets}}_{X,Y \coprod Z}"'] & & (Y \coprod X) \coprod Z \arrow[d, "\alpha^{\mathrm{Sets}}_{Y,X,Z}"] \\
(Y \coprod Z) \coprod X \arrow[dr, "\alpha^{\mathrm{Sets}}_{Y,Z,X}"'] & & Y \coprod (X \coprod Z) \arrow[dl, "\mathrm{id}_Y \coprod \sigma^{\mathrm{Sets}}_{X,Z}"] \\
& Y \coprod (Z \coprod X) &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
& (0, (0, x)) \arrow[dl, mapsto] & (0, (0, x)) \arrow[dr, mapsto] & \\
(0, x) \arrow[d, mapsto] & & & (0, (1, x)) \arrow[d, mapsto] \\
(1, x) \arrow[dr, mapsto] & & & (1, (0, x)) \arrow[dl, mapsto] \\
& (1, (1, x)) & (1, (1, x)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (0,(1,y)) \arrow[dl, mapsto] & (0,(1,y)) \arrow[dr, mapsto] & \\
(1,(0,y)) \arrow[d, mapsto] & & & (0,(0,y)) \arrow[d, mapsto] \\
(0,(0,y)) \arrow[dr, mapsto] & & & (0,y) \arrow[dl, mapsto] \\
 & (0,y) & (0,y) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (1,z) \arrow[dl, mapsto] & (1,z) \arrow[dr, mapsto] & \\
(1,(1,z)) \arrow[d, mapsto] & & & (1,z) \arrow[d, mapsto] \\
(0,(1,z)) \arrow[dr, mapsto] & & & (1,(1,z)) \arrow[dl, mapsto] \\
 & (1,(0,z)) & (1,(0,z)) &
\end{tikzcd}
$$

and thus the left hexagon identity is satisfied.

The Right Hexagon Identity: Let X, Y, and Z be sets. We have to show that the diagram

$$
\begin{tikzcd}
 & X \coprod (Y \coprod Z) \arrow[dl, "\left(\alpha^{\mathsf{Sets}}_{X,Y,Z}\right)^{-1}"'] \arrow[dr, "\mathrm{id}_X \coprod \sigma^{\mathsf{Sets}}_{Y,Z}"] & \\
(X \coprod Y) \coprod Z \arrow[d, "\sigma^{\mathsf{Sets}}_{X \coprod Y, Z}"'] & & X \coprod (Z \coprod Y) \arrow[d, "\left(\alpha^{\mathsf{Sets}}_{X,Z,Y}\right)^{-1}"] \\
Z \coprod (X \coprod Y) \arrow[dr, "\left(\alpha^{\mathsf{Sets}}_{Z,X,Y}\right)^{-1}"'] & & (X \coprod Z) \coprod Y \arrow[dl, "\sigma^{\mathsf{Sets}}_{X,Z} \coprod \mathrm{id}_Y"] \\
 & (Z \coprod X) \coprod Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & (0,x) \arrow[dl, mapsto] & (0,x) \arrow[dr, mapsto] & \\
(0,(0,x)) \arrow[d, mapsto] & & & (0,x) \arrow[d, mapsto] \\
(1,(0,x)) \arrow[dr, mapsto] & & & (0,(0,x)) \arrow[dl, mapsto] \\
 & (0,(1,x)) & (0,(1,x)) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (1,(0,y)) \arrow[dl, mapsto] & (1,(0,y)) \arrow[dr, mapsto] & \\
(0,(1,y)) \arrow[d, mapsto] & & & (1,(1,y)) \arrow[d, mapsto] \\
(1,(1,y)) \arrow[dr, mapsto] & & & (1,y) \arrow[dl, mapsto] \\
 & (1,y) & (1,y) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & (1,(1,z)) \arrow[dl, mapsto] & (1,(1,z)) \arrow[dr, mapsto] & \\
(1,z) \arrow[d, mapsto] & & & (1,(0,z)) \arrow[d, mapsto] \\
(0,z) \arrow[dr, mapsto] & & & (0,(1,z)) \arrow[dl, mapsto] \\
 & (0,(0,z)) & (0,(0,z)) &
\end{tikzcd}
$$

and thus the right hexagon identity is satisfied. □

5.3 The Bimonoidal Category of Sets, Products, and Coproducts

5.3.1 The Left Distributor

Definition 5.3.1.1.1. The **left distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_{\ell}^{\mathsf{Sets}} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \textstyle\coprod) \overset{\sim}{\Longrightarrow} \textstyle\coprod \circ (\times \times \times) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$

as in the diagram

$$\begin{tikzcd}
 & (\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[dr, "\boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] & \\
\mathsf{Sets} \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[ur, "\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}})"] \arrow[d, "\mathrm{id}_{\mathsf{Sets}} \times \coprod"'] & & (\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[d, "\times \times \times"] \\
\mathsf{Sets} \times \mathsf{Sets} \arrow[urr, Rightarrow, "\delta_{\ell}^{\mathsf{Sets}}"] \arrow[dr, "\times"'] & & \mathsf{Sets} \times \mathsf{Sets}, \arrow[dl, "\coprod"] \\
 & \mathsf{Sets} &
\end{tikzcd}$$

whose component

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets}} \colon X \times (Y \textstyle\coprod Z) \overset{\sim}{\dashrightarrow} (X \times Y) \textstyle\coprod (X \times Z)$$

at (X, Y, Z) is defined by

$$\delta_{\ell|X,Y,Z}^{\mathsf{Sets}}(x, a) \overset{\text{def}}{=} \begin{cases} (0, (x, y)) & \text{if } a = (0, y), \\ (1, (x, z)) & \text{if } a = (1, z) \end{cases}$$

for each $(x, a) \in X \times (Y \coprod Z)$.

Proof. Omitted. □

5.3.2 The Right Distributor

Definition 5.3.2.1.1. The **right distributor of the product of sets over the coproduct of sets** is the natural isomorphism

$$\delta_r^{\mathsf{Sets}}\colon \times \circ \left(\coprod \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ ((\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$

as in the diagram

$$\begin{tikzcd}
 & & (\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[drr, "\boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] & & \\
(\mathsf{Sets} \times \mathsf{Sets}) \times \mathsf{Sets} \arrow[urr, "(\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}}"] \arrow[d, "\coprod \times \mathrm{id}_{\mathsf{Sets}}"'] & & & & (\mathsf{Sets} \times \mathsf{Sets}) \times (\mathsf{Sets} \times \mathsf{Sets}) \arrow[d, "\times \times \times"] \\
\mathsf{Sets} \times \mathsf{Sets} \arrow[drr, "\times"'] \arrow[urrrr, Rightarrow, "\delta_r^{\mathsf{Sets}}"] & & & & \mathsf{Sets} \times \mathsf{Sets}, \arrow[dll, "\coprod"] \\
 & & \mathsf{Sets} & &
\end{tikzcd}$$

whose component

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z}\colon (X \coprod Y) \times Z \overset{\sim}{\dashrightarrow} (X \times Z) \coprod (Y \times Z)$$

at (X, Y, Z) is defined by

$$\delta^{\mathsf{Sets}}_{r|X,Y,Z}(a, z) \overset{\text{def}}{=} \begin{cases} (0, (x, z)) & \text{if } a = (0, x), \\ (1, (y, z)) & \text{if } a = (1, y) \end{cases}$$

for each $(a, z) \in (X \coprod Y) \times Z$.

Proof. Omitted. □

5.3.3 The Left Annihilator

Definition 5.3.3.1.1. The **left annihilator of the product of sets** is the natural isomorphism

$$\zeta_\ell^{\mathsf{Sets}}\colon \mathbb{0}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ \left(\mathrm{id}_{\mathrm{pt}} \times \boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \times \circ \left(\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$

as in the diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] \arrow[dd, phantom, "\overset{\zeta^{\mathsf{Sets}}_{\ell}}{\Longleftarrow}"] & \\
\mathrm{pt}\times\mathsf{Sets} \arrow[ur, "\mathrm{id}_{\mathrm{pt}}\times\epsilon^{\mathsf{Cats}_2}_{\mathsf{Sets}}"] \arrow[d, "\mathbb{0}^{\mathsf{Sets}}\times\mathrm{id}_{\mathsf{Sets}}"'] & & \mathrm{pt} \arrow[d, "\mathbb{0}^{\mathsf{Sets}}"] \\
\mathsf{Sets}\times\mathsf{Sets} \arrow[rr, "\times"'] & & \mathsf{Sets}
\end{tikzcd}$$

with components

$$\zeta^{\mathsf{Sets}}_{\ell|A}\colon \emptyset\times A \overset{\sim}{\dashrightarrow} \emptyset.$$

Proof. Omitted. For a partial proof, see [Pro25d]. □

5.3.4 The Right Annihilator

Definition 5.3.4.1.1. The **right annihilator of the product of sets** is the natural isomorphism

$$\zeta^{\mathsf{Sets}}_{r}\colon \mathbb{0}^{\mathsf{Sets}}\circ\mu^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}\circ\left(\epsilon^{\mathsf{Cats}_2}_{\mathsf{Sets}}\times\mathrm{id}_{\mathrm{pt}}\right)\overset{\sim}{\dashrightarrow}\times\circ\left(\mathrm{id}_{\mathsf{Sets}}\times\mathbb{0}^{\mathsf{Sets}}\right)$$

as in the diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, "\mu^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}"] \arrow[dd, phantom, "\overset{\zeta^{\mathsf{Sets}}_{r}}{\Longleftarrow}"] & \\
\mathsf{Sets}\times\mathrm{pt} \arrow[ur, "\epsilon^{\mathsf{Cats}_2}_{\mathsf{Sets}}\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_{\mathsf{Sets}}\times\mathbb{0}^{\mathsf{Sets}}"'] & & \mathrm{pt} \arrow[d, "\mathbb{0}^{\mathsf{Sets}}"] \\
\mathsf{Sets}\times\mathsf{Sets} \arrow[rr, "\times"'] & & \mathsf{Sets}
\end{tikzcd}$$

with components

$$\zeta^{\mathsf{Sets}}_{r|A}\colon A\times\emptyset \overset{\sim}{\dashrightarrow} \emptyset.$$

Proof. Omitted. For a partial proof, see [Pro25d]. □

5.3.5 The Bimonoidal Category of Sets, Products, and Coproducts

Proposition 5.3.5.1.1. The category Sets admits a closed symmetric bimonoidal category structure consisting of:

- *The Underlying Category.* The category Sets of pointed sets.
- *The Additive Monoidal Product.* The coproduct functor

$$\coprod \colon \mathsf{Sets} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

  of Item 1 of Definition 4.2.3.1.3.
- *The Multiplicative Monoidal Product.* The product functor

$$\times \colon \mathsf{Sets} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

  of Item 1 of Definition 4.1.3.1.3.
- *The Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}$$

  of Definition 5.1.3.1.1.
- *The Monoidal Zero.* The functor

$$\mathbb{0}^{\mathsf{Sets}} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}$$

  of Definition 5.1.3.1.1.
- *The Internal Hom.* The internal Hom functor

$$\mathsf{Sets} \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \longrightarrow \mathsf{Sets}$$

  of **??** of **??**.
- *The Additive Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\coprod \times \mathrm{id}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \coprod \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \coprod\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

  of Definition 5.2.3.1.1.

- *The Additive Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathbb{0}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \xRightarrow{\sim} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.4.1.1.

- *The Additive Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets},\coprod} \colon \coprod \circ \left(\mathrm{id} \times \mathbb{0}^{\mathsf{Sets}}\right) \xRightarrow{\sim} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.2.5.1.1.

- *The Additive Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets},\coprod} \colon \coprod \xRightarrow{\sim} \coprod \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.2.6.1.1.

- *The Multiplicative Associators.* The natural isomorphism

$$\alpha^{\mathsf{Sets}} \colon \times \circ (\times \times \mathrm{id}_{\mathsf{Sets}}) \xRightarrow{\sim} \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.4.1.1.

- *The Multiplicative Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right) \xRightarrow{\sim} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.5.1.1.

- *The Multiplicative Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}}\right) \xRightarrow{\sim} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}}$$

of Definition 5.1.6.1.1.

- *The Multiplicative Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}} \colon \times \xRightarrow{\sim} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets},\mathsf{Sets}}$$

of Definition 5.1.7.1.1.

- *The Left Distributor.* The natural isomorphism

$$\delta^{\mathsf{Sets}}_{\ell} \colon \times \circ (\mathrm{id}_{\mathsf{Sets}} \times \coprod) \xRightarrow{\sim} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ (\Delta_{\mathsf{Sets}} \times (\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}))$$

of Definition 5.3.1.1.1.

- *The Right Distributor.* The natural isomorphism

$$\delta_r^{\mathsf{Sets}}\colon \times \circ (\coprod \times \mathrm{id}_{\mathsf{Sets}}) \overset{\sim}{\Longrightarrow} \coprod \circ (\times \times \times) \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ ((\mathrm{id}_{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}) \times \Delta_{\mathsf{Sets}})$$

  of Definition 5.3.2.1.1.

- *The Left Annihilator.* The natural isomorphism

$$\zeta_\ell^{\mathsf{Sets}}\colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ \left(\mathrm{id}_{\mathrm{pt}} \times \boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}}\right) \overset{\sim}{\Longrightarrow} \times \circ \left(\mathbb{O}^{\mathsf{Sets}} \times \mathrm{id}_{\mathsf{Sets}}\right)$$

  of Definition 5.3.3.1.1.

- *The Right Annihilator.* The natural isomorphism

$$\zeta_r^{\mathsf{Sets}}\colon \mathbb{O}^{\mathsf{Sets}} \circ \boldsymbol{\mu}^{\mathsf{Cats}_2}_{4|\mathsf{Sets},\mathsf{Sets},\mathsf{Sets},\mathsf{Sets}} \circ \left(\boldsymbol{\epsilon}^{\mathsf{Cats}_2}_{\mathsf{Sets}} \times \mathrm{id}_{\mathrm{pt}}\right) \overset{\sim}{\dashrightarrow} \times \circ \left(\mathrm{id}_{\mathsf{Sets}} \times \mathbb{O}^{\mathsf{Sets}}\right)$$

  of Definition 5.3.4.1.1.

Proof. Omitted. □

Appendices

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Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Chapter 6

Pointed Sets

This chapter contains some foundational material on pointed sets.

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6.1 Pointed Sets

6.1.1 Foundations

Definition 6.1.1.1.1. A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 6.1.1.1.2. In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0] \colon \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

Example 6.1.1.1.3. The 0**-sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \overset{\mathrm{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

Example 6.1.1.1.4. The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\mathrm{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$**-modules**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0

Example 6.1.1.1.5. The **standard pointed set with $n + 1$ elements** is the pointed set $\langle n\rangle$ consisting of

- *The Underlying Set.* The set $\langle n\rangle$ defined by

$$\langle n\rangle \overset{\text{def}}{=} \{*\} \cup \{1, \ldots, n\}.$$

- *The Basepoint.* The element $*$ of $\langle n\rangle$.

6.1.2 Morphisms of Pointed Sets

Definition 6.1.2.1.1. A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

Remark 6.1.2.1.2. In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

6.1.3 The Category of Pointed Sets

Definition 6.1.3.1.1. The **category of pointed sets** is the category Sets_* defined equivalently as:

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**.
- The category Sets_* of **??**.

is also denoted $(\mathbb{F}_1, 0)$.

[4] *Further Terminology:* Also called a **pointed function**.

[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

Remark 6.1.3.1.2. In detail, the **category of pointed sets** is the category Sets_* where:

- *Objects.* The objects of Sets_* are pointed sets.
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets.
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)} \colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

  of Sets_* at (X, x_0) is defined by[6]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \stackrel{\text{def}}{=} \mathrm{id}_X .$$

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y, y_0), (Z, z_0)) \times \mathsf{Sets}_*((X, x_0), (Y, y_0)) \to \mathsf{Sets}_*((X, x_0), (Z, z_0))$$

  of Sets_* at $((X, x_0), (Y, y_0), (Z, z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

6.1.4 Elementary Properties of Pointed Sets

Proposition 6.1.4.1.1. Let (X, x_0) be a pointed set.

1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

   (a) Products, described as in Definition 6.2.3.1.1.

   (b) Pullbacks, described as in Definition 6.2.4.1.1.

[6] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

[7] Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

   (c) Equalisers, described as in Definition 6.2.5.1.1.

2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

   (a) Coproducts, described as in Definition 6.3.3.1.1.

   (b) Pushouts, described as in Definition 6.3.4.1.1;

   (c) Coequalisers, described as in Definition 6.3.5.1.1.

3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*\left(S^0, X\right) \cong X,$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0),$$

   again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

   between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

   (a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

   sends:

[8] The category Sets_* does admit a natural monoidal closed structure, however; see Tensor Products of Pointed Sets.

[9] In other words, the forgetful functor

$$忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.

i. A pointed set (X, x_0) to X.

ii. A pointed function

$$f : (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f : X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \overset{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1} : \mathsf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f : X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1} : (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. *Item 1, Completeness*: This follows from (the proofs) of Definitions 6.2.3.1.1, 6.2.4.1.1 and 6.2.5.1.1 and **??**.

Item 2, Cocompleteness: This follows from (the proofs) of Definitions 6.3.3.1.1, 6.3.4.1.1 and 6.3.5.1.1 and **??**.

Item 3, Failure To Be Cartesian Closed: See [MSE 2855868].

Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.

The isomorphism then

$$\mathbf{Sets}_*(S^0, X) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0} : S^0 \to X$, the basepoint of $\mathbf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.

Item 5, Relation to Partial Functions: See [MSE 884460]. □

6.1.5 Active and Inert Morphisms of Pointed Sets

Definition 6.1.5.1.1. Let $f : (X, x_0) \to (Y, y_0)$ be a morphism of pointed sets.

1. The morphism f is **active** if $f^{-1}(y_0) = x_0$.
2. The morphism f is **inert** if, for each $y \in Y$, the set $f^{-1}(y)$ has exactly one element.

Notation 6.1.5.1.2. We write $\mathrm{Sets}_*^{\mathrm{actv}}$ for the wide subcategory of Sets_* spanned by pointed sets and the active maps between them.

Example 6.1.5.1.3. Here are some examples of active and inert maps of pointed sets.

1. The map $\mu : \langle 2 \rangle \to \langle 1 \rangle$ given by

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 1 \\ 2 \arrow[ur, mapsto] & \\ * \arrow[r, mapsto] & * \end{tikzcd}$$

   is active but not inert.

2. The map $f : \langle 2 \rangle \to \langle 2 \rangle$ given by

$$\begin{tikzcd} 1 \arrow[r, mapsto] & 1 \\ 2 \arrow[dr, mapsto] & 2 \\ * \arrow[r, mapsto] & * \end{tikzcd}$$

   is inert but not active.

3. The map $f\colon \langle 3\rangle \longrightarrow \langle 1\rangle$ given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 \\
2 \arrow[ur, mapsto] & \\
3 \arrow[dr, mapsto] & \\
* \arrow[r, mapsto] & *
\end{tikzcd}$$

is neither inert nor active. However, it factors as $f = a \circ i$, where

$$\begin{aligned} i &\colon \langle 3\rangle \longrightarrow \langle 2\rangle, \\ a &\colon \langle 2\rangle \longrightarrow \langle 1\rangle \end{aligned}$$

are the morphisms of pointed sets given by

$$\begin{tikzcd}
1 \arrow[r, mapsto] & 1 & 1 \arrow[r, mapsto] & 1 \\
2 \arrow[r, mapsto] & 2 & 2 \arrow[ur, mapsto] & \\
3 \arrow[dr, mapsto] & & * \arrow[r, mapsto] & *, \\
* \arrow[r, mapsto] & * & &
\end{tikzcd}$$

with i being inert and a being active.

Proposition 6.1.5.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Active-Inert Factorisation.* Every morphism of pointed sets $f\colon (X, x_0) \longrightarrow (Y, y_0)$ factors uniquely as

$$f = a \circ i,$$

where:

(a) The map $i\colon (X, x_0) \longrightarrow (K, k_0)$ is an inert morphism of pointed sets

(b) The map $a\colon (K, k_0) \longrightarrow (Y, y_0)$ is an active morphism of pointed sets.

Moreover, this determines an orthogonal factorisation system in Sets_*.

Proof. *Item 1, Active-Inert Factorisation*: Let $f\colon X \longrightarrow Y$ be a morphism of pointed sets. We can factor f as

$$X \xrightarrow{i} K \xrightarrow{a} Y,$$

where:

- K is the pointed set given by

$$\begin{aligned} K &= \{x \in X \mid f(x) \neq y_0\} \cup \{x_0\} \\ &= \left(X \setminus f^{-1}(y_0)\right) \cup \{x_0\}; \end{aligned}$$

- $i\colon X \to K$ is the inert morphism of pointed sets given by

$$i(x) \overset{\text{def}}{=} \begin{cases} x & \text{if } x \in K, \\ x_0 & \text{otherwise} \end{cases}$$

  for each $x \in X$;

- $a\colon K \to Y$ is the active morphism of pointed sets given by

$$a(x) \overset{\text{def}}{=} f(x)$$

  for each $x \in K$.

Next, let

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

be a commutative diagram in Sets_*. Consider the morphism $\phi\colon Y \to A$ given by

$$\phi(y) = f\left(i^{-1}(y)\right)$$

for each $y \in Y$ (which is well-defined since, as i is inert, $i^{-1}(y)$ is a singleton for all $y \in Y$). We claim that ϕ is the unique diagonal filler in the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\exists!"', "\phi"] \\ A \arrow[r, "a"'] & B. \end{tikzcd}$$

Indeed, this diagram commutes, as we have

$$\begin{aligned} [\phi \circ i](x) &\overset{\text{def}}{=} \phi(i(x)) \\ &\overset{\text{def}}{=} f\left(i^{-1}(i(x))\right) \\ &= f(x) \end{aligned}$$

for each $x \in X$ and

$$\begin{aligned}[a \circ \phi](y) &\overset{\text{def}}{=} a(\phi(y)) \\ &\overset{\text{def}}{=} a\bigl(f\bigl(i^{-1}(y)\bigr)\bigr) \\ &\overset{\text{def}}{=} [a \circ f]\bigl(i^{-1}(y)\bigr) \\ &= [g \circ i]\bigl(i^{-1}(y)\bigr) \\ &\overset{\text{def}}{=} g\bigl(i\bigl(i^{-1}(y)\bigr)\bigr) \\ &\overset{\text{def}}{=} g(y)\end{aligned}$$

for each $y \in Y$. Moreover, given another morphism ψ such that the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, "g"] \arrow[dl, dashed, "\psi"] \\ A \arrow[r, "a"'] & B \end{tikzcd}$$

commutes, it follows that we must have $\psi = \phi$, since, given $y \in Y$, there exists a unique $x \in X$ such that $i(x) = y$, so we have

$$\begin{aligned}\psi(y) &= \psi(i(x)) \\ &= f(x) \\ &= f\bigl(i^{-1}(y)\bigr) \\ &\overset{\text{def}}{=} \phi(y).\end{aligned}$$

This finishes the proof. □

6.2 Limits of Pointed Sets

6.2.1 The Terminal Pointed Set

Definition 6.2.1.1.1. The **terminal pointed set** is the terminal object of Sets_* as in **??**.

Construction 6.2.1.1.2. Concretely, the **terminal pointed set** is the pair $\bigl((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\bigr)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X \colon (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_X(x) \overset{\text{def}}{=} \star$$

for each $x \in X$ and each $(X, x_0) \in \text{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\text{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad\qquad (\text{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X, x_0) \to (\text{pt}, \star)$$

making the diagram

$$(X, x_0) \xrightarrow[\exists!]{\phi} (\text{pt}, \star)$$

commute, namely $!_X$. □

6.2.2 Products of Families of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.2.2.1.1. The **product of** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ is the product of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_* as in **??**.

Construction 6.2.2.1.2. Concretely, the **product of** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ is the pair $\left(\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right), \left\{\text{pr}_i\right\}_{i\in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$.
- *The Cone.* The collection

  $$\left\{\text{pr}_i\colon \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i\in I}$$

  of maps given by

  $$\text{pr}_i\left(\left(x_j\right)_{j\in I}\right) \overset{\text{def}}{=} x_i$$

  for each $\left(x_j\right)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $\left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ is the categorical product of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} (P,*) \arrow[dr, "p_i"] & \\ \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd} (P,*) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & \left(X_i, x_0^i\right) \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i\in I} \\ &= \left(x_0^i\right)_{i\in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

Proposition 6.2.2.1.3. Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

6.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.2.3.1.1. The **product of** (X, x_0) **and** (Y, y_0) is the product of (X, x_0) and (Y, y_0) in Sets_* as in **??**.

Construction 6.2.3.1.2. Concretely, the **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned} \mathrm{pr}_1 &\colon (X \times Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

  defined by

  $$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} & (P, *) \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_1(*), p_2(*)) \\ &= (x_0, y_0), \end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.3.1.3. Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

   define functors

$$\begin{aligned} A \times - \colon \quad & \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \times B \colon \quad & \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ -_1 \times -_2 \colon \quad & \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \end{aligned}$$

   defined in the same way as the functors of Item 1 of Definition 4.1.3.1.3.

2. *Lack of Adjointness.* The functors $X \times -$ and $- \times Y$ do not admit right adjoints.

3. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0), \\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0), \end{aligned}$$

   natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of limits, **??** of **??**.
Item 2, *Lack of Adjointness*: See [MSE 2855868].
Item 3, *Associativity*: This follows from Item 4 of Definition 4.1.3.1.3.
Item 4, *Unitality*: This follows from Item 5 of Definition 4.1.3.1.3.
Item 5, *Commutativity*: This follows from Item 6 of Definition 4.1.3.1.3.
Item 6, *Symmetric Monoidality*: This follows from Item 14 of Definition 4.1.3.1.3.
□

6.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f: (X, x_0) \to (Z, z_0)$ and $g: (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

Definition 6.2.4.1.1. The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pullback of (X, x_0) and (Y, y_0) over (Z, z_0) along (f, g) in Sets_* as in **??**.

Construction 6.2.4.1.2. Concretely, the **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{pr}_1 &: (X \times_Z Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &: (X \times_Z Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

defined by

$$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} \left[f \circ \mathrm{pr}_1\right](x, y) &= f\left(\mathrm{pr}_1(x, y)\right) \\ &= f(x) \\ &= g(y) \\ &= g\left(\mathrm{pr}_2(x, y)\right) \\ &= \left[g \circ \mathrm{pr}_2\right](x, y), \end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd} (P, *) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\ & (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\ & (X, x_0) \arrow[r, "f"'] & (Z, z_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P, *) \to (X \times_Z Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} (P, *) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] \arrow[rd, dashed, "\phi", "\exists!"'] & & \\ & (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\ & (X, x_0) \arrow[r, "f"'] & (Z, z_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

Proposition 6.2.4.1.3. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$
\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} \left(X' \times_{Z'} Y', \left(x'_0, y'_0\right)\right)
$$

given by

$$
\xi(x, y) \stackrel{\text{def}}{=} (\phi(x), \psi(y))
$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$
\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & Y \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\
& X' \times_{Z'} Y' \arrow[rr, crossing over] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] \arrow[from=uu, crossing over] & & Z'
\end{tikzcd}
$$

commute.

2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[ddrr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
 & W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
 & W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & X \times_W (Y \times_V Z) \arrow[ddll] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
 & W & & V &
\end{tikzcd}$$

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned} X \times_X A &\cong A, \\ A \times_X X &\cong A, \end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

5. *Interaction With Products.* We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}$$

6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This is a special case of functoriality of co/limits,

?? of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.

Item 2, Associativity: This follows from Item 4 of Definition 6.2.4.1.3.

Item 3, Unitality: This follows from Item 6 of Definition 4.1.4.1.5.

Item 4, Commutativity: This follows from Item 7 of Definition 4.1.4.1.5.

Item 5, Interaction With Products: This follows from Item 10 of Definition 4.1.4.1.5.

Item 6, Symmetric Monoidality: This follows from Item 11 of Definition 4.1.4.1.5. □

6.2.5 Equalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.2.5.1.1. The **equaliser of** (f, g) is the equaliser of f and g in Sets_* as in **??**.

Construction 6.2.5.1.2. Concretely, the **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

  $$\mathrm{eq}(f, g)\colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E,*) \to (\mathrm{Eq}(f,g),x_0)$$

making the diagram

$$\begin{tikzcd}
(\mathrm{Eq}(f,g),x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \\
(E,*) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= e(*)\\ &= x_0,\end{aligned}$$

where we have used that e is a morphism of pointed sets. □

Proposition 6.2.5.1.3. Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(g,h), g\circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ \mathrm{eq}(g,h), h\circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))},$$

where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$\begin{tikzcd}
(X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0)
\end{tikzcd}$$

in Sets_*, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Item 1 of Definition 4.1.5.1.3.
Item 2, Unitality: This follows from Item 4 of Definition 4.1.5.1.3.
Item 3, Commutativity: This follows from Item 5 of Definition 4.1.5.1.3. □

6.3 Colimits of Pointed Sets

6.3.1 The Initial Pointed Set

Definition 6.3.1.1.1. The **initial pointed set** is the initial object of Sets_* as in **??**.

Construction 6.3.1.1.2. Concretely, the **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{\iota_X : (\mathrm{pt}, \star) \longrightarrow (X, x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$\iota_X(\star) \stackrel{\mathrm{def}}{=} x_0.$$

Proof. We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi : (\mathrm{pt}, \star) \longrightarrow (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \overset{\phi}{\underset{\exists!}{\dashrightarrow}} (X, x_0)$$

commute, namely ι_X. □

6.3.2 Coproducts of Families of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ be a family of pointed sets.

Definition 6.3.2.1.1. The **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$[11] is the coproduct of $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ in Sets_* as in **??**.

Construction 6.3.2.1.2. Concretely, the **coproduct of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$ is the pair $\left((\bigvee_{i\in I} X_i, p_0), \left\{\mathrm{inj}_i\right\}_{i\in I}\right)$ consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by

    $$\bigvee_{i\in I} X_i \stackrel{\text{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring

    $$\left(i, x_0^i\right) \sim \left(j, x_0^j\right)$$

    for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by

    $$\begin{aligned} p_0 &\stackrel{\text{def}}{=} \left[\left(i, x_0^i\right)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$

    for any $i, j \in I$.
- *The Cocone.* The collection

  $$\left\{\mathrm{inj}_i \colon \left(X_i, x_0^i\right) \to \left(\bigvee_{i\in I} X_i, p_0\right)\right\}_{i\in I}$$

  of morphism of pointed sets given by

  $$\mathrm{inj}_i(x) \stackrel{\text{def}}{=} (i, x)$$

  for each $x \in X_i$ and each $i \in I$.

[11] *Further Terminology:* Also called the **wedge sum of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i\in I}$.

Proof. We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\left\{(X_i, x_0^i)\right\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon \left(\bigvee_{i\in I} X_i, p_0\right) \to (C,*)$$

making the diagram

$$\begin{tikzcd} & (C,*) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_i\left(\left[\left(i, x_0^i\right)\right]\right) \\ &= *, \end{aligned}$$

as ι_i is a morphism of pointed sets. □

Proposition 6.3.2.1.3. Let $\left\{(X_i, x_0^i)\right\}_{i\in I}$ be a family of pointed sets.

1. *Functoriality.* The assignment $\left\{(X_i, x_0^i)\right\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a functor

$$\bigvee_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from **??** of **??**. □

6.3.3 Coproducts

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 6.3.3.1.1. The **coproduct of** (X, x_0) **and** (Y, y_0)[12] is the coproduct of (X, x_0) and (Y, y_0) in Sets_* as in **??**.

Construction 6.3.3.1.2. Concretely, the **coproduct of** (X, x_0) **and** (Y, y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:
  - *The Underlying Set.* The set $X \vee Y$ defined by

$$
\begin{aligned}
(X \vee Y, p_0) &\overset{\text{def}}{=} (X, x_0) \textstyle\coprod (Y, y_0)\\
&\cong \left(X \textstyle\coprod_{\mathrm{pt}} Y, p_0\right)\\
&\cong (X \textstyle\coprod Y/\sim, p_0),
\end{aligned}
$$

$$
\begin{tikzcd}
X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\
X \arrow[u] & \mathrm{pt}, \arrow[l, hook', "{[x_0]}"] \arrow[u, "{[y_0]}"']
\end{tikzcd}
$$

    where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.
  - *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$
\begin{aligned}
p_0 &\overset{\text{def}}{=} [(0, x_0)]\\
&= [(1, y_0)].
\end{aligned}
$$

- *The Cocone.* The morphisms of pointed sets

$$
\begin{aligned}
\mathrm{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0),\\
\mathrm{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),
\end{aligned}
$$

  given by

$$
\begin{aligned}
\mathrm{inj}_1(x) &\overset{\text{def}}{=} [(0, x)],\\
\mathrm{inj}_2(y) &\overset{\text{def}}{=} [(1, y)],
\end{aligned}
$$

  for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0)

[12] *Further Terminology:* Also called the **wedge sum of** (X, x_0) **and** (Y, y_0).

and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (C,*) & \\ (X,x_0) \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & (X \vee Y, p_0) & (Y,y_0) \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"] \end{tikzcd}$$

in Sets. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \vee Y, p_0) \to (C, *)$$

making the diagram

$$\begin{tikzcd} & (C,*) & \\ (X,x_0) \arrow[ur, bend left, "\iota_1"] \arrow[r, hook, "\mathrm{inj}_1"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y,y_0) \arrow[ul, bend right, "\iota_2"'] \arrow[l, hook', "\mathrm{inj}_2"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \phi \circ \mathrm{inj}_X &= \iota_X, \\ \phi \circ \mathrm{inj}_Y &= \iota_Y \end{aligned}$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0,x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1,y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0,x_0)]) \\ &= \iota_Y([(1,y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

Proposition 6.3.3.1.3. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee -\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ - \vee Y\colon \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*, \\ -_1 \vee -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\longrightarrow \mathsf{Sets}_*. \end{aligned}$$

2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned} (\mathrm{pt}, *) \vee (X, x_0) &\cong (X, x_0), \\ (X, x_0) \vee (\mathrm{pt}, *) &\cong (X, x_0), \end{aligned}$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

6. *The Fold Map.* We have a natural transformation

$$\nabla\colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & {} & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X\colon X \vee X \longrightarrow X$$

at X is given by

$$\nabla_X(p) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } p = [(0,x)], \\ x & \text{if } p = [(1,x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Associativity: Omitted.
Item 3, Unitality: Omitted.
Item 4, Commutativity: Omitted.
Item 5, Symmetric Monoidality: Omitted.
Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f\colon (X,x_0) \to (Y,y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i,x)]) &= \nabla_Y([(i,f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i,x)])) \\ &= [f \circ \nabla_X]([(i,x)]) \end{aligned}$$

for each $[(i,x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. □

6.3.4 Pushouts

Let (X,x_0), (Y,y_0), and (Z,z_0) be pointed sets and let $f\colon (Z,z_0) \to (X,x_0)$ and $g\colon (Z,z_0) \to (Y,y_0)$ be morphisms of pointed sets.

Definition 6.3.4.1.1. The **pushout of** (X,x_0) **and** (Y,y_0) **over** (Z,z_0) **along** (f,g) is the pushout of (X,x_0) and (Y,y_0) over (Z,z_0) along (f,g) in Sets_* as in **??**.

Construction 6.3.4.1.2. Concretely, the **pushout of** (X,x_0) **and** (Y,y_0) **over** (Z,z_0) **along** (f,g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\overset{\mathrm{def}}{=} [(0, x)]\\ \mathrm{inj}_2(y) &\overset{\mathrm{def}}{=} [(1, y)]\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}x_0 &= f(z_0),\\ y_0 &= g(z_0)\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.

We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}\left[\mathrm{inj}_1 \circ f\right](z) &= \mathrm{inj}_1(f(z))\\ &= [(0, f(z))]\\ &= [(1, g(z))]\\ &= \mathrm{inj}_2(g(z))\end{aligned}$$

$$= \left[\mathrm{inj}_2 \circ g\right](z),$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[dr, phantom, "\ulcorner", very near start] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists!"', "\phi"] \arrow[dr, phantom, "\ulcorner", very near start] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Definition 4.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)]) \\
&= \iota_1(x_0) \\
&= *,
\end{aligned}$$

or alternatively

$$\begin{aligned}
\phi(p_0) &= \phi([(1, y_0)]) \\
&= \iota_2(y_0) \\
&= *,
\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. □

Proposition 6.3.4.1.3. Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd}
& \bullet \\
\bullet & \bullet. \arrow[l] \arrow[u]
\end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, phantom, "\ulcorner"] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' \arrow[dr, phantom, "\ulcorner"] & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p_0')$$

given by

$$\xi(p) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

3. *Unitality*. We have isomorphisms of sets

$$\begin{tikzcd} A \arrow[r, equal] & A \\ X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, "\ulcorner", very near end] & X \arrow[u, "f"'] \end{tikzcd} \qquad \begin{aligned} X \textstyle\coprod_X A &\cong A, \\ A \textstyle\coprod_X X &\cong A, \end{aligned} \qquad \begin{tikzcd} A & X \arrow[l, "f"'] \\ X \arrow[u, equal] \arrow[ur, phantom, "\ulcorner", very near end] & X. \arrow[l, "f"] \arrow[u, equal] \end{tikzcd}$$

4. *Commutativity*. We have an isomorphism of sets

$$\begin{tikzcd} X \coprod_Z Y & Y \arrow[l] \\ X \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & Z, \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd} \qquad X \textstyle\coprod_Z Y \cong Y \coprod_Z X \qquad \begin{tikzcd} Y \coprod_Z X & X \arrow[l] \\ Y \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & Z. \arrow[l, "g"] \arrow[u, "f"'] \end{tikzcd}$$

5. *Interaction With Coproducts*. We have

$$X \textstyle\coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad \begin{tikzcd} X \vee Y & Y \arrow[l] \\ X \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook, "{[y_0]}"'] \end{tikzcd}$$

6. *Symmetric Monoidality*. The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, Associativity: This follows from Item 3 of Definition 4.2.4.1.6.
Item 3, Unitality: This follows from Item 5 of Definition 4.2.4.1.6.
Item 4, Commutativity: This follows from Item 6 of Definition 4.2.4.1.6.
Item 5, Interaction With Coproducts: Omitted.
Item 6, Symmetric Monoidality: Omitted. □

6.3.5 Coequalisers

Let $f, g \colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

Definition 6.3.5.1.1. The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Construction 6.3.5.1.2. The **coequaliser of** (f, g) is the pair $((\mathrm{CoEq}(f,g), [y_0]), \mathrm{coeq}(f,g))$ consisting of:

- *The Colimit.* The pointed set $(\mathrm{CoEq}(f,g), [y_0])$, where $\mathrm{CoEq}(f,g)$ is the coequaliser of f and g as in Definition 4.2.5.1.1.
- *The Cocone.* The map

  $$\mathrm{coeq}(f,g)\colon Y \twoheadrightarrow (\mathrm{CoEq}(f,g), [y_0])$$

  given by the quotient map, as in Item 2 of Definition 4.2.5.1.2.

Proof. We claim that $(\mathrm{CoEq}(f,g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](x) &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](f(x)) \\
&\overset{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](g(x)) \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](x)
\end{aligned}$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g), [y_0]) \\
& & (C, *)
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Definition 10.6.2.1.3 that there exists a unique map $\phi\colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd}
(X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g), [y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\
& & (C, *)
\end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned}\phi([y_0]) &= [\phi \circ \mathrm{coeq}(f, g)]([y_0])\\ &= c([y_0])\\ &= *,\end{aligned}$$

where we have used that c is a morphism of pointed sets. □

Proposition 6.3.5.1.3. Let (X, x_0) and (Y, y_0) be pointed sets and let $f, g, h\colon (X, x_0) \to (Y, y_0)$ be morphisms of pointed sets.

1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

   where $\mathrm{CoEq}(f, g, h)$ is the colimit of the diagram

$$\begin{tikzcd}(X, x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y, y_0)\end{tikzcd}$$

   in Sets_*.

2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, f) \cong B.$$

3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Item 1 of Definition 4.2.5.1.5.
Item 2, Unitality: This follows from Item 4 of Definition 4.2.5.1.5.
Item 3, Commutativity: This follows from Item 5 of Definition 4.2.5.1.5. □

6.4 Constructions With Pointed Sets

6.4.1 Free Pointed Sets

Let X be a set.

Definition 6.4.1.1.1. The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[13]

$$\begin{aligned} X^+ &\overset{\text{def}}{=} X \coprod \mathrm{pt} \\ &\overset{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 6.4.1.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \longrightarrow \mathsf{Sets}_*,$$

where:

  - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \overset{\text{def}}{=} X^+,$$

  where X^+ is the pointed set of Definition 6.4.1.1.1.

  - *Action on Morphisms.* For each morphism $f \colon X \longrightarrow Y$ of Sets, the image

$$f^+ \colon X^+ \longrightarrow Y^+$$

  of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

2. *Adjointness.* We have an adjunction

$$\left((-)^+ \dashv 忘\right) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*\left((X^+, \star_X), (Y, y_0)\right) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

[13] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity, specially

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Sets}_*, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \overset{\sim}{\dashrightarrow} (X \coprod Y)^+,$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \emptyset^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^+, (-)^+_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^+_{X,Y} \colon X^+ \wedge Y^+ \overset{\sim}{\dashrightarrow} (X \times Y)^+,$$
$$(-)^+_{\mathbb{1}} \colon S^0 \overset{\sim}{\dashrightarrow} \mathrm{pt}^+,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: We claim that $(-)^+$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

$$\mathrm{id}^+_X(x) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } x \in X, \\ \star_X & \text{if } x = \star_X, \end{cases}$$

for each $x \in X^+$, so $\mathrm{id}^+_X = \mathrm{id}_{X^+}$.

- *Preservation of Composition.* Given morphisms of sets

$$f \colon X \to Y,$$
$$g \colon Y \to Z,$$

when there are multiple free pointed sets involved in the current discussion.

we have

$$\begin{aligned}[g^+ \circ f^+](x) &\overset{\text{def}}{=} g^+(f^+(x)) \\ &\overset{\text{def}}{=} g^+(f(x)) \\ &\overset{\text{def}}{=} g(f(x)) \\ &\overset{\text{def}}{=} [g \circ f]^+(x)\end{aligned}$$

for each $x \in X$ and

$$\begin{aligned}[g^+ \circ f^+](\star_X) &\overset{\text{def}}{=} g^+(f^+(\star_X)) \\ &\overset{\text{def}}{=} g^+(\star_Y) \\ &\overset{\text{def}}{=} \star_Z \\ &\overset{\text{def}}{=} [g \circ f]^+(\star_X),\end{aligned}$$

so $(g \circ f)^+ = g^+ \circ f^+$.

This finishes the proof.

Item 2, *Adjointness*: We proceed in a few steps:

- *Map I.* We define a map

  $$\Phi_{X,Y} \colon \mathsf{Sets}_*(X^+, Y) \to \mathsf{Sets}(X, Y)$$

  by sending a morphism of pointed sets

  $$\xi \colon (X^+, \star_X) \to (Y, y_0)$$

  to the function

  $$\xi^\dagger \colon X \to Y$$

  given by

  $$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

  for each $x \in X$.

- *Map II.* We define a map

  $$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*(X^+, Y)$$

  given by sending a function $\xi \colon X \to Y$ to the morphism of pointed sets

  $$\xi^\dagger \colon (X^+, \star_X) \to (Y, y_0)$$

  defined by

  $$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

  for each $x \in X^+$.

- *Invertibility I.* Given a morphism of pointed sets

$$\xi\colon \left(X^{+}, \star_X\right) \to (Y, y_0),$$

we have

$$\begin{aligned}\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Psi_{X,Y}\left(\Phi_{X,Y}(\xi)\right) \\ &= \Psi_{X,Y}\left(\xi^{\dagger}\right) \\ &\stackrel{\text{def}}{=} \left[\!\left[x \mapsto \begin{cases} \xi^{\dagger}(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\right]\!\right] \\ &= \left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\right]\!\right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}_*(X^{+},Y)}\right](\xi).\end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*(X^{+},Y)}\,.$$

- *Invertibility II.* Given a map of sets $\xi\colon X \to Y$, we have

$$\begin{aligned}\left[\Phi_{X,Y} \circ \Psi_{X,Y}\right](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right) \\ &= \Phi_{X,Y}\left(\xi^{\dagger}\right) \\ &= \Phi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X \\ y_0 & \text{if } x = \star_X \end{cases}\right]\!\right]\right) \\ &= \left[\!\left[x \mapsto \xi(x)\right]\!\right] \\ &= \xi \\ &\stackrel{\text{def}}{=} \left[\mathrm{id}_{\mathsf{Sets}(X,Y)}\right](\xi).\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)}\,.$$

- *Naturality for* Φ*, Part I.* We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to \left(X', x_0'\right),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(X'^{,+}, Y) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets $\xi\colon X'^{,+} \longrightarrow Y$, we have

$$\begin{aligned}
\left[\Phi_{X,Y} \circ f^*\right](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= f^*\left(\Phi_{X',Y}(\xi)\right) \\
&= \left[f^* \circ \Phi_{X',Y}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets
$$g\colon (Y, y_0) \to \left(Y', y_0'\right),$$
the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(X^+, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*(X^+, Y'), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi^\dagger\colon X^+ \to Y,$$

we have

$$\begin{aligned}\left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi))\\ &= \Phi_{X,Y'}(g \circ \xi)\\ &= g \circ \xi\\ &= g \circ \Phi_{X,Y'}(\xi)\\ &= g_*(\Phi_{X,Y'}(\xi))\\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi).\end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'}$$

and the naturality diagram for Φ above indeed commutes.

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

This finishes the proof.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^+$ with respect to $\coprod$ and $\vee$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\sim} (X \coprod Y)^+$$

is given by

$$(-)^{+,\coprod}_{X,Y}(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X,\\ y & \text{if } z = [(1,y)] \text{ with } y \in Y,\\ \star_{X\coprod Y} & \text{if } z = [(0,\star_X)],\\ \star_{X\coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$(-)^{+,\coprod,-1}_{X,Y} \colon (X \coprod Y)^+ \xrightarrow{\sim} X^+ \vee Y^+$$

given by

$$(-)^{+,\coprod,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)],\\ [(1,y)] & \text{if } z = [(1,y)],\\ p_0 & \text{if } z = \star_{X\coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\coprod,\mathbb{1}}_{X,Y}\colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \emptyset^{+}$$

  is given by sending $\star_X$ to $\star_{\emptyset}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted.

Item 4, Symmetric Strong Monoidality With Respect to Smash Products: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\times$ and $\wedge$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{+}_{X,Y}\colon X^{+} \wedge Y^{+} \overset{\sim}{\dashrightarrow} (X \times Y)^{+}$$

  is given by

$$(-)^{+}_{X,Y}(x \wedge y) = \begin{cases} (x, y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

  for each $x \wedge y \in X^{+} \wedge Y^{+}$, with inverse

$$(-)^{+,-1}_{X,Y}\colon (X \times Y)^{+} \overset{\sim}{\dashrightarrow} X^{+} \wedge Y^{+}$$

  given by

$$(-)^{+,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x, y) \text{ with } (x, y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

  for each $z \in (X \times Y)^{+}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\mathbb{1}}_{X,Y}\colon S^{0} \overset{\sim}{\dashrightarrow} \mathrm{pt}^{+}$$

  is given by sending 0 to $\star_{\mathrm{pt}}$ and 1 to $\star$, where $\mathrm{pt}^{+} = \{\star, \star_{\mathrm{pt}}\}$.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted. □

6.4.2 Deleting Basepoints

Let (X, x_0) be a pointed set.

Definition 6.4.2.1.1. The **set with deleted basepoint associated to** X is the set X^- defined by

$$X^- \overset{\text{def}}{=} X \setminus \{x_0\}.$$

Proposition 6.4.2.1.2. Let (X, x_0) be a pointed set.

1. *Functoriality.* The assignment $(X, x_0) \mapsto X^-$ defines a functor

$$X^- \colon \mathsf{Sets}_*^{\mathrm{actv}} \longrightarrow \mathsf{Sets},$$

   where:

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*^{\mathrm{actv}})$, we have

   $$[(-)^-](X) \overset{\text{def}}{=} X^-,$$

   where X^- is the set of Definition 6.4.2.1.1.

   - *Action on Morphisms.* For each morphism $f \colon X \longrightarrow Y$ of $\mathsf{Sets}_*^{\mathrm{actv}}$, the image

   $$f^- \colon X^- \longrightarrow Y^-$$

   of f by $(-)^-$ is the map defined by

   $$f^-(x) \overset{\text{def}}{=} f(x)$$

   for each $x \in X^-$.

2. *Adjoint Equivalence.* We have an adjoint equivalence of categories

$$\big((-)^- \dashv (-)^+\big)\colon \quad \begin{tikzcd} \mathsf{Sets}_*^{\mathrm{actv}} \arrow[r, bend left, "(-)^-"] \arrow[r, phantom, "\perp_{\mathrm{eq}}"] & \mathsf{Sets}, \arrow[l, bend left, "(-)^+"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathsf{Sets}(X^-, Y) \cong \mathsf{Sets}_*\big(X, Y^+\big),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$, and by isomorphisms

$$\begin{aligned}(X^-)^+ &\cong X,\\ \big(Y^+\big)^- &\cong Y,\end{aligned}$$

   once again natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $Y \in \mathrm{Obj}(\mathsf{Sets})$.

3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-}, (-)^{-,\vee}, (-)^{-,\vee}_{\mathbb{1}}\right) \colon \left(\mathsf{Sets}^{\mathrm{actv}}_{*}, \vee, \mathrm{pt}\right), \longrightarrow \left(\mathsf{Sets}, \coprod, \emptyset\right),$$

being equipped with isomorphisms of pointed sets

$$(-)^{-,\vee}_{X,Y} \colon X^{-} \coprod Y^{-} \overset{\sim}{\dashrightarrow} (X \vee Y)^{-},$$
$$(-)^{-,\vee}_{\mathbb{1}} \colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^{-},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{-}, (-)^{-,\times}, (-)^{-,\times}_{\mathbb{1}}\right) \colon \left(\mathsf{Sets}^{\mathrm{actv}}_{*}, \wedge, S^{0}\right), \longrightarrow \left(\mathsf{Sets}, \times, \mathrm{pt}\right)$$

being equipped with isomorphisms of pointed sets

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \overset{\sim}{\dashrightarrow} (X \wedge Y)^{-},$$
$$(-)^{-}_{\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^{0}\right)^{-},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: We claim that $(-)^{-}$ is indeed a functor:

- *Preservation of Identities.* Let $X \in \mathrm{Obj}(\mathsf{Sets})$. We have

  $$\mathrm{id}^{-}_{X}(x) \overset{\mathrm{def}}{=} x$$

  for each $x \in X^{-}$, so $\mathrm{id}^{-}_{X} = \mathrm{id}_{X^{-}}$.

- *Preservation of Composition.* Given morphisms of pointed sets

  $$f \colon (X, x_0) \longrightarrow (Y, y_0),$$
  $$g \colon (Y, y_0) \longrightarrow (Z, z_0),$$

  we have

  $$\begin{aligned}[g^{-} \circ f^{-}](x) &\overset{\mathrm{def}}{=} g^{-}(f^{-}(x))\\ &\overset{\mathrm{def}}{=} g^{-}(f(x))\\ &\overset{\mathrm{def}}{=} g(f(x))\\ &\overset{\mathrm{def}}{=} [g \circ f]^{-}(x)\end{aligned}$$

  for each $x \in X$, so $(g \circ f)^{-} = g^{-} \circ f^{-}$.

This finishes the proof.

Item 2, Adjoint Equivalence: We proceed in a few steps:

1. *Map I.* We define a map

$$\Phi_{X,Y}\colon \mathsf{Sets}(X^-, Y) \to \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+)$$

by sending a map $\xi\colon X^- \to Y$ to the active morphism of pointed sets

$$\xi^\dagger\colon X \to Y^+$$

given by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X^-, \\ \star_Y & \text{if } x = x_0, \end{cases}$$

for each $x \in X$, where this morphism is indeed active since $\xi(x) \in Y = Y^+ \setminus \{\star_Y\}$ for all $x \in X^-$.

2. *Map II.* We define a map

$$\Psi_{X,Y}\colon \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+) \to \mathsf{Sets}(X^-, Y)$$

given by sending an active morphism of pointed sets $\xi\colon X \to Y^+$ to the map

$$\xi^\dagger\colon X^- \to Y$$

defined by

$$\xi^\dagger(x) \overset{\mathrm{def}}{=} \xi(x)$$

for each $x \in X^-$, which is indeed well-defined (in that $\xi(x) \in Y$ for all $x \in X^-$) since ξ is active.

3. *Invertibility I.* Given a map of sets $\xi\colon X^- \to Y$, we have

$$\begin{aligned}
\left[\Psi_{X,Y} \circ \Phi_{X,Y}\right](\xi) &\overset{\mathrm{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\
&\overset{\mathrm{def}}{=} \Psi_{X,Y}\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_Y & \text{if } x = x_0 \end{cases}\right]\!\right]\right) \\
&= [\![x \mapsto \xi(x)]\!] \\
&= \xi \\
&= \left[\mathrm{id}_{\mathsf{Sets}(X^-,Y)}\right](\xi).
\end{aligned}$$

Therefore we have

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X^-,Y)}\,.$$

4. *Invertibility II.* Given a morphism of pointed sets

$$\xi\colon (X,x_0)\to \left(Y^+,\star_Y\right),$$

we have

$$\begin{aligned}\left[\Phi_{X,Y}\circ\Psi_{X,Y}\right](\xi)&\overset{\text{def}}{=}\Phi_{X,Y}\left(\Psi_{X,Y}(\xi)\right)\\&=\Phi_{X,Y}(\llbracket x\mapsto \xi(x)\rrbracket)\\&=\llbracket x\mapsto\begin{cases}\xi(x)&\text{if }x\in X^-\\\star_Y&\text{if }x=x_0\end{cases}\rrbracket\\&=\xi\\&=\left[\mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\right](\xi).\end{aligned}$$

Therefore we have

$$\Phi_{X,Y}\circ\Psi_{X,Y}=\mathrm{id}_{\mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)}\,.$$

5. *Naturality for* Φ*, Part I.* We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0)\to\left(X',x_0'\right),$$

the diagram

$$\begin{tikzcd}\mathsf{Sets}\left(X^{\prime,-},Y\right) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X',Y^+) \arrow[d, "f^*"] \\ \mathsf{Sets}_*(X^-,Y) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X,Y^+)\end{tikzcd}$$

commutes. Indeed, given a map of sets $\xi\colon X'\to Y$, we have

$$\begin{aligned}\left[\Phi_{X,Y}\circ f^*\right](\xi)&=\Phi_{X,Y}(f^*(\xi))\\&=\Phi_{X,Y}(\xi\circ f)\\&=\llbracket x\mapsto\begin{cases}\xi(f(x))&\text{if }f(x)\in X^{\prime,-}\\\star_Y&\text{if }f(x)=x_0'\end{cases}\rrbracket\\&=f^*\left(\llbracket x'\mapsto\begin{cases}\xi(x')&\text{if }x'\in X^{\prime,-}\\\star_Y&\text{if }x'=x_0'\end{cases}\rrbracket\right)\\&=f^*\left(\Phi_{X',Y}(\xi)\right)\\&=\left[f^*\circ\Phi_{X',Y}\right](\xi).\end{aligned}$$

Therefore we have

$$\Phi_{X,Y} \circ f^* = f^* \circ \Phi_{X',Y},$$

and the naturality diagram for Φ above indeed commutes.

6. *Naturality for Φ, Part II.* We need to show that, given a morphism of pointed sets

$$g\colon (Y, y_0) \to (Y', y'_0),$$

the diagram

$$\begin{tikzcd} \mathsf{Sets}(X^-, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X, Y^+) \arrow[d, "g_*"] \\ \mathsf{Sets}(X^-, Y') \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}_*^{\mathrm{actv}}(X, Y'^{,+}) \end{tikzcd}$$

commutes. Indeed, given a map of sets $\xi\colon X^- \to Y$, we have

$$\begin{aligned} \left[\Phi_{X,Y'} \circ g_*\right](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\ &= \Phi_{X,Y'}(g \circ \xi) \\ &= \left[\!\left[x \mapsto \begin{cases} g(\xi(x)) & \text{if } x \in X^- \\ \star_{Y'} & \text{if } x = x_0 \end{cases}\right]\!\right] \\ &= g_*\left(\left[\!\left[x \mapsto \begin{cases} \xi(x) & \text{if } x \in X^- \\ \star_{Y} & \text{if } x = x_0 \end{cases}\right]\!\right]\right) \\ &= g_*(\Phi_{X,Y'}(\xi)) \\ &= \left[g_* \circ \Phi_{X,Y'}\right](\xi). \end{aligned}$$

Therefore we have

$$\Phi_{X,Y'} \circ g_* = g_* \circ \Phi_{X,Y'},$$

and the naturality diagram for Φ above indeed commutes.

7. *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Definition 11.9.7.1.2 that Ψ is also natural in each argument.

8. *Fully Faithfulness of $(-)^-$.* We aim to show that the assignment $f \mapsto f^-$ sets up a bijection

$$(-)^-_{X,Y}\colon \mathsf{Sets}_*^{\mathrm{actv}}(X, Y) \xrightarrow{\sim} \mathsf{Sets}(X^-, Y^-).$$

Indeed, the inverse map

$$(-)^{-,-1}_{X,Y} \colon \mathsf{Sets}(X^-, Y^-) \overset{\sim}{\dashrightarrow} \mathsf{Sets}^{\mathrm{actv}}_*(X, Y)$$

is given by sending a map of sets $f \colon X^- \to Y^-$ to the active morphism of pointed sets $f^\dagger \colon X \to Y$ defined by

$$f^\dagger(x) \overset{\mathrm{def}}{=} \begin{cases} f(x) & \text{if } x \in X^-, \\ y_0 & \text{if } x = x_0 \end{cases}$$

for each $x \in X$.

9. *Essential Surjectivity of* $(-)^-$. We need to show that, given an object $X \in \mathrm{Obj}(\mathsf{Sets})$, there exists some $X' \in \mathrm{Obj}\big(\mathsf{Sets}^{\mathrm{actv}}_*\big)$ such that $(X')^- \cong X$. Indeed, taking $X' = X^+$, we have

$$\begin{aligned} \big(X^+\big)^- &\overset{\mathrm{def}}{=} (X \cup \{\star_X\})^- \\ &\overset{\mathrm{def}}{=} (X \cup \{\star_X\}) \setminus \{\star_X\} \\ &= X, \end{aligned}$$

and thus we have in fact an *equality* $(X^+)^- = X$, showing $(-)^-$ to be essentially surjective.

10. *The Functor* $(-)^-$ *Is an Equivalence.* Since $(-)^-$ is fully faithful and essentially surjective, it is an equivalence by Item 1 of Definition 11.6.7.1.2.

This finishes the proof.
Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: We construct the strong monoidal structure on $(-)^-$ with respect to $\vee$ and $\coprod$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-,\vee}_{X,Y} \colon X^- \coprod Y^- \overset{\sim}{\dashrightarrow} (X \vee Y)^-$$

is given by

$$(-)^{-,\vee}_{X,Y}(z) = \begin{cases} [(0,x)] & \text{if } z = (0,x) \text{ with } x \in X, \\ [(1,y)] & \text{if } z = (1,y) \text{ with } y \in Y \end{cases}$$

for each $z \in X^- \coprod Y^-$, with inverse

$$(-)^{-,\vee,-1}_{X,Y} \colon (X \vee Y)^- \overset{\sim}{\dashrightarrow} X^- \coprod Y^-$$

given by

$$(-)^{-,\vee,-1}_{X,Y}(z) \overset{\text{def}}{=} \begin{cases} (0,x) & \text{if } z = [(0,x)], \\ (1,y) & \text{if } z = [(1,y)], \end{cases}$$

for each $z \in (X \vee Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{+,\vee,\mathbb{1}}_{X,Y} \colon \emptyset \overset{\sim}{\dashrightarrow} \mathrm{pt}^{-}$$

  is an equality.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{-}$ into a symmetric strong monoidal functor is omitted.

Item 4, *Symmetric Strong Monoidality With Respect to Smash Products*: We construct the strong monoidal structure on $(-)^{+}$ with respect to $\wedge$ and $\times$ as follows:

- *The Strong Monoidality Constraints.* The isomorphism

$$(-)^{-}_{X,Y} \colon X^{-} \times Y^{-} \overset{\sim}{\dashrightarrow} (X \wedge Y)^{-}$$

  is given by

$$(-)^{-}_{X,Y}(x,y) = x \wedge y$$

  for each $(x,y) \in X^{-} \times Y^{-}$, with inverse

$$(-)^{-,-1}_{X,Y} \colon (X \wedge Y)^{-} \overset{\sim}{\dashrightarrow} X^{-} \times Y^{-}$$

  given by

$$(-)^{-,-1}_{X,Y}(x \wedge y) \overset{\text{def}}{=} (x,y)$$

  for each $x \wedge y \in (X \wedge Y)^{-}$.

- *The Strong Monoidal Unity Constraint.* The isomorphism

$$(-)^{-,\mathbb{1}}_{X,Y} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \left(S^{0}\right)^{-}$$

  is given by sending $\star$ to 1.

The verification that these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ into a symmetric strong monoidal functor is omitted. □

Appendices

6.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

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3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Chapter 7

Tensor Products of Pointed Sets

In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

introduced in Section 7.5.1, defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f\colon X \wedge Y \to Z$.
- Maps of sets $f\colon X \times Y \to Z$ satisfying

  $$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ \rhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to $\wedge$, which turns out to endow Sets_* with a monoidal category structure (Definition 7.5.9.1.1), these do not admit invertible associators and unitors, but do endow Sets_* with the structure of a skew monoidal category, however (Definitions 7.3.8.1.1 and 7.4.8.1.1).

Finally, in addition to the tensor products $\lhd$, $\rhd$, and $\wedge$, we also have a "tensor product" of the form

$$\odot\colon \mathsf{Sets}\times\mathsf{Sets}_*\to\mathsf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned}\otimes_{k,\ell}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets})\times\mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets}),\\ \lhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets})\times\mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\\ \rhd_{i,k}\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets})\times\mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) &\to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\end{aligned}$$

where $k,\ell,i\in\mathbb{N}$ with $i\leq k-1$. Together with the Cartesian product $\times$ of Sets, the tensor products studied in this chapter form the cases:

- $(k,\ell)=(-1,-1)$ for the Cartesian product of Sets;
- $(k,\ell)=(0,-1)$ and $(-1,0)$ for the tensor of sets with pointed sets of Definition 7.2.1.1.1;
- $(i,k)=(-1,0)$ for the left and right tensor products of pointed sets of Sections 7.3 and 7.4;
- $(k,\ell)=(-1,-1)$ for the smash product of pointed sets of Section 7.5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in **??**, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k,\ell\leq 2$, and hence *all* cases since $\mathbb{E}_k$-monoids on Sets are the same as $\mathbb{E}_2$-monoids on Sets when $k\geq 2$.

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7.1 Bilinear Morphisms of Pointed Sets

7.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 7.1.1.1.1. A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[1,2]

(★) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

[1] *Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.
[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

Definition 7.1.1.1.2. The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

7.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 7.1.2.1.1. A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

satisfying the following condition:[3,4]

(★) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 7.1.2.1.2. The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

for each $y \in Y$.

[3] *Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

7.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

Definition 7.1.3.1.1. A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : X \times Y \to Z$$

that is both left bilinear and right bilinear.

Remark 7.1.3.1.2. In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f : (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[5,6]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

for each $x \in X$.

[5] *Slogan:* The map f is bilinear if it preserves basepoints in each argument.

[6] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

2. *Right Unital Bilinearity.* The diagram

$$
\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times\mathrm{pt} \arrow[ur, "\epsilon_X\times\mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X\times{[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}
$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

Definition 7.1.3.1.3. The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

7.2 Tensors and Cotensors of Pointed Sets by Sets

7.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 7.2.1.1.1. The **tensor of** (X, x_0) **by** A[7] is the tensor $A \odot (X, x_0)$[8] of (X, x_0) by A as in **??**.

Remark 7.2.1.1.2. In detail, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ satisfying the following universal property:

(★) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

This universal property is in turn equivalent to the following one:

[7] *Further Terminology:* Also called the **copower of** (X, x_0) **by** A.
[8] *Further Notation:* Often written $A \odot X$ for simplicity.

(★) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \;\middle|\; \begin{array}{l}\text{for each } a \in A, \text{ we} \\ \text{have } f(a, x_0) = k_0\end{array} \right\}.$$

Proof. We claim that we have a bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}(X, K)) \cong \mathsf{Sets}(A \times X, K)$$

of Item 2 of Definition 4.1.3.1.3:

- A map
$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$
in $\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$ gets sent to the map
$$\xi^\dagger \colon A \times X \to K$$
defined by
$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$
for each $(a, x) \in A \times X$, which indeed lies in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$, as we have
$$\begin{aligned} \xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &\stackrel{\text{def}}{=} k_0 \end{aligned}$$
for each $a \in A$, where we have used that $\xi_a \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map
$$\xi\colon A\times X\to K$$
in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A\times X,K)$ gets sent to the map
$$\begin{aligned}\xi^{\dagger}\colon A&\longrightarrow \mathsf{Sets}_*(X,K),\\ a&\longmapsto \left(\xi^{\dagger}_a\colon X\to K\right),\end{aligned}$$
where
$$\xi^{\dagger}_a\colon X\to K$$
is the map defined by
$$\xi^{\dagger}_a(x)\overset{\text{def}}{=}\xi(a,x)$$
for each $x\in X$, and indeed lies in $\mathsf{Sets}_*(X,K)$, as we have
$$\begin{aligned}\xi^{\dagger}_a(x_0)&\overset{\text{def}}{=}\xi(a,x_0)\\ &\overset{\text{def}}{=}k_0.\end{aligned}$$

This finishes the proof. □

Construction 7.2.1.1.3. Concretely, the **tensor of** (X,x_0) **by** A is the pointed set $A\odot(X,x_0)$ consisting of:

- *The Underlying Set.* The set $A\odot X$ given by
$$A\odot X\cong\bigvee_{a\in A}(X,x_0),$$
where $\bigvee_{a\in A}(X,x_0)$ is the wedge product of the A-indexed family $((X,x_0))_{a\in A}$ of Definition 6.3.2.1.1.
- *The Basepoint.* The point $[(a,x_0)]=[(a',x_0)]$ of $\bigvee_{a\in A}(X,x_0)$.

Proof. (Proven below in a bit.) □

Notation 7.2.1.1.4. We write $a\odot x$ for the element $[(a,x)]$ of
$$\begin{aligned}A\odot X&\cong\bigvee_{a\in A}(X,x_0)\\ &\overset{\text{def}}{=}\left(\coprod_{i\in I}X_i\right)\Big/\sim.\end{aligned}$$

Remark 7.2.1.1.5. Taking the tensor of any element of A with the base-point x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation $\sim$ on

$$\bigvee_{a \in A}(X, x_0) \stackrel{\text{def}}{=} \coprod_{a \in A} X/\sim$$

identifying (a, x_0) with (a', x_0), so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Map I.* We define a map

   $$\Phi_K \colon \mathsf{Sets}_*(A \odot X, K) \to \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

   by sending a morphism of pointed sets

   $$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

   to the map of sets

   $$\begin{aligned} \xi^\dagger \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

   where

   $$\xi_a \colon (X, x_0) \to (K, k_0)$$

   is the morphism of pointed sets defined by

   $$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

   for each $x \in X$. Note that we have

   $$\begin{aligned} \xi_a(x_0) &\stackrel{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0, \end{aligned}$$

   so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

2. *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \to \mathsf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \xi^\dagger(a \odot x_0) &\stackrel{\text{def}}{=} \xi_a(x_0) \\ &= k_0, \end{aligned}$$

where we have used that $\xi(a) \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

3. *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(A \odot X, K)} \, .$$

Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned} [\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \llbracket x \mapsto \xi(a \odot x) \rrbracket \rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket) \\ &= \llbracket a \odot x \mapsto \mathrm{ev}_x(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x') \rrbracket \rrbracket)) \rrbracket \\ &= \llbracket a \odot x \mapsto \mathrm{ev}_x(\llbracket x' \mapsto \xi(a \odot x') \rrbracket) \rrbracket \\ &= \llbracket a \odot x \mapsto \xi(a \odot x) \rrbracket \\ &= \xi. \end{aligned}$$

4. *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(X,K))}.$$

Indeed, given a morphism $\xi \colon A \to \mathsf{Sets}_*(X,K)$, we have

$$\begin{aligned}[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\ &= \Phi_K(\llbracket a \odot x \mapsto \xi_a(x) \rrbracket) \\ &= \llbracket a \mapsto \llbracket x \mapsto \xi_a(x) \rrbracket \rrbracket \\ &= \llbracket a \mapsto \xi(a) \rrbracket \\ &= \xi.\end{aligned}$$

5. *Naturality of Φ.* We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k'_0),$$

the diagram

$$\begin{tikzcd} \mathsf{Sets}_*(A \odot X, K) \arrow[r, "\Phi_K"] \arrow[d, "\phi_*"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X,K)) \arrow[d, "(\phi_*)_*"] \\ \mathsf{Sets}_*(A \odot X, K') \arrow[r, "\Phi_{K'}"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X,K')) \end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned}[\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\ &= \Phi_{K'}(\phi \circ \xi) \\ &= (\phi \circ \xi)^\dagger \\ &= \llbracket a \mapsto \phi \circ \xi(a \odot -) \rrbracket \\ &= \llbracket a \mapsto \phi_*(\xi(a \odot -)) \rrbracket \\ &= (\phi_*)_*(\llbracket a \mapsto \xi(a \odot - \rrbracket)) \\ &= (\phi_*)_*(\Phi_K(\xi)) \\ &= \left[(\phi_*)_* \circ \Phi_K\right](\xi).\end{aligned}$$

6. *Naturality of Ψ.* Since Φ is natural and Φ is a componentwise inverse to Ψ, it follows from Item 2 of Definition 11.9.7.1.2 that Ψ is also natural.

This finishes the proof. $\square$

Proposition 7.2.1.1.6. Let (X, x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X, x_0), $(A, (X, x_0))$ define functors
$$\begin{aligned} A \odot -\colon &\quad \mathsf{Sets}_* \quad \to \mathsf{Sets}_*, \\ - \odot X\colon &\quad \mathsf{Sets} \quad \to \mathsf{Sets}_*, \\ -_1 \odot -_2\colon &\ \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$
In particular, given:
   - A map of sets $f\colon A \to B$;
   - A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

   the induced map
$$f \odot \phi\colon A \odot X \to B \odot Y$$
is given by
$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$
for each $a \odot x \in A \odot X$.
2. *Adjointness I.* We have an adjunction
$$\left(- \odot X \dashv \mathsf{Sets}_*(X, -)\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "-\odot X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X,-)"] \end{tikzcd}$$
witnessed by a bijection
$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.
3. *Adjointness II.* We have an adjunctions
$$\left(A \odot - \dashv A \pitchfork -\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork-"] \end{tikzcd}$$
witnessed by a bijection
$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Colimit.* We have

$$A \odot X \cong \operatorname{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Interaction With Homs.* We have a natural isomorphism

$$\mathsf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathsf{Sets}_*(X, -).$$

7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\odot}_{X,Y}\colon \mathsf{Sets}_*(X, Y) \odot X \to Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\odot}_{X,Y}(f \odot x) \overset{\mathrm{def}}{=} f(x)$$

for each $f \odot x \in \mathsf{Sets}_*(X, Y) \odot X$.

8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$ and each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{coev}^{\odot}_{A,X}\colon A \to \mathsf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{coev}^{\odot}_{A,X}(a) \overset{\mathrm{def}}{=} [\![x \mapsto a \odot x]\!]$$

for each $a \in A$.

Proof. *Item 1*, *Functoriality*: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, *Adjointness I*: This is simply a rephrasing of Definition 7.2.1.1.1.
Item 3, *: Adjointness II*: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, *As a Weighted Colimit*: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.

Item 5, Iterated Tensors: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, Interaction With Homs: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, The Tensor Evaluation Map: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, The Tensor Coevaluation Map: This is the special case of **??** of **??** for $\mathcal{C} = \mathsf{Sets}_*$. □

7.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

Definition 7.2.2.1.1. The **cotensor of** (X, x_0) **by** A[9] is the cotensor $A \pitchfork (X, x_0)$[10] of (X, x_0) by A as in **??**.

Remark 7.2.2.1.2. In detail, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ satisfying the following universal property:

(★) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

This universal property is in turn equivalent to the following one:

(★) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \overset{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right\}.$$

Proof. This follows from the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ constructed in the proof of Definition 7.2.1.1.2. □

[9] *Further Terminology:* Also called the **power of** (X, x_0) **by** A.
[10] *Further Notation:* Often written $A \pitchfork X$ for simplicity.

Construction 7.2.2.1.3. Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \pitchfork X$ given by

$$A \pitchfork X \cong \bigwedge_{a \in A} (X, x_0),$$

  where $\bigwedge_{a \in A}(X, x_0)$ is the smash product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 7.6.1.1.1.

- *The Basepoint.* The point $[(x_0)_{a \in A}] = [(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a \in A}(X, x_0)$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Map I.* We define a map

$$\Phi_K \colon \mathsf{Sets}_*(K, A \pitchfork X) \to \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

   by sending a morphism of pointed sets

$$\xi \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a \in A}])$$

   to the map of sets

$$\begin{aligned} \xi^{\dagger} \colon A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a \colon K \to X), \end{aligned}$$

   where

$$\xi_a \colon (K, k_0) \to (X, x_0)$$

   is the morphism of pointed sets defined by

$$\xi_a(k) = \begin{cases} x^k_a & \text{if } \xi(k) \neq [(x_0)_{a \in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a \in A}] \end{cases}$$

   for each $k \in K$, where x^k_a is the ath component of $\xi(k) = \left[\left(x^k_a\right)_{a \in A}\right]$. Note that:

(a) The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have

$$\begin{aligned}\xi(k) &= \left[\left(x_a^k\right)_{a\in A}\right]\\ &= \left[\left(y_a^k\right)_{a\in A}\right]\end{aligned}$$

with $x_a^k \neq y_a^k$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x_{a_x}^k = y_{a_y}^k = x_0$. The equivalence relation $\sim$ on $\prod_{a\in A} X$ then forces

$$\begin{aligned}\left[\left(x_a^k\right)_{a\in A}\right] &= [(x_0)_{a\in A}],\\ \left[\left(y_a^k\right)_{a\in A}\right] &= [(x_0)_{a\in A}],\end{aligned}$$

however, and $\xi_a(k)$ is defined to be x_0 in this case.

(b) The map ξ_a is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since $\xi(k_0) = [(x_0)_{a\in A}]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the ath component of $[(x_0)_{a\in A}]$, is equal to x_0.

2. *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(K,X)) \to \mathsf{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned}\xi \colon A &\longrightarrow \mathsf{Sets}_*(K,X),\\ a &\longmapsto (\xi_a \colon K \to X),\end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

defined by

$$\xi^\dagger(k) \overset{\text{def}}{=} \left[(\xi_a(k))_{a\in A}\right]$$

for each $k \in K$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\xi^\dagger(k_0) &\overset{\text{def}}{=} \left[(\xi_a(k_0))_{a\in A}\right]\\ &= x_0,\end{aligned}$$

where we have used that $\xi_a \in \mathsf{Sets}_*(K,X)$ is a morphism of pointed sets for each $a \in A$.

3. *Invertibility I.* We claim that
$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(K, A \pitchfork X)}\,.$$
Indeed, given a morphism of pointed sets
$$\xi\colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$
we have
$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi))\\ &= \Psi_K([\![a \mapsto \xi_a]\!])\\ &= \Psi_K([\![a' \mapsto \xi_{a'}]\!])\\ &= [\![k \mapsto \left[(\mathrm{ev}_a([\![a' \mapsto \xi_{a'}(k)]\!]))_{a\in A}\right]]\!]\\ &= [\![k \mapsto \left[(\xi_a(k))_{a\in A}\right]]\!].\end{aligned}$$
Now, we have two cases:

(a) If $\xi(k) = [(x_0)_{a\in A}]$, we have
$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= [\![k \mapsto \left[(\xi_a(k))_{a\in A}\right]]\!]\\ &= [\![k \mapsto [(x_0)_{a\in A}]]\!]\\ &= [\![k \mapsto \xi(k)]\!]\\ &= \xi.\end{aligned}$$

(b) If $\xi(k) \neq [(x_0)_{a\in A}]$ and $\xi(k) = \left[\left(x^k_a\right)_{a\in A}\right]$ instead, we have
$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= [\![k \mapsto \left[(\xi_a(k))_{a\in A}\right]]\!]\\ &= [\![k \mapsto \left[\left(x^k_a\right)_{a\in A}\right]]\!]\\ &= [\![k \mapsto \xi(k)]\!]\\ &= \xi.\end{aligned}$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

4. *Invertibility II.* We claim that
$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A, \mathsf{Sets}_*(K,X))}\,.$$
Indeed, given a morphism $\xi\colon A \to \mathsf{Sets}_*(K, X)$, we have
$$\begin{aligned}[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi))\\ &= \Phi_K\left([\![k \mapsto \left[(\xi_a(k))_{a\in A}\right]]\!]\right)\\ &= [\![a \mapsto [\![k \mapsto \xi_a(k)]\!]]\!]\\ &= \xi\end{aligned}$$

5. *Naturality of* Ψ. We need to show that, given a morphism of pointed sets
$$\phi\colon (K,k_0)\to(K',k'_0),$$
the diagram
$$\begin{tikzcd}
\mathsf{Sets}(A,\mathsf{Sets}_*(K',X)) \arrow[r, "\Psi_{K'}"] \arrow[d, "(\phi^*)_*"'] & \mathsf{Sets}_*(K',A\pitchfork X) \arrow[d, "\phi^*"] \\
\mathsf{Sets}(A,\mathsf{Sets}_*(K,X)) \arrow[r, "\Psi_K"'] & \mathsf{Sets}_*(K,A\pitchfork X)
\end{tikzcd}$$
commutes. Indeed, given a map of sets
$$\begin{aligned}
\xi\colon A &\longrightarrow \mathsf{Sets}_*(K',X),\\
a &\longmapsto (\xi_a\colon K'\to X),
\end{aligned}$$
we have
$$\begin{aligned}
\left[\Psi_K\circ(\phi^*)_*\right](\xi) &= \Psi_K\big((\phi^*)_*(\xi)\big)\\
&= \Psi_K\big((\phi^*)_*(\llbracket a\mapsto \xi_a\rrbracket)\big)\\
&= \Psi_K((\llbracket a\mapsto \phi^*(\xi_a)\rrbracket))\\
&= \Psi_K((\llbracket a\mapsto \llbracket k\mapsto \xi_a(\phi(k))\rrbracket\rrbracket))\\
&= \llbracket k\mapsto \left[(\xi_a(\phi(k)))_{a\in A}\right]\rrbracket\\
&= \phi^*(\llbracket k'\mapsto \left[(\xi_a(k'))_{a\in A}\right]\rrbracket)\\
&= \phi^*(\Psi_{K'}(\xi))\\
&= [\phi^*\circ\Psi_{K'}](\xi).
\end{aligned}$$

6. *Naturality of* Φ. Since Ψ is natural and Ψ is a componentwise inverse to Φ, it follows from Item 2 of Definition 11.9.7.1.2 that Φ is also natural.

This finishes the proof. □

Proposition 7.2.2.1.4. Let (X,x_0) be a pointed set and let A be a set.

1. *Functoriality.* The assignments A, (X,x_0), $(A,(X,x_0))$ define functors
$$\begin{aligned}
A\pitchfork -\colon\quad & \mathsf{Sets}_* &&\to \mathsf{Sets}_*,\\
-\pitchfork X\colon\quad & \mathsf{Sets}^{\mathsf{op}} &&\to \mathsf{Sets}_*,\\
-_1\pitchfork -_2\colon\quad & \mathsf{Sets}^{\mathsf{op}}\times\mathsf{Sets}_* &&\to \mathsf{Sets}_*.
\end{aligned}$$
In particular, given:

- A map of sets $f : A \to B$;
- A pointed map $\phi : (X, x_0) \to (Y, y_0)$;

the induced map

$$f \odot \phi : A \pitchfork X \to B \pitchfork Y$$

is given by

$$[f \odot \phi]([(x_a)_{a \in A}]) \stackrel{\text{def}}{=} \left[\left(\phi\left(x_{f(a)}\right)\right)_{a \in A}\right]$$

for each $[(x_a)_{a \in A}] \in A \pitchfork X$.

2. *Adjointness I.* We have an adjunction

$$(- \pitchfork X \dashv \mathsf{Sets}_*(-, X)) : \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "-\pitchfork X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(-,X)"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_*^{\mathsf{op}}(A \pitchfork X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

i.e. by a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -) : \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork -"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. *As a Weighted Limit.* We have

$$A \pitchfork X \cong \lim^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \mathsf{Sets}_*(X, -) &\cong \mathsf{Sets}_*(A \odot X, -),\\ A \pitchfork \mathsf{Sets}_*(-, Y) &\cong \mathsf{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

7. *The Cotensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\pitchfork}_{X,Y}\colon X \to \mathsf{Sets}_*(X, Y) \pitchfork Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\pitchfork}_{X,Y}(x) \overset{\mathrm{def}}{=} \left[(f(x))_{f \in \mathsf{Sets}_*(X,Y)}\right]$$

for each $x \in X$.

8. *The Cotensor Coevaluation Map.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have a map

$$\mathrm{coev}^{\pitchfork}_{A,X}\colon A \to \mathsf{Sets}_*(A \pitchfork X, X),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $A \in \mathrm{Obj}(\mathsf{Sets})$, and given by

$$\mathrm{coev}^{\pitchfork}_{A,X}(a) \overset{\mathrm{def}}{=} [\![[(x_b)_{b \in A}] \mapsto x_a]\!]$$

for each $a \in A$.

Proof. *Item 1, Functoriality*: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$.
Item 2, Adjointness I: This is simply a rephrasing of Definition 7.2.2.1.1.
Item 3, : Adjointness II: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$.
Item 4, As a Weighted Limit: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$.
Item 5, Iterated Cotensors: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$.
Item 6, Commutativity With Homs: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$.
Item 7, The Cotensor Evaluation Map: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$.
Item 8, The Cotensor Coevaluation Map: This is the special case of **??** of **??** for $C = \mathsf{Sets}_*$. □

7.3 The Left Tensor Product of Pointed Sets

7.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.3.1.1.1. The **left tensor product of pointed sets** is the functor[11]

$$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}} \colon \mathsf{Sets}_* \times \mathsf{Sets} \overset{\sim}{\dashrightarrow} \mathsf{Sets} \times \mathsf{Sets}_*$ is the braiding of Cats_2, i.e. the functor witnessing the isomorphism

$$\mathsf{Sets}_* \times \mathsf{Sets} \cong \mathsf{Sets} \times \mathsf{Sets}_*.$$

- $\odot \colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Definition 7.2.1.1.6.

Remark 7.3.1.1.2. The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f \colon X \lhd Y \to Z$.
2. Maps of sets $f \colon X \times Y \to Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

Remark 7.3.1.1.3. The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \lhd Y, x_0 \lhd y_0), \iota)$ consisting of

[11] *Further Notation:* Also written $\lhd_{\mathsf{Sets}_*}$.

– A pointed set $(X \lhd Y, x_0 \lhd y_0)$;
– A left bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

satisfying the following universal property:

($\star$) Given another such pair $((Z, z_0), f)$ consisting of

* A pointed set (Z, z_0);
* A left bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

there exists a unique morphism of pointed sets $X \lhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd}
 & X \lhd Y \arrow[d, dashed, "\exists!"] \\
X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z
\end{tikzcd}$$

commute.

Construction 7.3.1.1.4. In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd Y, [x_0])$ consisting of:

- *The Underlying Set.* The set $X \lhd Y$ defined by

$$\begin{aligned} X \lhd Y &\overset{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0), \end{aligned}$$

  where $|Y|$ denotes the underlying set of (Y, y_0).

- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y \in Y}(X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

Proof. Since $\bigvee_{y \in Y}(X, x_0)$ is defined as the quotient of $\coprod_{y \in Y} X$ by the equivalence relation R generated by declaring $(y, x) \sim (y', x')$ if $x = x' = x_0$, we have, by **??**, a natural bijection

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}\left(\coprod_{y \in Y} X, Z\right),$$

where $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^R_{\mathsf{Sets}}\left(\coprod_{y\in Y} X, Z\right) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}\left(\coprod_{y\in Y} X, Z\right) \,\middle|\, \begin{array}{l}\text{for each } x, y \in X, \text{ if}\\ (y,x) \sim_R (y',x'), \text{ then}\\ f(y,x) = f(y',x')\end{array} \right\}.$$

However, the condition $(y, x) \sim_R (y', x')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. We have $x = x' = x_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}\left(\coprod_{y\in Y} X, Z\right)$ with a corresponding $\overline{f}\colon X \lhd Y \to Z$, the latter case above implies

$$\begin{aligned} f([(y, x_0)]) &= f([(y', x_0)]) \\ &= f([(y_0, x_0)]), \end{aligned}$$

and since $\overline{f}\colon X \lhd Y \to Z$ is a pointed map, we have

$$\begin{aligned} f([(y_0, x_0)]) &= \overline{f}([(y_0, x_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f\colon X \times Y \to Z$ satisfying the equality

$$f(x_0, y) = z_0$$

for each $y \in Y$, giving an equality

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. $\square$

Notation 7.3.1.1.5. We write[12] $x \lhd y$ for the element $[(y, x)]$ of

$$X \lhd Y \cong |Y| \odot X.$$

[12] *Further Notation:* Also written $x \lhd_{\mathsf{Sets}_*} y$.

Remark 7.3.1.1.6. Employing the notation introduced in Definition 7.3.1.1.5, we have

$$x_0 \lhd y_0 = x_0 \lhd y$$

for each $y \in Y$, and

$$x_0 \lhd y = x_0 \lhd y'$$

for each $y, y' \in Y$.

Proposition 7.3.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd Y$ define functors

$$\begin{aligned} X \lhd -\colon & \quad \mathsf{Sets}_* \quad \longrightarrow \mathsf{Sets}_*, \\ - \lhd Y\colon & \quad \mathsf{Sets}_* \quad \longrightarrow \mathsf{Sets}_*, \\ -_1 \lhd -_2\colon & \; \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\longrightarrow (A, a_0), \\ g\colon (Y, y_0) &\longrightarrow (B, b_0), \end{aligned}$$

   the induced map

$$f \lhd g\colon X \lhd Y \longrightarrow A \lhd B$$

   is given by

$$[f \lhd g](x \lhd y) \stackrel{\text{def}}{=} f(x) \lhd g(y)$$

   for each $x \lhd y \in X \lhd Y$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 7.3.2.1.1.

3. *Adjointness II*. The functor

$$X \lhd -\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

4. *Adjointness III*. We have a 忘-relative adjunction

$$(X \lhd - \dashv \mathsf{Sets}_*(X,-))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\lhd-"] \arrow[r, phantom, "\perp_{忘}"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X,-)"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X,Z))$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: This follows from the definition of $\lhd$ as a composition of functors (Definition 7.3.1.1.1).
Item 2, Adjointness I: This follows from Item 3 of Definition 7.2.1.1.6.
Item 3, Adjointness II: For $X \lhd -$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} X \lhd \mathrm{pt} &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $X \lhd -$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Definition 7.2.1.1.6. □

Remark 7.3.1.1.8. Here is some intuition on why $X \lhd -$ fails to be a left adjoint. Item 4 of Definition 7.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X,Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, \mathbf{Sets}_*(X,Z)),$$

also holds, which would give $X \lhd - \dashv \mathbf{Sets}_*(X,-)$. However, such a bijection would require every map

$$f\colon X \lhd Y \to Z$$

to satisfy

$$f(x \lhd y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \lhd y$. Thus $\mathbf{Sets}_*(X, -)$ can't be a right adjoint for $X \lhd -$, and as shown by Item 3 of Definition 7.3.1.1.7, no functor can.[13]

7.3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.3.2.1.1. The **left internal Hom**[14] **of pointed sets** is the functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘\times \mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Definition 7.2.2.1.4.

Remark 7.3.2.1.2. The left internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathsf{Sets}_*\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps $f \colon X \lhd Y \longrightarrow Z$.
2. Pointed maps $f \colon X \longrightarrow [Y, Z]^{\lhd}_{\mathsf{Sets}_*}$.

Remark 7.3.2.1.3. In detail, the **left internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\left([X, Y]^{\lhd}_{\mathsf{Sets}_*}, \left[(y_0)_{x \in X}\right]\right)$ consisting of:

[13] The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed sets of Definition 7.5.1.1.1. See Item 2 of Definition 7.5.1.1.10.

[14] For a proof that $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ is indeed the left internal Hom of Sets_* with respect to

- *The Underlying Set.* The set $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X, Y]^{\lhd}_{\mathsf{Sets}_*} &\stackrel{\text{def}}{=} |X| \pitchfork Y \\ &\cong \bigwedge_{x \in X} (Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $\left[(y_0)_{x \in X}\right]$ of $\bigwedge_{x \in X} (Y, y_0)$.

Proposition 7.3.2.1.4. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\lhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X, -]^{\lhd}_{\mathsf{Sets}_*} &\colon \quad \mathsf{Sets}_* \quad \to \mathsf{Sets}_*, \\ [-, Y]^{\lhd}_{\mathsf{Sets}_*} &\colon \quad \mathsf{Sets}_*^{\mathsf{op}} \quad \to \mathsf{Sets}_*, \\ [-_1, -_2]^{\lhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned}f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0),\end{aligned}$$

   the induced map

$$[f, g]^{\lhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\lhd}_{\mathsf{Sets}_*} \to [X, B]^{\lhd}_{\mathsf{Sets}_*}$$

   is given by

$$[f, g]^{\lhd}_{\mathsf{Sets}_*}\left(\left[(y_a)_{a \in A}\right]\right) \stackrel{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x \in X}\right]$$

   for each $\left[(y_a)_{a \in A}\right] \in [A, Y]^{\lhd}_{\mathsf{Sets}_*}$.

2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right) \colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_* \arrow[l, bend left, "{[Y, -]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd},$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$

3. *Adjointness II*. The functor

$$X \triangleleft -\colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1, Functoriality*: This follows from the definition of $[-,-]^{\triangleleft}_{\mathsf{Sets}_*}$ as a composition of functors (Definition 7.3.2.1.1).
Item 2, Adjointness I: This is a repetition of Item 2 of Definition 7.3.1.1.7, and is proved there.
Item 3, Adjointness II: This is a repetition of Item 3 of Definition 7.3.1.1.7, and is proved there. □

7.3.3 The Left Skew Unit

Definition 7.3.3.1.1. The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleleft}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\triangleleft}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

7.3.4 The Left Skew Associator

Definition 7.3.4.1.1. The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleleft}\colon \triangleleft \circ (\triangleleft \times \mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \triangleleft \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleleft) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\triangleleft"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\triangleleft\times\mathrm{id}"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleleft"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\triangleleft"'] \arrow[uur, phantom, "\Nearrow\alpha^{\mathsf{Sets}_*,\triangleleft}" description] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
(X \lhd Y) \lhd Z &\stackrel{\text{def}}{=} |Z| \odot (X \lhd Y)\\
&\stackrel{\text{def}}{=} |Z| \odot (|Y| \odot X)\\
&\cong \bigvee_{z\in Z} |Y| \odot X\\
&\cong \bigvee_{z\in Z}\left(\bigvee_{y\in Y} X\right)\\
&\to \bigvee_{[(z,y)]\in\bigvee_{z\in Z} Y} X\\
&\cong \bigvee_{[(z,y)]\in|Z|\odot Y} X\\
&\cong ||Z| \odot Y| \odot X\\
&\stackrel{\text{def}}{=} |Y \lhd Z| \odot X\\
&\stackrel{\text{def}}{=} X \lhd (Y \lhd Z),
\end{aligned}$$

where the map

$$\bigvee_{z\in Z}\left(\bigvee_{y\in Y} X\right) \to \bigvee_{(z,y)\in\bigvee_{z\in Z} Y} X$$

is given by $[(z,[(y,x)])] \mapsto [([(z,y)],x)]$.

Proof. (Proven below in a bit.) □

Remark 7.3.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}
[(z,[(y,x)])] &\stackrel{\text{def}}{=} [(z, x \lhd y)]\\
&\stackrel{\text{def}}{=} (x \lhd y) \lhd z
\end{aligned}$$

and

$$\begin{aligned}
[([(z,y)],x)] &\stackrel{\text{def}}{=} [(y \lhd z, x)]\\
&\stackrel{\text{def}}{=} x \lhd (y \lhd z).
\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y) \lhd z) \overset{\text{def}}{=} x \lhd (y \lhd z)$$

for each $(x \lhd y) \lhd z \in (X \lhd Y) \lhd Z$.

Remark 7.3.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y_0) \lhd z) \overset{\text{def}}{=} x \lhd (y_0 \lhd z).$$

However, by the definition of $\lhd$, we have $y_0 \lhd z = y_0 \lhd z'$ for all $z, z' \in Z$, preventing $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x_0 \lhd y_0) \lhd z_0) = x_0 \lhd (y_0 \lhd z_0).$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0), \\ g &\colon (Y, y_0) \to (Y', y'_0), \\ h &\colon (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \lhd Y) \lhd Z \arrow[r, "(f \lhd g) \lhd h"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"'] & (X' \lhd Y') \lhd Z' \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X',Y',Z'}"] \\ X \lhd (Y \lhd Z) \arrow[r, "f \lhd (g \lhd h)"'] & X' \lhd (Y' \lhd Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x \lhd y) \lhd z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \lhd g(y)) \lhd h(z) \arrow[d, mapsto] \\ x \lhd (y \lhd z) \arrow[r, mapsto] & f(x) \lhd (g(y) \lhd h(z)) \end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

7.3.5 The Left Skew Left Unitor

Definition 7.3.5.1.1. The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right)\overset{\sim}{\Longrightarrow}\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},\qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}"] \arrow[rd, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[ld, phantom, "\Swarrow\lambda^{\mathsf{Sets}_*,\lhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\lhd}_X\colon S^0\lhd X\to X$$

at $(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} S^0\lhd X &\cong |X|\odot S^0\\ &\cong \bigvee_{x\in X} S^0\\ &\to X, \end{aligned}$$

where $\bigvee_{x\in X} S^0\to X$ is the map given by

$$\begin{aligned} [(x,0)] &\mapsto x_0,\\ [(x,1)] &\mapsto x \end{aligned}$$

for each $x\in X$.

Proof. (Proven below in a bit.) □

Remark 7.3.5.1.2. In other words, $\lambda^{\mathsf{Sets}_*,\lhd}_X$ acts on elements as

$$\begin{aligned} \lambda^{\mathsf{Sets}_*,\lhd}_X(0\lhd x) &\overset{\text{def}}{=} x_0,\\ \lambda^{\mathsf{Sets}_*,\lhd}_X(1\lhd x) &\overset{\text{def}}{=} x \end{aligned}$$

for each $1\lhd x\in S^0\lhd X$.

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the left tensor product of pointed sets, see Item 2 of Definition 7.3.1.1.7.

Remark 7.3.5.1.3. The morphism $\lambda_X^{\mathsf{Sets}_*,\lhd}$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to S^0 \lhd X$$

given by

$$\phi_X(x) \overset{\text{def}}{=} 1 \lhd x$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi\right](x) &= \lambda_X^{\mathsf{Sets}_*,\lhd}(\phi(x))\\ &= \lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x)\\ &= x\\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}\right](1 \lhd x) &= \phi\left(\lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x)\right)\\ &= \phi(x)\\ &= 1 \lhd x\\ &= [\mathrm{id}_{S^0 \lhd X}](1 \lhd x),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}\right](0 \lhd x) &= \phi\left(\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x)\right)\\ &= \phi(x_0)\\ &= 1 \lhd x_0,\end{aligned}$$

where $0 \lhd x \neq 1 \lhd x_0$. Thus

$$\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd} \overset{?}{=} \mathrm{id}_{S^0 \lhd X}$$

holds for all elements in $S^0 \lhd X$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \colon S^0 \lhd X \to X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x_0) = x_0.$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
S^0 \lhd X \arrow[r, "\mathrm{id}_{S^0} \lhd f"] \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_X"'] & S^0 \lhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \lhd x \arrow[d, mapsto] & & 0 \lhd x \arrow[r, mapsto] & 0 \lhd f(x) \arrow[d, mapsto] \\
x_0 \arrow[r, mapsto] & f(x_0) & & y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \lhd x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \lhd f(x) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

7.3.6 The Left Skew Right Unitor

Definition 7.3.6.1.1. The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right), \qquad \begin{tikzcd}
\mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \arrow[dl, phantom, "\Nearrow \rho^{\mathsf{Sets}_*,\lhd}"] \\
 & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd S^0$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \lhd S^0, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 7.3.6.1.2. In other words, $\rho_X^{\mathsf{Sets}_*,\lhd}$ acts on elements as

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} x \lhd 1$$

for each $x \in X$.

Remark 7.3.6.1.3. The morphism $\rho_X^{\mathsf{Sets}_*,\lhd}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \lhd 0$ of $X \lhd S^0$ with $x \neq x_0$ are outside the image of $\rho_X^{\mathsf{Sets}_*,\lhd}$, which sends x to $x \lhd 1$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*,\lhd}(x_0) &= x_0 \lhd 1 \\ &= x_0 \lhd 0. \end{aligned}$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\lhd}"'] & Y \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\lhd}"] \\ X \lhd S^0 \arrow[r, "f \lhd \mathrm{id}_{S^0}"'] & Y \lhd S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \lhd 0 \arrow[r, mapsto] & f(x) \lhd 0
\end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

7.3.7 The Diagonal

Definition 7.3.7.1.1. The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd}
\mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\lhd}"] & \mathsf{Sets}_* \\
& \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\lhd"'] &
\end{tikzcd}$$

whose component

$$\Delta^{\lhd}_X\colon (X, x_0) \to (X \lhd X, x_0 \lhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\lhd}_X(x) \stackrel{\text{def}}{=} x \lhd x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\lhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \lhd x_0,$$

and thus $\Delta^{\lhd}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta^{\lhd}_X"'] & Y \arrow[d, "\Delta^{\lhd}_Y"] \\
X \lhd X \arrow[r, "f \lhd f"'] & Y \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x \lhd x \arrow[r, mapsto] & f(x) \lhd f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\lhd}$ to be natural. □

7.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to ◁

Proposition 7.3.8.1.1. The category Sets_* admits a left-closed left skew monoidal category structure consisting of:

- *The Underlying Category.* The category Sets_* of pointed sets.
- *The Left Skew Monoidal Product.* The left tensor product functor

  $$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.3.1.1.1.
- *The Left Internal Skew Hom.* The left internal Hom functor

  $$[-,-]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.3.2.1.1.
- *The Left Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*,\lhd} \colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.3.3.1.1.
- *The Left Skew Associators.* The natural transformation

  $$\alpha^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ (\lhd \times \mathsf{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd \circ (\mathsf{id}_{\mathsf{Sets}_*} \times \lhd) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 7.3.4.1.1.
- *The Left Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathsf{Sets}_*,\lhd} \colon \lhd \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathsf{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.3.5.1.1.

- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right)$$

  of Definition 7.3.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \lhd (X \lhd Y)) \lhd Z \arrow[dr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X\lhd Y,Z}"] & \\
((W \lhd X) \lhd Y) \lhd Z \arrow[ur, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y} \lhd \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{W\lhd X,Y,Z}"'] & & W \lhd ((X \lhd Y) \lhd Z) \arrow[d, "\mathrm{id}_W \lhd \alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"] \\
(W \lhd X) \lhd (Y \lhd Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y\lhd Z}"'] & & W \lhd (X \lhd (Y \lhd Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \lhd (x \lhd y)) \lhd z \arrow[dr, mapsto] & \\
((w \lhd x) \lhd y) \lhd z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \lhd ((x \lhd y) \lhd z) \arrow[d, mapsto] \\
(w \lhd x) \lhd (y \lhd z) \arrow[rr, mapsto] & & w \lhd (x \lhd (y \lhd z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.

The Left Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
\left(S^0 \lhd X\right) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{S^0,X,Y}"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & S^0 \lhd (X \lhd Y) \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"] \\
 & X \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x_0 \lhd y = x_0 \lhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
(1 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd (x \lhd y) \arrow[d, mapsto] \\
 & x \lhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Left Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"'] \arrow[dr, "\mathrm{id}_X \lhd \rho^{\mathsf{Sets}_*,\lhd}_Y"] & \\
(X \lhd Y) \lhd S^0 \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,S^0}"'] & X \lhd \left(Y \lhd S^0\right)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(x \lhd y) \lhd 1 \arrow[r, mapsto] & x \lhd (y \lhd 1)
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Left Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[r, equal] \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & X \lhd Y \\
\left(X \lhd S^0\right) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,S^0,Y}"'] & X \lhd \left(S^0 \lhd Y\right) \arrow[u, "\mathrm{id}_A \lhd \lambda^{\mathsf{Sets}_*,\lhd}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \lhd y \\
(x \lhd 1) \lhd y \arrow[r, mapsto] & x \lhd (1 \lhd y) \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,\lhd}_{S^0}"] \arrow[dr, equal] & S^0 \lhd S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{S^0}"] \\
 & S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd 1 \arrow[d, mapsto] \\
 & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd 1 \arrow[d, mapsto] \\
 & 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Left Skew Monoidal Left-Closedness: This follows from Item 2 of Definition 7.3.1.1.7. □

7.3.9 Monoids With Respect to the Left Tensor Product of Pointed Sets

Proposition 7.3.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ is isomorphic to the category of "monoids with left zero"[15] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \lhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets
$$\mu_A \colon A \lhd A \to A,$$
determining a left bilinear morphism of pointed sets
$$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$
- *The Unit Morphism.* A morphism of pointed sets
$$\eta_A \colon S^0 \to A$$
picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{tikzcd}
 & A \lhd (A \lhd A) \arrow[dr, "\mathrm{id}_A \lhd \mu_A"] & \\
(A \lhd A) \lhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \lhd}_{A,A,A}"] \arrow[d, "\mu_A \lhd \mathrm{id}_A"'] & & A \lhd A \arrow[d, "\mu_A"] \\
A \lhd A \arrow[rr, "\mu_A"'] & & A
\end{tikzcd}$$

[15] A monoid with left zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying
$$0_A a = 0_A$$
for each $a \in A$.

2. *Left Unitality.* The diagram

$$\begin{tikzcd} S^0 \lhd A \arrow[r, "\eta_A \times \mathrm{id}_A"] \arrow[dr, "\lambda_A^{\mathsf{Sets}_*, \lhd}"'] & A \lhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd} A \arrow[r, "\rho_A^{\mathsf{Sets}_*, \lhd}"] \arrow[d, equal] & A \lhd S^0 \arrow[d, "\mathrm{id}_A \times \eta_A"] \\ A & A \lhd A \arrow[l, "\mu_A"] \end{tikzcd}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd} (a \lhd b) \lhd c \arrow[d, mapsto] \\ ab \lhd c \arrow[r, mapsto] & (ab)c \end{tikzcd} \qquad \begin{tikzcd} & a \lhd (b \lhd c) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (a \lhd b) \lhd c & & a \lhd bc \arrow[d, mapsto] \\ & & a(bc) \end{tikzcd}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts:

(a) On $0 \lhd a$ as

$$\begin{tikzcd}
0 \lhd a \arrow[dr, mapsto] & \\
& 0_A
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 0_A \lhd a \arrow[d, mapsto] \\
& 0_A a.
\end{tikzcd}$$

(b) On $1 \lhd a$ as

$$\begin{tikzcd}
1 \lhd a \arrow[dr, mapsto] & \\
& a
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 1_A \lhd a \arrow[d, mapsto] \\
& 1_A a.
\end{tikzcd}$$

This gives

$$\begin{aligned} 1_A a &= a, \\ 0_A a &= 0_A \end{aligned}$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts as

$$\begin{tikzcd}
a \arrow[d, mapsto] \\
a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] & a \lhd 1 \arrow[d, mapsto] \\
a 1_A & a \lhd 1_A \arrow[l, mapsto]
\end{tikzcd}$$

This gives

$$a 1_A = a$$

for each $a \in A$.

Thus we see that monoids with respect to $\lhd$ are exactly monoids with left zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f \colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \lhd A \arrow[r, "f \lhd f"] \arrow[d, "\mu_A"'] & B \lhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

   commutes.

These act on elements as

$$\begin{tikzcd} a \lhd b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \lhd b \arrow[r, mapsto] & f(a) \lhd f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & \\ & 0_B \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & \\ & 1_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & f(1_A) \end{tikzcd}$$

giving

$$f(ab) = f(a)f(b),$$

$$f(0_A) = 0_B,$$
$$f(1_A) = 1_B,$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \lhd, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. □

7.4 The Right Tensor Product of Pointed Sets

7.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.4.1.1.1. The **right tensor product of pointed sets** is the functor[16]

$$\rhd : \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘\times \mathrm{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 : \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot : \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Definition 7.2.1.1.6.

Remark 7.4.1.1.2. The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f : X \rhd Y \to Z$.
2. Maps of sets $f : X \times Y \to Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

Remark 7.4.1.1.3. The right tensor product of pointed sets may be described as follows:

[16] *Further Notation:* Also written $\rhd_{\mathsf{Sets}_*}$.

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \rhd Y, x_0 \rhd y_0), \iota)$ consisting of
  - A pointed set $(X \rhd Y, x_0 \rhd y_0)$;
  - A right bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

  satisfying the following universal property:

  (★) Given another such pair $((Z, z_0), f)$ consisting of
  - ⋆ A pointed set (Z, z_0);
  - ⋆ A right bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

  there exists a unique morphism of pointed sets $X \rhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd}
 & X \rhd Y \arrow[d, dashed, "\exists!"] \\
X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z
\end{tikzcd}$$

  commute.

Construction 7.4.1.1.4. In detail, the **right tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \rhd Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \rhd Y$ defined by

$$\begin{aligned} X \rhd Y &\stackrel{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x \in X}(Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

Proof. Since $\bigvee_{y \in Y}(X, x_0)$ is defined as the quotient of $\coprod_{x \in X} Y$ by the equivalence relation R generated by declaring $(x, y) \sim (x', y')$ if $y = y' = y_0$, we have, by **??**, a natural bijection

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}\left(\coprod_{X \in X} Y, Z\right),$$

where $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}\left(\coprod_{x\in X} Y, Z\right) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}\left(\coprod_{x\in X} Y, Z\right) \middle| \begin{array}{l} \text{for each } x, y \in X, \text{if} \\ (x, y) \sim_R (x', y'), \text{then} \\ f(x, y) = f(x', y') \end{array} \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. We have $y = y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(\coprod_{x\in X} Y, Z)$ with a corresponding $\overline{f}\colon X \rhd Y \to Z$, the latter case above implies

$$\begin{aligned} f([(x, y_0)]) &= f([(x', y_0)]) \\ &= f([(x_0, y_0)]), \end{aligned}$$

and since $\overline{f}\colon X \rhd Y \to Z$ is a pointed map, we have

$$\begin{aligned} f([(x_0, y_0)]) &= \overline{f}([(x_0, y_0)]) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f\colon X \times Y \to Z$ satisfying the equality

$$f(x, y_0) = z_0$$

for each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. □

Notation 7.4.1.1.5. We write[17] $x \rhd y$ for the element $[(x, y)]$ of

$$X \rhd Y \cong |X| \odot Y.$$

[17] *Further Notation:* Also written $x \rhd_{\mathsf{Sets}_*} y$.

Remark 7.4.1.1.6. Employing the notation introduced in Definition 7.4.1.1.5, we have

$$x_0 \rhd y_0 = x \rhd y_0$$

for each $x \in X$, and

$$x \rhd y_0 = x' \rhd y_0$$

for each $x, x' \in X$.

Proposition 7.4.1.1.7. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd Y$ define functors

$$\begin{aligned} X \rhd -\colon & \quad \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ - \rhd Y\colon & \quad \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*, \\ -_1 \rhd -_2\colon & \quad \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\longrightarrow (A, a_0), \\ g\colon (Y, y_0) &\longrightarrow (B, b_0), \end{aligned}$$

   the induced map

$$f \rhd g\colon X \rhd Y \longrightarrow A \rhd B$$

   is given by

$$[f \rhd g](x \rhd y) \overset{\mathrm{def}}{=} f(x) \rhd g(y)$$

   for each $x \rhd y \in X \rhd Y$.

2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 7.4.2.1.1.

3. *Adjointness II*. The functor

$$- \rhd Y \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

does not admit a right adjoint.

4. *Adjointness III*. We have a 忘-relative adjunction

$$(- \rhd Y \dashv \mathsf{Sets}_*(Y,-))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \rhd Y"] \arrow[r, phantom, "\perp_{忘}"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y,Z))$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: This follows from the definition of $\rhd$ as a composition of functors (Definition 7.4.1.1.1).
Item 2, Adjointness I: This follows from Item 3 of Definition 7.2.1.1.6.
Item 3, Adjointness II: For $- \rhd Y$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} \mathrm{pt} \rhd X &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \rhd Y$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Definition 7.2.1.1.6. □

Remark 7.4.1.1.8. Here is some intuition on why $- \rhd Y$ fails to be a left adjoint. Item 4 of Definition 7.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y,Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y,Z)),$$

also holds, which would give $- \rhd Y \dashv \mathbf{Sets}_*(Y,-)$. However, such a bijection would require every map

$$f \colon X \rhd Y \to Z$$

to satisfy

$$f(x_0 \rhd y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \rhd y_0$. Thus $\mathbf{Sets}_*(Y, -)$ can't be a right adjoint for $- \rhd Y$, and as shown by Item 3 of Definition 7.4.1.1.7, no functor can.[18]

7.4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.4.2.1.1. The **right internal Hom**[19] **of pointed sets** is the functor

$$[-,-]^{\rhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork \colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Definition 7.2.2.1.4.

Remark 7.4.2.1.2. We have

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} = [-,-]^{\rhd}_{\mathsf{Sets}_*}.$$

Remark 7.4.2.1.3. The right internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathsf{Sets}_*\left(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

That is to say, the following data are in bijection:

1. Pointed maps $f \colon X \rhd Y \to Z$.

[18]The functor $\mathbf{Sets}_*(Y, -)$ is instead right adjoint to $- \wedge Y$, the smash product of pointed sets of Definition 7.5.1.1.1. See Item 2 of Definition 7.5.1.1.10.

[19]For a proof that $[-,-]^{\rhd}_{\mathsf{Sets}_*}$ is indeed the right internal Hom of Sets_* with respect to the right tensor product of pointed sets, see Item 2 of Definition 7.4.1.1.7.

2. Pointed maps $f\colon Y \longrightarrow [X, Z]^{\rhd}_{\mathsf{Sets}_*}$.

Remark 7.4.2.1.4. In detail, the **right internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\left([X, Y]^{\rhd}_{\mathsf{Sets}_*}, \left[(y_0)_{x \in X}\right]\right)$ consisting of:

- *The Underlying Set.* The set $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ defined by
$$\begin{aligned}[X, Y]^{\rhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x \in X} (Y, y_0),\end{aligned}$$
where $|X|$ denotes the underlying set of (X, x_0).
- *The Underlying Basepoint.* The point $\left[(y_0)_{x \in X}\right]$ of $\bigwedge_{x \in X}(Y, y_0)$.

Proposition 7.4.2.1.5. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\rhd}_{\mathsf{Sets}_*}$ define functors
$$\begin{aligned}[X, -]^{\rhd}_{\mathsf{Sets}_*} &\colon && \mathsf{Sets}_* && \longrightarrow \mathsf{Sets}_*,\\ [-, Y]^{\rhd}_{\mathsf{Sets}_*} &\colon && \mathsf{Sets}_*^{\mathsf{op}} && \longrightarrow \mathsf{Sets}_*,\\ [-_1, -_2]^{\rhd}_{\mathsf{Sets}_*} &\colon && \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* && \longrightarrow \mathsf{Sets}_*.\end{aligned}$$
In particular, given pointed maps
$$\begin{aligned}f &\colon (X, x_0) \longrightarrow (A, a_0),\\ g &\colon (Y, y_0) \longrightarrow (B, b_0),\end{aligned}$$
the induced map
$$[f, g]^{\rhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\rhd}_{\mathsf{Sets}_*} \longrightarrow [X, B]^{\rhd}_{\mathsf{Sets}_*}$$
is given by
$$[f, g]^{\rhd}_{\mathsf{Sets}_*}\left(\left[(y_a)_{a \in A}\right]\right) \overset{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x \in X}\right]$$
for each $\left[(y_a)_{a \in A}\right] \in [A, Y]^{\rhd}_{\mathsf{Sets}_*}$.
2. *Adjointness I.* We have an adjunction
$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right)\colon\quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X, Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$
natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 7.4.2.1.1.

3. *Adjointness II.* The functor

$$- \triangleright Y \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

   does not admit a right adjoint.

Proof. *Item 1, Functoriality*: This follows from the definition of $[-,-]^{\triangleright}_{\mathsf{Sets}_*}$ as a composition of functors (Definition 7.4.2.1.1).
Item 2, Adjointness I: This is a repetition of Item 2 of Definition 7.4.1.1.7, and is proved there.
Item 3, Adjointness II: This is a repetition of Item 3 of Definition 7.4.1.1.7, and is proved there. □

7.4.3 The Right Skew Unit

Definition 7.4.3.1.1. The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\triangleright}_{\mathsf{Sets}_*} \overset{\text{def}}{=} S^0.$$

7.4.4 The Right Skew Associator

Definition 7.4.4.1.1. The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright\right) \Longrightarrow \triangleright \circ \left(\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & (\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[dr, "\triangleright\times\mathrm{id}"] & \\
\mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\mathrm{id}\times\triangleright"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\triangleright"'] \arrow[urr, phantom, "\Uparrow \alpha^{\mathsf{Sets}_*,\triangleright}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z} \colon X \triangleright (Y \triangleright Z) \longrightarrow (X \triangleright Y) \triangleright Z$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
X \rhd (Y \rhd Z) &\stackrel{\text{def}}{=} |X| \odot (Y \rhd Z) \\
&\stackrel{\text{def}}{=} |X| \odot (|Y| \odot Z) \\
&\cong \bigvee_{x \in X} (|Y| \odot Z) \\
&\cong \bigvee_{x \in X} \left(\bigvee_{y \in Y} Z \right) \\
&\to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z \\
&\cong \bigvee_{[(x,y)] \in |X| \odot Y} Z \\
&\cong ||X| \odot Y| \odot Z \\
&\stackrel{\text{def}}{=} |X \rhd Y| \odot Z \\
&\stackrel{\text{def}}{=} (X \rhd Y) \rhd Z,
\end{aligned}$$

where the map

$$\bigvee_{x \in X} \left(\bigvee_{y \in Y} Z \right) \to \bigvee_{[(x,y)] \in \bigvee_{x \in X} Y} Z$$

is given by $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$.

Proof. (Proven below in a bit.) □

Remark 7.4.4.1.2. Unwinding the notation for elements, we have

$$\begin{aligned}
[(x, [(y, z)])] &\stackrel{\text{def}}{=} [(x, y \rhd z)] \\
&\stackrel{\text{def}}{=} x \rhd (y \rhd z)
\end{aligned}$$

and

$$\begin{aligned}
[([(x, y)], z)] &\stackrel{\text{def}}{=} [(x \rhd y, z)] \\
&\stackrel{\text{def}}{=} (x \rhd y) \rhd z.
\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*, \rhd}_{X,Y,Z}(x \rhd (y \rhd z)) \stackrel{\text{def}}{=} (x \rhd y) \rhd z$$

for each $x \rhd (y \rhd z) \in X \rhd (Y \rhd Z)$.

Remark 7.4.4.1.3. Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x \rhd (y_0 \rhd z)) \overset{\text{def}}{=} (x \rhd y_0) \rhd z.$$

However, by the definition of $\rhd$, we have $x \rhd y_0 = x' \rhd y_0$ for all $x, x' \in X$, preventing $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z} : X \rhd (Y \rhd Z) \to (X \rhd Y) \rhd Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x_0 \rhd (y_0 \rhd z_0)) = (x_0 \rhd y_0) \rhd z_0.$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \to \left(X', x'_0\right), \\ g &: (Y, y_0) \to \left(Y', y'_0\right), \\ h &: (Z, z_0) \to \left(Z', z'_0\right) \end{aligned}$$

the diagram

$$\begin{tikzcd} X \rhd (Y \rhd Z) \arrow[r, "f \rhd (g \rhd h)"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}"'] & X' \rhd (Y' \rhd Z') \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X',Y',Z'}"] \\ (X \rhd Y) \rhd Z \arrow[r, "(f \rhd g) \rhd h"'] & (X' \rhd Y') \rhd Z' \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd (y \rhd z) \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd (g(y) \rhd h(z)) \arrow[d, mapsto] \\ (x \rhd y) \rhd z \arrow[r, mapsto] & (f(x) \rhd g(y)) \rhd h(z) \end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

7.4.5 The Right Skew Left Unitor

Definition 7.4.5.1.1. The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$
\lambda^{\mathsf{Sets}_*,\rhd}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \rhd \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right)
\qquad
\begin{tikzcd}
\mathrm{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\rhd"] \\
& \mathsf{Sets}_*, \arrow[u, phantom, "\lambda^{\mathsf{Sets}_*,\rhd}"]
\end{tikzcd}
$$

whose component

$$\lambda^{\mathsf{Sets}_*,\rhd}_X \colon X \to S^0 \rhd X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$
\begin{aligned}
X &\to X \vee X \\
&\cong |S^0| \odot X \\
&\cong S^0 \rhd X,
\end{aligned}
$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

Remark 7.4.5.1.2. In other words, $\lambda^{\mathsf{Sets}_*,\rhd}_X$ acts on elements as

$$\lambda^{\mathsf{Sets}_*,\rhd}_X(x) \overset{\text{def}}{=} [(1, x)]$$

i.e. by

$$\lambda^{\mathsf{Sets}_*,\rhd}_X(x) \overset{\text{def}}{=} 1 \rhd x$$

for each $x \in X$.

Remark 7.4.5.1.3. The morphism $\lambda^{\mathsf{Sets}_*,\rhd}_X$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0 \rhd x$ of $S^0 \rhd X$ with $x \neq x_0$ are outside the image of $\lambda^{\mathsf{Sets}_*,\rhd}_X$, which sends x to $1 \rhd x$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda^{\mathsf{Sets}_*,\rhd}_X \colon X \to S^0 \rhd X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\lambda_X^{\mathsf{Sets}_*,\triangleright}(x_0) &= 1 \triangleright x_0\\ &= 0 \triangleright x_0.\end{aligned}$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*,\triangleright}"'] & Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*,\triangleright}"] \\ S^0 \triangleright X \arrow[r, "\mathrm{id}_{S^0} \triangleright f"'] & S^0 \triangleright Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ 1 \triangleright x \arrow[r, mapsto] & 1 \triangleright f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. □

7.4.6 The Right Skew Right Unitor

Definition 7.4.6.1.1. The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright}\colon \triangleright \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*,\triangleright}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho_X^{\mathsf{Sets}_*,\rhd}\colon X \rhd S^0 \longrightarrow X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X \rhd S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x\in X} S^0 \\ &\to X, \end{aligned}$$

where $\bigvee_{x\in X} S^0 \longrightarrow X$ is the map given by

$$\begin{aligned} [(x,0)] &\mapsto x_0, \\ [(x,1)] &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. (Proven below in a bit.) □

Remark 7.4.6.1.2. In other words, $\rho_X^{\mathsf{Sets}_*,\rhd}$ acts on elements as

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 0) &\stackrel{\text{def}}{=} x_0, \\ \rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each $x \rhd 1 \in X \rhd S^0$.

Remark 7.4.6.1.3. The morphism $\rho_X^{\mathsf{Sets}_*,\rhd}$ is almost invertible, with its would-be-inverse

$$\phi_X\colon X \longrightarrow X \rhd S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \rhd 1$$

for each $x \in X$. Indeed, we have

$$\begin{aligned} \left[\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi\right](x) &= \rho_X^{\mathsf{Sets}_*,\rhd}(\phi(x)) \\ &= \rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1) \\ &= x \\ &= [\mathrm{id}_X](x) \end{aligned}$$

so that

$$\rho_X^{\mathsf{Sets}_*,\rhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}\right](x \rhd 1) &= \phi\left(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 1)\right)\\ &= \phi(x)\\ &= x \rhd 1\\ &= [\mathrm{id}_{X \rhd S^0}](x \rhd 1),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd}\right](x \rhd 0) &= \phi\left(\rho_X^{\mathsf{Sets}_*,\rhd}(x \rhd 0)\right)\\ &= \phi(x_0)\\ &= 1 \rhd x_0,\end{aligned}$$

where $x \rhd 0 \neq 1 \rhd x_0$. Thus

$$\phi \circ \rho_X^{\mathsf{Sets}_*,\rhd} \stackrel{?}{=} \mathrm{id}_{X \rhd S^0}$$

holds for all elements in $X \rhd S^0$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\rhd} : X \rhd S^0 \longrightarrow X$$

is indeed a morphism of pointed sets as we have

$$\rho_X^{\mathsf{Sets}_*,\rhd}(x_0 \rhd 0) = x_0.$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \longrightarrow (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \rhd S^0 \arrow[r, "f \rhd \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\rhd}"'] & Y \rhd S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\rhd}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \rhd 0 \arrow[r, mapsto] & f(x) \rhd 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \triangleright 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \triangleright 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. □

7.4.7 The Diagonal

Definition 7.4.7.1.1. The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\triangleright} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \triangleright \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\triangleright}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\triangleright"'] & \end{tikzcd}$$

whose component

$$\Delta^{\triangleright}_X \colon (X, x_0) \to (X \triangleright X, x_0 \triangleright x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\triangleright}_X(x) \stackrel{\text{def}}{=} x \triangleright x$$

for each $x \in X$.

Proof. *Being a Morphism of Pointed Sets*: We have

$$\Delta^{\triangleright}_X(x_0) \stackrel{\text{def}}{=} x_0 \triangleright x_0,$$

and thus $\Delta^{\triangleright}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\triangleright}_X"'] & Y \arrow[d, "\Delta^{\triangleright}_Y"] \\ X \triangleright X \arrow[r, "f \triangleright f"'] & Y \triangleright Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \triangleright x \arrow[r, mapsto] & f(x) \triangleright f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\triangleright}$ to be natural. □

7.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to ▷

Proposition 7.4.8.1.1. The category Sets_* admits a right-closed right skew monoidal category structure consisting of:

- *The Underlying Category.* The category Sets_* of pointed sets.
- *The Right Skew Monoidal Product.* The right tensor product functor

  $$\triangleright \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 7.4.1.1.1.
- *The Right Internal Skew Hom.* The right internal Hom functor

  $$[-,-]^{\triangleright}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 7.4.2.1.1.
- *The Right Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*,\triangleright} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 7.4.3.1.1.
- *The Right Skew Associators.* The natural transformation

  $$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 7.4.4.1.1.
- *The Right Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right)$$

  of Definition 7.4.5.1.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.4.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & & W \triangleright ((X \triangleright Y) \triangleright Z) \arrow[drr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X\triangleright Y,Z}"] & & \\
W \triangleright (X \triangleright (Y \triangleright Z)) \arrow[urr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y} \triangleright \mathrm{id}_Z"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W\triangleright X,Y,Z}"'] & & & & (W \triangleright (X \triangleright Y)) \triangleright Z \arrow[dl, "\mathrm{id}_W \triangleright \alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"] \\
 & (W \triangleright X) \triangleright (Y \triangleright Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y\triangleright Z}"'] & & ((W \triangleright X) \triangleright Y) \triangleright Z &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & & w \triangleright ((x \triangleright y) \triangleright z) \arrow[drr, mapsto] & & \\
w \triangleright (x \triangleright (y \triangleright z)) \arrow[urr, mapsto] \arrow[dr, mapsto] & & & & (w \triangleright (x \triangleright y)) \triangleright z \arrow[dl, mapsto] \\
 & (w \triangleright x) \triangleright (y \triangleright z) \arrow[rr, mapsto] & & ((w \triangleright x) \triangleright y) \triangleright z &
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.
The Right Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \triangleright Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\triangleright}_{X \triangleright Y}"'] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\triangleright}_{X} \triangleright \mathrm{id}_Y"] & \\
S^0 \triangleright (X \triangleright Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\triangleright}_{S^0,X,Y}"'] & \left(S^0 \triangleright X\right) \triangleright Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \triangleright y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
1 \triangleright (x \triangleright y) \arrow[r, mapsto] & (1 \triangleright x) \triangleright y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.
The Right Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \triangleright \left(Y \triangleright S^0\right) \arrow[r, "\mathrm{id}_X \triangleright \rho^{\mathsf{Sets}_*,\triangleright}_{Y}"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{S^0,X,Y}"'] & (X \triangleright Y) \triangleright S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\triangleright}_{X \triangleright Y}"] \\
& X \triangleright Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \triangleright (y \triangleright 0) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \triangleright y) \triangleright 0 \arrow[d, mapsto] \\
& x \triangleright y_0 = x_0 \triangleright y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \triangleright (y \triangleright 1) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \triangleright y) \triangleright 1 \arrow[d, mapsto] \\
& x \triangleright y
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Right Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd} X \rhd Y \arrow[r, equal] \arrow[d, "\mathrm{id}_X \rhd \lambda^{\mathsf{Sets}_*,\rhd}_Y"'] & X \rhd Y \\ X \rhd (S^0 \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,S^0,Y}"'] & (X \rhd S^0) \rhd Y \arrow[u, "\rho^{\mathsf{Sets}_*,\rhd}_X \rhd \mathrm{id}_Y"'] \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \rhd y \\ x \rhd (1 \rhd y) \arrow[r, mapsto] & (x \rhd 1) \rhd y \arrow[u, mapsto] \end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd} S^0 \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_{S^0}"] \arrow[dr, equal] & S^0 \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{S^0}"] \\ & S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 0 \arrow[d, mapsto] \\ & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 1 \arrow[d, mapsto] \\ & 1 \end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

Right Skew Monoidal Right-Closedness: This follows from Item 2 of Definition 7.4.1.1.7. □

7.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

Proposition 7.4.9.1.1. The category of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ is isomorphic to the category of "monoids with right zero"[20] and morphisms between them.

Proof. *Monoids on* $(\mathsf{Sets}_*, \rhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \rhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \rhd A \to A,$$

  determining a right bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{tikzcd}
 & A \rhd (A \rhd A) \arrow[dr, "\mathrm{id}_A \rhd \mu_A"] & \\
(A \rhd A) \rhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \rhd}_{A,A,A}"] \arrow[d, "\mu_A \rhd \mathrm{id}_A"'] & & A \rhd A \arrow[d, "\mu_A"] \\
A \rhd A \arrow[rr, "\mu_A"'] & & A
\end{tikzcd}$$

[20]A monoid with right zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
A \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_A"] \arrow[d, equal] & S^0 \rhd A \arrow[d, "\eta_A \times \mathrm{id}_A"] \\
A & A \rhd A \arrow[l, "\mu_A"]
\end{tikzcd}$$

commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
A \rhd S^0 \arrow[r, "\mathrm{id}_A \times \eta_A"] \arrow[dr, "\rho^{\mathsf{Sets}_*,\rhd}_A"'] & A \rhd A \arrow[d, "\mu_A"] \\
& A
\end{tikzcd}$$

commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd}
& & a \rhd (b \rhd c) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(a \rhd b) \rhd c \arrow[d, mapsto] & (a \rhd b) \rhd c & & a \rhd bc \arrow[d, mapsto] \\
ab \rhd c \arrow[r, mapsto] & (ab)c & & a(bc)
\end{tikzcd}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts as

$$
\begin{tikzcd}
a \arrow[d, mapsto] \\
a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, mapsto] & 1 \rhd a \arrow[d, mapsto] \\
1_A a & 1_A \rhd a \arrow[l, mapsto]
\end{tikzcd}
$$

This gives

$$1_A a = a$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts:

   (a) On $1 \rhd 0$ as

$$
\begin{tikzcd}
1 \rhd 0 \arrow[dr, mapsto] & \\
 & 0_A
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rhd 0 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 0_A \arrow[d, mapsto] \\
 & a0_A.
\end{tikzcd}
$$

   (b) On $a \rhd 1$ as

$$
\begin{tikzcd}
a \rhd 1 \arrow[dr, mapsto] & \\
 & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \rhd 1 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 1_A \arrow[d, mapsto] \\
 & a1_A.
\end{tikzcd}
$$

   This gives

$$
\begin{aligned}
a1_A &= a, \\
a0_A &= 0_A
\end{aligned}
$$

   for each $a \in A$.

Thus we see that monoids with respect to $\rhd$ are exactly monoids with right zero.
Morphisms of Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f : (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \rhd A \arrow[r, "f \rhd f"] \arrow[d, "\mu_A"'] & B \rhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

   commutes.

These act on elements as

$$\begin{tikzcd} a \rhd b \arrow[d, mapsto] & \\ ab \arrow[r, mapsto] & f(ab) \end{tikzcd} \qquad \begin{tikzcd} a \rhd b \arrow[r, mapsto] & f(a) \rhd f(b) \arrow[d, mapsto] \\ & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & \\ & 0_B \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & \\ & 1_B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & f(1_A) \end{tikzcd}$$

giving

$$f(ab) = f(a)f(b),$$

$$f(0_A) = 0_B,$$
$$f(1_A) = 1_B,$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero.
Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \rhd, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. □

7.5 The Smash Product of Pointed Sets

7.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.5.1.1.1. The **smash product of** (X, x_0) **and** (Y, y_0)[21] is the pointed set $X \wedge Y$[22] satisfying the bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Remark 7.5.1.1.2. That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f \colon X \wedge Y \to Z$.
- Maps of sets $f \colon X \times Y \to Z$ satisfying

$$f(x_0, y) = z_0,$$
$$f(x, y_0) = z_0$$

  for each $x \in X$ and each $y \in Y$.

Remark 7.5.1.1.3. The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of

[21] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[22] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the

– A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
– A bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

satisfying the following universal property:

(★) Given another such pair $((Z, z_0), f)$ consisting of
  * A pointed set (Z, z_0);
  * A bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

Construction 7.5.1.1.4. Concretely, the smash product of (X, x_0) and (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of:

- *The Underlying Set.* The set $X \wedge Y$ defined by
$$X \wedge Y \cong (X \times Y)/\sim_R,$$
where $\sim_R$ is the equivalence relation on $X \times Y$ obtained by declaring
$$\begin{aligned}(x_0, y) &\sim_R (x_0, y'),\\ (x, y_0) &\sim_R (x', y_0)\end{aligned}$$
for each $x, x' \in X$ and each $y, y' \in Y$.
- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

smash product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

Proof. By **??**, we have a natural bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z),$$

where $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$ is the set

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) \overset{\mathrm{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \text{ if} \\ (x, y) \sim_R (x', y'), \text{ then} \\ f(x, y) = f(x', y') \end{array} \right\}.$$

However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. The following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ with a corresponding $\overline{f} \colon X \wedge Y \to Z$, the latter case above implies

$$\begin{aligned} f(x_0, y) &= f(x, y_0) \\ &= f(x_0, y_0), \end{aligned}$$

and since $\overline{f} \colon X \wedge Y \to Z$ is a pointed map, we have

$$\begin{aligned} f(x_0, y_0) &= \overline{f}(x_0, y_0) \\ &= z_0. \end{aligned}$$

Thus the elements f in $\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f \colon X \times Y \to Z$ satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z),$$

gives our desired natural bijection, finishing the proof. □

Remark 7.5.1.1.5. It is also somewhat common to write

$$X \wedge Y \overset{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

Construction 7.5.1.1.6. Alternatively, the smash product of (X, x_0) and (Y, y_0) may be constructed as the pointed set $X \wedge Y$ given by

$$\begin{aligned} X \wedge Y &\cong \bigvee_{x \in X^-} Y \\ &\cong \bigvee_{y \in Y^-} X. \end{aligned}$$

Proof. Indeed, since $X \cong \bigvee_{x \in X^-} S^0$, we have

$$\begin{aligned} X \wedge Y &\cong \left(\bigvee_{x \in X^-} S^0 \right) \wedge Y \\ &\cong \bigvee_{x \in X^-} S^0 \wedge Y \\ &\cong \bigvee_{x \in X^-} Y, \end{aligned}$$

where we have used that $\wedge$ preserves colimits in both variables via **??** for the second isomorphism above, since it has right adjoints in both variables by Item 2.

A similar proof applies to the isomorphism $X \wedge Y \cong \bigvee_{y \in Y^-} X$. □

Notation 7.5.1.1.7. We write $x \wedge y$ for the element $[(x, y)]$ of $X \wedge Y \cong X \times Y/{\sim}$.

Remark 7.5.1.1.8. Employing the notation introduced in Definition 7.5.1.1.7, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

Example 7.5.1.1.9. Here are some examples of smash products of pointed sets.

1. *Smashing With pt.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} \mathrm{pt} \wedge X &\cong \mathrm{pt}, \\ X \wedge \mathrm{pt} &\cong \mathrm{pt}. \end{aligned}$$

2. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets

$$\begin{aligned} S^0 \wedge X &\cong X, \\ X \wedge S^0 &\cong X. \end{aligned}$$

Proposition 7.5.1.1.10. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors

$$\begin{aligned} X \wedge -\colon &\quad \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \wedge Y\colon &\quad \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ -_1 \wedge -_2\colon &\quad \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\to (A, a_0), \\ g\colon (Y, y_0) &\to (B, b_0), \end{aligned}$$

   the induced map

$$f \wedge g\colon X \wedge Y \to A \wedge B$$

   is given by

$$[f \wedge g](x \wedge y) \overset{\mathrm{def}}{=} f(x) \wedge g(y)$$

   for each $x \wedge y \in X \wedge Y$.

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X \wedge -"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X, -)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "- \wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y, -)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, where the pushout is taken in Sets, and the embedding $\iota\colon X \vee Y \hookrightarrow X \times Y$ is defined following Definition 7.5.1.1.5.

5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$\begin{aligned}X \wedge (Y \vee Z) &\cong (X \wedge Y) \vee (X \wedge Z),\\ (X \vee Y) \wedge Z &\cong (X \wedge Z) \vee (Y \wedge Z),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. Item 1, Functoriality: The map $f \wedge g$ comes from Item 4 of Definition 10.6.2.1.3 via the map

$$f \wedge g \colon X \times Y \to A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where $\sim_R$ is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y)/{\sim_R}$$

in Definition 7.5.1.1.4. The condition defining $\sim$ is that at least one of the following conditions is satisfied:

1. We have $x = x'$ and $y = y'$;
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

1. In the first case, we clearly have

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

   since $x = x'$ and $y = y'$.

2. If $x = x_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y)\\ &= a_0 \wedge g(y)\\ &= a_0 \wedge g(y')\\ &= f(x_0) \wedge g(y')\\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

3. If $x = x_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y)\\ &= a_0 \wedge g(y)\end{aligned}$$

$$
\begin{aligned}
&= a_0 \wedge b_0 \\
&= f(x') \wedge b_0 \\
&= f(x') \wedge g(y_0) \\
&\overset{\text{def}}{=} [f \wedge g](x', y_0).
\end{aligned}
$$

4. If $y = y_0$ and $x' = x_0$, we have

$$
\begin{aligned}
[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0) \\
&= f(x) \wedge b_0 \\
&= a_0 \wedge b_0 \\
&= a_0 \wedge g(y') \\
&= f(x_0) \wedge g(y') \\
&\overset{\text{def}}{=} [f \wedge g](x_0, y').
\end{aligned}
$$

5. If $y = y_0$ and $y' = y_0$, we have

$$
\begin{aligned}
[f \wedge g](x, y_0) &\overset{\text{def}}{=} f(x) \wedge g(y_0) \\
&= f(x) \wedge b_0 \\
&= f(x') \wedge b_0 \\
&= f(x) \wedge g(y_0) \\
&\overset{\text{def}}{=} [f \wedge g](x', y_0).
\end{aligned}
$$

Thus $f \wedge g$ is well-defined. Next, we claim that $\wedge$ preserves identities and composition:

- *Preservation of Identities.* We have

$$
\begin{aligned}
[\mathrm{id}_X \wedge \mathrm{id}_Y](x \wedge y) &\overset{\text{def}}{=} \mathrm{id}_X(x) \wedge \mathrm{id}_Y(y) \\
&= x \wedge y \\
&= [\mathrm{id}_{X \wedge Y}](x \wedge y)
\end{aligned}
$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$
\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y}.
$$

- *Preservation of Composition.* Given pointed maps

$$
\begin{aligned}
f &\colon (X, x_0) \to (X', x'_0), \\
h &\colon (X', x'_0) \to (X'', x''_0),
\end{aligned}
$$

$$g\colon (Y, y_0) \to (Y', y_0'),$$
$$k\colon (Y', y_0') \to (Y'', y_0''),$$

we have

$$\begin{aligned}
[(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\
&\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\
&\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\
&\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y)
\end{aligned}$$

for each $x \wedge y \in X \wedge Y$, and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.

Item 2, Adjointness: We prove only the adjunction $- \wedge Y \dashv \mathbf{Sets}_*(Y, -)$, witnessed by a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathbf{Sets}_*(X, -)$ is similar. We claim we have a bijection

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, impliying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(X \times Y, Z) \cong \mathsf{Sets}(X, \mathsf{Sets}(Y, Z))$$

of Item 2 of Definition 4.1.3.1.3:

- A map
$$\xi\colon X \times Y \to Z$$
in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ gets sent to the pointed map
$$\begin{aligned}
\xi^\dagger\colon (X, x_0) &\longrightarrow (\mathbf{Sets}_*(Y, Z), \Delta_{z_0}), \\
x &\longmapsto \left(\xi^\dagger_x\colon Y \to Z\right),
\end{aligned}$$
where $\xi^\dagger_x\colon Y \to Z$ is the map defined by
$$\xi^\dagger_x(y) \stackrel{\text{def}}{=} \xi(x, y)$$
for each $y \in Y$, where:

– The map $\xi^\dagger$ is indeed pointed, as we have

$$\begin{aligned}\xi^\dagger_{x_0}(y) &\overset{\text{def}}{=} \xi(x_0, y)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

for each $y \in Y$. Thus $\xi^\dagger_{x_0} = \Delta_{z_0}$ and $\xi^\dagger$ is pointed.

– The map $\xi^\dagger_x$ indeed lies in $\mathsf{Sets}_*(Y, Z)$, as we have

$$\begin{aligned}\xi^\dagger_x(y_0) &\overset{\text{def}}{=} \xi(x, y_0)\\ &\overset{\text{def}}{=} z_0.\end{aligned}$$

- Conversely, a map

$$\begin{aligned}\xi\colon (X, x_0) &\to \big(\mathsf{Sets}_*(Y, Z), \Delta_{z_0}\big),\\ x &\longmapsto (\xi_x\colon Y \to Z),\end{aligned}$$

in $\mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathsf{Sets}_*(Y, Z))$ gets sent to the map

$$\xi^\dagger\colon X \times Y \to Z$$

defined by

$$\xi^\dagger(x, y) \overset{\text{def}}{=} \xi_x(y)$$

for each $(x, y) \in X \times Y$, which indeed lies in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$, as:

– *Left Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x_0, y) &\overset{\text{def}}{=} \xi_{x_0}(y)\\ &\overset{\text{def}}{=} \Delta_{z_0}(y)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

for each $y \in Y$, since $\xi_{x_0} = \Delta_{z_0}$ as ξ is assumed to be a pointed map.

– *Right Bilinearity.* We have

$$\begin{aligned}\xi^\dagger(x, y_0) &\overset{\text{def}}{=} \xi_x(y_0)\\ &\overset{\text{def}}{=} z_0\end{aligned}$$

for each $x \in X$, since $\xi_x \in \mathsf{Sets}_*(Y, Z)$ is a morphism of pointed sets.

This finishes the proof.
Item 3, Enriched Adjointness: This follows from Item 2 and **??** of **??**.
Item 4, As a Pushout: Following the description of Definition 4.2.4.1.3, we have

$$\mathrm{pt} \coprod_{X \vee Y} (X \times Y) \cong (\mathrm{pt} \times (X \times Y))/\sim,$$

where $\sim$ identifies the elemenet $\star$ in pt with all elements of the form (x_0, y) and (x, y_0) in $X \times Y$. Thus Item 4 of Definition 10.6.2.1.3 coupled with Definition 7.5.1.1.8 then gives us a well-defined map

$$\mathrm{pt} \coprod_{X \vee Y} (X \times Y) \to X \wedge Y$$

via $[(\star, (x, y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \to \mathrm{pt} \coprod_{X \vee Y} (X \times Y)$$

given by $x \wedge y \mapsto [(\star, (x, y))]$.
Item 5, Distributivity Over Wedge Sums: This follows from Definition 7.5.9.1.1, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_* (Definition 6.3.3.1.1).

□

7.5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

Definition 7.5.2.1.1. The **internal Hom**[23] **of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$[24] consisting of:

- *The Underlying Set.* The set $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ of morphisms of pointed sets from (X, x_0) to (Y, y_0).

- *The Basepoint.* The element

  $$\Delta_{y_0} \colon (X, x_0) \to (Y, y_0)$$

  of $\mathsf{Sets}_*((X, x_0), (Y, y_0))$ given by

  $$\Delta_{y_0}(x) \stackrel{\text{def}}{=} y_0$$

  for each $x \in X$.

[23] For a proof that $\mathbf{Sets}_*$ is indeed the internal Hom of Sets_* with respect to the smash product of pointed sets, see Item 2 of Definition 7.5.1.1.10.
[24] *Further Notation:* Also written $\mathbf{Hom}_{\mathbf{Sets}_*}(X, Y)$.

Proposition 7.5.2.1.2. Let (X, x_0) and (Y, y_0) be pointed sets.

1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathbf{Sets}_*(X, Y)$ define functors

$$\begin{aligned}\mathbf{Sets}_*(X, -)\colon & \quad \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ \mathbf{Sets}_*(-, Y)\colon & \quad \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ \mathbf{Sets}_*(-_1, -_2)\colon & \quad \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f\colon (X, x_0) &\to (A, a_0),\\ g\colon (Y, y_0) &\to (B, b_0),\end{aligned}$$

   the induced map

$$\mathbf{Sets}_*(f, g)\colon \mathbf{Sets}_*(A, Y) \to \mathbf{Sets}_*(X, B)$$

   is given by

$$[\mathbf{Sets}_*(f, g)](\phi) \overset{\text{def}}{=} g \circ \phi \circ f$$

   for each $\phi \in \mathbf{Sets}_*(A, Y)$.

2. *Adjointness.* We have adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

   witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\ \mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

   natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -))\colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\begin{aligned}\mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),\\ \mathbf{Sets}_*(X \wedge Y, Z) &\cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),\end{aligned}$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: This follows from Item 1 of Definition 4.3.5.1.2 and from the equalities

$$\begin{aligned}g \circ \Delta_{y_0} &= \Delta_{z_0},\\ \Delta_{y_0} \circ f &= \Delta_{y_0}\end{aligned}$$

for morphisms $f\colon (K, k_0) \longrightarrow (X, x_0)$ and $g\colon (Y, y_0) \longrightarrow (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2, Adjointness: This is a repetition of Item 2 of Definition 7.5.1.1.10, and is proved there.

Item 3, Enriched Adjointness: This is a repetition of Item 3 of Definition 7.5.1.1.10, and is proved there. □

7.5.3 The Monoidal Unit

Definition 7.5.3.1.1. The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*}\colon \mathsf{pt} \longrightarrow \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

7.5.4 The Associator

Definition 7.5.4.1.1. The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge \circ \left(\wedge \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \wedge \circ \left(\mathrm{id}_{\mathsf{Sets}_*} \times \wedge\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*},$$

as in the diagram

$$
\begin{tikzcd}
 & & \mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\wedge"] & \\
(\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[dr, "\wedge\times\mathrm{id}"'] & & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dl, "\wedge"] \\
 & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[r, "\wedge"'] \arrow[ur, phantom, "\Uparrow \alpha^{\mathsf{Sets}_*}"] & \mathsf{Sets}_*, &
\end{tikzcd}
$$

whose component

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z} \colon (X \wedge Y) \wedge Z \overset{\sim}{\dashrightarrow} X \wedge (Y \wedge Z)$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x \wedge y) \wedge z) \overset{\text{def}}{=} x \wedge (y \wedge z)$$

for each $(x \wedge y) \wedge z \in (X \wedge Y) \wedge Z$.

Proof. Well-Definedness: Let $[((x, y), z)] = [((x', y'), z')]$ be an element in $(X \wedge Y) \wedge Z$. Then either:

1. We have $x = x'$, $y = y'$, and $z = z'$.

2. Both of the following conditions are satisfied:

   (a) We have $x = x_0$ or $y = y_0$ or $z = z_0$.

   (b) We have $x' = x_0$ or $y' = y_0$ or $z' = z_0$.

In the first case, $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ clearly sends both elements to the same element in $X \wedge (Y \wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0 \wedge y_0) \wedge z_0$ of $(X \wedge Y) \wedge Z$, which gets sent to the basepoint $x_0 \wedge (y_0 \wedge z_0)$ of $X \wedge (Y \wedge Z)$.

Being a Morphism of Pointed Sets: As just mentioned, we have

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x_0 \wedge y_0) \wedge z_0) \overset{\text{def}}{=} x_0 \wedge (y_0 \wedge z_0),$$

and thus $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is a morphism of pointed sets.

Invertibility: The inverse of $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is given by the morphism

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \colon X \wedge (Y \wedge Z) \overset{\sim}{\dashrightarrow} (X \wedge Y) \wedge Z$$

defined by

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}(x \wedge (y \wedge z)) \overset{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.

Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &: (X, x_0) \to (X', x_0'), \\ g &: (Y, y_0) \to (Y', y_0'), \\ h &: (Z, z_0) \to (Z', z_0') \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "(f \wedge g) \wedge h"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X',Y',Z'}"] \\ X \wedge (Y \wedge Z) \arrow[r, "f \wedge (g \wedge h)"'] & X' \wedge (Y' \wedge Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x \wedge y) \wedge z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \wedge g(y)) \wedge h(z) \arrow[d, mapsto] \\ x \wedge (y \wedge z) \arrow[r, mapsto] & f(x) \wedge (g(y) \wedge h(z)) \end{tikzcd}$$

and hence indeed commutes, showing α^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since α^{Sets_*} is natural and $\alpha^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to α^{Sets_*}, it follows from Item 2 of Definition 11.9.7.1.2 that $\alpha^{\mathsf{Sets}_*,-1}$ is also natural. Thus α^{Sets_*} is a natural isomorphism. □

7.5.5 The Left Unitor

Definition 7.5.5.1.1. The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}_*}\colon \wedge\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times \mathrm{id}_{\mathsf{Sets}_*}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}\qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Sets}_*}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*}_X\colon S^0\wedge X \overset{\sim}{\dashrightarrow} X$$

at $X\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} 0\wedge x &\mapsto x_0,\\ 1\wedge x &\mapsto x \end{aligned}$$

for each $x\in X$.

Proof. *Well-Definedness*: Let $[(x,y)]=[(x',y')]$ be an element in $S^0\wedge X$. Then either:

1. We have $x=x'$ and $y=y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x=0$ or $y=x_0$.
   (b) We have $x'=0$ or $y'=x_0$.

In the first case, $\lambda^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $0\wedge x_0$ of $S^0\wedge X$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\lambda^{\mathsf{Sets}_*}_X(0\wedge x_0)\overset{\mathrm{def}}{=}x_0,$$

and thus $\lambda^{\mathsf{Sets}_*}_X$ is a morphism of pointed sets.

Invertibility: The inverse of $\lambda^{\mathsf{Sets}_*}_X$ is the morphism

$$\lambda^{\mathsf{Sets}_*,-1}_X \colon X \overset{\sim}{\dashrightarrow} S^0 \wedge X$$

defined by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x) \overset{\text{def}}{=} 1 \wedge x$$

for each $x \in X$. Indeed:

1. *Invertibility I.* We have

$$\begin{aligned}\left[\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X\right](0 \wedge x) &= \lambda^{\mathsf{Sets}_*,-1}_X\left(\lambda^{\mathsf{Sets}_*}_X(0 \wedge x)\right)\\ &= \lambda^{\mathsf{Sets}_*,-1}_X(x_0)\\ &= 1 \wedge x_0\\ &= 0 \wedge x,\end{aligned}$$

   and

$$\begin{aligned}\left[\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X\right](1 \wedge x) &= \lambda^{\mathsf{Sets}_*,-1}_X\left(\lambda^{\mathsf{Sets}_*}_X(1 \wedge x)\right)\\ &= \lambda^{\mathsf{Sets}_*,-1}_X(x)\\ &= 1 \wedge x\end{aligned}$$

   for each $x \in X$, and thus we have

$$\lambda^{\mathsf{Sets}_*,-1}_X \circ \lambda^{\mathsf{Sets}_*}_X = \mathrm{id}_{S^0 \wedge X}\,.$$

2. *Invertibility II.* We have

$$\begin{aligned}\left[\lambda^{\mathsf{Sets}_*}_X \circ \lambda^{\mathsf{Sets}_*,-1}_X\right](x) &= \lambda^{\mathsf{Sets}_*}_X\left(\lambda^{\mathsf{Sets}_*,-1}_X(x)\right)\\ &= \lambda^{\mathsf{Sets}_*,-1}_X(1 \wedge x)\\ &= x\end{aligned}$$

   for each $x \in X$, and thus we have

$$\lambda^{\mathsf{Sets}_*}_X \circ \lambda^{\mathsf{Sets}_*,-1}_X = \mathrm{id}_X\,.$$

This shows $\lambda^{\mathsf{Sets}_*}_X$ to be invertible.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \longrightarrow (Y, y_0),$$

the diagram

$$\begin{tikzcd}
S^0 \wedge X \arrow[r, "\mathrm{id}_{S^0} \wedge f"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_X"'] & S^0 \wedge Y \arrow[d, "\lambda^{\mathsf{Sets}_*}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \wedge x \arrow[d, mapsto] & & 0 \wedge x \arrow[r, mapsto] & 0 \wedge f(x) \arrow[d, mapsto] \\
x_0 \arrow[r, mapsto] & f(x_0) & & y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \wedge x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \wedge f(x) \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*}$ to be a natural transformation.

Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets}_*}$ is natural and $\lambda^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}_*}$, it follows from Item 2 of Definition 11.9.7.1.2 that $\lambda^{\mathsf{Sets}_*,-1}$ is also natural. Thus $\lambda^{\mathsf{Sets}_*}$ is a natural isomorphism. □

7.5.6 The Right Unitor

Definition 7.5.6.1.1. The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}_*} \colon \wedge \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd}
\mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*}"] \\
& \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\rho_X^{\mathsf{Sets}_*}\colon X \wedge S^0 \xrightarrow{\sim} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x \end{aligned}$$

for each $x \in X$.

Proof. Well-Definedness: Let $[(x,y)] = [(x',y')]$ be an element in $X \wedge S^0$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = 0$.
   (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge 0$ of $X \wedge S^0$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\rho_X^{\mathsf{Sets}_*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus $\rho_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: The inverse of $\rho_X^{\mathsf{Sets}_*}$ is the morphism

$$\rho_X^{\mathsf{Sets}_*,-1}\colon X \xrightarrow{\sim} X \wedge S^0$$

defined by

$$\rho_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

1. *Invertibility I*. We have

$$\begin{aligned} \left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 0) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 0)\right) \\ &= \rho_X^{\mathsf{Sets}_*,-1}(x_0) \\ &= x_0 \wedge 1 \\ &= x \wedge 0, \end{aligned}$$

and

$$\begin{aligned}\left[\rho^{\mathsf{Sets}_*,-1}_X \circ \rho^{\mathsf{Sets}_*}_X\right](x \wedge 1) &= \rho^{\mathsf{Sets}_*,-1}_X\left(\rho^{\mathsf{Sets}_*}_X(x \wedge 1)\right)\\ &= \rho^{\mathsf{Sets}_*,-1}_X(x)\\ &= x \wedge 1\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho^{\mathsf{Sets}_*,-1}_X \circ \rho^{\mathsf{Sets}_*}_X = \mathrm{id}_{X \wedge S^0}.$$

2. *Invertibility II*. We have

$$\begin{aligned}\left[\rho^{\mathsf{Sets}_*}_X \circ \rho^{\mathsf{Sets}_*,-1}_X\right](x) &= \rho^{\mathsf{Sets}_*}_X\left(\rho^{\mathsf{Sets}_*,-1}_X(x)\right)\\ &= \rho^{\mathsf{Sets}_*,-1}_X(x \wedge 1)\\ &= x\end{aligned}$$

for each $x \in X$, and thus we have

$$\rho^{\mathsf{Sets}_*}_X \circ \rho^{\mathsf{Sets}_*,-1}_X = \mathrm{id}_X.$$

This shows $\rho^{\mathsf{Sets}_*}_X$ to be invertible.

Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \wedge S^0 \arrow[r, "f \wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*}_X"'] & Y \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \wedge 0 \arrow[r, mapsto] & f(x) \wedge 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \wedge 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing ρ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since ρ^{Sets_*} is natural and $\rho^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to ρ^{Sets_*}, it follows from Item 2 of Definition 11.9.7.1.2 that $\rho^{\mathsf{Sets}_*,-1}$ is also natural. Thus ρ^{Sets_*} is a natural isomorphism. □

7.5.7 The Symmetry

Definition 7.5.7.1.1. The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\wedge"] \arrow[dr, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\overset{\Vert}{\underset{\Downarrow}{\sigma^{\mathsf{Sets}_*}}}"] & \mathsf{Sets}_*, \\ & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}_*}_{X,Y} \colon X \wedge Y \overset{\sim}{\dashrightarrow} Y \wedge X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ is defined by

$$\sigma^{\mathsf{Sets}_*}_{X,Y}(x \wedge y) \overset{\text{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof. *Well-Definedness*: Let $[(x,y)] = [(x',y')]$ be an element in $X \wedge Y$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\sigma_X^{\mathsf{Sets}_*}(x_0 \wedge y_0) \stackrel{\text{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: The inverse of $\sigma_{X,Y}^{\mathsf{Sets}_*}$ is given by the morphism

$$\sigma_{X,Y}^{\mathsf{Sets}_*,-1} \colon Y \wedge X \overset{\sim}{\dashrightarrow} X \wedge Y$$

defined by

$$\sigma_{X,Y}^{\mathsf{Sets}_*,-1}(y \wedge x) \stackrel{\text{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.
Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "f \wedge g"] \arrow[d, "\sigma_{X,Y}^{\mathsf{Sets}_*}"'] & A \wedge B \arrow[d, "\sigma_{A,B}^{\mathsf{Sets}_*}"] \\ Y \wedge X \arrow[r, "g \wedge f"'] & B \wedge A \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge g(y) \arrow[d, mapsto] \\ y \wedge x \arrow[r, mapsto] & g(y) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since σ^{Sets_*} is natural and $\sigma^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to σ^{Sets_*}, it follows from Item 2 of Definition 11.9.7.1.2 that $\sigma^{\mathsf{Sets}_*,-1}$ is also natural. Thus σ^{Sets_*} is a natural isomorphism. □

7.5.8 The Diagonal

Definition 7.5.8.1.1. The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\wedge}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \wedge \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta^{\wedge}_X\colon (X, x_0) \to (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta^{\wedge}_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow ((X \times X)/\sim, [(x_0, x_0)]) \\ &\overset{\text{def}}{=\!=} (X \wedge X, x_0 \wedge x_0) \end{aligned}$$

in Sets_*, and thus by

$$\Delta^{\wedge}_X(x) \overset{\text{def}}{=} x \wedge x$$

for each $x \in X$.

Proof. *Being a Morphism of Pointed Sets*: We have

$$\Delta^{\wedge}_X(x_0) \overset{\text{def}}{=} x_0 \wedge x_0,$$

and thus $\Delta^{\wedge}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\wedge}_X"'] & Y \arrow[d, "\Delta^{\wedge}_Y"] \\ X \wedge X \arrow[r, "f \wedge f"'] & Y \wedge Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \wedge x \arrow[r, mapsto] & f(x) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\wedge}$ to be natural. □

Proposition 7.5.8.1.2. Let $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

1. *Monoidality.* The diagonal

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},$$

of the smash product of pointed sets is a monoidal natural transformation:

(a) *Compatibility With Strong Monoidality Constraints.* For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes.

(b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd} \qquad \begin{tikzcd} S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0}"] \\ & S^0 \end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned} \Delta^{\wedge}_{S^0} &= \lambda^{\mathsf{Sets}_*,-1}_{S^0} \\ &= \rho^{\mathsf{Sets}_*,-1}_{S^0}, \end{aligned}$$

where we recall that the equalities

$$\begin{aligned}\lambda^{\mathsf{Sets}_*}_{S^0} &= \rho^{\mathsf{Sets}_*}_{S^0},\\ \lambda^{\mathsf{Sets}_*,-1}_{S^0} &= \rho^{\mathsf{Sets}_*,-1}_{S^0}\end{aligned}$$

are always true in any monoidal category by **??** of **??**.

2. *The Diagonal of the Unit.* The component

$$\Delta^{\wedge}_{S^0} \colon S^0 \overset{\sim}{\dashrightarrow} S^0 \wedge S^0$$

of $\Delta^{\wedge}$ at S^0 is an isomorphism.

Proof. *Item 1, Monoidality*: We claim that $\Delta^{\wedge}$ is indeed monoidal:

1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X\wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[dr, mapsto] & (x \wedge x) \wedge (y \wedge y) \arrow[d, mapsto] \\ & (x \wedge y) \wedge (x \wedge y) \end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 7.5.5.1.1, the inverse of the left unitor of Sets_* with respect to to the smash product of pointed sets at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\lambda^{\mathsf{Sets}_*,-1}_{X}(x) \overset{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\lambda^{\mathsf{Sets}_*,-1}_{S^0}(0) \overset{\text{def}}{=} 1 \wedge 0,$$

$$\lambda^{\mathsf{Sets}_*,-1}_{S^0}(1) \stackrel{\text{def}}{=} 1 \wedge 1.$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\Delta^{\wedge}_{S^0}(0) \stackrel{\text{def}}{=} 0 \wedge 0,$$
$$\Delta^{\wedge}_{S^0}(1) \stackrel{\text{def}}{=} 1 \wedge 1,$$

it follows that we indeed have $\Delta^{\wedge}_{S^0} = \lambda^{\mathsf{Sets}_*,-1}_{S^0}$.

This finishes the proof.
Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets_* with respect to $\wedge$, proved in the proof of Definition 7.5.5.1.1 for the left unitor or the proof of Definition 7.5.6.1.1 for the right unitor. □

7.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

Proposition 7.5.9.1.1. The category Sets_* admits a closed monoidal category with diagonals structure consisting of:

- *The Underlying Category.* The category Sets_* of pointed sets.
- *The Monoidal Product.* The smash product functor

  $$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Item 1 of Definition 7.5.1.1.10.
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Sets}_* \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

  of Item 1 of Definition 7.5.2.1.2.
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathrm{pt} \longrightarrow \mathsf{Sets}_*$$

  of Definition 7.5.3.1.1.
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}_*} \colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \xRightarrow{\sim} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 7.5.4.1.1.

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.5.5.1.1.

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}_*}\colon \wedge \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.5.6.1.1.

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}_*}\colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 7.5.7.1.1.

- *The Diagonals.* The monoidal natural transformation

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 7.5.8.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \wedge (X \wedge Y)) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{W,X\wedge Y,Z}"] & \\
((W \wedge X) \wedge Y) \wedge Z \arrow[ur, "\alpha^{\mathsf{Sets}_*}_{W,X,Y} \wedge \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{W\wedge X,Y,Z}"'] & & W \wedge ((X \wedge Y) \wedge Z) \arrow[d, "\mathrm{id}_W \wedge \alpha^{\mathsf{Sets}_*}_{X,Y,Z}"] \\
(W \wedge X) \wedge (Y \wedge Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{W,X,Y\wedge Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \wedge (x \wedge y)) \wedge z \arrow[dr, mapsto] & \\
((w \wedge x) \wedge y) \wedge z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \wedge ((x \wedge y) \wedge z) \arrow[d, mapsto] \\
(w \wedge x) \wedge (y \wedge z) \arrow[rr, mapsto] & & w \wedge (x \wedge (y \wedge z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.
The Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
(X \wedge S^0) \wedge Y \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{X,S^0,Y}"] \arrow[dr, "\rho^{\mathsf{Sets}_*}_X \wedge \mathrm{id}_Y"'] & & X \wedge (S^0 \wedge Y) \arrow[dl, "\mathrm{id}_X \wedge \lambda^{\mathsf{Sets}_*}_Y"] \\
 & X \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[dr, mapsto] & \\
 & x_0 \wedge y
\end{tikzcd}
\qquad
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[r, mapsto] & x \wedge (0 \wedge y) \arrow[dl, mapsto] \\
x \wedge y_0 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
(x \wedge 1) \wedge y \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \wedge (1 \wedge y) \arrow[dl, mapsto] \\
 & x \wedge y, &
\end{tikzcd}$$

and thus we see that the triangle identity is satisfied.

The Left Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (X \wedge Y) \wedge Z \arrow[dl, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] \arrow[dr, "\beta^{\mathsf{Sets}_*}_{X,Y} \wedge \mathrm{id}_Z"] & \\
X \wedge (Y \wedge Z) \arrow[d, "\beta^{\mathsf{Sets}_*}_{X,Y\wedge Z}"'] & & (Y \wedge X) \wedge Z \arrow[d, "\alpha^{\mathsf{Sets}_*}_{Y,X,Z}"] \\
(Y \wedge Z) \wedge X \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{Y,Z,X}"'] & & Y \wedge (X \wedge Z) \arrow[dl, "\mathrm{id}_Y \wedge \beta^{\mathsf{Sets}_*}_{X,Z}"] \\
 & Y \wedge (Z \wedge X) &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (x \wedge y) \wedge z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
x \wedge (y \wedge z) \arrow[d, mapsto] & & (y \wedge x) \wedge z \arrow[d, mapsto] \\
(y \wedge z) \wedge x \arrow[dr, mapsto] & & y \wedge (x \wedge z) \arrow[dl, mapsto] \\
 & y \wedge (z \wedge x) &
\end{tikzcd}$$

and thus we see that the left hexagon identity is satisfied.

The Right Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

We have to show that the diagram

$$\begin{tikzcd}[row sep=small, column sep=tiny]
 & X \wedge (Y \wedge Z) \arrow[dl, "\left(\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\right)^{-1}"'] \arrow[dr, "\mathrm{id}_X \wedge \beta^{\mathsf{Sets}_*}_{Y,Z}"] & \\
(X \wedge Y) \wedge Z \arrow[dd, "\beta^{\mathsf{Sets}_*}_{X \wedge Y,Z}"'] & & X \wedge (Z \wedge Y) \arrow[dd, "\left(\alpha^{\mathsf{Sets}_*}_{X,Z,Y}\right)^{-1}"] \\
 & & \\
Z \wedge (X \wedge Y) \arrow[dr, "\left(\alpha^{\mathsf{Sets}_*}_{Z,X,Y}\right)^{-1}"'] & & (X \wedge Z) \wedge Y \arrow[dl, "\beta^{\mathsf{Sets}_*}_{X,Z} \wedge \mathrm{id}_Y"] \\
 & (Z \wedge X) \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}[row sep=small, column sep=tiny]
 & x \wedge (y \wedge z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(x \wedge y) \wedge z \arrow[dd, mapsto] & & x \wedge (z \wedge y) \arrow[dd, mapsto] \\
 & & \\
z \wedge (x \wedge y) \arrow[dr, mapsto] & & (x \wedge z) \wedge y \arrow[dl, mapsto] \\
 & (z \wedge x) \wedge y &
\end{tikzcd}$$

and thus we see that the right hexagon identity is satisfied.

Monoidal Closedness: This follows from Item 2 of Definition 7.5.1.1.10.

Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Definition 7.5.8.1.2. □

7.5.10 The Universal Property of $\left(\mathsf{Sets}_*, \wedge, S^0\right)$

Theorem 7.5.10.1.1. The symmetric monoidal structure on the category Sets_* of Definition 7.5.9.1.1 is uniquely determined by the following requirements:

1. *Existence of an Internal Hom.* The tensor product

$$\otimes_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of Sets_* admits an internal Hom $[-_1, -_2]_{\mathsf{Sets}_*}$.

2. *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} \cong S^0$.

More precisely, the full subcategory of the category $\mathcal{M}^{\mathrm{cld}}_{\mathbb{E}_\infty}(\mathsf{Sets}_*)$ of **??** spanned by the closed symmetric monoidal categories $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, [-_1, -_2]_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ satisfying Items 1 and 2 is contractible (i.e. equivalent to the punctual category).

Proof. *Unwinding the Statement*: Let $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, [-_1, -_2]_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda', \rho', \sigma')$ be a closed symmetric monoidal category satisfying Items 1 and 2. We need to show that the identity functor

$$\mathrm{id}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \longrightarrow \mathsf{Sets}_*$$

admits a *unique* closed symmetric monoidal functor structure

$$\begin{aligned} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*} &\colon X \otimes_{\mathsf{Sets}_*} Y \overset{\sim}{\dashrightarrow} X \wedge Y, \\ \mathrm{id}^{\mathrm{Hom}}_{\mathsf{Sets}_*} &\colon [X, Y]_{\mathsf{Sets}_*} \overset{\sim}{\dashrightarrow} \mathsf{Sets}_*(X, Y), \\ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} &\colon \mathbb{1}_{\mathsf{Sets}_*} \overset{\sim}{\dashrightarrow} S^0, \end{aligned}$$

making it into a symmetric monoidal strongly closed isomorphism of categories from $(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, [-_1, -_2]_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda', \rho', \sigma')$ to the closed symmetric monoidal category $(\mathsf{Sets}_*, \times, \mathsf{Sets}_*(-_1, -_2), \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*})$ of Definition 7.5.9.1.1.

Constructing an Isomorphism $[-_1, -_2]_{\mathsf{Sets}_*} \cong \mathsf{Sets}_*(-_1, -_2)$: By **??**, we have a natural isomorphism

$$\mathsf{Sets}_*(S^0, [-_1, -_2]_{\mathsf{Sets}_*}) \cong \mathsf{Sets}_*(-_1, -_2).$$

By Item 4 of Definition 6.1.4.1.1, we also have a natural isomorphism

$$\mathsf{Sets}_*(S^0, [-_1, -_2]_{\mathsf{Sets}_*}) \cong [-_1, -_2]_{\mathsf{Sets}_*}.$$

Composing both natural isomorphisms, we obtain a natural isomorphism

$$\mathsf{Sets}_*(-_1, -_2) \cong [-_1, -_2]_{\mathsf{Sets}_*}.$$

Given $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we will write

$$\mathrm{id}^{\mathrm{Hom}}_{X,Y} \colon \mathsf{Sets}_*(X, Y) \overset{\sim}{\dashrightarrow} [X, Y]_{\mathsf{Sets}_*}$$

for the component of this isomorphism at (X, Y).

Constructing an Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: Since $\otimes_{\mathsf{Sets}_*}$ is adjoint in each variable to $[-_1, -_2]_{\mathsf{Sets}_*}$ by assumption and $\wedge$ is adjoint in each variable to $\mathsf{Sets}_*(-_1, -_2)$ by Item 2 of Definition 4.3.5.1.2, uniqueness of adjoints (**??**) gives us natural isomorphisms

$$X \otimes_{\mathsf{Sets}_*} - \cong X \wedge -,$$
$$- \otimes_{\mathsf{Sets}_*} Y \cong Y \wedge -.$$

By **??**, we then have $\otimes_{\mathsf{Sets}_*} \cong \wedge$. We will write

$$\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \colon X \otimes_{\mathsf{Sets}_*} Y \overset{\sim}{\dashrightarrow} X \wedge Y$$

for the component of this isomorphism at (X, Y).

Alternative Construction of an Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: Alternatively, we may construct a natural isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$ as follows:

1. Let $X \in \mathrm{Obj}(\mathsf{Sets}_*)$.

2. Since $\otimes_{\mathsf{Sets}_*}$ is part of a closed monoidal structure, it preserves colimits in each variable by **??**.

3. Since $X \cong \bigvee_{x\in X^-} S^0$ and $\otimes_{\mathsf{Sets}_*}$ preserves colimits in each variable, we have

$$\begin{aligned} X \otimes_{\mathsf{Sets}_*} Y &\cong \left(\bigvee_{x\in X^-} S^0\right) \otimes_{\mathsf{Sets}_*} Y \\ &\cong \bigvee_{x\in X^-} \left(S^0 \otimes_{\mathsf{Sets}_*} Y\right) \\ &\cong \bigvee_{x\in X^-} Y \\ &\cong \bigvee_{x\in X^-} S^0 \wedge Y \\ &\cong \left(\bigvee_{x\in X^-} S^0\right) \wedge Y \\ &\cong X \wedge Y, \end{aligned}$$

   naturally in $Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, where we have used that S^0 is the monoidal unit for $\otimes_{\mathsf{Sets}_*}$. Thus $X \otimes_{\mathsf{Sets}_*} - \cong X \wedge -$ for each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$.

4. Similarly, $- \otimes_{\mathsf{Sets}_*} Y \cong - \wedge Y$ for each $Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

5. By **??**, we then have $\otimes_{\mathsf{Sets}_*} \cong \wedge$.

Below, we'll show that if a natural isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$ exists, then it must be unique. This will show that the isomorphism constructed above is equal to the isomorphism $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \colon X \otimes_{\mathsf{Sets}_*} Y \to X \wedge Y$ from before.

Constructing an Isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}_*} \to S^0$: We define an isomorphism $\mathrm{id}^{\otimes}_{\mathbb{1}} \colon \mathbb{1}_{\mathsf{Sets}_*} \to S^0$ as the composition

$$\mathbb{1}_{\mathsf{Sets}_*} \underset{\sim}{\overset{\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}}{\dashrightarrow}} \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \underset{\sim}{\overset{\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*}}}{\dashrightarrow}} \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \underset{\sim}{\overset{\lambda'_{S^0}}{\dashrightarrow}} S^0$$

in Sets_*.

Monoidal Left Unity of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$\begin{tikzcd}
& S^0 \otimes_{\mathsf{Sets}_*} X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X}"] & S^0 \wedge X \arrow[dr, "\lambda^{\mathsf{Sets}_*}_X"] & \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_X"] \arrow[rrr, "\lambda'_X"'] & & & X
\end{tikzcd}$$

commutes. To this end, we will first show that the diagram

$$\begin{tikzcd}
& S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, phantom, "(\dagger)"] & S^0 \wedge S^0 \arrow[dr, "\lambda^{\mathsf{Sets}_*}_{S^0}"] & \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[rrr, "\lambda'_{S^0}"'] & {} & & S^0,
\end{tikzcd}$$

corresponding to the case $X = S^0$, commutes. Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,-1}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] \arrow[dd, equal, bend right=60] \arrow[dr, phantom, "(1)"] & (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0,S^0}"] \arrow[dr, phantom, "(2)"] & (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\lambda'_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0,S^0}"] \arrow[dr, phantom, "(3)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[r, bend left, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}} \wedge \mathrm{id}_{S^0}"] \arrow[r, bend right, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"'] \arrow[r, phantom, "(4)"] \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] \arrow[dr, phantom, "(5)"] & (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0} \wedge \mathrm{id}_{S^0}"] \arrow[dr, phantom, "(6)"] & (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \wedge S^0 \arrow[r, "\lambda'_{S^0} \wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"] \arrow[dr, phantom, "(7)"] & S^0 \wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0} = \lambda^{\mathsf{Sets}_*}_{S^0}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[urr, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"] \arrow[rr, equal] & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\lambda'_{S^0}"'] & S^0,
\end{tikzcd}$$

whose boundary diagram corresponds to the diagram (†) above. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$.
- Subdiagram (4) commutes by **??**.
- Subdiagram (5) commutes by the naturality of $\rho^{\mathsf{Sets}_*,-1}$.
- Subdiagram (6) commutes trivially.
- Subdiagram (7) commutes by the naturality of ρ^{Sets_*}, where the equality $\rho^{\mathsf{Sets}_*}_{S^0} = \lambda^{\mathsf{Sets}_*}_{S^0}$ comes from **??**.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram (†) above. As a result, the diagram

$$\begin{tikzcd}
& S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, phantom, "(\ddagger)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] & \\
S^0 \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\lambda'^{,-1}_{S^0}"'] & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0
\end{tikzcd}$$

also commutes. Now, let $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, and consider the diagram

$$\begin{tikzcd}
& S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[dd, "\mathrm{id}_{S^0} \wedge [x]"] \arrow[drr, phantom, "(\ddagger)"] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0}"] \arrow[dd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} [x]"] & \\
S^0 \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrrr, "\lambda'^{,-1}_{S^0}"] \arrow[dd, "{[x]}"'] \arrow[dr, phantom, "(3)"] & & \arrow[d, phantom, "(1)"] & \arrow[dr, phantom, "(4)"] & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}_*}} \wedge [x]"] \\
& S^0 \wedge X \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X}"] \arrow[drr, phantom, "(2)"] & (5) & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X"] & \\
X \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_X"] \arrow[rrrr, "\lambda'^{,-1}_X"] & & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X.
\end{tikzcd}$$

Since:

- Subdiagram (5) commutes by the naturality of $\lambda'^{,-1}$.

- Subdiagram (‡) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathsf{Sets}_*,-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & & S^0 \wedge X \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X}"] & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_X"] & \\
S^0 \arrow[r, "{[x]}"] & X \arrow[ur, "\lambda^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\lambda'^{,-1}_X"'] & & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X
\end{tikzcd}$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{aligned}
\lambda'^{,-1}_X(x) &= \left[\lambda'^{,-1}_X \circ [x]\right](1) \\
&= \left[\left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \lambda^{\mathsf{Sets}_*,-1}_X \circ [x]\right](1) \\
&= \left[\left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \lambda^{\mathsf{Sets}_*,-1}_X\right](x)
\end{aligned}$$

for each $x \in X$, and thus we have

$$\lambda'^{,-1}_X = \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \lambda^{\mathsf{Sets}_*,-1}_X.$$

Taking inverses then gives

$$\lambda'_X = \lambda^{\mathsf{Sets}_*}_X \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X} \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_X\right),$$

showing that the diagram

$$\begin{tikzcd}
 & S^0 \otimes_{\mathsf{Sets}_*} X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X}"] & S^0 \wedge X \arrow[dr, "\lambda^{\mathsf{Sets}_*}_X"] & \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} X \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_X"] \arrow[rrr, "\lambda'_X"'] & & & X
\end{tikzcd}$$

indeed commutes.

Braidedness of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the

diagram

$$\begin{tikzcd}
X \otimes_{\mathsf{Sets}_*} Y \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y}"] \arrow[d, "\sigma'_{X,Y}"'] & X \wedge Y \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"] \\
Y \otimes_{\mathsf{Sets}_*} X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,X}"'] & Y \wedge X
\end{tikzcd}$$

commutes. To this end, we will first show that the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, "\sigma'_{S^0,S^0}"'] \arrow[dr, phantom, "(\dagger)"] & S^0 \wedge S^0 \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,S^0}"] \\
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"'] & S^0 \wedge S^0
\end{tikzcd}$$

commutes. To that end, we will first show that the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(\ddagger)"] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0
\end{tikzcd}$$

commutes, and, to this end, we will first show that the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[dr, phantom, "(\S)"] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[u, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[r, "\rho'_{S^0}"'] & S^0
\end{tikzcd}$$

commutes. Indeed, consider the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrrr, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[dddd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"'] \arrow[ddrrr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] & & & & S^0 \wedge S^0 \arrow[dddd, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\
 & (3) & (1) & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0}"] & \\
(2) & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[uul, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[rr, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[dr, "\rho'_{\mathbb{1}_{\mathsf{Sets}_*}} = \lambda'_{\mathbb{1}_{\mathsf{Sets}_*}}"] & & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[u, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[ddr, "\lambda'_{S^0}"] \arrow[r, phantom, "(4)"] & \\
 & (6) & \mathbb{1}_{\mathsf{Sets}_*} \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] & (5) & \\
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[uur, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[rrrr, "\rho'_{S^0}"'] & & & & S^0
\end{tikzcd}
$$

whose boundary diagram corresponds to diagram (§) above. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagrams (2) and (3) commute by the functoriality of $\otimes$;
- Subdiagram (4) commutes by the left monoidal unity of $\left(\mathrm{id}^{\otimes}, \mathrm{id}^{\otimes}_{\mathbb{1}}\right)$, which we proved above;
- Subdiagram (5) commutes by the naturality of λ';
- Subdiagram (6) commutes by the naturality of ρ', where the equality $\rho'_{\mathbb{1}_{\mathsf{Sets}_*}} = \lambda'_{\mathbb{1}_{\mathsf{Sets}_*}}$ comes from **??**;

it follows that the boundary diagram, i.e. diagram (§), also commutes.

Next, consider the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[rrr, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dr, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[ddd, "\rho'_{S^0}"] \arrow[dddd, bend right=60, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] & & & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[ddd, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dddd, bend left=60, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \\
 & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[u, phantom, "(2)"] \arrow[dd, phantom, "(\S)"] & S^0 \wedge S^0 \arrow[ur, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}}"'] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \arrow[dr, phantom, "(3)"] & \\
 & & S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[dll, equal] \arrow[d, phantom, "(4)"] & \\
S^0 \arrow[rrr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}}"] \arrow[d, "\lambda'^{,-1}_{S^0}"] \arrow[drrr, phantom, "(5)"] \arrow[u, phantom, "(1)"] & & & \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[u, phantom, "(6)"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & & & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0
\end{tikzcd}
$$

whose boundary diagram corresponds to the diagram (‡) above. Since:

- Subdiagrams (1) and (6) commute by **??**;
- Subdiagram (2) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagram (§) commutes, as was shown above;
- Subdiagram (3) commutes by the naturality of $\lambda^{\mathsf{Sets}_*}$;
- Subdiagram (4) commutes trivially;
- Subdiagram (5) commutes by Item 2c of Item 2 of Definition 13.1.1.1.4, whose proof uses only the left monoidal unity of $(\mathrm{id}^{\otimes}, \mathrm{id}^{\otimes}_{\mathbb{1}})$, which has been proven above;

it follows that the boundary diagram, i.e. diagram (‡), also commutes.

Next, consider the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\mathrm{id}^{\otimes}_{S^0,S^0}"] \arrow[ddd, "\sigma'_{S^0,S^0}"'] \arrow[dr, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] \arrow[ddd, phantom, shift left=12ex, "(2)"] & & & S^0 \wedge S^0 \arrow[dl, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}}"] \arrow[ddd, "\sigma^{\mathsf{Sets}_*}_{S^0,S^0}"] \arrow[ddd, phantom, shift right=12ex, "(3)"'] \\
& S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}^{\otimes}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dr, phantom, "(\ddagger)"] \arrow[r, phantom, shift left=8ex, "(1)"] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] & \\
& \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[r, phantom, shift right=8ex, "(4)"'] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[dr, "\mathrm{id}^{\otimes}_{\mathbb{1}} \wedge \mathrm{id}_{S^0}"] & \\
S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrr, "\mathrm{id}^{\otimes}_{S^0,S^0}"'] & & & S^0 \wedge S^0,
\end{tikzcd}
$$

whose boundary diagram corresponds to the diagram (†). Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;
- Subdiagram (2) commutes by the naturality of σ' and the fact that $\mathrm{id}^{\otimes}_{\mathbb{1}}$ is invertible;
- Subdiagram (‡) commutes as proved above;
- Subdiagram (3) commutes by the naturality of σ^{Sets_*} and the fact that $\mathrm{id}^{\otimes}_{\mathbb{1}}$ is invertible;
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;

it follows that the boundary diagram, i.e. diagram (†) also commutes.

Taking inverses for the diagram (†), we see that the diagram

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] \arrow[dr, phantom, "(\P)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\sigma'^{,-1}_{S^0,S^0}"] \\
S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"'] & S^0 \otimes_{\mathsf{Sets}_*} S^0
\end{tikzcd}$$

commutes as well. Now, let $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, let $y \in Y$, and consider the diagram

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dr, "{[y]\wedge[x]}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[y]\otimes_{\mathsf{Sets}_*}[x]}"] \arrow[dd, "\sigma'^{,-1}_{S^0,S^0}" near start] & \\
& Y \wedge X \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,A}" near start] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{A,Y}" near start] & & Y \otimes_{\mathsf{Sets}_*} X \arrow[dd, "\sigma'^{,-1}_{A,Y}"] \\
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}" near start] \arrow[dr, "{[x]\wedge[y]}"'] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[x]\otimes_{\mathsf{Sets}_*}[y]}"] & \\
& X \wedge Y \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|A,Y}"'] & & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}$$

which we partition into subdiagrams as follows:

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dr, "{[y]\wedge[x]}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[y]\otimes_{\mathsf{Sets}_*}[x]}"] \arrow[d, phantom, "(1)"] & \\
& Y \wedge X \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{X,Y}"] \arrow[d, phantom, "(2)"'] & & Y \otimes_{\mathsf{Sets}_*} X \arrow[dd, "\sigma'^{,-1}_{X,Y}"] \\
S^0 \wedge S^0 \arrow[dr, "{[x]\wedge[y]}"'] & & (3) & \\
& X \wedge Y \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y}"'] & & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}
\qquad
\begin{tikzcd}
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dd, "\sigma^{\mathsf{Sets}_*,-1}_{S^0,S^0}"'] \arrow[ddrr, phantom, "(\P)"] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[y]\otimes_{\mathsf{Sets}_*}[x]}"] \arrow[dd, "\sigma'^{,-1}_{S^0,S^0}"] & \\
& & & Y \otimes_{\mathsf{Sets}_*} X \arrow[dd, "\sigma'^{,-1}_{X,Y}"] \arrow[dl, phantom, "(4)"] \\
S^0 \wedge S^0 \arrow[rr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[dr, "{[x]\wedge[y]}"'] \arrow[drr, phantom, "(5)"] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "{[x]\otimes_{\mathsf{Sets}_*}[y]}"] & \\
& X \wedge Y \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y}"'] & & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}$$

Since:

- Subdiagram (2) commutes by the naturality of $\sigma^{\mathsf{Sets}_*,-1}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.
- Subdiagram (¶) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\sigma'^{,-1}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}$.

it follows that the diagram

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[dr, "{[y]\wedge[x]}"] & & \\
& Y \wedge X \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,X}"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"'] & Y \otimes_{\mathsf{Sets}_*} X \arrow[d, "\sigma'_{X,Y}"] \\
& X \wedge Y \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y}"'] & X \otimes_{\mathsf{Sets}_*} Y
\end{tikzcd}$$

commutes. We then have

$$\begin{aligned}
\left[\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \circ \sigma^{\mathsf{Sets}_*,-1}_{X,Y}\right](y,x) &= \left[\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \circ \sigma^{\mathsf{Sets}_*,-1}_{X,Y} \circ ([y] \wedge [x])\right](1,1) \\
&= \left[\sigma'^{,-1}_{X,Y} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X} \circ ([y] \wedge [x])\right](1,1) \\
&= \left[\sigma'^{,-1}_{X,Y} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X}\right](y,x)
\end{aligned}$$

for each $(y,x) \in Y \wedge X$, and thus we have

$$\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \circ \sigma^{\mathsf{Sets}_*,-1}_{X,Y} = \sigma'^{,-1}_{X,Y} \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,X}.$$

Taking inverses then gives

$$\sigma^{\mathsf{Sets}_*}_{X,Y} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} = \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,X} \circ \sigma'_{X,Y},$$

showing that the diagram

$$\begin{tikzcd}
A \otimes_{\mathsf{Sets}_*} B \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|A,B}"] \arrow[d, "\sigma'_{A,B}"'] & A \wedge B \arrow[d, "\sigma^{\mathsf{Sets}_*}_{A,B}"] \\
B \otimes_{\mathsf{Sets}_*} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|B,A}"'] & B \wedge A
\end{tikzcd}$$

indeed commutes.

Monoidal Right Unity of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$
\begin{tikzcd}
 & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,S^0}"] & X \wedge S^0 \arrow[dr, "\rho^{\mathsf{Sets}_*}_X"] & \\
X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[ur, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[rrr, "\rho'_X"'] & & & X
\end{tikzcd}
$$

commutes. To this end, we will first show that the diagram

$$
\begin{tikzcd}
 & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, phantom, "(\dagger)"] & S^0 \wedge S^0 \arrow[dr, "\rho^{\mathsf{Sets}_*}_{S^0}"] & \\
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[rrr, "\rho'_{S^0}"'] & {} & & S^0,
\end{tikzcd}
$$

corresponding to the case $X = S^0$, commutes. First, notice that we may write

$$\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}} : S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \to \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0$$

as the composition

$$
\begin{aligned}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} &\xrightarrow{\mathrm{id}^{\otimes}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}} S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \\
&\xrightarrow{\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}} \mathbb{1}_{\mathsf{Sets}_*} \\
&\xrightarrow{\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}} \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \\
&\xrightarrow{\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}} \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0.
\end{aligned}
$$

Indeed, we may write this composition as part of the diagram

$$
\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}^{\otimes}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(1)"] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[dr, phantom, "(2)"] & \mathbb{1}_{\mathsf{Sets}_*} \arrow[dl, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0,
\end{tikzcd}
$$

which commutes since:

- Subdiagram (1) commutes by the braidedness of $\mathrm{id}^{\otimes}$, as proved above.
- Subdiagram (2) commutes by **??**.

Next, consider the diagram

$$\begin{tikzcd}
S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(1)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \arrow[dr, phantom, "(2)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \lambda'_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"] \arrow[dr, phantom, "(3)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \\
S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}_{S^0} \wedge \rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*}}"'] \arrow[dr, phantom, "(4)"] & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[r, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \arrow[dr, phantom, "(5)"] & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[r, "\mathrm{id}_{S^0} \wedge \lambda'_{S^0}"] \arrow[d, "\lambda^{\mathsf{Sets}_*}_{\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0}"] \arrow[dr, phantom, "(6)"] & S^0 \wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0} = \rho^{\mathsf{Sets}_*}_{S^0}"] \\
\mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\rho^{\mathsf{Sets}_*,-1}_{\mathbb{1}_{\mathsf{Sets}_*}}"] & \mathbb{1}_{\mathsf{Sets}_*} \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & \mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\lambda'_{S^0}"] & S^0,
\end{tikzcd}$$

whose boundary diagram corresponds to the diagram (†) above, since the composition in red is equal to $\sigma'_{S^0,\mathbb{1}_{\mathsf{Sets}_*}}$ as proved above, and then the composition in red composed with λ'_{S^0} is equal to ρ'_{S^0} by **??**. In this diagram:

- Subdiagrams (1), (2), and (3) commute by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$.
- Subdiagrams (4), (5), and (6) commute by the naturality of $\lambda^{\mathsf{Sets}_*}$, where the equality $\lambda^{\mathsf{Sets}_*}_{S^0} = \rho^{\mathsf{Sets}_*}_{S^0}$ comes from **??**.

Since all subdiagrams commute, so does the boundary diagram, i.e. the diagram (†) above. As a result, the diagram

$$\begin{tikzcd}
 & S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[d, phantom, "(\ddagger)"] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] & \\
S^0 \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\rho'^{,-1}_{S^0}"'] & & & S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}.
\end{tikzcd}$$

also commutes. Now, let $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, and consider the

diagram

$$\begin{tikzcd}
 & S^0 \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \arrow[dd, "\mathrm{id}_{S^0} \wedge [x]"] \arrow[r, phantom, "(\dagger)", shift right=4ex] & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[dd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} [x]"] & \\
S^0 \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[rrr, "\rho'^{,-1}_{S^0}"] \arrow[dd, "{[x]}"'] \arrow[dr, phantom, "(3)"] & & \arrow[l, phantom, "(1)"] \arrow[dr, phantom, "(4)"] & S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[dd, "\mathrm{id}_{\mathbb{1}_{\mathsf{Sets}_*}} \wedge [x]"] \\
 & X \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,S^0}"] \arrow[r, phantom, "(5)", shift left=4ex] \arrow[r, phantom, "(2)", shift right=4ex] & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] & \\
X \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\rho'^{,-1}_X"] & & & X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}.
\end{tikzcd}$$

Since:

- Subdiagram (5) commutes by the naturality of $\rho'^{,-1}$.
- Subdiagram ($\dagger$) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}$.
- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (3) commutes by the naturality of $\rho^{\mathsf{Sets}_*,-1}$.

it follows that the diagram

$$\begin{tikzcd}
 & & X \wedge S^0 \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,S^0}"] & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] & \\
S^0 \arrow[r, "{[x]}"] & X \arrow[ur, "\rho^{\mathsf{Sets}_*,-1}_X"] \arrow[rrr, "\rho'^{,-1}_X"'] & & & X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}
\end{tikzcd}$$

Here's a step-by-step showcase of this argument: [Link]. We then have

$$\begin{aligned}
\rho'^{,-1}_X(a) &= \left[\rho'^{,-1}_X \circ [x]\right](1) \\
&= \left[\left(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \rho^{\mathsf{Sets}_*,-1}_X \circ [x]\right](1) \\
&= \left[\left(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \rho^{\mathsf{Sets}_*,-1}_X\right](a)
\end{aligned}$$

for each $a \in X$, and thus we have

$$\rho'^{,-1}_X = \left(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,X} \circ \rho^{\mathsf{Sets}_*,-1}_X.$$

Taking inverses then gives

$$\rho'_X = \rho^{\mathsf{Sets}_*}_X \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,X} \circ \left(\mathrm{id}_X \wedge \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}\right),$$

showing that the diagram

$$\begin{tikzcd}
 & X \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,S^0}"] & X \wedge S^0 \arrow[dr, "\rho^{\mathsf{Sets}_*}_X"] & \\
X \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[ur, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[rrr, "\rho'_X"'] & & & X
\end{tikzcd}$$

indeed commutes.

Monoidality of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: We have to show that the diagram

$$\begin{tikzcd}
 & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] \arrow[dr, "\alpha'_{X,Y,Z}"] & \\
(X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X\wedge Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,Z}"] \\
(X \wedge Y) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y\wedge Z}"] \\
 & X \wedge (Y \wedge Z) &
\end{tikzcd}$$

commutes. To this end, we will first prove that the diagram

$$\begin{tikzcd}
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[dr, "\alpha'_{S^0,S^0,S^0}"] & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0\wedge S^0,S^0}"'] \arrow[rr, phantom, "(\dagger)"] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \\
(S^0 \wedge S^0) \wedge S^0 \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,S^0,S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0\wedge S^0}"] \\
 & S^0 \wedge (S^0 \wedge S^0) &
\end{tikzcd}$$

commutes, and, to that end, we will first show that the diagram

$$
\begin{tikzcd}
 & (S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*},S^0}"'] \arrow[rr, phantom, "(\ddagger)" near end, shift right=8ex] & & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \wedge S^0 \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \\
 & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) &
\end{tikzcd}
$$

commutes. Indeed, consider the diagram

$$
\begin{tikzcd}
 & & (S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[dl, "(\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[ddr, "\rho'_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[ddrr, "\alpha'_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & & \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[dddd, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*},S^0}"'] \arrow[r, phantom, "(1)"] & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[r, phantom, "(2)"] & & \arrow[d, phantom, "(3)"] & \\
 & (S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, "\rho^{\mathsf{Sets}_*}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge S^0,S^0}"] \arrow[dr, phantom, "(5)"] & & S^0 \otimes_{\mathsf{Sets}_*} S^0 \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \lambda'^{,-1}_{S^0}"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"'] \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[dr, phantom, "(9)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"] \\
\arrow[r, phantom, "(4)"] & (S^0 \wedge S^0) \wedge S^0 \arrow[r, bend left, "\rho^{\mathsf{Sets}_*}_{S^0} \wedge \mathrm{id}_{S^0}"] \arrow[r, bend right, "\rho^{\mathsf{Sets}_*}_{S^0 \wedge S^0}"'] \arrow[r, phantom, "(6)"] \arrow[dl, "(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}) \wedge \mathrm{id}_{S^0}"'] \arrow[d, phantom, "(7)"] & S^0 \wedge S^0 \arrow[d, "\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}"] \arrow[r, phantom, "(8)"] & S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*} \arrow[r, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \rho^{\mathsf{Sets}_*,-1}_{S^0}"] \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[d, phantom, "(11)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[ddll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"] \\
(S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \wedge S^0 \arrow[rr, "\rho^{\mathsf{Sets}_*}_{S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}}"] \arrow[drr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & \arrow[d, phantom, "(10)"] & S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*} \arrow[d, "\mathrm{id}_{S^0} \wedge \rho^{\mathsf{Sets}_*}_{S^0}"] & & \\
 & & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) & &
\end{tikzcd}
$$

whose boundary diagram corresponds to diagram (‡) above. Since:

- Subdiagrams (1), (4), (5), (8), and (11) commute by the naturality of $\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}$;

- Subdiagram (2) commutes by the right monoidal unity of $\left(\mathrm{id}^{\otimes}_{\mathsf{Sets}_*}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*}\right)$;
- Subdiagram (3) commutes by the triangle identity for $(\alpha', \lambda', \rho')$;
- Subdiagram (6) commutes by **??**;
- Subdiagram (7) commutes by the naturality of ρ^{Sets_*};
- Subdiagram (9) commutes by **??**;
- Subdiagram (10) commutes by **??**;

it follows that the boundary diagram, i.e. diagram (‡), also commutes. Consider now the diagram

$$\begin{tikzcd}
 & & & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dlll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[d, "\left(\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}\right) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] \arrow[drrr, "\alpha'_{S^0,S^0,S^0}"] & & & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, phantom, ""] \arrow[ddd, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge S^0,S^0}"'] \arrow[drr, "\left(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}\right) \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"] & & {} \arrow[u, phantom, "(1)"] & (S^0 \otimes_{\mathsf{Sets}_*} \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*}} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[dr, "\alpha'_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & {} \arrow[u, phantom, "(2)"] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dll, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}\right)"'] \arrow[ddd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0}"] \\
 & {} \arrow[d, phantom, "(3)"] & (S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \otimes_{\mathsf{Sets}_*} S^0 \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*},S^0}"] & {} \arrow[d, phantom, "(‡)"] & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \otimes_{\mathsf{Sets}_*} S^0) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|\mathbb{1}_{\mathsf{Sets}_*},S^0}"'] & {} \arrow[d, phantom, "(4)"] & \\
 & {} & (S^0 \wedge \mathbb{1}_{\mathsf{Sets}_*}) \wedge S^0 \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{S^0,\mathbb{1}_{\mathsf{Sets}_*},S^0}"] & {} & S^0 \otimes_{\mathsf{Sets}_*} (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0}"'] & {} & \\
(S^0 \wedge S^0) \wedge S^0 \arrow[urr, "\left(\mathrm{id}_{S^0} \wedge \mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*}\right) \wedge \mathrm{id}_{S^0}"] \arrow[drrr, "\alpha^{\mathsf{Sets}_*}_{S^0,S^0,S^0}"'] & & {} \arrow[d, phantom, "(5)"] & S^0 \wedge (\mathbb{1}_{\mathsf{Sets}_*} \wedge S^0) \arrow[d, "\mathrm{id}_{S^0} \wedge \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0}\right)"] & {} \arrow[d, phantom, "(6)"] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[ull, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}_*} \wedge \mathrm{id}_{S^0}\right)"'] \arrow[dlll, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|S^0,S^0 \wedge S^0}"] \\
 & & {} & S^0 \wedge (S^0 \wedge S^0) & {} & &
\end{tikzcd}$$

whose boundary corresponds to diagram (†) above. Since:

- Subdiagrams (1), (3), (4), and (6) commute by the naturality of $\mathrm{id}^{\otimes}_{Sets_*}$;
- Subdiagram (‡) commutes, as proved above;

- Subdiagram (2) commutes by the naturality of α';
- Subdiagram (5) commutes by the naturality of α^{Sets_*};

it follows that the boundary diagram, i.e. diagram (†), also commutes. Taking inverses on the diagram (†), we see that the diagram

$$\begin{tikzcd}
 & S^0 \wedge \left(S^0 \wedge S^0\right) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0\wedge S^0}"] & \\
\left(S^0 \wedge S^0\right) \wedge S^0 \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0\wedge S^0,S^0}"'] \arrow[rr, phantom, "(\dagger)"] & & S^0 \otimes_{\mathsf{Sets}_*} \left(S^0 \wedge S^0\right) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0}"] \\
\left(S^0 \wedge S^0\right) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} \left(S^0 \otimes_{\mathsf{Sets}_*} S^0\right) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"] \\
 & \left(S^0 \otimes_{\mathsf{Sets}_*} S^0\right) \otimes_{\mathsf{Sets}_*} S^0 &
\end{tikzcd}$$

commutes as well. Now, let $X, Y, Z \in \mathrm{Obj}(\mathsf{Sets}_*)$, let $x \in X$, let $y \in Y$, let $z \in Z$, and consider the diagram

$$\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0\wedge S^0}"] \arrow[rrrr, "{[x]\wedge([y]\wedge[z])}"] & & & & X \wedge (Y \wedge Z) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}"] & \\
\left(S^0 \wedge S^0\right) \wedge S^0 \arrow[d, "\mathrm{id}^{\otimes,-1}_{S^0\wedge S^0,S^0}"'] \arrow[rrrr, "{([x]\wedge[y])\wedge[z]}"] & & S^0 \otimes_{\mathsf{Sets}_*} \left(S^0 \wedge S^0\right) \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{S^0,S^0}"] \arrow[rrrr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\wedge[z])}"] & & (X \wedge Y) \wedge Z \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z}"] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}"] \\
\left(S^0 \wedge S^0\right) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] \arrow[rrrr, "{([x]\wedge[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & S^0 \otimes_{\mathsf{Sets}_*} \left(S^0 \otimes_{\mathsf{Sets}_*} S^0\right) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"] \arrow[rrrr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\otimes_{\mathsf{Sets}_*}[z])}"] & & (X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] & & X \otimes_{\mathsf{Sets}_*} \left(Y \otimes_{\mathsf{Sets}_*} Z\right) \arrow[dl, "\alpha'^{,-1}_{X,Y,Z}"] \\
 & \left(S^0 \otimes_{\mathsf{Sets}_*} S^0\right) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rrrr, "{([x]\otimes_{\mathsf{Sets}_*}[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & & & \left(X \otimes_{\mathsf{Sets}_*} Y\right) \otimes_{\mathsf{Sets}_*} Z &
\end{tikzcd}$$

which we partition into subdiagrams as follows:

$$
\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0 \wedge S^0}"] & \\
(S^0 \wedge S^0) \wedge S^0 \arrow[dd, "\mathrm{id}^{\otimes,-1}_{S^0 \wedge S^0,S^0}"'] \arrow[ddrr, phantom, "(\dagger)"] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[dd, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{S^0,S^0}"'] \\
 & & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] & & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"] \\
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S^0 \wedge (S^0 \wedge S^0) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{S^0,S^0,S^0}"'] \arrow[rr, "{[x]\wedge([y]\wedge[z])}"] \arrow[drr, phantom, "(1)"] & & X \wedge (Y \wedge Z) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}"] \\
(S^0 \wedge S^0) \wedge S^0 \arrow[rr, "{([x]\wedge[y])\wedge[z]}"] \arrow[d, "\mathrm{id}^{\otimes,-1}_{S^0 \wedge S^0,S^0}"'] \arrow[drr, phantom, "(2)"] & & (X \wedge Y) \wedge Z \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z}"] & \\
(S^0 \wedge S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, "{([x]\wedge[y])\otimes_{\mathsf{Sets}_*}[z]}"] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}_{S^0}"'] & & (X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"] & \\
 & (S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, "{([x]\otimes_{\mathsf{Sets}_*}[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z
\end{tikzcd}
$$

(3)

$$
\begin{tikzcd}
S^0 \wedge (S^0 \wedge S^0) \arrow[dr, "\mathrm{id}^{\otimes,-1}_{S^0,S^0 \wedge S^0}"'] \arrow[rr, "{[x]\wedge([y]\wedge[z])}"] \arrow[drr, phantom, "(4)"] & & X \wedge (Y \wedge Z) \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}"] & \\
 & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \wedge S^0) \arrow[rr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\wedge[z])}"] \arrow[d, "\mathrm{id}_{S^0} \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{S^0,S^0}"'] \arrow[drr, phantom, "(5)"] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}"] \\
 & S^0 \otimes_{\mathsf{Sets}_*} (S^0 \otimes_{\mathsf{Sets}_*} S^0) \arrow[dl, "\alpha'^{,-1}_{S^0,S^0,S^0}"'] \arrow[rr, "{[x]\otimes_{\mathsf{Sets}_*}([y]\otimes_{\mathsf{Sets}_*}[z])}"] \arrow[drr, phantom, "(6)"] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z). \arrow[dl, "\alpha'^{,-1}_{X,Y,Z}"] \\
(S^0 \otimes_{\mathsf{Sets}_*} S^0) \otimes_{\mathsf{Sets}_*} S^0 \arrow[rr, "{([x]\otimes_{\mathsf{Sets}_*}[y])\otimes_{\mathsf{Sets}_*}[z]}"] & & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z &
\end{tikzcd}
$$

Since:

- Subdiagram (1) commutes by the naturality of $\alpha^{\mathsf{Sets}_*,-1}$.
- Subdiagram (2) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (3) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (†) commutes, as proved above.
- Subdiagram (4) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (5) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*}$.
- Subdiagram (6) commutes by the naturality of $\alpha'^{,-1}$.

it follows that the diagram

$$
\begin{tikzcd}
 & S^0 \wedge \left(S^0 \wedge S^0\right) \arrow[d, "{[x]\wedge([y]\wedge[z])}"] & \\
 & X \wedge (Y \wedge Z) \arrow[dl, "\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}"'] \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[d, "\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}"] \\
(X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dr, "\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] & & X \otimes_{\mathsf{Sets}_*} \left(Y \otimes_{\mathsf{Sets}_*} Z\right) \arrow[dl, "\alpha'^{,-1}_{X,Y,Z}"] \\
 & \left(X \otimes_{\mathsf{Sets}_*} Y\right) \otimes_{\mathsf{Sets}_*} Z &
\end{tikzcd}
$$

also commutes. We then have

$$
\begin{aligned}
\Big[\Big(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z\Big) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z} & \\
\circ\, \alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}\Big](x,(y,z)) &= \Big[\Big(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z\Big) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z} \\
&\qquad \circ\, \alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \circ ([x] \wedge ([y] \wedge [z]))\Big](1,(1,1)) \\
&= \Big[\alpha'^{,-1}_{X,Y,Z} \circ \Big(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}\Big) \\
&\qquad \circ\, \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z} \circ ([x] \wedge ([y] \wedge [z]))\Big](1,(1,1)) \\
&= \Big[\alpha'^{,-1}_{X,Y,Z} \circ \Big(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}\Big) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}\Big](x,(y,z))
\end{aligned}
$$

for each $(x, (y, z)) \in X \wedge (Y \wedge Z)$, and thus we have

$$\left(\mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X\wedge Y,Z} \circ \alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} = \alpha'^{,-1}_{X,Y,Z} \circ \left(\mathrm{id}_X \wedge \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|Y,Z}\right) \circ \mathrm{id}^{\otimes,-1}_{\mathsf{Sets}_*|X,Y\wedge Z}.$$

Taking inverses then gives

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z} \circ \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X\wedge Y,Z} \circ \left(\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z\right) = \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y\wedge Z} \circ \left(\mathrm{id}_X \wedge \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,Z}\right) \circ \alpha'_{X,Y,Z},$$

showing that the diagram

$$\begin{tikzcd}
 & (X \otimes_{\mathsf{Sets}_*} Y) \otimes_{\mathsf{Sets}_*} Z \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y} \otimes_{\mathsf{Sets}_*} \mathrm{id}_Z"'] \arrow[dr, "\alpha'_{X,Y,Z}"] & \\
(X \wedge Y) \otimes_{\mathsf{Sets}_*} Z \arrow[d, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X\wedge Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \otimes_{\mathsf{Sets}_*} Z) \arrow[d, "\mathrm{id}_X \otimes_{\mathsf{Sets}_*} \mathrm{id}^{\otimes}_{\mathsf{Sets}_*|Y,Z}"] \\
(X \wedge Y) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & & X \otimes_{\mathsf{Sets}_*} (Y \wedge Z) \arrow[dl, "\mathrm{id}^{\otimes}_{\mathsf{Sets}_*|X,Y\wedge Z}"] \\
 & X \wedge (Y \wedge Z) &
\end{tikzcd}$$

indeed commutes.

Uniqueness of the Isomorphism $\otimes_{\mathsf{Sets}_*} \cong \wedge$: Let $\phi, \psi \colon -_1 \otimes_{\mathsf{Sets}_*} -_2 \Rightarrow -_1 \wedge -_2$ be natural isomorphisms. Since these isomorphisms are compatible with the unitors of Sets_* with respect to $\wedge$ and $\otimes$ (as shown above), we have

$$\begin{aligned}
\lambda'_Y &= \lambda^{\mathsf{Sets}_*}_Y \circ \phi_{S^0,Y} \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right),\\
\lambda'_Y &= \lambda^{\mathsf{Sets}_*}_Y \circ \psi_{S^0,Y} \circ \left(\mathrm{id}^{\otimes}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right).
\end{aligned}$$

Postcomposing both sides with $\lambda^{\mathsf{Sets}_*,-1}_Y$ and then precomposing both sides with $\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y$ gives

$$\begin{aligned}
\lambda^{\mathsf{Sets}_*,-1}_Y \circ \lambda'_Y \circ \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right) &= \phi_{S^0,Y},\\
\lambda^{\mathsf{Sets}_*,-1}_Y \circ \lambda'_Y \circ \left(\mathrm{id}^{\otimes,-1}_{\mathbb{1}|\mathsf{Sets}} \otimes_{\mathsf{Sets}} \mathrm{id}_Y\right) &= \psi_{S^0,Y},
\end{aligned}$$

and thus we have

$$\phi_{S^0,Y} = \psi_{S^0,Y}$$

for each $Y \in \mathrm{Obj}(\mathsf{Sets}_*)$. Now, let $x \in X$ and consider the naturality diagrams

$$\begin{tikzcd} S^0 \wedge Y \arrow[r, "{[x]\wedge \mathrm{id}_Y}"] \arrow[d, "\phi_{S^0,Y}"'] & X \wedge Y \arrow[d, "\phi_{X,Y}"] \\ S^0 \otimes_{\mathsf{Sets}_*} Y \arrow[r, "{[x]\otimes_{\mathsf{Sets}_*}\mathrm{id}_Y}"'] & X \otimes_{\mathsf{Sets}_*} Y \end{tikzcd} \qquad \begin{tikzcd} S^0 \wedge Y \arrow[r, "{[x]\wedge \mathrm{id}_Y}"] \arrow[d, "\psi_{S^0,Y}"'] & X \wedge Y \arrow[d, "\psi_{X,Y}"] \\ S^0 \otimes_{\mathsf{Sets}_*} Y \arrow[r, "{[x]\otimes_{\mathsf{Sets}_*}\mathrm{id}_Y}"'] & X \otimes_{\mathsf{Sets}_*} Y \end{tikzcd}$$

for ϕ and ψ with respect to the morphisms $[x]$ and id_Y. Having shown that $\phi_{S^0,Y} = \psi_{S^0,Y}$, we have

$$\begin{aligned} \phi_{X,Y}(x,y) &= \left[\phi_{X,Y} \circ ([x] \wedge \mathrm{id}_Y)\right](1,y) \\ &= \left[\left([x] \otimes_{\mathsf{Sets}_*} \mathrm{id}_Y\right) \circ \phi_{S^0,Y}\right](1,y) \\ &= \left[\left([x] \otimes_{\mathsf{Sets}_*} \mathrm{id}_Y\right) \circ \psi_{S^0,Y}\right](1,y) \\ &= \left[\psi_{X,Y} \circ ([x] \wedge \mathrm{id}_Y)\right](1,y) \\ &= \psi_{X,Y}(x,y) \end{aligned}$$

for each $(x,y) \in X \wedge Y$. Therefore we have

$$\phi_{X,Y} = \psi_{X,Y}$$

for each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ and thus $\phi = \psi$, showing the isomorphism $\otimes_{\mathsf{Sets}_*} \cong \times$ to be unique. □

Corollary 7.5.10.1.2. The symmetric monoidal structure on the category Sets_* of Definition 7.5.9.1.1 is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The tensor product
$$\otimes_{\mathsf{Sets}_*} \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$
of Sets_* preserves colimits separately in each variable.
2. *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} \cong S^0$.

More precisely, the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathsf{Sets}_*)$ of **??** spanned by the symmetric monoidal categories $\left(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*}\right)$ satisfying Items 1 and 2 is contractible.

Proof. Since Sets_* is locally presentable (**??**), it follows from **??** that Definition 7.5.10.1.2 is equivalent to the existence of an internal Hom as in Item 1 of Definition 7.5.10.1.1. The result then follows from Definition 7.5.10.1.1. □

Corollary 7.5.10.1.3. The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure, i.e. the full subcategory of the category $\mathcal{M}_{\mathbb{E}_\infty}(\mathsf{Sets}_*)$ of **??** spanned by the symmetric monoidal categories $\left(\mathsf{Sets}_*, \otimes_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*}\right)$ with respect to which $(-)^+$ admits a symmetric monoidal structure is contractible.

Proof. Let $\left(\otimes_{\mathsf{Sets}_*}, \mathbb{1}_{\mathsf{Sets}_*}, \lambda^{\mathsf{Sets}_*}, \rho^{\mathsf{Sets}_*}, \sigma^{\mathsf{Sets}_*}\right)$ be a symmetric monoidal structure on Sets_* such that $(-)^+$ admits a symmetric monoidal structure with respect to $\otimes_{\mathsf{Sets}_*}$ and $\wedge$. We have isomorphisms

$$\begin{aligned}
X \otimes_{\mathsf{Sets}_*} Y &\cong (X^-)^+ \otimes_{\mathsf{Sets}_*} (Y^-)^+ \\
&\cong (X^- \times Y^-)^+ \\
&\cong (X^-)^+ \wedge (Y^-)^+ \\
&\cong X \wedge Y,
\end{aligned}$$

all natural in X and Y. Now, since $\wedge$ preserves colimits in both variables and $\otimes_{\mathsf{Sets}_*} \cong \wedge$, it follows that $\otimes_{\mathsf{Sets}_*}$ also preserves colimits in both variables, so the result then follows from Definition 7.5.10.1.2. □

7.5.11 Monoids With Respect to the Smash Product of Pointed Sets

Proposition 7.5.11.1.1. The category of monoids on $\left(\mathsf{Sets}_*, \wedge, S^0\right)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof. See **??**, in particular **??**, **??**, and **??**. □

7.5.12 Comonoids With Respect to the Smash Product of Pointed Sets

Proposition 7.5.12.1.1. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of Item 4 of Definition 6.4.1.1.2 lifts to an equivalence of categories

$$\begin{aligned}
\mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt}) \\
&\cong \mathsf{Sets}.
\end{aligned}$$

Proof. We follow [PS19, Lemma 2.4].
Faithfulness: Given morphisms $f, g\colon X \longrightarrow Y$, if $f^+ = g^+$, then we have

$$\begin{aligned} f(x) &\overset{\text{def}}{=} f^+(x) \\ &= g^+(x) \\ &\overset{\text{def}}{=} g(x) \end{aligned}$$

for each $x \in X^+$, and thus $f = g$, showing $(-)^+$ to be faithful.
Fullness: Let $f\colon X^+ \longrightarrow Y^+$ be a morphism of comonoids in Sets_*. By counitality, the diagram

$$\begin{tikzcd} X^+ \arrow[rr, "f"] \arrow[dr, "\epsilon^+_X"'] & & Y^+ \arrow[dl, "\epsilon^+_Y"] \\ & S^0 & \end{tikzcd}$$

commutes. If $f(x) = \star_Y$ for $x \neq \star_X$, the commutativity of this diagram then gives

$$\begin{aligned} 1 &= \epsilon^+_X(x) \\ &= \epsilon^+_Y(f(x)) \\ &= \epsilon^+_Y(\star_Y) \\ &= 0, \end{aligned}$$

which is a contradiction. Thus f is an active morphism of pointed sets, so there exists a map f^- such that $(f^-)^+ = f$ (Item 1 of Definition 6.4.2.1.2).
Essential Surjectivity: Let $(X, \Delta_X, \epsilon_X)$ be a comonoid in Sets_*. We claim that

$$\Delta_X(x) = x \wedge x$$

for each $x \in X$ with $x \neq \star_X$. Indeed:

- Suppose that $x \neq \star_X$ and write $\Delta_X(x) = x_1 \wedge x_2$.
- Since $\mathrm{id}_X \wedge \epsilon_X$ is pointed, we have
$$[\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) = \star_{X \wedge S^0}.$$
- The counitality condition for Δ_X, corresponding to the commutativity of the diagram
$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[dr, "\rho^{\mathsf{Sets}_*,-1}_X"'] & X \wedge X \arrow[d, "\mathrm{id}_X \wedge \epsilon_X"] \\ & X \wedge S^0 \end{tikzcd}$$

gives

$$\begin{aligned} x \wedge 1 &= \rho^{\mathsf{Sets}_*,-1}_X(x) \\ &= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\ &= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x)) \\ &= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\ &= \star_{X \wedge S^0}, \end{aligned}$$

which is a contradiction. Thus $x_1 \neq \star_X$.

- Similarly, if $x \neq \star_X$, then $x_2 \neq \star_X$.
- Next, we claim that $\epsilon_X(x_2) = 1$, as otherwise we would have

$$\begin{aligned} \star_{X \wedge S^0} &= x_1 \wedge 0 \\ &= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\ &= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x)) \\ &= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\ &= \rho^{\mathsf{Sets}_*,-1}_X(x) \\ &= x \wedge 1, \end{aligned}$$

a contradiction. Thus $\epsilon_X(x_2) = 1$.

- Similarly, if $x \neq \star_X$, then $\epsilon_X(x_1) = 1$.
- Now, since Δ_X is counital, we have

$$\begin{aligned} x \wedge 1 &= \rho^{\mathsf{Sets}_*,-1}_X(x) \\ &= [\mathrm{id}_X \wedge \epsilon_X \circ \Delta_X](x) \\ &= [\mathrm{id}_X \wedge \epsilon_X](\Delta_X(x)) \\ &= [\mathrm{id}_X \wedge \epsilon_X](x_1 \wedge x_2) \\ &= x_1 \wedge 1, \end{aligned}$$

so $x = x_1$.

- Similarly, $x = x_2$, and we are done.

Next, notice that $X \cong \epsilon_X^{-1}(0) \coprod \epsilon_X^{-1}(1)$, and let $x \in \epsilon_X^{-1}(0)$. We then have

$$\begin{aligned} [(\mathrm{id}_X \wedge \epsilon_X) \circ \Delta_X](x) &= [\mathrm{id}_X \wedge \epsilon_X](x \wedge x) \\ &= x \wedge 0 \end{aligned}$$

$$= \star_{X \wedge S^0}.$$

The counitality condition for Δ_X then gives $x = \star_X$, so $\epsilon_X^{-1}(0) = \{\star_X\}$. Thus we have $\left(\epsilon_X^{-1}(1)\right)^+ \cong X$, and this isomorphism is compatible with the comonoid structures when equipping $\epsilon_X^{-1}(1)$ with its unique comonoid structure. This shows that $(-)^+$ is essentially surjective.

Equivalence: Since $(-)^+$ is fully faithful and essentially surjective, it is an equivalence by Item 1b of Item 1 of Definition 11.6.7.1.2. □

7.6 Miscellany

7.6.1 The Smash Product of a Family of Pointed Sets

Let $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$ be a family of pointed sets.

Definition 7.6.1.1.1. The **smash product of the family** $\left\{\left(X_i, x_0^i\right)\right\}_{i \in I}$ is the pointed set $\bigwedge_{i \in I} X_i$ consisting of:

- *The Underlying Set*. The set $\bigwedge_{i \in I} X_i$ defined by

$$\bigwedge_{i \in I} X_i \stackrel{\text{def}}{=} \left(\prod_{i \in I} X_i\right) \Big/ \sim,$$

  where $\sim$ is the equivalence relation on $\prod_{i \in I} X_i$ obtained by declaring

$$(x_i)_{i \in I} \sim (y_i)_{i \in I}$$

  if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i$.

- *The Basepoint*. The element $[(x_0)_{i \in I}]$ of $\bigwedge_{i \in I} X_i$.

Appendices

7.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Part III

Relations

Chapter 8

Relations

This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 8.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 8.1.2).
3. The various kinds of categories that relations form, namely:
   (a) A category (Section 8.3.2).
   (b) A monoidal category (Section 8.3.3).
   (c) A 2-category (Section 8.3.4).
   (d) A double category (Section 8.3.5).
4. The various categorical properties of the 2-category of relations, including:
   (a) The self-duality of Rel and **Rel** (Definition 8.5.1.1.1).
   (b) Identifications of equivalences and isomorphisms in **Rel** with bijections (Definition 8.5.2.1.2).
   (c) Identifications of adjunctions in **Rel** with functions (Definition 8.5.3.1.1).
   (d) Identifications of monads in **Rel** with preorders (**??**).
   (e) Identifications of comonads in **Rel** with subsets (**??**).
   (f) A description of the monoids and comonoids in **Rel** with respect to the Cartesian product (Definition 8.5.9.1.1).

(g) Characterisations of monomorphisms in Rel (Definition 8.5.10.1.1).

(h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (Definition 8.5.11.1.1).

(i) Characterisations of epimorphisms in Rel (Definition 8.5.12.1.1).

(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Definition 8.5.13.1.1).

(k) The partial co/completeness of Rel (Definition 8.5.14.1.1).

(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (Definition 15.2.1.1.7).

(m) The closedness of **Rel** (Definition 8.5.19.1.1).

(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Definition 8.5.20.1.1).

5. The adjoint pairs

$$R_! \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_* \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R \colon A \nrightarrow B$, as well as the properties of $R_!$, R_{-1}, R^{-1}, and R_* (Section 8.7).

Of particular note are the following points:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_! \dashv f^{-1} \dashv f_*$ induced by a function $f \colon A \to B$ studied in Section 4.6.

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Definition 8.7.2.1.3).

(c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

$$R_! \dashv R_{-1} = R^{-1} \dashv R_*$$

reduces to the triple adjunction

$$f_! \dashv f^{-1} \dashv f_*$$

from Section 4.6.

(d) The pairs $R_! \dashv R_{-1}$ and $R^{-1} \dashv R_*$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (**??**).

6. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(C, \mathcal{D})$ appearing in the definition of a relative monad (Sections 8.8 and 8.9).

This chapter is under revision. TODO:

1. Replicate Section 8.5 for apartness composition
2. Revise Section 8.7
3. Add subsection "A Six Functor Formalism for Sets, Part 2", now with relations, building upon Section 8.7.
4. Replicate Section 8.7 for apartness composition
5. Revise sections on skew monoidal structures on $\mathbf{Rel}(A, B)$
6. Replicate the sections on skew monoidal structures on $\mathbf{Rel}(A, B)$ for apartness composition.
7. Explore relative co/monads in Rel, defined to be co/monoids in $\mathbf{Rel}(A, B)$ with its left/right skew monoidal structures of Sections 8.8 and 8.9
8. functional total relations defined with "satisfying the following equivalent conditions:"

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8.1 Relations

8.1.1 Foundations

Let A and B be sets.

Definition 8.1.1.1.1. A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is equivalently:

1. A subset R of $A \times B$.
2. A function from $A \times B$ to $\{\text{true}, \text{false}\}$.
3. A function from A to $\mathcal{P}(B)$.

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B.
[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

4. A function from B to $\mathcal{P}(A)$.

5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

6. A continuous morphism of posets from $(\mathcal{P}(B), \supset)$ to $(\mathcal{P}(A), \supset)$.

Proof. (We will prove that Items 1 to 6 are indeed equivalent in a bit.) □

Remark 8.1.1.1.2. We may think of a relation $R\colon A \nrightarrow B$ as a function from A to B that is *multivalued*, assigning to each element a in A a set $R(a)$ of elements of B, thought of as the *set of values of R at a*.

Note that this includes also the possibility of R having no value at all on a given $a \in A$ when $R(a) = \emptyset$.

Remark 8.1.1.1.3. Another way of stating the equivalence between Items 1 to 5 of Definition 8.1.1.1.1 is by saying that we have bijections of sets

$$\begin{aligned}
\{\text{relations from } A \text{ to } B\} &\cong \mathcal{P}(A \times B) \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)) \\
&\cong \mathsf{Sets}(B, \mathcal{P}(A)) \\
&\cong \mathsf{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B)) \\
&\cong \mathsf{Pos}^{\mathcal{C}}(\mathcal{P}(B), \mathcal{P}(A))
\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$, where $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are endowed with the poset structure given by inclusion.

Proof. We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This is a special case of Items 2 and 3 of Definition 4.5.1.1.4.

- *Item 2* $\iff$ *Item 3:* This follows from the bijections

  $$\begin{aligned}
  \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\})) \\
  &\cong \mathsf{Sets}(A, \mathcal{P}(B)),
  \end{aligned}$$

  where the last bijection is from Items 2 and 3 of Definition 4.5.1.1.4.

- *Item 2* $\iff$ *Item 4:* This follows from the bijections

  $$\begin{aligned}
  \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\})) \\
  &\cong \mathsf{Sets}(B, \mathcal{P}(A)),
  \end{aligned}$$

  where again the last bijection is from Items 2 and 3 of Definition 4.5.1.1.4.

- *Item 2* $\iff$ *Item 5*: This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X : X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$, as in Definition 4.4.5.1.1. In particular, the bijection

$$\mathsf{Sets}(A, \mathcal{P}(B)) \cong \mathsf{Pos}^{\mathcal{J}}(\mathcal{P}(A), \mathcal{P}(B))$$

is given by extending each $f : A \to \mathcal{P}(B)$ in $\mathsf{Sets}(A, \mathcal{P}(B))$ from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X, recovering the direct image function $f_! : \mathcal{P}(A) \to \mathcal{P}(B)$ of f of Definition 4.6.1.1.1.

- *Item 5* $\iff$ *Item 6*: Omitted.

This finishes the proof. □

Notation 8.1.1.1.4. Let A and B be sets and let $R : \;\;\nrightarrow B$ be a relation from A to B.

1. We write $\mathrm{Rel}(A, B)$ for the set of relations from A to B.
2. We write $\mathbf{Rel}(A, B)$ for the sub-poset of $(\mathcal{P}(A \times B), \subset)$ spanned by the relations from A to B.
3. Given $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.
4. When viewing R as a function

$$R : A \times B \to \{\mathsf{t}, \mathsf{f}\},$$

we write R^b_a for the value of R at (a, b).[3]

Proposition 8.1.1.1.5. Let A and B be sets and let $R, S : A \nrightarrow B$ be relations.

1. *End Formula for the Set of Inclusions of Relations.* We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \cong \int_{a \in A} \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, S^b_a\right).$$

[3]The choice to write R^b_a in place of R^a_b is to keep the notation consistent with the notation we will later employ for profunctors in **??**.

Proof. Item 1, End Formula for the Set of Inclusions of Relations: Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B}\mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a\in A \text{ and each } b\in B, \\ & \text{we have } \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^b_a, S^b_a) \cong \mathrm{pt} \\ \emptyset & \text{otherwise.}\end{cases}$$

Since we have $\mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}(R^b_a, S^b_a) = \{\mathrm{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathrm{false}$ or $R^b_a = S^b_a = \mathrm{true}$, we get

$$\int_{a\in A}\int_{b\in B}\mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^b_a, S^b_a\right) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a\in A \text{ and each } b\in B, \\ & \text{if } a\sim_R b\text{, then } a\sim_S b, \\ \emptyset & \text{otherwise.}\end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R\subset S, \\ \emptyset & \text{otherwise.}\end{cases}$$

Since $(a\sim_R b \implies a\sim_S b)$ iff $R\subset S$, the two sets above are isomorphic. This finishes the proof. □

8.1.2 Relations as Decategorifications of Profunctors

Remark 8.1.2.1.1. The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}}\times\mathcal{C}\to\mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A\times B\to\{\mathrm{true},\mathrm{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats}\to\mathsf{Cats}$ restricts to the identity endofunctor on Sets.
- The values that profunctors and relations take are analogous:

– A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

of sets, with profunctors taking values on it.

– A set is enriched over the set

$$\{\mathsf{true}, \mathsf{false}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

of classical truth values, with relations taking values on it.

Remark 8.1.2.1.2. Extending Definition 8.1.2.1.1, the equivalent definitions of relations in Definition 8.1.1.1.1 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$.
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{CoPSh}(\mathcal{C})$.
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.
5. A limit-preserving functor $\mathfrak{p}\colon \mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}} \to \mathsf{CoPSh}(\mathcal{C})^{\mathsf{op}}$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\end{aligned}$$

  and

$$\begin{aligned}\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 2 and 4 follows from the universal properties of:

  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$, as stated and proved in Definition 4.4.5.1.1.
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ\colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in **??** of Definition 12.1.4.1.3.

- The equivalence between Items 3 and 5 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free completion of X via the characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

    of X into $\mathcal{P}(X)$, as stated and proved in Definition 4.4.6.1.1.
  - The category $\mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$ of copresheaves on a category $\mathcal{D}$ as the free completion of $\mathcal{D}$ via the dual Yoneda embedding

$$\text{Ұ}\colon \mathcal{D} \hookrightarrow \mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$$

    of $\mathcal{D}$ into $\mathsf{CoPSh}(\mathcal{D})^{\mathsf{op}}$, as stated and proved in **??** of Definition 12.1.4.1.3.

8.1.3 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 8.1.3.1.1. The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

1. Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a,c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

2. Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\begin{aligned}(S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{b\in B} S^{-1}_{b} \times R^{b}_{-2} \\ &= \bigvee_{b\in B} S^{-1}_{b} \times R^{b}_{-2},\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of Definition 3.2.2.1.3.

3. Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \stackrel{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, "\chi_B"'] \arrow[d, phantom, shift left=4, "\Swarrow"] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] & \end{tikzcd}$$

where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}\left[\mathrm{Lan}_{\chi_B}(S)\right](V) &\cong \int^{b\in B} \chi_{\mathcal{P}(B)}(\chi_b, V) \odot S(b) \\ &\cong \int^{b\in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b\in B} \chi_V(b) \odot S(b) \\ &\cong \bigcup_{b\in V} S(b)\end{aligned}$$

for each $V \in \mathcal{P}(B)$, so we have[4]

$$\begin{aligned}[S \diamond R](a) &\stackrel{\text{def}}{=} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{b\in R(a)} S(b).\end{aligned}$$

for each $a \in A$.

[4]That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i\in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i\in I} = \left\{\{c_{j_i}\}_{j_i\in J_i}\right\}_{i\in I}$ in C.

Remark 8.1.3.1.2. You might wonder what happens if we instead define an alternative composition of relations $\diamond'$ via right Kan extensions. In this case, we would take the right Kan extension of S along the dual characteristic embedding $B \hookrightarrow \mathcal{P}(B)^{\mathsf{op}}$:

$$S \diamond' R \overset{\text{def}}{=} \mathrm{Ran}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, hook, "\chi_B"'] & \mathcal{P}(C). \\ A \arrow[r, "R"'] & \mathcal{P}(B)^{\mathsf{op}} \arrow[ur, "\mathrm{Ran}_{\chi'_B}(S)"'] \arrow[u, phantom, shift left=3, "\Nearrow"] & \end{tikzcd}$$

In this case, we would have[5]

$$[S \diamond' R](a) \overset{\text{def}}{=} \bigcap_{b \in R(a)} S(b).$$

This alternative composition turns out to actually be a different kind of structure: it's an internal right Kan extension in Rel, namely $\mathrm{Ran}_{R^\dagger}(S)$ — see Section 8.5.17.

Example 8.1.3.1.3. Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq. \end{aligned}$$

Proposition 8.1.3.1.4. Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

[5]If we replace $R(a)$ with $B \setminus R(a)$, defining

$$S \mathbin{\square} R \overset{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b),$$

we instead obtain the apartness composition of relations; see Section 8.1.4.

1. *Functoriality.* The assignments $R, S, (R, S) \mapsto S \diamond R$ define functors

$$\begin{aligned} S \diamond -\colon \quad & \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C), \\ - \diamond R\colon \quad & \mathbf{Rel}(B, C) \longrightarrow \mathbf{Rel}(A, C), \\ -_1 \diamond -_2\colon \quad & \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C). \end{aligned}$$

   In particular, given relations

$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \underset{S_2}{\overset{S_1}{\rightrightarrows}} C,$$

   if $R_1 \subset R_2$ and $S_1 \subset S_2$, then $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

   That is, we have

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)$$

   for each $a \in A$.

3. *Unitality.* We have

$$\begin{aligned} \Delta_B \diamond R &= R, \\ R \diamond \Delta_A &= R. \end{aligned}$$

   That is, we have

$$\begin{aligned} \bigcup_{b \in R(a)} \{b\} &= R(a), \\ \bigcup_{a \in \{a\}} R(a) &= R(a) \end{aligned}$$

   for each $a \in A$.

4. *Relation to Apartness Composition of Relations.* We have

$$\begin{aligned} (S \diamond R)^{\mathsf{c}} &= S^{\mathsf{c}} \,\square\, R^{\mathsf{c}}, \\ (S \,\square\, R)^{\mathsf{c}} &= S^{\mathsf{c}} \diamond R^{\mathsf{c}}, \end{aligned}$$

   where $(-)^{\mathsf{c}}$ is the complement functor of Section 4.3.11. In particular, $\diamond$ is a special case of apartness composition of relations, as we have

$$S \diamond R = (S^{\mathsf{c}} \,\square\, R^{\mathsf{c}})^{\mathsf{c}}.$$

   This is also compatible with units, as we have $\Delta_A^{\mathsf{c}} = \nabla_A$.

5. *Linear Distributivity.* We have inclusions of relations

$$\begin{aligned} T \diamond (S \mathbin{\square} R) &\subset (T \diamond S) \mathbin{\square} R, \\ (T \mathbin{\square} S) \diamond R &\subset T \mathbin{\square} (S \diamond R). \end{aligned}$$

   That is, we have

$$\begin{aligned} T\left(\bigcap_{b \in B \setminus R(a)} S(b)\right) &\subset \bigcap_{b \in B \setminus R(a)} T(S(b)) \\ \bigcup_{b \in R(a)} \bigcap_{c \in C \setminus S(b)} T(c) &\subset \bigcap_{c \in C \setminus S(R(a))} T(c) \end{aligned}$$

   or, unwinding the expression for $S(R(a))$, we have

$$\begin{aligned} \bigcup_{c \in \bigcap_{b \in B \setminus R(a)} S(b)} T(c) &\subset \bigcap_{b \in B \setminus R(a)} \bigcup_{c \in S(b)} T(c) \\ \bigcup_{b \in R(a)} \bigcap_{c \in C \setminus S(b)} T(c) &\subset \bigcap_{c \in C \setminus \bigcup_{b \in R(a)} S(b)} T(c) \end{aligned}$$

   for each $a \in A$.

6. *Interaction With Converses.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

7. *Interaction With Ranges and Domains.* We have

$$\begin{aligned} \mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R), \\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S). \end{aligned}$$

Proof. *Item 1, Functoriality*: We have

$$\begin{aligned} S_1 \diamond R_1 &\overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{, such} \\ \text{that } a \sim_{R_1} b \text{ or } b \sim_{S_1} c \end{array} \right\} \\ &\subset \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{, such} \\ \text{that } a \sim_{R_2} b \text{ or } b \sim_{S_2} c \end{array} \right\} \\ &\overset{\text{def}}{=} S_2 \diamond R_2. \end{aligned}$$

This finishes the proof.

Item 2, Associativity, Proof I: Indeed, we have

$$\begin{aligned}
(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{c \in C} T_c^{-1} \times S^c_{-_2}\right) \diamond R \\
&\stackrel{\text{def}}{=} \int^{b \in B} \left(\int^{c \in C} T_c^{-1} \times S^c_b\right) \times R^b_{-_2} \\
&= \int^{b \in B} \int^{c \in C} (T_c^{-1} \times S^c_b) \times R^b_{-_2} \\
&= \int^{c \in C} \int^{b \in B} (T_c^{-1} \times S^c_b) \times R^b_{-_2} \\
&= \int^{c \in C} \int^{b \in B} T_c^{-1} \times \left(S^c_b \times R^b_{-_2}\right) \\
&= \int^{c \in C} T_c^{-1} \times \left(\int^{b \in B} S^c_b \times R^b_{-_2}\right) \\
&\stackrel{\text{def}}{=} \int^{c \in C} T_c^{-1} \times (S \diamond R)^c_{-_2} \\
&\stackrel{\text{def}}{=} T \diamond (S \diamond R).
\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   - We have $a \sim_R b$;
   - We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
     - We have $b \sim_S c$;
     - We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   - We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
     - We have $a \sim_R b$;
     - We have $b \sim_S c$;
   - We have $c \sim_T d$;

both of which are equivalent to the statement

($\star$) There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 2, Associativity, Proof II: Using Item 3 of Definition 8.1.3.1.1, we have

$$\begin{aligned}[(T \diamond S) \diamond R](a) &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} (T \diamond S)(b) \\ &\stackrel{\text{def}}{=} \bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c)\end{aligned}$$

on the one hand and

$$\begin{aligned}[T \diamond (S \diamond R)](a) &\stackrel{\text{def}}{=} \bigcup_{c \in [S \diamond R](a)} T(c) \\ &\stackrel{\text{def}}{=} \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c)\end{aligned}$$

on the other, so we want to prove an equality of the form

$$\bigcup_{b \in R(a)} \bigcup_{c \in S(b)} T(c) = \bigcup_{c \in \bigcup_{b \in R(a)} S(b)} T(c).$$

This then follows from an application of Item 2 of Definition 4.3.6.1.2 in which we consider $X = D$, consider $\mathcal{P}(\mathcal{P}(\mathcal{P}(D)))$, take $U = U_c = T(c)$, take A to be

$$A_b \stackrel{\text{def}}{=} \{T(c) \in \mathcal{P}(D) \mid c \in S(b)\},$$

and then finally take

$$\begin{aligned}\mathcal{A} &\stackrel{\text{def}}{=} \{A_b \in \mathcal{P}(\mathcal{P}(D)) \mid b \in R(a)\} \\ &\stackrel{\text{def}}{=} \{\{T(c) \in \mathcal{P}(D) \mid c \in S(b)\} \mid b \in R(a)\}.\end{aligned}$$

Indeed, we have

$$\begin{aligned}\bigcup_{A \in \mathcal{A}} \left(\bigcup_{U \in A} U \right) &= \bigcup_{A_b \in \mathcal{A}} \left(\bigcup_{c \in S(b)} T(c) \right) \\ &= \bigcup_{b \in R(a)} \left(\bigcup_{c \in S(b)} T(c) \right)\end{aligned}$$

and

$$\bigcup_{U \in \bigcup_{A \in \mathcal{A}} A} U = \bigcup_{U_c \in \bigcup_{b \in R(a)} A_b} U_c$$

$$
\begin{aligned}
&= \bigcup_{T(c)\in\bigcup_{b\in R(a)} A_b} T(c)\\
&= \bigcup_{c\in\bigcup_{b\in R(a)} S(b)} T(c).
\end{aligned}
$$

This finishes the proof.

Item 3, *Unitality*: Indeed, we have

$$
\begin{aligned}
\Delta_B \diamond R &\stackrel{\text{def}}{=} \int^{b\in B} (\Delta_B)^{-1}_{b} \times R^{b}_{-2}\\
&= \bigvee_{b\in B} (\Delta_B)^{-1}_{b} \times R^{b}_{-2}\\
&= \bigvee_{\substack{b\in B\\ b=-_1}} R^{b}_{-2}\\
&= R^{-1}_{-2},
\end{aligned}
$$

and

$$
\begin{aligned}
R \diamond \Delta_A &\stackrel{\text{def}}{=} \int^{a\in A} R^{-1}_{a} \times (\Delta_A)^{a}_{-2}\\
&= \bigvee_{a\in B} R^{-1}_{a} \times (\Delta_A)^{a}_{-2}\\
&= \bigvee_{\substack{a\in B\\ a=-_2}} R^{-1}_{a}\\
&= R^{-1}_{-2}.
\end{aligned}
$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\Delta_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  - We have $a \sim_b B$.
  - There exists some $b' \in B$ such that:
    - We have $a \sim_R b'$
    - We have $b' \sim_{\Delta_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \Delta_A = R$$
witnesses the equivalence of the following two statements:

  - There exists some $a' \in A$ such that:
    * We have $a \sim_{\Delta_B} a'$, i.e. $a = a'$.
    * We have $a' \sim_R b$
  - We have $a \sim_b B$.

Item 4, Relation to Apartness Composition of Relations: This is a repetition of Item 4 of Definition 8.1.4.1.3 and is proved there.
Item 5, Linear Distributivity: This is a repetition of Item 5 of Definition 8.1.4.1.3 and is proved there.
Item 6, Interaction With Converses: This is a repetition of Item 3 of Definition 8.1.5.1.3 and is proved there.
Item 7, Interaction With Ranges and Domains: We have

$$\begin{aligned}
\mathrm{dom}(S \diamond R) &\overset{\text{def}}{=} \{a \in A \mid a \sim_{S\diamond R} c \text{ for some } c \in C\}, \\
&= \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists some } b \in B \text{ and } c \in C \\ \text{such that } a \sim_R b \text{ and } b \sim_R c\end{array} \right\}, \\
&\subset \left\{ a \in A \;\middle|\; \begin{array}{l}\text{there exists some } b \in B \\ \text{such that } a \sim_R b\end{array} \right\}, \\
&\overset{\text{def}}{=} \mathrm{dom}(R)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{range}(S \diamond R) &\overset{\text{def}}{=} \{c \in C \mid a \sim_{S\diamond R} c \text{ for some } a \in A\}, \\
&= \left\{ c \in C \;\middle|\; \begin{array}{l}\text{there exists some } a \in A \text{ and } b \in B \\ \text{such that } a \sim_R b \text{ and } b \sim_R c\end{array} \right\}, \\
&\subset \left\{ c \in C \;\middle|\; \begin{array}{l}\text{there exists some } b \in B \\ \text{such that } b \sim_S c\end{array} \right\}, \\
&\overset{\text{def}}{=} \mathrm{range}(S).
\end{aligned}$$

This finishes the proof. □

8.1.4 Apartness Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

Definition 8.1.4.1.1. The **apartness composition of** R **and** S is the relation $S \mathbin{\square} R$ defined as follows:

- Viewing relations as subsets of $A \times C$, we define

$$S \mathbin{\square} R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_R b \text{ or } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times C \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned} (S \mathbin{\square} R)^{-1}_{-2} &\overset{\text{def}}{=} \int_{b \in B} S^{-1}_{b} \coprod R^{b}_{-2} \\ &= \bigwedge_{b \in B} S^{-1}_{b} \coprod R^{b}_{-2}, \end{aligned}$$

where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 3.2.2.1.3.

- Viewing relations as functions $A \to \mathcal{P}(C)$, we define

$$[S \mathbin{\square} R](a) \overset{\text{def}}{=} \bigcap_{b \in B \setminus R(a)} S(b)$$

for each $a \in A$.

Example 8.1.4.1.2. Here are some examples of apartness composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \mathbin{\square} \geq &= \emptyset, \\ \geq \mathbin{\square} \leq &= \emptyset. \end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* Let $A = B = C = \mathbb{R}$. We have

$$\begin{aligned} \leq \mathbin{\square} \leq &= \emptyset, \\ \geq \mathbin{\square} \geq &= \emptyset. \end{aligned}$$

3. *Equality and Inequality.* Let $A = B = C = \mathbb{Z}$. We have

$$\begin{aligned} = \mathbin{\square} \neq &= =, \\ \neq \mathbin{\square} = &= =. \end{aligned}$$

4. *Subset Inclusion.* Let X be a set with at least three elements and consider the relations $\subset$ and $\supset$ in $\mathcal{P}(X)$. We have

$$\supset \mathbin{\square} \subset \; = \{(U, V) \in \mathcal{P}(X) \mid U = \emptyset \text{ or } V = \emptyset\}.$$

Proposition 8.1.4.1.3. Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

1. *Functoriality.* The assignments $R, S, (R, S) \mapsto S \mathbin{\square} R$ define functors

$$\begin{aligned} S \mathbin{\square} -\colon & \quad \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C), \\ - \mathbin{\square} R\colon & \quad \mathbf{Rel}(B, C) \longrightarrow \mathbf{Rel}(A, C), \\ -_1 \mathbin{\square} -_2\colon & \quad \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \longrightarrow \mathbf{Rel}(A, C). \end{aligned}$$

   In particular, given relations

$$\begin{tikzcd} A \arrow[r, shift left, "R_1"] \arrow[r, shift right, "R_2"'] & B \arrow[r, shift left, "S_1"] \arrow[r, shift right, "S_2"'] & C, \end{tikzcd}$$

   if $R_1 \subset R_2$ and $S_1 \subset S_2$, then $S_1 \mathbin{\square} R_1 \subset S_2 \mathbin{\square} R_2$.

2. *Associativity.* We have

$$(T \mathbin{\square} S) \mathbin{\square} R = T \mathbin{\square} (S \mathbin{\square} R).$$

3. *Unitality.* We have

$$\begin{aligned} \nabla_B \mathbin{\square} R &= R, \\ R \mathbin{\square} \nabla_A &= R. \end{aligned}$$

4. *Relation to Composition of Relations.* We have

$$\begin{aligned} (S \mathbin{\square} R)^{\mathsf{c}} &= S^{\mathsf{c}} \diamond R^{\mathsf{c}}, \\ (S \diamond R)^{\mathsf{c}} &= S^{\mathsf{c}} \mathbin{\square} R^{\mathsf{c}}, \end{aligned}$$

   where $(-)^{\mathsf{c}}$ is the complement functor of Section 4.3.11. In particular, $\square$ is a special case of composition of relations, as we have

$$S \mathbin{\square} R = (S^{\mathsf{c}} \diamond R^{\mathsf{c}})^{\mathsf{c}}.$$

   This is also compatible with units, as we have $\nabla_A^{\mathsf{c}} = \Delta_A$.

5. *Linear Distributivity.* We have inclusions of relations

$$T \diamond (S \mathbin{\square} R) \subset (T \diamond S) \mathbin{\square} R,$$
$$(T \mathbin{\square} S) \diamond R \subset T \mathbin{\square} (S \diamond R).$$

6. *Interaction With Converses.* We have

$$(S \mathbin{\square} R)^{\dagger} = R^{\dagger} \mathbin{\square} S^{\dagger}.$$

Proof. *Item 1*, *Functoriality*: We have

$$\begin{aligned} S_1 \mathbin{\square} R_1 &\overset{\text{def}}{=} \left\{ (a, c) \in A \times C \,\middle|\, \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_{R_1} b \text{ or } b \sim_{S_1} c \end{array} \right\} \\ &\subset \left\{ (a, c) \in A \times C \,\middle|\, \begin{array}{l} \text{for each } b \in B \text{, we have} \\ a \sim_{R_2} b \text{ or } b \sim_{S_2} c \end{array} \right\} \\ &\overset{\text{def}}{=} S_2 \mathbin{\square} R_2. \end{aligned}$$

This finishes the proof.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned} (T \mathbin{\square} S) \mathbin{\square} R &\overset{\text{def}}{=} \left(\int_{c \in C} T_c^{-1} \coprod S_{-2}^{c} \right) \mathbin{\square} R \\ &\overset{\text{def}}{=} \int_{b \in B} \left(\int_{c \in C} T_c^{-1} \coprod S_b^{c} \right) \coprod R_{-2}^{b} \\ &= \int_{b \in B} \int_{c \in C} (T_c^{-1} \coprod S_b^{c}) \coprod R_{-2}^{b} \\ &= \int_{c \in C} \int_{b \in B} (T_c^{-1} \coprod S_b^{c}) \coprod R_{-2}^{b} \\ &= \int_{c \in C} \int_{b \in B} T_c^{-1} \coprod \left(S_b^{c} \coprod R_{-2}^{b} \right) \\ &= \int_{c \in C} T_c^{-1} \coprod \left(\int_{b \in B} S_b^{c} \coprod R_{-2}^{b} \right) \\ &\overset{\text{def}}{=} \int_{c \in C} T_c^{-1} \coprod (S \mathbin{\square} R)_{-2}^{c} \\ &\overset{\text{def}}{=} T \mathbin{\square} (S \mathbin{\square} R). \end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

- We have $a \sim_{(T\square S)\square R} d$, i.e. there exists some $b \in B$ such that:
  - We have $a \sim_R b$;
  - We have $b \sim_{T\square S} d$, i.e. there exists some $c \in C$ such that:
    * We have $b \sim_S c$;
    * We have $c \sim_T d$;
- We have $a \sim_{T\square(S\square R)} d$, i.e. there exists some $c \in C$ such that:
  - We have $a \sim_{S\square R} c$, i.e. there exists some $b \in B$ such that:
    * We have $a \sim_R b$;
    * We have $b \sim_S c$;
  - We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, *Unitality*: Indeed, we have

$$\begin{aligned}
\nabla_B \square R &\stackrel{\text{def}}{=} \int_{b\in B} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2} \\
&= \bigwedge_{b\in B} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2} \\
&= \left(\bigwedge_{\substack{b\in B\\ b=-_1}} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2} \right) \wedge \left(\bigwedge_{\substack{b\in B\\ b\neq -_1}} (\nabla_B)^{-1}_{b} \coprod R^{b}_{-2} \right) \\
&= \left((\nabla_B)^{-1}_{-_1} \coprod R^{-_1}_{-_2} \right) \wedge \left(\bigwedge_{\substack{b\in B\\ b\neq -_1}} \mathsf{t} \coprod R^{b}_{-2} \right) \\
&= \left(\mathsf{f} \coprod R^{-_1}_{-_2} \right) \wedge \left(\bigwedge_{\substack{b\in B\\ b\neq -_1}} \mathsf{t} \right) \\
&= R^{-_1}_{-_2} \wedge \mathsf{t} \\
&= R^{-_1}_{-_2},
\end{aligned}$$

and

$$R \square \nabla_A \stackrel{\text{def}}{=} \int_{a\in A} R^{-_1}_{a} \coprod (\nabla_A)^{a}_{-_2}$$

$$
\begin{aligned}
&= \bigwedge_{a\in A} R_a^{-1} \coprod (\nabla_A)^a_{-2}\\
&= \left(\bigwedge_{\substack{a\in A\\ a=-2}} R_a^{-1} \coprod (\nabla_A)^a_{-2}\right) \wedge \left(\bigwedge_{\substack{a\in A\\ a\neq -2}} R_a^{-1} \coprod (\nabla_A)^a_{-2}\right)\\
&= \left(R_{-2}^{-1} \coprod (\nabla_A)^{-2}_{-2}\right) \wedge \left(\bigwedge_{\substack{a\in A\\ a\neq -2}} R_a^{-1} \coprod \mathsf{t}\right)\\
&= \left(R_{-2}^{-1} \coprod \mathsf{f}\right) \wedge \left(\bigwedge_{\substack{a\in A\\ a\neq -2}} \mathsf{t}\right)\\
&= R_{-2}^{-1} \wedge \mathsf{t}\\
&= R_{-2}^{-1},
\end{aligned}
$$

This finishes the proof.

Item 4, *Relation to Composition of Relations*: We proceed in a few steps.

- We have $a \sim_{(S\square R)^{\mathsf{c}}} b$ iff $a \nsim_{S\square R} b$.
- We have $a \nsim_{S\square R} b$ iff the assertion "for each $b \in B$, we have $a \sim_R b$ or $b \sim_S c$" is false.
- That happens iff there exists some $b \in B$ such that $a \nsim_R b$ and $b \nsim_S c$.
- That happens iff there exists some $b \in B$ such that $a \sim_{R^{\mathsf{c}}} b$ and $b \sim_{S^{\mathsf{c}}} c$.

The second equality then follows from the first one by Item 3 of Definition 4.3.11.1.2.

Item 5, *Linear Distributivity*: We have

$$
\begin{aligned}
T \diamond (S \square R) &\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{there exists some } c \in C \text{ such}\\ \text{that } a \sim_{S\square R} c \text{ and } c \sim_T d\end{array} \right\}\\
&\overset{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{there exists some } c \in C \text{ such that}\\ c \sim_T d \text{ and, for each } b \in B,\\ \text{we have } a \sim_R b \text{ or } b \sim_S c\end{array} \right\}\\
&= \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l}\text{the following conditions are satisfied:}\\ \quad 1.\ \text{For each } b \in B, \text{ we have } a \sim_R b \text{ or } b \sim_S c.\\ \quad 2.\ \text{There exists } c \in C \text{ such that } c \sim_T d.\end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\subset \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B\text{, at least one of the} \\ \text{following conditions is satisfied:} \\ \quad \text{1. We have } a \sim_R b. \\ \quad \text{2. There exists } c \in C \text{ such that } b \sim_S c \\ \quad\quad \text{and } c \sim_T d. \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B\text{, we have} \\ a \sim_R b \text{ or there exists some } c \in C \\ \text{such that } b \sim_S c \text{ and } c \sim_T d \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } b \in B\text{, we have} \\ a \sim_R b \text{ or } b \sim_{T \diamond S} d \end{array} \right\} \\
&\stackrel{\text{def}}{=} (T \diamond S) \mathbin{\square} R
\end{aligned}
$$

and

$$
\begin{aligned}
(T \mathbin{\square} S) \diamond R &\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_{T \square S} d \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and, for each } c \in C\text{,} \\ \text{we have } b \sim_S c \text{ or } c \sim_T d \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ satisfying} \\ \text{the following conditions:} \\ \quad \text{1. We have } a \sim_R b. \\ \quad \text{2. For each } c \in C\text{, we have } b \sim_S c \\ \quad\quad \text{or } c \sim_T d. \end{array} \right\} \\
&\subset \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C\text{, at least one of the} \\ \text{following conditions is satisfied:} \\ \quad \text{1. We have } c \sim_T d. \\ \quad \text{2. There exists some } b \in B \text{ such that} \\ \quad\quad \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C\text{, we have } c \sim_T d \\ \text{or there exists some } b \in B\text{, such that} \\ \text{we have } a \sim_R b \text{ and } b \sim_S c \end{array} \right\} \\
&\stackrel{\text{def}}{=} \left\{ (d,a) \in D \times A \;\middle|\; \begin{array}{l} \text{for each } c \in C\text{, we have} \\ a \sim_{S \diamond R} c \text{ or } c \sim_T d \end{array} \right\} \\
&\subset T \mathbin{\square} (S \diamond R).
\end{aligned}
$$

This finishes the proof.

Item 6, Interaction With Converses: This is a repetition of Item 4 of Definition 8.1.5.1.3 and is proved there. □

8.1.5 The Converse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 8.1.5.1.1. The **converse of** R[6] is the relation $R^{\dagger}$ defined as follows:

- Viewing relations as subsets, we define

$$R^{\dagger} \stackrel{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } a \sim_R b\}.$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$[R^{\dagger}]^a_b \stackrel{\text{def}}{=} R^b_a$$

  for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define[7]

$$R^{\dagger}(b) \stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\}$$

  for each $b \in B$.

Example 8.1.5.1.2. Here are some examples of converses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^{\dagger} = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^{\dagger} = \leq$.
3. *Functions.* Let $f : A \to B$ be a function. We have

$$\begin{aligned} \mathrm{Gr}(f)^{\dagger} &= f^{-1}, \\ (f^{-1})^{\dagger} &= \mathrm{Gr}(f), \end{aligned}$$

   where $\mathrm{Gr}(f)$ and f^{-1} are the relations of Sections 8.2.2 and 8.2.3.

Proposition 8.1.5.1.3. Let $R : A \nrightarrow B$ and $S : B \nrightarrow C$ be relations.

---

[6] *Further Terminology:* Also called the **opposite of** R or the **transpose of** R.

[7] Note that $R^{\dagger}(b) = R^{-1}(\{b\})$.

1. *Functoriality.* The assignment $R \mapsto R^{\dagger}$ defines a functor (i.e. morphism of posets)
$$(-)^{\dagger} \colon \mathbf{Rel}(A, B) \to \mathbf{Rel}(B, A).$$
In other words, given relations $R, S \colon A \nrightarrow B$, we have:
   (★) If $R \subset S$, then $R^{\dagger} \subset S^{\dagger}$.
2. *Interaction With Ranges and Domains.* We have
$$\begin{aligned}\mathrm{dom}\left(R^{\dagger}\right) &= \mathrm{range}(R),\\ \mathrm{range}\left(R^{\dagger}\right) &= \mathrm{dom}(R).\end{aligned}$$
3. *Interaction With Composition.* We have
$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$
4. *Interaction With Apartness Composition.* We have
$$(S \mathbin{\square} R)^{\dagger} = R^{\dagger} \mathbin{\square} S^{\dagger}.$$
5. *Invertibility.* We have
$$\left(R^{\dagger}\right)^{\dagger} = R.$$
6. *Identity I.* We have
$$\Delta_A^{\dagger} = \Delta_A.$$
7. *Identity II.* We have
$$\nabla_A^{\dagger} = \nabla_A.$$

Proof. Item 1, *Functoriality*: We have
$$\begin{aligned}R^{\dagger} &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}\\ &\subset \{a \in A \mid b \in S(a)\}\\ &\overset{\text{def}}{=} S^{\dagger}.\end{aligned}$$
This finishes the proof.
Item 2, *Interaction With Ranges and Domains*: We have
$$\mathrm{dom}\left(R^{\dagger}\right) \overset{\text{def}}{=} \{b \in B \mid b \sim_{R^{\dagger}} a \text{ for some } a \in A\}$$

$$
\begin{aligned}
&= \{b \in B \mid a \sim_R b \text{ for some } a \in A\} \\
&\overset{\text{def}}{=} \operatorname{range}(R)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{range}\left(R^{\dagger}\right) &\overset{\text{def}}{=} \{a \in A \mid b \sim_{R^{\dagger}} a \text{ for some } b \in B\} \\
&= \{a \in A \mid a \sim_R b \text{ for some } b \in B\} \\
&\overset{\text{def}}{=} \operatorname{dom}(R).
\end{aligned}
$$

This finishes the proof.
Item 3, Interaction With Composition: We have

$$
\begin{aligned}
(S \diamond R)^{\dagger} &\overset{\text{def}}{=} \left\{(c, a) \in C \times A \,\middle|\, c \sim_{(S \diamond R)^{\dagger}} a\right\} \\
&= \{(c, a) \in C \times A \mid a \sim_{S \diamond R} c\} \\
&= \left\{(c, a) \in C \times A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array}\right\} \\
&= \left\{(c, a) \in C \times A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } b \sim_{R^{\dagger}} a \text{ and } c \sim_{S^{\dagger}} b\end{array}\right\} \\
&= \left\{(c, a) \in C \times A \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } c \sim_{S^{\dagger}} b \text{ and } b \sim_{R^{\dagger}} a\end{array}\right\} \\
&\overset{\text{def}}{=} R^{\dagger} \diamond S^{\dagger}.
\end{aligned}
$$

This finishes the proof.
Item 4, Interaction With Apartness Composition: We have

$$
\begin{aligned}
(S \mathbin{\square} R)^{\dagger} &\overset{\text{def}}{=} \left\{(c, a) \in C \times A \,\middle|\, c \sim_{(S \square R)^{\dagger}} a\right\} \\
&= \{(c, a) \in C \times A \mid a \sim_{S \square R} c\} \\
&= \left\{(c, a) \in C \times A \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have} \\ a \sim_R b \text{ or } b \sim_S c\end{array}\right\} \\
&= \left\{(c, a) \in C \times A \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have} \\ b \sim_{R^{\dagger}} a \text{ or } c \sim_{S^{\dagger}} b\end{array}\right\} \\
&= \left\{(c, a) \in C \times A \,\middle|\, \begin{array}{l}\text{for each } b \in B, \text{ we have} \\ c \sim_{S^{\dagger}} b \text{ or } b \sim_{R^{\dagger}} a\end{array}\right\} \\
&\overset{\text{def}}{=} R^{\dagger} \mathbin{\square} S^{\dagger}.
\end{aligned}
$$

This finishes the proof.
Item 5, Invertibility: We have

$$\begin{aligned}\left(R^{\dagger}\right)^{\dagger} &\stackrel{\text{def}}{=} \{(a,b)\in A\times B \mid b \sim_{R^{\dagger}} a\}\\ &= \{(a,b)\in A\times B \mid a \sim_{R} b\}\\ &\stackrel{\text{def}}{=} R.\end{aligned}$$

This finishes the proof.
Item 6, Identity I: We have

$$\begin{aligned}\Delta_A^{\dagger} &\stackrel{\text{def}}{=} \left\{(a,b)\in A\times A \,\middle|\, a \sim_{\Delta_A} b\right\}\\ &= \{(a,b)\in A\times A \mid a = b\}\\ &= \Delta_A.\end{aligned}$$

This finishes the proof.
Item 7, Identity II: We have

$$\begin{aligned}\nabla_A^{\dagger} &\stackrel{\text{def}}{=} \left\{(a,b)\in A\times A \,\middle|\, a \sim_{\nabla_A} b\right\}\\ &= \{(a,b)\in A\times A \mid a \neq b\}\\ &= \nabla_A.\end{aligned}$$

This finishes the proof. □

8.2 Examples of Relations

8.2.1 Elementary Examples of Relations

Example 8.2.1.1.1. The **trivial relation on** A **and** B is the relation $\sim_{\text{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\text{triv}} \stackrel{\text{def}}{=} A \times B.$$

2. As a function from $A \times B$ to $\{\text{true}, \text{false}\}$, the relation $\sim_{\text{triv}}$ is the constant function

$$\Delta_{\text{true}} \colon A \times B \longrightarrow \{\text{true}, \text{false}\}$$

   from $A \times B$ to $\{\text{true}, \text{false}\}$ taking the value true.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{triv}}$ is the function

$$\Delta_{\mathsf{true}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} B$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

Example 8.2.1.1.2. The **cotrivial relation on** A **and** B is the relation $\sim_{\mathrm{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\mathrm{cotriv}} \overset{\mathrm{def}}{=} \emptyset.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value false.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{cotriv}}$ is the function

$$\Delta_{\mathsf{false}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{false}}(a) \overset{\mathrm{def}}{=} \emptyset$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \nsim_R b$ for each $a \in A$ and each $b \in B$.

Example 8.2.1.1.3. The characteristic relation χ_X on X of Definition 4.5.3.1.1:

1. As a subset of $X \times X$, we have

$$\begin{aligned} \sim_{\chi_X} &\overset{\mathrm{def}}{=} \Delta_X \\ &\overset{\mathrm{def}}{=} \{(x, x) \in X \times X\}. \end{aligned}$$

2. As a function from $X \times X$ to $\{\text{true}, \text{false}\}$, we have

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } x = y, \\ \text{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

3. As a function from X to $\mathcal{P}(X)$, we have

$$\chi_X(x) \overset{\text{def}}{=} \{x\}$$

for each $x \in X$.

Example 8.2.1.1.4. The **antidiagonal relation on** X is the relation ∇_X defined equivalently as follows:

1. As a subset of $X \times X$, we have

$$\begin{aligned} \sim_{\nabla_X} &\overset{\text{def}}{=} \nabla_X \\ &\overset{\text{def}}{=} X \setminus \Delta_X \\ &= \{(x, y) \in X \times X \mid x \neq y\}. \end{aligned}$$

2. As a function from $X \times X$ to $\{\text{true}, \text{false}\}$, we have

$$\nabla_X(x, y) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } a \neq b, \\ \text{false} & \text{if } a = b \end{cases}$$

for each $x, y \in X$.

3. As a function from X to $\mathcal{P}(X)$, we have

$$\nabla_X(x) \overset{\text{def}}{=} X \setminus \{x\}$$

for each $x \in X$.

Example 8.2.1.1.5. Partial functions may be viewed (or defined) as being exactly those relations which are functional; see Section 10.1.1.

Example 8.2.1.1.6. Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in* $\mathbb{Q}$. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-}\colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 8.2.1.1.7. The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \overset{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \,\middle|\, k \in \mathbb{Z}\right\}$$

for each $a + bi \in \mathbb{C}$.

Example 8.2.1.1.8. See [Wik25] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

8.2.2 The Graph of a Function

Let $f\colon A \to B$ be a function.

Definition 8.2.2.1.1. The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[8]

- Viewing relations from A to B as subsets of $A \times B$, we define

$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$

- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\mathrm{Gr}(f)^b_a \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

  for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition

$$A \xrightarrow{f} B \overset{\chi_B}{\hookrightarrow} \mathcal{P}(B).$$

[8] *Further Terminology and Notation:* When $f = \mathrm{id}_A$, we write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, calling

Proposition 8.2.2.1.2. Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

   $$\mathrm{Gr}\colon \mathsf{Sets} \to \mathsf{Rel}$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

     $$\mathrm{Gr}(A) \stackrel{\mathrm{def}}{=} A.$$

   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

     $$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A,B) \to \underbrace{\mathsf{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\mathrm{def}}{=}\mathsf{Rel}(A,B)}$$

     of Gr at (A, B) is defined by

     $$\mathrm{Gr}_{A,B}(f) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(f),$$

     where $\mathrm{Gr}(f)$ is the graph of f as in Definition 8.2.2.1.1.

   In particular, the following statements are true:

   - *Preservation of Identities.* We have

     $$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

     for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

   - *Preservation of Composition.* We have

     $$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

     for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

2. *Adjointness.* We have an adjunction

   $$(\mathrm{Gr} \dashv \mathcal{P}_!)\colon\quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathcal{P}_!"] \end{tikzcd}$$

   witnessed by a bijection of sets

   $$\mathsf{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

   natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Rel})$.

3. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \longrightarrow \mathrm{Rel}$ of Item 1 preserves colimits.

4. *Adjointness Inside* **Rel**. We have an internal adjunction

$$
(\mathrm{Gr}(f) \dashv f^{-1})\colon \quad
\begin{tikzcd}
A \arrow[r, bend left, "\mathrm{Gr}(f)", "+"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}", "+"{marking}]
\end{tikzcd}
$$

   in **Rel**, where f^{-1} is the inverse of f of Definition 8.2.3.1.1.

5. *Interaction With Converses.* We have

$$
\begin{aligned}
\mathrm{Gr}(f)^{\dagger} &= f^{-1},\\
\left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f).
\end{aligned}
$$

6. *Characterisations.* Let $R\colon A \rightarrow\!\!\!\!\!+ B$ be a relation. The following conditions are equivalent:

   (a) There exists a function $f\colon A \longrightarrow B$ such that $R = \mathrm{Gr}(f)$.

   (b) The relation R is total and functional.

   (c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

   (d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness: This is a repetition of Definition 4.4.4.1.1, and is proved there.
Item 3, Cocontinuity: This follows from Item 2 and **??**.
Item 4, Adjointness Inside **Rel**: We need to check that there are inclusions

$$
\begin{aligned}
\chi_A &\subset f^{-1} \diamond \mathrm{Gr}(f),\\
\mathrm{Gr}(f) \diamond f^{-1} &\subset \chi_B.
\end{aligned}
$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

---

it the **graph of** A.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.
Item 5, Interaction With Converses: Omitted.
Item 6, Characterisations: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of Definition 8.5.2.1.2.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton. Since the conditions
  - $R(a) \cap V \neq \emptyset$;
  - $R(a) \subset V$;

  are equivalent when $R(a)$ is a singleton, it follows that the sets

  $$\begin{aligned} R^{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}, \\ R_{-1}(V) &\stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\} \end{aligned}$$

  are equal for all $V \in \mathcal{P}(B)$.
- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality*. We proceed in a few steps:
    * If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$.
    * But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction.
    * Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality*. If $R^{-1} = R_{-1}$, then we have

    $$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, so R is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Definition 8.5.3.1.1.

This finishes the proof. □

8.2.3 The Inverse of a Function

Let $f: A \to B$ be a function.

Definition 8.2.3.1.1. The **inverse of** f is the relation $f^{-1}: B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define
$$f^{-1} \stackrel{\text{def}}{=} \left\{ \left(b, f^{-1}(b)\right) \in B \times A \,\middle|\, a \in A \right\},$$
where
$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$
for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$, we define
$$\left[f^{-1}\right]_a^b \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$
for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define
$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$
for each $b \in B$.

Proposition 8.2.3.1.2. Let $f: A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto A, f \mapsto f^{-1}$ defines a functor
$$(-)^{-1}: \mathsf{Sets} \to \mathsf{Rel}$$
where
   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
   $$\left[(-)^{-1}\right](A) \stackrel{\text{def}}{=} A.$$
   - *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
   $$(-)^{-1}_{A,B}: \mathsf{Sets}(A, B) \to \mathsf{Rel}(A, B)$$
   of $(-)^{-1}$ at (A, B) is defined by
   $$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$
   where f^{-1} is the inverse of f as in Definition 8.2.3.1.1.

In particular, the following statements are true:

- *Preservation of Identities.* We have
$$\mathrm{id}_A^{-1} = \chi_A$$
for each $A \in \mathrm{Obj}(\mathsf{Sets})$.
- *Preservation of Composition.* We have
$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$
for pair of functions $f : A \to B$ and $g : B \to C$.

2. *Adjointness Inside* **Rel**. We have an adjunction
$$(\mathrm{Gr}(f) \dashv f^{-1}) \colon \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$
in **Rel**.

3. *Interaction With Converses of Relations.* We have
$$\begin{aligned}(f^{-1})^{\dagger} &= \mathrm{Gr}(f),\\ \mathrm{Gr}(f)^{\dagger} &= f^{-1}.\end{aligned}$$

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Adjointness Inside **Rel**: This is a repetition of Item 4 of Definition 8.2.2.1.2 and is proved there.
Item 3, Interaction With Converses of Relations: This is a repetition of Item 5 of Definition 8.2.2.1.2 and is proved there. □

8.2.4 Representable Relations

Let A and B be sets.

Definition 8.2.4.1.1. Let $f : A \to B$ and $g : B \to A$ be functions.[9]

[9] More generally, given functions
$$\begin{aligned}f &: A \to C,\\ g &: B \to D\end{aligned}$$

1. The **representable relation associated to** f is the relation $\chi_f \colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g \colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g \times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

8.3 Categories of Relations

8.3.1 The Category of Relations Between Two Sets

Definition 8.3.1.1.1. The **category of relations from** A **to** B is the category $\mathbf{Rel}(A, B)$ defined by[10]

$$\mathbf{Rel}(A, B) \stackrel{\text{def}}{=} \mathbf{Rel}(A, B)_{\mathsf{pos}},$$

where $\mathbf{Rel}(A, B)_{\mathsf{pos}}$ is the posetal category associated to the poset $\mathbf{Rel}(A, B)$ of Item 2 of Definition 8.1.1.1.4 and Definition 11.2.7.1.1.

8.3.2 The Category of Relations

Definition 8.3.2.1.1. The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}(A, B) \stackrel{\text{def}}{=} \mathrm{Rel}(A, B).$$

---

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

[10] Here we choose to abuse notation by writing $\mathbf{Rel}(A, B)$ instead of $\mathbf{Rel}(A, B)_{\mathsf{pos}}$ for

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \longrightarrow \mathsf{Rel}(A, A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Definition 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B) \longrightarrow \mathsf{Rel}(A, C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \overset{\mathrm{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

8.3.3 The Closed Symmetric Monoidal Category of Relations

8.3.3.1 The Monoidal Product

Definition 8.3.3.1.1. The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\times(A, B) \overset{\mathrm{def}}{=} A \times B,$$

  where $A \times B$ is the Cartesian product of sets of Definition 4.1.3.1.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} \colon \mathsf{Rel}(A, B) \times \mathsf{Rel}(C, D) \longrightarrow \mathsf{Rel}(A \times C, B \times D)$$

of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$\begin{aligned} R &\colon A \nrightarrow B, \\ S &\colon C \nrightarrow D \end{aligned}$$

to the relation

$$R \times S \colon A \times C \nrightarrow B \times D$$

of Definition 9.2.6.1.1.

8.3.3.2 The Monoidal Unit

Definition 8.3.3.2.1. The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \to \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \overset{\text{def}}{=} \mathrm{pt}$$

of Rel.

8.3.3.3 The Associator

Definition 8.3.3.3.1. The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathrm{id}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Rel}\times(\mathsf{Rel}\times\mathsf{Rel}) \arrow[dr, "\mathrm{id}\times(\times)", "\shortmid" marking] & \\
(\mathsf{Rel}\times\mathsf{Rel})\times\mathsf{Rel} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\sim"'] \arrow[d, "(\times)\times\mathrm{id}"', "\shortmid" marking] & & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times", "\shortmid" marking] \\
\mathsf{Rel}\times\mathsf{Rel} \arrow[rr, "\times"', "\shortmid" marking] \arrow[ur, phantom, "\Uparrow\alpha^{\mathsf{Rel}}"] & & \mathsf{Rel},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} \colon (A \times B) \times C \nrightarrow A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

8.3.3.4 The Left Unitor

Definition 8.3.3.4.1. The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Leftarrow \lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_A \colon \mathbb{1}_{\mathsf{Rel}} \times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a = b$.

8.3.3.5 The Right Unitor

Definition 8.3.3.5.1. The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Leftarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

8.3.3.6 The Symmetry

Definition 8.3.3.6.1. The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}}\colon \times \Longrightarrow \times \circ \sigma^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\Downarrow \sigma^{\mathsf{Rel}}"] & \mathsf{Rel}, \\ & \mathsf{Rel}\times\mathsf{Rel} \arrow[ur, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B}\colon A\times B \longrightarrow B\times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

8.3.3.7 The Internal Hom

Definition 8.3.3.7.1. The **internal Hom of** Rel is the functor

$$\mathrm{Rel}\colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

defined

- On objects by sending $A, B \in \mathrm{Obj}(\mathsf{Rel})$ to the set $\mathrm{Rel}(A, B)$ of **??** of **??**.
- On morphisms by pre/post-composition defined as in Definition 8.1.3.1.1.

Proposition 8.3.3.7.2. Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

1. *Adjointness.* We have adjunctions

$$(A\times - \dashv \mathrm{Rel}(A, -))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(-\times B \dashv \mathrm{Rel}(B, -))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathrm{Rel}(B, C)),\\ \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathrm{Rel}(A, C)),\end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathrm{Rel})$.

Proof. *Item 1, Adjointness*: Indeed, we have

$$\begin{aligned}\mathrm{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathrm{Sets}(A \times B \times C, \{\mathrm{true}, \mathrm{false}\})\\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C)\\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, \mathrm{Rel}(B, C)),\end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathrm{Rel}(A, C))$. □

8.3.3.8 The Closed Symmetric Monoidal Category of Relations

Proposition 8.3.3.8.1. The category Rel admits a closed symmetric monoidal category structure consisting of[11]

- *The Underlying Category.* The category Rel of sets and relations of Definition 8.3.2.1.1.
- *The Monoidal Product.* The functor

  $$\times \colon \mathrm{Rel} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 8.3.3.1.1.
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Rel} \colon \mathrm{Rel}^{\mathrm{op}} \times \mathrm{Rel} \longrightarrow \mathrm{Rel}$$

  of Definition 8.3.3.7.1.
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathrm{Rel}} \colon \mathrm{pt} \longrightarrow \mathrm{Rel}$$

  of Definition 8.3.3.2.1.

---

the posetal category of relations from A to B, even though the same notation is used for the poset of relations from A to B.

[11] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in

- *The Associators.* The natural isomorphism
$$\alpha^{\mathrm{Rel}}\colon \times\circ(\times\times\mathrm{id}_{\mathrm{Rel}})\overset{\sim}{\Longrightarrow}\times\circ(\mathrm{id}_{\mathrm{Rel}}\times\times)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$
of Definition 8.3.3.3.1.

- *The Left Unitors.* The natural isomorphism
$$\lambda^{\mathrm{Rel}}\colon \times\circ\left(\mathbb{1}^{\mathrm{Rel}}\times\mathrm{id}_{\mathrm{Rel}}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$
of Definition 8.3.3.4.1.

- *The Right Unitors.* The natural isomorphism
$$\rho^{\mathrm{Rel}}\colon \times\circ\left(\mathsf{id}\times\mathbb{1}^{\mathrm{Rel}}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$
of Definition 8.3.3.5.1.

- *The Symmetry.* The natural isomorphism
$$\sigma^{\mathrm{Rel}}\colon \times\overset{\sim}{\Longrightarrow}\times\circ\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathrm{Rel},\mathrm{Rel}}$$
of Definition 8.3.3.6.1.

Proof. Omitted. □

8.3.4 The 2-Category of Relations

Definition 8.3.4.1.1. The **2-category of relations** is the locally posetal 2-category Rel where

- *Objects.* The objects of Rel are sets.
- **Hom**-*Objects.* For each $A,B\in\mathrm{Obj}(\mathsf{Sets})$, we have
$$\begin{aligned}\mathrm{Hom}_{\mathsf{Rel}}(A,B)&\overset{\mathrm{def}}{=}\mathbf{Rel}(A,B)\\&\overset{\mathrm{def}}{=}(\mathrm{Rel}(A,B),\subset).\end{aligned}$$
- *Identities.* For each $A\in\mathrm{Obj}(\mathsf{Rel})$, the unit map
$$\mathbb{1}^{\mathsf{Rel}}_{A}\colon \mathrm{pt}\to\mathbf{Rel}(A,A)$$
of Rel at A is defined by
$$\mathrm{id}^{\mathsf{Rel}}_{A}\overset{\mathrm{def}}{=}\chi_A(-_1,-_2),$$
where $\chi_A(-_1,-_2)$ is the characteristic relation of A of Definition 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[12]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

 of $\mathbf{Rel}$ at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \overset{\mathrm{def}}{=} S \diamond R$$

 for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

8.3.5 The Double Category of Relations

8.3.5.1 The Double Category of Relations

Definition 8.3.5.1.1. The **double category of relations** is the locally posetal double category $\mathrm{Rel}^{\mathrm{dbl}}$ where

- *Objects.* The objects of $\mathrm{Rel}^{\mathrm{dbl}}$ are sets.

- *Vertical Morphisms.* The vertical morphisms of $\mathrm{Rel}^{\mathrm{dbl}}$ are maps of sets $f \colon A \to B$.

- *Horizontal Morphisms.* The horizontal morphisms of $\mathrm{Rel}^{\mathrm{dbl}}$ are relations $R \colon A \nrightarrow X$.

- *2-Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd}$$

 of $\mathrm{Rel}^{\mathrm{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

fact given by the disjoint union of sets; see ??.

[12] That this is indeed a morphism of posets is proven in ?? of Definition 8.1.3.1.4.

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 8.3.5.2.1.
- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \stackrel{\text{def}}{=} \mathrm{id}_A\,.$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\mathrm{id}_R}}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \end{tikzcd}$$

of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 8.3.5.3.1.
- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$

- *Vertical Composition of 2-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 8.3.5.4.1.
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.5.1.
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.6.1.
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ are defined as in Definition 8.3.5.7.1.

8.3.5.2 Horizontal Identities

Definition 8.3.5.2.1. The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\Big(\mathsf{Rel}^{\mathsf{dbl}}_0\Big)$, we have

$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2).$$

- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\mathbb{1}_f}}"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "|"{marking}] & B \end{tikzcd}$$

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of Item 1 of Definition 4.5.3.1.3.

8.3.5.3 Horizontal Composition

Definition 8.3.5.3.1. The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \stackrel{\text{def}}{=} S \diamond R,$$

where $S \diamond R$ is the composition of R and S of Definition 8.1.3.1.1.

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\alpha}}"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\beta}}"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "|" marking] & Z \end{tikzcd}$$

  of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathrm{true}, \mathrm{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathrm{true}, \mathrm{false}\} \end{tikzcd}$$

  of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S \odot R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\|}{\underset{\Downarrow}{\beta \odot \alpha}}"] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "|" marking] & Z \end{tikzcd}$$

  of α and β is the inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\mathrm{true}, \mathrm{false}\}. \end{tikzcd}$$

Proof. The inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R)$$

follows from the fact that the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  - We have $f(a) \sim_T y$.

  - We have $y \sim_U h(c)$.

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  - We have $a \sim_R b$.
  - We have $b \sim_S c$.

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

This finishes the proof. □

8.3.5.4 Vertical Composition of 2-Morphisms

Definition 8.3.5.4.1. The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\alpha}}"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\beta}}"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\text{true}, \text{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\text{true}, \text{false}\} \arrow[d, "\mathrm{id}_{\{\text{true},\text{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\text{true}, \text{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h \circ f"'] \arrow[dr, phantom, "\overset{\Vert}{\underset{\Downarrow}{\beta \circ \alpha}}"] & X \arrow[d, "k \circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

Proof. The inclusion

$$T \circ [(h \circ f) \times (k \circ g)] \subset R$$

follows from the fact that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

This finishes the proof. $\square$

8.3.5.5 The Associators

Definition 8.3.5.5.1. For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "\shortmid"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "\shortmid"{marking}] & B \arrow[r, "S"', "\shortmid"{marking}] & C \arrow[r, "T"', "\shortmid"{marking}] & D \end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[13]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T \diamond S) \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{45}{$=$}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T \diamond (S \diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

8.3.5.6 The Left Unitors

Definition 8.3.5.6.1. For each horizontal morphism $R \colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R} \colon \mathbb{1}_B \odot R \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[14]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\rotatebox{45}{$=$}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

[13] As proved in Item 2 of Definition 8.1.3.1.4.

[14] As proved in Item 3 of Definition 8.1.3.1.4.

8.3.5.7 The Right Unitors

Definition 8.3.5.7.1. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R : R \odot \mathbb{1}_A \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "|" marking] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|" marking] \arrow[d, phantom, "\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|" marking] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[15]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R\diamond\chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

8.4 Categories of Relations With Apartness Composition

8.4.1 The Category of Relations With Apartness Composition

Definition 8.4.1.1.1. The **category of relations with apartness composition** is the category $\mathsf{Rel}^{\square}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\square}$ are sets.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}^{\square}(A, B) \overset{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel}^{\square})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}^{\square}}_A : \mathrm{pt} \to \mathrm{Rel}(A, A)$$

[15] As proved in Item 3 of Definition 8.1.3.1.4.

of $\mathrm{Rel}^{\square}$ at A is defined by

$$\mathrm{id}_A^{\mathrm{Rel}^{\square}} \stackrel{\text{def}}{=} \nabla_A(-_1, -_2),$$

where $\nabla_A(-_1, -_2)$ is the antidiagonal relation of A of Definition 8.2.1.1.4.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathrm{Rel}^{\square})$, the composition map

$$\circ_{A,B,C}^{\mathrm{Rel}^{\square}} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, C)$$

of $\mathrm{Rel}^{\square}$ at (A, B, C) is defined by

$$S \circ_{A,B,C}^{\mathrm{Rel}^{\square}} R \stackrel{\text{def}}{=} S \mathbin{\square} R$$

for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.4.1.1.

Proposition 8.4.1.1.2. The functor

$$(-)^{\mathsf{c}} \colon \mathrm{Rel} \longrightarrow \mathrm{Rel}^{\square}$$

given by the identity on objects and by $R \mapsto R^{\mathsf{c}}$ on morphisms is an isomorphism of categories.

Proof. By Item 4 of Definition 8.1.4.1.3, we see that $(-)^{\mathsf{c}}$ is indeed a functor.

By Item 1 of Definition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\mathsf{c}} \colon \mathrm{Rel}(A, B) \longrightarrow \mathrm{Rel}(A, B)$$

defined by the assignment $R \mapsto R^{\mathsf{c}}$ is a bijection by Item 3 of Definition 4.3.11.1.2. Thus $(-)^{\mathsf{c}}$ is an isomorphism of categories. □

8.4.2 The 2-Category of Relations With Apartness Composition

Definition 8.4.2.1.1. The **2-category of relations with apartness composition** is the locally posetal 2-category $\mathbf{Rel}$ where

- *Objects.* The objects of $\mathbf{Rel}$ are sets.
- **Hom**-*Objects.* For each $A, B \in \mathrm{Obj}(\mathrm{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A, B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A, B)\\ &\stackrel{\text{def}}{=} (\mathrm{Rel}(A, B), \subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}_A^{\mathsf{Rel}} \colon \mathrm{pt} \to \mathbf{Rel}(A, A)$$

of Rel at A is defined by

$$\mathrm{id}_A^{\mathsf{Rel}} \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Definition 8.2.1.1.3.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map[16]

$$\circ_{A,B,C}^{\mathsf{Rel}} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

of Rel at (A, B, C) is defined by

$$S \circ_{A,B,C}^{\mathsf{Rel}} R \stackrel{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 8.1.3.1.1.

Proposition 8.4.2.1.2. The functor

$$(-)^{\mathsf{c}} \colon \mathsf{Rel} \to \mathsf{Rel}^{\square,\mathrm{co}}$$

given by the identity on objects and by $R \mapsto R^{\mathsf{c}}$ on 1-morphisms is a 2-isomorphism of 2-categories.

Proof. By Item 4 of Definition 8.1.4.1.3, we see that $(-)^{\mathsf{c}}$ is indeed a functor. By Item 1 of Definition 4.3.11.1.2, it is also a 2-functor.

By **??**, it suffices to show that $(-)^{\mathsf{c}}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Definition 8.4.1.1.1.
- Bijective on 2-morphisms, which follows from Item 1 of Definition 4.3.11.1.2.

Thus $(-)^{\mathsf{c}}$ is indeed a 2-isomorphism of categories. □

[16] That this is indeed a morphism of posets is proven in **??** of Definition 8.1.4.1.3.

8.4.3 The Linear Bicategory of Relations

Definition 8.4.3.1.1. The **linear bicategory of relations** is the linear bicategory consisting of:

- *The Underlying Bicategory I.* The bicategory Rel of Definition 8.3.4.1.1.
- *The Underlying Bicategory II.* The bicategory Rel of Definition 8.4.2.1.1.
- *Linear Distributors.* The inclusions

$$\begin{aligned}\delta^{\ell}_{R,S,T}&\colon T\diamond(S\mathbin{\square}R)\hookrightarrow(T\diamond S)\mathbin{\square}R,\\ \delta^{r}_{R,S,T}&\colon (T\mathbin{\square}S)\diamond R\hookrightarrow T\mathbin{\square}(S\diamond R)\end{aligned}$$

  of Item 5 of Definition 8.1.4.1.3.

Proof. Since Rel and $\mathrm{Rel}^{\square}$ are locally posetal, the commutativity of the coherence conditions for linear bicategories follows automatically (**??** of **??**). □

8.4.4 Other Categorical Structures With Apartness Composition

Remark 8.4.4.1.1. It seems apartness composition fails to form the following categorical structures:

- *Monoidal Category With Products.* Products don't seem to endow $\mathrm{Rel}^{\square}$ with a monoidal structure.
- *Monoidal Category With Coproducts.* Coproducts also don't seem to endow $\mathrm{Rel}^{\square}$ with a monoidal structure.
- *Double Categorical Structure.* It seems the apartness composition of relations doesn't form a double category in a natural[17] way.

8.5 Properties of the 2-Category of Relations

8.5.1 Self-Duality

Proposition 8.5.1.1.1. The 2-/category of relations is self-dual:

[17] I.e. such that the composition of vertical morphisms is the usual composition of functions, as in Sets.

1. *Self-Duality I*. We have an isomorphism

$$\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$$

of categories.

2. *Self-Duality II*. We have a 2-isomorphism

$$\mathbf{Rel}^{\mathsf{op}} \cong \mathbf{Rel}$$

of 2-categories.

Proof. *Item 1, Self-Duality I*: We claim that the functor

$$(-)^{\dagger} \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories. Note that this is indeed a functor by Items 3 and 6 of Definition 8.1.5.1.3.

By Item 1 of Definition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\dagger} \colon \mathsf{Rel}(A, B) \to \mathsf{Rel}(B, A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Definition 8.1.5.1.3, showing $(-)^{\dagger}$ to be fully faithful.

Item 2, Self-Duality II: We claim that the 2-functor

$$(-)^{\dagger} \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Definition 8.1.5.1.3, is an isomorphism of categories.

By **??**, it suffices to show that $(-)^{\dagger}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Definition 8.1.5.1.3.

Thus $(-)^{\dagger}$ is indeed a 2-isomorphism of categories. □

8.5.2 Isomorphisms and Equivalences

Let $R \colon A \nrightarrow B$ be a relation from A to B.

Lemma 8.5.2.1.1. The conditions below are row-wise equivalent:

Condition	Inclusion
R is functional	$R \diamond R^{\dagger} \subset \Delta_B$
R is total	$\Delta_A \subset R^{\dagger} \diamond R$
R is injective	$R^{\dagger} \diamond R \subset \Delta_A$
R is surjective	$\Delta_B \subset R \diamond R^{\dagger}$

Proof. *Functionality Is Equivalent to $R \diamond R^{\dagger} \subset \Delta_B$*: The condition $R \diamond R^{\dagger} \subset \Delta_B$ unwinds to

(★) For each $b, b' \in B$, if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$.

Since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, the condition says that $a \sim_R b$ and $a \sim_R b'$ imply $b = b'$. This is precisely the condition for R to be functional.
Totality Is Equivalent to $\Delta_A \subset R^{\dagger} \diamond R$: The condition $\Delta_A \subset R^{\dagger} \diamond R$ unwinds to

(★) For each $a, a' \in A$, if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^{\dagger}} a'$.

Since $b \sim_{R^{\dagger}} a'$ is the same as $a' \sim_R b$, the condition says that for each $a \in A$, there is some $b \in B$ with $b \in R(a)$, so $R(a) \neq \emptyset$. This is precisely the condition for R to be total.
Injectivity Is Equivalent to $R^{\dagger} \diamond R \subset \Delta_A$: The condition $R^{\dagger} \diamond R \subset \Delta_A$ unwinds to

(★) For each $a, a' \in A$, if there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^{\dagger}} a'$, then $a = a'$.

Since $b \sim_{R^{\dagger}} a'$ is the same as $a' \sim_R b$, the condition says that for each $b \in B$, if $a \sim_R b$ and $a' \sim_R b$, then $a = a'$. This is precisely the condition for R to be injective.
Surjectivity Is Equivalent to $\Delta_B \subset R \diamond R^{\dagger}$: The condition $\Delta_B \subset R \diamond R^{\dagger}$ unwinds to

(★) For each $b, b' \in B$, if $b = b'$, then there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$.

Since $b \sim_{R^\dagger} a$ is the same as $a \sim_R b$, the condition says that for each $b \in B$, there is some $a \in A$ with $b \in R(a)$, so $R^{-1}(b) \neq \emptyset$. This is precisely the condition for R to be surjective. □

Proposition 8.5.2.1.2. The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms
   $$\begin{aligned} R^{-1} \diamond R &\cong \Delta_A, \\ R \diamond R^{-1} &\cong \Delta_B. \end{aligned}$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have
   $$\begin{aligned} R^{-1} \diamond R &= \Delta_A, \\ R \diamond R^{-1} &= \Delta_B. \end{aligned}$$

3. There exists a bijection $f\colon A \xrightarrow{\sim} B$ with $R = \mathrm{Gr}(f)$.

Proof. We claim that Items 1 to 3 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.

- *Item 2* $\implies$ *Item 3:* We proceed in a few steps:

  – First, note that the equalities in Item 2 imply $R \dashv R^{-1}$ and thus, by Definition 8.5.3.1.1, there exists a function $f_R\colon A \to B$ associated to R.

  – By Definition 8.5.2.1.1, f_R is a bijection.

- *Item 3* $\implies$ *Item 2:* By Item 4 of Definition 8.2.2.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions
  $$\begin{aligned} \Delta_A &\subset f^{-1} \diamond \mathrm{Gr}(f), \\ \mathrm{Gr}(f) \diamond f^{-1} &\subset \Delta_B. \end{aligned}$$
  If f is bijective, then the reverse inclusions are also true by Definition 8.5.2.1.1.

This finishes the proof. □

8.5.3 Internal Adjunctions

Let A and B be sets.

Proposition 8.5.3.1.1. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\},$$

with every adjunction in Rel being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof. We proceed step by step:

1. *From Adjunctions in* Rel *to Functions.* An adjunction in Rel from A to B consists of a pair of relations

$$R\colon A \mathrel{-\!\!\!\mapsto} B,$$
$$S\colon B \mathrel{-\!\!\!\mapsto} A,$$

   together with inclusions

$$\Delta_A \subset S \diamond R,$$
$$R \diamond S \subset \Delta_B.$$

   By Definition 8.5.2.1.1, R is total and functional. In particular, $R(a)$ is a singleton for all $a \in A$. Defining f_R such that $f_R(a)$ is the unique element of $R(a)$ then gives us our desired function, forming a map

$$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

   Moreover, by uniqueness of adjoints (**??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* Rel. By Item 4 of Definition 8.2.2.1.2, every function $f\colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions}\\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel}\\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This follows form the fact that we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S}\colon A \to B$, we have

$$\begin{aligned} \mathrm{Gr}\big(f_{R,S}\big) &= R, \\ f^{-1}_{R,S} &= S. \end{aligned}$$

We check these explicitly:

- $Gr\big(f_{R,S}\big) = R$. We have
$$\begin{aligned} \mathrm{Gr}\big(f_{R,S}\big) &\overset{\text{def}}{=} \big\{\big(a, f_{R,S}(a)\big) \in A \times B \,\big|\, a \in A\big\} \\ &\overset{\text{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\} \\ &= R. \end{aligned}$$
- $f^{-1}_{R,S} = S$. We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* We proceed in a few steps.
    * Since $\Delta_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
    * Since $a \sim_R b$ and R is functional, we have $k = b$.
    * Thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* We proceed in a few steps.
    * First note that, since R is total, we have $a \sim_R b'$ for some $b' \in B$.
    * Since $R \diamond S \subset \Delta_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$.
    * Thus $a \sim_R b$.

  Having show this, we now have
$$\begin{aligned} f^{-1}_{R,S}(b) &\overset{\text{def}}{=} \big\{a \in A \,\big|\, f_{R,S}(a) = b\big\} \\ &\overset{\text{def}}{=} \{a \in A \mid a \sim_R b\} \\ &= \{a \in A \mid b \sim_S a\} \\ &\overset{\text{def}}{=} S(b). \end{aligned}$$
  for each $b \in B$, and thus $f^{-1}_{R,S} = S$.

This finishes the proof. □

8.5.4 Internal Monads

Let X be a set.

Proposition 8.5.4.1.1. We have a natural identification[18]

$$\left\{\begin{matrix}\text{Monads in}\\ \mathsf{Rel}\text{ on }X\end{matrix}\right\} \cong \{\text{Preorders on }X\}.$$

Proof. A monad in Rel on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\mu_R&\colon R \diamond R \subset R,\\ \eta_R&\colon \Delta_X \subset R\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
\Delta_X \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R", "|" marking] \arrow[dr, equal, "\lambda^{\mathsf{Rel}}_R"'] & R \diamond R \arrow[d, "\mu_R", "-" marking] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \diamond (R \diamond R) \arrow[dr, "\mathrm{id}_R \diamond \mu_R", "|" marking] & \\
(R \diamond R) \diamond R \arrow[ur, equal, "\alpha^{\mathsf{Rel}}_{R,R,R}"] \arrow[d, "\mu_R \diamond \mathrm{id}_R"', "-" marking] & & R \diamond R \arrow[d, "\mu_R", "-" marking] \\
R \diamond R \arrow[rr, "\mu_R"', "|" marking] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \Delta_X \arrow[r, "\mathrm{id}_R \diamond \eta_R", "|" marking] \arrow[dr, equal, "\rho^{\mathsf{Rel}}_R"'] & R \diamond R \arrow[d, "\mu_R", "-" marking] \\
& R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (**??** of **??**), and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $x, z \in X$, if there exists some $y \in Y$ such that $x \sim_R y$ and $y \sim_R z$, then $x \sim_R z$.
2. For each $x \in X$, we have $x \sim_R x$.

These are exactly the requirements for R to be a preorder (**??**). Conversely, any preorder $\preceq$ gives rise to a pair of maps $\mu_{\preceq}$ and $\eta_{\preceq}$, forming a monad on X. □

Example 8.5.4.1.2. Let $R\colon A \nrightarrow B$ be a relation.

[18] See also **??** for an extension of this correspondence to "relative monads in Rel".

1. The codensity monad $\mathrm{Ran}_R(R) \colon B \nrightarrow B$ is given by

$$[\mathrm{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b) \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(R)"] \\ A \arrow[ur, "R"] \arrow[r, "R"'] \arrow[ur, phantom, shift right=3, "\Downarrow"] & B \end{tikzcd}$$

   for each $b \in B$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Ran}_R(R)} \colon B \times B \to \{\mathsf{t}, \mathsf{f}\}$$

   on B obtained by declaring $b \preceq_{\mathrm{Ran}_R(R)} b'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a \sim_R b'$.
   (b) We have $R^{-1}(b) \subset R^{-1}(b')$.

2. The dual codensity monad $\mathrm{Rift}_R(R) \colon A \nrightarrow A$ is given by

$$[\mathrm{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\} \qquad \begin{tikzcd} & A \arrow[d, "R"] \\ A \arrow[ur, dashed, "\mathrm{Rift}_R(R)"] \arrow[r, "R"'] \arrow[ur, phantom, shift right=3, "\Downarrow"] & B \end{tikzcd}$$

   for each $a \in A$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Rift}_R(R)} \colon A \times A \to \{\mathsf{t}, \mathsf{f}\}$$

   on A obtained by declaring $a \preceq_{\mathrm{Rift}_R(R)} a'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a' \sim_R b$.
   (b) We have $R(a') \subset R(a)$.

8.5.5 Internal Comonads

Let X be a set.

Proposition 8.5.5.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in} \\ \textbf{Rel} \text{ on } X\end{matrix}\right\} \cong \{\text{Subsets of } X\}.$$

Proof. A comonad in **Rel** on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\Delta_R\colon R &\subset R \diamond R,\\ \epsilon_R\colon R &\subset \Delta_X\end{aligned}$$

making the diagrams

$$\begin{tikzcd} R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equals, "\lambda^{\mathsf{Rel},-1}_R"'] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "\shortmid"{marking}] \\ & \Delta_X \diamond R \end{tikzcd}
\qquad
\begin{tikzcd} & R \diamond R \arrow[r, "\mathrm{id}_R \diamond \Delta_R", "\shortmid"{marking}] & R \diamond (R \diamond R) \arrow[d, equals, "\alpha^{\mathsf{Rel},-1}_{R,R,R}"] \\ R \arrow[ur, "\Delta_R", "\shortmid"{marking}] \arrow[r, "\Delta_R"', "\shortmid"{marking}] & R \diamond R \arrow[r, "\Delta_R \diamond \mathrm{id}_R"', "\shortmid"{marking}] & (R \diamond R) \diamond R \end{tikzcd}
\qquad
\begin{tikzcd} R \arrow[r, "\Delta_R", "\shortmid"{marking}] \arrow[dr, equals, "\rho^{\mathsf{Rel},-1}_R"'] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "\shortmid"{marking}] \\ & R \diamond \Delta_X \end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (**??** of **??**), and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $x, y \in X$, if $x \sim_R y$, then there exists some $k \in X$ such that $x \sim_R k$ and $k \sim_R y$.

2. For each $x, y \in X$, if $x \sim_R y$, then $x = y$.

The second condition implies that $R \subset \Delta_X$, so R must be a subset of X. Taking $k = y$ in the first condition above then shows it to be trivially satisfied. Conversely, any subset U of X satisfies $U \subset \Delta_X$, defining a comonad as above. □

Example 8.5.5.1.2. Let $f\colon A \to B$ be a function.

1. The density comonad $\mathrm{Lan}_f(f)\colon B \nrightarrow B$ is given by

$$\left[\mathrm{Lan}_f(f)\right](b) = \bigcup_{a \in f^{-1}(b)} f(a) \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Lan}_f(f)", "\shortmid"{marking}] \\ A \arrow[ur, "f", "\shortmid"{marking}] \arrow[r, "f"', "\shortmid"{marking}] \arrow[ur, phantom, shift right=3, "\Uparrow"] & B \end{tikzcd}$$

for each $b \in B$. Thus, it corresponds to the image $\mathrm{Im}(f)$ of f as a subset of B.

2. The dual density comonad $\mathrm{Lift}_{f^{\dagger}}(f^{\dagger}) \colon A \mathrel{\mapstochar\mkern-5mu\rightarrow} A$ is given by

$$\left[\mathrm{Lift}_{f^{\dagger}}\left(f^{\dagger}\right)\right](b) = \bigcup_{a \in f^{-1}(b)} f(a) \qquad \begin{tikzcd} & B \arrow[d, "f^{\dagger}", "\mapstochar" marking] \\ B \arrow[ur, dashed, "\mathrm{Lift}_{f^{\dagger}}(f^{\dagger})", "\mapstochar" marking] \arrow[r, "f^{\dagger}"', "\mapstochar" marking] & A \arrow[u, phantom, "\Uparrow", shift right=3] \end{tikzcd}$$

for each $b \in B$. Thus, it also corresponds to the image $\mathrm{Im}(f)$ of f as a subset of B.

8.5.6 Modules Over Internal Monads

Let A be a set.

Proposition 8.5.6.1.1. Let $\preceq_A$ be a preorder on A, viewed also as an internal monad on A via Definition 8.5.4.1.1.

1. *Left Modules.* We have a natural identification

$$\{\text{Left modules over } \preceq_A\} \cong \left\{\begin{array}{l} \text{Relations } R\colon B \mathrel{\mapstochar\mkern-5mu\rightarrow} A \text{ such that,} \\ \text{for each } b \in B, \text{ the set } R(b) \text{ is} \\ \text{upward-closed in } A \end{array}\right\}.$$

2. *Right Modules.* We have a natural identification

$$\{\text{Right modules over } \preceq_A\} \cong \left\{\begin{array}{l} \text{Relations } R\colon A \mathrel{\mapstochar\mkern-5mu\rightarrow} B \text{ such that,} \\ \text{for each } b \in B, \text{ the set } R^{-1}(b) \text{ is} \\ \text{downward-closed in } A \end{array}\right\}.$$

3. *Bimodules.* We have a natural identification

$$\{\text{Bimodules over } \preceq_A\} \cong \left\{\begin{array}{l} \text{Quadruples } (B, C, R, S) \text{ such that:} \\ \text{1. For each } b \in B, \text{ the set } R(b) \text{ is} \\ \quad \text{upward-closed in } A. \\ \text{2. For each } c \in C, \text{ the set } S^{-1}(c) \text{ is} \\ \quad \text{downward-closed in } A. \end{array}\right\}.$$

Proof. Item 1, Left Modules: A left module over $\preceq_A$ in Rel consists of a relation $R\colon B \mathrel{\mapstochar\mkern-5mu\rightarrow} A$ together with an inclusion

$$\alpha_B \colon \preceq_A \diamond R \subset R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (**??** of **??**), and hence all that is left is the data of the inclusion α_B. This corresponds to the following condition:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that $b \sim_R a$ and $a \preceq_a a'$, then $b \sim_R a'$.

This condition is equivalent to $R(b)$ being downward-closed for all $b \in B$.

Item 2, Right Modules: The proof is dual to Item 1, and is therefore omitted.

Item 3, Bimodules: Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (**??** of **??**), and hence a bimodule is just a left module along with a right module. □

8.5.7 Comodules Over Internal Comonads

Let A be a set.

Proposition 8.5.7.1.1. Let U be a subset of A, viewed also as an internal comonad on A via Definition 8.5.5.1.1.

1. *Left Comodules.* We have a natural identification

$$\{\text{Left comodules over } U\} \cong \left\{\begin{matrix}\text{Relations } R\colon B \nrightarrow A \text{ such that,}\\ \text{for each } b \in B\text{, we have } R(b) \subset U\end{matrix}\right\}.$$

2. *Right Comodules.* We have a natural identification

$$\{\text{Right comodules over } U\} \cong \left\{\begin{matrix}\text{Relations } R\colon A \nrightarrow B \text{ such that,}\\ \text{for each } b \in B\text{, we have } R^{-1}(b) \subset U\end{matrix}\right\}.$$

3. *Bicomodules.* We have a natural identification

$$\{\text{Bicomodules over } U\} \cong \left\{\begin{matrix}\text{Quadruples } (B, C, R, S) \text{ such that:}\\ \text{1. For each } b \in B\text{, we have } R(b) \subset U\\ \text{2. For each } c \in C\text{, we have } S^{-1}(c) \subset U\end{matrix}\right\}.$$

Proof. Item 1, Left Comodules: A left comodule over U in **Rel** consists of a relation $R\colon B \nrightarrow A$ together with an inclusion

$$R \subset U \diamond R$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (**??** of **??**), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

(★) For each $b \in B$, if $b \sim_R a$, then there exists some $a' \in A$ such that $b \sim_R a'$ and $a' \sim_U a$.

Since $a' \sim_U a$ is true if $a = a'$ and $a \in U$, this condition ends up being equivalent to $R(b) \subset U$.

Item 2, Right Comodules: A right comodule over U in **Rel** consists of a relation $R \colon A \nrightarrow B$ together with an inclusion

$$R \subset R \diamond U$$

making appropriate diagrams commute. Since **Rel** is locally posetal, however, the commutativity of the diagrams in question is automatic (**??** of **??**), and hence all that is left is the data of the inclusion. This corresponds to the following condition:

(★) For each $a \in A$, if $a \sim_R b$, then there exists some $x \in A$ such that $a \sim_U x$ and $x \sim_R b$.

Since $a \sim_U x$ is true if $a = x$ and $a \in U$, this condition ends up being equivalent to $R^{-1}(b) \subset U$.

Item 3, Bicomodules: Since **Rel** is locally posetal, the diagram encoding the compatibility conditions for a bimodule commutes automatically (**??** of **??**), and hence a bicomodule is just a left comodule along with a right comodule. □

8.5.8 Eilenberg–Moore and Kleisli Objects

Let X be a set.

Proposition 8.5.8.1.1. Let R be a preorder on X, viewed as an internal monad on X via Definition 8.5.4.1.1.

1. *Eilenberg–Moore Objects in* **Rel**. The Eilenberg–Moore object for R exists iff it is an equivalence relation, in which case it is the quotient $X/{\sim_R}$ of X by R.

2. *Kleisli Objects in* **Rel**. [...]

Proof. Omitted. □

8.5.9 Co/Monoids

Remark 8.5.9.1.1. The monoids in **Rel** with respect to the Cartesian monoidal structure of Definition 8.3.3.8.1 are called *hypermonoids*, and their theory is explored in **??**. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in **??**.

8.5.10 Monomorphisms

In this section we characterise the epimorphisms in the category Rel, following **??**.

Proposition 8.5.10.1.1. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is a monomorphism in Rel.
2. The direct image function
$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.
3. The codirect image function
$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$
associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that
$$\begin{aligned} a &\sim_R b,\\ a' &\sim_R b, \end{aligned}$$
then $a = a'$.

Proof. First Proof of the Equivalence of Items 1 to 3: Firstly note that Items 2 and 3 are equivalent by Item 7 of Definition 8.7.1.1.4. We then claim that Items 1 and 2 are also equivalent:

- *Item 1 $\implies$ Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram
$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\nrightarrow} B.$$
By Definition 8.7.1.1.3, we have
$$\begin{aligned} R_!(U) &= R \diamond U,\\ R_!(V) &= R \diamond V. \end{aligned}$$
Now, if $R \diamond U = R \diamond V$, i.e. $R_!(U) = R_!(V)$, then $U = V$ since R is assumed to be a monomorphism, showing $R_!$ to be injective.

- *Item 2* $\Longrightarrow$ *Item 1:* Conversely, suppose that $R_!$ is injective, consider the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "S", "\shortmid"{marking}] \arrow[r, shift right, "T"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

and suppose that $R \diamond S = R \diamond T$. Note that, since $R_!$ is injective, given a diagram of the form

$$\begin{tikzcd} \mathrm{pt} \arrow[r, shift left, "U", "\shortmid"{marking}] \arrow[r, shift right, "V"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

if $R_!(U) = R \diamond U = R \diamond V = R_!(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[x]}", "\shortmid"{marking}] & X \arrow[r, shift left, "S", "\shortmid"{marking}] \arrow[r, shift right, "T"', "\shortmid"{marking}] & A \arrow[r, "R", "\shortmid"{marking}] & B, \end{tikzcd}$$

where we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each $x \in X$. Thus $S = T$ and R is a monomorphism.

Second Proof of the Equivalence of Items 1 to 3: A more abstract proof can also be given, following [MSE 350788]:

- *Item 1* $\Longrightarrow$ *Item 2:* Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathrm{Sets}$ maps R to $R_!$ by Definition 8.7.1.1.3.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to $R_!$, it follows that $R_!$ is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that $R_!$ is injective.
- *Item 2* $\Longrightarrow$ *Item 1:* Assume that $R_!$ is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathrm{Sets}$ maps R to $R_!$ by Definition 8.7.1.1.3.

- Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that $R_!$ is a monomorphism.
- Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
- Since $R_!$ is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to $R_!$, it follows that R is also a monomorphism.

Proof of the Second Half of Definition 8.5.10.1.1: Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\mathrm{pt} \underset{[a']}{\overset{[a]}{\rightrightarrows}} A \xrightarrow{R} B.$$

Then:

- Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R \diamond [a]} b$.
- Similarly, $\star \sim_{R \diamond [a']} b$.
- Thus $R \diamond [a] = R \diamond [a']$.
- Since R is a monomorphism, we have $[a] = [a']$, so $a = a'$.

Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$

then $a = a'$.

consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and let $x \in X$ and $a \in A$ such that $x \sim_S a$.

- Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$.
- In this case, we have $x \sim_{R \diamond S} b$, and since $R \diamond S = R \diamond T$, we have also $x \sim_{R \diamond T} b$.

- Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$.
- However, since $a \sim_R b$ and $a' \sim_R b$, we must have $a = a'$.
- Thus $x \sim_T a$ as well.
- A similar argument shows that if $x \sim_T a$, then $x \sim_S a$.
- Thus $S = T$ and it follows that R is a monomorphism.

This finishes the proof. □

8.5.11 2-Categorical Monomorphisms

In this section we characterise (for now, some of) the 2-categorical monomorphisms in $\mathbf{Rel}$, following Section 14.1.

Proposition 8.5.11.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Representably Faithful Morphisms in* $\mathbf{Rel}$. Every morphism of $\mathbf{Rel}$ is a representably faithful morphism.
2. *Representably Full Morphisms in* $\mathbf{Rel}$. The following conditions are equivalent:
   (a) The morphism $R\colon A \nrightarrow B$ is a representably full morphism.
   (b) For each pair of relations $S, T\colon X \rightrightarrows A$, the following condition is satisfied:
      ($\star$) If $R \diamond S \subset R \diamond T$, then $S \subset T$.
   (c) The direct image functor
   $$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
   of Item 2 of Definition 8.7.1.1.5 is full.
   (d) The codirect image functor
   $$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$
   of Item 2 of Definition 8.7.4.1.4 is full.
   (e) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.
   (f) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

3. *Representably Fully Faithful Morphisms in* **Rel**. Every representably full morphism in **Rel** is a representably fully faithful morphism.

Proof. *Item 1, Representably Faithful Morphisms in* **Rel**: The relation R is a representably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R_! \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

is faithful, i.e. iff the morphism

$$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$. However, $\mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T)$ is either empty or a singleton, in either case of which the map $R_{*|S,T}$ is necessarily injective.
Item 2, Representably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a representably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

  $$R_! \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

  is full, i.e. iff the morphism

  $$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$, i.e. iff, whenever $R \diamond S \subset R \diamond T$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2e:* This is also simply a matter of unwinding definitions: The functor

  $$R_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

  $$R_{*|U,V} \colon \mathrm{Hom}_{\mathcal{P}(A)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(B)}(R_!(U), R_!(V))$$

  is surjective, i.e. iff whenever $R_!(U) \subset R_!(V)$, we also necessarily have $U \subset V$.

- *Item 2d* $\iff$ *Item 2f:* This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2e given above.

- *Item 2e* $\implies$ *Item 2f*: Suppose that the following condition is true:

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  We need to show that the condition

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  – Suppose we have $U, V \in \mathcal{P}(A)$ with $R_*(U) \subset R_*(V)$.
  – By Item 7 of Definition 8.7.4.1.3, we have

  $$\begin{aligned} R_*(U) &= B \setminus R_!(A \setminus U), \\ R_*(V) &= B \setminus R_!(A \setminus V). \end{aligned}$$

  – By Item 1 of Definition 4.3.10.1.2 we have $R_!(A \setminus V) \subset R_!(A \setminus U)$.
  – By assumption, we then have $A \setminus V \subset A \setminus U$.
  – By Item 1 of Definition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\implies$ *Item 2e*: Suppose that the following condition is true:

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  We need to show that the condition

  (★) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  – Suppose we have $U, V \in \mathcal{P}(A)$ with $R_!(U) \subset R_!(V)$.
  – By Item 7 of Definition 8.7.1.1.4, we have

  $$\begin{aligned} R_!(U) &= B \setminus R_*(A \setminus U), \\ R_!(V) &= B \setminus R_*(A \setminus V). \end{aligned}$$

  – By Item 1 of Definition 4.3.10.1.2 we have $R_*(A \setminus V) \subset R_*(A \setminus U)$.
  – By assumption, we then have $A \setminus V \subset A \setminus U$.
  – By Item 1 of Definition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\implies$ *Item 2e*: Consider the diagram

  $$X \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\rightarrow} B,$$

  and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

  $$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\rightarrow} B,$$

  if $R_!(U) = R \diamond U \subset R \diamond V = R_!(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

  $$\mathrm{pt} \overset{[x]}{\rightarrow} X \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\rightarrow} B,$$

  for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

  $$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

  for each $x \in X$, implying $S \subset T$.

- *Item 2e* $\implies$ *Item 2b*: Let $U, V \in \mathcal{P}(A)$ and consider the diagram

  $$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\rightarrow} B.$$

  By Definition 8.7.1.1.3, we have

  $$R_!(U) = R \diamond U,$$
  $$R_!(V) = R \diamond V.$$

  Now, if $R_!(U) \subset R_!(V)$, i.e. $R \diamond U \subset R \diamond V$, then $U \subset V$ by assumption.

Item 3, Representably Fully Faithful Morphisms in **Rel**: This follows from Items 1 and 2. □

Question 8.5.11.1.2. Item 2 of Definition 8.5.11.1.1 gives a characterisation of the representably full morphisms in **Rel**.

Are there other nice characterisations of these?

This question also appears as [MO 467527].

8.5.12 Epimorphisms

In this section we characterise the epimorphisms in the category Rel, following **??**.

Proposition 8.5.12.1.1. Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

1. The relation R is an epimorphism in Rel.
2. The weak inverse image function
$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.
3. The strong inverse image function
$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.
4. The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":
   ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is total and an epimorphism, then it satisfies the following equivalent conditions:

1. For each $b \in B$, there exists some $a \in A$ such that $a \sim_R b$.
2. We have $\mathrm{Im}(R) = B$.

Proof. First Proof of the Equivalence of Items 1 to 3: Firstly note that Items 2 and 3 are equivalent by Item 7 of Definition 8.7.2.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram
$$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$
By Definition 8.7.1.1.3, we have
$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$
Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\implies$ *Item 1*: Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \overset{R}{\relbar\joinrel\mapsto} B \overset{S}{\underset{T}{\rightrightarrows}} X,$$

and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \overset{R}{\relbar\joinrel\mapsto} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\relbar\joinrel\mapsto} B \overset{S}{\underset{T}{\rightrightarrows}} X \overset{[x]}{\relbar\joinrel\mapsto} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

for each $x \in X$. Thus $S = T$ and R is an epimorphism.

Second Proof of the Equivalence of Items 1 to 3: A more abstract proof can also be given, following [MSE 350788]:

- *Item 1* $\implies$ *Item 2*: Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Definition 8.7.3.1.2.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$ to monomorphisms in Sets.
  - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is injective.

- *Item 2* $\implies$ *Item 1:* Assume that R^{-1} is injective.
    - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt})\colon \mathrm{Rel}^{\mathrm{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Definition 8.7.3.1.2.
    - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is a monomorphism.
    - Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects monomorphisms.
    - That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathrm{op}}$.
    - The monomorphisms $\mathrm{Rel}^{\mathrm{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
    - Since R^{-1} is a monomorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

Proof of the Equivalence of Items 2 and 4: We claim that Items 2 and 4 are equivalent, following [`MO 350788`]:

- *Item 2* $\implies$ *Item 4:* We proceed in two steps.
    - Since $B\setminus\{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$.
    - Taking some $a \in R^{-1}(B)\setminus R^{-1}(B \setminus \{b\})$, we obtain an element of A such that $R(a) = \{b\}$.
- *Item 4* $\implies$ *Item 2:* We proceed in a few steps.
    - Let $U, V \subset B$ with $U \neq V$.
    - Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V.
    - Let then $b \in U \setminus V$.
    - By assumption, there exists an $a \in A$ with $R(a) = \{b\}$.
    - Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, so $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Proof of the Second Half of Definition 8.5.12.1.1: Finally, we prove the second part of the statement. Assume R is a total epimorphism in Rel and consider the diagram

$$A \overset{R}{\nrightarrow} B \underset{T}{\overset{S}{\rightrightarrows}} \{0, 1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$.

- Since R is total, we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ for all $a \in A$, and no element of A is related to 1 by $S \diamond R$ or $T \diamond R$.
- Thus $S \diamond R = T \diamond R$.
- Since R is an epimorphism, we have $S = T$.
- But by the definition of T, this implies $\mathrm{Im}(R) = B$.

This finishes the proof. □

8.5.13 2-Categorical Epimorphisms

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following Section 14.2.

Proposition 8.5.13.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a corepresentably faithful morphism.

2. *Corepresentably Full Morphisms in* **Rel**. The following conditions are equivalent:

   (a) The morphism $R\colon A \nrightarrow B$ is a corepresentably full morphism.

   (b) For each pair of relations $S, T\colon X \rightrightarrows A$, the following condition is satisfied:

   ($\star$) If $S \diamond R \subset T \diamond R$, then $S \subset T$.

   (c) The functor

   $$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   is full.

   (d) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

(e) The functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full.

(f) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

3. *Corepresentably Fully Faithful Morphisms in* **Rel**. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

Proof. *Item 1, Corepresentably Faithful Morphisms in* **Rel**: The relation R is a corepresentably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$. However, $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, in either case of which the map $R^*_{S,T}$ is necessarily injective.

Item 2, Corepresentably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b*: This is simply a matter of unwinding definitions: The relation R is a corepresentably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

  is full, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$, i.e. iff, whenever $S \diamond R \subset T \diamond R$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d*: This is also simply a matter of unwinding definitions: The functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

$$R^{-1}_{U,V}\colon \mathrm{Hom}_{\mathcal{P}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(A)}\bigl(R^{-1}(U), R^{-1}(V)\bigr)$$

  is surjective, i.e. iff whenever $R^{-1}(U) \subset R^{-1}(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f*: This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.
- *Item 2d* $\implies$ *Item 2f*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  - Suppose we have $U, V \in \mathcal{P}(B)$ with $R_{-1}(U) \subset R_{-1}(V)$.
  - By Item 7 of Definition 8.7.2.1.3, we have
  $$\begin{aligned} R_{-1}(U) &= B \setminus R^{-1}(A \setminus U), \\ R_{-1}(V) &= B \setminus R^{-1}(A \setminus V). \end{aligned}$$
  - By Item 1 of Definition 4.3.10.1.2 we have $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$.
  - By assumption, we then have $A \setminus V \subset A \setminus U$.
  - By Item 1 of Definition 4.3.10.1.2 again, we have $U \subset V$.
- *Item 2f* $\implies$ *Item 2d*: Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  - Suppose we have $U, V \in \mathcal{P}(B)$ with $R^{-1}(U) \subset R^{-1}(V)$.
  - By Item 7 of Definition 8.7.3.1.3, we have
  $$\begin{aligned} R^{-1}(U) &= B \setminus R_{-1}(A \setminus U), \\ R^{-1}(V) &= B \setminus R_{-1}(A \setminus V). \end{aligned}$$
  - By Item 1 of Definition 4.3.10.1.2 we have $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$.
  - By assumption, we then have $A \setminus V \subset A \setminus U$.

  - By Item 1 of Definition 4.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\implies$ *Item 2d:* Consider the diagram

$$A \xrightarrow{R} B \overset{S}{\underset{T}{\rightrightarrows}} X,$$

and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form

$$A \xrightarrow{R} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$A \xrightarrow{R} B \overset{S}{\underset{T}{\rightrightarrows}} X \xrightarrow{[x]} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2e* $\implies$ *Item 2b:* Let $U, V \in \mathcal{P}(B)$ and consider the diagram

$$A \xrightarrow{R} B \overset{U}{\underset{V}{\rightrightarrows}} \mathrm{pt}.$$

By Definition 8.7.1.1.3, we have

$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$

Now, if $R^{-1}(U) \subset R^{-1}(V)$, i.e. $U \diamond R \subset V \diamond R$, then $U \subset V$ by assumption.

Item 3, Corepresentably Fully Faithful Morphisms in **Rel**: This follows from Items 1 and 2. □

Question 8.5.13.1.2. Item 2 of Definition 8.5.13.1.1 gives a characterisation of the corepresentably full morphisms in **Rel**.

Are there other nice characterisations of these?

This question also appears as [`MO 467527`].

8.5.14 Co/Limits

Proposition 8.5.14.1.1. This will be properly written later on.

Proof. Omitted. □

8.5.15 Internal Left Kan Extensions

Proposition 8.5.15.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Internal Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

2. *Characterisation of Relations Admitting Internal Left Kan Extensions Along Them.* The following conditions are equivalent:

   (a) The left Kan extension

   $$\mathrm{Lan}_R\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 8.2.2.1.1) for some function f.

Proof. Item 1, Non-Existence of All Internal Left Kan Extensions in **Rel**: By Item 2, it suffices to take a relation that doesn't have a left adjoint.
Item 2, Characterisation of Relations Admitting Left Kan Extensions Along Them: This proof is mostly due to Tim Campion, via [MO 460693].

- We may view precomposition

  $$- \diamond R\colon \mathrm{Rel}(B, C) \to \mathrm{Rel}(A, C)$$

  with $R\colon A \nrightarrow B$ as a cocontinuous functor from $\mathcal{P}(B \times C)$ to $\mathcal{P}(A \times C)$ (via Item 5 of Definition 8.1.1.1.1).

- By the adjoint functor theorem (**??**), this map has a left adjoint iff it preserves limits.

- If $C = \emptyset$, this holds trivially.

- Otherwise, C admits pt as a retract, and we reduce to the case $C = \mathrm{pt}$ via **??**.

- For the case $C = \mathrm{pt}$, a relation $T\colon B \relbar\joinrel\mapstochar\joinrel\rightarrow \mathrm{pt}$ is the same as a subset of B, and $- \diamond R$ becomes the weak inverse image functor R^{-1} of Section 8.7.3.
- Now, again by the adjoint functor theorem, R^{-1} preserves limits exactly when it has a left adjoint.
- Finally R^{-1} has a left adjoint precisely when $R = \mathrm{Gr}(f)$ for f a function (Item 8 of Definition 8.7.3.1.3).

This finishes the proof. □

Example 8.5.15.1.2. Given a function $f\colon A \to B$, the left Kan extension

$$\mathrm{Lan}_f\colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

along f exists by Item 2 of Definition 8.5.15.1.1. Explicitly, given a relation $R\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow X$, the left Kan extension

$$\mathrm{Lan}_f(R)\colon B \relbar\joinrel\mapstochar\joinrel\rightarrow X, \qquad \begin{tikzcd} & B \arrow[d, dashed, "\shortmid" marking, "\mathrm{Lan}_f(R)"] \\ A \arrow[ur, "f", "\shortmid" marking] \arrow[r, "R"', "\shortmid" marking] \arrow[ur, phantom, shift right=3, "\Nearrow"] & X \end{tikzcd}$$

may be described as follows:

1. We declare $b \sim_{\mathrm{Lan}_f(R)} x$ iff there exists some $a \in R$ such that $b = f(a)$ and $a \sim_R x$.
2. We have[19]

$$\left[\mathrm{Lan}_f(R)\right](b) = \bigcup_{a \in f^{-1}(b)} R(a)$$

for each $b \in B$.

Remark 8.5.15.1.3. Following Definition 8.5.15.1.2, given a relation $R\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ and a relation $F\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow X$, we could perhaps try to define an "honorary" left Kan extension

$$\mathrm{Lan}'_R(F)\colon B \relbar\joinrel\mapstochar\joinrel\rightarrow X$$

[19] Cf. Item 3 of Definition 8.5.17.1.2.

by

$$\left[\mathrm{Lan}'_F(F)\right](b) \overset{\mathrm{def}}{=} \bigcup_{a \in R^{-1}(b)} F(a)$$

for each $b \in B$.

The failure of $\mathrm{Lan}'_R(F)$ to be a Kan extension can then be seen as follows. Let $G\colon B \nrightarrow X$ be a relation. If $\mathrm{Lan}'_R(F)$ were a left Kan extension, then the following conditions ***would be*** equivalent:

1. For each $b \in B$, we have $\bigcup_{a \in R^{-1}(b)} F(a) \subset G(b)$.
2. For each $a \in A$, we have $F(a) \subset \bigcup_{b \in R(a)} G(b)$.

The issue is two-fold:

- *Totality.* If R isn't total, then the implication Item 1 $\Longrightarrow$ Item 2 fails.
- *Functionality.* If R isn't functional, then the implication Item 2 $\Longrightarrow$ Item 1 fails.

Question 8.5.15.1.4. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan extension[20]

$$\mathrm{Lan}_S(R)\colon B \nrightarrow X$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

8.5.16 Internal Left Kan Lifts

Proposition 8.5.16.1.1. Let $R\colon A \nrightarrow B$ be a relation.

1. *Non-Existence of All Internal Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.
2. *Characterisation of Relations Admitting Internal Left Kan Lifts Along Them.* The following conditions are equivalent:

   (a) The left Kan lift

   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X, B) \to \mathbf{Rel}(X, A)$$

   along R exists.

[20] Specifically for R and S, not Lan_S the functor.

(b) The relation R admits a right adjoint in Rel.

(c) The relation R is of the form f^{-1} (as in Definition 8.2.3.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Internal Left Kan Lifts in* Rel: By Item 2, it suffices to take a relation that doesn't have a right adjoint.

Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them: This proof is dual to that of Item 2 of Definition 8.5.15.1.1, and is therefore omitted. □

Example 8.5.16.1.2. Given a function $f\colon A \to B$, the left Kan lift

$$\mathrm{Lift}_{f^\dagger}\colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

along $f^\dagger$ exists by Item 2 of Definition 8.5.16.1.1. Explicitly, given a relation $R\colon X \nrightarrow A$, the left Kan lift

$$\mathrm{Lift}_{f^\dagger}(R)\colon X \nrightarrow B, \qquad \begin{tikzcd} & B \arrow[d, "f^\dagger", "\mid" marking] \\ X \arrow[ur, dashed, "\mathrm{Lift}_{f^\dagger}(R)", "\mid" marking] \arrow[r, "R"', "\mid" marking] \arrow[r, phantom, shift left=3, "\Uparrow"] & A. \end{tikzcd}$$

is given by

$$\begin{aligned} \left[\mathrm{Lift}_f(R)\right](x) &= [\mathrm{Gr}(f) \diamond R](a) \\ &= \bigcup_{a \in R(x)} f(a) \end{aligned}$$

for each $x \in X$.

Question 8.5.16.1.3. Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan lift[21]

$$\mathrm{Lift}_S(R)\colon X \nrightarrow A$$

exists in terms of properties of R and S?

This question also appears as [`MO 461592`].

[21]Specifically for R and S, not Lift_S the functor.

8.5.17 Internal Right Kan Extensions

Let A, B, and X be sets and let $R \colon A \nrightarrow B$ and $F \colon A \nrightarrow X$ be relations.

Motivation 8.5.17.1.1. We want to understand internal right Kan extensions in **Rel**, which look like this:

$$\begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(F)", "\shortmid"' marking] \arrow[dl, phantom, "\Swarrow"] \\ A \arrow[ur, "R", "\shortmid" marking] \arrow[r, "F"', "\shortmid" marking] & X. \end{tikzcd}$$

Note in particular here that $F \colon A \nrightarrow X$ is a relation from A to X. These will form a functor

$$\mathrm{Ran}_R \colon \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X)$$

that is right adjoint to the precomposition by R functor

$$R^* \colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X).$$

Proposition 8.5.17.1.2. The internal right Kan extension of F along R is the relation $\mathrm{Ran}_R(F)$ described as follows:

1. Viewing relations from B to X as subsets of $B \times X$, we have

$$\mathrm{Ran}_R(F) = \left\{ (b, x) \in B \times X \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, if } a \sim_R b, \\ \text{then we have } a \sim_F x \end{array} \right\}.$$

2. Viewing relations as functions $B \times X \to \{\mathsf{true}, \mathsf{false}\}$, we have

$$\begin{aligned} (\mathrm{Ran}_R(F))^{-1}_{-2} &= \int_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, F^{-1}_a\right) \\ &= \bigwedge_{a \in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, F^{-1}_a\right), \end{aligned}$$

   where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 3.2.2.1.3.

3. Viewing relations as functions $B \to \mathcal{P}(X)$, we have

$$\mathrm{Ran}_R(F) = \mathrm{Ran}_{\chi'_A}(F) \circ R^{-1}, \qquad \begin{tikzcd} & A \arrow[r, "F"] \arrow[d, hook, "\chi_A"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{P}(X), \\ B \arrow[r, "R^{-1}"'] & \mathcal{P}(A)^{\mathsf{op}} \arrow[ur, "\mathrm{Ran}_{\chi_A}(F)"'] & \end{tikzcd}$$

where $\mathrm{Ran}_{\chi'_B}(F)$ is computed by the formula

$$\begin{aligned}
\left[\mathrm{Ran}_{\chi'_A}(F)\right](V) &\cong \int_{a\in A} \chi_{\mathcal{P}(A)^{\mathrm{op}}}(V,\chi_a) \pitchfork F(a)\\
&\cong \int_{a\in A} \chi_{\mathcal{P}(A)}(\chi_a, V) \pitchfork F(a)\\
&\cong \int_{a\in A} \chi_V(a) \pitchfork F(a)\\
&\cong \bigcap_{a\in A} \chi_V(a) \pitchfork F(a)\\
&\cong \bigcap_{a\in V} F(a)
\end{aligned}$$

for each $V \in \mathcal{P}(B)$, so we have

$$[\mathrm{Ran}_R(F)](b) = \bigcap_{a\in R^{-1}(b)} F(a)$$

for each $b \in B$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(F\diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((F\diamond R)^x_a, T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} F^x_b \times R^b_a\right), T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^x_b \times R^b_a, T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right)\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right)\\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^x_b, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^x_a\right)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(F, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, T^{-1}_a\right)\right)
\end{aligned}$$

naturally in each $F \in \mathbf{Rel}(B,X)$ and each $T \in \mathbf{Rel}(A,X)$, showing that

$$\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, T^{-1}_a\right)$$

is right adjoint to the precomposition functor $-\diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Definition 8.1.1.1.5.
2. Definition 8.1.3.1.1.
3. **??** of **??**.
4. Definition 3.2.2.1.5.
5. **??** of **??**.
6. **??** of **??**.
7. Item 1 of Definition 8.1.1.1.5.

This finishes the proof. □

Example 8.5.17.1.3. Here are some examples of internal right Kan extensions of relations.

1. *Orthogonal Complements.* Let $A = B = X = \mathcal{V}$ be an inner product space, and let $R = F = \perp$ be the orthogonality relation, so that we have

$$\begin{aligned} R(v) &= v^{\perp} \\ F(u) &= u^{\perp}, \end{aligned}$$

for each $u, v \in \mathcal{V}$, where

$$v^{\perp} \stackrel{\text{def}}{=} \{u \in V \mid v \perp u\}$$

is the orthogonal complement of v. The right Kan extension $\mathrm{Ran}_R(F)$ is then given by

$$\begin{aligned} [\mathrm{Ran}_R(F)](v) &= \bigcap_{u \in R^{-1}(v)} F(u) \\ &= \bigcap_{\substack{u \in V \\ u \perp v}} u^{\perp} \\ &= \left(v^{\perp}\right)^{\perp}, \end{aligned}$$

the double orthogonal complement. In particular:

- If $\mathcal{V}$ is finite-dimensional, then $[\mathrm{Ran}_R(F)](v) = \mathrm{Span}(v)$.
- If $\mathcal{V}$ is a Hilbert space, then $[\mathrm{Ran}_R(F)](v) = \overline{\mathrm{Span}(v)}$.

2. *Galois Connections and Closure Operators.* Let:

   - $B = X = (P, \preceq_P)$ and $A = (Q, \preceq_Q)$ be posets;
   - (f, g) be a Galois connection (adjunction) between P and Q;
   - $R, F \colon Q \rightrightarrows P$ be the relations defined by

   $$\begin{aligned} R(q) &\overset{\text{def}}{=} \{p \in P \mid q \preceq_Q f(p)\}, \\ F(q) &\overset{\text{def}}{=} \{p \in P \mid p \preceq_P g(q)\} \end{aligned}$$

   for each $q \in Q$.

   We have

   $$\begin{aligned} [\mathrm{Ran}_R(F)](p) &= \bigcap_{q \in R^{-1}(p)} F(q) \\ &= \bigcap_{\substack{q \in Q \\ q \preceq_Q f(p)}} \{p \in P \mid p \preceq_P g(q)\} \\ &= \{p \in P \mid p \preceq_P g(f(q))\} \\ &= \downarrow g(f(p)), \end{aligned}$$

   the down set of $g(f(p))$. In other words, $\mathrm{Ran}_R(F)$ is the closure operator on P associated with the Galois connection (f, g).

Proposition 8.5.17.1.4. Let A, B, C and X be sets and let $R \colon A \nrightarrow B$, $S \colon B \nrightarrow C$, and $F \colon A \nrightarrow X$ be relations.

1. *Functoriality.* The assignments $R, F, (R, F) \mapsto \mathrm{Ran}_R(F)$ define functors

   $$\begin{aligned} \mathrm{Ran}_{(-)}(F) \colon & \quad \mathbf{Rel}(A, B)^{\mathsf{op}} \to \mathbf{Rel}(B, X), \\ \mathrm{Ran}_R \colon & \quad \mathbf{Rel}(A, X) \to \mathbf{Rel}(B, X), \\ \mathrm{Ran}_{(-_1)}(-_2) \colon & \quad \mathbf{Rel}(A, X) \times \mathbf{Rel}(A, B)^{\mathsf{op}} \to \mathbf{Rel}(B, X). \end{aligned}$$

   In other words, given relations

   $$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \qquad A \underset{F_2}{\overset{F_1}{\rightrightarrows}} X,$$

   if $R_1 \subset R_2$ and $F_1 \subset F_2$, then $\mathrm{Ran}_{R_2}(F_1) \subset \mathrm{Ran}_{R_1}(F_2)$.

2. *Interaction With Composition.* We have

$$\mathrm{Ran}_{S\diamond R}(F) = \mathrm{Ran}_S(\mathrm{Ran}_R(F))$$

and an equality

$$\begin{tikzcd}
 & & C \arrow[dd, dashed, "\mathrm{Ran}_S(\mathrm{Ran}_R(F))"] \arrow[dl, phantom, "\Downarrow"] \\
 & B \arrow[ur, "S"] \arrow[dr, dashed, "\mathrm{Ran}_R(F)"'] \arrow[d, phantom, "\Downarrow"] & \\
A \arrow[ur, "R"] \arrow[rr, "F"'] & {} & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & C \arrow[dd, dashed, "\mathrm{Ran}_{S\diamond R}(F)"] \\
 & B \arrow[ur, "S"] \arrow[d, phantom, "\Downarrow"] & \\
A \arrow[ur, "R"] \arrow[rr, "F"'] & {} & X
\end{tikzcd}$$

of pasting diagrams in **Rel**.

3. *Interaction With Converses.* We have

$$\mathrm{Ran}_R(F)^{\dagger} = \mathrm{Rift}_{R^{\dagger}}\left(F^{\dagger}\right).$$

4. *Interaction With Weak Inverse Images.* We have

$$[\mathrm{Ran}_R(F)]^{-1}(x) = \left\{b \in B \,\middle|\, R^{-1}(b) \subset F^{-1}(x)\right\}$$

for each $x \in X$.

Proof. *Item 1*, *Functoriality*: We have

$$\begin{aligned}
\left[\mathrm{Ran}_{R_2}(F_1)\right](b) &= \bigcap_{a\in R_2^{-1}(b)} F_1(a) \\
&\subset \bigcap_{a\in R_1^{-1}(b)} F_1(a) \\
&\subset \bigcap_{a\in R_1^{-1}(b)} F_2(a) \\
&= \left[\mathrm{Ran}_{R_1}(F_2)\right](b)
\end{aligned}$$

for each $b \in B$, so we therefore have $\mathrm{Ran}_{R_2}(F_1) \subset \mathrm{Ran}_{R_1}(F_2)$.
Item 2, *Interaction With Composition*: This holds in a general bicategory with the necessary right Kan extensions, being therefore a special case of **??**.

Item 3, Interaction With Converses: We have

$$\begin{aligned}\left[\mathrm{Rift}_{R^\dagger}\left(F^\dagger\right)\right](x) &= \left\{b \in B \,\middle|\, R^\dagger(b) \subset F^\dagger(x)\right\}\\ &= \left\{b \in B \,\middle|\, R^{-1}(b) \subset F^{-1}(x)\right\}\\ &= \mathrm{Ran}_R(F)^{-1}(x)\\ &= \mathrm{Ran}_R(F)^\dagger(x)\end{aligned}$$

where we have used Definition 8.5.18.1.2 and Item 4.
Item 4, Interaction With Weak Inverse Images: We proceed in a few steps.

- We have $b \in [\mathrm{Ran}_R(F)]^{-1}(x)$ iff, for each $a \in R^{-1}(b)$, we have $b \in F(a)$.
- This holds iff, for each $a \in R^{-1}(b)$, we have $a \in F^{-1}(b)$.
- This holds iff $R^{-1}(b) \subset F^{-1}(b)$.

This finishes the proof. □

8.5.18 Internal Right Kan Lifts

Let A, B, and X be sets and let $R\colon A \nrightarrow B$ and $F\colon X \nrightarrow B$ be relations.

Motivation 8.5.18.1.1. We want to understand internal right Kan lifts in **Rel**, which look like this:

$$\begin{tikzcd} & A \arrow[d, "R", "\mapstochar"' marking] \\ X \arrow[ur, dashed, "\mathrm{Rift}_R(F)", "\times" marking] \arrow[r, "F"', "\mapstochar" marking] & B. \arrow[ul, phantom, "\Swarrow"] \end{tikzcd}$$

Note in particular here that $F\colon B \nrightarrow X$ is a relation from B to X. These will form a functor

$$\mathrm{Rift}_R\colon \mathbf{Rel}(X, B) \to \mathbf{Rel}(X, A)$$

that is right adjoint to the postcomposition by R functor

$$R_*\colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B).$$

Proposition 8.5.18.1.2. The internal right Kan lift of F along R is the relation $\mathrm{Rift}_R(F)$ described as follows:

1. Viewing relations from X to A as subsets of $X \times A$, we have
$$\mathrm{Rift}_R(F) = \left\{ (x,a) \in X \times A \,\middle|\, \begin{array}{l} \text{for each } b \in B, \text{if } a \sim_R b, \\ \text{then we have } x \sim_F b \end{array} \right\}.$$

2. Viewing relations as functions $X \times A \to \{\mathsf{true}, \mathsf{false}\}$, we have
$$\begin{aligned} (\mathrm{Rift}_R(F))^{-1}_{-2} &= \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, F^b_{-2}\right) \\ &= \bigwedge_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, F^b_{-2}\right), \end{aligned}$$
where the meet $\bigwedge$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 3.2.2.1.3.

3. Viewing relations as functions $X \to \mathcal{P}(A)$, we have
$$[\mathrm{Rift}_R(F)](x) = \{a \in A \mid R(a) \subset F(x)\}$$
for each $a \in A$.

Proof. We have
$$\begin{aligned} \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond F, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((R\diamond F)^b_x, T^b_x\right) \\ &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^b_a \times F^a_x\right), T^b_x\right) \\ &\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a \times F^a_x, T^b_x\right) \\ &\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right) \\ &\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right) \\ &\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(F^a_x, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right) \\ &\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(F, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, T^b_{-2}\right)\right) \end{aligned}$$
naturally in each $F \in \mathbf{Rel}(X,A)$ and each $T \in \mathbf{Rel}(X,B)$, showing that
$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, F^b_{-2}\right)$$
is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Definition 8.1.1.1.5.
2. Definition 8.1.3.1.1.
3. ?? of ??.
4. Definition 3.2.2.1.5.
5. ?? of ??.
6. ?? of ??.
7. Item 1 of Definition 8.1.1.1.5.

This finishes the proof. □

Example 8.5.18.1.3. Here are some examples of internal right Kan lifts of relations.

1. *Pullbacks.* Let $p\colon A \to B$ and $f\colon X \to B$ be functions. We have

$$\begin{aligned}\left[\mathrm{Rift}_{\mathrm{Gr}(p)}(\mathrm{Gr}(f))\right](x) &= \{a \in A \mid [\mathrm{Gr}(p)](a) \subset [\mathrm{Gr}(f)](x)\}\\ &= \{a \in A \mid p(a) = f(x)\}.\end{aligned}$$

Thus, as a subset of $X \times A$, the right Kan lift $\mathrm{Rift}_{\mathrm{Gr}(p)}(\mathrm{Gr}(f))$ corresponds precisely to the pullback $X \times_B A$ of X and A along p and f of Section 4.1.4.

Proposition 8.5.18.1.4. Let A, B, C and X be sets and let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $F\colon X \nrightarrow B$ be relations.

1. *Functoriality.* The assignments $R, F, (R,F) \mapsto \mathrm{Rift}_R(F)$ define functors

$$\begin{aligned}\mathrm{Rift}_{(-)}(F)\colon &\qquad \mathbf{Rel}(A,B)^{\mathrm{op}} \to \mathbf{Rel}(B,X),\\ \mathrm{Rift}_R\colon &\qquad \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X),\\ \mathrm{Rift}_{(-_1)}(-_2)\colon &\ \mathbf{Rel}(A,X) \times \mathbf{Rel}(A,B)^{\mathrm{op}} \to \mathbf{Rel}(B,X).\end{aligned}$$

In other words, given relations

$$A \underset{R_2}{\overset{R_1}{\rightrightarrows}} B \qquad A \underset{F_2}{\overset{F_1}{\rightrightarrows}} X,$$

if $R_1 \subset R_2$ and $F_1 \subset F_2$, then $\mathrm{Rift}_{R_2}(F_1) \subset \mathrm{Rift}_{R_1}(F_2)$.

2. *Interaction With Composition.* We have

$$\mathrm{Rift}_{S\diamond R}(F) = \mathrm{Rift}_R(\mathrm{Ran}_S(F))$$

and an equality

$$\begin{tikzcd} & A \arrow[d, "R"] \\ & B \arrow[d, "S"] \arrow[u, phantom, "\Leftarrow"] \\ X \arrow[r, "F"'] \arrow[uur, dashed, "\mathrm{Rift}_R(\mathrm{Rift}_S(F))"] \arrow[ur, dashed, "\mathrm{Rift}_S(F)"'] & C \arrow[u, phantom, "\Leftarrow"] \end{tikzcd} = \begin{tikzcd} & A \arrow[d, "R"] \\ & B \arrow[d, "S"] \\ X \arrow[r, "F"'] \arrow[uur, dashed, "\mathrm{Rift}_{S\diamond R}(F)"] & C \arrow[uu, phantom, "\Leftarrow"] \end{tikzcd}$$

of pasting diagrams in **Rel**.

3. *Interaction With Converses.* We have

$$\mathrm{Rift}_R(F)^\dagger = \mathrm{Ran}_{R^\dagger}\left(F^\dagger\right).$$

4. *Interaction With Weak Inverse Images.* We have

$$\mathrm{Rift}_R(F)^\dagger = \mathrm{Ran}_{\chi'_B}\left(F^\dagger\right) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "F^\dagger"] \arrow[d, "\chi_B"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{P}(X), \\ A \arrow[r, "R"'] & \mathcal{P}(B)^{\mathsf{op}} \arrow[ur, bend right, "\mathrm{Ran}_{\chi_A}\left(F^{-1}\right)"'] & \end{tikzcd}$$

where $\mathrm{Ran}_{\chi_A}\left(F^\dagger\right)$ is computed by the formula

$$\begin{aligned} \left[\mathrm{Ran}_{\chi_A}\left(F^\dagger\right)\right](U) &\cong \int_{a\in A} \chi_{\mathcal{P}(B)^{\mathsf{op}}}(U, \chi_a) \pitchfork F^\dagger(a) \\ &\cong \int_{a\in A} \chi_{\mathcal{P}(B)}(\chi_a, U) \pitchfork F^{-1}(a) \\ &\cong \int_{a\in A} \chi_U(a) \pitchfork F(a) \\ &\cong \bigcap_{a\in A} \chi_U(a) \pitchfork F(a) \\ &\cong \bigcap_{a\in U} F(a) \end{aligned}$$

for each $U \in \mathcal{P}(A)$, so we have

$$[\mathrm{Rift}_R(F)]^{-1}(a) = \bigcap_{b \in R(a)} F^{-1}(b)$$

for each $a \in A$.

Proof. *Item 1, Functoriality*: We have

$$\begin{aligned}\left[\mathrm{Rift}_{R_2}(F_1)\right](x) &= \{a \in A \mid R_2(a) \subset F_1(x)\}\\ &\subset \{a \in A \mid R_1(a) \subset F_1(x)\}\\ &\subset \{a \in A \mid R_1(a) \subset F_2(x)\}\\ &= \mathrm{Rift}_{R_1}(F_2)\end{aligned}$$

for each $x \in X$, so we therefore have $\mathrm{Rift}_{R_2}(F_1) \subset \mathrm{Rift}_{R_1}(F_2)$.
Item 2, Interaction With Composition: This holds in a general bicategory with the necessary right Kan lifts, being therefore a special case of **??**.
Item 3, Interaction With Converses: This follows from Item 3 of Definition 8.5.17.1.4 by duality.
Item 4, Interaction With Weak Inverse Images: We proceed in a few steps.

- We have $x \in \mathrm{Rift}_R(F)^{\dagger}(a)$ iff $a \in \mathrm{Rift}_R(F)(x)$.
- This holds iff $R(a) \subset F(x)$.
- This holds iff, for each $b \in R(a)$, we have $b \in F(x)$.
- This holds iff, for each $b \in R(a)$, we have $x \in F^{-1}(b)$.
- This holds iff $x \in \bigcap_{b \in R(a)} F^{-1}(b)$.

This finishes the proof. □

8.5.19 Closedness

Proposition 8.5.19.1.1. The 2-category $\mathbf{Rel}$ is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_! \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\mathbf{Rel}(S \diamond R, T) \cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),$$
$$\mathbf{Rel}(R \diamond U, V) \cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

Proof. This follows from **????**. □

8.5.20 Rel as a Category of Free Algebras

Proposition 8.5.20.1.1. We have an isomorphism of categories

$$\mathsf{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}_!}(\mathsf{Sets}),$$

where $\mathcal{P}_!$ is the powerset monad of **??**.

Proof. Omitted. □

8.6 Properties of the 2-Category of Relations With Apartness Composition

8.6.1 Self-Duality

Proposition 8.6.1.1.1. The 2-/category of relations with apartness-composition-is self-dual:

1. *Self-Duality I.* We have an isomorphism

   $$\left(\mathsf{Rel}^{\square}\right)^{\mathsf{op}} \cong \mathsf{Rel}^{\square}$$

   of categories.

2. *Self-Duality II.* We have a 2-isomorphism

   $$\left(\mathbf{Rel}^{\square}\right)^{\mathsf{op}} \cong \mathbf{Rel}^{\square}$$

   of 2-categories.

Proof. *Item 1, Self-Duality I*: We claim that the functor

$$(-)^{\dagger} \colon \left(\mathsf{Rel}^{\square}\right)^{\mathsf{op}} \to \mathsf{Rel}^{\square}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories. Note that this is indeed a functor by Items 4 and 7 of Definition 8.1.5.1.3.

By Item 1 of Definition 11.6.8.1.3, it suffices to show that $(-)^{\dagger}$ is bijective on objects (which follows by definition) and fully faithful. Indeed, the map

$$(-)^{\dagger}\colon \mathrm{Rel}(A,B) \to \mathrm{Rel}(B,A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Definition 8.1.5.1.3, showing $(-)^{\dagger}$ to be fully faithful.

Item 2, Self-Duality II: We claim that the 2-functor

$$(-)^{\dagger}\colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Definition 8.1.5.1.3, is an isomorphism of categories.

By **??**, it suffices to show that $(-)^{\dagger}$ is:

- Bijective on objects, which follows by definition.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Definition 8.1.5.1.3.

Thus $(-)^{\dagger}$ is indeed a 2-isomorphism of categories. □

8.6.2 Isomorphisms and Equivalences

Let $R\colon A \nrightarrow B$ be a relation from A to B, and recall that $R^{\mathsf{c}} \stackrel{\text{def}}{=} B \times A \setminus R$.

Lemma 8.6.2.1.1. The conditions below are row-wise equivalent:

Condition	Inclusion
R^{c} is functional	$\nabla_B \subset R \mathbin{\square} R^{\dagger}$
R^{c} is total	$R \mathbin{\square} R^{\dagger} \subset \nabla_A$
R^{c} is injective	$\nabla_A \subset R^{\dagger} \mathbin{\square} R$
R^{c} is surjective	$R^{\dagger} \mathbin{\square} R \subset \nabla_B$

Proof. This follows from Definition 8.5.2.1.1 and Item 4 of Definition 8.1.4.1.3. For instance:

- Suppose we have $R \mathbin{\square} R^{\dagger} \subset \nabla_B$.
- Taking complements, we obtain $\nabla_B^{\mathsf{c}} \subset \left(R \mathbin{\square} R^{\dagger}\right)^{\mathsf{c}}$.
- Applying Item 4 of Definition 8.1.4.1.3, this becomes $\Delta_B \subset R^{\mathsf{c}} \diamond \left(R^{\dagger}\right)^{\mathsf{c}}$.
- Then, by Definition 8.5.2.1.1, this is equivalent to R^{c} being total.

The proof of the other equivalences is similar, and thus omitted. □

Remark 8.6.2.1.2. The statements in Definition 8.6.2.1.1 unwind to the following:

Inclusion	Quantifier	Condition
$\nabla_B \subset R \mathbin{\square} R^{\dagger}$	For each $b_1, b_2 \in B$	If $b_1 \neq b_2$, then, for each $a \in A$, we have $a \sim_R b_1$ or $a \sim_R b_2$.
$R \mathbin{\square} R^{\dagger} \subset \nabla_B$	For each $b_1, b_2 \in B$	If, for each $a \in A$, $a \sim_R b_1$ or $a \sim_R b_2$, then $b_1 \neq b_2$.
$\nabla_A \subset R^{\dagger} \mathbin{\square} R$	For each $a_1, a_2 \in A$	If $a_1 \neq a_2$, then, for each $b \in B$, we have $a_1 \sim_R b$ or $a_2 \sim_R b$.
$R^{\dagger} \mathbin{\square} R \subset \nabla_A$	For each $a_1, a_2 \in A$	If, for each $b \in B$, $a_1 \sim_R b$ or $a_2 \sim_R b$, then $a_1 \neq a_2$.

Equivalently:

Inclusion	Quantifier	If	Then
$\nabla_B \subset R \mathbin{\square} R^{\dagger}$	For each $b_1, b_2 \in B$	$b_1 \neq b_2$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$
$R \mathbin{\square} R^{\dagger} \subset \nabla_B$	For each $b_1, b_2 \in B$	$R^{-1}(b_1) \cup R^{-1}(b_2) = A$	$b_1 \neq b_2$
$\nabla_A \subset R^{\dagger} \mathbin{\square} R$	For each $a_1, a_2 \in A$	$a_1 \neq a_2$	$R(a_1) \cup R(a_2) = B$
$R^{\dagger} \mathbin{\square} R \subset \nabla_A$	For each $a_1, a_2 \in A$	$R(a_1) \cup R(a_2) = B$	$a_1 \neq a_2$

Proposition 8.6.2.1.3. The following conditions are equivalent:

1. The relation $R\colon A \nrightarrow B$ is an equivalence in $\mathbf{Rel}^{\square}$, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A together with isomorphisms

$$R^{-1} \mathbin{\square} R \cong \nabla_A,$$
$$R \mathbin{\square} R^{-1} \cong \nabla_B.$$

2. The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e.:

   (★) There exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

$$R^{-1} \mathbin{\square} R = \nabla_A,$$
$$R \mathbin{\square} R^{-1} = \nabla_B.$$

3. There exists a bijection $f\colon B \xrightarrow{\sim} A$ with $R^{\mathsf{c}} = f^{-1}$.

Proof. This follows from Definition 8.5.2.1.2 and Item 4 of Definition 8.1.4.1.3. □

8.6.3 Internal Adjunctions

Let A and B be sets.

Proposition 8.6.3.1.1. We have a natural bijection

$$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel}^{\square} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } B \text{ to } A\end{matrix}\right\},$$

with every adjunction in $\mathbf{Rel}^{\square}$ being of the form $\left(f^{-1}\right)^{\mathsf{c}} \dashv \mathrm{Gr}(f)^{\mathsf{c}}$ for some function $f\colon B \to A$.

Proof. This follows from Definition 8.5.3.1.1 and Item 4 of Definition 8.1.4.1.3. □

8.6.4 Internal Monads

Let X be a set.

Proposition 8.6.4.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Monads in} \\ \mathbf{Rel}^{\square} \text{ on } X\end{matrix}\right\} \cong \{\text{Subsets of } X\}.$$

Proof. This follows from Definition 8.6.4.1.1 and Item 4 of Definition 8.1.4.1.3. □

8.6.5 Internal Comonads

Let X be a set.

Proposition 8.6.5.1.1. We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \mathsf{Rel}^{\square}\text{ on }X\end{matrix}\right\} \cong \{\text{Strict total orders on } X\}.$$

Proof. A comonad in $\mathsf{Rel}^{\square}$ on X consists of a relation $R\colon X \nrightarrow X$ together with maps

$$\begin{aligned}\Delta_R &\colon R \subset R \mathbin{\square} R,\\ \epsilon_R &\colon R \subset \nabla_X\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid" marking] \arrow[dr, equal, "\lambda^{\mathsf{Rel}^{\square},-1}_R"'] & R \mathbin{\square} R \arrow[d, "\epsilon_R \mathbin{\square} \mathrm{id}_R", "\shortmid" marking] \\
& \nabla_X \mathbin{\square} R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \mathbin{\square} R \arrow[dr, "\mathrm{id}_R \mathbin{\square} \Delta_R", "\shortmid" marking] & \\
R \arrow[ur, "\Delta_R", "\shortmid" marking] \arrow[d, "\Delta_R"', "\shortmid" marking] & & R \mathbin{\square} (R \mathbin{\square} R) \arrow[d, equal, "\alpha^{\mathsf{Rel}^{\square},-1}_{R,R,R}"] \\
R \mathbin{\square} R \arrow[rr, "\Delta_R \mathbin{\square} \mathrm{id}_R"', "\shortmid" marking] & & (R \mathbin{\square} R) \mathbin{\square} R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "\shortmid" marking] \arrow[dr, equal, "\rho^{\mathsf{Rel}^{\square},-1}_R"'] & R \mathbin{\square} R \arrow[d, "\mathrm{id}_R \mathbin{\square} \epsilon_R", "\shortmid" marking] \\
& R \mathbin{\square} \nabla_X
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic (**??** of **??**), and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $x, z \in X$, if $x \sim_R z$, then, for each $y \in X$, we have $x \sim_R y$ or $y \sim_R z$.

2. For each $x, y \in X$, if $x \sim_R y$, then $x \neq y$.

Replacing $\sim_R$ with $<_R$ and taking the contrapositive of each condition, we obtain:

1. For each $x, z \in X$, if there exists some $y \in X$ such that $x <_R y$ and $y <_R z$, then $x <_R z$.

2. For each $x \in X$, we have $x \not<_R x$.

These are exactly the requirements for R to be a strict linear order (**??**). Conversely, any strict linear order $<_R$ gives rise to a pair of maps $\Delta_{<_R}$ and $\epsilon_{<_R}$, forming a comonad on X. □

Example 8.6.5.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. The codensity monad $\mathrm{Ran}_R(R)\colon B \nrightarrow B$ is given by

$$[\mathrm{Ran}_R(R)](b) = \bigcap_{a \in R^{-1}(b)} R(b) \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(R)", "\shortmid" marking] \\ A \arrow[ur, "R", "\shortmid" marking] \arrow[r, "R"', "\shortmid" marking] \arrow[ur, phantom, shift left=3, "\Downarrow"] & B \end{tikzcd}$$

   for each $b \in B$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Ran}_R(R)}\colon B \times B \to \{\mathsf{t}, \mathsf{f}\}$$

   on B obtained by declaring $b \preceq_{\mathrm{Ran}_R(R)} b'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a \sim_R b'$.

   (b) We have $R^{-1}(b) \subset R^{-1}(b')$.

2. The dual codensity monad $\mathrm{Rift}_R(R)\colon A \nrightarrow A$ is given by

$$[\mathrm{Rift}_R(R)](a) = \{a' \in A \mid R(a') \subset R(a)\} \qquad \begin{tikzcd} & A \arrow[d, "R", "\shortmid" marking] \\ A \arrow[ur, dashed, "\mathrm{Rift}_R(R)", "\shortmid" marking] \arrow[r, "R"', "\shortmid" marking] \arrow[ur, phantom, shift right=3, "\Downarrow"] & B \end{tikzcd}$$

   for each $a \in A$. Thus, it corresponds to the preorder

$$\preceq_{\mathrm{Rift}_R(R)}\colon A \times A \to \{\mathsf{t}, \mathsf{f}\}$$

   on A obtained by declaring $a \preceq_{\mathrm{Rift}_R(R)} a'$ iff the following equivalent conditions are satisfied:

   (a) For each $a \in A$, if $a \sim_R b$, then $a' \sim_R b$.

   (b) We have $R(a') \subset R(a)$.

8.6.6 Modules Over Internal Monads

8.6.7 Comodules Over Internal Comonads

8.6.8 Eilenberg–Moore and Kleisli Objects

8.6.9 Monomorphisms

8.6.10 2-Categorical Monomorphisms

8.6.11 Epimorphisms

8.6.12 2-Categorical Epimorphisms

8.6.13 Co/Limits

This will be expanded later on.

8.6.14 Internal Left Kan Extensions

8.6.15 Internal Left Kan Lifts

8.6.16 Internal Right Kan Extensions

8.6.17 Internal Right Kan Lifts

8.6.18 Coclosedness

8.7 The Adjoint Pairs $R_! \dashv R_{-1}$ and $R^{-1} \dashv R_*$

8.7.1 Direct Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.1.1.1. The **direct image function associated to** R is the function[22]

$$R_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[23]

$$R_!(U) \overset{\text{def}}{=} \bigcup_{a \in U} R(a)$$

[22] *Further Notation:* Also written simply $R\colon \mathcal{P}(X) \to \mathcal{P}(Y)$.

[23] *Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

$$= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\}$$

for each $U \in \mathcal{P}(X)$.

Warning 8.7.1.1.2. Notation for direct images between powersets is tricky; see Definition 4.6.1.1.3. Here we'll try to align our notation for relations with that for functions.

Remark 8.7.1.1.3. Identifying subsets of X with relations from pt to X via Item 3 of Definition 4.4.1.1.4, we see that the direct image function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(\mathrm{pt}, X)} \to \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(\mathrm{pt}, Y)}$$

defined by

$$R_!(U) \overset{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(X)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} X \overset{R}{\nrightarrow} Y.$$

Proposition 8.7.1.1.4. Let $R \colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_! \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$$

   where

   - *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[R_!](U) \overset{\text{def}}{=} R_!(U).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(X)$:
     - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1}) \colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

   witnessed by:

(a) Units and counits of the form

$$\begin{aligned}\mathrm{id}_{\mathcal{P}(X)} &\hookrightarrow R_{-1} \circ R_!,\\ R_! \circ R_{-1} &\hookrightarrow \mathrm{id}_{\mathcal{P}(Y)},\end{aligned}$$

having components of the form

$$\begin{aligned}U &\subset R_{-1}(R_!(U)),\\ R_!(R_{-1}(V)) &\subset V\end{aligned}$$

indexed by $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$

(b) A bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$. In particular:

($\star$) The following conditions are equivalent:
- We have $R_!(U) \subset V$.
- We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_!\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$\begin{aligned}R_!(U) \cup R_!(V) &= R_!(U \cup V),\\ R_!(\emptyset) &= \emptyset,\end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_!\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$\begin{aligned}R_!(U \cap V) &\subset R_!(U) \cap R_!(V),\\ R_!(X) &\subset Y,\end{aligned}$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$R_{*|U,V}^{\otimes} : R_!(U) \cup R_!(V) \xrightarrow{=} R_!(U \cup V),$$
$$R_{*|\mathbb{1}}^{\otimes} : \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(X)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$R_{*|U,V}^{\otimes} : R_!(U \cap V) \subset R_!(U) \cap R_!(V),$$
$$R_{*|\mathbb{1}}^{\otimes} : R_!(X) \subset Y,$$

natural in $U, V \in \mathcal{P}(X)$.

7. *Relation to Codirect Images.* We have

$$R_!(U) = Y \setminus R_*(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Codirect Images: The proof proceeds in the same way as in the case of functions (**??** of Definition 4.6.1.1.5): applying Item 7 of Definition 8.7.4.1.3 to $A \setminus U$, we have

$$R_*(X \setminus U) = Y \setminus R_!(X \setminus (X \setminus U))$$

$$= Y \setminus R_!(U).$$

Taking complements, we then obtain

$$\begin{aligned} R_!(U) &= Y \setminus (Y \setminus R_!(U)), \\ &= Y \setminus R_*(X \setminus U), \end{aligned}$$

which finishes the proof. □

Proposition 8.7.1.1.5. Let $R \colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathrm{Rel}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathrm{Rel}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have[24]

$$(\chi_X)_! = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R \colon X \nrightarrow Y$ and $S \colon Y \nrightarrow C$, we have[25]

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "R_!"] \arrow[dr, "(S \diamond R)_!"'] & \mathcal{P}(Y) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

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[24] That is, the postcomposition function

$$(\chi_X)_! \colon \mathrm{Rel}(\mathrm{pt}, X) \to \mathrm{Rel}(\mathrm{pt}, X)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt}, X)}$.

[25] That is, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, X) \arrow[r, "R_!"] \arrow[dr, "(S \diamond R)_!"'] & \mathrm{Rel}(\mathrm{pt}, Y) \arrow[d, "S_!"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_X)_!(U) &\stackrel{\text{def}}{=} \bigcup_{a\in U} \chi_X(a)\\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} \{a\}\\ &= U\\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{P}(X)}(U)\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_! = \mathrm{id}_{\mathcal{P}(X)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\stackrel{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a)\\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} S(R(a))\\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} S_!(R(a))\\ &= S_!\left(\bigcup_{a\in U} R(a)\right)\\ &\stackrel{\text{def}}{=} S_!(R_!(U))\\ &\stackrel{\text{def}}{=} [S_! \circ R_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(X)$, where we used Item 3 of Definition 8.7.1.1.4. Thus $(S \diamond R)_! = S_! \circ R_!$. □

8.7.2 Strong Inverse Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.2.1.1. The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[26]

$$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in X \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(Y)$.

[26] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 8.7.2.1.2. Identifying subsets of Y with relations from pt to Y via Item 3 of Definition 4.4.1.1.4, we see that the inverse image function associated to R is equivalently the function

$$R_{-1} \colon \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(\mathrm{pt}, Y)} \to \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(\mathrm{pt}, X)}$$

defined by

$$R_{-1}(V) \overset{\mathrm{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & X \arrow[d, "R", "\mid" marking] \\ \mathrm{pt} \arrow[ur, dashed, "\mathrm{Rift}_R(V)", "\mid" marking] \arrow[r, "V"', "\mid" marking] & Y, \arrow[ul, phantom, "\Swarrow"] \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\mathrm{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b \in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, V^b_{-_2}\right), \end{aligned}$$

where we have used **??**.

Proof. We have

$$\begin{aligned} \mathrm{Rift}_R(V) &\cong \int_{b \in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, V^b_{-_2}\right) \\ &= \left\{ a \in X \;\middle|\; \int_{b \in Y} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, V^b_\star\right) = \mathsf{true} \right\} \\ &= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{for each } b \in Y, \text{ at least one of the} \\ \text{following conditions hold:} \\ \quad 1. \ \text{We have } R^b_a = \mathsf{false} \\ \quad 2. \ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathsf{true} \\ \qquad \text{(b) We have } V^b_\star = \mathsf{true} \end{array} \right\} \end{aligned}$$

$$= \left\{ a \in X \;\middle|\; \begin{array}{l} \text{for each } b \in Y\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(a) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } b \in V \end{array} \right\}$$

$$\begin{aligned} &= \{a \in X \mid \text{for each } b \in R(a)\text{, we have } b \in V\} \\ &= \{a \in X \mid R(a) \subset V\} \\ &\overset{\text{def}}{=} R_{-1}(V). \end{aligned}$$

This finishes the proof. □

Proposition 8.7.2.1.3. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(Y)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(Y)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_! \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "R_!"] \arrow[r, phantom, "\perp"] & \mathcal{P}(Y), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}(R_!(U), V) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

(★) The following conditions are equivalent:
  - We have $R_!(U) \subset V$.

   - We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i\in I} U_i\right),$$

   natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

   natural in $U, V \in \mathcal{P}(Y)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_{-1}(U_i),$$

   natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{gathered} R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(Y) = Y, \end{gathered}$$

   natural in $U, V \in \mathcal{P}(Y)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

   being equipped with inclusions

$$\begin{gathered} R^{\otimes}_{-1|U,V} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ R^{\otimes}_{-1|\mathbb{1}} \colon \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

   natural in $U, V \in \mathcal{P}(Y)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon R_{-1}(X) \xrightarrow{=} Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = X \setminus R^{-1}(Y \setminus V)$$

for each $V \in \mathcal{P}(Y)$.

8. *Interaction With Weak Inverse Images II.* Let $R \colon X \nrightarrow Y$ be a relation from X to Y.

 (a) If R is a total relation, then we have an inclusion of sets

 $$R_{-1}(V) \subset R^{-1}(V)$$

 natural in $V \in \mathcal{P}(Y)$.

 (b) If R is total and functional, then the above inclusion is in fact an equality.

 (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Weak Inverse Images I: We claim we have an equality

$$R_{-1}(Y \setminus V) = X \setminus R^{-1}(V).$$

Indeed, we have

$$R_{-1}(Y \setminus V) = \{a \in X \mid R(a) \subset Y \setminus V\},$$
$$X \setminus R^{-1}(V) = \{a \in X \mid R(a) \cap V = \emptyset\}.$$

Taking $V = Y \setminus V$ then implies the original statement.
Item 8, Interaction With Weak Inverse Images II: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Definition 8.2.2.1.2. □

Proposition 8.7.2.1.4. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1}\colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_X)_{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{array}{ccc} \mathcal{P}(C) & \xrightarrow{S_{-1}} & \mathcal{P}(Y) \\ & {\scriptstyle (S\diamond R)_{-1}} \searrow & \downarrow {\scriptstyle R_{-1}} \\ & & \mathcal{P}(X). \end{array}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_X)_{-1}(U) &\overset{\text{def}}{=} \{a \in X \mid \chi_X(a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in X \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_{-1} = \mathrm{id}_{\mathcal{P}(X)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_{-1}(U) &\overset{\text{def}}{=} \{a \in X \mid [S \diamond R](a) \subset U\} \\ &\overset{\text{def}}{=} \{a \in X \mid S(R(a)) \subset U\} \\ &\overset{\text{def}}{=} \{a \in X \mid S_!(R(a)) \subset U\} \\ &= \{a \in X \mid R(a) \subset S_{-1}(U)\} \\ &\overset{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\overset{\text{def}}{=} [R_{-1} \circ S_{-1}](U) \end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Definition 8.7.2.1.3, which implies that the conditions

- We have $S_!(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

8.7.3 Weak Inverse Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.3.1.1. The **weak inverse image function associated to** R[27] is the function

$$R^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$$

defined by[28]

$$R^{-1}(V) \overset{\text{def}}{=} \{a \in X \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(Y)$.

Remark 8.7.3.1.2. Identifying subsets of Y with relations from Y to pt via Item 3 of Definition 4.4.1.1.4, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(Y,\mathrm{pt})} \to \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(X,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \overset{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(X)$, where $R \diamond V$ is the composition

$$X \overset{R}{\nrightarrow} Y \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} V \diamond R \\ &\overset{\text{def}}{=} \int^{b \in Y} V^{-1}_{b} \times R^{b}_{-2}. \end{aligned}$$

[27] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[28] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Proof. We have

$$\begin{aligned}
V \diamond R &\stackrel{\text{def}}{=} \int^{b\in Y} V^{-1}_{b} \times R^{b}_{-_2} \\
&= \left\{ a \in X \,\middle|\, \int^{b\in Y} V^{\star}_{b} \times R^{b}_{a} = \mathsf{true} \right\} \\
&= \left\{ a \in X \,\middle|\, \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } V^{\star}_{b} = \mathsf{true} \\ \quad 2.\ \text{We have } R^{b}_{a} = \mathsf{true} \end{array} \right\} \\
&= \left\{ a \in X \,\middle|\, \begin{array}{l} \text{there exists } b \in Y \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \in V \\ \quad 2.\ \text{We have } b \in R(a) \end{array} \right\} \\
&= \{a \in X \mid \text{there exists } b \in V \text{ such that } b \in R(a)\} \\
&= \{a \in X \mid R(a) \cap V \neq \emptyset\} \\
&\stackrel{\text{def}}{=} R^{-1}(V)
\end{aligned}$$

This finishes the proof. □

Proposition 8.7.3.1.3. Let $R\colon X \rightarrow\!\!\!\!\!+ Y$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}\colon (\mathcal{P}(Y), \subset) \to (\mathcal{P}(X), \subset)$$

   where

   - *Action on Objects.* For each $V \in \mathcal{P}(Y)$, we have

$$\left[R^{-1}\right](V) \stackrel{\text{def}}{=} R^{-1}(V).$$

   - *Action on Morphisms.* For each $U, V \in \mathcal{P}(Y)$:
     – If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_*\right)\colon\quad \begin{tikzcd} \mathcal{P}(Y) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "R_*"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}\big(R^{-1}(U), V\big) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

(★) The following conditions are equivalent:
- We have $R^{-1}(U) \subset V$.
- We have $U \subset R_*(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(Y)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(X) &\subset Y, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R^{-1,\otimes}_{U,V} &\colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V), \\ R^{-1,\otimes}_{\mathbb{1}} &\colon \emptyset \xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(Y)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V} \colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}} \colon R^{-1}(X) \subset Y,$$

natural in $U, V \in \mathcal{P}(Y)$.

7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = X \setminus R_{-1}(Y \setminus V)$$

for each $V \in \mathcal{P}(Y)$.

8. *Interaction With Strong Inverse Images II.* Let $R \colon X \nrightarrow Y$ be a relation from X to Y.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(Y)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Strong Inverse Images I: This follows from Item 7 of Definition 8.7.2.1.3.
Item 8, Interaction With Strong Inverse Images II: This was proved in Item 8 of Definition 8.7.2.1.3. □

Proposition 8.7.3.1.4. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function
$$(-)^{-1}\colon \mathrm{Rel}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$
2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function
$$(-)^{-1}\colon \mathrm{Rel}(X, Y) \to \mathsf{Pos}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$
3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have[29]
$$(\chi_X)^{-1} = \mathrm{id}_{\mathcal{P}(X)}.$$
4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have[30]
$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(Y) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(X). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Item 5 of Definition 11.1.4.1.2.
Item 4, Interaction With Composition: This follows from Item 2 of Definition 11.1.4.1.2. □

[29] That is, the postcomposition
$$(\chi_X)^{-1}\colon \mathrm{Rel}(\mathrm{pt}, X) \to \mathrm{Rel}(\mathrm{pt}, X)$$
is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},X)}$.

[30] That is, we have
$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt}, Y) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt}, X). \end{tikzcd}$$

8.7.4 Codirect Images

Let X and Y be sets and let $R\colon X \nrightarrow Y$ be a relation.

Definition 8.7.4.1.1. The **codirect image function associated to** R is the function

$$R_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

defined by[31,32]

$$\begin{aligned} R_*(U) &\overset{\text{def}}{=} \left\{ b \in Y \;\middle|\; \begin{array}{l} \text{for each } a \in X, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in Y \;\middle|\; R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(X)$.

Remark 8.7.4.1.2. Identifying subsets of Y with relations from pt to Y via Item 3 of Definition 4.4.1.1.4, we see that the codirect image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(X)}_{\cong \mathrm{Rel}(X,\mathrm{pt})} \to \underbrace{\mathcal{P}(Y)}_{\cong \mathrm{Rel}(Y,\mathrm{pt})}$$

defined by

$$R_*(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & Y \arrow[d, dashed, "\mathrm{Ran}_R(U)", "\shortmid"{marking}] \\ X \arrow[ur, "R", "\shortmid"{marking}] \arrow[r, "U"', "\shortmid"{marking}] & \mathrm{pt}, \arrow[from=1-2, to=2-1, phantom, "\Swarrow"] \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R_a^{-2}, U_a^{-1}\right), \end{aligned}$$

where we have used **??**.

[31] *Further Terminology:* The set $R_*(U)$ is called the **codirect image of** U **by** R.

[32] We also have

$$R_*(U) = Y \setminus R_!(X \setminus U);$$

Proof. We have

$$
\begin{aligned}
\mathrm{Ran}_R(V) &\cong \int_{a\in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{-2}_a, U^{-1}_a\right) \\
&= \left\{ b \in Y \,\middle|\, \int_{a\in X} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^{b}_a, U^{\star}_a\right) = \mathsf{true} \right\} \\
&= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{for each } a \in X \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R^b_a = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^b_a = \mathsf{true} \\ \qquad \text{(b) We have } U^{\star}_a = \mathsf{true} \end{array} \right\} \\
&= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{for each } a \in X \text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(X) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\
&= \left\{ b \in Y \,\middle|\, \begin{array}{l} \text{for each } a \in X \text{, if we have} \\ b \in R(a) \text{, then } a \in U \end{array} \right\} \\
&= \left\{ b \in Y \,\middle|\, R^{-1}(b) \subset U \right\} \\
&\stackrel{\text{def}}{=} R^{-1}(U).
\end{aligned}
$$

This finishes the proof. □

Proposition 8.7.4.1.3. Let $R \colon X \nrightarrow Y$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_* \colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(Y), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(X)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_*\right): \quad \begin{tikzcd} \mathcal{P}(Y) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "R_*"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(X)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(X)}(U, R_*(V)),$$

natural in $U \in \mathcal{P}(X)$ and $V \in \mathcal{P}(Y)$, i.e. such that:

(★) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_*(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_*(U_i) \subset R_*\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have inclusions

$$R_*(U) \cup R_*(V) \subset R_*(U \cup V),$$
$$\emptyset \subset R_*(\emptyset),$$

natural in $U, V \in \mathcal{P}(X)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(X)^{\times I}$. In particular, we have equalities

$$R_*(U \cap V) = R_*(U) \cap R_*(V),$$
$$R_*(X) = Y,$$

natural in $U, V \in \mathcal{P}(X)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The codirect image function of Item 1 has a symmetric lax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cup, \emptyset) \to (\mathcal{P}(Y), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} : R_*(U) \cup R_*(V) \subset R_*(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} : \emptyset \subset R_*(\emptyset),$$

natural in $U, V \in \mathcal{P}(X)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(X), \cap, X) \to (\mathcal{P}(Y), \cap, Y),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} : R_*(U \cap V) \xrightarrow{=} R_*(U) \cap R_*(V),$$
$$R_{!|\mathbb{1}}^{\otimes} : R_*(X) \xrightarrow{=} Y,$$

natural in $U, V \in \mathcal{P}(X)$.

7. *Relation to Direct Images.* We have

$$R_*(U) = Y \setminus R_!(X \setminus U)$$

for each $U \in \mathcal{P}(X)$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: This follows from Item 7 of Definition 8.7.1.1.4. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Item 16 of Definition 4.6.3.1.7).

We claim that $R_*(U) = Y \setminus R_!(X \setminus U)$:

- *The First Implication.* We claim that

  $$R_*(U) \subset Y \setminus R_!(X \setminus U).$$

  Let $b \in R_*(U)$. We need to show that $b \notin R_!(X \setminus U)$, i.e. that there is no $a \in X \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_*(U)$).

  Thus $b \in Y \setminus R_!(X \setminus U)$.

- *The Second Implication.* We claim that

  $$Y \setminus R_!(X \setminus U) \subset R_*(U).$$

  Let $b \in Y \setminus R_!(X \setminus U)$. We need to show that $b \in R_*(U)$, i.e. that $R^{-1}(b) \subset U$.

  Since $b \notin R_!(X \setminus U)$, there exists no $a \in X \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

  Thus $b \in R_*(U)$.

This finishes the proof. □

Proposition 8.7.4.1.4. Let $R\colon X \nrightarrow Y$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

   $$(-)_*\colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}(\mathcal{P}(X), \mathcal{P}(Y)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

   $$(-)_*\colon \mathsf{Sets}(X, Y) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(X), \subset), (\mathcal{P}(Y), \subset)).$$

3. *Interaction With Identities.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

   $$(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{P}(X)}.$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon X \nrightarrow Y$ and $S\colon Y \nrightarrow C$, we have

   $$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(X) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(Y) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_X)_*(U) &\stackrel{\text{def}}{=} \left\{a \in X \,\middle|\, \chi_X^{-1}(a) \subset U\right\} \\ &\stackrel{\text{def}}{=} \{a \in X \mid \{a\} \subset U\} \\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(X)$. Thus $(\chi_X)_* = \mathrm{id}_{\mathcal{P}(X)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_*(U) &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\} \\ &\stackrel{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\big(R^{-1}(c)\big) \subset U\right\} \\ &= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_*(U)\right\} \\ &\stackrel{\text{def}}{=} R_*(S_*(U)) \\ &\stackrel{\text{def}}{=} [R_* \circ S_*](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Definition 8.7.4.1.3, which implies that the conditions

- We have $S^{-1}\big(R^{-1}(c)\big) \subset U$.
- We have $R^{-1}(c) \subset S_*(U)$.

are equivalent. Thus $(S \diamond R)_* = S_* \circ R_*$. □

8.7.5 Functoriality of Powersets

Proposition 8.7.5.1.1. The assignment $X \mapsto \mathcal{P}(X)$ defines functors[33]

$$\begin{aligned}\mathcal{P}_! \colon \mathsf{Rel} &\longrightarrow \mathsf{Sets}, \\ \mathcal{P}_{-1} \colon \mathsf{Rel}^{\mathsf{op}} &\longrightarrow \mathsf{Sets}, \\ \mathcal{P}^{-1} \colon \mathsf{Rel}^{\mathsf{op}} &\longrightarrow \mathsf{Sets}, \\ \mathcal{P}_* \colon \mathsf{Rel} &\longrightarrow \mathsf{Sets}\end{aligned}$$

where

see Item 7 of Definition 8.7.4.1.3.

[33]The functor $\mathcal{P}_! \colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 2 of Definition 8.2.2.1.2.

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned}
\mathcal{P}_!(X) &\overset{\text{def}}{=} \mathcal{P}(X),\\
\mathcal{P}_{-1}(X) &\overset{\text{def}}{=} \mathcal{P}(X),\\
\mathcal{P}^{-1}(X) &\overset{\text{def}}{=} \mathcal{P}(X),\\
\mathcal{P}_*(X) &\overset{\text{def}}{=} \mathcal{P}(X).
\end{aligned}$$

- *Action on Morphisms.* For each morphism $R \colon X \nrightarrow Y$ of Rel, the images

$$\begin{aligned}
\mathcal{P}_!(R) &\colon \mathcal{P}(X) \to \mathcal{P}(Y),\\
\mathcal{P}_{-1}(R) &\colon \mathcal{P}(Y) \to \mathcal{P}(X),\\
\mathcal{P}^{-1}(R) &\colon \mathcal{P}(Y) \to \mathcal{P}(X),\\
\mathcal{P}_*(R) &\colon \mathcal{P}(X) \to \mathcal{P}(Y)
\end{aligned}$$

  of R by $\mathcal{P}_!$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_*$ are defined by

$$\begin{aligned}
\mathcal{P}_!(R) &\overset{\text{def}}{=} R_!,\\
\mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1},\\
\mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1},\\
\mathcal{P}_*(R) &\overset{\text{def}}{=} R_*,
\end{aligned}$$

  as in Definitions 8.7.1.1.1, 8.7.2.1.1, 8.7.3.1.1 and 8.7.4.1.1.

Proof. This follows from Items 3 and 4 of Definition 8.7.1.1.5, Items 3 and 4 of Definition 8.7.2.1.4, Items 3 and 4 of Definition 8.7.3.1.4, and Items 3 and 4 of Definition 8.7.4.1.4. □

8.7.6 Functoriality of Powersets: Relations on Powersets

Let X and Y be sets and let $R \colon X \nrightarrow Y$ be a relation.

Definition 8.7.6.1.1. The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R) \colon \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$$

defined by[34]

$$\mathcal{P}(R)^{V}_{U} \overset{\text{def}}{=} \mathbf{Rel}\big(\chi_{\mathrm{pt}}, V \diamond R \diamond U\big)$$

for each $U \in \mathcal{P}(X)$ and each $V \in \mathcal{P}(Y)$.

Remark 8.7.6.1.2. In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$.
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathsf{true}$, i.e. we have

$$\int^{a \in X} \int^{b \in Y} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathsf{true}.$$

- There exists some $a \in X$ and some $b \in Y$ such that:
  - We have $U^{a}_{\star} = \mathsf{true}$.
  - We have $R^{b}_{a} = \mathsf{true}$.
  - We have $V^{\star}_{b} = \mathsf{true}$.
- There exists some $a \in X$ and some $b \in Y$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

Proposition 8.7.6.1.3. The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathrm{Rel} \longrightarrow \mathrm{Rel}.$$

Proof. Omitted. □

[34]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"', "+"{marking}] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}", "+"{marking}] & X \arrow[r, "R"', "+"{marking}] & Y \arrow[r, "V"', "+"{marking}] & \mathrm{pt}. \end{tikzcd}$$

8.8 The Left Skew Monoidal Structure on $\mathrm{Rel}(A, B)$

8.8.1 The Left Skew Monoidal Product

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

Definition 8.8.1.1.1. The **left J-skew monoidal product of** $\mathrm{Rel}(A, B)$ is the functor

$$\lhd_J : \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\text{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "J", "\shortmid"{marking}] & B. \\ A \arrow[r, "R"', "\shortmid"{marking}] \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] & B \arrow[ul, phantom, "\Leftarrow"] \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')} : \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \lhd_J R, S' \lhd_J R')$$

  of $\lhd_J$ at $((R, S), (R', S'))$ is defined by[35]

$$\beta \lhd_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S", "\shortmid"{marking}] \arrow[r, bend right, "S'"', "\shortmid"{marking}] \arrow[r, phantom, "\beta\Downarrow"] \arrow[d, "J", "\shortmid"{marking}] & B \\ A \arrow[ur, dashed, bend left=50, "\mathrm{Rift}_J(R)", "\shortmid"{marking}] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"', "\shortmid"{marking}] \arrow[r, bend left=0, "R", "\shortmid"{marking}] \arrow[r, bend right, "R'"', "\shortmid"{marking}] \arrow[r, phantom, shift right=3, "\alpha\Downarrow"] & B \arrow[ul, phantom, "\Leftarrow"] \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[35]Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset$

8.8.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

Definition 8.8.2.1.1. The **left J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} : \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\triangleleft_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

8.8.3 The Left Skew Associators

Let A and B be sets and let $J : A \nrightarrow B$ be a relation.

Definition 8.8.3.1.1. The **left J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J} : \triangleleft_J \circ (\triangleleft_J \times \mathsf{id}) \Longrightarrow \triangleleft_J \circ (\mathsf{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Rel}(A,B) \times (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \arrow[drr, "\mathsf{id}\times\triangleleft_J"] & & \\
(\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \times \mathbf{Rel}(A,B) \arrow[urr, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[dr, "\triangleleft_J\times\mathsf{id}"'] & & \arrow[d, phantom, "\Uparrow \alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[dl, "\triangleleft_J"] \\
 & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[rr, "\triangleleft_J"'] & {} & \mathbf{Rel}(A,B), &
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} : \underbrace{(T \triangleleft_J S) \triangleleft_J R}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T \triangleleft_J (S \triangleleft_J R)}_{\overset{\text{def}}{=} T \diamond \mathrm{Rift}_J(S \diamond \mathrm{Rift}_J(R))}$$

$S' \triangleleft_J R'$.

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} \stackrel{\text{def}}{=} \mathrm{id}_T \diamond \gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R) \hookrightarrow \mathrm{Rift}_J\big(S \diamond \mathrm{Rift}_J(R)\big)$$

is the inclusion adjunct to the inclusion

$$\epsilon_S \star \mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J \diamond \mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R)}_{\stackrel{\text{def}}{=} J_!(\mathrm{Rift}_J(S) \diamond \mathrm{Rift}_J(R))} \hookrightarrow S \diamond \mathrm{Rift}_J(R)$$

under the adjunction $J_! \dashv \mathrm{Rift}_J$, where $\epsilon\colon J \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.4.1.1. The **left J-skew left unitor of Rel**(A, B) is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}\right) \Longrightarrow \lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathbf{Rel}(A,B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
 & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R\colon \underbrace{J \triangleleft_J R}_{\stackrel{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \stackrel{\text{def}}{=} \epsilon_R,$$

where $\epsilon\colon J_! \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.5 The Left Skew Right Unitors

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

Definition 8.8.5.1.1. The **left J-skew right unitor of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\right)$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B) \times \mathsf{pt} \arrow[r, "\mathsf{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Uparrow \rho^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}_R\colon R \hookrightarrow \underbrace{R \triangleleft_J J}_{\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} R \diamond \chi_A \\
&\overset{\mathrm{id}_R \diamond \eta_{\chi_A}}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J_!(\chi_A)) \\
&\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J \diamond \chi_A) \\
&\overset{\sim}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J) \\
&\overset{\text{def}}{=} R \triangleleft_J J,
\end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(A,A)} \Longrightarrow \mathrm{Rift}_J \circ J_!$ is the unit of the adjunction $J_! \dashv \mathrm{Rift}_J$.

8.8.6 The Left Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Proposition 8.8.6.1.1. The category $\mathbf{Rel}(A, B)$ admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of **??** of **??**.
- *The Left Skew Monoidal Product.* The left J-skew monoidal product
$$\triangleleft_J \colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$
of Definition 8.8.1.1.1.
- *The Left Skew Monoidal Unit.* The functor
$$\mathbb{1}^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \mathrm{pt} \to \mathbf{Rel}(A, B)$$
of Definition 8.8.2.1.1.
- *The Left Skew Associators.* The natural transformation
$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \triangleleft_J \circ (\triangleleft_J \times \mathrm{id}) \Longrightarrow \triangleleft_J \circ (\mathrm{id} \times \triangleleft_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$
of Definition 8.8.3.1.1.
- *The Left Skew Left Unitors.* The natural transformation
$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \triangleleft_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathrm{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$
of Definition 8.8.4.1.1.
- *The Left Skew Right Unitors.* The natural transformation
$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\right)$$
of Definition 8.8.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic (**??** of **??**), and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a left skew monoidal category. □

8.9 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

8.9.1 The Right Skew Monoidal Product

Definition 8.9.1.1.1. The **right J-skew monoidal product of** $\mathrm{Rel}(A,B)$ is the functor

$$\triangleright_J \colon \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \longrightarrow \mathbf{Rel}(A,B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, we have

$$S \triangleright_J R \overset{\mathrm{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mid"{marking}] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\mid"{marking}] & B. \\ & A \arrow[u, "J", "\mid"{marking}] \arrow[ur, "S"'] \arrow[u, phantom, shift left=4ex, "\Uparrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')} \colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

  of $\triangleright_J$ at $((S,R),(S',R'))$ is defined by[36]

$$\beta \triangleright_J \alpha \overset{\mathrm{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'", "\mid"{marking}] \arrow[r, bend right, "R"', "\mid"{marking}] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[r, dashed, bend left, "\mathrm{Ran}_J(S')", "\mid"{marking}] \arrow[r, dashed, bend right, "\mathrm{Ran}_J(S)"', "\mid"{marking}] \arrow[r, phantom, "\mathrm{Ran}_J(\beta)\Uparrow"] & B \\ & A \arrow[u, "J", "\mid"{marking}] \arrow[ur, "S'"] \arrow[ur, bend right, "S"'] \arrow[ur, phantom, shift right=2ex, "\beta\Uparrow"] & \end{tikzcd}$$

  for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S,S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,R')$.

8.9.2 The Right Skew Monoidal Unit

Definition 8.9.2.1.1. The **right J-skew monoidal unit of** $\mathrm{Rel}(A,B)$ is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A,B)$$

[36]Since $\mathbf{Rel}(A,B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \triangleright_J R \subset$

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

8.9.3 The Right Skew Associators

Definition 8.9.3.1.1. The **right J-skew associator of** $\mathrm{Rel}(A, B)$ is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ \left(\mathsf{id} \times \triangleright_J\right) \Longrightarrow \triangleright_J \circ \left(\triangleright_J \times \mathsf{id}\right) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
 & (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \times \mathbf{Rel}(A,B) \arrow[dr, "\triangleright_J \times \mathsf{id}"] & \\
\mathbf{Rel}(A,B) \times (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\mathsf{id} \times \triangleright_J"'] \arrow[rr, phantom, "\Uparrow \alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[rr, "\triangleright_J"'] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R} \colon \underbrace{T \triangleright_J \left(S \triangleright_J R\right)}_{\overset{\text{def}}{=}\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\diamond R} \hookrightarrow \underbrace{\left(T \triangleright_J S\right) \triangleright_J R}_{\overset{\text{def}}{=}\mathrm{Ran}_J\left(\mathrm{Ran}_J(T)\diamond S\right)\diamond R}$$

at (T, S, R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J\left(\mathrm{Ran}_J(T) \diamond S\right)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=}J^*\left(\mathrm{Ran}_J(T)\diamond\mathrm{Ran}_J(S)\right)} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.4 The Right Skew Left Unitors

Definition 8.9.4.1.1. The **right J-skew left unitor of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id}\right),$$

as in the diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathbf{Rel}(A, B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[r, phantom, shift right=6ex, "\Nearrow \lambda^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleright_J"] \\
& \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_R \colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\stackrel{\mathrm{def}}{=} \mathrm{Ran}_J(J)\diamond R}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} \chi_B \diamond R \\
&\xRightarrow{\eta_{\chi_B} \diamond \mathrm{id}_R} \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \\
&\stackrel{\mathrm{def}}{=} \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R \\
&\overset{\sim}{\Longrightarrow} \mathrm{Ran}_J(J) \diamond R \\
&\stackrel{\mathrm{def}}{=} R \triangleright_J J,
\end{aligned}$$

where $\eta \colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.5 The Right Skew Right Unitors

Definition 8.9.5.1.1. The **right J-skew right unitor of** $\mathbf{Rel}(A, B)$ is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B) \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\rhd_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[dr, phantom, "\overset{\rho^{\mathbf{Rel}(A,B),\rhd_J}}{\Longleftarrow}"] & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B), \arrow[d, "\rhd_J"] \\
& \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\rhd_J}_{S} \colon \underbrace{S \rhd_J J}_{\overset{\mathrm{def}}{=} \mathrm{Ran}_J(S) \diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\rhd_J}_{S} \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon \colon J^* \circ \mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

8.9.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Proposition 8.9.6.1.1. The category $\mathbf{Rel}(A, B)$ admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of **??** of **??**.

- *The Right Skew Monoidal Product.* The right J-skew monoidal product

$$\lhd_J \colon \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,B)$$

  of Definition 8.9.1.1.1.

- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J} \colon \mathrm{pt} \to \mathbf{Rel}(A,B)$$

  of Definition 8.9.2.1.1.

$S' \rhd_J R'$.

- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ (\mathrm{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathrm{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 8.9.3.1.1.

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathrm{id}\right)$$

  of Definition 8.9.4.1.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ \left(\mathrm{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 8.9.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic (**??** of **??**), and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a right skew monoidal category. □

Appendices

8.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories

12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Chapter 9

Constructions With Relations

This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category Rel (Definition 15.2.1.1.8).

2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, converse relations, composition of relations, and collages (Section 9.2).

This chapter is under revision. TODO:

1. Rename range to image

2. Co/limits in Rel.

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9.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

9.2 More Constructions With Relations

9.2.1 The Domain and Range of a Relation

Let A and B be sets.

Definition 9.2.1.1.1. Let $R\colon A \nrightarrow B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\mathrm{def}}{=} \left\{ a \in A \;\middle|\; \begin{aligned}&\text{there exists some } b \in B\\ &\text{such that } a \sim_R b\end{aligned} \right\}.$$

[1]Following **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned}\chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}\left(R^b_a\right) && (a \in A)\\ &\cong \bigvee_{b \in B} R^b_a,\\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}\left(R^b_a\right) && (b \in B)\\ &\cong \bigvee_{a \in A} R^b_a,\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$ of Definition 3.2.2.1.3.

[2]Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned}\mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y))\\ &\cong \bigcup_{y \in Y} R(y),\\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x))\end{aligned}$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

9.2.2 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 9.2.2.1.1. The **union of** R **and** S[3] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[4]

$$R \cup S \overset{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cup S](a) \overset{\text{def}}{=} R(a) \cup S(a)$$

  for each $a \in A$.

Proposition 9.2.2.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Converses.* We have

$$(R \cup S)^{\dagger} = R^{\dagger} \cup S^{\dagger}.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. *Item 1*, *Interaction With Converses*: Clear.
Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:

$$\cong \bigcup_{x \in X} R(x),$$

[3] *Further Terminology:* Also called the **binary union of** R **and** S, for emphasis.
[4] This is the same as the union of R and S as subsets of $A \times B$.

- There exists some $b \in B$ such that:
  - $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

  or

  - $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

- The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
  - There exists some $b \in B$ such that:
    - $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

    and

    - $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

9.2.3 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

Definition 9.2.3.1.1. The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[5]
$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{there exists some } i \in I\\ \text{such that } a \sim_{R_i} b\end{array} \right\}.$$
- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define
$$\left[\bigcup_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$
for each $a \in A$.

Proposition 9.2.3.1.2. Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Converses.* We have
$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof. Item 1, *Interaction With Converses*: Clear. □

[5]This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

9.2.4 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 9.2.4.1.1. The **intersection of** R **and** S[6] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[7]

$$R \cap S \stackrel{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

  for each $a \in A$.

Proposition 9.2.4.1.2. Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Converses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. *Item 1*, *Interaction With Converses*: Clear.
Item 2, *Interaction With Composition*: Unwinding the definitions, we see that:

- The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
  - There exists some $b \in B$ such that:
    * $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

    and
    * $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
- The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:

[6] *Further Terminology:* Also called the **binary intersection of** R **and** S, for emphasis.
[7] This is the same as the intersection of R and S as subsets of $A \times B$.

- There exists some $b \in B$ such that:
  - $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

  and
  - $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. □

9.2.5 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 9.2.5.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[8]

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i \right](a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

  for each $a \in A$.

Proposition 9.2.5.1.2. Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

1. *Interaction With Converses.* We have

$$\left(\bigcap_{i \in I} R_i \right)^{\dagger} = \bigcap_{i \in I} R_i^{\dagger}.$$

Proof. *Item 1, Interaction With Converses*: Clear. □

[8]This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.

9.2.6 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

Definition 9.2.6.1.1. The **product of** R **and** S[9] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[10]

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

  in Sets, i.e. by

$$[R \times S](a, x) \overset{\text{def}}{=} R(a) \times S(x)$$

  for each $(a, x) \in A \times X$.

Proposition 9.2.6.1.2. Let A, B, X, and Y be sets.

1. *Interaction With Converses.* Let

$$R\colon A \nrightarrow A,$$
$$S\colon X \nrightarrow X$$

   We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$R_1\colon A \nrightarrow B,$$
$$S_1\colon B \nrightarrow C,$$
$$R_2\colon X \nrightarrow Y,$$
$$S_2\colon Y \nrightarrow Z$$

   be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

[9] *Further Terminology:* Also called the **binary product of** R **and** S, for emphasis.

[10] That is, $R \times S$ is the relation given by declaring $(a, x) \sim_{R \times S} (b, y)$ iff $a \sim_R b$ and

Proof. *Item 1, Interaction With Converses*: Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(R \times S)^\dagger} (b, y)$ iff:
  - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
    - We have $b \sim_R a$;
    - We have $y \sim_S x$;
- We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
  - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
    - We have $b \sim_R a$;
    - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

- We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
  - We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
    - There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
    - There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
- We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
  - There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
    - We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
    - We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

9.2.7 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i \colon A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 9.2.7.1.1. The **product of the family** $\{R_i\}_{i\in I}$ is the relation $\prod_{i\in I} R_i$ from $\prod_{i\in I} A_i$ to $\prod_{i\in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i\in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i\in I} R_i \stackrel{\text{def}}{=} \left\{ (a_i, b_i)_{i\in I} \in \prod_{i\in I} (A_i \times B_i) \,\middle|\, \begin{array}{l}\text{for each } i \in I,\\ \text{we have } a_i \sim_{R_i} b_i\end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i\in I} R_i\right]((a_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} R_i(a_i)$$

  for each $(a_i)_{i\in I} \in \prod_{i\in I} R_i$.

9.2.8 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 9.2.8.1.1. The **collage of** R[11] is the poset $\mathbf{Coll}(R) \stackrel{\text{def}}{=} (\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)})$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by

$$\mathrm{Coll}(R) \stackrel{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order

$$\preceq_{\mathbf{Coll}(R)} \colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$

  on $\mathrm{Coll}(R)$ defined by

$$\preceq(a, b) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

Notation 9.2.8.1.2. We write $\mathsf{Pos}_{/\Delta^1}(A, B)$ for the category defined as the

$x \sim_S y$.

[11] *Further Terminology:* Also called the **cograph of** R.

pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{ev}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{ev}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{ev}_{[0]}"] \arrow[dr, "\mathrm{ev}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

Remark 9.2.8.1.3. In detail, $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category where:

- *Objects.* An object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of
  - A poset X;
  - A morphism $\phi_X : X \to \Delta^1$;

  such that we have

$$\phi_X^{-1}(0) = A,$$
$$\phi_X^{-1}(1) = B.$$

- *Morphisms.* A morphism of $\mathsf{Pos}_{/\Delta^1}(A, B)$ from (X, ϕ_X) to (Y, ϕ_Y) is a morphism of posets $f : X \to Y$ making the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\phi_X"'] & & Y \arrow[dl, "\phi_Y"] \\
 & \Delta^1 &
\end{tikzcd}$$

  commute.

Proposition 9.2.8.1.4. Let A and B be sets and let $R : A \nrightarrow B$ be a relation from A to B.

1. *Functoriality.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor

$$\mathbf{Coll}\colon \mathbf{Rel}(A,B) \longrightarrow \mathsf{Pos}_{/\Delta^1}(A,B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, we have

$$[\mathbf{Coll}](R) \overset{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

for each $R \in \mathbf{Rel}(A,B)$, where

  – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 9.2.8.1.1.
  – The morphism $\phi_R\colon \mathbf{Coll}(R) \longrightarrow \Delta^1$ is given by

$$\phi_R(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

  for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}\colon \ \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \longrightarrow \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of $\mathbf{Coll}$ at (R,S) is given by sending an inclusion

$$\iota\colon R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \longrightarrow \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \overset{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[12]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Equivalence*: Omitted. □

Appendices

[12]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$,

9.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

---

so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Chapter 10

Conditions on Relations

This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

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10.1 Functional and Total Relations

10.1.1 Functional Relations

Let A and B be sets.

Definition 10.1.1.1.1. A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

Proposition 10.1.1.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is functional.

   (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b:* Let $(b, b') \in B \times B$. We need to show that

$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\implies$ *Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  – Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.

  – Since $R \diamond R^{\dagger} \subset \chi_B$, we have

$$\left[R \diamond R^{\dagger}\right](b, b') \preceq_{\{\mathrm{t},\mathrm{f}\}} \chi_B(b, b'),$$

  and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^{\dagger}\right](b, b') = \mathsf{true}$, and thus $\chi_B(b, b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

10.1.2 Total Relations

Let A and B be sets.

Definition 10.1.2.1.1. A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

Proposition 10.1.2.1.2. Let $R\colon A \nrightarrow B$ be a relation.

1. *Characterisations.* The following conditions are equivalent:

   (a) The relation R is total.

   (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. Item 1, Characterisations: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\implies$ *Item 1b:* We have to show that, for each $(a, a') \in A$, we have
  $$\chi_A(a, a') \preceq_{\{\mathrm{t},\mathrm{f}\}} \left[R^\dagger \diamond R\right](a, a'),$$
  i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b* $\implies$ *Item 1a:* Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have
  $$\{a\} \subset \left[R^\dagger \diamond R\right](a),$$
  implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. □

10.2 Reflexive Relations

10.2.1 Foundations

Let A be a set.

Definition 10.2.1.1.1. A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.

Remark 10.2.1.1.2. In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 10.2.1.1.3. Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 10.2.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^{\dagger}$.

2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

10.2.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 10.2.2.1.1. The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[2] satisfying the following universal property:[3]

(★) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

Construction 10.2.2.1.2. Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A, A), \chi_A)$[4], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

Proof. Clear. □

[2] *Further Notation:* Also written R^{refl}.
[3] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.
[4] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$.

Proposition 10.2.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathrm{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 10.2.2.1.1.
Item 2, The Reflexive Closure of a Reflexive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.2.2.1.4. □

10.3 Symmetric Relations

10.3.1 Foundations

Let A be a set.

Definition 10.3.1.1.1. A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

Remark 10.3.1.1.2. In detail, a relation R is symmetric if it satisfies the following condition:

(★) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

Definition 10.3.1.1.3. Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 10.3.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Clear. □

10.3.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 10.3.2.1.1. The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5] satisfying the following universal property:[6]

(★) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

[5] *Further Notation:* Also written R^{symm}.
[6] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Construction 10.3.2.1.2. Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof. Clear. □

Proposition 10.3.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 10.3.2.1.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.3.1.1.4. □

10.4 Transitive Relations

10.4.1 Foundations

Let A be a set.

Definition 10.4.1.1.1. A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

Remark 10.4.1.1.2. In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 10.4.1.1.3. Let A be a set.

1. The **set of transitive relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

Proposition 10.4.1.1.4. Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^{\dagger}$.

[7] Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: See [MSE 2096272].[8] □

10.4.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 10.4.2.1.1. The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\text{trans}}$[9] satisfying the following universal property:[10]

(★) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\text{trans}} \subset \sim_S$.

Construction 10.4.2.1.2. Concretely, $\sim_R^{\text{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[11], being given by

$$\begin{aligned} R^{\text{trans}} &\overset{\text{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\text{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\text{def}}{=} \left\{ (a, b) \in A \times B \,\middle|\, \begin{aligned} &\text{there exists some } (x_1, \ldots, x_n) \in R^{\times n} \\ &\text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{aligned} \right\}. \end{aligned}$$

Proof. Clear. □

[8] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

- If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
  - There is some $b \in A$ such that:
    - $a \sim_R b$;
    - $b \sim_S c$;
  - There is some $d \in A$ such that:
    - $c \sim_R d$;
    - $d \sim_S e$.

[9] *Further Notation:* Also written R^{trans}.

[10] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

Proposition 10.4.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}\left(R^{\mathrm{trans}}, S\right) \cong \mathbf{Rel}(R,S),$$

   natural in $R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{trans}}(A,A)\right)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\text{poss.}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 10.4.2.1.1.
Item 2, The Transitive Closure of a Transitive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.

Item 4, Interaction With Inverses: We have

$$\begin{aligned}\left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty}\left(R^{\dagger}\right)^{\diamond n}\\ &= \bigcup_{n=1}^{\infty}\left(R^{\diamond n}\right)^{\dagger}\\ &= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger}\\ &= \left(R^{\mathrm{trans}}\right)^{\dagger},\end{aligned}$$

where we have used, respectively:

- Definition 10.4.2.1.2.
- ?? of Definition 8.1.3.1.4.
- ?? of Definition 9.2.3.1.2.
- Definition 10.4.2.1.2.

This finishes the proof.

Item 5, Interaction With Composition: This follows from Item 2 of Definition 10.4.1.1.4. □

10.5 Equivalence Relations

10.5.1 Foundations

Let A be a set.

Definition 10.5.1.1.1. A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

Example 10.5.1.1.2. The **kernel of a function** $f\colon A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

[12] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Item 4 of Definition 8.2.2.1.2.

Definition 10.5.1.1.3. Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A,B)$ of $\mathrm{Rel}(A,B)$ spanned by the equivalence relations.

2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A,B)$ of $\mathbf{Rel}(A,B)$ spanned by the equivalence relations.

10.5.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 10.5.2.1.1. The **equivalence closure**[14] of $\sim_R$ is the relation $\sim_R^{\mathrm{eq}}$[15] satisfying the following universal property:[16]

(★) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{eq}} \subset \sim_S$.

Construction 10.5.2.1.2. Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$
\begin{aligned}
R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\
&= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \\
&= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1,\ldots,x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1.\ \text{The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad\ \text{for each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2.\ \text{We have } a = b. \end{array} \right\}.
\end{aligned}
$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 10.2.2.1.1, 10.3.2.1.1 and 10.4.2.1.1), we see that it suffices to prove that:

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written R^{eq}.

[16] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing

1. The symmetric closure of a reflexive relation is still reflexive.

2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

Proposition 10.5.2.1.3. Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{eq}} \dashv 忘\right)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

   natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 10.5.2.1.1.
Item 2, The Equivalence Closure of an Equivalence Relation: Clear.
Item 3, Idempotency: This follows from Item 2. □

10.6 Quotients by Equivalence Relations

10.6.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 10.6.1.1.1. The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned}[t] [a] &\overset{\mathrm{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. && \text{(since } R \text{ is symmetric)} \end{aligned}$$

10.6.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 10.6.2.1.1. The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 10.6.2.1.2. The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by
$$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$
but we could equally well define
$$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$
instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[17]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

Proposition 10.6.2.1.3. Let $f\colon X \to Y$ be a function and let R be a relation on X.

1. *As a Coequaliser.* We have an isomorphism of sets
$$X/{\sim_R^{\mathrm{eq}}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$
where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

R.

[17]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see **??**.

2. *As a Pushout.* We have an isomorphism of sets[18]

$$X/\sim^{\mathrm{eq}}_{R} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/\sim^{\mathrm{eq}}_{R} & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \arrow[ul, phantom, very near end, "\ulcorner"] \end{tikzcd}$$

where $\sim^{\mathrm{eq}}_{R}$ is the equivalence relation generated by $\sim_R$.

3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]
$$X/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map
   $$\overline{f}\colon X/\sim_R \to Y$$

[18]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \cong X \times_{X/\sim^{\mathrm{eq}}_{R}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d] \\ X \arrow[r] & X/\sim^{\mathrm{eq}}_{R}. \end{tikzcd}$$

[19]*Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{CoIm}(f)$.

[20]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\mathrm{Ker}(f)\colon X \nrightarrow X,$$
$$\mathrm{Im}(f) \subset Y$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$(\mathrm{Gr}(f) \dashv f^{-1})\colon \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Item 4 of Definition 8.2.2.1.2.

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$

commute.

(b) We have $R \subset \mathrm{Ker}(f)$.

(c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/\sim_R \arrow[ur, dashed, "\exists!", "\overline{f}"'] & \end{tikzcd}$$

commute.

6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/\sim_R, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \stackrel{\mathrm{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

(a) The map $\overline{f}$ is an injection.

 (b) We have $R = \mathrm{Ker}(f)$.
 (c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

 (a) The map $f \colon X \to Y$ is surjective.
 (b) The map $\overline{f} \colon X/{\sim_R} \to Y$ is surjective.

9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

 (a) The map f satisfies the equivalent conditions of Item 4:
 - There exists a map
$$\overline{f} \colon X/{\sim_R^{\mathrm{eq}}} \to Y$$
making the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R^{\mathrm{eq}}} \arrow[ur, dotted, "\exists", "\overline{f}"'] & \end{tikzcd}$$
commute.
 - For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

 (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. *Item 1, As a Coequaliser*: Omitted.
Item 2, As a Pushout: Omitted.
Item 3, The First Isomorphism Theorem for Sets: Clear.
Item 4, Descending Functions to Quotient Sets, I: See [Pro25n].
Item 5, Descending Functions to Quotient Sets, II: See [Pro25aa].
Item 6, Descending Functions to Quotient Sets, III: This follows from Items 5 and 6.
Item 7, Descending Functions to Quotient Sets, IV: See [Pro25m].
Item 8, Descending Functions to Quotient Sets, V: See [Pro25l].
Item 9, Descending Functions to Quotient Sets, VI: The implication Item 8a $\Longrightarrow$ Item 8b is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

- The following conditions are satisfied:
  - We have $x \sim_R x_1$ or $x_1 \sim_R x$;
  - We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
  - We have $y \sim_R x_n$ or $x_n \sim_R y$;
- We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show. □

Appendices

10.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Part IV

Categories

Chapter 11

Categories

This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 11.1).

2. Examples of categories (Section 11.2).

3. The quadruple adjunction $\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}$ between the category of categories and the category of sets (Section 11.3).

4. Groupoids, categories in which all morphisms admit inverses (Section 11.4).

5. Functors (Section 11.5).

6. The conditions one may impose on functors in decreasing order of importance:

   (a) Section 11.6 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.

   (b) Section 11.7 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 11.6, such as being dominant, being a monomorphism, being pseudomonic, etc.

   (c) Section 11.8 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.

7. Natural transformations (Section 11.9).

8. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 11.10).

This chapter is under active revision. TODO:

- Fix categories having an underlying set of objects by having them have an underlying setoid of objects (not necessarily by definition, as that'll likely be bothersome; at least Section 11.3 should be fixed and several remarks should be added at several points). Related: Definition 11.3.1.1.2

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11.1 Categories

11.1.1 Foundations

Definition 11.1.1.1.1. A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**.
- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\mathbb{1}^{\mathcal{C}}_{A} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$

  called the **unit map of $\mathcal{C}$ at A**, determining a morphism

  $$\mathrm{id}_A \colon A \to A$$

of $\mathcal{C}$, called the **identity morphism of** A.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets

  $$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C),$$

  called the **composition map of** $\mathcal{C}$ **at** (A,B,C).

such that the following conditions are satisfied:

1. *Associativity.* The diagram

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Hom}_{\mathcal{C}}(C,D) \times (\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)} \times \circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(B,C)) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$

   commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have
   $$(f \circ g) \circ h = f \circ (g \circ h).$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_B \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,B) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

   commutes, i.e. for each morphism $f \colon A \to B$ of $\mathcal{C}$, we have

   $$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)} \times \mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[drr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & & \\
\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[rr, "\circ^{\mathcal{C}}_{A,A,B}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A \to B$ of $\mathcal{C}$, we have

$$f \circ \mathrm{id}_A = f.$$

Notation 11.1.1.1.2. Let $\mathcal{C}$ be a category.

1. We also write $\mathcal{C}(A,B)$ for $\mathrm{Hom}_{\mathcal{C}}(A,B)$.
2. We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

Definition 11.1.1.1.3. Let κ be a regular cardinal. A category $\mathcal{C}$ is

1. **Locally small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a set.
2. **Locally essentially small** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A,B)/\{\text{isomorphisms}\}$$

is a set.
3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.
4. κ**-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C}) < \kappa$.

11.1.2 Subcategories

Let $\mathcal{C}$ be a category.

Definition 11.1.2.1.1. A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathcal{A}}(A,B) \subset \mathrm{Hom}_{\mathcal{C}}(A,B).$$

3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have

$$\mathbb{1}_A^{\mathcal{A}} = \mathbb{1}_A^{C}.$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have

$$\circ_{A,B,C}^{\mathcal{A}} = \circ_{A,B,C}^{C}.$$

Definition 11.1.2.1.2. A subcategory $\mathcal{A}$ of C is **full** if the canonical inclusion functor $\mathcal{A} \to C$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion

$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{C}(A, B)$$

is surjective (and thus bijective).

Definition 11.1.2.1.3. A subcategory $\mathcal{A}$ of a category C is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[1]

Definition 11.1.2.1.4. A subcategory $\mathcal{A}$ of C is **wide**[2] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(C)$.

11.1.3 Skeletons of Categories

Definition 11.1.3.1.1. A[3] **skeleton** of a category C is a full subcategory $\mathsf{Sk}(C)$ with one object from each isomorphism class of objects of C.

Definition 11.1.3.1.2. A category C is **skeletal** if $C \cong \mathsf{Sk}(C)$.[4]

Proposition 11.1.3.1.3. Let C be a category.

1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(C)$ always exists.
2. *Pseudofunctoriality.* The assignment $C \mapsto \mathsf{Sk}(C)$ defines a pseudofunctor
$$\mathsf{Sk} \colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$

[1] That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(C)$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.
[2] *Further Terminology:* Also called **lluf**.
[3] Due to Item 3 of Definition 11.1.3.1.3, which states that any two skeletons of a category are equivalent, we often refer to any such full subcategory $\mathsf{Sk}(C)$ of C as *the* skeleton of C.
[4] That is, C is **skeletal** if isomorphic objects of C are equal.

3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.

4. *Inclusions of Skeletons Are Equivalences.* The inclusion

$$\iota_{\mathcal{C}} \colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof. *Item 1, Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
Item 2, Pseudofunctoriality: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
Item 3, Uniqueness Up to Equivalence: Clear.
Item 4, Inclusions of Skeletons Are Equivalences: Clear. □

11.1.4 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

Definition 11.1.4.1.1. Let $f \colon A \to B$ and $g \colon B \to C$ be morphisms of $\mathcal{C}$.

1. The **precomposition function associated to** f is the function

$$f^* \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

2. The **postcomposition function associated to** g is the function

$$g_* \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Proposition 11.1.4.1.2. Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f \colon A \to B$ and $g \colon B \to C$ be morphisms of $\mathcal{C}$.

1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B, D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A, D). \end{tikzcd}$$

2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

3. *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{aligned} [g \circ f] &= g_* \circ [f], \\ [g \circ f] &= f^* \circ [g], \end{aligned} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

4. *Interaction With Composition III.* We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathrm{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathrm{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathrm{id} \times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times\mathrm{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

5. *Interaction With Identities.* We have

$$\begin{aligned} (\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}, \\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}. \end{aligned}$$

Proof. *Item 1, Interaction Between Precomposition and Postcomposition*: Clear.
Item 2, Interaction With Composition I: Clear.
Item 3, Interaction With Composition II: Clear.
Item 4, Interaction With Composition III: Clear.
Item 5, Interaction With Identities: Clear. □

11.2 Examples of Categories

11.2.1 The Empty Category

Example 11.2.1.1.1. The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have
$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset.$$
- *Morphisms.* We have
$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\text{def}}{=} \emptyset.$$
- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

11.2.2 The Punctual Category

Example 11.2.2.1.1. The **punctual category**[5] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathrm{pt}) \overset{\text{def}}{=} \{\star\}.$$
- *Morphisms.* The unique Hom-set of pt is defined by
$$\mathrm{Hom}_{\mathrm{pt}}(\star,\star) \overset{\text{def}}{=} \{\mathrm{id}_\star\}.$$
- *Identities.* The unit map
$$\mathbb{1}^{\mathrm{pt}}_\star \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star,\star)$$
of pt at $\star$ is defined by
$$\mathrm{id}^{\mathrm{pt}}_\star \overset{\text{def}}{=} \mathrm{id}_\star .$$
- *Composition.* The composition map
$$\circ^{\mathrm{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathrm{pt}}(\star,\star) \times \mathrm{Hom}_{\mathrm{pt}}(\star,\star) \longrightarrow \mathrm{Hom}_{\mathrm{pt}}(\star,\star)$$
of pt at $(\star,\star,\star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

[5] *Further Terminology:* Also called the **singleton category**.

11.2.3 Monoids as One-Object Categories

Example 11.2.3.1.1. We have an isomorphism of categories[6]

$$\mathsf{Mon} \cong \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathbf{B}\colon \mathsf{Mon} \to \mathsf{Cats}$ of **??** of **??**, exhibiting monoids as exactly those categories having a single object.

Proof. Omitted. □

11.2.4 Ordinal Categories

Example 11.2.4.1.1. The ***n*th ordinal category** is the category $\mathbb{n}$ where[7]

[6]This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{\mathsf{2disc}} \cong \mathrm{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{\mathsf{2disc}}}{\times} \mathsf{Cats}_{\mathsf{2},*}, \qquad \begin{tikzcd} \mathsf{Mon}_{\mathsf{2disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_{\mathsf{2},*} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{\mathsf{2disc}} \end{tikzcd}$$

between the discrete 2-category $\mathsf{Mon}_{\mathsf{2disc}}$ on Mon and the 2-category of pointed categories with one object.

[7]In other words, $\mathbb{n}$ is the category associated to the poset

$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$

The category $\mathbb{n}$ for $n \geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (**??**): we have isomorphisms of categories

$$\begin{aligned} \mathbb{1} &\cong \mathbb{0} \star \mathbb{0}, \\ \mathbb{2} &\cong \mathbb{1} \star \mathbb{0} \\ &\cong (\mathbb{0} \star \mathbb{0}) \star \mathbb{0}, \\ \mathbb{3} &\cong \mathbb{2} \star \mathbb{0} \\ &\cong (\mathbb{1} \star \mathbb{0}) \star \mathbb{0} \\ &\cong ((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \\ \mathbb{4} &\cong \mathbb{3} \star \mathbb{0} \\ &\cong (\mathbb{2} \star \mathbb{0}) \star \mathbb{0} \end{aligned}$$

- *Objects.* We have

$$\mathrm{Obj}(\mathbb{n}) \overset{\mathrm{def}}{=} \{[0], \ldots, [n]\}.$$

- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have

$$\mathrm{Hom}_{\mathbb{n}}([i],[j]) \overset{\mathrm{def}}{=} \begin{cases} \left\{\mathrm{id}_{[i]}\right\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$

- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map

$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$

  of $\mathbb{n}$ at $[i]$ is defined by

$$\mathrm{id}^{\mathbb{n}}_{[i]} \overset{\mathrm{def}}{=} \mathrm{id}_{[i]}\,.$$

- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\mathbb{n})$, the composition map

$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$

  of $\mathbb{n}$ at $([i],[j],[k])$ is defined by

$$\begin{gathered} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} = \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) = ([i] \to [k]). \end{gathered}$$

11.2.5 The Walking Arrow

Definition 11.2.5.1.1. The **walking arrow** is the category $\mathbb{1}$ defined as the first ordinal category.

Remark 11.2.5.1.2. In detail, the walking arrow is the category $\mathbb{1}$ where:

- *Objects.* We have $\mathrm{Obj}(\mathbb{1}) = \{0, 1\}$.

$$\begin{aligned} &\cong ((\mathbb{1} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0} \\ &\cong (((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}, \end{aligned}$$

and so on.

- *Morphisms.* We have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{1}}(0,0) &= \{\mathrm{id}_0\},\\ \mathrm{Hom}_{\mathbb{1}}(1,1) &= \{\mathrm{id}_1\},\\ \mathrm{Hom}_{\mathbb{1}}(0,1) &= \{f_{01}\},\\ \mathrm{Hom}_{\mathbb{1}}(1,0) &= \emptyset.\end{aligned}$$

- *Identities and Composition.* The identities and composition of $\mathbb{1}$ are completely determined by the unitality and associativity axioms for $\mathbb{1}$.

11.2.6 More Examples of Categories

Example 11.2.6.1.1. Here we list some of the other categories appearing throughout this work.

1. The category Sets_* of pointed sets of Definition 6.1.3.1.1.
2. The category Rel of sets and relations of Definition 8.3.2.1.1.
3. The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of **??**.
4. The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**.
5. The category ISets of indexed sets of **??**.
6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**.
7. The category $\mathsf{FibSets}$ of fibred sets of **??**.
8. Categories of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ as in Definition 11.10.1.1.1.
9. The category of categories Cats of Definition 11.10.2.1.1.
10. The category of groupoids Grpd of Definition 11.10.4.1.1.

11.2.7 Posetal Categories

Definition 11.2.7.1.1. Let $(X, \preceq_X)$ be a poset.

1. The **posetal category associated to** $(X, \preceq_X)$ is the category X_{pos} where

- *Objects.* We have

$$\mathrm{Obj}(X_{\mathrm{pos}}) \overset{\mathrm{def}}{=} X.$$

- *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathrm{pos}})$, we have

$$\mathrm{Hom}_{X_{\mathrm{pos}}}(a, b) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$

- *Identities.* For each $a \in \mathrm{Obj}(X_{\mathrm{pos}})$, the unit map

$$\mathbb{1}_a^{X_{\mathrm{pos}}} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathrm{pos}}}(a, a)$$

  of X_{pos} at a is given by the identity map.

- *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathrm{pos}})$, the composition map

$$\circ_{a,b,c}^{X_{\mathrm{pos}}} \colon \mathrm{Hom}_{X_{\mathrm{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathrm{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathrm{pos}}}(a, c)$$

  of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.

2. A category $\mathcal{C}$ is **posetal**[8] if $\mathcal{C}$ is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

Proposition 11.2.7.1.2. Let $(X, \preceq_X)$ be a poset and let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathrm{pos}}$ defines a functor

$$(-)_{\mathrm{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$

2. *Fully Faithfulness.* The functor $(-)_{\mathrm{pos}}$ of Item 1 is fully faithful.

3. *Characterisations.* The following conditions are equivalent:

   (a) The category $\mathcal{C}$ is posetal.

   (b) For each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have $f = g$.

[8] *Further Terminology:* Also called a **thin** category or a $(0, 1)$**-category**.

4. *Automatic Commutativity of Diagrams.* Every diagram in a posetal category commutes.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Fully Faithfulness*: Omitted.
Item 3, *Characterisations*: Clear.
Item 4, *Automatic Commutativity of Diagrams*: This follows from the fact that if C is posetal, then there's at most one morphism between any two objects. □

11.3 The Quadruple Adjunction With Sets

11.3.1 Statement

Let C be a category.

Proposition 11.3.1.1.1. We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=70, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}, \arrow[l, bend right=70, "\pi_0"'] \arrow[l, bend left=15, "\mathsf{Obj}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(\pi_0(C), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(C, X_{\mathsf{disc}}), \\ \mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathsf{Obj}(C)), \\ \mathrm{Hom}_{\mathsf{Sets}}(\mathsf{Obj}(C), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(C, X_{\mathsf{indisc}}), \end{aligned}$$

natural in $C \in \mathsf{Obj}(\mathsf{Cats})$ and $X \in \mathsf{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0 \colon \mathsf{Cats} \to \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 11.3.2.2.1.

- The functor
$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.

- The functor
$$\mathsf{Obj}\colon \mathsf{Cats} \longrightarrow \mathsf{Sets},$$
the **object functor**, is the functor sending a category to its set of objects.

- The functor
$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \longrightarrow \mathsf{Cats},$$
the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

Proof. Omitted. □

Warning 11.3.1.1.2. (This is a stub, to be revised and expanded upon later.)

The discrete category functor of Definition 11.3.1.1.1 lifts to a 2-functor, but it fails to preserve 2-categorical colimits, and hence lacks a right 2-adjoint. For instance, the 2-pushout of $\mathrm{pt} \longleftarrow S^0 \longrightarrow \mathrm{pt}$ in $\mathsf{Sets}_{\mathsf{disc}}$ is pt, but in Cats_2 it is given by $\mathrm{B}\mathbb{Z}$.

11.3.2 Connected Components and Connected Categories

11.3.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 11.3.2.1.1. A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[9]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

11.3.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

Definition 11.3.2.2.1. The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

Proposition 11.3.2.2.2. Let $\mathcal{C}$ be a category.

[9] In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\sim$

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0 \colon \mathsf{Cats} \longrightarrow \mathsf{Sets}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=20, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathsf{Obj}"] \end{tikzcd}$$

3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\begin{aligned} \pi_0\left(\mathcal{C} \coprod \mathcal{D}\right) &\cong \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}), \\ \pi_0\left(\mathcal{C} \coprod\nolimits_{\mathcal{E}} \mathcal{D}\right) &\cong \pi_0(\mathcal{C}) \coprod\nolimits_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}), \\ \pi_0\left(\mathrm{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) &\cong \mathrm{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathsf{Obj}(\mathsf{Cats})$.

5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right) \colon \left(\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}\right) \longrightarrow \left(\mathsf{Sets}, \coprod, \emptyset\right),$$

being equipped with isomorphisms

$$\begin{aligned} \pi_{0|\mathcal{C},\mathcal{D}}^{\coprod} \colon \pi_0(\mathcal{C}) \coprod \pi_0(\mathcal{D}) &\overset{\sim}{\dashrightarrow} \pi_0\left(\mathcal{C} \coprod \mathcal{D}\right), \\ \pi_{0|\mathbb{1}}^{\coprod} \colon \emptyset &\overset{\sim}{\dashrightarrow} \pi_0(\emptyset_{\mathsf{cat}}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\times}, \pi_{0|\mathbb{1}}^{\times}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\sim} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\times} \colon \mathrm{pt} \xrightarrow{\sim} \pi_0(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Definition 11.3.1.1.1.
Item 3, Interaction With Groupoids: Clear.
Item 4, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 6, Symmetric Strong Monoidality With Respect to Products: Clear. □

11.3.2.3 Connected Categories

Definition 11.3.2.3.1. A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[10,11]

11.3.3 Discrete Categories

Definition 11.3.3.1.1. Let X be a set.

1. The **discrete category on** X is the category X_{disc} where

   - *Objects.* We have

$$\mathrm{Obj}(X_{\mathsf{disc}}) \stackrel{\text{def}}{=} X.$$

   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have

$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \stackrel{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$

with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

[10] *Further Terminology:* A category is **disconnected** if it is not connected.

[11] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathrm{disc}})$, the unit map
$$\mathbb{1}^{X_{\mathrm{disc}}}_A : \mathrm{pt} \to \mathrm{Hom}_{X_{\mathrm{disc}}}(A, A)$$
of X_{disc} at A is defined by
$$\mathrm{id}^{X_{\mathrm{disc}}}_A \overset{\mathrm{def}}{=} \mathrm{id}_A .$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathrm{disc}})$, the composition map
$$\circ^{X_{\mathrm{disc}}}_{A,B,C} : \mathrm{Hom}_{X_{\mathrm{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathrm{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathrm{disc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$\mathrm{id}_A \circ \mathrm{id}_A \overset{\mathrm{def}}{=} \mathrm{id}_A .$$

2. A category C is **discrete** if it is equivalent to X_{disc} for some set X.

Proposition 11.3.3.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathrm{disc}}$ defines a functor
$$(-)_{\mathrm{disc}} : \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathrm{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathrm{indisc}}) : \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, "(-)_{\mathrm{disc}}" description, shift left=1] \arrow[r, bend right=50, "(-)_{\mathrm{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=50, "\pi_0"'] \arrow[l, "\mathrm{Obj}" description, shift left=1] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure
$$\left((-)_{\mathrm{disc}}, (-)^{\coprod}_{\mathrm{disc}}, (-)^{\coprod}_{\mathrm{disc}|\mathbb{1}}\right) : (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}),$$
being equipped with isomorphisms
$$(-)^{\coprod}_{\mathrm{disc}|X,Y} : X_{\mathrm{disc}} \coprod Y_{\mathrm{disc}} \xrightarrow{\sim} (X \coprod Y)_{\mathrm{disc}},$$
$$(-)^{\coprod}_{\mathrm{disc}|\mathbb{1}} : \emptyset_{\mathsf{cat}} \xrightarrow{\sim} \emptyset_{\mathrm{disc}},$$
natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \overset{\sim}{\dashrightarrow} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} \colon \mathrm{pt} \overset{\sim}{\dashrightarrow} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Definition 11.3.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 4, Symmetric Strong Monoidality With Respect to Products: Clear. □

11.3.4 Indiscrete Categories

Definition 11.3.4.1.1. Let X be a set.

1. The **indiscrete category on** X[12] is the category X_{indisc} where
   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{indisc}}) \overset{\mathrm{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
   $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\overset{\mathrm{def}}{=} \{[A] \to [B]\}\\ &\cong \mathrm{pt}.\end{aligned}$$
   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
   $$\mathbb{1}^{X_{\mathsf{indisc}}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
   of X_{indisc} at A is defined by
   $$\mathrm{id}^{X_{\mathsf{indisc}}}_A \overset{\mathrm{def}}{=} \{[A] \to [A]\}.$$

[12] *Further Terminology:* Sometimes called the **chaotic category on** X.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map

$$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B,C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A,B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A,C)$$

of X_{disc} at (A, B, C) is defined by

$$([B] \to [C]) \circ ([A] \to [B]) \stackrel{\text{def}}{=} ([A] \to [C]).$$

2. A category C is **indiscrete** if it is equivalent to X_{indisc} for some set X.

Proposition 11.3.4.1.2. Let X be a set.

1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor

$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathsf{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats} \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathsf{Obj}" description] \end{tikzcd}$$

3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{indisc}|X,Y} \colon X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\sim} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\times}_{\mathsf{indisc}|\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\sim} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Definition 11.3.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Products: Clear. □

11.4 Groupoids

11.4.1 Isomorphisms

Let C be a category.

Definition 11.4.1.1.1. A morphism $f\colon A \longrightarrow B$ of C is an **isomorphism** if there exists a morphism $f^{-1}\colon B \longrightarrow A$ of C such that

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A\,. \end{aligned}$$

Notation 11.4.1.1.2. We write $\mathrm{Iso}_C(A, B)$ for the set of all isomorphisms in C from A to B.

11.4.2 Groupoids

Definition 11.4.2.1.1. A **groupoid** is a category in which every morphism is an isomorphism.

Example 11.4.2.1.2. The isomorphism of categories of Definition 11.2.3.1.1 restricts to an isomorphism

$$\mathsf{Grp} \cong \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Grpd}, \qquad \begin{tikzcd} \mathsf{Grp} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Grpd} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

where Grpd is the full subcategory of Cats spanned by the groupoids.

In other words, we have an identification

$$\{\text{Groups}\} \cong \{\text{One-object groupoids}\}.$$

11.4.3 The Groupoid Completion of a Category

Let C be a category.

Definition 11.4.3.1.1. The **groupoid completion of** C[13] is the pair $(\mathrm{K}_0(C), \iota_C)$ consisting of

[13] *Further Terminology:* Also called the **Grothendieck groupoid of** C or the

- A groupoid $\mathrm{K}_0(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[14]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

Construction 11.4.3.1.2. Concretely, the groupoid completion of $\mathcal{C}$ is the Gabriel–Zisman localisation $\mathrm{Mor}(\mathcal{C})^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$; see **??**.

(To be expanded upon later on.)

Proof. Omitted. □

Proposition 11.4.3.1.3. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor

$$\mathrm{K}_0 : \mathsf{Cats} \to \mathsf{Grpd}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathrm{K}_0 : \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\mathrm{K}_0 \dashv \iota) : \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, shift left, "\iota"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

Grothendieck groupoid completion of $\mathcal{C}$.

[14] See Item 5 of Definition 11.4.3.1.3 for an explicit construction.

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Definition 11.4.4.1.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd} \arrow[l, hook', "\iota" description] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd} \arrow[l, hook', shift left, "\iota"] \end{tikzcd},$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Definition 11.4.4.1.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, phantom, shift right=3, "\perp_2"] \arrow[r, bend right=50, "\mathsf{Core}"'] & \mathsf{Grpd} \arrow[l, hook', "\iota" description] \end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd}
\mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\
\mathsf{sSets} \arrow[r, "|-|"'] & \mathbb{T} \arrow[u, "\Pi_{\leq 1}"']
\end{tikzcd}$$

commutes up to natural isomorphism.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure
$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right) \colon \left(\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}\right) \to \left(\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}}\right)$$
being equipped with isomorphisms
$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} \colon \mathrm{K}_0(\mathcal{C}) \coprod \mathrm{K}_0(\mathcal{D}) \xrightarrow{\sim} \mathrm{K}_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\coprod} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\sim} \mathrm{K}_0(\emptyset_{\mathsf{cat}}),$$
natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure
$$\left(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$
being equipped with isomorphisms
$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \xrightarrow{\sim} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\times} \colon \mathrm{pt} \xrightarrow{\sim} \mathrm{K}_0(\mathrm{pt}),$$
natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Classifying Spaces: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 7, Symmetric Strong Monoidality With Respect to Products: Omitted. □

11.4.4 The Core of a Category

Let $\mathcal{C}$ be a category.

Definition 11.4.4.1.1. The **core** of $\mathcal{C}$ is the pair $(\mathsf{Core}(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{Core}(\mathcal{C})$;
- A functor $\iota_{\mathcal{C}} \colon \mathsf{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathsf{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

Notation 11.4.4.1.2. We also write $\mathcal{C}^{\simeq}$ for $\mathsf{Core}(\mathcal{C})$.

Construction 11.4.4.1.3. The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[15]

1. *Objects.* We have

$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \overset{\text{def}}{=} \mathrm{Obj}(\mathcal{C}).$$

2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

Proof. This follows from the fact that functors preserve isomorphisms (Item 1 of Definition 11.5.1.1.6). □

Proposition 11.4.4.1.4. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor

$$\mathsf{Core} \colon \mathsf{Cats} \to \mathsf{Grpd}.$$

[15] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor

$$\mathsf{Core}\colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

3. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Definition 11.4.3.1.3, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Definition 11.4.3.1.3, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=3, "\perp_2"] \arrow[r, phantom, shift right=3, "\perp_2"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathsf{Obj}(\mathsf{Grpd})$.

5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\begin{aligned} \mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \times \mathsf{Core}(\mathcal{D}) &\xrightarrow{\sim} \mathsf{Core}(\mathcal{C} \times \mathcal{D}), \\ \mathsf{Core}^{\times}_{\mathbb{1}} \colon \mathrm{pt} &\xrightarrow{\sim} \mathsf{Core}(\mathrm{pt}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right) \colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \textstyle\coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\begin{aligned} \mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} \colon \mathsf{Core}(\mathcal{C}) \textstyle\coprod \mathsf{Core}(\mathcal{D}) &\xrightarrow{\sim} \mathsf{Core}(\mathcal{C} \textstyle\coprod \mathcal{D}), \\ \mathsf{Core}^{\coprod}_{\mathbb{1}} \colon \emptyset_{\mathsf{cat}} &\xrightarrow{\sim} \mathsf{Core}(\emptyset_{\mathsf{cat}}), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathsf{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Products: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted. □

11.5 Functors

11.5.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.5.1.1.1. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ **from** $\mathcal{C}$ **to** $\mathcal{D}$[16] consists of:

1. *Action on Objects.* A map of sets
$$F\colon \operatorname{Obj}(\mathcal{C}) \to \operatorname{Obj}(\mathcal{D}),$$
called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \operatorname{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{D}}(F(A),F(B)),$$
called the **action on morphisms of** F **at** (A,B)[17].

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \operatorname{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \operatorname{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \operatorname{Hom}_{\mathcal{D}}(F(A),F(A)) \end{tikzcd}$$
commutes, i.e. we have
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}\,.$$

2. *Preservation of Composition.* For each $A, B, C \in \operatorname{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd} \operatorname{Hom}_{\mathcal{C}}(B,C) \times \operatorname{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C}\times F_{A,B}"'] & \operatorname{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\ \operatorname{Hom}_{\mathcal{D}}(F(B),F(C)) \times \operatorname{Hom}_{\mathcal{D}}(F(A),F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \operatorname{Hom}_{\mathcal{D}}(F(A),F(C)) \end{tikzcd}$$
commutes, i.e. for each composable pair (g,f) of morphisms of $\mathcal{C}$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

[16] *Further Terminology:* Also called a **covariant functor**.

[17] *Further Terminology:* Also called **action on Hom-sets of** F **at** (A,B).

Notation 11.5.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of **??**.

1. Given a functor
$$F\colon \mathcal{C} \to \mathcal{D},$$
we also write F_A for $F(A)$.

2. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$
we also write F^A for $F(A)$.

3. Given a functor
$$F\colon \mathcal{C} \times \mathcal{C} \to \mathcal{D},$$
we also write $F_{A,B}$ for $F(A, B)$.

4. Given a functor
$$F\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D},$$
we also write F^A_B for $F(A, B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F\colon \mathcal{C} \to \mathcal{D}$.

Notation 11.5.1.1.3. Following the notation $[\![x \mapsto f(x)]\!]$ for a function $f\colon X \to Y$ introduced in Definition 3.1.1.1.2, we will sometimes denote a functor $F\colon \mathcal{C} \to \mathcal{D}$ by
$$F \overset{\text{def}}{=} [\![A \mapsto F(A)]\!],$$
specially when the action on morphisms of F is clear from its action on objects.

Example 11.5.1.1.4. The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{id}_{\mathcal{C}}(A) \overset{\text{def}}{=} A.$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on mor-

phisms

$$(\mathrm{id}_{\mathcal{C}})_{A,B}\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \underbrace{\operatorname{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A),\mathrm{id}_{\mathcal{C}}(B))}_{\overset{\text{def}}{=}\operatorname{Hom}_{\mathcal{C}}(A,B)}$$

of $\mathrm{id}_{\mathcal{C}}$ at (A,B) is defined by

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \overset{\text{def}}{=} \mathrm{id}_{\operatorname{Hom}_{\mathcal{C}}(A,B)}\,.$$

Proof. Preservation of Identities: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \overset{\text{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions: For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g\circ f) &\overset{\text{def}}{=} g\circ f\\ &\overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g)\circ \mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof. □

Definition 11.5.1.1.5. The **composition** of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is the functor $G\circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

  $$[G\circ F](A) \overset{\text{def}}{=} G(F(A)).$$

- *Action on Morphisms.* For each $A,B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

  $$(G\circ F)_{A,B}\colon \operatorname{Hom}_{\mathcal{C}}(A,B) \to \operatorname{Hom}_{\mathcal{E}}\big(G_{F_A}, G_{F_B}\big)$$

  of $G\circ F$ at (A,B) is defined by

  $$[G\circ F](f) \overset{\text{def}}{=} G(F(f)).$$

Proof. Preservation of Identities: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned}G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F)\\ &= \mathrm{id}_{G_{F_A}}\,. && \text{(functoriality of } G)\end{aligned}$$

Preservation of Composition: For each composable pair (g,f) of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}G_{F_{g\circ f}} &= G_{F_g\circ F_f} && \text{(functoriality of } F)\\ &= G_{F_g}\circ G_{F_f}. && \text{(functoriality of } G)\end{aligned}$$

This finishes the proof. □

Proposition 11.5.1.1.6. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[18]

Proof. *Item 1, Preservation of Isomorphisms*: Indeed, we have

$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\big(f^{-1} \circ f\big) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$

and

$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\big(f \circ f^{-1}\big) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$

showing $F(f)$ to be an isomorphism. □

11.5.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of **??**.

Definition 11.5.2.1.1. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

Remark 11.5.2.1.2. In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets

   $$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$

   called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map

   $$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)),$$

   called the **action on morphisms of** F **at** (A, B).

[18] When the converse holds, we call F *conservative*, see Definition 11.6.4.1.1.

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A), F(A))
\end{tikzcd}$$

commutes, i.e. we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)} .$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \arrow[dr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \arrow[ur, "F_{B,C} \times F_{A,B}"] \arrow[d, "\circ^{\mathcal{C}}_{A,B,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(B), F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C), F(B)) \arrow[d, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
\mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C), F(A))
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

Remark 11.5.2.1.3. Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F \colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

11.5.3 Forgetful Functors

Definition 11.5.3.1.1. There isn't a precise definition of a **forgetful functor**.

Remark 11.5.3.1.2. Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Definitions 11.5.3.1.3 and 11.5.3.1.4).

Example 11.5.3.1.3. Examples of forgetful functors that forget structure include:

1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \longrightarrow \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.

2. *Forgetting Topologies.* The functor $\mathbb{T} \longrightarrow \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.

3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \longrightarrow \mathsf{Sets}$ sending a K-fibred set $\phi_X \colon X \longrightarrow K$ to the set X, forgetting the map ϕ_X and the base set K.

Example 11.5.3.1.4. Examples of forgetful functors that forget properties include:

1. *Forgetting Commutativity.* The inclusion functor $\iota \colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.

2. *Forgetting Inverses.* The inclusion functor $\iota \colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

Notation 11.5.3.1.5. Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$忘 \colon \mathsf{Grp} \to \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Definition 11.5.3.1.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

1. 忘れる, transcribed as *wasureru*, meaning *to forget*.

2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful functor*.
3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.
4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánz▯*, meaning *forgetful functor*.

Remark 11.5.3.1.6. Here we collect the pronunciation of the words in Definition 11.5.3.1.5 for accuracy and completeness.

1. Pronunciation of 忘れる:
   - See here.
   - IPA broad transcription: [wäsɯɾe̞ɾɯ].
   - IPA narrow transcription: [ɰ̟▯äsɨ▯ɾe̞ɾɯ̟▯].
2. Pronunciation of 忘却関手: Pronunciation:
   - See here.
   - IPA broad transcription: [bo̞ːkʲäkɯ kã̠ɰ̃ɕɯ].
   - IPA narrow transcription: [bo̞ːkʲäkɯ̟▯ kã̠ɰ̃ɕɯ▯].
3. Pronunciation of 忘记:
   - See here.
   - Broad IPA transcription: [wɑŋtɕi].
   - Sinological IPA transcription: [wɑŋ$^{51-53}$t͡ɕi^{51}].
4. Pronunciation of 遗忘函子:
   - See here.
   - Broad IPA transcription: [iwɑŋ xänt͡sz̩ɨ].
   - Sinological IPA transcription: [i^{35}wɑŋ^{51} xän^{35}t͡sz̩$^{214-21(4)}$].

11.5.4 The Natural Transformation Associated to a Functor

Definition 11.5.4.1.1. Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[19]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathrm{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[rr, "F^{\mathrm{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\Uparrow F^{\dagger}"] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\mathrm{def}}{=} F_{A,B}.$$

Proof. The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^*\circ\psi_*=\psi_*\circ\phi^*"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\ \mathrm{Hom}_{\mathcal{D}}(F_X,F_Y) \arrow[r, "F(\phi)^*\circ F(\psi)_*=F(\psi)_*\circ F(\phi)^*"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_B), \end{tikzcd}$$

acting on elements as

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \psi\circ f\circ\phi \arrow[d, mapsto] \\ F(f) \arrow[r, mapsto] & F(\psi)\circ F(f)\circ F(\psi) = F(\psi\circ f\circ\phi) \end{tikzcd}$$

commutes, which follows from the functoriality of F. ☐

[19]This is the 1-categorical version of **??** of **??**.

Proposition 11.5.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F)$ associated to F is a natural isomorphism.

   (b) The functor F is fully faithful.

2. *Interaction With Composition.* We have an equality of pasting diagrams

$$
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[r, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[r, "G^{\mathsf{op}}\times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] \arrow[dl, phantom, "\Nearrow F^{\dagger}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \arrow[dll, phantom, "\Nearrow G^{\dagger}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "(G\circ F)^{\mathsf{op}}\times(G\circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\Nearrow (G\circ F)^{\dagger}"] & \mathcal{E}^{\mathsf{op}} \times \mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
$$

   in $\mathsf{Cats}_{\mathsf{2}}$, i.e. we have

$$
(G \circ F)^{\dagger} = \left(G^{\dagger} \star \mathrm{id}_{F^{\mathsf{op}}\times F}\right) \circ F^{\dagger}.
$$

3. *Interaction With Identities.* We have

$$
\mathrm{id}_{\mathcal{C}}^{\dagger} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},
$$

   i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$.

Proof. *Item 1, Interaction With Natural Isomorphisms*: Clear.
Item 2, Interaction With Composition: Clear.
Item 3, Interaction With Identities: Clear. □

11.6 Conditions on Functors

11.6.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.1.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **faithful** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$
F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)
$$

of F at (A, B) is injective.

Proposition 11.6.1.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Composition.* If F and G are faithful, then so is $G \circ F$.

2. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is faithful.

   (c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism in Cats_2 in the sense of Definition 14.1.1.1.1.

3. *Interaction With Precomposition I.* Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

   (a) If F is faithful, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   can fail to be faithful.

   (b) Conversely, if the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful, then F ***can fail*** to be faithful.

4. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

5. *Interaction With Precomposition III.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is faithful.

(b) For each $\mathcal{X} \in \text{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

(c) For each $\mathcal{X} \in \text{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is monadic.

(d) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism in Cats_2 in the sense of Definition 14.2.1.1.1.

(e) The components

$$\eta_G \colon G \Longrightarrow \text{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \text{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \text{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \text{Ran}_F$ are all monomorphisms.

(f) The components

$$\epsilon_G \colon \text{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \text{Lan}_F \circ F^* \Longrightarrow \text{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\text{Lan}_F \dashv F^*$ are all epimorphisms.

(g) The functor F is dominant (Definition 11.7.1.1.1), i.e. every object of $\mathcal{D}$ is a retract of some object in $\text{Im}(F)$:

(★) For each $B \in \text{Obj}(\mathcal{D})$, there exist:
- An object A of $\mathcal{C}$;
- A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;
- A morphism $r \colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \text{id}_B$.

Proof. Item 1, Interaction With Composition: Since the map

$$(G \circ F)_{A,B} \colon \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

defined as the composition

$$\mathrm{Hom}_{C}(A,B) \xrightarrow{F_{A,B}} \mathrm{Hom}_{\mathcal{D}}(F_A,F_B) \xrightarrow{G_{F(A),F(B)}} \mathrm{Hom}_{\mathcal{D}}(G_{F_A},G_{F_B}),$$

is a composition of injective functions, it follows from **??** that it is also injective. Therefore $G \circ F$ is faithful.

Item 2, Interaction With Postcomposition: Omitted.

Item 3, Interaction With Precomposition I: See [MSE 733163] for Item 3a. Item 3b follows from Item 4 and the fact that there are essentially surjective functors that are not faithful.

Item 4, Interaction With Precomposition II: Omitted, but see https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html for a formalised proof.

Item 5, Interaction With Precomposition III: We claim Items 5a to 5g are equivalent:

- *Items 5a and 5d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Definition 14.2.1.1.1.
- *Items 5a to 5c and 5g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 5a and 5g.
- *Items 5a, 5e and 5f Are Equivalent:* See **??** of **??**.

This finishes the proof. □

11.6.2 Full Functors

Let C and $\mathcal{D}$ be categories.

Definition 11.6.2.1.1. A functor $F\colon C \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(C)$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{C}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A,F_B)$$

of F at (A,B) is surjective.

Proposition 11.6.2.1.2. Let $F\colon C \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors.

1. *Interaction With Composition.* If F and G are full, then so is $G \circ F$.
2. *Interaction With Postcomposition I.* If F is full, then the postcomposi-

tion functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

can fail to be full.

3. *Interaction With Postcomposition II.* If, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is full, then F is also full.

4. *Interaction With Precomposition I.* If F is full, then the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be full.

5. *Interaction With Precomposition II.* If, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is full, then F ***can fail*** to be full.

6. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is full (and also faithful by Item 4 of Definition 11.6.1.1.2).

7. *Interaction With Precomposition IV.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is full.

   (b) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably full morphism in Cats_2 in the sense of Definition 14.2.1.1.1.

(c) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

(d) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

(e) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of C;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following condition:

(★) For each $A \in \mathrm{Obj}(C)$ and each pair of morphisms

$$\begin{aligned} r &\colon F(A) \to B, \\ s &\colon B \to F(A) \end{aligned}$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. *Item 1, Interaction With Composition*: Since the map

$$(G \circ F)_{A,B} \colon \mathrm{Hom}_C(A, B) \to \mathrm{Hom}_{\mathcal{D}}\big(G_{F_A}, G_{F_B}\big),$$

defined as the composition

$$\mathrm{Hom}_C(A, B) \xrightarrow{F_{A,B}} \mathrm{Hom}_{\mathcal{D}}(F_A, F_B) \xrightarrow{G_{F(A),F(B)}} \mathrm{Hom}_{\mathcal{D}}\big(G_{F_A}, G_{F_B}\big),$$

is a composition of surjective functions, it follows from **??** that it is also surjective. Therefore $G \circ F$ is full.

Item 2, Interaction With Postcomposition I: We follow the proof (completely formalised in cubical Agda!) given by Naïm Camille Favier in [**favier:postcompose-not-full**]. Let C be the category where:

- *Objects.* We have $\mathrm{Obj}(C) = \{A, B\}$.
- *Morphisms.* We have

$$\begin{aligned}\mathrm{Hom}_C(A, A) &= \{e_A, \mathrm{id}_A\},\\ \mathrm{Hom}_C(B, B) &= \{e_B, \mathrm{id}_B\},\\ \mathrm{Hom}_C(A, B) &= \{f, g\},\\ \mathrm{Hom}_C(B, A) &= \emptyset.\end{aligned}$$

- *Composition.* The nontrivial compositions in C are the following:

$$\begin{aligned}e_A \circ e_A &= \mathrm{id}_A, & f \circ e_A &= g, & e_B \circ f &= f,\\ e_B \circ e_B &= \mathrm{id}_B, & g \circ e_A &= f, & e_B \circ g &= g.\end{aligned}$$

We may picture C as follows:

$$\begin{tikzcd} A \arrow[loop left, "e_A"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[loop right, "e_B"] \end{tikzcd}.$$

Next, let $\mathcal{D}$ be the walking arrow category $\mathbb{1}$ of Definition 11.2.5.1.1 and let $F : C \to \mathbb{1}$ be the functor given on objects by

$$\begin{aligned}F(A) &= 0,\\ F(B) &= 1\end{aligned}$$

and on non-identity morphisms by

$$\begin{aligned}F(f) &= f_{01}, & F(e_A) &= \mathrm{id}_0,\\ F(g) &= f_{01}, & F(e_B) &= \mathrm{id}_1\,.\end{aligned}$$

Finally, let $X = \mathrm{B}\mathbb{Z}_{/2}$ be the walking involution and let $\iota_A, \iota_B : \mathrm{B}\mathbb{Z}_{/2} \rightrightarrows C$ be the inclusion functors from $\mathrm{B}\mathbb{Z}_{/2}$ to C with

$$\begin{aligned}\iota_A(\bullet) &= A,\\ \iota_B(\bullet) &= B.\end{aligned}$$

Since every morphism in $\mathbb{1}$ has a preimage in C by F, the functor F is full. Now, for F_* to be full, the map

$$\begin{aligned} F_{*|\iota_A,\iota_B}\colon \mathrm{Nat}(\iota_A, \iota_B) &\longrightarrow \mathrm{Nat}(F \circ \iota_A, F \circ \iota_B) \\ \alpha &\longmapsto \mathrm{id}_F \star \alpha \end{aligned}$$

would need to be surjective. However, as we will show next, we have

$$\begin{aligned} \mathrm{Nat}(\iota_A, \iota_B) &= \emptyset, \\ \mathrm{Nat}(F \circ \iota_A, F \circ \iota_B) &\cong \mathrm{pt}, \end{aligned}$$

so this is impossible:

- *Proof of* $\mathrm{Nat}(\iota_A, \iota_B) = \emptyset$*:* A natural transformation $\alpha\colon \iota_A \Rightarrow \iota_B$ consists of a morphism

  $$\alpha\colon \underbrace{\iota_A(\bullet)}_{=A} \to \underbrace{\iota_B(\bullet)}_{=B}$$

  in C making the diagram

  $$\begin{tikzcd} \iota_A(\bullet) \arrow[r, "\iota_A(e)"] \arrow[d, "\alpha"'] & \iota_A(\bullet) \arrow[d, "\alpha"] \\ \iota_B(\bullet) \arrow[r, "\iota_B(e)"'] & \iota_B(\bullet) \end{tikzcd}$$

  commute for each $e \in \mathrm{Hom}_{\mathrm{B}\mathbb{Z}_{/2}}(\bullet, \bullet) \cong \mathbb{Z}_{/2}$. We have two cases:

  1. If $\alpha = f$, the naturality diagram for the unique nonidentity element of $\mathbb{Z}_{/2}$ is given by

     $$\begin{tikzcd} A \arrow[r, "e_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "e_B"'] & B. \end{tikzcd}$$

     However, $e_B \circ f = f$ and $f \circ e_A = g$, so this diagram does not commute.
  2. If $\alpha = g$, the naturality diagram for the unique nonidentity

element of $\mathbb{Z}_{/2}$ is given by

$$\begin{tikzcd} A \arrow[r, "e_A"] \arrow[d, "g"'] & A \arrow[d, "g"] \\ B \arrow[r, "e_B"'] & B. \end{tikzcd}$$

However, $e_B \circ g = g$ and $g \circ e_A = f$, so this diagram does not commute.

As a result, there are no natural transformations from ι_A to ι_B.

- *Proof of Nat*$(F \circ \iota_A, F \circ \iota_B) \cong$ *pt:* A natural transformation

$$\beta \colon F \circ \iota_A \Rightarrow F \circ \iota_B$$

consists of a morphism

$$\beta \colon \underbrace{[F \circ \iota_A](\bullet)}_{=0} \to \underbrace{[F \circ \iota_B](\bullet)}_{=1}$$

in $\mathbb{1}$ making the diagram

$$\begin{tikzcd} {[F \circ \iota_A](\bullet)} \arrow[r, "{[F\circ\iota_A](e)}"] \arrow[d, "\beta"'] & {[F \circ \iota_A](\bullet)} \arrow[d, "\beta"] \\ {[F \circ \iota_B](\bullet)} \arrow[r, "{[F\circ\iota_B](e)}"'] & {[F \circ \iota_B](\bullet)} \end{tikzcd}$$

commute for each $e \in \mathrm{Hom}_{\mathrm{B}\mathbb{Z}_{/2}}(\bullet, \bullet) \cong \mathbb{Z}_{/2}$. Since the only morphism from 0 to 1 in $\mathbb{1}$ is f_{01}, we must have $\beta = f_{01}$ if such a transformation were to exist, and in fact it indeed does, as in this case the naturality diagram above becomes

$$\begin{tikzcd} 0 \arrow[r, "\mathrm{id}_0"] \arrow[d, "f_{01}"'] & 0 \arrow[d, "f_{01}"] \\ 1 \arrow[r, "\mathrm{id}_1"'] & 1 \end{tikzcd}$$

for each $e \in \mathbb{Z}_{/2}$, and this diagram indeed commutes, making β into a natural transformation.

This finishes the proof.
Item 3, Interaction With Postcomposition II: Taking $\mathcal{X} = \mathrm{pt}$, it follows by assumption that the functor

$$F_*\colon \mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \to \mathsf{Fun}(\mathrm{pt}, \mathcal{D})$$

is full. However, by Item 5 of Definition 11.10.1.1.2, we have isomorphisms of categories

$$\mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C},$$
$$\mathsf{Fun}(\mathrm{pt}, \mathcal{D}) \cong \mathcal{D}$$

and the diagram

$$\begin{tikzcd} \mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \arrow[r, "F_*"] \arrow[d, dashed, "\wr"] & \mathsf{Fun}(\mathrm{pt}, \mathcal{D}) \arrow[d, dashed, "\wr"'] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

commutes. It then follows from Item 1 that F is full.
Item 4, Interaction With Precomposition I: Omitted.
Item 5, Interaction With Precomposition II: See [BS10, p. 47].
Item 6, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 7, Interaction With Precomposition IV: We claim Items 7a to 7e are equivalent:

- *Items 7a and 7b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Definition 14.2.2.1.1.
- *Items 7a, 7c and 7d Are Equivalent:* See **??** of **??**.
- *Items 7a and 7e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof. □

Question 11.6.2.1.3. Item 7 of Definition 11.6.2.1.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones?

This question also appears as [`MO 468121b`].

11.6.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.3.1.1. A functor $F: \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}: \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A,B) is bijective.

Proposition 11.6.3.1.2. Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$ be functors.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is fully faithful.

   (b) We have a pullback square

$$\mathsf{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D} \times \mathcal{D}} \mathsf{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathsf{Arr}(\mathcal{C}) \arrow[r, "\mathsf{Arr}(F)"] \arrow[d, "\mathrm{src} \times \mathrm{tgt}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src} \times \mathrm{tgt}"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "F \times F"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

   in Cats.

2. *Interaction With Composition.* If F and G are fully faithful, then so is $G \circ F$.

3. *Conservativity.* If F is fully faithful, then F is conservative.

4. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

5. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

6. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F: \mathcal{C} \to \mathcal{D}$ is fully faithful.

   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*: \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is fully faithful.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably fully faithful morphism in Cats_2 in the sense of Definition 14.1.3.1.1.

7. *Interaction With Precomposition I*. If F is fully faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be fully faithful.

8. *Interaction With Precomposition II*. If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful, then F ***can fail*** to be fully faithful (and in fact it can also fail to be either full or faithful).

9. *Interaction With Precomposition III*. If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

10. *Interaction With Precomposition IV*. The following conditions are equivalent:

(a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

(b) The precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$$

is fully faithful.

(c) The functor

$$\mathrm{Lan}_F\colon \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D}, \mathsf{Sets})$$

is fully faithful.

(d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Definition 14.2.3.1.1.

(e) The functor F is absolutely dense.

(f) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

(g) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

(h) The natural transformation

$$\alpha \colon \mathrm{Lan}_{h_F}\left(h^F\right) \Longrightarrow h$$

with components

$$\alpha_{B',B} \colon \int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi, \psi)]) = \psi \circ \phi$$

is a natural isomorphism.

(i) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have $r_B \circ s_B = \mathrm{id}_B$.

ii. For each morphism $f\colon B' \longrightarrow B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

in $\int^{A \in C} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Interaction With Composition: Since the map

$$(G \circ F)_{A,B}\colon \operatorname{Hom}_C(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

defined as the composition

$$\operatorname{Hom}_C(A, B) \xrightarrow{F_{A,B}} \operatorname{Hom}_{\mathcal{D}}(F_A, F_B) \xrightarrow{G_{F(A),F(B)}} \operatorname{Hom}_{\mathcal{D}}(G_{F_A}, G_{F_B}),$$

is a composition of bijective functions, it follows from **??** that it is also bijective. Therefore $G \circ F$ is fully faithful.
Item 3, Conservativity: This is a repetition of Item 2 of Definition 11.6.4.1.2, and is proved there.
Item 4, Essential Injectivity: Omitted.
Item 5, Interaction With Co/Limits: Omitted.
Item 6, Interaction With Postcomposition: This follows from Item 2 of Definition 11.6.1.1.2 and Item 1 of Definition 11.6.2.1.2.
Item 7, Interaction With Precomposition I: See [`MSE 733161`] for an example of a fully faithful functor whose precomposition with which fails to be full.
Item 8, Interaction With Precomposition II: See [`MSE 749304`, Item 3].
Item 9, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 10, Interaction With Precomposition IV: We claim Items 10a to 10i are equivalent:

- *Items 10a and 10d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Definition 14.2.3.1.1.
- *Items 10a, 10f and 10g Are Equivalent:* See **??** of **??**.
- *Items 10a to 10c Are Equivalent:* This follows from [Low15, Proposition A.1.5].
- *Items 10a, 10e, 10h and 10i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

11.6.4 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[20]

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Proposition 11.6.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is conservative.

   (b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.

2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

3. *Interaction With Precomposition.* The following conditions are equivalent:

   (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is conservative.

   (b) The equivalent conditions of Item 5 of Definition 11.6.1.1.2 are satisfied.

Proof. *Item 1, Characterisations*: This follows from Item 1 of Definition 11.5.1.1.6.

Item 2, Interaction With Fully Faithfulness: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\big(f \circ f^{-1}\big). \end{aligned}$$

[20] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

Similarly, $F(\mathrm{id}_A) = F(f^{-1} \circ f)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative. □

Question 11.6.4.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

(★) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

11.6.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

(★) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

Question 11.6.5.1.2. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

11.6.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.6.6.1.1. A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[21] if it satisfies the following condition:

($\star$) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

Question 11.6.6.1.2. Is there a characterisation of functors $F \colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially surjective?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

11.6.7 Equivalences of Categories

Definition 11.6.7.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of functors

$$F \colon \mathcal{C} \to \mathcal{D},$$
$$G \colon \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms

$$\eta \colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F,$$
$$\epsilon \colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}.$$

[21] *Further Terminology:* Also called an **eso** functor, meaning *essentially surjective on objects*.

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunction.

Proposition 11.6.7.1.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[22], then the following conditions are equivalent:[23]

   (a) The functor F is an equivalence of categories.

   (b) The functor F is fully faithful and essentially surjective.

   (c) The induced functor
   $$\restriction FSk(\mathcal{C}) \colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$
   is an *isomorphism* of categories.

   (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is an equivalence of categories.

   (e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
   $$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$
   is an equivalence of categories.

2. *Two-Out-of-Three.* Let
$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G \circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$
be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

[22] Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[23] In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of excluded middle.

3. *Stability Under Composition.* Let

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[24]

5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

   (a) The functor F is an equivalence of groupoids.

   (b) The following conditions are satisfied:

      i. The functor F induces a bijection

      $$\pi_0(F)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

      of sets.

      ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

      $$F_{x,x}\colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

      is an isomorphism of groups.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b:* Clear.

2. *Item 1b* $\implies$ *Item 1a:* Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B\colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j\colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B\colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta\colon \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

[24] More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint

3. *Item 1a* $\implies$ *Item 1c*: This follows from Item 4 of Definition 11.1.3.1.3.
4. *Item 1c* $\implies$ *Item 1a*: Omitted.
5. *Items 1a, 1d and 1e Are Equivalent:* This follows from **??**.

This finishes the proof of Item 1.
Item 2, Two-Out-of-Three: Omitted.
Item 3, Stability Under Composition: Clear.
Item 4, Equivalences vs.Adjoint Equivalences: See [Rie16, Proposition 4.4.5].
Item 5, Interaction With Groupoids: See [nLa25a, Proposition 4.4]. □

11.6.8 Isomorphisms of Categories

Definition 11.6.8.1.1. An **isomorphism of categories** is a pair of functors

$$F: \mathcal{C} \to \mathcal{D},$$
$$G: \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

Example 11.6.8.1.2. Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

Proposition 11.6.8.1.3. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:
   (a) The functor F is an isomorphism of categories.
   (b) The functor F is fully faithful and bijective on objects.
   (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
   $$F^*: \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   is an isomorphism of categories.

(d) For each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an isomorphism of categories.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Definition 11.6.7.1.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

11.7 More Conditions on Functors

11.7.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.1.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

Proposition 11.7.1.1.2. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I\colon \mathcal{X} \to \mathcal{C}$ be a functor.

1. *Interaction With Right Whiskering.* If I is full and dominant, then the map
$$-\star \mathrm{id}_I\colon\ \mathrm{Nat}(F,G)\to \mathrm{Nat}(F\circ I,G\circ I)$$
is a bijection.

2. *Interaction With Adjunctions.* Let $(F,G)\colon \mathcal{C}\rightleftarrows \mathcal{D}$ be an adjunction.

   (a) If F is dominant, then G is faithful.

   (b) The following conditions are equivalent:

      i. The functor G is full.

      ii. The restriction
$$\restriction G\mathrm{Im}_F\colon \mathrm{Im}(F)\to \mathcal{C}$$
      of G to $\mathrm{Im}(F)$ is full.

Proof. *Item 1, Interaction With Right Whiskering*: See [DFH75, Proposition 1.4].
Item 2, Interaction With Adjunctions: See [DFH75, Proposition 1.7]. □

Question 11.7.1.1.3. Is there a characterisation of functors $F\colon \mathcal{C}\to\mathcal{D}$ such that:

1. For each $\mathcal{X}\in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor
$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X})\to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$
is dominant?

2. For each $\mathcal{X}\in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor
$$F_*\colon \mathsf{Fun}(\mathcal{X},\mathcal{C})\to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$
is dominant?

This question also appears as [MO 468121a].

11.7.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

---

equivalences (F,G,η',ϵ) and (F,G,η,ϵ').

Definition 11.7.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**).

Proposition 11.7.2.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is a monomorphism of categories.

   (b) The functor F is injective on objects and morphisms, i.e. F is injective on objects and the map

   $$F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$$

   is injective.

Proof. *Item 1*, *Characterisations*: Omitted. □

Question 11.7.2.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is a monomorphism of categories?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

   $$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is a monomorphism of categories?

This question also appears as [MO 468121a].

11.7.3 Epimorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**).

Proposition 11.7.3.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:[25]

   (a) The functor F is a epimorphism of categories.

   (b) For each morphism $f\colon A \longrightarrow B$ of $\mathcal{D}$, we have a diagram

$$\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"'] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}$$

   in $\mathcal{D}$ satisfying the following conditions:

   i. We have $f = \alpha_0 \circ \phi_1$.

   ii. We have $f = \psi_m \circ \alpha_{2m}$.

   iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.

2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

Proof. *Item 1*, *Characterisations*: See [Isb68].
Item 2, *Surjectivity on Objects*: Omitted. □

Question 11.7.3.1.3. Is there a characterisation of functors $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \longrightarrow \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

   is an epimorphism of categories?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \longrightarrow \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

   is an epimorphism of categories?

This question also appears as [MO 468121a].

[25] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

11.7.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

   if we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

   then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

   there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

   such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Proposition 11.7.4.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

(a) The functor F is pseudomonic.

(b) The functor F satisfies the following conditions:

i. The functor F is faithful, i.e. for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is injective.

ii. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the restriction

$$F^{\mathrm{iso}}_{A,B} \colon \mathrm{Iso}_{\mathcal{C}}(A, B) \to \mathrm{Iso}_{\mathcal{D}}(F_A, F_B)$$

of the action on morphisms of F at (A, B) to isomorphisms is surjective.

(c) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{D}} \mathcal{C}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

in Cats_2 up to equivalence.

(d) We have an isocomma square of the form

$$\mathcal{C} \overset{\text{eq.}}{\cong} \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathsf{Arr}(\mathcal{D})} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] \arrow[dr, phantom, "\Leftrightarrow"] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathsf{Arr}(F)"] \\ \mathcal{D} \arrow[r, hook] & \mathsf{Arr}(\mathcal{D}) \end{tikzcd}$$

in Cats_2 up to equivalence.

(e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition[26] functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudomonic.

[26] Asking the precomposition functors

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

2. *Conservativity.* If F is pseudomonic, then F is conservative.

3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

Proof. *Item 1*, *Characterisations*: Omitted.
Item 2, *Conservativity*: Omitted.
Item 3, *Essential Injectivity*: Omitted. □

11.7.5 Pseudoepic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.7.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Definition 11.7.5.1.2.

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F .$$

Proposition 11.7.5.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:
   (a) The functor F is pseudoepic.
   (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor
   $$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$
   given by precomposition by F is pseudomonic.
   (c) We have an isococomma square of the form
   $$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] \arrow[ur, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$
   in Cats_2 up to equivalence.
2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 11.7.1.1.1).

Proof. Item 1, Characterisations: Omitted.
Item 2, Dominance: If F is pseudoepic, then

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 5g of Item 5 of Definition 11.6.1.1.2, this is equivalent to requiring F to be dominant. □

Question 11.7.5.1.3. Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Definition 11.7.4.1.2?

This question also appears as [MO 321971].

Question 11.7.5.1.4. A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?

This question also appears as [MO 468334].

Question 11.7.5.1.5. Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^{*}\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is pseudoepic?

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_{*}\colon \mathsf{Fun}(\mathcal{X},\mathcal{C}) \to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$

is pseudoepic?

This question also appears as [MO 468121a].

11.8 Even More Conditions on Functors

11.8.1 Injective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.1.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **injective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

Proposition 11.8.1.1.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Characterisations.* The following conditions are equivalent:

   (a) The functor F is injective on objects.

   (b) The functor F is an isocofibration in Cats_2.

Proof. *Item 1*, *Characterisations*: Omitted. □

11.8.2 Surjective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.2.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

11.8.3 Bijective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.3.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **bijective on objects**[27] if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

11.8.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.4.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 11.8.4.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

Question 11.8.4.1.3. Is there a characterisation of functors representably faithful on cores?

[27] *Further Terminology:* Also called a **bo** functor.

11.8.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.5.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{D}))$$

is full.

Remark 11.8.5.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F\circ\phi \overset{\sim}{\Longrightarrow} F\circ\psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 11.8.5.1.3. Is there a characterisation of functors representably full on cores?

This question also appears as [MO 468121a].

11.8.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.6.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faith-**

ful on cores if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_* \colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

Remark 11.8.6.1.2. In detail, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Definitions 11.8.4.1.2 and 11.8.5.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

   with α and β natural isomorphisms, if we have $\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta$, then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta \colon F \circ \phi \xRightarrow{\sim} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of $\mathcal{C}$, there exists a natural isomorphism

$$\alpha \colon \phi \xRightarrow{\sim} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

   of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

Question 11.8.6.1.3. Is there a characterisation of functors representably fully faithful on cores?

11.8.7 Functors Corepresentably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.7.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

Remark 11.8.7.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

Question 11.8.7.1.3. Is there a characterisation of functors corepresentably faithful on cores?

11.8.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.8.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

Remark 11.8.8.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F\,.$$

Question 11.8.8.1.3. Is there a characterisation of functors corepresentably full on cores?

This question also appears as [MO 468121a].

11.8.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 11.8.9.1.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X},\mathcal{D}))$$

is fully faithful.

Remark 11.8.9.1.2. In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Definitions 11.8.7.1.2 and 11.8.8.1.2, i.e.:

1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F .$$

Question 11.8.9.1.3. Is there a characterisation of functors corepresentably fully faithful on cores?

11.9 Natural Transformations

11.9.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 11.9.1.1.1. A **transformation**[28] $\alpha\colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

Notation 11.9.1.1.2. We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

Remark 11.9.1.1.3. We have an isomorphism

$$\mathrm{Trans}(F, G) \cong \prod_{A\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(F_A, G_A).$$

Proof. Clear. □

[28] *Further Terminology:* Also called an **unnatural transformation** for emphasis.

11.9.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 11.9.2.1.1. A **natural transformation** $\alpha \colon F \Rightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A \colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f \colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

commutes.

Remark 11.9.2.1.2. Let $\alpha \colon F \Rightarrow G$ be a natural transformation.

1. For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \to G_A$ is called the **component of** α **at** A.

2. We denote natural transformations such as α in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}. \end{tikzcd}$$

Notation 11.9.2.1.3. We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

Definition 11.9.2.1.4. Two natural transformations $\alpha, \beta \colon F \Rightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

11.9.3 Examples of Natural Transformations

Example 11.9.3.1.1. The **identity natural transformation** $\mathrm{id}_F \colon F \Rightarrow F$ **of** F is the natural transformation consisting of the collection

$$\{(\mathrm{id}_F)_A \colon F(A) \to F(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

defined by

$$(\mathrm{id}_F)_A \stackrel{\text{def}}{=} \mathrm{id}_{F(A)}$$

for each $A \in \mathrm{Obj}(C)$.

Proof. The naturality condition for id_F is the requirement that, for each morphism $f\colon A \to B$ of C, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\
F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

commutes. This follows from unitality of the composition of $\mathcal{D}$, as we have

$$\begin{aligned}
F(f) \circ \mathrm{id}_{F(A)} &= F(f) \\
&= \mathrm{id}_{F(B)} \circ F(f),
\end{aligned}$$

where we have applied unitality twice. □

Example 11.9.3.1.2. Let A and B be monoids and let $f, g\colon A \rightrightarrows B$ be morphisms of monoids. Applying the delooping construction of **??**, we obtain functors $\mathsf{B}f, \mathsf{B}g\colon \mathsf{B}A \rightrightarrows \mathsf{B}B$. We then have

$$\mathrm{Nat}(\mathsf{B}f, \mathsf{B}g) \cong \left\{ b \in B \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ we} \\ \text{have } bf(a) = g(a)b\end{array} \right\}.$$

Proof. Unwinding the definitions in this case, we see that a transformation α from $\mathsf{B}f$ to $\mathsf{B}g$ consists of a collection

$$\{\alpha_\bullet\colon \bullet \to \bullet\}_{\bullet \in \mathrm{Obj}(\mathsf{B}A)}$$

of morphisms of $\mathsf{B}B$ indexed by $\mathrm{Obj}(\mathsf{B}A)$. Since $\mathrm{Obj}(\mathsf{B}A) = \mathrm{pt}$ and the morphisms of $\mathsf{B}B$ are precisely the elements of B, it follows that α corresponds precisely to the data of an element $b \in B$. Now, a transformation $[b]\colon \mathsf{B}f \Rightarrow \mathsf{B}g$ is natural precisely if, for each $a \in \mathrm{Hom}_{\mathsf{B}A}(\bullet, \bullet) \stackrel{\text{def}}{=} A$, the diagram

$$\begin{tikzcd}
\mathsf{B}f(\bullet) \arrow[r, "\mathsf{B}f(a)"] \arrow[d, "{[b]_\bullet}"'] & \mathsf{B}f(\bullet) \arrow[d, "{[b]_\bullet}"] \\
\mathsf{B}g(\bullet) \arrow[r, "\mathsf{B}g(a)"'] & \mathsf{B}g(\bullet)
\end{tikzcd}$$

commutes. Unwinding the definitions, we see that this diagram is given by

$$\begin{tikzcd} \bullet \arrow[r, "f(a)"] \arrow[d, "b"'] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "g(a)"'] & \bullet, \end{tikzcd}$$

and hence corresponds precisely to the condition $g(a)b = bf(a)$. □

11.9.4 Vertical Composition of Natural Transformations

Definition 11.9.4.1.1. The **vertical composition** of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

is the natural transformation $\beta \circ \alpha\colon F \Longrightarrow H$ consisting of the collection

$$\left\{(\beta \circ \alpha)_A\colon F(A) \to H(A)\right\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \overset{\text{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof. The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\ H(A) \arrow[r, "H(f)"'] & H(B) \end{tikzcd}$$

commutes. Since

- Subdiagram (1) commutes by the naturality of α.
- Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. □

Proposition 11.9.4.1.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function

$$\circ_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$

2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
& \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Nat}(F,G) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Nat}(F,G)} \times [\mathrm{id}_F]"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G) \times \mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F_1"{name=U}] \arrow[r, "F_2"{description, name=M}] \arrow[r, bend right=50, "F_3"'{name=L}] & \mathcal{D} \arrow[r, bend left=50, "G_1"{name=U2}] \arrow[r, "G_2"{description, name=M2}] \arrow[r, bend right=50, "G_3"'{name=L2}] & \mathcal{E}
\arrow[from=U, to=M, Rightarrow, "\alpha"']
\arrow[from=M, to=L, Rightarrow, "\alpha'"']
\arrow[from=U2, to=M2, Rightarrow, "\beta"']
\arrow[from=M2, to=L2, Rightarrow, "\beta'"']
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1*, *Functionality*: Clear.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned}
((\gamma \circ \beta) \circ \alpha)_A &\overset{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\
&\overset{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\
&= \gamma_A \circ (\beta_A \circ \alpha_A)
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\
&\overset{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A
\end{aligned}
$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 3, *Unitality*: We have

$$
\begin{aligned}
(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A \\
&= \alpha_A, \\
(\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F \\
&= \alpha_A
\end{aligned}
$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, *Middle Four Exchange*: This is proved in Item 4 of Definition 11.9.5.1.3.

□

11.9.5 Horizontal Composition of Natural Transformations

Definition 11.9.5.1.1. The **horizontal composition**[29,30] of two natural transformations $\alpha \colon F \Longrightarrow G$ and $\beta \colon H \Longrightarrow K$ as in the diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E}
\end{tikzcd}
$$

of α and β is the natural transformation

$$
\beta \star \alpha \colon (H \circ F) \Longrightarrow (K \circ G),
$$

as in the diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\Downarrow\beta\star\alpha"] & \mathcal{E},
\end{tikzcd}
$$

[29] *Further Terminology:* Also called the **Godement product** of α and β.
[30] Horizontal composition forms a map

$$
\star_{(F,H),(G,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).
$$

consisting of the collection

$$\left\{(\beta \star \alpha)_A : H(F(A)) \to K(G(A))\right\}_{A \in \mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\stackrel{\text{def}}{=} \beta_{G(A)} \circ H(\alpha_A) \\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

Proof. First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A : F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd} H(F(A)) \arrow[r, "H(F(f))"] \arrow[d, "H(\alpha_A)"'] \arrow[dr, phantom, "(1)"] & H(F(B)) \arrow[d, "H(\alpha_B)"] \\ H(G(A)) \arrow[r, "H(G(f))" description] \arrow[d, "\beta_{G(A)}"'] \arrow[dr, phantom, "(2)"] & H(G(B)) \arrow[d, "\beta_{G(B)}"] \\ K(G(A)) \arrow[r, "K(G(f))"'] & K(G(B)) \end{tikzcd}$$

commutes. Since

- Subdiagram (1) commutes by the naturality of α.
- Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[31]

□

Definition 11.9.5.1.2. Let

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a diagram in Cats_2.

1. The **left whiskering of α with G** is the natural transformation[32]

$$\mathrm{id}_G \star \alpha \colon G \circ \phi \Longrightarrow G \circ \psi.$$

2. The **right whiskering of α with F** is the natural transformation[33]

$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

Proposition 11.9.5.1.3. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \longrightarrow \mathrm{Nat}(H \circ F, K \circ G).$$

2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd} \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\ \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1) \end{tikzcd}$$

[31] *Reference:* [Bor94, Proposition 1.3.4].

[32] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations in this work.

[33] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these notations in this work.

commutes, i.e. given natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma\Downarrow"] & \mathcal{F}, \end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

3. *Interaction With Identities.* Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G \circ F}\,.$$

4. *Middle Four Exchange.* Let $F_1, F_2, F_3\colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3\colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}[column sep=tiny] (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(G_1, G_2)) \times (\mathrm{Nat}(F_2, F_3) \times \mathrm{Nat}(F_1, F_2)) \arrow[rr, dashed, leftrightarrow, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2, G_3) \times \mathrm{Nat}(F_2, F_3)) \times (\mathrm{Nat}(G_1, G_2) \times \mathrm{Nat}(F_1, F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\ \mathrm{Nat}(G_1, G_3) \times \mathrm{Nat}(F_1, F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\ & \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) & \end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\alpha'\Downarrow"] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\beta\Downarrow"] \arrow[r, phantom, shift right=3, "\beta'\Downarrow"] & \mathcal{E} \end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1*, *Functionality*: Clear.
Item 2, *Associativity*: Omitted.
Item 3, *Interaction With Identities*: We have

$$
\begin{aligned}
(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
&\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \\
&\overset{\text{def}}{=} (\mathrm{id}_{G \circ F})_A
\end{aligned}
$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, *Middle Four Exchange*: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$
\begin{tikzcd}
& & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
& & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}
$$

The top composition

$$
\begin{tikzcd}
& & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
& & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}
$$

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1(\alpha'_A)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2(\alpha'_A)"'] & &
\end{tikzcd}$$

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad \begin{tikzcd}
G_1(F_2(A)) \arrow[r, "G_1(\alpha'_A)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\
G_2(F_2(A)) \arrow[r, "G_2(\alpha'_A)"'] & G_2(F_3(A))
\end{tikzcd}$$

for $\beta\colon G_1 \Longrightarrow G_2$ at $\alpha'_A\colon F_2(A) \to F_3(A)$, and thus commutes. Thus we have

$$((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A = ((\beta' \star \alpha') \circ (\beta \star \alpha))_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof. $\square$

11.9.6 Properties of Natural Transformations

Proposition 11.9.6.1.1. Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[34]

1. A natural transformation $\alpha\colon F \Longrightarrow G$.

[34] Taken from [MO 64365].

2. A functor $[\alpha] \colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$
\begin{tikzcd}
& \mathcal{D} \\
\mathcal{C} \arrow[ur, "F"] \arrow[dr, "G"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}^{\mathbb{1}}. \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\
& \mathcal{D}
\end{tikzcd}
$$

3. A functor $[\alpha] \colon \mathcal{C} \times \mathbb{1} \to \mathcal{D}$ filling the diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[dr, "F"] & \\
\mathcal{C} \times \mathbb{1} \arrow[u, "\mathrm{ev}_0"] \arrow[d, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\
\mathcal{C} \arrow[ur, "G"'] &
\end{tikzcd}
$$

Proof. From Item 1 to Item 2 and Back: We may identify $\mathcal{D}^{\mathbb{1}}$ with $\mathrm{Arr}(\mathcal{D})$. Given a natural transformation $\alpha \colon F \Longrightarrow G$, we have a functor

$$
\begin{aligned}
[\alpha] \colon \mathcal{C} &\longrightarrow \mathcal{D}^{\mathbb{1}} \\
A &\longmapsto \alpha_A \\
(f \colon A \to B) &\longmapsto \left(
\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\
G_A \arrow[r, "G_f"'] & G_B
\end{tikzcd}
\right)
\end{aligned}
$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back: This follows from Item 3 of Definition 11.10.1.1.2.

□

11.9.7 Natural Isomorphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 11.9.7.1.1. A natural transformation $\alpha \colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1} \colon G \Longrightarrow F$ such that

$$\begin{aligned} \alpha^{-1} \circ \alpha &= \mathrm{id}_F, \\ \alpha \circ \alpha^{-1} &= \mathrm{id}_G . \end{aligned}$$

Proposition 11.9.7.1.2. Let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

1. *Characterisations.* The following conditions are equivalent:
   (a) The natural transformation α is a natural isomorphism.
   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \to G_A$ is an isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations.* Let $\alpha^{-1} \colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} \alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)}, \\ \alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)} . \end{aligned}$$

   Then α^{-1} is a natural transformation.

Proof. Item 1, *Characterisations*: The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2.

Item 2, *Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations*: The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the

diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B), \end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. □

11.10 Categories of Categories

11.10.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

Definition 11.10.1.1.1. The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[35] is the category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$[36] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$.
- *Morphisms.* For each $F, G \in \mathsf{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, we have

$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F, G) \stackrel{\text{def}}{=} \mathrm{Nat}(F, G).$$

[35] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.
[36] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C}, \mathcal{D}]$.

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the unit map
$$\mathbb{1}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \colon \mathrm{pt} \longrightarrow \mathrm{Nat}(F, F)$$
of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at F is given by
$$\mathrm{id}_F^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \overset{\mathrm{def}}{=} \mathrm{id}_F,$$
where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Definition 11.9.3.1.1.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D}))$, the composition map
$$\circ_{F,G,H}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \longrightarrow \mathrm{Nat}(F, H)$$
of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ at (F, G, H) is given by
$$\beta \circ_{F,G,H}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})} \alpha \overset{\mathrm{def}}{=} \beta \circ \alpha,$$
where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Definition 11.9.4.1.2.

Proposition 11.10.1.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define functors
$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -) \colon &\quad \mathsf{Cats} \longrightarrow \mathsf{Cats}, \\ \mathsf{Fun}(-, \mathcal{D}) \colon &\quad \mathsf{Cats}^{\mathsf{op}} \longrightarrow \mathsf{Cats}, \\ \mathsf{Fun}(-_1, -_2) \colon &\quad \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \longrightarrow \mathsf{Cats}. \end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C}, \mathcal{D})$ define 2-functors
$$\begin{aligned} \mathsf{Fun}(\mathcal{C}, -) \colon &\quad \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2, \\ \mathsf{Fun}(-, \mathcal{D}) \colon &\quad \mathsf{Cats}_2^{\mathsf{op}} \longrightarrow \mathsf{Cats}_2, \\ \mathsf{Fun}(-_1, -_2) \colon &\quad \mathsf{Cats}_2^{\mathsf{op}} \times \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2. \end{aligned}$$

3. *Adjointness.* We have adjunctions
$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$
$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd},$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -)) \colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),\\ \mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),\end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

6. *Objectwise Computation of Co/Limits.* Let

$$D \colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\begin{aligned}\lim(D)_A &\cong \lim_{i \in \mathcal{I}}(D_i(A)),\\ \operatorname{colim}(D)_A &\cong \operatorname*{colim}_{i \in \mathcal{I}}(D_i(A)),\end{aligned}$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

8. *Monomorphisms and Epimorphisms.* Let $\alpha\colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

   (a) The natural transformation

   $$\alpha\colon F \Longrightarrow G$$

   is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

   (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

   $$\alpha_A\colon F_A \to G_A$$

   is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Punctual Categories: Omitted.
Item 6, Objectwise Computation of Co/Limits: Omitted.
Item 7, Interaction With Co/Completeness: This follows from **??**.
Item 8, Monomorphisms and Epimorphisms: Omitted. □

11.10.2 The Category of Categories and Functors

Definition 11.10.2.1.1. The **category of (small) categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

  $$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \overset{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

  $$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

  of Cats at $\mathcal{C}$ is defined by

  $$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

  where $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Definition 11.5.1.1.4.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

  $$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

  $$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 11.5.1.1.5.

Proposition 11.10.2.1.2. Let $\mathcal{C}$ be a category.

1. *Co/Completeness.* The category Cats is complete and cocomplete.
2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathrm{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof. *Item 1, Co/Completeness*: Omitted.
Item 2, Cartesian Monoidal Structure: Omitted. □

11.10.3 The 2-Category of Categories, Functors, and Natural Transformations

Definition 11.10.3.1.1. The **2-category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.
- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

  $$\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

  $$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathrm{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

  $$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

- *Action on Objects.* For each object $(G, F) \in \mathrm{Obj}\big(\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})\big)$, we have
$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F.$$
- *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$ of $\mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have
$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$
where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 11.9.5.1.1.

Proposition 11.10.3.1.2. Let $\mathcal{C}$ be a category.

1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof. Item 1, Co/Completeness: Omitted. □

11.10.4 The Category of Groupoids

Definition 11.10.4.1.1. The **category of (small) groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

11.10.5 The 2-Category of Groupoids

Definition 11.10.5.1.1. The 2-**category of (small) groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

11.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Chapter 12

Presheaves and the Yoneda Lemma

This chapter contains some material about presheaves and the Yoneda lemma.

This chapter is under revision. TODO:

1. Subsection properties of categories of copresheaves
2. Adjointness of tensor product of functors
3. Limit of category of elements (instead of colimit)
4. Category of elements where objects are natural transformations $\mathcal{F} \Rightarrow h_X$ instead of the other way around. Is this related to Isbell duality?
5. Motivate the proof of the Yoneda lemma as in Martin's comment here: https://mathoverflow.net/questions/130883/are-there-proofs-that-you-feel-you-did-not-understand-for-a-long-time#comment360113_131050
6. Add discussion of universal properties
7. Add $h_{g\circ f} = h_g \circ h_f$ to properties of representable natural transformations

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12.1 Presheaves

12.1.1 Foundations

Let C be a category.

Definition 12.1.1.1.1. A **presheaf on** C is a functor $\mathcal{F}: C^{\mathsf{op}} \to \mathsf{Sets}$.

Example 12.1.1.1.2. Presheaves on the delooping $\mathsf{B}A$ of a monoid A are precisely the left A-sets; see **??**.

Definition 12.1.1.1.3. A **morphism of presheaves** on C from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation $\alpha\colon \mathcal{F} \Rightarrow \mathcal{G}$.

Definition 12.1.1.1.4. The **category of presheaves on** $\mathcal{C}$ is the category $\mathsf{PSh}(\mathcal{C})$[1] defined by

$$\mathsf{PSh}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}\Big(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets}\Big).$$

Remark 12.1.1.1.5. In detail, the **category of presheaves on** $\mathcal{C}$ is the category $\mathsf{PSh}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{PSh}(\mathcal{C})$ are presheaves on $\mathcal{C}$ as in Definition 12.1.1.1.1.
- *Morphisms.* The morphisms of $\mathsf{PSh}(\mathcal{C})$ are morphisms of presheaves as in Definition 12.1.1.1.3, i.e. we have

  $$\mathrm{Hom}_{\mathsf{PSh}(\mathcal{C})}(\mathcal{F}, \mathcal{G}) \overset{\text{def}}{=} \mathrm{Nat}(\mathcal{F}, \mathcal{G})$$

  for each $\mathcal{F}, \mathcal{G} \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$.
- *Identities.* For each $\mathcal{F} \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$, the unit map

  $$\mathbb{1}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}} \colon \mathrm{pt} \longrightarrow \mathrm{Nat}(\mathcal{F}, \mathcal{F})$$

  of $\mathsf{PSh}(\mathcal{C})$ at $\mathcal{F}$ is defined by

  $$\mathrm{id}^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F}} \overset{\text{def}}{=} \mathrm{id}_{\mathcal{F}},$$

  where $\mathrm{id}_{\mathcal{F}} \colon \mathcal{F} \Longrightarrow \mathcal{F}$ is the identity natural transformation of Definition 11.9.3.1.1.
- *Composition.* For each $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathsf{Obj}(\mathsf{PSh}(\mathcal{C}))$, the composition map

  $$\circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} \colon \mathrm{Nat}(\mathcal{G}, \mathcal{H}) \times \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Nat}(\mathcal{F}, \mathcal{H})$$

  of $\mathsf{PSh}(\mathcal{C})$ at $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined by

  $$\beta \circ^{\mathsf{PSh}(\mathcal{C})}_{\mathcal{F},\mathcal{G},\mathcal{H}} \alpha \overset{\text{def}}{=} \beta \circ \alpha,$$

  where $\beta \circ \alpha \colon \mathcal{F} \Longrightarrow \mathcal{H}$ is the vertical composition of α and β of Definition 11.9.4.1.1.

[1] *Further Notation:* Also written $\widehat{\mathcal{C}}$ in some parts of the literature.

12.1.2 Representable Presheaves

Let C be a category.

Definition 12.1.2.1.1. Let $A \in \mathrm{Obj}(C)$.

1. The **representable presheaf associated to** A is the presheaf

$$h_A\colon C^{\mathrm{op}} \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(C)$, we have

$$h_A(X) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_C(X, A).$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(C)$, the action on morphisms

$$h_{A|X,Y}\colon \mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_{\mathsf{Sets}}(h_A(Y), h_A(X))$$

of h_A at (X, Y) is given by sending a morphism

$$f\colon X \to Y$$

of C to the map of sets

$$h_A(f)\colon \underbrace{h_A(Y)}_{\stackrel{\mathrm{def}}{=}\mathrm{Hom}_C(Y,A)} \to \underbrace{h_A(X)}_{\stackrel{\mathrm{def}}{=}\mathrm{Hom}_C(X,A)}$$

defined by

$$h_A(f) \stackrel{\mathrm{def}}{=} f^*,$$

where f^* is the precomposition by f morphism of Item 1 of Definition 11.1.4.1.1.

2. A **representing object** for a presheaf $\mathcal{F}\colon C^{\mathrm{op}} \to \mathsf{Sets}$ on C is an object A of C such that we have $\mathcal{F} \cong h_A$.

3. A presheaf $\mathcal{F}\colon C^{\mathrm{op}} \to \mathsf{Sets}$ on C is **representable** if $\mathcal{F}$ admits a representing object.

Example 12.1.2.1.2. The representable presheaf on the delooping $\mathrm{B}A$ of a monoid A associated to the unique object $\bullet$ of $\mathrm{B}A$ is the left regular representation of A of **??**.

Proposition 12.1.2.1.3. Let $\mathcal{F} : C^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf. If there exist $A, B \in \mathrm{Obj}(C)$ such that we have natural isomorphisms

$$h_A \cong \mathcal{F},$$
$$h_B \cong \mathcal{F},$$

then $A \cong B$.

Proof. By composing the isomorphisms $h_A \cong \mathcal{F} \cong h_B$, we get a natural isomorphism $h_A \cong h_B$. By Item 2 of Definition 12.1.4.1.3, we have $A \cong B$. □

12.1.3 Representable Natural Transformations

Let C be a category, let $A, B \in \mathrm{Obj}(C)$, and let $f : A \to B$ be a morphism of C.

Definition 12.1.3.1.1. The **representable natural transformation associated to** f is the natural transformation

$$h_f : h_A \Rightarrow h_B$$

consisting of the collection

$$\left\{ h_{f|X} : \underbrace{h_A(X)}_{\overset{\text{def}}{=} \mathrm{Hom}_C(X,A)} \to \underbrace{h_B(X)}_{\overset{\text{def}}{=} \mathrm{Hom}_C(X,B)} \right\}_{X \in \mathrm{Obj}(C)}$$

with

$$h_{f|X} \overset{\text{def}}{=} f_*,$$

where f_* is the postcomposition by f morphism of Item 2 of Definition 11.1.4.1.1.

12.1.4 The Yoneda Embedding

Definition 12.1.4.1.1. The **Yoneda embedding of** C[2] is the functor[3]

$$よ_C : C \hookrightarrow \mathsf{PSh}(C)$$

where

[2] *Further Terminology:* Also called the **covariant Yoneda embedding** to distinguish it from the contravariant Yoneda embedding of Definition 12.2.5.1.1.

[3] *Further Notation:* Also written $h_{(-)}$, or simply よ.

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$よ_{\mathcal{C}}(A) \overset{\text{def}}{=} h_A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$よ_{\mathcal{C}|A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}(h_A, h_B)$$

of $よ_{\mathcal{C}}$ at (A, B) is given by

$$よ_{\mathcal{C}|A,B}(f) \overset{\text{def}}{=} h_f$$

for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, where h_f is the representable natural transformation associated to f of Definition 12.1.3.1.1.

Remark 12.1.4.1.2. The notation よ for the Yoneda embedding was first introduced in [JS17]. The symbol よ is the hiragana for *yo*, and comes from "Yoneda" in Nobuo Yoneda (米田信夫).

It is pronounced *yo* but without letting the "o" in *yo* sound like an o-u diphthong:

- See here.
- IPA transcription: [jo̞].

Proposition 12.1.4.1.3. Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The Yoneda embedding

$$よ_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

is fully faithful.

2. *Preservation and Reflection of Isomorphisms.* The Yoneda embedding

$$よ_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

preserves and reflects isomorphisms, i.e. given $A, B \in \mathrm{Obj}(\mathcal{C})$, the following conditions are equivalent:

(a) We have $A \cong B$.

(b) We have $h_A \cong h_B$.

3. *Density*. The Yoneda embedding

$$\text{よ}_{\mathcal{C}} \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

is dense.

4. *Interaction With Density Comonads*. We have

$$\mathrm{Lan}_{\text{よ}}(\text{よ}) \cong \mathrm{id}_{\mathsf{PSh}(\mathcal{C})}, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}}(\text{よ})"] \\ \mathcal{C} \arrow[ur, "\text{よ}_{\mathcal{C}}"] \arrow[r, "\text{よ}_{\mathcal{C}}"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathsf{PSh}(\mathcal{C}). \end{tikzcd}$$

5. *Interaction With Codensity Monads*. We have

$$\mathrm{Ran}_{\text{よ}}(\text{よ}) \cong \mathrm{Spec} \circ \mathrm{O},$$

where Spec and O are the functors of **??**.

Proof. *Item 1*, *Fully Faithfulness*: Let $A, B \in \mathrm{Obj}(\mathcal{C})$. Applying the Yoneda lemma (Definition 12.1.5.1.1) to the functor h_B (i.e. in the case $\mathcal{F} = h_B$), we have

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \cong \mathrm{Nat}(h_A, h_B),$$

and the natural isomorphism

$$\xi_{A,B} \colon h_B(A) \Rightarrow \mathrm{Nat}(h_A, h_B)$$

witnessing this bijection is given by

$$\begin{aligned} \xi_{A,B}(g)_X &\stackrel{\mathrm{def}}{=} h^X_g \\ &\stackrel{\mathrm{def}}{=} g_* \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathcal{C})$ and each $g \in h^X_B$, i.e. we have $\xi_{A,B} = \text{よ}_{\mathcal{C}|A,B}$. Thus $\text{よ}_{\mathcal{C}}$ is fully faithful.

Item 2, *Preservation and Reflection of Isomorphisms*: This follows from Item 1 of Definition 11.5.1.1.6 and Item 3 of Definition 11.6.3.1.2.

Item 3, *Density*: Omitted.

Item 4, *Interaction With Density Comonads*: Omitted.

Item 5, *Interaction With Codensity Monads*: Omitted. □

12.1.5 The Yoneda Lemma

Let $\mathcal{F} \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$.

Theorem 12.1.5.1.1. We have a bijection

$$\mathrm{Nat}(h_A, \mathcal{F}) \cong \mathcal{F}(A),$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big) \cong \mathcal{F}.$$

Proof. The Transformation $ev\colon$ $Nat\big(h_{(-)}, \mathcal{F}\big) \Rightarrow \mathcal{F}$: Let

$$\mathrm{ev}\colon \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big) \Rightarrow \mathcal{F}$$

be the transformation consisting of the collection

$$\{\mathrm{ev}_A\colon \mathrm{Nat}(h_A, \mathcal{F}) \to \mathcal{F}(A)\}_{A\in\mathrm{Obj}(\mathcal{C})}$$

with

$$\mathrm{ev}_A(\alpha) = \alpha_A(\mathrm{id}_A)$$

for each $\alpha \in \mathrm{Nat}(h_A, \mathcal{F})$, where α_A is the component

$$\alpha_A\colon \mathrm{Hom}_{\mathcal{C}}(A, A) \to \mathcal{F}(A)$$

of α at A.

The Transformation $\xi\colon \mathcal{F} \Rightarrow Nat\big(h_{(-)}, \mathcal{F}\big)$: Let

$$\xi\colon \mathcal{F} \Rightarrow \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big)$$

be the transformation consisting of the collection

$$\{\xi_A\colon \mathcal{F}(A) \to \mathrm{Nat}(h_A, \mathcal{F})\}_{A\in\mathrm{Obj}(\mathcal{C})},$$

where ξ_A is the map sending an element $\phi \in \mathcal{F}(A)$ to the transformation

$$\xi_A(\phi)\colon h_A \Rightarrow \mathcal{F}$$

(which we will show is natural in a bit) consisting of the collection

$$\big\{\xi_A(\phi)_X\colon h_A(X) \to \mathcal{F}(X)\big\}_{X\in\mathrm{Obj}(\mathcal{C})},$$

with

$$\xi_A(\phi)_X(f) \stackrel{\mathrm{def}}{=} [\mathcal{F}(f)](\phi)$$

for each $f \in h_A(X)$, where

$$\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(X)$$

is the image of f by $\mathcal{F}$.

Naturality of $\xi_A(\phi)\colon h_A \Rightarrow \mathcal{F}$: The transformation

$$\xi_A(\phi)\colon h_A \Rightarrow \mathcal{F}$$

is indeed natural, as the diagram

$$\begin{tikzcd} h_A^Y \arrow[r, "f^*"] \arrow[d, "\xi_A(\phi)_Y"'] & h_A^X \arrow[d, "\xi_A(\phi)_X"] \\ \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

commutes for each morphism $f\colon X \to Y$ of C, acting on elements as

$$\begin{tikzcd} h \arrow[d, mapsto] & & h \arrow[r, mapsto] & h \circ f \arrow[d, mapsto] \\ {[\mathcal{F}(h)](\phi)} \arrow[r, mapsto] & {[\mathcal{F}(f)]([\mathcal{F}(h)](\phi))} & & {[\mathcal{F}(h \circ f)(\phi)],} \end{tikzcd}$$

where we have

$$[\mathcal{F}(f)]([\mathcal{F}(h)](\phi)) = [\mathcal{F}(h \circ f)(\phi)]$$

by the functoriality of $\mathcal{F}$.

Naturality of ev: $\mathit{Nat}\big(h_{(-)}, \mathcal{F}\big) \Rightarrow \mathcal{F}$: Let $f\colon X \to Y$ be a morphism of C. We claim the naturality diagram

$$\begin{tikzcd} \mathrm{Nat}(h_Y, \mathcal{F}) \arrow[r, "(h_f)^*"] \arrow[d, "\mathrm{ev}_Y"'] & \mathrm{Nat}(h_X, \mathcal{F}) \arrow[d, "\mathrm{ev}_X"] \\ \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

for ev at f, acting on elements as

$$\begin{tikzcd} \alpha \arrow[d, mapsto] & & \alpha \arrow[r, mapsto] & \alpha \circ h_f \arrow[d, mapsto] \\ \alpha_Y(\mathrm{id}_Y) \arrow[r, mapsto] & {[\mathcal{F}(f)](\alpha_Y(\mathrm{id}_Y))} & & {\big[\alpha \circ h_f\big]_X(\mathrm{id}_X),} \end{tikzcd}$$

commutes. Indeed:

- We have

$$\begin{aligned}\left[\alpha \circ h_f\right]_X(\mathrm{id}_X) &\stackrel{\text{def}}{=} \left[\alpha_X \circ h_{f|X}\right](\mathrm{id}_X)\\ &\stackrel{\text{def}}{=} [\alpha_X \circ f_*](\mathrm{id}_X)\\ &\stackrel{\text{def}}{=} \alpha_X(f_*(\mathrm{id}_X))\\ &\stackrel{\text{def}}{=} \alpha_X(f).\end{aligned}$$

- Applying the naturality diagram

$$\begin{tikzcd} h^Y_Y \arrow[r, "f^*"] \arrow[d, "\alpha_Y"'] & h^X_Y \arrow[d, "\alpha_X"] \\ \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

of $\alpha\colon h_Y \Rightarrow \mathcal{F}$ at $f\colon X \to Y$ to the element id_Y of h^Y_Y, we have

$$\begin{tikzcd} \mathrm{id}_Y \arrow[d, mapsto] & & \mathrm{id}_Y \arrow[r, mapsto] & f \arrow[d, mapsto] \\ \alpha_Y(\mathrm{id}_Y) \arrow[r, mapsto] & {[\mathcal{F}(f)](\alpha_Y(\mathrm{id}_Y))} & & \alpha_X(f), \end{tikzcd}$$

showing that we have

$$[\mathcal{F}(f)](\alpha_Y(\mathrm{id}_Y)) = \alpha_X(f).$$

Thus the naturality diagram for ev at f commutes, and ev is natural.
Naturality of $\xi\colon \mathcal{F} \Rightarrow \mathrm{Nat}\big(h_{(-)}, \mathcal{F}\big)$: Let $f\colon X \to Y$ be a morphism of C. We claim the naturality diagram

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\xi_Y"'] & \mathcal{F}(X) \arrow[d, "\xi_X"] \\ \mathrm{Nat}(h_Y, \mathcal{F}) \arrow[r, "(h_f)^*"'] & \mathrm{Nat}(h_X, \mathcal{F}) \end{tikzcd}$$

for ξ at f, acting on elements as

$$\begin{tikzcd} \phi \arrow[d, mapsto] & & \phi \arrow[r, mapsto] & {[\mathcal{F}(f)](\phi)} \arrow[d, mapsto] \\ \xi_Y(\phi) \arrow[r, mapsto] & \xi_Y(\phi) \circ h_f & & \xi_X([\mathcal{F}(f)](\phi)) \end{tikzcd}$$

commutes. Indeed, for each $X \in \mathrm{Obj}(C)$ and each $g \in h^A_X$, we have

$$\begin{aligned}
\left[\xi_Y(\phi) \circ h_f\right]_X(g) &\stackrel{\text{def}}{=} \left[\xi_Y(\phi)_X \circ h_{f|X}\right](g)\\
&\stackrel{\text{def}}{=} \left[\xi_Y(\phi)_X \circ f_*\right](g)\\
&\stackrel{\text{def}}{=} \xi_Y(\phi)_X(f_*(g))\\
&\stackrel{\text{def}}{=} \xi_Y(\phi)_X(f \circ g)\\
&\stackrel{\text{def}}{=} [\mathcal{F}(f \circ g)](\phi)
\end{aligned}$$

and

$$\begin{aligned}
[\xi_X([\mathcal{F}(f)](\phi))]_X(g) &\stackrel{\text{def}}{=} \mathcal{F}(g)([\mathcal{F}(f)](\phi))\\
&= [\mathcal{F}(f \circ g)](\phi),
\end{aligned}$$

where we have used the functoriality of $\mathcal{F}$. Thus $\xi_Y(\phi) \circ h_f$ and $\xi_X([\mathcal{F}(f)](\phi))$ are equal, and the naturality diagram for ξ at f above commutes, showing ξ to be natural.

Invertibility I: $ev \circ \xi = id_{\mathcal{F}}$: We claim that $\mathrm{ev} \circ \xi = \mathrm{id}_{\mathcal{F}}$, i.e. that we have

$$(\mathrm{ev} \circ \xi)_A = \mathrm{id}_{\mathcal{F}(A)}$$

for each $A \in \mathrm{Obj}(C)$. Indeed, we have

$$\begin{aligned}
[\mathrm{ev} \circ \xi]_A(\phi) &\stackrel{\text{def}}{=} [\mathrm{ev}_A \circ \xi_A](\phi)\\
&\stackrel{\text{def}}{=} \mathrm{ev}_A(\xi_A(\phi))\\
&\stackrel{\text{def}}{=} \xi_A(\phi)_A(\mathrm{id}_A)\\
&\stackrel{\text{def}}{=} [\mathcal{F}(\mathrm{id}_A)](\phi)\\
&= \left[\mathrm{id}_{\mathcal{F}(A)}\right](\phi)
\end{aligned}$$

for each $\phi \in \mathcal{F}(A)$.

Invertibility II: $\xi \circ ev = id_{\mathrm{Nat}(h_{(-)},\mathcal{F})}$: We claim that $\xi \circ \mathrm{ev} = \mathrm{id}_{\mathrm{Nat}(h_{(-)},\mathcal{F})}$, i.e. that we have

$$(\xi \circ \mathrm{ev})_A = \mathrm{id}_{\mathrm{Nat}(h_A,\mathcal{F})}$$

for each $A \in \mathrm{Obj}(C)$. Indeed:

- We have

$$\begin{aligned}
[\xi \circ \mathrm{ev}]_A(\alpha) &\stackrel{\text{def}}{=} [\xi_A \circ \mathrm{ev}_A](\alpha)\\
&\stackrel{\text{def}}{=} \xi_A(\mathrm{ev}_A(\alpha))\\
&\stackrel{\text{def}}{=} \xi_A(\alpha_A(\mathrm{id}_A))
\end{aligned}$$

  for each $\alpha \in \mathrm{Nat}(h_A, \mathcal{F})$.

- For each $X \in \mathrm{Obj}(\mathcal{C})$, we have

  $$\xi_A(\alpha_A(\mathrm{id}_A))_X = \alpha_X,$$

  since we have

  $$\begin{aligned}\xi_A(\alpha_A(\mathrm{id}_A))_X(f) &\overset{\text{def}}{=} [\mathcal{F}(f)](\alpha_A(\mathrm{id}_A))\\ &\overset{(\dagger)}{=} \alpha_X(f)\end{aligned}$$

  for each $f \in h_A(X)$, where the equality marked with $(\dagger)$ follows from the commutativity of the naturality diagram

  $$\begin{tikzcd} h^A_A \arrow[r, "f_*"] \arrow[d, "\alpha_A"'] & h^A_X \arrow[d, "\alpha_X"] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"'] & \mathcal{F}(X) \end{tikzcd}$$

  of α at $f\colon A \to X$, which acts on id_A as

  $$\begin{tikzcd} \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\ \alpha_A(\mathrm{id}_A) \arrow[r, mapsto] & {[\mathcal{F}(f)](\alpha_A(\mathrm{id}_A)) = \alpha_X(f).} \end{tikzcd}$$

This finishes the proof. □

12.1.6 Properties of Categories of Presheaves

Proposition 12.1.6.1.1. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{PSh}(\mathcal{C})$ defines a functor

   $$\mathsf{PSh}\colon \mathsf{Cats} \to \mathsf{Cats}$$

   up to some set-theoretic considerations.[4]

[4]For instance:

- The Cats in the source of PSh could be small categories, and then the Cats in the right would be locally small categories.

2. *Interaction With Slice Categories.* Let $X \in \mathrm{Obj}(C)$. We have an equivalence of categories

$$\mathsf{PSh}(C_{/X}) \overset{\text{eq.}}{\cong} \mathsf{PSh}(C)_{/h_X}.$$

3. *Interaction With Categories of Elements.* Let $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(C))$. We have an equivalence of categories

$$\mathsf{PSh}\left(\int_C \mathcal{F}\right) \overset{\text{eq.}}{\cong} \mathsf{PSh}(C)_{/\mathcal{F}}.$$

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Interaction With Slice Categories*: Omitted.
Item 3, *Interaction With Categories of Elements*: Omitted. □

12.2 Copresheaves

12.2.1 Foundations

Let C be a category.

Definition 12.2.1.1.1. A **copresheaf on** C is a functor $F : C \to \mathsf{Sets}$.

Example 12.2.1.1.2. Copresheaves on the delooping $\mathbf{B}A$ of a monoid A are precisely the right A-sets; see **??**.

Definition 12.2.1.1.3. A **morphism of copresheaves** on C from F to G is a natural transformation $\alpha : F \Rightarrow G$.

Definition 12.2.1.1.4. The **category of copresheaves on** C is the category $\mathsf{CoPSh}(C)$ defined by

$$\mathsf{CoPSh}(C) \overset{\text{def}}{=} \mathsf{Fun}(C, \mathsf{Sets}).$$

Remark 12.2.1.1.5. In detail, the **category of copresheaves on** C is the category $\mathsf{CoPSh}(C)$ where

- *Objects.* The objects of $\mathsf{CoPSh}(C)$ are copresheaves on C as in Definition 12.2.1.1.1.

- The Cats in the source of PSh could be locally small categories, and then the Cats on the right would be large categories.

- *Morphisms.* The morphisms of $\mathsf{CoPSh}(\mathcal{C})$ are morphisms of copresheaves as in Definition 12.2.1.1.3, i.e. we have
$$\mathrm{Hom}_{\mathsf{CoPSh}(\mathcal{C})}(F,G) \stackrel{\text{def}}{=} \mathrm{Nat}(F,G)$$
for each $F, G \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$.

- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the unit map
$$\mathbb{1}_F^{\mathsf{CoPSh}(\mathcal{C})}\colon \mathrm{pt} \longrightarrow \mathrm{Nat}(F,F)$$
of $\mathsf{CoPSh}(\mathcal{C})$ at F is defined by
$$\mathrm{id}_F^{\mathsf{CoPSh}(\mathcal{C})} \stackrel{\text{def}}{=} \mathrm{id}_F,$$
where $\mathrm{id}_F\colon F \Longrightarrow F$ is the identity natural transformation of Definition 11.9.3.1.1.

- *Composition.* For each $F, G, H \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$, the composition map
$$\circ_{F,G,H}^{\mathsf{CoPSh}(\mathcal{C})}\colon \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \longrightarrow \mathrm{Nat}(F,H)$$
of $\mathsf{CoPSh}(\mathcal{C})$ at (F,G,H) is defined by
$$\beta \circ_{F,G,H}^{\mathsf{CoPSh}(\mathcal{C})} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$
where $\beta \circ \alpha\colon F \Longrightarrow H$ is the vertical composition of α and β of Definition 11.9.4.1.1.

12.2.2 Corepresentable Copresheaves

Let $\mathcal{C}$ be a category.

Definition 12.2.2.1.1. Let $A \in \mathrm{Obj}(\mathcal{C})$.

1. The **corepresentable copresheaf associated to** A is the copresheaf
$$h^A\colon \mathcal{C} \to \mathsf{Sets}$$
where

In general, one can systematise and formalise this using Grothendieck universes.

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathcal{C})$, we have

$$h^A(X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, X).$$

- *Action on Morphisms.* For each $X, Y \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$h^A_{X,Y} \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{Sets}}\Big(h^A(X), h^A(Y)\Big)$$

of h^A at (X, Y) is given by sending a morphism

$$f \colon X \to Y$$

of $\mathcal{C}$ to the map of sets

$$h^A(f) \colon \underbrace{h^A(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,X)} \to \underbrace{h^A(Y)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A,Y)}$$

defined by

$$h^A(f) \stackrel{\text{def}}{=} f_*,$$

where f_* is the postcomposition by f morphism of Item 2 of Definition 11.1.4.1.1.

2. A **corepresenting object** for a copresheaf $F \colon \mathcal{C} \to \mathsf{Sets}$ on $\mathcal{C}$ is an object A of $\mathcal{C}$ such that we have $F \cong h^A$.

3. A copresheaf $F \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ on $\mathcal{C}$ is **corepresentable** if F admits a corepresenting object.

Example 12.2.2.1.2. The corepresentable copresheaf on the delooping $\mathsf{B}A$ of a monoid A associated to the unique object $\bullet$ of $\mathsf{B}A$ is the right regular representation of A of **??**.

Proposition 12.2.2.1.3. Let $F \colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf. If there exist $A, B \in \mathrm{Obj}(\mathcal{C})$ such that we have natural isomorphisms

$$h^A \cong F,$$
$$h^B \cong F,$$

then $A \cong B$.

Proof. By composing the isomorphisms $h^A \cong F \cong h^B$, we get a natural isomorphism $h^A \cong h^B$. By Item 2 of Definition 12.2.4.1.2, we have $A \cong B$. $\square$

12.2.3 Corepresentable Natural Transformations

Let $\mathcal{C}$ be a category, let $A, B \in \mathrm{Obj}(\mathcal{C})$, and let $f\colon A \to B$ be a morphism of $\mathcal{C}$.

Definition 12.2.3.1.1. The **corepresentable natural transformation associated to** f is the natural transformation

$$h^f\colon h^B \Rightarrow h^A$$

consisting of the collection

$$\left\{ h^f_X\colon \underbrace{h^B(X)}_{\stackrel{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(B,X)} \to \underbrace{h^A(X)}_{\stackrel{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,X)} \right\}_{X\in\mathrm{Obj}(\mathcal{C})}$$

with

$$h^f_X \stackrel{\mathrm{def}}{=} f^*,$$

where f_* is the precomposition by f morphism of Item 1 of Definition 11.1.4.1.1.

12.2.4 The Contravariant Yoneda Embedding

Definition 12.2.4.1.1. The **contravariant Yoneda embedding of** $\mathcal{C}$ is the functor[5]

$$よ_{\mathcal{C}}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

  $$よ_{\mathcal{C}}(A) \stackrel{\mathrm{def}}{=} h^A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

  $$よ_{\mathcal{C}|A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Nat}\left(h^B, h^A\right)$$

  of $よ_{\mathcal{C}}$ at (A, B) is given by

  $$よ_{\mathcal{C}|A,B}(f) \stackrel{\mathrm{def}}{=} h^f$$

  for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, where h^f is the corepresentable natural transformation associated to f of Definition 12.2.3.1.1.

[5] *Further Notation:* Also written $h^{(-)}$, or simply $よ$.

Proposition 12.2.4.1.2. Let $\mathcal{C}$ be a category.

1. *Fully Faithfulness.* The contravariant Yoneda embedding

$$\Upsilon_{\mathcal{C}} \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})$$

   is fully faithful.

2. *Preservation and Reflection of Isomorphisms.* The contravariant Yoneda embedding

$$\Upsilon_{\mathcal{C}} \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{CoPSh}(\mathcal{C})$$

   preserves and reflects isomorphisms, i.e. given $A, B \in \mathrm{Obj}(\mathcal{C})$, the following conditions are equivalent:

   (a) We have $A \cong B$.

   (b) We have $h^A \cong h^B$.

Proof. *Item 1, Fully Faithfulness*: The proof is dual to that of Item 1 of Definition 12.1.4.1.3, and is therefore omitted.
Item 2, Preservation and Reflection of Isomorphisms: This follows from Item 1 of Definition 11.5.1.1.6 and Item 3 of Definition 11.6.3.1.2. □

12.2.5 The Contravariant Yoneda Lemma

Let $F \colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf on $\mathcal{C}$.

Theorem 12.2.5.1.1. We have a bijection

$$\mathrm{Nat}\Big(h^A, F\Big) \cong F(A),$$

natural in $A \in \mathrm{Obj}(\mathcal{C})$, determining a natural isomorphism of functors

$$\mathrm{Nat}\Big(h^{(-)}, F\Big) \cong F.$$

Proof. The proof is dual to that of Definition 12.1.5.1.1, and is therefore omitted. □

12.3 Restricted Yoneda Embeddings and Yoneda Extensions

12.3.1 Foundations

let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

Definition 12.3.1.1.1. The **restricted Yoneda embedding associated to** F is the functor

$$\text{よ}_F \colon \mathcal{D} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

defined as the composition

$$\mathcal{D} \xhookrightarrow{\text{よ}_{\mathcal{D}}} \mathsf{PSh}(\mathcal{D}) \xrightarrow{F^{\mathsf{op},*}} \mathsf{PSh}(\mathcal{C}).$$

Remark 12.3.1.1.2. In detail, the **restricted Yoneda embedding associated to** F is the functor

$$\text{よ}_F \colon \mathcal{D} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{D})$, we have

$$\begin{aligned}\text{よ}_F(A) &\overset{\text{def}}{=} h_A \circ F^{\mathsf{op}}\\ &\overset{\text{def}}{=} h_A^{F(-)}.\end{aligned}$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{D})$, the action on morphisms

$$\text{よ}_{F|A,B} \colon \mathrm{Hom}_{\mathcal{D}}(A, B) \to \mathrm{Nat}\left(h_A^{F(-)}, h_B^{F(-)}\right)$$

  of よ_F at (A, B) is given by

$$\begin{aligned}\text{よ}_{F|A,B}(f) &\overset{\text{def}}{=} h_f^{F(-)}\\ &\overset{\text{def}}{=} h_f \star \mathrm{id}_{F^{\mathsf{op}}}\end{aligned}$$

  for each $f \in \mathrm{Hom}_{\mathcal{D}}(A, B)$, where h_f is the representable natural transformation associated to f of Definition 12.1.3.1.1.

Example 12.3.1.1.3. Here are some examples of restricted Yoneda embeddings.

1. *The Nerve Functor.* Let

$$\iota \colon \mathbb{\Delta} \hookrightarrow \mathsf{Cats}$$

   be the functor given by $[n] \to \mathbb{n}$. Then the restricted Yoneda embedding

$$\text{よ}_\iota \colon \mathsf{Cats} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\text{def}}{=}\mathsf{sSets}}$$

   of ι is given by the nerve functor $\mathrm{N}_\bullet$ of **??**.

2. *The Singular Simplicial Set Associated to a Topological Space.* Let

$$\iota\colon \mathbb{\Delta} \hookrightarrow \mathbb{T}$$

be the functor given by $[n] \to |\Delta^n|$. Then the restricted Yoneda embedding

$$よ_\iota\colon \mathbb{T} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\stackrel{\text{def}}{=}\mathsf{sSets}}$$

of ι is given by the singular simplicial set functor $\mathrm{Sing}_\bullet$ of **??**.

3. *The Coherent Nerve Functor.* Let

$$\iota\colon \mathbb{\Delta} \hookrightarrow \mathsf{sCats}$$

be the functor given by $[n] \to \mathsf{Path}(\Delta^n)$, where $\mathsf{Path}(\Delta^n)$ is the simplicial category of **??**. Then the restricted Yoneda embedding

$$よ_\iota\colon \mathsf{sCats} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\stackrel{\text{def}}{=}\mathsf{sSets}}$$

of ι is given by the coherent nerve functor $\mathrm{N}^{\mathrm{hc}}_\bullet$ of **??**.

4. *Kan's Ex Functor.* Let

$$\mathrm{sd}\colon \mathbb{\Delta} \hookrightarrow \mathsf{sSets}$$

be the functor given by $[n] \to \mathrm{Sd}(\Delta^n)$, where $\mathrm{Sd}(\Delta^n)$ is the barycentric subdivision of Δ^n of **??**. Then the restricted Yoneda embedding

$$よ_{\mathrm{sd}}\colon \mathsf{sSets} \to \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\stackrel{\text{def}}{=}\mathsf{sSets}}$$

of sd is given by Kan's Ex functor of **??**.

Proposition 12.3.1.1.4. let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

1. *Interaction With Fully Faithfulness.* The following conditions are equivalent:

   (a) The restricted Yoneda embedding $よ_F$ is fully faithful.

   (b) The functor F is dense (**??**).

2. *As a Left Kan Extension.* We have a natural isomorphism of functors

$$\text{よ}_F \cong \mathrm{Lan}_F(\text{よ}), \qquad \begin{tikzcd} & \mathcal{D} \arrow[d, dashed, "\text{よ}_F"] \\ C \arrow[ur, "F"] \arrow[r, "\text{よ}_C"'] \arrow[ur, phantom, shift right=3, "\Nearrow"] & \mathsf{PSh}(C). \end{tikzcd}$$

Proof. Item 1, *Interaction With Fully Faithfulness*: Omitted.
Item 2, *As a Left Kan Extension*: Omitted. □

12.3.2 The Yoneda Extension Functor

Let $F \colon C \to \mathcal{D}$ be a functor with C small and $\mathcal{D}$ cocomplete.

Definition 12.3.2.1.1. The **Yoneda extension functor associated to** F is the left Kan extension

$$\mathrm{Lan}_{\text{よ}}(F) \colon \mathsf{PSh}(C) \to \mathcal{D}, \qquad \begin{tikzcd} & \mathsf{PSh}(C) \arrow[d, dashed, "\mathrm{Lan}_{\text{よ}}(F)"] \\ C \arrow[ur, "\text{よ}_C"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\Nearrow"] & \mathcal{D}. \end{tikzcd}$$

Example 12.3.2.1.2. Here are some examples of Yoneda extensions.

1. *The Homotopy Category Functor.* Let

   $$\iota \colon \mathbb{\Delta} \hookrightarrow \mathsf{Cats}$$

   be the functor given by $[n] \to \mathbb{n}$. Then the Yoneda extension

   $$\mathrm{Lan}_{\text{よ}}(\iota) \colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\stackrel{\text{def}}{=} \mathsf{sSets}} \to \mathsf{Cats}$$

   of ι is given by the homotopy category functor Ho of **??**.

2. *The Geometric Realisation Functor.* Let

   $$\iota \colon \mathbb{\Delta} \hookrightarrow \mathbb{T}$$

   be the functor given by $[n] \to |\Delta^n|$. Then the Yoneda extension

   $$\mathrm{Lan}_{\text{よ}}(\iota) \colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\stackrel{\text{def}}{=} \mathsf{sSets}} \to \mathbb{T}$$

   of ι is given by the geometric realisation functor $|-|$ of **??**.

3. *The Path Simplicial Category Functor.* Let

$$\iota\colon \mathbb{\Delta} \hookrightarrow \mathsf{sCats}$$

be the functor given by $[n] \to \mathsf{Path}(\Delta^n)$, where $\mathsf{Path}(\Delta^n)$ is the simplicial category of **??**. Then the Yoneda extension

$$\mathrm{Lan}_{よ}(\iota)\colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\mathrm{def}}{=}\mathsf{sSets}} \to \mathsf{sCats}$$

of ι is given by the path simplicial category functor Path of **??**.

4. *The Barycentric Subdivision Functor.* Let

$$\mathrm{sd}\colon \mathbb{\Delta} \hookrightarrow \mathsf{sSets}$$

be the functor given by $[n] \to \mathsf{Sd}(\Delta^n)$, where $\mathsf{Sd}(\Delta^n)$ is the barycentric subdivision of Δ^n of **??**. Then the Yoneda extension

$$\mathrm{Lan}_{よ}(\mathrm{sd})\colon \underbrace{\mathsf{PSh}(\mathbb{\Delta})}_{\overset{\mathrm{def}}{=}\mathsf{sSets}} \to \mathsf{sSets}$$

of sd is given by the barycentric subdivision functor Sd of **??**.

Proposition 12.3.2.1.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor with $\mathcal{C}$ small and $\mathcal{D}$ cocomplete.

1. *Functoriality.* The assignment $F \mapsto \mathrm{Lan}_{よ}(F)$ defines a functor

$$\mathrm{Lan}_{よ}\colon \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \to \mathsf{Fun}(\mathsf{PSh}(\mathcal{C}), \mathcal{D}).$$

2. *Adjointness.* We have an adjunction[6]

$$\big(\mathrm{Lan}_{よ}(F) \dashv よ_F\big)\colon \quad \begin{tikzcd} \mathsf{PSh}(\mathcal{C}) \arrow[r, bend left, "\mathrm{Lan}_{よ}(F)"] \arrow[r, phantom, "\perp"] & \mathcal{D}, \arrow[l, bend left, "よ_F"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathcal{D}}\big(\big[\mathrm{Lan}_{よ}(F)\big](\mathcal{F}), D\big) \cong \mathrm{Nat}\big(\mathcal{F}, よ_F(D)\big),$$

natural in $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$ and $D \in \mathrm{Obj}(\mathcal{D})$.

[6] Applying Item 2 of Definition 12.3.1.1.4, we see that this adjunction has the form

3. *Interaction With the Yoneda Embedding.* We have a natural isomorphism of functors

$$\mathrm{Lan}_{よ}(F) \circ よ_{C} \cong F, \qquad \begin{tikzcd} & \mathsf{PSh}(C) \arrow[d, dashed, "\mathrm{Lan}_{よ}(F)"] \\ C \arrow[ur, "よ_{C}"] \arrow[r, "F"'] & \mathcal{D}. \end{tikzcd}$$

4. *As a Coend.* We have

$$\begin{aligned}\left[\mathrm{Lan}_{よ}(F)\right](\mathcal{F}) &\cong \int^{A\in C} \mathrm{Nat}(h_A, \mathcal{F}) \odot F(A)\\ &\cong \int^{A\in C} \mathcal{F}(A) \odot F(A)\end{aligned}$$

for each $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(C))$.

5. *Interaction With Tensors of Presheaves With Functors.* We have a natural isomorphism

$$\mathrm{Lan}_{よ}(F) \cong (-) \odot_{C} F,$$

natural in $F \in \mathrm{Obj}(\mathsf{Fun}(C, \mathcal{D}))$.

6. *Interaction With Finite Limits.* Let $F\colon C \to \mathsf{Sets}$ be a functor. The following conditions are equivalent:

   (a) The functor F preserves finite limits.

   (b) The functor $\mathrm{Lan}_{よ}(F)$ preserves finite limits.

   (c) The category of elements $\int_C F$ of F is cofiltered.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Adjointness: Omitted.
Item 3, Interaction With the Yoneda Embedding: This follows from **??** of **??**.
Item 4, As a Coend: This follows from **??** of **??** and Definition 12.1.5.1.1.
Item 5, Interaction With Tensors of Presheaves With Functors: This follows from Item 4.
Item 6, Interaction With Finite Limits: See [**coend-calculus**]. □

$\mathrm{Lan}_{よ}(F) \dashv \mathrm{Lan}_F(よ)$.

12.4 Functor Tensor Products

12.4.1 The Tensor Product of Presheaves With Copresheaves

Let $\mathcal{C}$ be a category, let $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on $\mathcal{C}$, and let $G\colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf on $\mathcal{C}$.

Definition 12.4.1.1.1. The **tensor product** of $\mathcal{F}$ with G is the set $\mathcal{F} \boxtimes_{\mathcal{C}} G$[7] defined by

$$\mathcal{F} \boxtimes_{\mathcal{C}} G \overset{\text{def}}{=} \int^{A\in\mathcal{C}} \mathcal{F}(A) \times G(A).$$

Remark 12.4.1.1.2. In other words, the tensor product of $\mathcal{F}$ with G is the set $\mathcal{F} \boxtimes_{\mathcal{C}} G$ defined as the coend of the functor

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{F}\times G} \mathsf{Sets} \times \mathsf{Sets} \xrightarrow{\times} \mathsf{Sets},$$

which is equivalently the composition

$$\times \circ (\mathcal{F} \times G) \cong \mathcal{F} \diamond F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "F", "\shortmid" marking] \arrow[dr, "\times\circ(\mathcal{F}\times G)"', "\shortmid" marking] & \mathrm{pt} \arrow[d, "\mathcal{F}", "\shortmid" marking] \\ & \mathcal{C} \end{tikzcd}$$

in Prof.

Example 12.4.1.1.3.

Proposition 12.4.1.1.4. Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignments $\mathcal{F}, G, (\mathcal{F}, G) \mapsto \mathcal{F} \boxtimes_{\mathcal{C}} G$ define functors

$$\begin{aligned} \mathcal{F} \boxtimes_{\mathcal{C}} -\colon \quad & \mathsf{PSh}(\mathcal{C}) \to \mathsf{Sets}, \\ - \boxtimes_{\mathcal{C}} G\colon \quad & \mathsf{CoPSh}(\mathcal{C}) \to \mathsf{Sets}, \\ -_1 \boxtimes_{\mathcal{C}} -_2\colon \quad & \mathsf{PSh}(\mathcal{C}) \times \mathsf{CoPSh}(\mathcal{C}) \to \mathsf{Sets}. \end{aligned}$$

2. *As a Composition of Profunctors.* Let $\mathcal{C}$ be a category and let:
   - $\mathcal{F}\colon \mathrm{pt} \nrightarrow \mathcal{C}$ be a presheaf on $\mathcal{C}$, viewed as a profunctor.

[7] *Further Notation:* Also written simply $\mathcal{F} \boxtimes G$.

- $F\colon \mathcal{C} \nrightarrow \mathrm{pt}$ be a copresheaf on $\mathcal{C}$, viewed as a profunctor.

We have a natural isomorphism of profunctors

$$\mathcal{F} \boxtimes_{\mathcal{C}} F \cong F \diamond \mathcal{F}, \qquad \begin{tikzcd} & \mathcal{C} \arrow[d, "F", "\shortmid"{marking}] \\ \mathrm{pt} \arrow[ur, "\mathcal{F}", "\shortmid"{marking}] \arrow[r, "\mathcal{F}\boxtimes_{\mathcal{C}}F"', "\shortmid"{marking}] \arrow[ur, phantom, shift left=3, "\Leftrightarrow"] & \mathrm{pt}, \end{tikzcd}$$

natural in $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$ and $F \in \mathrm{Obj}(\mathsf{CoPSh}(\mathcal{C}))$.

3. *Interaction With Representable Presheaves.* Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We have a bijection of sets

$$\mathcal{F} \boxtimes_{\mathcal{C}} h^X \cong \mathcal{F}(X),$$

natural in $X \in \mathrm{Obj}(\mathcal{C})$, giving a natural isomorphism of functors

$$\mathcal{F} \boxtimes_{\mathcal{C}} h^{(-)} \cong \mathcal{F}, \qquad \begin{tikzcd} & \mathsf{CoPSh}(\mathcal{C}) \arrow[d, "\mathcal{F}\boxtimes_{\mathcal{C}}-"] \\ \mathcal{C}^{\mathrm{op}} \arrow[ur, "\text{よ}_{\mathcal{C}}"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, shift left=3, "\Leftrightarrow"] & \mathsf{Sets}. \end{tikzcd}$$

4. *Interaction With Corepresentable Copresheaves.* Let G be a copresheaf on $\mathcal{C}$. We have a bijection of sets

$$h_X \boxtimes_{\mathcal{C}} G \cong G(X),$$

natural in $X \in \mathrm{Obj}(\mathcal{C})$, giving a natural isomorphism of functors

$$h_{(-)} \boxtimes_{\mathcal{C}} G \cong G, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, "-\boxtimes_{\mathcal{C}}G"] \\ \mathcal{C} \arrow[ur, "\text{よ}_{\mathcal{C}}"] \arrow[r, "G"'] \arrow[ur, phantom, shift left=3, "\Leftrightarrow"] & \mathsf{Sets}. \end{tikzcd}$$

5. *Interaction With Yoneda Extensions.* Let $G\colon \mathcal{C} \to \mathsf{Sets}$ be a copresheaf on $\mathcal{C}$. We have a natural isomorphism

$$\mathrm{Lan}_{\text{よ}}(G) \cong (-) \boxtimes_{\mathcal{C}} G, \qquad \begin{tikzcd} & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed, "(-)\boxtimes_{\mathcal{C}}G"] \\ \mathcal{C} \arrow[ur, "\text{よ}_{\mathcal{C}}"] \arrow[r, "G"'] \arrow[ur, phantom, shift left=3, "\Rightarrow"] & \mathsf{Sets}, \end{tikzcd}$$

natural in $G \in \mathrm{Obj}(\mathsf{CoPSh}(C))$.

6. *Interaction With Contravariant Yoneda Extensions.* Let $\mathcal{F} : C^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on C. We have a natural isomorphism

$$\mathrm{Lan}_{よ}(\mathcal{F}) \cong \mathcal{F} \boxtimes_C (-), \qquad \begin{tikzcd} & \mathsf{CoPSh}(C) \arrow[d, dashed, "\mathcal{F} \boxtimes_C (-)"] \\ C^{\mathsf{op}} \arrow[ur, "よ_C"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, shift left=4, "\Uparrow"] & \mathsf{Sets}, \end{tikzcd}$$

natural in $\mathcal{F} \in \mathrm{Obj}(\mathsf{PSh}(C))$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, As a Composition of Profunctors: Clear.
Item 3, Interaction With Representable Presheaves: This follows from **??**.
Item 4, Interaction With Corepresentable Copresheaves: This follows from **??**.
Item 5, Interaction With Yoneda Extensions: This is a special case of Item 5 of Definition 12.3.2.1.3.
Item 6, Interaction With Contravariant Yoneda Extensions: This is a special case of **??** of **??**. □

12.4.2 The Tensor of a Presheaf With a Functor

Let C be a category, let $\mathcal{D}$ be a category with coproducts, let $\mathcal{F} : C^{\mathsf{op}} \to \mathsf{Sets}$ be a presheaf on C, and let $G : C \to \mathcal{D}$ be a functor.

Definition 12.4.2.1.1. The **tensor** of $\mathcal{F}$ with G is the object $\mathcal{F} \odot_C G$[8] of $\mathcal{D}$ defined by

$$\mathcal{F} \odot_C G \overset{\text{def}}{=} \int^{A \in C} \mathcal{F}(A) \odot G(A).$$

Remark 12.4.2.1.2. In other words, the tensor of $\mathcal{F}$ with G is the object $\mathcal{F} \odot_C G$ of $\mathcal{D}$ defined as the coend of the functor

$$C^{\mathsf{op}} \times C \xrightarrow{\mathcal{F} \times G} \mathsf{Sets} \times \mathcal{D} \xrightarrow{\odot} \mathcal{D}.$$

Proposition 12.4.2.1.3. Let C be a category.

1. *Functoriality.* The assignments $\mathcal{F}, G, (\mathcal{F}, G) \mapsto \mathcal{F} \odot_C G$ define

[8] *Further Notation:* Also written simply $\mathcal{F} \odot G$.

functors

$$\begin{aligned}\mathcal{F}\odot_{C}-\colon && \mathsf{PSh}(C) &\to \mathcal{D},\\ -\odot_{C}G\colon && \mathsf{Fun}(C,\mathcal{D}) &\to \mathcal{D},\\ -_1\odot_{C}-_2\colon && \mathsf{PSh}(C)\times\mathsf{Fun}(C,\mathcal{D}) &\to \mathcal{D}.\end{aligned}$$

2. *Interaction With Corepresentable Copresheaves.* We have an isomorphism

   $$h_X\odot_{C}G\cong G(X),$$

   natural in $X\in\mathrm{Obj}(C)$, giving a natural isomorphism of functors

   $$h_{(-)}\odot_{C}G\cong G.$$

3. *Interaction With Yoneda Extensions.* We have a natural isomorphism

   $$\mathrm{Lan}_{\text{よ}}(G)\cong(-)\odot_{C}G,$$

   natural in $G\in\mathrm{Obj}(\mathsf{Fun}(C,\mathcal{D}))$.

Proof. *Item 1*, *Functoriality*: Omitted.
Item 2, *Interaction With Corepresentable Copresheaves*: This follows from **??**.
Item 3, *Interaction With Yoneda Extensions*: This is a repetition of Item 5 of Definition 12.3.2.1.3, and is proved there. □

12.4.3 The Tensor of a Copresheaf With a Functor

Let C be a category, let $\mathcal{D}$ be a category with coproducts, let $F\colon C\to\mathsf{Sets}$ be a copresheaf on C, and let $G\colon C^{\mathsf{op}}\to\mathcal{D}$ be a functor.

Definition 12.4.3.1.1. The **tensor** of F with G is the set $F\odot_{C}G$[9] defined by

$$F\odot_{C}G\overset{\text{def}}{=}\int^{A\in C}F(A)\odot G(A).$$

Remark 12.4.3.1.2. In other words, the tensor of F with G is the object $F\odot_{C}G$ of $\mathcal{D}$ defined as the coend of the functor

$$C^{\mathsf{op}}\times C\overset{\sim}{\dashrightarrow}C\times C^{\mathsf{op}}\xrightarrow{F\times G}\mathsf{Sets}\times\mathcal{D}\xrightarrow{\odot}\mathcal{D}.$$

Proposition 12.4.3.1.3. Let C be a category.

[9] *Further Notation:* Also written simply $F\odot G$.

1. *Functoriality.* The assignments $F, G, (F, G) \mapsto F \odot_{\mathcal{C}} G$ define functors

$$\begin{aligned} F \odot_{\mathcal{C}} -\colon & \quad \mathsf{CoPSh}(\mathcal{C}) && \to \mathcal{D}, \\ - \odot_{\mathcal{C}} \mathcal{G}\colon & \quad \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}) && \to \mathcal{D}, \\ -_1 \odot_{\mathcal{C}} -_2\colon & \quad \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}) \times \mathsf{CoPSh}(\mathcal{C}) && \to \mathcal{D}. \end{aligned}$$

2. *Interaction With Corepresentable Copresheaves.* We have an isomorphism

$$h^X \odot_{\mathcal{C}} G \cong G(X),$$

natural in $X \in \mathsf{Obj}(\mathcal{C})$, giving a natural isomorphism of functors

$$h^{(-)} \odot_{\mathcal{C}} G \cong G.$$

3. *Interaction With Contravariant Yoneda Extensions.* We have a natural isomorphism

$$\mathrm{Lan}_{\text{よ}}(G) \cong G \odot_{\mathcal{C}} (-),$$

natural in $G \in \mathsf{Obj}(\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}))$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, Interaction With Representable Presheaves: This follows from **??**.
Item 2, Interaction With Corepresentable Copresheaves: This follows from **??**.
??, *Interaction With Yoneda Extensions*: Omitted.
Item 3, Interaction With Contravariant Yoneda Extensions: Omitted. □

Appendices

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Monoidal Categories

Chapter 13

Constructions With Monoidal Categories

This chapter contains some material on constructions with monoidal categories.

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13.1 Moduli Categories of Monoidal Structures

13.1.1 The Moduli Category of Monoidal Structures on a Category

Let $\mathcal{C}$ be a category.

Definition 13.1.1.1.1. The **moduli category of monoidal structures on** $\mathcal{C}$ is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ defined by

$$\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{pt} \times_{\mathsf{Cats}} \mathsf{MonCats}, \qquad \begin{tikzcd} \mathcal{M}_{\mathbb{E}_1}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{MonCats} \arrow[d, "忘"] \\ \mathsf{pt} \arrow[r, "{[\mathcal{C}]}"'] & \mathsf{Cats}. \end{tikzcd}$$

Remark 13.1.1.1.2. In detail, **the moduli category of monoidal structures on** $\mathcal{C}$ is the category $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ where:

- *Objects.* The objects of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ are monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ whose underlying category is $\mathcal{C}$.
- *Morphisms.* A morphism from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ is a strong monoidal functor structure

$$\begin{gathered} \mathrm{id}^{\otimes}_{\mathcal{C}} \colon A \boxtimes_{\mathcal{C}} B \overset{\sim}{\dashrightarrow} A \otimes_{\mathcal{C}} B, \\ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}} \colon \mathbb{1}'_{\mathcal{C}} \overset{\sim}{\dashrightarrow} \mathbb{1}_{\mathcal{C}} \end{gathered}$$

  on the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.
- *Identities.* For each $M \overset{\text{def}}{=} (\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}) \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,M} \colon \mathsf{pt} \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M, M)$$

  of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at M is defined by

$$\mathrm{id}^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M} \overset{\text{def}}{=} \left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right),$$

  where $\left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right)$ is the identity monoidal functor of $\mathcal{C}$ of **??**.

- *Composition.* For each $M, N, P \in \mathrm{Obj}(\mathcal{M}_{\mathbb{E}_1}(\mathcal{C}))$, the composition map

$$\circ^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,N,P} \colon \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(N,P) \times \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M,N) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}(M,P)$$

of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ at (M,N,P) is defined by

$$\left(\mathrm{id}^{\otimes,\prime}_{\mathcal{C}}, \mathrm{id}^{\otimes,\prime}_{\mathbb{1}|\mathcal{C}}\right) \circ^{\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})}_{M,N,P} \left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right) \overset{\mathrm{def}}{=} \left(\mathrm{id}^{\otimes,\prime}_{\mathcal{C}} \circ \mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes,\prime}_{\mathbb{1}|\mathcal{C}} \circ \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right).$$

Remark 13.1.1.1.3. In particular, a morphism in $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ to $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ satisfies the following conditions:

1. *Naturality.* For each pair $f \colon A \to X$ and $g \colon B \to Y$ of morphisms of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
A \boxtimes_{\mathcal{C}} B \arrow[r, "f \boxtimes_{\mathcal{C}} g"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B}"'] & X \boxtimes_{\mathcal{C}} Y \arrow[d, "\mathrm{id}^{\otimes}_{X,Y}"] \\
A \otimes_{\mathcal{C}} B \arrow[r, "f \otimes_{\mathcal{C}} g"'] & X \otimes_{\mathcal{C}} Y
\end{tikzcd}$$

commutes.

2. *Monoidality.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] \arrow[dr, "\alpha^{\mathcal{C},\prime}_{A,B,C}"] & \\
(A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"'] & & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[dr, "\alpha^{\mathcal{C}}_{A,B,C}"'] & & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[dl, "\mathrm{id}^{\otimes}_{A, B \otimes_{\mathcal{C}} C}"] \\
 & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) &
\end{tikzcd}$$

commutes.

3. *Left Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & \mathbb{1}_{\mathcal{C}} \boxtimes_{\mathcal{C}} A \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}'_{\mathcal{C}},A}"] & \mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} A \arrow[dr, "\lambda^{\mathcal{C}}_{A}"] & \\
\mathbb{1}'_{\mathcal{C}} \boxtimes_{\mathcal{C}} A \arrow[ur, "\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_{\mathcal{C}} \mathrm{id}_A"] \arrow[rrr, "\lambda^{\mathcal{C},\prime}_{A}"'] & & & A
\end{tikzcd}$$

commutes.

4. *Right Monoidal Unity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & A \boxtimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[r, "\mathrm{id}^{\otimes}_{A,\mathbb{1}'_{\mathcal{C}}}"] & A \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \arrow[dr, "\rho^{\mathcal{C}}_{A}"] & \\
A \boxtimes_{\mathcal{C}} \mathbb{1}'_{\mathcal{C}} \arrow[ur, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[rrr, "\rho^{\mathcal{C},\prime}_{A}"'] & & & A
\end{tikzcd}$$

commutes.

Proposition 13.1.1.1.4. Let $\mathcal{C}$ be a category.

1. *Extra Monoidality Conditions.* Let $\left(\mathrm{id}^{\otimes}_{\mathcal{C}}, \mathrm{id}^{\otimes}_{\mathbb{1}|\mathcal{C}}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(\mathcal{C})$ from $\left(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}}\right)$ to $\left(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime}\right)$.

   (a) The diagram

$$\begin{tikzcd}
(A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"] \arrow[d, "\mathrm{id}^{\otimes}_{A \boxtimes_{\mathcal{C}} B, C}"'] & (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[d, "\mathrm{id}^{\otimes}_{A \otimes_{\mathcal{C}} B, C}"] \\
(A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C"'] & (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C
\end{tikzcd}$$

   commutes.

   (b) The diagram

$$\begin{tikzcd}
A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[r, "\mathrm{id}_A \boxtimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A, B \boxtimes_{\mathcal{C}} C}"'] & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}^{\otimes}_{A, B \otimes_{\mathcal{C}} C}"] \\
A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[r, "\mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"'] & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)
\end{tikzcd}$$

   commutes.

2. *Extra Monoidal Unity Constraints.* Let $\left(\mathrm{id}^{\otimes}_{C}, \mathrm{id}^{\otimes}_{\mathbb{1}|C}\right)$ be a morphism of $\mathcal{M}_{\mathbb{E}_1}(C)$ from $\left(C, \otimes_C, \mathbb{1}_C, \alpha^C, \lambda^C, \rho^C\right)$ to $\left(C, \boxtimes_C, \mathbb{1}'_C, \alpha^{C,\prime}, \lambda^{C,\prime}, \rho^{C,\prime}\right)$.

   (a) The diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^C_{\mathbb{1}'_C}"'] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

   commutes.

   (b) The diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^C_{\mathbb{1}'_C}"'] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

   commutes.

   (c) The diagram

$$\begin{tikzcd}
\mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"'] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^C_{\mathbb{1}'_C}"] \\
\mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

   commutes.

   (d) The diagram

$$\begin{tikzcd}
\mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^C_{\mathbb{1}'_C}"] \\
\mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

   commutes.

3. *Mixed Associators.* Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha^{\mathcal{C}}, \lambda^{\mathcal{C}}, \rho^{\mathcal{C}})$ and $(\mathcal{C}, \boxtimes_{\mathcal{C}}, \mathbb{1}'_{\mathcal{C}}, \alpha^{\mathcal{C},\prime}, \lambda^{\mathcal{C},\prime}, \rho^{\mathcal{C},\prime})$ be monoidal structures on $\mathcal{C}$ and let

$$\mathrm{id}^{\otimes}_{-_1,-_2} \colon -_1 \boxtimes_{\mathcal{C}} -_2 \to -_1 \otimes_{\mathcal{C}} -_2$$

be a natural transformation.

   (a) If there exists a natural transformation

$$\alpha^{\otimes}_{A,B,C} \colon (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \to A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C)$$

   making the diagrams

$$\begin{tikzcd} (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\otimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A\otimes_{\mathcal{C}} B,C}"'] & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_A \otimes_{\mathcal{C}} \mathrm{id}^{\otimes}_{B,C}"] \\ (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C}}_{A,B,C}"'] & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \end{tikzcd}$$

   and

$$\begin{tikzcd} (A \boxtimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C},\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_{\mathcal{C}} \mathrm{id}_C"'] & A \boxtimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}_{A,B\boxtimes_{\mathcal{C}} C}"] \\ (A \otimes_{\mathcal{C}} B) \boxtimes_{\mathcal{C}} C \arrow[r, "\alpha^{\otimes}_{A,B,C}"'] & A \otimes_{\mathcal{C}} (B \boxtimes_{\mathcal{C}} C) \end{tikzcd}$$

   commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

   (b) If there exists a natural transformation

$$\alpha^{\boxtimes}_{A,B,C} \colon (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \to A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C)$$

   making the diagrams

$$\begin{tikzcd} (A \boxtimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\boxtimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \otimes_{\mathcal{C}} \mathrm{id}_C"'] & A \boxtimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \arrow[d, "\mathrm{id}^{\otimes}_{A,B\otimes_{\mathcal{C}} C}"] \\ (A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{C}} C \arrow[r, "\alpha^{\mathcal{C}}_{A,B,C}"'] & A \otimes_{\mathcal{C}} (B \otimes_{\mathcal{C}} C) \end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_C B) \boxtimes_C C \arrow[r, "\alpha^{C,\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A\boxtimes_C B,C}"'] & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes}_{A,B,C}"'] & A \boxtimes_C (B \otimes_C C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

(c) If there exists a natural transformation

$$\alpha^{\boxtimes,\otimes}_{A,B,C} \colon (A \boxtimes_C B) \otimes_C C \to A \otimes_C (B \boxtimes_C C)$$

making the diagrams

$$\begin{tikzcd}
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes,\otimes}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C"'] & A \otimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[r, "\alpha^{C}_{A,B,C}"'] & A \otimes_C (B \otimes_C C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A \boxtimes_C B) \boxtimes_C C \arrow[r, "\alpha^{C,\prime}_{A,B,C}"] \arrow[d, "\mathrm{id}^{\otimes}_{A\boxtimes_C B,C}"'] & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}^{\otimes}_{A,B\boxtimes_C C}"] \\
(A \boxtimes_C B) \otimes_C C \arrow[r, "\alpha^{\boxtimes,\otimes}_{A,B,C}"'] & A \otimes_C (B \boxtimes_C C)
\end{tikzcd}$$

commute, then the natural transformation $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

Proof. Item 1, Extra Monoidality Conditions: We claim that Items 1a and 1b are indeed true:

1. *Proof of Item 1a:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms $\mathrm{id}^{\otimes}_{A,B}$ and id_C.

2. *Proof of Item 1b:* This follows from the naturality of $\mathrm{id}^{\otimes}$ with respect to the morphisms id_A and $\mathrm{id}^{\otimes}_{B,C}$.

This finishes the proof.

Item 2, Extra Monoidal Unity Constraints: We claim that Items 2a and 2b are indeed true:

1. *Proof of Item 1a:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[dr, "\mathrm{id}_{\mathbb{1}_C} \otimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[dd, "\lambda^C_{\mathbb{1}'_C}"'] & & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[d, "\mathrm{id}_{\mathbb{1}_C} \boxtimes_C \mathrm{id}^{\otimes}_{\mathbb{1}}"] \arrow[dl, phantom, "(1)"] \arrow[ddd, bend left=60, "\rho^{C,\prime}_{\mathbb{1}_C}"] & \\
& \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dr, equal] & \mathbb{1}_C \boxtimes_C \mathbb{1}_C \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[dl, phantom, "(2)"] & \\
\mathbb{1}'_C \arrow[rr, phantom, "(3)"] \arrow[drr, "\mathrm{id}^{\otimes}_{\mathbb{1}}"] & & \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[d, "\lambda^C_{\mathbb{1}_C} = \rho^C_{\mathbb{1}_C}"] \arrow[r, phantom, "(4)"] & \, \\
& & \mathbb{1}_C &
\end{tikzcd}$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_C$;

- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of λ^C, where the equality $\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the right monoidal unity of $\left(\mathrm{id}_C, \mathrm{id}^{\otimes}_C, \mathrm{id}^{\otimes}_{C|\mathbb{1}}\right)$;

so does the boundary diagram, and we are done.

2. *Proof of Item 1b:* Indeed, consider the diagram

$$
\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[rr, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}'_C,\mathbb{1}_C}"] \arrow[ddd, "\rho^C_{\mathbb{1}'_C}"'] \arrow[dr, "\mathrm{id}^{\otimes}_{\mathbb{1}} \otimes_C \mathrm{id}_{\mathbb{1}_C}"] & & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[d, "\mathrm{id}^{\otimes}_{\mathbb{1}} \boxtimes_C \mathrm{id}_{\mathbb{1}_C}"] \arrow[dddd, bend left=60, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \arrow[dl, phantom, "(1)"] \\
& \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[ddr, equal] & \mathbb{1}_C \boxtimes_C \mathbb{1}_C \arrow[dd, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}_C}"] \arrow[ddl, phantom, "(2)"] \\
& & \\
\mathbb{1}'_C \arrow[ddrr, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] \arrow[rr, phantom, "(3)"] & & \mathbb{1}_C \otimes_C \mathbb{1}_C \arrow[dd, "\rho^C_{\mathbb{1}_C} = \lambda^C_{\mathbb{1}_C}"] \arrow[r, phantom, "(4)"] & {} \\
& & \\
& & \mathbb{1}_C
\end{tikzcd}
$$

whose boundary diagram is the diagram whose commutativity we wish to prove. Since:

- Subdiagram (1) commutes by the naturality of $\mathrm{id}^{\otimes,-1}_{C}$;
- Subdiagram (2) commutes trivially;
- Subdiagram (3) commutes by the naturality of ρ^{C}, where the equality $\rho^{C}_{\mathbb{1}_C} = \lambda^{C}_{\mathbb{1}_C}$ comes from **??**;
- Subdiagram (4) commutes by the left monoidal unity of $\left(\mathrm{id}_C, \mathrm{id}^{\otimes}_{C}, \mathrm{id}^{\otimes}_{C|\mathbb{1}}\right)$;

so does the boundary diagram, and we are done.

3. *Proof of Item 2c:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"'] \arrow[dr, phantom, "(1)"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C.
\end{tikzcd}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1b;

it follows that the diagram

$$\begin{tikzcd}
\mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] & \mathbb{1}'_C \boxtimes_C \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}'_C \otimes_C \mathbb{1}_C \arrow[d, "\rho^{C}_{\mathbb{1}'_C}"] \\
& \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}'_C
\end{tikzcd}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}$ is an isomorphism, it follows that the diagram $(\dagger)$ also commutes, and we are done.

4. *Proof of Item 2d:* Indeed, consider the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\lambda^{C}_{\mathbb{1}'_C}"'] \arrow[dr, phantom, "(1)"] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^{C}_{\mathbb{1}'_C}"] \\
\mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

Since:

- The boundary diagram commutes trivially;
- Subdiagram (1) commutes by Item 1a;

it follows that the diagram

$$\begin{tikzcd}
\mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] & \mathbb{1}_C \boxtimes_C \mathbb{1}'_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}_C,\mathbb{1}'_C}"] \arrow[d, "\rho^{C,\prime}_{\mathbb{1}_C}"'] \arrow[dr, phantom, "(\dagger)"] & \mathbb{1}_C \otimes_C \mathbb{1}'_C \arrow[d, "\lambda^{C}_{\mathbb{1}'_C}"] \\
 & \mathbb{1}_C \arrow[r, "\mathrm{id}^{\otimes,-1}_{\mathbb{1}}"'] & \mathbb{1}_C
\end{tikzcd}$$

commutes. But since $\mathrm{id}^{\otimes,-1}_{\mathbb{1}}$ is an isomorphism, it follows that the diagram $(\dagger)$ also commutes, and we are done.

This finishes the proof.

Item 3, Mixed Associators: We claim that Items 3a to 3c are indeed true:

1. *Proof of Item 3a:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$

of Item 2 of Definition 13.1.1.1.3 as follows:

$$
\begin{tikzcd}
& & (A \boxtimes_C B) \boxtimes_C C \arrow[dll, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[dd, "\mathrm{id}^{\otimes}_{A \boxtimes_C B, C}"] \arrow[drr, "\alpha^{C}_{A,B,C}"] & & \\
(A \otimes_C B) \boxtimes_C C \arrow[dd, "\mathrm{id}^{\otimes}_{A \otimes_C B, C}"'] & (1) & & (2) & A \boxtimes_C (B \boxtimes_C C) \arrow[dd, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
& & (A \boxtimes_C B) \otimes_C C \arrow[dll, "\mathrm{id}^{\otimes}_{A,B} \otimes_C \mathrm{id}_C"'] \arrow[drr, "\alpha^{\otimes}_{A,B,C}"] & & \\
(A \otimes_C B) \otimes_C C \arrow[drr, "\alpha^{C,\prime}_{A,B,C}"'] & & (3) & & A \boxtimes_C (B \otimes_C C) \arrow[dll, "\mathrm{id}^{\otimes}_{A, B \otimes_C C}"] \\
& & A \otimes_C (B \otimes_C C) & &
\end{tikzcd}
$$

Since:

- Subdiagram (1) commutes by Item 1a of Item 1.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by assumption.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

2. *Proof of Item 3b:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of Item 2 of Definition 13.1.1.1.3 as follows:

$$
\begin{tikzcd}
& & (A \boxtimes_C B) \boxtimes_C C \arrow[dll, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[drr, "\alpha^{C}_{A,B,C}"] & & \\
(A \otimes_C B) \boxtimes_C C \arrow[dd, "\mathrm{id}^{\otimes}_{A \otimes_C B, C}"'] \arrow[drr, "\alpha^{\boxtimes}_{A,B,C}"] & & (1) & & A \boxtimes_C (B \boxtimes_C C) \arrow[dd, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \arrow[dll, "\mathrm{id}^{\otimes}_{A, B \boxtimes_C C}"'] \\
& & A \otimes_C (B \boxtimes_C C) \arrow[dd, "\mathrm{id}_A \otimes_C \mathrm{id}^{\otimes}_{B,C}"] & & \\
(A \otimes_C B) \otimes_C C \arrow[drr, "\alpha^{C,\prime}_{A,B,C}"'] & (2) & & (3) & A \boxtimes_C (B \otimes_C C) \arrow[dll, "\mathrm{id}^{\otimes}_{A, B \otimes_C C}"] \\
& & A \otimes_C (B \otimes_C C) & &
\end{tikzcd}
$$

Since:

- Subdiagram (1) commutes by assumption.
- Subdiagram (2) commutes by assumption.
- Subdiagram (3) commutes by Item 1b of Item 1.

it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

3. *Proof of Item 3c:* We may partition the monoidality diagram for $\mathrm{id}^{\otimes}$ of Item 2 of Definition 13.1.1.1.3 as follows:

$$\begin{tikzcd}
 & (A \boxtimes_C B) \boxtimes_C C \arrow[dl, "\mathrm{id}^{\otimes}_{A,B} \boxtimes_C \mathrm{id}_C"'] \arrow[dr, "\alpha^{C}_{A,B,C}"] & \\
(A \otimes_C B) \boxtimes_C C \arrow[d, "\mathrm{id}^{\otimes}_{A\otimes_C B,C}"'] \arrow[drr, "\alpha^{\boxtimes,\otimes}_{A,B,C}" description] & (1) & A \boxtimes_C (B \boxtimes_C C) \arrow[d, "\mathrm{id}_A \boxtimes_C \mathrm{id}^{\otimes}_{B,C}"] \\
(A \otimes_C B) \otimes_C C \arrow[dr, "\alpha^{C,\prime}_{A,B,C}"'] & (2) & A \boxtimes_C (B \otimes_C C) \arrow[dl, "\mathrm{id}^{\otimes}_{A,B\otimes_C C}"] \\
 & A \otimes_C (B \otimes_C C) &
\end{tikzcd}$$

Since subdiagrams (1) and (2) commute by assumption, it follows that the boundary diagram also commutes, i.e. $\mathrm{id}^{\otimes}$ satisfies the monoidality condition of Item 2 of Definition 13.1.1.1.3.

This finishes the proof. □

13.1.2 The Moduli Category of Braided Monoidal Structures on a Category

13.1.3 The Moduli Category of Symmetric Monoidal Structures on a Category

13.2 Moduli Categories of Closed Monoidal Structures

13.3 Moduli Categories of Refinements of Monoidal Structures

13.3.1 The Moduli Category of Braided Refinements of a Monoidal Structure

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5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

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8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Part VI

Bicategories

Chapter 14

Types of Morphisms in Bicategories

In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 14.1 and 14.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 14.1.10.1.1) and of a *pseudoepic morphism* (Definition 14.2.10.1.1), although the other notions introduced in Sections 14.1 and 14.2 are also interesting on their own.

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14.1 Monomorphisms in Bicategories

14.1.1 Representably Faithful Morphisms

Let C be a bicategory.

Definition 14.1.1.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is faithful.

Remark 14.1.1.1.2. In detail, f is representably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

Example 14.1.1.1.3. Here are some examples of representably faithful morphisms.

1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Item 2 of Definition 11.6.1.1.2.

2. *Representably Faithful Morphisms in* $\mathbf{Rel}$. Every morphism of $\mathbf{Rel}$ is representably faithful; see Item 1 of Definition 8.5.11.1.1.

14.1.2 Representably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.2.1.1. A 1-morphism $f: A \to B$ of $\mathcal{C}$ is **representably full**[2] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_* : \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

Remark 14.1.2.1.2. In detail, f is representably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta : f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha : \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

Definition 14.1.1.1.3.

[2] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of Definition 14.1.2.1.3.

of C such that we have an equality

$$
\begin{tikzcd}
X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B
\end{tikzcd}
$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 14.1.2.1.3. Here are some examples of representably full morphisms.

1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Item 1 of Definition 11.6.2.1.2.

2. *Representably Full Morphisms in* $\mathbf{Rel}$. The representably full morphisms in $\mathbf{Rel}$ are characterised in Item 2 of Definition 8.5.11.1.1.

14.1.3 Representably Fully Faithful Morphisms

Let C be a bicategory.

Definition 14.1.3.1.1. A 1-morphism $f : A \to B$ of C is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 14.1.1.1.1) and representably full (Definition 14.1.2.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f_* : \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is fully faithful.

Remark 14.1.3.1.2. In detail, f is representably fully faithful if the conditions in Definition 14.1.1.1.2 and Definition 14.1.2.1.2 hold:

[3] *Further Terminology:* Also called simply a **fully faithful morphism**, based on Item 1 of Definition 14.1.3.1.3.

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Example 14.1.3.1.3. Here are some examples of representably fully faithful morphisms.

1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Item 6 of Definition 11.6.3.1.2.

2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Item 3 of Definition 8.5.11.1.1) with the representably full morphisms in **Rel**, which are characterised in Item 2 of Definition 8.5.11.1.1.

14.1.4 Morphisms Representably Faithful on Cores

Let C be a bicategory.

Definition 14.1.4.1.1. A 1-morphism $f\colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X,A)) \to \mathsf{Core}(\mathsf{Hom}_C(X,B))$$

given by postcomposition by f is faithful.

Remark 14.1.4.1.2. In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

14.1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

Definition 14.1.5.1.1. A 1-morphism $f\colon A \to B$ of C is **representably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_C(X,A)) \to \mathsf{Core}(\mathsf{Hom}_C(X,B))$$

given by postcomposition by f is full.

Remark 14.1.5.1.2. In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f\circ\phi \overset{\sim}{\Longrightarrow} f\circ\psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.6 Morphisms Representably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

Definition 14.1.6.1.1. A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 14.1.5.1.1) and representably full on cores (Definition 14.1.4.1.1).
2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

   given by postcomposition by f is fully faithful.

Remark 14.1.6.1.2. In detail, f is representably fully faithful on cores if the conditions in Definition 14.1.4.1.2 and Definition 14.1.5.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

14.1.7 Representably Essentially Injective Morphisms

Let C be a bicategory.

Definition 14.1.7.1.1. A 1-morphism $f\colon A \to B$ of C is **representably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is essentially injective.

Remark 14.1.7.1.2. In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ of C, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

14.1.8 Representably Conservative Morphisms

Let C be a bicategory.

Definition 14.1.8.1.1. A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

Remark 14.1.8.1.2. In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi \colon X \rightrightarrows A$ and each 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow"] & A \end{tikzcd}$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha \colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f \circ \phi"] \arrow[r, bend right, "f \circ \psi"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_f \star \alpha"] & B \end{tikzcd}$$

is a 2-isomorphism, then so is α.

14.1.9 Strict Monomorphisms

Let C be a bicategory.

Definition 14.1.9.1.1. A 1-morphism $f \colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathrm{Hom}_C(X, A) \to \mathrm{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathrm{Hom}_C(X, A)) \to \mathrm{Obj}(\mathrm{Hom}_C(X, B))$$

is injective.

Remark 14.1.9.1.2. In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

Example 14.1.9.1.3. Here are some examples of strict monomorphisms.

1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Item 1 of Definition 11.7.2.1.2.

2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Definition 8.5.10.1.1.

14.1.10 Pseudomonic Morphisms

Let C be a bicategory.

Definition 14.1.10.1.1. A 1-morphism $f\colon A \to B$ of C is **pseudomonic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is pseudomonic.

Remark 14.1.10.1.2. In detail, a 1-morphism $f\colon A \to B$ of C is pseudomonic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

   if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

   of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

Proposition 14.1.10.1.3. Let $f\colon A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) The morphism f is representably full on cores and representably faithful.

   (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \\ A \arrow[r, "F"'] \arrow[ur, phantom, "\Leftrightarrow"] & B \end{tikzcd}$$

   in C up to equivalence.

2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

   (a) The morphism f is pseudomonic.

   (b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1}\pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1}\pitchfork F"] \\ B \arrow[r, hook] \arrow[ur, phantom, "\Leftrightarrow"] & \mathbb{1} \pitchfork B \end{tikzcd}$$

   in C up to equivalence.

Proof. *Item 1*, *Characterisations*: Omitted.
Item 2, *Interaction With Cotensors*: Omitted. □

14.2 Epimorphisms in Bicategories

14.2.1 Corepresentably Faithful Morphisms

Let C be a bicategory.

Definition 14.2.1.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably**

faithful if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is faithful.

Remark 14.2.1.1.2. In detail, f is corepresentably faithful if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

Example 14.2.1.1.3. Here are some examples of corepresentably faithful morphisms.

1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Item 5 of Definition 11.6.1.1.2.

2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Item 1 of Definition 8.5.13.1.1.

14.2.2 Corepresentably Full Morphisms

Let C be a bicategory.

Definition 14.2.2.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably full** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathsf{Hom}_C(B, X) \to \mathsf{Hom}_C(A, X)$$

given by precomposition by f is full.

Remark 14.2.2.1.2. In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

Example 14.2.2.1.3. Here are some examples of corepresentably full morphisms.

1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Item 7 of Definition 11.6.2.1.2.

2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in **??** of Definition 8.5.11.1.1.

14.2.3 Corepresentably Fully Faithful Morphisms

Let C be a bicategory.

Definition 14.2.3.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 14.2.2.1.1) and corepresentably faithful (Definition 14.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is fully faithful.

[4] *Further Terminology:* Corepresentably fully faithful morphisms have also been called

Remark 14.2.3.1.2. In detail, f is corepresentably fully faithful if the conditions in Definition 14.2.1.1.2 and Definition 14.2.2.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of C, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

   of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

Example 14.2.3.1.3. Here are some examples of corepresentably fully faithful morphisms.

lax epimorphisms in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Item 10 of Definition 11.6.3.1.2.

2. *Corepresentably Fully Faithful Morphisms in* $\mathbf{Rel}$. The corepresentably fully faithful morphisms of $\mathbf{Rel}$ coincide (Item 3 of Definition 8.5.13.1.1) with the corepresentably full morphisms in $\mathbf{Rel}$, which are characterised in Item 2 of Definition 8.5.13.1.1.

14.2.4 Morphisms Corepresentably Faithful on Cores

Let C be a bicategory.

Definition 14.2.4.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

given by precomposition by f is faithful.

Remark 14.2.4.1.2. In detail, f is corepresentably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

14.2.5 Morphisms Corepresentably Full on Cores

Let C be a bicategory.

Definition 14.2.5.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

given by precomposition by f is full.

Remark 14.2.5.1.2. In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

14.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let C be a bicategory.

Definition 14.2.6.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 14.2.5.1.1) and corepresentably faithful on cores (Definition 14.2.1.1.1).

2. For each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_C(B,X)) \to \mathsf{Core}(\mathsf{Hom}_C(A,X))$$

given by precomposition by f is fully faithful.

Remark 14.2.6.1.2. In detail, f is corepresentably fully faithful on cores if the conditions in Definition 14.2.4.1.2 and Definition 14.2.5.1.2 hold:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta \colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of C, there exists a 2-isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

14.2.7 Corepresentably Essentially Injective Morphisms

Let C be a bicategory.

Definition 14.2.7.1.1. A 1-morphism $f \colon A \to B$ of C is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^* \colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is essentially injective.

Remark 14.2.7.1.2. In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi \colon B \rightrightarrows X$ of C, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

14.2.8 Corepresentably Conservative Morphisms

Let C be a bicategory.

Definition 14.2.8.1.1. A 1-morphism $f\colon A \to B$ of C is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is conservative.

Remark 14.2.8.1.2. In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow"] & X \end{tikzcd}$$

of C, if the 2-morphism

$$\alpha \star \mathrm{id}_f\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi \circ f"] \arrow[r, bend right, "\psi \circ f"'] \arrow[r, phantom, "\overset{\|}{\underset{\Downarrow}{\alpha\star\mathrm{id}_f}}"] & X \end{tikzcd}$$

is a 2-isomorphism, then so is α.

14.2.9 Strict Epimorphisms

Let C be a bicategory.

Definition 14.2.9.1.1. A 1-morphism $f\colon A \to B$ is a **strict epimorphism in** C if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B, X) \to \mathrm{Hom}_C(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathrm{Hom}_C(B, X)) \to \mathrm{Obj}(\mathrm{Hom}_C(A, X))$$

is injective.

Remark 14.2.9.1.2. In detail, f is a strict epimorphism if, for each diagram in C of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

Example 14.2.9.1.3. Here are some examples of strict epimorphisms.

1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Item 1 of Definition 11.7.3.1.2.

2. *Strict Epimorphisms in* $\mathbf{Rel}$. The strict epimorphisms in $\mathbf{Rel}$ are characterised in Definition 8.5.12.1.1.

14.2.10 Pseudoepic Morphisms

Let C be a bicategory.

Definition 14.2.10.1.1. A 1-morphism $f\colon A \to B$ of C is **pseudoepic** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f^*\colon \mathrm{Hom}_C(B,X) \to \mathrm{Hom}_C(A,X)$$

given by precomposition by f is pseudomonic.

Remark 14.2.10.1.2. In detail, a 1-morphism $f\colon A \to B$ of C is pseudoepic if it satisfies the following conditions:

1. For all diagrams in C of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

   if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

   then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(C)$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

   of C, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f .$$

Proposition 14.2.10.1.3. Let $f : A \to B$ be a 1-morphism of C.

1. *Characterisations.* The following conditions are equivalent:

   (a) The morphism f is pseudoepic.

   (b) The morphism f is corepresentably full on cores and corepresentably faithful.

   (c) We have an isococomma square of the form

$$B \overset{\text{eq.}}{\cong} B \overset{\leftrightarrow}{\coprod}_A B, \qquad \begin{tikzcd} B & B \arrow[l, "\mathrm{id}_B"'] \\ B \arrow[u, "\mathrm{id}_B"] & A \arrow[l, "F"] \arrow[u, "F"'] \arrow[ul, phantom, "\Leftrightarrow"] \end{tikzcd}$$

   in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted. □

Appendices

14.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Part VII

Extra Part

Chapter 15

Notes

This chapter contains some notes.

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15.1 TikZ Code for Commutative Diagrams

In this section we gather some useful examples of `tikzcd` code for commutative diagrams.

15.1.1 Product Diagram With Circular Arrows

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

in the preamble, as well as

```
\tikzcdset{
    productArrows/.style args={#1#2#3}{
    execute at end picture={
        % FIRST ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=0,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-1-a] (\tikzcdmatrixname-1-2.east) -- (\tikzcdmat
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
        \path[name path=curve-1-b] (\tikzcdmatrixname-1-2) arc[start angle=90,
        \fill [name intersections={of=curve-1-a and curve-
1-b}] (intersection-2);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.east);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-
2.center);
        \draw let
            \p1 = ($\left(intersection-2\right) - \left(arc-
center\right)$), % \p1 is the vector from the arc's centre to the intersection
2 for the 2nd intersection)
```

```
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degre
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees from
          in [->] (intersection-2) -- ++(\n2:0.1pt);
        % SECOND ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=180,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-2-a] (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatr
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
        \path[name path=curve-2-b] (\tikzcdmatrixname-1-2) arc[start angle=90,e
        \fill [name intersections={of=curve-2-a and curve-
2-b}] (intersection-2);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.west);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-
2.center);
        \draw let
            \p1 = ($\left(intersection-2\right) - \left(arc-
center\right)$), % \p1 is the vector from the arc's centre to the intersection
2 for the 2nd intersection)
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degre
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees from
          in [<-] (intersection-2) -- ++(\n2:0.1pt);
          % Labels
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=180,radius
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=0,radius=#
    }
  }
}
```

The code

```
\begin{tikzcd}[row sep={4.5*\the\DL,between origins}, column sep={4.5*\the\DL,b
```

```
    {}% Don't remove this line, it's important!
    \&
    P
    \arrow[d,"\phi"'{pos=0.475},"\exists!"{pos=0.475}, dashed]
    \&
    {}% Don't remove this line, it's important!
    \\
    A
    \&
    A\times B
    \arrow[l,"\pr_{1}"{pos=0.425},two heads]
    \arrow[r,"\pr_{2}"'{pos=0.425},two heads]
    \&
    B
\end{tikzcd}
```

will then produce the following diagram:

$$\begin{tikzcd}
 & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\
A & A \times B \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & B
\end{tikzcd}$$

15.1.2 Coproduct Diagram With Circular Arrows

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

in the preamble, as well as

```
\tikzcdset{
    coproductArrows/.style args={#1#2#3}{
    execute at end picture={
        % FIRST ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
```

```
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=0,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-1-a] (\tikzcdmatrixname-1-2.east) -- (\tikzcdmatr
2-2.center) -- (\tikzcdmatrixname-2-3.north) -- (\tikzcdmatrixname-
1-3.center) -- cycle;
        \path[name path=curve-1-b] (\tikzcdmatrixname-1-2) arc[start angle=90,e
        \fill [name intersections={of=curve-1-a and curve-
1-b}] (intersection-1);
        % Step 2.2: Find the angle at which to place the arrowhead
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.east);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-
2.center);
        \draw let
            \p1 = ($\left(intersection-1\right) - \left(arc-
center\right)$), % \p1 is the vector from the arc's centre to the intersection
2 for the 2nd intersection)
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degre
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees fron
          in [<-] (intersection-1) -- ++(\n2:0.1pt);
        % SECOND ARROW
        % Step 1: Draw arrow body
        \begin{scope}
            \clip (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatrixname-
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
            \path[draw,line width=rule_thickness] (\tikzcdmatrixname-
1-2) arc[start angle=90,end angle=180,radius=#1];
        \end{scope}
        % Step 2: Draw arrow head
        % Step 2.1: Find the point at which to place the arrowhead
        \path[name path=curve-2-a] (\tikzcdmatrixname-1-2.west) -- (\tikzcdmatr
2-2.center) -- (\tikzcdmatrixname-2-1.north) -- (\tikzcdmatrixname-
1-1.center) -- cycle;
        \path[name path=curve-2-b] (\tikzcdmatrixname-1-2) arc[start angle=90,e
        \fill [name intersections={of=curve-2-a and curve-
2-b}] (intersection-1);
        % Step 2.2: Find the angle at which to place the arrowhead
```

```
        \coordinate (arc-start) at (\tikzcdmatrixname-1-2.west);
        \coordinate (arc-center) at (\tikzcdmatrixname-2-
2.center);
        \draw let
            \p1 = ($\left(intersection-1\right) - \left(arc-
center\right)$), % \p1 is the vector from the arc's centre to the intersection
2 for the 2nd intersection)
            \n1 = {atan2(\y1, \x1)}, % \n1 is the angle of that vector in degr
            \n2 = {\n1 - 90} % \n2 is the angle of the tangent (90 degrees fro
          in [->] (intersection-1) -- ++(\n2:0.1pt);
          % Labels
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=180,radiu
          \path (\tikzcdmatrixname-1-2) arc[start angle=90,end angle=0,radius=
    }
  }
}
```

The code

```
\begin{tikzcd}[row sep={4.5*\the\DL,between origins}, column sep={4.5*\the\DL,
    {}% Don't remove this line, it's important!
    \&
    C
    \arrow[from=d,"\phi","\exists!"', dashed]
    \&
    {}% Don't remove this line, it's important!
    \\
    A
    \&
    A\icoprod B
    \arrow[from=l,"\inj_{1}"',hook]
    \arrow[from=r,"\inj_{2}",hook']
    \&
    B
\end{tikzcd}
```

will then produce the following diagram:

$$\begin{tikzcd}
 & C & \\
A \arrow[r, hook, "\mathrm{inj}_1"'] \arrow[ur, bend left, "\iota_1"] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[l, hook', "\mathrm{inj}_2"] \arrow[ul, bend right, "\iota_2"']
\end{tikzcd}$$

15.1.3 Cube Diagram

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

The code

```
\begin{tikzcd}[row sep={4.0*\the\DL,between origins}, column sep={4.0*\the\DL,
    1
    \&
    \&
    2
    \&
    \\
    \&
    1'
    \&
    \&
    2'
    \\
    3
    \&
    \&
    4
    \&
    \\
    \&
    3'
    \&
    \&
    4'
    % 1-Arrows
    % First Square
    \arrow[from=1-1,to=3-1,"f"']%
    \arrow[from=3-1,to=3-3,"h"{description,pos=0.25}]%
    \arrow[from=1-1,to=1-3,"g"]%
    \arrow[from=1-3,to=3-3,"k"{description,pos=0.25}]%
    % Second Square
    \arrow[from=2-2,to=4-2,"f'"{description,pos=0.3},crossing over]%
    \arrow[from=4-2,to=4-4,"h'"']%
```

```
    \arrow[from=2-2,to=2-4,"g'"{description,pos=0.3},crossing over]%
    \arrow[from=2-4,to=4-4,"k'"]%
    % Connecting Arrows
    \arrow[from=1-1,to=2-2,"c_{1}"description]%
    \arrow[from=1-3,to=2-4,"c_{2}"]%
    \arrow[from=3-1,to=4-2,"c_{3}"']%
    \arrow[from=3-3,to=4-4,"c_{4}"description]%
\end{tikzcd}
```

will produce the following diagram:

$$
\begin{tikzcd}
1 \arrow[rr, "g"] \arrow[dd, "f"'] \arrow[dr, "c_1" description] & & 2 \arrow[dd, "k" description] \arrow[dr, "c_2"] & \\
& 1' \arrow[rr, "g'" description, crossing over] & & 2' \arrow[dd, "k'"] \\
3 \arrow[rr, "h" description] \arrow[dr, "c_3"'] & & 4 \arrow[dr, "c_4" description] & \\
& 3' \arrow[rr, "h'"'] \arrow[from=uu, "f'" description, crossing over] & & 4'
\end{tikzcd}
$$

15.1.4 Cube Diagram With Labelled Faces

Define

```
\newlength{\DL}
\setlength{\DL}{0.9em}
```

The code

```
\begin{tikzcd}[row sep={4.0*\the\DL,between origins}, column sep={4.0*\the\DL,
    1
    \&
    \&
    2
    \&
    \\
    \&
    1'
    \&
    \&
```

```
    2'
    \\
    3
    \&
    \&
    \&
    \\
    \&
    3'
    \&
    \&
    4'
    % 1-Arrows
    % First Square
    \arrow[from=1-1,to=3-1,"f"']%
    \arrow[from=1-1,to=1-3,"g"]%
    % Second Square
    \arrow[from=2-2,to=4-2,"f'"{description},crossing over]%
    \arrow[from=4-2,to=4-4,"h'"']%
    \arrow[from=2-2,to=2-4,"g'"{description},crossing over]%
    \arrow[from=2-4,to=4-4,"k'"]%
    % Connecting Arrows
    \arrow[from=1-1,to=2-2,"c_{1}"description]%
    \arrow[from=1-3,to=2-4,"c_{2}"]%
    \arrow[from=3-1,to=4-2,"c_{3}"']%
    % Subdiagrams
    \arrow[from=2-2,to=1-3,"\scriptstyle(1)"{rotate=-0.3,xslant=-
0.903569337,yslant=0,xscale=7.0341,yscale=4.4454,xscale=0.225,yscale=0.225},ph
    \arrow[from=3-1,to=2-2,"\scriptstyle(2)"{rotate=-44.6,xslant=-
0.965688775,yslant=0,xscale=8.6931,yscale=8.2852,xscale=0.15,yscale=0.15},phan
    \arrow[from=4-2,to=2-4,"\scriptstyle(3)"{rotate=0,xslant=0,yslant=0,xscale=
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep={4.0*\the\DL,between origins}, column sep={4.0*\the\DL,
    1
    \&
    \&
    2
    \&
    \\
```

```
    \&
    \&
    \&
    2'
    \\
    3
    \&
    \&
    4
    \&
    \\
    \&
    3'
    \&
    \&
    4'
    % 1-Arrows
    % First Square
    \arrow[from=1-1,to=3-1,"f"']%
    \arrow[from=3-1,to=3-3,"h"{description}]%
    \arrow[from=1-1,to=1-3,"g"]%
    \arrow[from=1-3,to=3-3,"k"{description}]%
    % Second Square
    \arrow[from=4-2,to=4-4,"h'"']%
    \arrow[from=2-4,to=4-4,"k'"]%
    % Connecting Arrows
    \arrow[from=1-3,to=2-4,"c_{2}"]%
    \arrow[from=3-1,to=4-2,"c_{3}"']%
    \arrow[from=3-3,to=4-4,"c_{4}"description]%
    % Subdiagrams
    \arrow[from=1-1,to=3-3,"\scriptstyle(4)"{rotate=0,xslant=0,yslant=0,xscale=
    \arrow[from=3-3,to=2-4,"\scriptstyle(5)"{rotate=-44.6,xslant=-
0.965688775,yslant=0,xscale=8.6931,yscale=8.2852,xscale=0.15,yscale=0.15},phan
    \arrow[from=4-2,to=3-3,"\scriptstyle(6)"{rotate=-0.3,xslant=-
0.903569337,yslant=0,xscale=7.0341,yscale=4.4454,xscale=0.225,yscale=0.225},ph
\end{tikzcd}
```

will produce the following diagram:

$$
\begin{tikzcd}
1 \arrow[rr, "g"] \arrow[dr, "c_1"] \arrow[ddd, phantom, "(2)"] \arrow[dd, "f"'] \arrow[drrr, phantom, "(1)"] & & 2 \arrow[dr, "c_2"] & \\
& 1' \arrow[rr, "g'" description] \arrow[dd, "f'" description] & & 2' \arrow[dd, "k'"] \\
3 \arrow[dr, "c_3"'] & & (3) & \\
& 3' \arrow[rr, "h'"'] & & 4'
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rr, "g"] \arrow[dd, "f"'] & & 2 \arrow[dr, "c_2"] \arrow[dd, "k" description] \arrow[ddr, phantom, "(5)"] & \\
& (4) & & 2' \arrow[dd, "k'"] \\
3 \arrow[rr, "h" description] \arrow[dr, "c_3"'] \arrow[drrr, phantom, "(6)"] & & 4 \arrow[dr, "c_4" description] & \\
& 3' \arrow[rr, "h'"'] & & 4'
\end{tikzcd}
$$

15.1.5 Pentagon Diagram

Define

```
\newlength{\ThreeCm}
\setlength{\ThreeCm}{3.0cm}
```

The code

```
\begin{tikzcd}[row sep={0*\the\DL,between origins}, column sep={0*\the\DL,betw
    \&[0.30901699437\ThreeCm]
    \&[0.5\ThreeCm]
    A\otimes_{R}(A\otimes_{R}A)
    \&[0.5\ThreeCm]
    \&[0.30901699437\ThreeCm]
    \\[0.58778525229\ThreeCm]
    \left(A\otimes_{R}A\right)\otimes_{R}A
    \&[0.30901699437\ThreeCm]
    \&[0.5\ThreeCm]
    \&[0.5\ThreeCm]
    \&[0.30901699437\ThreeCm]
    A\otimes_{R}A
    \\[0.95105651629\ThreeCm]
    \&[0.30901699437\ThreeCm]
    A\otimes_{R}A
    \&[0.5\ThreeCm]
    \&[0.5\ThreeCm]
    A
    \&[0.30901699437\ThreeCm]
```

```
    % 1-Arrows
    % Left Boundary
    \arrow[from=2-1,to=1-3,"\alpha^{\Mod_{R}}_{A,A,A}"{pos=0.4125}]%
    \arrow[from=1-3,to=2-5,"\id_{A}\otimes_{R}\mu^{\times}_{A}"{pos=0.6}]%
    \arrow[from=2-5,to=3-4,"\mu^{\times}_{A}"{pos=0.425}]%
    % Right Boundary
    \arrow[from=2-1,to=3-2,"\mu^{\times}_{A}\otimes_{R}\id_{A}"'{pos=0.425}]%
    \arrow[from=3-2,to=3-4,"\mu^{\times}_{A}"']%
\end{tikzcd}
```

will produce the following pentagon diagram:

$$\begin{tikzcd}
& A \otimes_R (A \otimes_R A) \arrow[dr, "\mathrm{id}_A \otimes_R \mu^{\times}_A"] & \\
(A \otimes_R A) \otimes_R A \arrow[ur, "\alpha^{\mathsf{Mod}_R}_{A,A,A}"] \arrow[d, "\mu^{\times}_A \otimes_R \mathrm{id}_A"'] & & A \otimes_R A \arrow[d, "\mu^{\times}_A"] \\
A \otimes_R A \arrow[rr, "\mu^{\times}_A"'] & & A
\end{tikzcd}$$

To make the diagram larger, one could use e.g.

```
\newlength{\FourCm}
\setlength{\FourCm}{2.0cm}
```

and replace all instances of `\ThreeCm` with `\FourCm` in the code above.

15.1.6 Hexagon Diagram

Define

```
\newlength{\OneCmPlusHalf}
\setlength{\OneCmPlusHalf}{1.5cm}
```

The code

```
\begin{tikzcd}[row sep={0.0*\the\DL,between origins}, column sep={0.0*\the\DL,
    \&[0.86602540378\OneCmPlusHalf]
    1
    \&[0.86602540378\OneCmPlusHalf]
```

```
    \\[0.5\OneCmPlusHalf]
    2
    \&[0.86602540378\OneCmPlusHalf]
    \&[0.86602540378\OneCmPlusHalf]
    3
    \\[\OneCmPlusHalf]
    4
    \&[0.86602540378\OneCmPlusHalf]
    \&[0.86602540378\OneCmPlusHalf]
    5
    \\[0.5\OneCmPlusHalf]
    \&[0.86602540378\OneCmPlusHalf]
    6
    \&[0.86602540378\OneCmPlusHalf]
    % 1-Arrows
    % Left Boundary
    \arrow[from=1-2,to=2-1,"L_{1}"']%
    \arrow[from=2-1,to=3-1,"L_{2}"']%
    \arrow[from=3-1,to=4-2,"L_{3}"']%
    % Right Boundary
    \arrow[from=1-2,to=2-3,"R_{1}"]%
    \arrow[from=2-3,to=3-3,"R_{2}"]%
    \arrow[from=3-3,to=4-2,"R_{3}"]%
\end{tikzcd}
```

will produce the following hexagon diagram:

$$\begin{tikzcd}
& 1 \arrow[dl, "L_1"'] \arrow[dr, "R_1"] & \\
2 \arrow[d, "L_2"'] & & 3 \arrow[d, "R_2"] \\
4 \arrow[dr, "L_3"'] & & 5 \arrow[dl, "R_3"] \\
& 6 &
\end{tikzcd}$$

To make the diagram larger, one could use e.g.

```
\newlength{\TwoCm}
\setlength{\TwoCm}{2.0cm}
```

and replace all instances of `\OneCmPlusHalf` with `\TwoCm` in the code above.

15.1.7 Double Square Diagram

Define

```
\newlength{\DL}
\setlength{\DL}{0.9cm}
```

The code

```
\begin{tikzcd}[row sep={10.0*\the\DL,between origins}, column sep={10.0*\the\DL
    \bullet
    \&
    \&
    \&
    \bullet
    \\
    \&
    \bullet
    \&
    \bullet
    \&
    \\
    \&
    \bullet
    \&
    \bullet
    \&
    \\
    \bullet
    \&
    \&
    \&
    \bullet
    % Arrows
    % Outer Square
    \arrow[from=1-1,to=1-4]%
    \arrow[from=1-4,to=4-4]%
    %
    \arrow[from=1-1,to=4-1]%
    \arrow[from=4-1,to=4-4]%
    % Inner Square
    \arrow[from=2-2,to=2-3]%
```

```
    \arrow[from=2-3,to=3-3]%
    %
    \arrow[from=2-2,to=3-2]%
    \arrow[from=3-2,to=3-3]%
    % Connecting Arrows
    \arrow[from=1-1,to=2-2]%
    \arrow[from=1-4,to=2-3]%
    \arrow[from=3-2,to=4-1]%
    \arrow[from=3-3,to=4-4]%
    % Subdiagrams
    \arrow[from=2-2,to=3-3,"\scriptstyle(1)",phantom,yshift=10.0*\the\DL]%
    \arrow[from=2-2,to=3-2,"\scriptstyle(2)",phantom,xshift=-
5.0*\the\DL]%
    \arrow[from=2-2,to=3-3,"\scriptstyle(3)",phantom]%
    \arrow[from=2-3,to=3-3,"\scriptstyle(4)",phantom,xshift=5.0*\the\DL]%
    \arrow[from=2-2,to=3-3,"\scriptstyle(5)",phantom,yshift=-
10.0*\the\DL]%
\end{tikzcd}
```

will produce the following double square diagram:

$$\begin{tikzcd}
\bullet \arrow[rrr] \arrow[ddd] \arrow[dr] & & & \bullet \arrow[ddd] \arrow[dl] \\
& \bullet \arrow[r] \arrow[d] \arrow[r, phantom, "(1)", yshift=10ex] \arrow[dr, phantom, "(3)"] \arrow[d, phantom, "(2)", xshift=-5ex] & \bullet \arrow[d] \arrow[d, phantom, "(4)", xshift=5ex] & \\
& \bullet \arrow[r] \arrow[dl] \arrow[r, phantom, "(5)", yshift=-10ex] & \bullet \arrow[dr] & \\
\bullet \arrow[rrr] & & & \bullet
\end{tikzcd}$$

15.1.8 Double Hexagon Diagram

Define

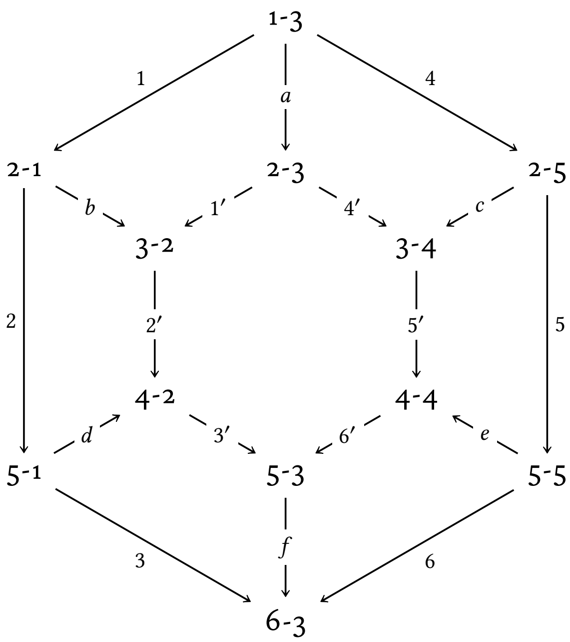
```
\newlength{\OneCm}
\setlength{\OneCm}{1.0cm}
```

The code

```
\begin{tikzcd}[row sep={0.0*\the\DL,between origins}, column sep={0.0*\the\DL,b
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{1-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \\[2.0*\OneCm]
    \text{2-1}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{2-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{2-5}
    \\[1.0*\OneCm]
    \&[1.73205081*\OneCm]
    \text{3-2}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{3-4}
    \&[1.73205081*\OneCm]
    \\[2.0*\OneCm]
    \&[1.73205081*\OneCm]
    \text{4-2}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{4-4}
    \&[1.73205081*\OneCm]
    \\[1.0*\OneCm]
    \text{5-1}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{5-3}
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{5-5}
    \\[2.0*\OneCm]
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    \text{6-3}
```

```
    \&[1.73205081*\OneCm]
    \&[1.73205081*\OneCm]
    % Arrows
    \arrow[from=1-3,to=2-1,"1"']%
    \arrow[from=2-1,to=5-1,"2"']%
    \arrow[from=5-1,to=6-3,"3"']%
    %
    \arrow[from=1-3,to=2-5,"4"]%
    \arrow[from=2-5,to=5-5,"5"]%
    \arrow[from=5-5,to=6-3,"6"]%
    %
    \arrow[from=2-3,to=3-2,"1'"description]%
    \arrow[from=3-2,to=4-2,"2'"description]%
    \arrow[from=4-2,to=5-3,"3'"description]%
    %
    \arrow[from=2-3,to=3-4,"4'"description]%
    \arrow[from=3-4,to=4-4,"5'"description]%
    \arrow[from=4-4,to=5-3,"6'"description]%
    %
    \arrow[from=1-3,to=2-3,"a"description]%
    \arrow[from=2-1,to=3-2,"b"description]%
    \arrow[from=2-5,to=3-4,"c"description]%
    \arrow[from=5-1,to=4-2,"d"description]%
    \arrow[from=5-5,to=4-4,"e"description]%
    \arrow[from=5-3,to=6-3,"f"description]%
\end{tikzcd}
```

will produce the following double hexagon diagram:



To make the diagram larger, one could use e.g.

```
\newlength{\TwoCm}
\setlength{\TwoCm}{2.0cm}
```

and replace all instances of `\OneCm` with `\TwoCm` in the code above.

15.2 Retired Tags

15.2.1 Relations

Old Tag 15.2.1.1.1. The content of this tag has been moved to Definition 8.1.1.1.1.

Old Tag 15.2.1.1.2. The original statement of this tag was false.

Old Tag 15.2.1.1.3. The original statement of this tag was false.

Old Tag 15.2.1.1.4. This was a question. Now an explicit description is available as **??**.

Old Tag 15.2.1.1.5. This was a question. Now an explicit description is available as **??**.

Old Tag 15.2.1.1.6. This tag is obsolete; see Sections 8.5.15 to 8.5.18 instead.

Old Tag 15.2.1.1.7. This tag is obsolete; see Sections 8.5.15 to 8.5.18 instead.

Old Tag 15.2.1.1.8. This tag is obsolete; see Sections 8.5.15 to 8.5.18 instead.

15.2.2 Pointed Sets

Old Tag 15.2.2.1.1. The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

Old Tag 15.2.2.1.2. The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

15.2.3 Tensor Products of Pointed Sets

Old Tag 15.2.3.1.1. Absorbed into Section 7.5.10.

Old Tag 15.2.3.1.2. Absorbed into Section 7.5.10.

Old Tag 15.2.3.1.3. Absorbed into Section 7.5.10.

Old Tag 15.2.3.1.4. Absorbed into Section 7.5.10.

15.2.4 Categories

Old Tag 15.2.4.1.1. We denote natural transformations in diagrams as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}. \end{tikzcd}$$

(This tag has been removed and is now part of Definition 11.9.2.1.2.)

Old Tag 15.2.4.1.2. (This Tag was an item of Definition 11.6.2.1.2, but has since been removed because its statement is incorrect. Naïm Camille Favier provided a counterexample, and the corrected statements now appear as Items 2 and 3 of Definition 11.6.2.1.2.)

1. *Interaction With Postcomposition.* The following conditions are equivalent:

   (a) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is full.

(b) For each $\mathcal{X} \in \mathsf{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is full.

(c) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in Cats_2 in the sense of Definition 14.1.2.1.1.

15.3 Miscellany

15.3.1 List of Things To Explore/Add

Here we list things to be explored in or added to this work in the future. This is a very quick and dirty list; some items may not be fully intelligible.

Remark 15.3.1.1.1. Set Theory:

1. `https://math.stackexchange.com/questions/200389/show-that-the-set-of-all-finite-subsets-of-mathbbn-is-countable`
2. `https://mathoverflow.net/a/479528`
3. `https://www.maths.ed.ac.uk/~tl/ast/ast.pdf`

Type Theory:

1. `https://mathoverflow.net/questions/497570/universes-dont-need-to-be-indexed-by-natural-numbers`

Pointed sets:

1. Universal properties (plural!) of the left tensor product of pointed sets
2. Universal properties (plural!) of the right tensor product of pointed sets

Relations:

1. Internal fibrations in **Rel**, like discrete fibrations and Street fibrations
2. Return to Eilenberg–Moore and Kleisli objects in **Rel** once the general theory has been set up for internal monads

Spans:

1. https://arxiv.org/abs/2505.22832
2. Spans: study certain compositions of spans like composing $B \xleftarrow{f} A = A$ and $A = A \xleftarrow{g} B$ into a span $B \xleftarrow{f} A \xleftarrow{g} B$
3. Comparison *double functor* from Span to Rel and vice versa
4. Apartness composition for spans and alternate compositions for spans in general
5. non-Cartesian analogue of spans
   (a) View spans as morphisms $S \longrightarrow A \times B$ and consider instead morphisms $S \longrightarrow A \otimes_C B$
6. Record the universal property of the bicategory of spans of https://ncatlab.org/nlab/show/span
7. https://ncatlab.org/nlab/show/span+trace
8. Cospans.
9. Multispans.

Un/Straightening for Indexed and Fibred Sets:

1. Analogue of adjoints for Grothendieck construction for indexed and fibred sets
2. Write proper sections on straightening for lax functors from Sets to Rel or Span (displayed sets)
3. co/units for un/straightening adjunction

Categories:

1. https://www.numdam.org/actas/SE/, https://www.numdam.org/journals/CTGDC/
2. https://www.numdam.org/item/CTGDC_1966__8__A5_0.pdf
3. https://mathoverflow.net/questions/493931/is-the-category-of-posets-locally-cartesian-closed

4. From Keith: Presheaves on a topological space X valued in $\{\mathrm{t}, \mathrm{f}\}$
   (a) They are the same as collections of open subsets of X
   (b) They are sheaves iff that collection is closed under union
   (c) Their sheafification is the closure of that collection under unions
5. https://arxiv.org/abs/2504.20949
6. Notion of equality that is weaker than equivalence but stronger than adjunction
7. Tangent categories, Beck modules, categorical derivations
8. Flat functors
9. Is the classifying space of a category isomorphic to Ex^{∞} of the nerve of the category? If so, an intuition for having an initial/terminal object implying being homotopically contractible is that taking the free ∞-groupoid generated by that identifies every object with the terminal one.
10. https://en.wikipedia.org/wiki/Category_algebra
11. simple objects
12. https://mathoverflow.net/questions/442212/properties-of-categorical-zeta-function
13. Polynomial functors, https://ncatlab.org/nlab/show/polynomial+functor, https://arxiv.org/abs/2312.00990
14. https://ncatlab.org/nlab/show/simple+object
15. https://mathoverflow.net/questions/442212/properties-of-categorical-zeta-function
16. https://arxiv.org/abs/2409.17489
17. https://mathoverflow.net/a/478644
18. Posetal category associated to a poset as a right adjoint
19. "Presetal category" associated to a preordered set

20. Vopenka's principle simplifies stuff in the theory of locally presentable categories. If we build categories using type theory or HoTT, what stuff from vopenka holds?

21. Are pseudoepic functors those functors whose restricted Yoneda embedding is pseudomonic and Yoneda preserves absolute colimits?

22. Absolutely dense functors enriched over $\mathbb{R}^{+}$ apparently reduce to topological density

23. Is there a reasonable notion of category homology? It is very common for the geometric realisation of a category to be contractible (e.g. having an initial or terminal object), but maybe some notion of directed homology could work here

24. Nerves of categories:

    (a) Dihedral and symmetric nerves of categories via groupoids (define them first for groupoids and then Kan extend along $\mathsf{Grpd} \hookrightarrow \mathsf{Cats}$)

        i. Same applies to twisted nerves

    (b) Cyclic nerve of a category

    (c) Crossed Simplicial Group Categorical Nerves, `https://arxiv.org/abs/1603.08768`

25. Define contractible categories and add a discussion of universal properties as stating that certain categories are contractible. (Example of non-unique isomorphisms as e.g. being a group of order 5 corresponds to all objects being isomorphic but the category not being contractible)

26. Expand Definition 11.4.3.1.2 and add a proof to it.

27. Sections and retractions; retracts, `https://ncatlab.org/nlab/show/retract`.

28. Groupoid cardinality

    (a) `https://mathoverflow.net/questions/376175/category-theory-and-arithmetical-identities/376223#376223`

(b) https://mathoverflow.net/questions/420088/groupoid-cardinality-of-the-class-of-abelian-p-groups?rq=1

(c) https://mathoverflow.net/questions/363292/what-is-the-groupoid-cardinality-of-the-category-of-vector-spaces-over-a-finite

(d) The groupoid cardinality of the core of the category of finite sets is e. What is the groupoid cardinality of the core of $\mathsf{FinSets}_G$?

(e) groupoid cardinality of the core of the category of finite G-sets, https://www.arxiv.org/pdf/2502.03585

(f) https://ncatlab.org/nlab/show/groupoid+cardinality

(g) https://arxiv.org/abs/2104.11399

(h) https://terrytao.wordpress.com/2017/04/13/counting-objects-up-to-isomorphism-groupoid-cardinality/

(i) https://arxiv.org/abs/0809.2130

(j) https://qchu.wordpress.com/2012/11/08/groupoid-cardinality/

(k) https://mathoverflow.net/questions/363292/what-is-the-groupoid-cardinality-of-the-category-of-vector-spaces-over-a-finite

29. combinatorial species

(a) https://ncatlab.org/nlab/show/Schur+functor

i. Equivalence between twisted commutative algebras and algebras on categories of polynomial functors, https://mathweb.ucsd.edu/~ssam/talks/2014/ihp-tca.pdf

(b) https://mathoverflow.net/questions/22462/what-are-some-examples-of-interesting-uses-of-the-theory-of-combinatorial-specie

(c) https://en.wikipedia.org/wiki/Combinatorial_species

30. Leinster's the eventual image, https://arxiv.org/abs/2210.00302

(a) Telescope notation $\mathrm{tel}_\phi(X) \overset{\text{def}}{=} \mathrm{colim}\left(X \xrightarrow{\phi} X \xrightarrow{\phi}\xrightarrow{\phi} \cdots\right)$ introduced in https://arxiv.org/abs/2505.06979

31. https://ncatlab.org/nlab/show/separable+functor

32. Dagger categories:

    (a) https://en.wikipedia.org/wiki/Dagger_category
    (b) https://ncatlab.org/nlab/show/dagger+category
    (c) Dagger compact categories, https://en.wikipedia.org/wiki/Dagger_compact_category
    (d) https://mathoverflow.net/questions/220032/are-dagger-categories-truly-evil
    (e) generalisation of dagger categories to categories with duality, i.e. categories $\mathcal{C}$ together with a functor $\dagger\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$
        i. Perhaps with the additional condition that $\dagger \circ \dagger = \mathrm{id}$
        ii. categories with involutions in general

Regular Categories:

1. https://arxiv.org/pdf/2004.08964.pdf.

2. Internal relations

Types of Morphisms in Categories:

1. https://mathoverflow.net/questions/490476/duality-of-injectivity-surjectivity-of-precomposition-map for motivation of monomorphisms/epimorphisms

2. Characterisation of epimorphisms in the category of fields, https://math.stackexchange.com/q/4941660

3. Strong epimorphisms

4. Behaviour in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$, e.g. pointwise sections vs. sections in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.

5. Faithful functors from balanced categories are conservative

6. Natural cotransformations:

(a) If there is a natural transformation between functors between categories, taking nerves gives a homotopy equivalence (or something like that). What happens for natural cotransformations?

(b) Natural transformations come with a vertical composition map

$$\circ \colon \coprod_{G \in \mathsf{Fun}(\mathcal{C},\mathcal{D})} \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \longrightarrow \mathrm{Nat}(F,H).$$

As Morgan Rogers shows here, there's no vertical cocomposition map of the form

$$\mathrm{CoNat}(F,H) \longrightarrow \prod_{G \in \mathsf{Fun}(\mathcal{C},\mathcal{D})} \mathrm{CoNat}(G,H) \times \mathrm{CoNat}(F,G)$$

or of the form

$$\mathrm{CoNat}(F,H) \longrightarrow \prod_{G \in \mathsf{Fun}(\mathcal{C},\mathcal{D})} \mathrm{CoNat}(G,H) \coprod \mathrm{CoNat}(F,G)$$

for natural cotransformations.

(c) Cap product for CoNat and Nat

i. recovers map $\mathrm{Z}(G) \times \mathrm{Cl}(G) \longrightarrow \mathrm{Cl}(G)$.

(d) What is the geometric realisation of $\mathrm{CoTrans}(F,G)$?

i. Related: `https://mathoverflow.net/questions/89753/geometric-realization-of-hochschild-complex`

(e) What is the totalisation of $\mathrm{Trans}(F,G)$?

i. If we view sets as discrete topological spaces, what are the homotopy/homology groups of it? The nLab says this (`https://ncatlab.org/nlab/show/totalization`):

> The homotopy groups of the totalization of a cosimplicial space are computed by a Bousfield-Kan spectral sequence.
> The homology groups by an Eilenberg-Moore spectral sequence.

(f) Abstract

Adjunctions:

1. Relative adjunctions: message Alyssa asking for her notes
2. Adjunctions, units, counits, and fully faithfulness as in https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints.
3. Morphisms between adjunctions and bicategory $\mathsf{Adj}(\mathcal{C})$.
4. https://ncatlab.org/nlab/show/transformation+of+adjoints

Presheaves and the Yoneda Lemma:

1. https://mathoverflow.net/questions/498069/products-and-coproducts-in-the-category-of-elements-of-a-presheaf
2. Yoneda extension along $よ_{\mathcal{D}} \circ F \colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$, giving a functor left adjoint to the precomposition functor $F^* \colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$.
3. Consider the diagram

$$\begin{tikzcd}
 & \mathsf{PSh}(\mathcal{C}) \arrow[d, dashed] \arrow[dr, dashed] & \\
\mathcal{C} \arrow[ur, hook] \arrow[r] & \mathcal{D} \arrow[r, hook] & \mathsf{PSh}(\mathcal{D})
\end{tikzcd}$$

4. Does the functor tensor product admit a right adjoint ("Hom") in some sense?
5. Yoneda embedding preserves limits
6. universal objects and universal elements
7. adjoints to the Yoneda embedding and total categories
8. The co-Yoneda lemma: co/presheaves are colimits of co/representables
9. Properties of categories of copresheaves
10. Contravariant restricted Yoneda embedding
11. Contravariant Yoneda extensions

12. Make table of $\mathrm{Lift}_{\text{よ}}(\text{よ})$, $\mathrm{Ran}_{\text{よ}}(\text{よ})$, $\mathrm{Ran}_{\text{よ}}(\rotatebox[origin=c]{180}{よ})$, etc.
13. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: https://qchu.wordpress.com/2015/05/17/generators/
14. Tensor product of functors and relation to profunctors
15. rifts and rans and lifts and lans involving yoneda in Cats and Prof
16. Tensor product of functors and relation to rifts and rans of profunctors

Isbell Duality:

1. enriched Isbell over walking chain complex
2. Isbell self-dual presheaves for Lawvere metric spaces; when

$$f(x) = \sup_{x \in X}\left(\left|f(x) - \sup_{y \in X}(|f(y) - \mathrm{d}_X(y, x)|)\right|\right)$$

   holds.
3. https://ncatlab.org/nlab/show/Fr%C3%B6licher+spaces+and+Isbell+envelopes
4. https://ncatlab.org/nlab/show/envelope+of+an+adjunction
5. https://ncatlab.org/nlab/show/nucleus+of+a+profunctor
6. https://ncatlab.org/nlab/show/nuclear+adjunction
7. https://ncatlab.org/nlab/show/fixed+point+of+an+adjunction
8. **Important:** I should reconsider going with the notation O and Spec. Although a bit common in the (somewhat scarce) literature on Isbell duality, I have doubts regarding how useful/nice of a choice O and Spec are, and whether there are better choices of notation for them.
9. Interaction with $\times$, Hom, $F_!$, F^*, and F_*

10. Interactions between presheaves and copresheaves:
    (a) Natural transformations from a presheaf to a copresheaf and vice versa
    (b) Mixed Day convolution?
11. Isbell duality for monoids:
    (a) Set up a dictionary between properties of $\mathsf{Sets}_A^{\mathsf{L}}$ or $\mathsf{Sets}_A^{\mathsf{R}}$ and properties of A
    (b) Do the same for O given by $A \mapsto \mathsf{Sets}_A^{\mathsf{L}}(X, A)$
    (c) Do the same for Spec given by $A \mapsto \mathsf{Sets}_A^{\mathsf{R}}(X, A)$
    (d) Do the same for $\mathsf{O} \circ \mathsf{Spec}$
    (e) Do the same for $\mathsf{Spec} \circ \mathsf{O}$
    (f) Algebras for $\mathsf{Spec} \circ \mathsf{O}$
    (g) Coalgebras for $\mathsf{O} \circ \mathsf{Spec}$
12. Properties of Spec (e.g. fully faithfulness) vs. properties of C
13. Properties of O (e.g. fully faithfulness) vs. properties of C
14. co/unit being monomorphism/epimorphism
15. reflexive completion
16. Isbell duality for simplicial sets; what's the reflexive completion?
17. Isbell envelope
18. What does Isbell duality look like, when Cat(Aop,Set) is identified with the category of discrete opfibrations over A, using A.5.14?
19. Generalizations of Isbell duality:
    (a) Monoidal Isbell duality: monoidality for Isbell adjunction with day convolution (6.3 of coend cofriend)
    (b) Isbell duality with sheaves
    (c) Isbell duality with Lawvere theories, product preserving functors or whatever
    (d) Isbell duality for profunctors

i. In view of **??** of **??**, can we just use right Kan lifts/extensions?

ii. Right Kan lift/extension of Hom functors (there's probably a version of the Yoneda lemma here)

  A. What is $\mathrm{Rift}_F(\mathrm{Hom}_C)$
  B. What is $\mathrm{Ran}_F(\mathrm{Hom}_C)$
  C. What is $\mathrm{Rift}_{\mathrm{Hom}_C}(F)$
  D. What is $\mathrm{Ran}_{\mathrm{Hom}_C}(F)$
  E. What is $\mathrm{Lift}_F(\mathrm{Hom}_C)$
  F. What is $\mathrm{Lan}_F(\mathrm{Hom}_C)$
  G. What is $\mathrm{Lift}_{\mathrm{Hom}_C}(F)$
  H. What is $\mathrm{Lan}_{\mathrm{Hom}_C}(F)$

20. Tensor product of functors and Isbell duality

  (a) What is $\mathcal{F} \boxtimes_C \mathsf{O}(\mathcal{F})$?

  (b) What is $\mathsf{Spec}(F) \boxtimes_C F$?

  (c) I think there is a canonical morphism

$$\mathcal{F} \boxtimes_C \mathsf{O}(\mathcal{F}) \longrightarrow \mathrm{Tr}(C).$$

  By the way, what is $\mathrm{Tr}(\mathbb{\Delta})$? What is $\mathrm{Tr}(\mathsf{B}A)$? What about $\mathrm{Nat}(\mathrm{id}_C, \mathrm{id}_C)$ for $C = \mathsf{B}A$ or $C = \mathbb{\Delta}$

21. Isbell with coends:

  (a) $\mathrm{Hom}(F(A), h_A)$ but it's a coend

  (b) Conatural transformations and all that

22. Co/limit preservation for O/Spec

23. Isbell duality for N vs. $\mathrm{N} + \mathrm{N}$

24. What do we get if we replace $\mathsf{O} \overset{\text{def}}{=} \mathrm{Nat}(-, h_X)$ by $\mathrm{Nat}^{[W]}(-, h_X)$, and in particular by $\mathrm{DiNat}(-, h_X)$?

Species:

1. Joyal–Street's q-species; via promonoidal structures `https://arxiv.org/pdf/1201.2991#page=22`

2. associators, braidings, unitors; $\mathbb{F}_q^n \to \mathbb{F}_q^n$ centre of $\mathrm{GL}_n(\mathbb{F}_q)$ trick

3. group completion of $\mathcal{GL}(\mathbb{F}_q)$ as algebraic k-theory

Constructions With Categories:

1. https://arxiv.org/abs/2504.21764
2. Comparison between pseudopullbacks and isocomma categories: the “evident” functor $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$ is essentially surjective and full, but not faithful in general.
3. Quotients of categories by actions of monoidal categories
   (a) Quotients of categories by actions of monoids $\mathrm{B}A$
   (b) Quotients of categories by actions of monoids A_{disc}
   (c) Lax, oplax, pseudo, strict, etc. quotients of categories
   (d) lax Kan extensions along $\mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{D}$ for $\mathcal{C} \to \mathcal{D}$ a monoidal functor
4. Quotient of $\mathsf{Fun}(\mathrm{B}A, \mathcal{C})$ by the A-action.
   (a) This is used to build the cycle and p-cycle categories from the paracycle category.
   (b) The quotient of $\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C})$ by the $\mathbb{N}$-action should act as a kind of cyclic directed loop space of $\mathcal{C}$
5. $\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C})$ as a homotopy pullback in Cats_2
   (a) $\mathsf{Fun}(\mathrm{B}\mathbb{Z}, \mathcal{C})$ as a homotopy pullback in Grpd_2
   (b) Free loop space objects

Limits and colimits:

1. adjunction between co/product and diagonal; abstract version of Item 3 and Item 2
2. Examples of kan extensions along functors of the form $\mathsf{FinSets} \hookrightarrow \mathsf{Sets}$
3. Initial/terminal objects as left/right adjoints to $!_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{pt}$.
4. A small cocomplete category is a poset, https://mathoverflow.net/questions/108737/small-categories-and-completeness

5. Co/limits in $\mathsf{B}A$, including e.g. co/equalisers in $\mathsf{B}A$
6. Add the characterisations of absolutely dense functors given in **??** to Item 10.
7. Absolutely dense functors, https://ncatlab.org/nlab/show/absolutely+dense+functor. Also theorem 1.1 here: http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf.
8. Dense functors, codense functors, and absolutely codense functors.
9. van kampen colimits

Completions and cocompletions:

1. https://mathoverflow.net/questions/429003/manifolds-as-cauchy-completed-objects
2. what is the conservative cocompletion of smooth manifolds? Is it related to diffeological spaces?
3. what is the conservative completion of smooth manifolds? Is it related to diffeological spaces?
4. what is the conservative bicompletion of smooth manifolds? Is it related to diffeological spaces?
5. completion of a category under exponentials
6. https://mathoverflow.net/questions/468897/cocompletion-without-cocontinuous-functors
7. The free cocompletion of a category;
8. The free completion of a category;
9. The free completion under finite products;
10. The free cocompletion under finite coproducts;
11. The free bicompletion of a category;
12. The free bicompletion of a category under nonempty products and nonempty coproducts (https://ncatlab.org/nlab/show/free+bicompletion);
13. Cauchy completions

14. Dedekind–MacNeille completions
15. Isbell completion (https://ncatlab.org/nlab/show/reflexive+completion)
16. Isbell envelope

Ends and Coends:

1. motivate co/ends as co/limits of profunctors
2. Ask Fosco about whether composition of dinatural transformations into higher dinaturals could be useful for https://arxiv.org/abs/2409.10237
3. Cyclic co/ends
   (a) Try to mimic the construction given in Haugseng for the cycle, paracycle, cube, etc. categories
   (b) cyclotomic stuff for cyclic co/ends
      i. Check out Ayala–Mazel-Gee–Rozenblyum's *Symmetries of the cyclic nerve*
      ii. isogenetic $\mathbb{N}^{\times}$-action (what the fuck does this mean?)
4. After stating the co/ends

$$\begin{aligned} &\int^{A\in C} h_A \odot \mathcal{F}^A, && \int_{A\in C} \mathsf{Sets}\Big(h_A, \mathcal{F}^A\Big),\\ &\int^{A\in C} h^A \odot F_A, && \int_{A\in C} \mathsf{Sets}\Big(h^A, F_A\Big) \end{aligned}$$

in the co/end version of the Yoneda lemma, add a remark explaining what the co/ends

$$\begin{aligned} &\int_{A\in C} h_A \odot \mathcal{F}^A, && \int^{A\in C} \mathsf{Sets}\Big(h_A, \mathcal{F}^A\Big),\\ &\int_{A\in C} h^A \odot F_A, && \int^{A\in C} \mathsf{Sets}\Big(h^A, F_A\Big) \end{aligned}$$

and the co/ends

$$\begin{aligned} &\int^{A\in C} \mathcal{F}^A \odot h_A, && \int_{A\in C} \mathsf{Sets}\left(\mathcal{F}^A, h_A\right), \\ &\int^{A\in C} F_A \odot h^A, && \int_{A\in C} \mathsf{Sets}\left(F_A, h^A\right), \\ &\int_{A\in C} \mathcal{F}^A \odot h_A, && \int^{A\in C} \mathsf{Sets}\left(\mathcal{F}^A, h_A\right), \\ &\int_{A\in C} F_A \odot h^A, && \int^{A\in C} \mathsf{Sets}\left(F_A, h^A\right) \end{aligned}$$

are.

5. ends $C \to \mathcal{D}$ with $\odot$ is a special case of ends for a certain enrichment over $\mathcal{D}$
6. try to figure out what the end/coend

$$\int^{X\in C} h^A_X \times h^X_B, \qquad \int_{X\in C} h^A_X \times h^X_B$$

are for $C = \mathrm{B}A$. (I think the coend is like tensor product of A as a left A-set with it as a right A-set)

7. Cyclic ends
8. Dihedral ends
9. Does Haugseng's constructions give a way to define cyclic co/homology with coefficients in a bimodule?
10. Category of elements of dinatural transformation classifier
11. Examples of co/ends: `https://mathoverflow.net/a/461814`
12. Cofinality for co/ends, `https://mathoverflow.net/questions/353876`
13. "Fourier transforms" as in `https://arxiv.org/pdf/1501.02503#page=168` or `https://tetrapharmakon.github.io/stuff/itaca.pdf`

Weighted/diagonal category theory:

1. co/ends as centre/trace-infused co/limits: compare the co/end of Hom_C with the co/limit of Hom_C

2. Codensity W-weighted monads, $\mathrm{Ran}_F^{[W]}(F)$;
3. Codensity diagonal monads, $\mathrm{DiRan}_F(F)$;

Profunctors:

1. Apartness defines a composition for relations, but its analogue

$$\mathfrak{q} \mathbin{\square} \mathfrak{p} \stackrel{\mathrm{def}}{=} \int_{A \in C} \mathfrak{p}_A^{-1} \coprod \mathfrak{q}_{-2}^{A}$$

   fails to be unital for profunctors with the unit h_{-}^{A}. Is it unital for some other unit? Is there a less obvious analogue of apartness composition for profunctors? Or maybe does Prof equipped with $\square$ and units h_{-}^{A} form a skew bicategory?

   Is $\Delta_{\emptyset}$ a unit?

2. Figure what monoidal category structures on Sets induce associative and unital compositions on Prof.
3. https://mathoverflow.net/questions/470213/a-distributor-between-categories-induces-a-distributor-between-their-categories
4. Different compositions for profunctors from monoidal structures on the category of sets (e.g. https://mathoverflow.net/questions/155939/what-other-monoidal-structures-exist-on-the-category-of-sets)
5. Nucleus of a profunctor;
6. Isbell duality for profunctors:
   (a) https://mathoverflow.net/questions/259525/isbell-duality-for-profunctors
   (b) https://mathoverflow.net/questions/260322/the-mathfrak-l-functor-on-textsfprof
   (c) https://mathoverflow.net/questions/262462/again-on-the-mathfrak-l-functor-on-mathsfprof

Centres and Traces of Categories:

1. $\mathrm{K}_0(\mathsf{Fun}(\mathrm{B}\mathbb{N}, C))$ vs. $\pi_0(\mathsf{Fun}(\mathrm{B}\mathbb{N}, C))$ vs. $\mathrm{Tr}(C)$, and how these are generalisations of conjugacy classes for monoids

2. Explicitly work out the trace and $\pi_0\mathsf{Fun}(\mathrm{B}\mathbb{N}, -)$ for monoids with few elements.

3. $[1_A]$ can contain more than one element. An example is $\mathsf{Sets}(\mathbb{N}, \mathbb{N})$ and the maps given by

$$\{0, 1, 2, 3, \ldots\} \mapsto \{0, 0, 1, 2, \ldots\},$$
$$\{0, 1, 2, 3, \ldots\} \mapsto \{2, 3, 4, 5, \ldots\}.$$

   Show also that if $c \in [1_A]$, then c is idempotent.

4. Drinfeld centre

5. trace of the symmetric simplex category; it's probably different from that of $\mathsf{FinSets}$

6. Trace of Rep_G and interaction with induction, restriction, etc.

7. $\pi_0(\mathrm{B}\mathbb{N}, \mathrm{B}A)$, $K(\mathrm{B}\mathbb{N}, \mathrm{B}A)$, and $\mathrm{Tr}(\mathrm{B}\mathbb{N}, \mathrm{B}A)$ as concepts of conjugacy for monoids, their equivalents for categories, and comparison with traces

8. Comparison between $\pi_0(\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C}))$ and $K(\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C}))$

9. Lax, oplax, pseudo, and strict trace of simplex 2-category

10. duality over Γ might give a map from product of a monoid with a set to $\mathrm{Tr}(\Gamma)$

11. Studying the set $\mathrm{Nat}(\mathrm{id}_{\mathcal{C}}, F)$ as a notion of categorical trace:

    (a) Ganter–Kapranov define the trace of a 1-endomorphism $f \colon A \to A$ in a 2-category C to be the set $\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_A, f)$;

       i. https://arxiv.org/abs/math/0602510
       ii. https://golem.ph.utexas.edu/string/archives/000757.html
       iii. https://ncatlab.org/nlab/show/categorical+trace

    We should study this notion in detail, and also study $\mathrm{Nat}(F, \mathrm{id}_{\mathcal{C}})$ as well as $\mathrm{CoNat}(\mathrm{id}_{\mathcal{C}}, F)$ and $\mathrm{CoNat}(F, \mathrm{id}_{\mathcal{C}})$.

12. Centre of bicategories

13. Lax centres and lax traces

14. Examples of traces:
    (a) Discrete categories
    (b) Posets
        i. $\mathsf{Open}(X)$
    (c) Trace of small but non-finite categories:
        i. Sets
        ii. $\mathsf{Rep}(G)$
        iii. category of finite groups
        iv. category of finite abelian groups
        v. category of finite p-groups for fixed p
        vi. category of finite p-groups for all p
        vii. category of finite fields
        viii. category of finite topological spaces
        ix. category of finite [insert a mathematical object here]
15. When is the trace of a groupoid just the disjoint sum of sets of conjugacy classes?
16. Set-theoretical issues when defining traces
    (a) Sets is a large category, and yet we can speak of its centre

$$\begin{aligned} \mathrm{Z}(\mathsf{Sets}) &\overset{\text{def}}{=} \int_{A\in\mathsf{Sets}} \mathsf{Sets}(X, X) \\ &\cong \mathsf{Nat}(\mathrm{id}_{\mathsf{Sets}}, \mathrm{id}_{\mathsf{Sets}}) \\ &\cong \mathrm{pt}. \end{aligned}$$

    Is there a way to do the same for the trace of sets, or otherwise work with traces of large categories?
17. Understand how traces are defined via universal properties in Xinwen Zhu's Geometric Satake, categorical traces, and arithmetic of Shimura varieties.
18. trace as an $\mathsf{Obj}(C)$-indexed set
    (a) properties, functoriality, etc.
19. Maybe actually call $\mathsf{Fun}(\mathsf{B}\mathbb{N}, C)$ the categorical directed loop space of C?

20. Cyclic version of $\mathsf{Fun}(\mathrm{B}\mathbb{N}, C)$

21. Traces of categories, nerves of categories, and the cycle category

Categorical Hochschild Homology:

1. To any functor we have an associated natural transformation (Definition 11.5.4.1.1). Do we have sharp transformations associated to natural transformation?

2. build Hochschild co/simplicial set and study its homotopy groups

3. $\mathsf{Fun}(\mathrm{B}\mathbb{N}, X_\bullet)$ vs. $\mathsf{Fun}(\Delta^1/\partial\Delta^1, X_\bullet)$

   (a) Their π_0's vs. the π_0's of $\mathrm{Hom}_{X_\bullet}(x, x)$, of $\mathrm{Hom}^{\mathrm{L}}_{X_\bullet}(x, x)$, and $\mathrm{Hom}^{\mathrm{R}}_{X_\bullet}(x, x)$.

Monoidal Categories:

1. `https://mathoverflow.net/questions/380302`

2. Analogue of Picard rings for dualisable objects

3. Moduli of associators, braidings, etc. for species, q-species

4. When is the left Kan extension along a fully faithful functor of monoidal categories a strong monoidal functor?

5. Interaction between Day convolution and Isbell duality

6. general theory for lifting pseudomonads from Cat to Prof along the equipment embedding

7. definition of prostrength on a functor between promonoidal categories, differential 2-rigs fosco

8. Promonoidal structure in `https://arxiv.org/pdf/1201.2991#page=22`

9. Day convolution as a colimit over category of factorizations $F(A)\otimes_C G(B) \to V$

10. Day convolution with respect to Cartesian monoidal structure is Cartesian monoidal. There's an easy proof of this with coend Yoneda

11. `https://mathoverflow.net/questions/491234`

12. https://mathoverflow.net/questions/488426/adjunction-of-monoidal-closed-categories
13. https://arxiv.org/abs/2502.02532
14. Does the forgetful functor $忘\colon \mathsf{IdemMon}(\mathcal{C}) \to \mathsf{Mon}(\mathcal{C})$ admit a left adjoint? What about $忘\colon \mathsf{IdemMon}(\mathcal{C}) \to \mathcal{C}$?
15. Clifford algebras in monoidal categories
16. Exterior algebras in monoidal categories
    (a) https://mathoverflow.net/questions/70607/exterior-powers-in-tensor-categories
    (b) https://mathoverflow.net/questions/127476/analogy-between-the-exterior-power-and-the-power-set
    (c) https://mathoverflow.net/questions/182476/delignes-exterior-power
    (d) martin brandenburg's phd thesis
17. Different monoidal products in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$ and their distributivity
    (a) Composition
    (b) Pointwise product
    (c) Day convolution
    (d) Relative monad version of Day convolution
18. Classification of monoidal structures on $\mathbb{\Delta}$
19. Classification of monoidal structures on Λ
20. Tensor Categories, 8.5.4
21. https://ncatlab.org/nlab/show/monoidal+action+of+a+monoidal+category
22. https://arxiv.org/abs/2203.16351
23. Para construction
24. Drinfeld center; Symmetric center; JY's books on bimonoidal categories

25. Picard and Brauer 2-groups

    (a) Differential Picard and Brauer Groups via $\mathsf{Fun}(\mathrm{B}\mathbb{N}, \mathsf{Mod}_R)$.
    (b) Brauer and Picard groups of $(\mathsf{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$
    (c) Brauer and Picard groups of $\mathsf{Rep}(G)$
    (d) Brauer and Picard groups of Sets
    (e) Brauer and Picard groups of $\mathsf{Ch}_{\mathbb{Z}}(R)$
    (f) Brauer and Picard groups of $\mathsf{Shv}(X)$
    (g) Brauer and Picard groups of dgMod_R

26. Explore examples in which Day convolution gives weird things, like $\mathsf{Fun}(\mathrm{B}\mathbb{Z}_{/n}, \mathsf{Sets})$.

27. Day convolution is a left Kan extension; explore the right Kan extension

28. Further develop the theory of moduli categories of monoidal structures

29. Picard group

    (a) Picard group for Day convolution. A special case is one of Kaplansky's conjectures, https://en.wikipedia.org/wiki/Kaplansky%27s_conjectures, about units of group rings

30. Day convolution between representable and an arbitrary presheaf $\mathcal{F}$ — can we prove something nice using the colimit formula for $\mathcal{F}$ in terms of representables?

31. Notion of braided monoidal categories in which the braiding is not an isomorphism. Relation to https://arxiv.org/abs/1307.5969

32. Proving a certain diagram between free monoidal categories commutes involves Fermat's little theorem. Can we reverse this and prove Fermat's little theorem from the commutativty of that diagram?

33. https://nilesjohnson.net/notes/grPic-P2S.pdf

34. Proof that monoidal equivalences F of monoidal categories auto-

matically admit monoidal natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong F^{-1} \circ F$ and $\mathrm{id}_{\mathcal{D}} \cong F \circ F^{-1}$.

35. Proof that category with products is monoidal under the Cartesian monoidal structure, [MO 382264].

36. Explore 2-categorical algebra:

    (a) Find a construction of the free 2-group on a monoidal category. Apply it to the multiplicative structure on the category of finite sets and permutations, as well as to the multiplicative structure on the 1-truncation of the sphere spectrum, and try to figure out whether this looks like a categorification of $\mathbb{Q}$.

    (b) What is the free 2-group on $(\mathbb{\Delta}, \oplus, [0])$?

37. Categorify the preorder $\leq$ on $\mathbb{N}$ to a promonad $\mathfrak{p}$ on the groupoid of finite sets and permutations $\mathbb{F}$:

    (a) A preorder is a monad in Rel

    (b) A promonad is a monad in Prof.

    (c) There's a promonad $\mathfrak{p}$ in $\mathbb{F}$ defined by

    $$\mathfrak{p}(m, n) \overset{\text{def}}{=} \{\text{surjections from } \{1, \ldots, m\} \text{ to } \{1, \ldots, n\}\}$$

    This promonad categorifies $\leq$ in that its values are the witnesses to the fact that m is bigger than n (i.e. surjections).

    (d) Figure out whether this promonad extends to the 1-truncation of the sphere spectrum, and perhaps to other categorified analogues of monoids/groups/rings.

38. https://arxiv.org/abs/1307.5969

39. https://arxiv.org/abs/1306.3215

40. https://mathoverflow.net/questions/477219/reference-for-the-monoidal-category-structure-x-otimes-y-x-y-x-times-y

41. Include an explicit proof of Item 14

42. Include an explicit proof of Item 6

43. Definition 4.1.3.1.4

44. obstruction theory for braided enhancements of monoidal categories, using the "moduli category of braided enhancements"

45. Define symmetric and exterior algebras internal to braided monoidal categories

    (a) https://mathoverflow.net/questions/471372/is-there-an-alternating-power-functor-on-braided-monoidal-categories

    (b) https://arxiv.org/abs/math/0504155

46. https://mathoverflow.net/q/382364

47. https://mathoverflow.net/q/471490

48. Concepts of bicategories applied to monoidal categories (e.g. internal adjunctions lead to dualisable objects)

49. Involutive Category Theory

50. https://mathoverflow.net/questions/474662/the-analogy-between-dualizable-categories-and-compact-hausdorff-spaces

Bimonoidal Categories:

1. Bimonoidal structures on the category of species

2. Include an explicit proof of Item 15

Six Functor Formalisms:

1. Michael Shulman:

    > A lot of the "six functor formalism" makes sense in the context of an arbitrary indexed monoidal category (= monoidal fibration), particularly with cartesian base. In particular, I studied the external tensor product in this generality in my paper on Framed bicategories and monoidal fibrations.
    >
    > The internal-hom of powersets in particular, with Ø as a dualizing object, is well-known in constructive mathematics and topos theory, where powersets are in general a Heyting algebra rather than a Boolean algebra.

Morgan Rogers:

> I second this: you're discovering (and making pleasingly explicit, I might add) a special case of "thin category theory": a lot of what you've discovered will work for posets, with the powerset replaced with the frame of downsets :D

2. A six functor formalism for monoids
3. https://mathoverflow.net/questions/258159/yoga-of-six-functors-for-group-representations
4. Is the 1-categorical analogue of six functor formalisms given by Mann interesting?
   (a) Mann defines:
   > A six functor formalism is an ∞-functor $f\colon \mathsf{Corr}(C,E) \to \mathsf{Cats}_\infty$ such that $-\otimes A$, f^*, and $f_!$ admit right adjoints

   (b) Is the notion
   > A 1-categorical six functor formalism is a (lax?) 2-functor $f\colon \mathsf{Corr}(C,E) \to \mathsf{Cats}_2$ (or should Cats be the target?) such that $-\otimes A$, f^*, and $f_!$ admit right adjoints

   interesting?
5. Interaction of the six functors with Kan extensions (e.g. how the left Kan extension of $-\otimes A$ may interact with the other functors)
6. Contexts like Wirthmuller Grothendieck etc
7. formalisation by cisinski and deglise
8. How do the following examples fit?
   (a) base change between $C_{/X}$ and $C_{/Y}$
   (b) $f_! \dashv f_* \dashv f^*$ adjunction between powersets
   (c) $f_! \dashv f_* \dashv f^*$ adjunction between $\mathsf{Span}(\mathrm{pt}, A)$ and $\mathsf{Span}(\mathrm{pt}, B)$
   (d) quadruple adjunction between powersets induced by a relation

(e) adjunctions between categories of presheaves induced by a functor or a profunctor

(f) Adjunction between left A-sets and left B-sets

Do they have exceptional $f^!$? Is there a notion of Fourier–Mukai transform for them? What kind of compatibility conditions (proper base change, etc.) do we have?

Skew Monoidal Categories:

1. https://arxiv.org/abs/2506.06847

2. Try to come up with examples of skew monoidal categories by twisting a tensor product $A \otimes B$ into $T(A) \otimes B$. Related idea: product of G-sets but twisted on the left by an automorphism of G, so that $(ag, b) \sim (a, gb)$ becomes $(a\phi(g), b) \sim (a, gb)$.

3. Skew monoidal category induced from G-sets in analogy to Rel

4. Free monoidal category on a skew monoidal category

5. Skew monoidal structures associated to a locally Cartesian closed category

6. Does the $\mathbb{E}_1$ tensor product of monoids admit a skew monoidal category structure?

7. Is there a (right?) skew monoidal category structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ using right Kan extensions instead of left Kan extensions?

8. Similarly, are there skew monoidal category structures on the subcategory of $\mathbf{Rel}(A, B)$ spanned by the functions using left Kan extensions and left Kan lifts?

9. Add example: $\mathcal{C}$ with coproducts, take $\mathcal{C}_{X/}$ and define

$$\left(X \xrightarrow{f} A\right) \oplus \left(X \xrightarrow{g} B\right) \overset{\text{def}}{=} \left[X \to X \coprod X \xrightarrow{f \coprod g} A \coprod B\right]$$

10. Duals:

(a) Dualisable objects in monoidal categories and traces of endomorphisms of them, including also examples for monoidal categories which are not autonomous/rigid, such as $(\mathsf{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{id}_{\mathcal{C}})$.

(b) compact closed categories

(c) star autonomous categories

(d) Chu construction

(e) Balanced monoidal categories, https://ncatlab.org/nlab/show/balanced+monoidal+category

(f) Traced monoidal categories, https://ncatlab.org/nlab/show/traced+monoidal+category

11. Invertible objects and Picard groupoids

12. https://mathoverflow.net/questions/155939/what-other-monoidal-structures-exist-on-the-category-of-sets

13. Free braided monoidal category with a braided monoid: https://ncatlab.org/nlab/show/vine

14. https://golem.ph.utexas.edu/category/2024/08/skew_monoidal_categories_throu.html

Fibred Category Theory:

1. https://arxiv.org/abs/2402.11644

2. https://categorytheory.zulipchat.com/#narrow/channel/229136-theory.3A-category-theory/topic/A.20.22change.20of.20variables.22.20for.20the.20Grothendieck.20construction/near/495776958

3. Internal **Hom** in categories of co/Cartesian fibrations.

4. *Tensor structures on fibered categories* by Luca Terenzi: https://arxiv.org/abs/2401.13491. Check also the other papers by Luca Terenzi.

5. https://ncatlab.org/nlab/show/cartesian+natural+transformation (this is a cartesian morphism in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ apparently)

6. CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

Operads and Multicategories:

1. Simplicial lists in operad theory I

Monads:

1. Relative monads: message Alyssa asking for her notes
2. https://ncatlab.org/nlab/show/adjoint+monad
3. Kantorovich monad (https://ncatlab.org/nlab/show/Kantorovich+monad) and probability monads in general, https://ncatlab.org/nlab/show/monads+of+probability%2C+measures%2C+and+valuations.

Enriched Categories:

1. $\mathcal{V}$-matrices

Bicategories:

1. Bicategories of Lax Fractions, https://arxiv.org/abs/2507.12044
2. Linear bicategories, https://ncatlab.org/nlab/show/linear+bicategory
   (a) Linearly distributive category, https://ncatlab.org/nlab/show/linearly+distributive+category
   (b) Diagrammatic Algebra of First Order Logic
   (c) Constructing linear bicategories
   (d) Introduction to linear bicategories
3. Allegories, https://ncatlab.org/nlab/show/allegory
4. Skew bicategories
5. Bigroupoid cardinality
6. Bicategory where objects are groups and a morphism $G \nrightarrow H$ is a representation of $G^{\mathsf{op}} \times H$. (I.e. functors $\mathsf{B}G^{\mathsf{op}} \times \mathsf{B}H \to \mathsf{Vect}_k$).
7. Relative monads internal to a bicategory
8. Bicategory of monoid actions

9. https://arxiv.org/abs/0809.1760

10. $\mathsf{Rel}_G \stackrel{\text{def}}{=} \mathsf{Fun}(\mathbf{B}G, \mathsf{Rel})$

11. Rel but for Ab, where morphisms are pairings of the form $A \otimes_{\mathbb{Z}} B \to \mathbb{Z}$.

12. 2-dimensional co/limits in 2-category of categories and adjoint functors

13. Category of equivalence classes

    (a) Given a category $\mathcal{C}$, we have a set $\mathrm{K}_0(\mathcal{C})$ of isomorphism classes of objects

    (b) Given a bicategory $\mathcal{C}$, there should be a category $\mathrm{K}_0(\mathcal{C})$ with $\mathrm{Hom}_{\mathrm{K}_0(\mathcal{C})}(A, B) \stackrel{\text{def}}{=} \mathrm{K}_0(\mathsf{Hom}_{\mathcal{C}}(A, B))$

    (c) The set $\mathrm{K}_0^{\mathrm{eq}}(\mathcal{C})$ of equivalence classes of objects of $\mathcal{C}$ should then satisfy

$$\mathrm{K}_0^{\mathrm{eq}}(\mathcal{C}) \cong \mathrm{K}_0(\mathrm{K}_0(\mathcal{C})).$$

14. bicategory of chain complexes, section "Second Example: Differential Complexes of an Abelian Category" on Gabriel–Zisman's calculus of fractions

15. 2-vector spaces

16. Morita equivalence is equivalence internal to bimod

17. https://mathoverflow.net/questions/478867/2-category-structure-on-modr

18. Bicategories of matrices, as in Street's Variation through enrichment, also https://arxiv.org/abs/2410.18877

19. https://mathoverflow.net/a/86933

20. What are the internal 2-adjunctions in the fundamental 2-groupoid of a space?

21. 2-category structure on Mod_R, where a 2-morphism is a commutative square. Characterisation of adjuntions therein

22. Cook up a very large list of examples of bicategories, like the ones

I made for the AI problems. In particular, find an interesting bicategory of representations qualitatively different from the one I described in the Epoch AI problem

23. 2-category structure on category of R-algebras as enriched Mod_R-categories

24. Let C be a bicategory, let $A, B \in \mathrm{Obj}(C)$, and let $F, G \in \mathrm{Obj}(\mathsf{Hom}_C(A, B))$.

    (a) Does precomposition with $\lambda^C_{A|F}\colon \mathrm{id}_A \circ F \Rightarrow F$ induce an isomorphism of sets

    $$\mathrm{Hom}_{\mathsf{Hom}_C(A,B)}(F, G) \cong \mathrm{Hom}_{\mathsf{Hom}_C(A,B)}(F \circ \mathrm{id}_A, G)$$

    for each $F, G \in \mathrm{Obj}(\mathsf{Hom}_C(A, B))$?

    (b) Similarly, do we have an induced isomorphism of the form

    $$\mathrm{Hom}_{\mathsf{Hom}_C(A,B)}(F, G) \cong \mathrm{Hom}_{\mathsf{Hom}_C(A,B)}(F, \mathrm{id}_B \circ G)$$

    and so on?

25. Are there two Duskin nerve functors? (lax/oplax/etc.?)

26. Interaction with cotransformations:

    (a) Can we abstract the structure provided to Cats_2 by natural cotransformations?

    (b) Are there analogues of cotransformations for **Rel**, Span, BiMod, MonAct, etc.?

    (c) Perhaps this might also make sense as a 1-categorical definition, e.g. comorphisms of groups from A to B as $\mathsf{Sets}(A, B)$ quotiented by $f(ab) \sim f(a)f(b)$.

27. Consider developing the analogue of traces for endomorphisms of dualisable objects in monoidal categories to the setting of bicategories, including e.g. the trace of a category as a trace internal to Prof.

28. Centres of bicategories (lax, strict, etc.)

29. Concepts of monoidal categories applied to bicategories (e.g. traces)

30. Internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].

31. Comonads in the bicategory of profunctors.

32. 2-limit of id, id: $\mathsf{Sets} \rightrightarrows \mathsf{Sets}$ is $\mathrm{B}\mathbb{Z}$, https://mathoverflow.net/questions/209904/van-kampen-colimits?rq=1#comment520288_209904

33. https://mathoverflow.net/questions/473527/universal-property-of-2-presheaves-and-pseudo-lax-colax-natural-transformations

34. https://mathoverflow.net/questions/473526/free-cocompletion-of-a-2-category-under-pseudo-colimits-lax-colimits-and-colax

Types of Morphisms in Bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.

2. Statements like "coequifiers are lax epimorphisms", Item 2 of Examples 2.4 of https://arxiv.org/abs/2109.09836, along with most of the other statements/examples there.

3. Dense, absolutely dense, etc. morphisms in bicategories

Internal adjunctions:

1. https://www.google.com/search?q=mate+of+an+adjunction

2. Moreover, by uniqueness of adjoints (**??** of **??**), this implies also that $S = f^{-1}$.

3. define bicategory $\mathsf{Adj}(C)$

4. walking monad

5. proposition: 2-functors preserve unitors and associators

6. https://ncatlab.org/nlab/show/2-category+of+adjunctions. Is there a 3-category too?

7. https://ncatlab.org/nlab/show/free+monad

8. https://ncatlab.org/nlab/show/CatAdj

9. https://ncatlab.org/nlab/show/Adj
10. $\mathsf{Adj}(\mathsf{Adj}(\mathcal{C}))$
11. Examples of internal adjunctions
    (a) Internal adjunctions in Mod.
    (b) Internal adjunctions in $\mathsf{PseudoFun}(\mathcal{C}, \mathcal{D})$.
    (c) Internal adjunctions in $\mathsf{LaxFun}(\mathcal{C}, \mathcal{D})$.
    (d) Internal adjunctions in 2-categories related to fibrations.

2-Categorical Limits:

1. https://sorilee.github.io/posts/strict-bilimit-and-its-proper-examples

Double Categories:

1. Ehresmann
2. https://arxiv.org/abs/2505.08766
3. https://arxiv.org/abs/2504.18065
4. https://arxiv.org/abs/2504.11099
5. Pinwheel/Yojouhan diagrams and compositionality, section on nLab at https://ncatlab.org/nlab/show/double+category

Homological Algebra:

1. https://arxiv.org/abs/2505.08321
2. https://mathoverflow.net/questions/418676/derived-functor-of-functor-tensor-product
3. https://math.stackexchange.com/questions/3665036/higher-chain-homotopies

Topos theory:

1. https://arxiv.org/abs/2505.08766
2. https://arxiv.org/abs/2304.05338
3. https://arxiv.org/abs/2503.20664

4. https://arxiv.org/abs/2204.08351
5. https://arxiv.org/abs/2404.12313
6. https://www.teses.usp.br/teses/disponiveis/45/45131/tde-31082023-163143/en.php
7. https://teses.usp.br/teses/disponiveis/45/45131/tde-24042019-195658/pt-br.php
8. https://mathoverflow.net/q/479496
9. Grothendieck topologies on BA
10. Enriched Grothendieck topologies
    (a) Borceux–Quintero, https://www.numdam.org/item/CTGDC_1996__37_2_145_0/
    (b) https://arxiv.org/abs/2405.19529
11. Cotopos theory:
    (a) Copresheaves and copresheaf cotopoi
    (b) Elementary cotopoi
        i. https://mathoverflow.net/questions/474287/intuition-for-the-internal-logic-of-a-cotopos
        ii. https://mathoverflow.net/questions/394098/what-is-a-cotopos

            In case you haven't seen it yet, Grothendieck studies (pseudo) cotopos in pursuing stacks

Formal category theory:

1. Yosegi boxes https://arxiv.org/abs/1901.01594

Homotopical Algebra:

1. https://arxiv.org/abs/2109.07803

Simplicial stuff:

1. https://arxiv.org/abs/2503.13663

2. https://www.math.univ-paris13.fr/~harpaz/quasi_unital.pdf
   (a) slogan: geometric definition of ∞-categories should be geometric for identities too
   (b) In an ∞-category, define a **quasi-unit** to be a 1-morphism f such that

   $$[f]_*\colon \ \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Spaces})}(\mathrm{Hom}_{\mathcal{C}}(X,A)\ \mathrm{Hom}_{\mathcal{C}}(X,B)),$$
   $$[f]^*\colon \ \mathrm{Hom}_{\mathsf{Ho}(\mathsf{Spaces})}(\mathrm{Hom}_{\mathcal{C}}(B,X)\ \mathrm{Hom}_{\mathcal{C}}(A,X))$$

   are the identity in $\mathsf{Ho}(\mathsf{Spaces})$. Explore equivalent conditions,
   (c) https://arxiv.org/abs/1606.05669
   (d) https://arxiv.org/abs/1702.08696
3. https://arxiv.org/abs/math/0507116, https://arxiv.org/abs/2503.11338
4. https://arxiv.org/abs/2302.02484 and https://arxiv.org/abs/2411.19751
5. Internal adjunctions in Δ are the same as Galois connections between $[n]$ and $[m]$.
6. https://mathoverflow.net/q/478461
7. draw coherence for lax functors using the diagram for Δ^2
8. characterisation of simplicial sets such that left, right, and two-sided homotopies agree
9. every continuous simplicial set arises as the nerve of a poset.
10. Functor sd is convolution of $よ_{\Delta}$ with itself; see https://arxiv.org/pdf/1501.02503.pdf#page=109
11. Extra degeneracies
    (a) https://www.google.com/search?client=firefox-b-d&q=augmented+simplicial+objects+with+extra+degeneracies

(b) https://leanprover-community.github.io/mathlib_docs/algebraic_topology/extra_degeneracy.html

12. Comparison between $\Delta^1/\partial\Delta^1$ and $\mathrm{B}\mathbb{N}$

∞-Categories:

1. https://arxiv.org/abs/2505.22640
2. https://arxiv.org/abs/2410.17102
3. https://arxiv.org/abs/2410.02578, https://scholar.colorado.edu/concern/graduate_thesis_or_dissertations/st74cr650, https://arxiv.org/abs/2206.00849
4. https://mathoverflow.net/questions/479716/non-strictly-unital-functors-of-infinity-categories
5. https://mathoverflow.net/questions/472253/whats-the-localization-of-the-infty-category-of-categories-under-inverting-f

Condensed Mathematics:

1. https://golem.ph.utexas.edu/category/2020/03/pyknoticity_versus_cohesivenes.html#c057724
2. https://golem.ph.utexas.edu/category/2020/03/pyknoticity_versus_cohesivenes.html#c057810
3. https://maths.anu.edu.au/news-events/events/universal-property-category-condensed-sets
4. https://grossack.site/2024/07/03/life-in-johnstones-topological-topos
5. https://grossack.site/2024/07/03/topological-topos-2-algebras
6. https://grossack.site/2024/07/03/topological-topos-3-bonus-axioms
7. https://terrytao.wordpress.com/2025/04/23/stonean-spaces-projective-objects-the-riesz-representation-theorem-and-possibly-condensed-mathematics/

Monoids:

1. https://mathoverflow.net/questions/278429/
2. Homological algebra of A-sets, https://arxiv.org/abs/1503.02309
3. Catalan monoids, https://arxiv.org/abs/1309.6120
4. https://mathoverflow.net/questions/438305/grothendieck-group-of-the-fibonacci-monoid
5. https://math.stackexchange.com/questions/2662005/how-much-of-a-group-g-is-determined-by-the-category-of-g-sets
6. https://math.stackexchange.com/a/4996051/603207, https://arxiv.org/abs/1006.5687
7. Six functor formalism for monoids, following Section 4.6.4, but in which $\cap$ and $[-,-]$ are replaced with Day convolution.
8. Monoid $(\{1,\ldots,n\} \cup \infty, \gcd)$. The element ∞ can be replaced by $p_1^{\min(e_1^1,\ldots,e_1^m)} \cdots p_k^{\min\left(e_k^1,\ldots,e_k^m\right)}$.
9. Universal property of localisation of monoids as a left adjoint to the forgetful functor $C \to \mathcal{D}$, where:
   - C is the category whose objects are pairs (A, S) with A a monoid and S a submonoid of A.
   - $\mathcal{D}$ is the category whose objects are pairs (A, S) with A a monoid and S a submonoid of A which is also a group.

   Explore this also for localisations of rings

   Explore if we can define field spectra with an approach like this
10. Adjunction between monoids and monoids with zero corresponding to $(-)^- \dashv (-)^+$
11. Rock paper scissors as an example of a non-associative operation
12. https://mathoverflow.net/questions/438305/grothendieck-group-of-the-fibonacci-monoid

13. Witt monoid, https://www.google.com/search?q=Witt+monoid

14. semi-direct product of monoids, https://ncatlab.org/nlab/show/semidirect+product+group

15. morphisms of monoids as natural transformation between left A-sets over A and B_A.

16. Figure out if 2-morphisms of monoids coming from $\mathsf{Fun}^{\otimes}(A_{\mathsf{disc}}, B_{\mathsf{disc}})$, $\mathsf{PseudoFun}(\mathsf{B}A, \mathsf{B}B)$, etc. are interesting

17. Write sections on the quotient and set of fixed points of a set by a monoid action

18. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:

    (a) A morphism $f\colon A \to B$ of semigroups is an epimorphism.

    (b) For each $b \in B$, one of the following conditions is satisfied:

    - We have $f(a) = b$.
    - There exist some $m \in \mathbb{N}_{\geq 1}$ and two factorisations

    $$\begin{aligned} b &= a_0 y_1, \\ b &= x_m a_{2m} \end{aligned}$$

    connected by relations

    $$\begin{aligned} a_0 &= x_1 a_1, \\ a_1 y_1 &= a_2 y_2, \\ x_1 a_2 &= x_2 a_3, \\ a_{2m-1} y_m &= a_{2m} \end{aligned}$$

    such that, for each $1 \leq i \leq m$, we have $a_i \in \mathrm{Im}(f)$.

    Wikipedia says in https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem:

    > For monoids, this theorem can be written more concisely:

19. Representation theory of monoids

(a) https://mathoverflow.net/questions/37115/why-arent-representations-of-monoids-studied-so-much
(b) Representation theory of groups associated to monoids (groups of units, group completions, etc.)

Monoid Actions:

1. https://link.springer.com/book/10.1007/978-3-642-11297-3

2. https://ncatlab.org/schreiber/files/EquivariantInfinityBundles_220809.pdf has some interesting things, like a fully faithful embedding of $\mathsf{Mon}(\mathsf{Sets}^{\mathsf{L}}_{A})$ into $\mathsf{Mon}_{/A}$ whose essential image is given by those monoids of the form $X \rtimes_{\alpha} A$.

3. $f_! \dashv f^* \dashv f_*$ adjunction
   (a) Is it related to the Kan extensions adjunction for $f\colon \mathsf{B}A \longrightarrow \mathsf{B}B$ and the categories $\mathsf{Sets}^{\mathsf{L}}_{A} \cong \mathsf{PSh}(\mathsf{B}A^{\mathsf{op}}, \mathsf{Sets})$ and $\mathsf{Sets}^{\mathsf{L}}_{B} \cong \mathsf{PSh}(\mathsf{B}B^{\mathsf{op}}, \mathsf{Sets})$?
   (b) Is it related to the cobase change adjunction of https://ncatlab.org/nlab/show/base+change? Maybe we can take a morphism of monoids $f\colon A \longrightarrow B$ and consider B^{L}_{A} as a left A-set, and then $(\mathsf{Sets}^{\mathsf{L}}_{A})_{A/}$ and $(\mathsf{Sets}^{\mathsf{L}}_{A})_{B^{\mathsf{L}}_{A}/}$

4. https://arxiv.org/abs/2112.10198

5. double category of monoid actions

6. Analogue of Brauer groups for A-sets

7. Hochschild homology for A-sets

Group Theory:

1. https://mathoverflow.net/questions/45651/is-there-a-q-analog-to-the-braid-group

2. https://johncarlosbaez.wordpress.com/2025/03/27/the-mcgee-group/

3. https://bookstore.ams.org/memo-1-2/

4. https://link.springer.com/book/10.1007/978-3-662-59144-4

5. https://en.wikipedia.org/wiki/Tits_group

6. https://en.wikipedia.org/wiki/Group_of_Lie_type

7. https://mathoverflow.net/questions/251769/what-meanings-does-chevalley-group-have

8. https://encyclopediaofmath.org/wiki/Chevalley_group

9. https://en.wikipedia.org/wiki/Group_of_Lie_type

10. MO: cardinality of $\mathrm{Cl}\big(\mathrm{Aut}\big(\mathrm{GL}_n(\mathbb{F}_q)\big)\big)$

11. https://math.stackexchange.com/questions/4419869/do-the-groups-operatornamesl-operatornamepgl-and-operatornamepsl

12. https://groupprops.subwiki.org/wiki/Order_formulas_for_linear_groups

13. https://groupprops.subwiki.org/wiki/Order_of_semidirect_product_is_product_of_orders

14. https://groupprops.subwiki.org/wiki/Central_automorphism_group_of_general_linear_group

15. https://groupprops.subwiki.org/wiki/Automorphism_group_of_general_linear_group_over_a_field

16. https://groupprops.subwiki.org/wiki/Inner-centralizing_automorphism

17. https://math.stackexchange.com/questions/2519372/number-of-conjugacy-classes-for-the-modular-group

18. $\mathrm{GL}_n(K)$ for K a skew field

19. https://arxiv.org/abs/1212.6157, https://arxiv.org/abs/0708.1608, https://en.wikipedia.org/wiki/Wild_problem, https://www.google.com/search?q=matrix+pair+problem, https://arxiv.org/abs/2007.09242, https://mathoverflow.net/questions/291815/rational-canonical-form-over-mathbbz-pk-mathbbz, https://mathoverflow.net/questions/291815/rational-canonical-form-over-mathbbz-pk-mathbbz

20. https://link.springer.com/book/10.1007/978-981-13-2895-4

21. https://ysharifi.wordpress.com/2022/09/14/automorphisms-of-dihedral-groups/

22. https://en.wikipedia.org/wiki/PSL(2,7)

23. https://arxiv.org/abs/2304.08617

24. https://johncarlosbaez.wordpress.com/2016/03/22/the-involute-of-a-cubical-parabola/#comment-78884

25. https://arxiv.org/abs/0904.1876

26. finite subgroups of $\mathrm{SU}(2)$, and viewing them as groups of rotations and such

27. https://arxiv.org/abs/1201.2363

28. https://ncatlab.org/nlab/show/group+extension#SchreierTheory, https://ncatlab.org/nlab/show/nonabelian+cohomology, https://ncatlab.org/nlab/show/nonabelian+group+cohomology

29. https://en.wikipedia.org/wiki/Fibonacci_group

30. Study the functoriality properties of $G \mapsto \mathrm{Aut}(G)$ via functoriality of ends

31. Is $\sum_{[g]\in\mathrm{Cl}(G)} \frac{1}{|g|}$ an interesting invariant of G?

32. Idempotent endomorphism $f\colon A \longrightarrow A$ is the same as a decomposition $A \cong B \oplus C$ via $B \cong \mathrm{Im}(f)$ and $C \cong \mathrm{Ker}(f)$.

    (a) https://mathstrek.blog/2015/03/02/idempotents-and-decomposition/

33. https://math.stackexchange.com/questions/34271/order-of-general-and-special-linear-groups-over-finite-fields

Linear Algebra:

1. Size of conjugacy class $[A]$ of $A \in \mathrm{GL}_n(\mathbb{F}_q)$ is given by $\#\mathrm{GL}_n(\mathbb{F}_q)$

divided by the centralizer $\mathrm{Z}_{\mathrm{GL}_n(\mathbb{F}_q)}(A)$ of A in $\mathrm{GL}_n(\mathbb{F}_q)$, whose order is given by

$$\begin{aligned}\#\mathrm{Z}_{\mathrm{GL}_n(\mathbb{F}_q)}(A) &= \prod_{i=1}^{k} \#\mathrm{GL}_{r_i}(\mathbb{F}_q) \\ &= q^{\sum_{i=1}^{k}\binom{r_i}{2}} \prod_{i=1}^{k}\prod_{j=0}^{r_i-1}(q^{r_i-j}-1)\end{aligned}$$

if A is diagonalisable with eigenvalues $\lambda_1, \ldots, \lambda_k$ having multiplicities $r_1, \ldots, r_k$. More generally, see https://groupprops.subwiki.org/wiki/Conjugacy_class_size_formula_in_general_linear_group_over_a_finite_field

2. https://en.wikipedia.org/wiki/Semilinear_map
3. conjugacy for $\mathrm{GL}_n(\mathbb{F}_q)$, https://mathoverflow.net/a/104457
4. https://en.wikipedia.org/wiki/Dieudonn%C3%A9_determinant, https://ncatlab.org/nlab/show/Dieudonn%C3%A9+determinant#Dieudonne
5. https://ncatlab.org/nlab/show/Pfaffian
6. https://math.stackexchange.com/questions/1715249/the-number-of-subspaces-over-a-finite-field
7. https://math.stackexchange.com/questions/70801/how-many-k-dimensional-subspaces-there-are-in-n-dimensional-vector-space-over
8. https://en.wikipedia.org/wiki/Gaussian_binomial_coefficient
9. https://en.wikipedia.org/wiki/List_of_q-analogs

Noncommutative Algebra:

1. https://arxiv.org/abs/1608.08140
2. https://arxiv.org/abs/2401.12884
3. https://ncatlab.org/nlab/show/dihedral+homology

4. https://www.sciencedirect.com/science/article/pii/0022404995000836

5. https://arxiv.org/abs/2008.11569, https://www.lakeheadu.ca/sites/default/files/uploads/77/docs/Cox%20Daniel.pdf

Commutative Algebra:

1. If $M \in \mathrm{Pic}(R)$, then $\mathrm{Aut}(M) \cong R^{\times}$.

2. https://math.stackexchange.com/questions/637918/

3. https://categorytheory.zulipchat.com/#narrow/stream/411257-theory.3A-mathematics/topic/Big.20Witt.20ring

4. https://math.stackexchange.com/questions/535623/how-many-irreducible-factors-does-xn-1-have-over-finite-field

5. Derivations between morphisms of R-algebras, after https://mathoverflow.net/questions/434488

   (a) Namely, a derivation from a morphism $f\colon A \to B$ of R-algebras to a morphism $g\colon A \to B$ of R-algebras is a map $D\colon B \to B$ such that we have

   $$D(ab) = g(a)D(b) + D(a)f(b)$$

   for each $a, b \in B$.

Hyper Algebra:

1. https://arxiv.org/abs/2205.02362

2. http://www.numdam.org/item/SD_1959-1960__13_1_A9_0/

3. https://www.worldscientific.com/worldscibooks/10.1142/13652#t=aboutBook

Coalgebra:

1. https://mathoverflow.net/questions/483668/textrepd-4-and-its-three-fiber-functors

Topological Algebra:

1. https://golem.ph.utexas.edu/category/2014/08/holy_crap_do_you_know_what_a_c.html
2. https://categorytheory.zulipchat.com/#narrow/channel/411257-theory.3A-mathematics/topic/topological.20rings.20and.20fields
3. https://mathoverflow.net/q/477757
4. https://math.stackexchange.com/questions/2593556/galois-theory-for-topological-fields

Differential Graded Algebras:

1. https://mathoverflow.net/questions/476150/constructing-an-adjunction-between-algebras-and-differential-graded-algebras

Topology:

1. Topologies on $\mathcal{P}(\mathcal{P}(X))$, https://mathoverflow.net/questions/496630/topological-analogues-of-gromov-hausdorff-convergence
2. https://mathoverflow.net/questions/255912/what-is-the-structure-associated-to-almost-everywhere-convergence
3. https://arxiv.org/abs/2504.12965
4. https://mathoverflow.net/questions/485669/exponential-law-for-topological-spaces-for-the-topology-of-pointwise-convergence and comments therein
5. This paper has some cool references on convergence spaces: https://arxiv.org/abs/2410.18245
6. https://arxiv.org/abs/2402.12316
7. Write about the 6-functor formalism for sheaves on topological spaces and for topological stacks, with lots of examples.
   (a) MO question titled *6-functor formalism for topological stacks*: https://mathoverflow.net/q/471758

Measure Theory:

1. https://mathoverflow.net/questions/126994/beck-chevalley-for-measures
2. https://mathoverflow.net/questions/483726
3. https://en.wikipedia.org/wiki/Valuation_%28measure_theory%29
4. There's a theorem saying that there does not exist an infinite-dimensional "Lebesgue" measure, i.e. (from https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure):

   > Let X be an infinite-dimensional, separable Banach space. Then, the only locally finite and translation invariant Borel measure μ on X is a trivial measure. Equivalently, there is no locally finite, strictly positive, and translation invariant measure on X.

   What kind of measures exist/not exist that satisfy all conditions above except being locally finite?
5. https://ncatlab.org/nlab/show/categories+of+measure+theory
6. Functions $f_!$, f^*, and f_* between spaces of (probability) measures on probability/measurable spaces, mimicking how a map of sets $f\colon X \to Y$ induces morphisms of sets $f_!$, f^*, and f_* between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.
7. Analogies between representable presheaves and the Yoneda lemma on the one hand and Dirac probability measures on the other hand
   (a) Universal property of the embedding of a space X into the space of probability measures on X
   (b) Same question but for distributions
   (c) non-symmetric metric on space of probability measures where we define $\mathrm{d}(\mu, \nu)$ to be the measure given by
   $$U \mapsto \int_U \rho_\mu \, \mathrm{d}\nu,$$
   where ρ_μ is the probability density of μ. Can we make this idea work?

8. https://arxiv.org/abs/0801.2250

9. https://mathoverflow.net/questions/325861

   > *In particular, I came across a PhD thesis by Martial Agueh. I thought it was interesting because it explicitly investigated the geodesics of Wasserstein space to produce solutions to a type of parabolic PDE.*

Probability Theory:

1. https://en.wikipedia.org/wiki/Wiener_sausage

2. https://link.springer.com/book/10.1007/978-3-319-20828-2

3. https://arxiv.org/abs/2406.10676

4. Lévy's forgery theorem

5. https://www.epatters.org/wiki/stats-ml/categorical-probability-theory

6. https://ncatlab.org/nlab/show/category-theoretic+approaches+to+probability+theory

7. Categorical probability theory

8. https://golem.ph.utexas.edu/category/2024/08/introduction_to_categorical_pr.html

9. https://arxiv.org/abs/1109.1880

10. Connection between fractional differential operators and stochastic processes with jumps

Statistics:

1. https://towardsdatascience.com/t-test-from-application-to-theory-5e5051b0f9dc

Metric Spaces:

1. Lawvere metric spaces: object of $\mathcal{V}$-natural transformations corresponds to $\inf(\mathrm{d}(f(x), g(x)))$.

2. Does the assignment $d(x, y) \mapsto d(x, y)/(1 + d(x, y))$ constructing a bounded metric from a metric be given a universal property?

3. Explore Lawvere metric spaces in a comprehensive manner
4. metric $\mathrm{lcm}(x, y)/\mathrm{gcd}(x, y)$ on $\mathbb{N}$, https://mathoverflow.net/questions/461588/. What shape do balls on $\mathbb{N} \times \mathbb{N}$ have with respect to this metric?
5. https://golem.ph.utexas.edu/category/2023/05/metric_spaces_as_enriched_categories_ii.html
6. Simon Willerton's work on the Legendre–Fenchel transform:
   (a) https://golem.ph.utexas.edu/category/2014/04/enrichment_and_the_legendrefen.html
   (b) https://golem.ph.utexas.edu/category/2014/05/enrichment_and_the_legendrefen_1.html
   (c) https://arxiv.org/abs/1501.03791

Special Functions:

1. https://en.wikipedia.org/wiki/Dickson_polynomial

p-Adic Analysis:

1. https://arxiv.org/abs/2503.08909
2. Analysis of functions $\mathbb{Z}_p \to \mathbb{Q}_q$, $\mathbb{Q}_p \to \mathbb{Q}_q$, $\mathbb{Z}_p \to \mathbb{C}_q$, etc.
   (a) https://siegelmaxwellc.wordpress.com/publications-pre-prints/

Partial Differential Equations:

1. Moduli of PDEs
   (a) https://arxiv.org/abs/2312.05226, https://arxiv.org/abs/2406.16825
   (b) https://arxiv.org/abs/2304.08671, https://arxiv.org/abs/2404.07931
   (c) https://arxiv.org/abs/2507.07937
2. https://en.wikipedia.org/wiki/Homotopy_principle
3. https://mathoverflow.net/questions/125166/wild-solutions-of-the-heat-equation-how-to-graph-them

4. https://math.stackexchange.com/questions/2112841/difference-between-linear-semilinear-and-quasilinear-pdes/5036699#5036699
5. Proof of the smoothing property of the heat equation via:
   (a) Feynman–Kac formula
   (b) Radon–Nikodym + Wiener process has Gaussian as PDF
   (c) Convolution of locally integrable with smooth is smooth
6. Geometry of PDEs:
   (a) https://mathoverflow.net/questions/457268/pdes-and-algebraic-varieties
   (b) Can we build a kind of algebraic geometry of PDEs starting with the notion of the zero locus of a differential operator?
      i. https://ncatlab.org/nlab/show/diffiety

Functional Analysis:

1. https://www.numdam.org/item/SE_1957-1958__1__A3_0/
2. https://thenumb.at/Functions-are-Vectors/
3. Tate vector spaces
4. Analytic sheaves, https://mathoverflow.net/questions/484408/literature-on-fr%c3%a9chet-quasi-coherent-sheaves
5. https://mathscinet.ams.org/mathscinet/article?mr=1257171
6. Vidav–Palmer theorem
7. In the Hilbert space $\ell^2(\mathbb{N};\mathbb{C})$, the operator $(x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}$ admits $(x_n)_{n\in\mathbb{N}} \mapsto (0, x_0, x_1, \ldots)$ as its adjoint.
8. https://arxiv.org/abs/2110.06300

Lie algebras:

1. Pre-Lie algebras
2. Post-Lie algebras

3. https://arxiv.org/abs/2504.05929

Modular Representation Theory:

1. https://en.wikipedia.org/wiki/Deligne%E2%80%93Lusztig_theory
2. https://math.stackexchange.com/questions/167979/representation-of-cyclic-group-over-finite-field
3. https://math.stackexchange.com/questions/153429/irreducible-representations-of-a-cyclic-group-over-a-field-of-prime-order

Homotopy theory:

1. https://mathoverflow.net/questions/495229
2. https://ncatlab.org/nlab/show/Moore+path+category, https://mathoverflow.net/questions/486905/has-the-path-category-of-a-topological-space-been-studied/487212#487212
3. https://ncatlab.org/nlab/show/group+actions+on+spheres, https://www.maths.ed.ac.uk/~v1ranick/papers/wall7.pdf, https://math.stackexchange.com/questions/1575798/which-groups-act-freely-on-sn, https://arxiv.org/abs/math/0212280.
4. Pascal's triangle via homology of n-tori, https://topospaces.subwiki.org/wiki/Homology_of_torus
5. Conditions on morphisms of spaces $f\colon X \to Y$ such that $f^*\colon [Y,K] \to [X,K]$ or $f_*\colon [K,X] \to [K,Y]$ are injective/surjective (so, epi/-monomorphisms in $\mathsf{Ho}(\mathbb{T}))$ or other conditions.

Algebraic Geometry:

1. Galois points, https://bdtd.ibict.br/vufind/Record/USP_c5e6638812a74657c40fcd402a894514
2. https://arxiv.org/abs/2407.09256

Differential Geometry:

1. https://en.wikipedia.org/wiki/Spherical_3-manifold

2. functor of points approach to differential geometry

Number Theory:

1. https://math.stackexchange.com/questions/10233/uses-of-quadratic-reciprocity-theorem/10719#10719

2. https://mathoverflow.net/questions/120067/what-do-theta-functions-have-to-do-with-quadratic-reciprocity

Classical Mechanics:

1. Koopman–von Neumann formalism

2. Relativistic Lagrangian and Hamiltonian mechanics

Quantum Mechanics:

1. https://ncatlab.org/nlab/show/geometrical+formulation+of+quantum+mechanics

Quantum Field Theory:

1. https://arxiv.org/abs/2309.15913 and https://arxiv.org/abs/2311.09284

2. The current ongoing work on higher gauge theory, specially Christian Saemann's

3. The recent work about determining the value of the strong coupling constant in the long-distance range, some pointers and keywords for this are available at this scientific american article.

Combinatorics:

1. Catalan numbers, https://mathstrek.blog/2012/02/19/power-series-and-generating-functions-ii-formal-power-series/

Other:

1. https://arxiv.org/abs/2202.00084

2. Are sedenions and higher useful for anything?

3. https://mathstodon.xyz/@pschwahn/113388126188923908

4. Tambara functors, https://arxiv.org/abs/2410.23052
5. 2-vector spaces
6. 2-term chain complexes. They form a 2-category and middle-four exchange holds, the proof using the fact that we have

$$h_1 \circ \alpha + \beta \circ g_2 = k_1 \circ \alpha + \beta \circ f_2,$$

   which uses the chain homotopy identities

$$\begin{aligned} d_V \circ \alpha &= g_2 - f_2, \\ -\beta \circ d_V &= h_1 - k_1. \end{aligned}$$

   Can we modify this to work for usual chain complexes, seeking an answer to https://mathoverflow.net/questions/424268? What seems to make things go wrong in that case is that the chain homotopy identities are replaced with

$$\begin{aligned} \alpha_{n+1} \circ d_n^V + d_{n-1}^W \circ \alpha_n &= g_n - f_n, \\ \beta_{n+1} \circ d_n^V + d_{n-1}^W \circ \beta_n &= k_n - h_n. \end{aligned}$$

7. https://arxiv.org/abs/1402.2600
8. https://grossack.site/blog
9. Classifying space of $\mathbb{Q}_p$
10. https://www.valth.eu/proc.htm
11. Construction of $\mathbb{R}$ via slopes:
    (a) http://maths.mq.edu.au/~street/EffR.pdf
    (b) https://arxiv.org/abs/math/0301015
    (c) Pierre Colmez's comment “Et si on remplace $\mathbb{Z}$ par $\mathbb{Q}$, on obtient les adèles.”
    (d) I wonder if one could apply an analogue of this construction to the sphere spectrum and obtain a kind of spectral version of the real numbers, as in e.g. following the spirit of https://mathoverflow.net/questions/443018.
12. https://arxiv.org/abs/2406.04936

13. https://mathoverflow.net/a/471510
14. https://mathoverflow.net/questions/279478/the-category-theory-of-span-enriched-categories-2-segal-spaces/448523#448523
15. The works of David Kern, https://dskern.github.io/writings
16. https://qchu.wordpress.com/
17. https://aroundtoposes.com/
18. https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor
19. https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration
20. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
21. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html (Isbell conjugacy and the reflexive completion)
22. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
23. The works of Philip Saville, https://philipsaville.co.uk/
24. https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html
25. https://mathoverflow.net/q/463855 (One-object lax transformations)
26. https://ncatlab.org/nlab/show/analytic+completion+of+a+ring
27. https://en.wikipedia.org/wiki/Quaternionic_analysis
28. https://arxiv.org/abs/2401.15051 (The Norm Functor over Schemes)
29. https://mathoverflow.net/questions/407291/ (Adjunctions with respect to profunctors)

30. https://mathoverflow.net/a/462726 (Prof is free completion of Cats under right extensions)

31. there's some cool stuff in https://arxiv.org/abs/2312.00990 (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.

32. https://ncatlab.org/nlab/show/adjoint+lifting+theorem

33. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

General TODO:

1. https://arxiv.org/abs/2108.11952

2. https://mathoverflow.net/questions/483243/is-there-a-theory-of-completions-of-semirings-similar-to-i-adic-completions-of

3. https://mathoverflow.net/questions/9218/probabilistic-proofs-of-analytic-facts

4. https://x.com/cihanpoststhms

5. Special graded rings, https://mathoverflow.net/questions/403448/in-search-of-lost-graded-rings

    (a) https://arxiv.org/abs/1209.5122

6. Counterexamples in category theory

7. https://math.stackexchange.com/questions/279347/counterexample-math-books

8. Browse MO questions/answers for interesting ideas/topics

9. Change Longrightarrow to Rightarrow where appropriate

10. Try to minimize the amount of footnotes throughout the project. There should be no long footnotes.

Appendices

15.A Other Chapters

Preliminaries

1. Introduction
2. A Guide to the Literature

Sets

3. Sets
4. Constructions With Sets
5. Monoidal Structures on the Category of Sets
6. Pointed Sets
7. Tensor Products of Pointed Sets

Relations

8. Relations
9. Constructions With Relations
10. Conditions on Relations

Categories

11. Categories
12. Presheaves and the Yoneda Lemma

Monoidal Categories

13. Constructions With Monoidal Categories

Bicategories

14. Types of Morphisms in Bicategories

Extra Part

15. Notes

Bibliography

[MO 119454] user30818. *Category and the axiom of choice*. MathOverflow. url: https://mathoverflow.net/q/119454 (cit. on p. 683).

[MO 321971] Ivan Di Liberti. *Characterization of pseudo monomorphisms and pseudo epimorphisms in Cat*. MathOverflow. url: https://mathoverflow.net/q/321971 (cit. on pp. 19, 693).

[MO 350788] Peter LeFanu Lumsdaine. *Epimorphisms of relations*. MathOverflow. url: https://mathoverflow.net/q/455260 (cit. on p. 537).

[MO 382264] Neil Strickland. *Proof that a cartesian category is monoidal*. MathOverflow. url: https://mathoverflow.net/q/382264 (cit. on p. 828).

[MO 460693] Tim Campion. *Answer to "Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I"*. MathOverflow. url: https://mathoverflow.net/q/460693 (cit. on p. 542).

[MO 461592] Emily. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. MathOverflow. url: https://mathoverflow.net/q/461592 (cit. on pp. 18, 544, 545).

[MO 467527] Emily. *What are the 2-categorical mono/epimorphisms in the 2-category of relations?* MathOverflow. url: https://mathoverflow.net/q/467527 (cit. on pp. 18, 534, 541).

[MO 468121a] Emily. *Characterisations of functors F such that F^* or F_* is [property], e.g. faithful, conservative, etc*. MathOverflow. url: https://mathoverflow.net/q/468125 (cit. on pp. 18, 19, 681, 682, 687–689, 694, 696, 699).

[MO 468121b] Emily. *Looking for a nice characterisation of functors F whose precomposition functor F^* is full*. MathOverflow. url: https://mathoverflow.net/q/468121 (cit. on pp. 18, 675).

[MO 468334] Emily. *Is a pseudomonic and pseudoepic functor necessarily an equivalence of categories?* MathOverflow. url: https://mathoverflow.net/q/468334 (cit. on pp. 19, 694).

[MO 484647] Zhen Lin. *Reference for Basic Multicategory Theory*. MathOverflow. url: https://mathoverflow.net/q/484647 (cit. on p. 21).

[MO 490557] Maxime Ramzi. *Answer to "What theorems or insights are well known for ∞-categories but not well known for categories?"* MathOverflow. url: https://mathoverflow.net/a/490557 (cit. on p. 20).

[MO 494959] Emily. *Gaps in the category theory literature you'd like to see filled*. MathOverflow. url: https://mathoverflow.net/q/494959 (cit. on p. 20).

[MO 495003] Fosco. *Answer to "Gaps in the category theory literature you'd like to see filled"*. MathOverflow. url: https://mathoverflow.net/a/495003 (cit. on p. 22).

[MO 497309] Noah Snyder. *Answer to "Gaps in the category theory literature you'd like to see filled"*. MathOverflow. url: https://mathoverflow.net/q/497309 (cit. on p. 22).

[MO 497419] Nick Hu. *Answer to "Gaps in the category theory literature you'd like to see filled"*. MathOverflow. url: https://mathoverflow.net/q/497419 (cit. on p. 22).

[MO 64365] Giorgio Mossa. *Natural transformations as categorical homotopies*. MathOverflow. url: https://mathoverflow.net/q/64365 (cit. on p. 711).

[MSE 1465107] kilian. *Equivalence of categories and axiom of choice*. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/1465107 (cit. on p. 683).

[MSE 2096272] Akiva Weinberger. *Is composition of two transitive relations transitive? If not, can you give me a counterexample?* Mathematics Stack Exchange. url: https://math.stackexchange.com/q/2096272 (cit. on p. 618).

[MSE 267365] J. B. *Show that the powerset partial order is a cartesian closed category*. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/267365 (cit. on p. 166).

[MSE 267469] Zhen Lin. *Show that the powerset partial order is a cartesian closed category*. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/267469 (cit. on p. 132).

[MSE 2719059] Vinny Chase. *$\mathcal{P}(X)$ with symmetric difference as addition as a vector space over $\mathbb{Z}_2$*. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/2719059 (cit. on p. 146).

[MSE 2855868] Qiaochu Yuan. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. url: https://math.stackexchange.com/q/2855868 (cit. on pp. 292, 301).

[MSE 350788] Qiaochu Yuan. *Mono's and epi's in the category Rel?* Mathematics Stack Exchange. url: https://math.stackexchange.com/q/350788 (cit. on pp. 529, 536).

[MSE 733161] Stefan Hamcke. *Precomposition with a faithful functor*. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/733161 (cit. on p. 679).

[MSE 733163] Zhen Lin. *Precomposition with a faithful functor*. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/733163 (cit. on p. 669).

[MSE 749304] Zhen Lin. *If the functor on presheaf categories given by precomposition by F is ff, is F full? faithful?* Mathematics Stack Exchange. url: https://math.stackexchange.com/q/749304 (cit. on p. 679).

[MSE 884460] Martin Brandenburg. *Why are the category of pointed sets and the category of sets and partial functions "essentially the same"?* Mathematics Stack Exchange. url: https://math.stackexchange.com/q/884460 (cit. on p. 293).

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. issn: 1201-561X (cit. on pp. 669, 675, 679, 779).

[Bor94] Francis Borceux. *Handbook of Categorical Algebra I*. Vol. 50. Encyclopedia of Mathematics and its Applications. Basic Category Theory. Cambridge University Press, Cambridge, 1994, pp. xvi+345. isbn: 0-521-44178-1 (cit. on p. 708).

[BS10] John C. Baez and Michael Shulman. "Lectures on n-Categories and Cohomology". In: *Towards higher categories*. Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 1–68. doi: 10.1007/978-1-4419-1524-5_1. url: https://doi.org/10.1007/978-1-4419-1524-5_1 (cit. on pp. 29, 675).

[Cie97] Krzysztof Ciesielski. *Set Theory for the Working Mathematician*. Vol. 39. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. xii+236. isbn: 0-521-59441-3; 0-521-59465-0. doi: 10.1017/CBO9781139173131. url: https://doi.org/10.1017/CBO9781139173131 (cit. on p. 94).

[DFH75] Aristide Deleanu, Armin Frei, and Peter Hilton. "Idempotent Triples and Completion". In: *Math. Z.* 143 (1975), pp. 91–104. issn: 0025-5874,1432-1823. doi: 10.1007/BF01173053. url: https://doi.org/10.1007/BF01173053 (cit. on p. 687).

[DS06] Eduardo Dubuc and Ross Street. "Dinatural Transformations". In: *Reports of the Midwest Category Seminar IV*. Springer. 2006, pp. 126–137 (cit. on p. 20).

[Fre09] Jonas Frey. *On the 2-Categorical Duals of (Full and) Faithful Functors*. https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=4c289321d622f8fcf947e7a7cfd1bdf Archived at https://web.archive.org/web/20240331195546/https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca33 July 2009. url: https://citeseerx.ist.psu.edu/document?repid=rep1%5C&type=pdf%5C&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca33 (cit. on pp. 669, 679).

[GJO24] Nick Gurski, Niles Johnson, and Angélica M. Osorno. "The Symmetric Monoidal 2-Category of Permutative Categories". In: *High. Struct.* 8.1 (2024), pp. 244–320. issn: 2209-0606 (cit. on p. 20).

[Isb68] John R. Isbell. “Epimorphisms and Dominions. III”. In: *Amer. J. Math.* 90 (1968), pp. 1025–1030. issn: 0002-9327,1080-6377. doi: 10.2307/2373286. url: https://doi.org/10.2307/2373286 (cit. on p. 689).

[JS17] Theo Johnson-Freyd and Claudia Scheimbauer. “(Op)lax Natural Transformations, Twisted Quantum Field Theories, and “Even Higher” Morita Categories”. In: *Adv. Math.* 307 (2017), pp. 147–223. issn: 0001-8708,1090-2082. doi: 10.1016/j.aim.2016.11.014. url: https://doi.org/10.1016/j.aim.2016.11.014 (cit. on p. 726).

[JY21] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, Oxford, 2021, pp. xix+615. isbn: 978-0-19-887138-5; 978-0-19-887137-8. doi: 10.1093/oso/9780198871378.001.0001. url: https://doi.org/10.1093/oso/9780198871378.001.0001 (cit. on pp. 3, 10, 835).

[Low15] Zhen Lin Low. *Notes on Homotopical Algebra*. Nov. 2015. url: https://zll22.user.srcf.net/writing/homotopical-algebra/2015-11-10-Main.pdf (cit. on p. 679).

[Lur25] Jacob Lurie. *Kerodon*. https://kerodon.net. 2025 (cit. on p. 11).

[Mac98] Saunders Mac Lane. *Categories for the Working Mathematician*. Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314. isbn: 0-387-98403-8 (cit. on p. 10).

[nLa25a] nLab Authors. *Groupoid*. https://ncatlab.org/nlab/show/groupoid. Oct. 2025 (cit. on p. 685).

[nLa25b] nLab Authors. *Interactions of Images and Pre-images with Unions and Intersections*. https://ncatlab.org/nlab/show/interactions+of+images+and+pre-images+with+unions+and+intersections. Oct. 2025 (cit. on p. 222).

[nLab23] nLab Authors. *Skeleton*. 2025. url: https://ncatlab.org/nlab/show/skeleton (cit. on p. 637).

[Pro25a] Proof Wiki Contributors. *Cartesian Product Distributes Over Set Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Set_Difference (cit. on p. 51).

[Pro25b] Proof Wiki Contributors. *Cartesian Product Distributes Over Symmetric Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Symmetric_Difference (cit. on p. 51).

[Pro25c] Proof Wiki Contributors. *Cartesian Product Distributes Over Union — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Union (cit. on p. 51).

[Pro25d] Proof Wiki Contributors. *Cartesian Product Is Empty Iff Factor Is Empty — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Cartesian_Product_is_Empty_iff_Factor_is_Empty (cit. on p. 282).

[Pro25e] Proof Wiki Contributors. *Cartesian Product of Intersections — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Cartesian_Product_of_Intersections (cit. on p. 51).

[Pro25f] Proof Wiki Contributors. *Characteristic Function of Intersection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Characteristic_Function_of_Intersection (cit. on p. 133).

[Pro25g] Proof Wiki Contributors. *Characteristic Function of Set Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Characteristic_Function_of_Set_Difference (cit. on p. 137).

[Pro25h] Proof Wiki Contributors. *Characteristic Function of Symmetric Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Characteristic_Function_of_Symmetric_Difference (cit. on p. 146).

[Pro25i] Proof Wiki Contributors. *Characteristic Function of Union — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Characteristic_Function_of_Union (cit. on p. 127).

[Pro25j] Proof Wiki Contributors. *Complement of Complement — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Complement_of_Complement (cit. on p. 139).

[Pro25k] Proof Wiki Contributors. *Complement of Preimage equals Preimage of Complement — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Complement_of_Preimage_equals_Preimage_of_Complement (cit. on p. 209).

[Pro25l] Proof Wiki Contributors. *Condition For Mapping from Quotient Set To Be A Surjection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Surjection (cit. on p. 626).

[Pro25m] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be An Injection— Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Injection (cit. on p. 626).

[Pro25n] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be Well-Defined — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Well-Defined (cit. on p. 626).

[Pro25o] Proof Wiki Contributors. *De Morgan's Laws (Set Theory) — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory) (cit. on pp. 137, 139).

[Pro25p] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)/Set Difference/Difference with Union — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory)/Set_Difference/Difference_with_Union (cit. on p. 137).

[Pro25q] Proof Wiki Contributors. *Equivalence of Definitions of Symmetric Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Equivalence_of_Definitions_of_Symmetric_Difference (cit. on p. 145).

[Pro25r] Proof Wiki Contributors. *Image of Intersection Under Mapping — Proof Wiki*. 2025. url: https://proofwiki.

org/wiki/Image_of_Intersection_under_Mapping (cit. on pp. 133, 200, 201).

[Pro25s] Proof Wiki Contributors. *Image of Set Difference Under Mapping — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Image_of_Set_Difference_under_Mapping (cit. on pp. 137, 200).

[Pro25t] Proof Wiki Contributors. *Image of Union Under Mapping — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Image_of_Union_under_Mapping (cit. on pp. 127, 200, 201).

[Pro25u] Proof Wiki Contributors. *Intersection Distributes Over Symmetric Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_Distributes_over_Symmetric_Difference (cit. on p. 146).

[Pro25v] Proof Wiki Contributors. *Intersection Is Associative — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_is_Associative (cit. on p. 132).

[Pro25w] Proof Wiki Contributors. *Intersection Is Commutative — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_is_Commutative (cit. on p. 132).

[Pro25x] Proof Wiki Contributors. *Intersection With Empty Set — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_with_Empty_Set (cit. on p. 132).

[Pro25y] Proof Wiki Contributors. *Intersection With Set Difference Is Set Difference With Intersection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_with_Set_Difference_is_Set_Difference_with_Intersection (cit. on p. 137).

[Pro25z] Proof Wiki Contributors. *Intersection With Subset Is Subset — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_with_Subset_is_Subset (cit. on p. 132).

[Pro25aa] Proof Wiki Contributors. *Mapping From Quotient Set When Defined Is Unique — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Mapping_from_Quotient_Set_when_Defined_is_Unique (cit. on p. 626).

[Pro25ab] Proof Wiki Contributors. *Preimage of Intersection Under Mapping — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Preimage_of_Intersection_under_Mapping (cit. on pp. 133, 209, 210).

[Pro25ac] Proof Wiki Contributors. *Preimage of Set Difference Under Mapping — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Preimage_of_Set_Difference_under_Mapping (cit. on pp. 137, 209).

[Pro25ad] Proof Wiki Contributors. *Preimage of Union Under Mapping — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Preimage_of_Union_under_Mapping (cit. on pp. 127, 209, 210).

[Pro25ae] Proof Wiki Contributors. *Quotient Mapping Is Coequalizer — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Quotient_Mapping_is_Coequalizer (cit. on p. 86).

[Pro25af] Proof Wiki Contributors. *Set Difference as Intersection With Complement — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_as_Intersection_with_Complement (cit. on p. 137).

[Pro25ag] Proof Wiki Contributors. *Set Difference as Symmetric Difference With Intersection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_as_Symmetric_Difference_with_Intersection (cit. on p. 137).

[Pro25ah] Proof Wiki Contributors. *Set Difference Is Right Distributive Over Union — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_is_Right_Distributive_over_Union (cit. on p. 137).

[Pro25ai] Proof Wiki Contributors. *Set Difference Over Subset — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_over_Subset (cit. on p. 137).

[Pro25aj] Proof Wiki Contributors. *Set Difference With Empty Set Is Self — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_with_Empty_Set_is_Self (cit. on p. 137).

[Pro25ak] Proof Wiki Contributors. *Set Difference With Self Is Empty Set — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_with_Self_is_Empty_Set (cit. on p. 137).

[Pro25al] Proof Wiki Contributors. *Set Difference With Set Difference Is Union of Set Difference With Intersection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_with_Set_Difference_is_Union_of_Set_Difference_with_Intersection (cit. on p. 137).

[Pro25am] Proof Wiki Contributors. *Set Difference With Subset Is Superset of Set Difference — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_with_Subset_is_Superset_of_Set_Difference (cit. on p. 137).

[Pro25an] Proof Wiki Contributors. *Set Difference With Union — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Difference_with_Union (cit. on p. 137).

[Pro25ao] Proof Wiki Contributors. *Set Intersection Distributes Over Union — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Intersection_Distributes_over_Union (cit. on pp. 127, 133).

[Pro25ap] Proof Wiki Contributors. *Set Intersection Is Idempotent — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Intersection_is_Idempotent (cit. on p. 133).

[Pro25aq] Proof Wiki Contributors. *Set Intersection Preserves Subsets — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Intersection_Preserves_Subsets (cit. on p. 132).

[Pro25ar] Proof Wiki Contributors. *Set Union Is Idempotent — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Union_is_Idempotent (cit. on p. 127).

[Pro25as] Proof Wiki Contributors. *Set Union Preserves Subsets — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Set_Union_Preserves_Subsets (cit. on p. 127).

[Pro25at] Proof Wiki Contributors. *Symmetric Difference Is Associative — Proof Wiki*. 2025. url: https://proofwiki.org/

wiki/Symmetric_Difference_is_Associative (cit. on p. 146).

[Pro25au] Proof Wiki Contributors. *Symmetric Difference Is Commutative — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_is_Commutative (cit. on p. 146).

[Pro25av] Proof Wiki Contributors. *Symmetric Difference of Complements — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_of_Complements (cit. on p. 146).

[Pro25aw] Proof Wiki Contributors. *Symmetric Difference on Power Set Forms Abelian Group — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_on_Power_Set_forms_Abelian_Group (cit. on p. 146).

[Pro25ax] Proof Wiki Contributors. *Symmetric Difference With Complement — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_with_Complement (cit. on p. 146).

[Pro25ay] Proof Wiki Contributors. *Symmetric Difference With Empty Set — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_with_Empty_Set (cit. on p. 146).

[Pro25az] Proof Wiki Contributors. *Symmetric Difference With Intersection Forms Ring — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_with_Intersection_forms_Ring (cit. on p. 146).

[Pro25ba] Proof Wiki Contributors. *Symmetric Difference With Self Is Empty Set — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_with_Self_is_Empty_Set (cit. on p. 146).

[Pro25bb] Proof Wiki Contributors. *Symmetric Difference With Union Does Not Form Ring — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Symmetric_Difference_with_Union_does_not_form_Ring (cit. on p. 144).

[Pro25bc] Proof Wiki Contributors. *Symmetric Difference With Universe — Proof Wiki*. 2025. url: https://proofwiki.

org/wiki/Symmetric_Difference_with_Universe (cit. on p. 146).

[Pro25bd] Proof Wiki Contributors. *Union as Symmetric Difference With Intersection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Union_as_Symmetric_Difference_with_Intersection (cit. on p. 127).

[Pro25be] Proof Wiki Contributors. *Union Distributes Over Intersection — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Union_Distributes_over_Intersection (cit. on pp. 127, 133).

[Pro25bf] Proof Wiki Contributors. *Union Is Associative — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Union_is_Associative (cit. on p. 127).

[Pro25bg] Proof Wiki Contributors. *Union Is Commutative — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Union_is_Commutative (cit. on p. 127).

[Pro25bh] Proof Wiki Contributors. *Union of Symmetric Differences — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Union_of_Symmetric_Differences (cit. on p. 146).

[Pro25bi] Proof Wiki Contributors. *Union With Empty Set — Proof Wiki*. 2025. url: https://proofwiki.org/wiki/Union_with_Empty_Set (cit. on p. 127).

[PS19] Maximilien Péroux and Brooke Shipley. "Coalgebras in Symmetric Monoidal Categories of Spectra". In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. issn: 1532-0073. doi: 10.4310/HHA.2019.v21.n1.a1. url: https://doi.org/10.4310/HHA.2019.v21.n1.a1 (cit. on p. 457).

[Rie16] Emily Riehl. *Category Theory in Context*. Aurora Dover Modern Math Originals. Dover Publications, Inc., Mineola, NY, 2016, pp. xvii+240. isbn: 978-0-486-80903-8; 0-486-80903-X (cit. on pp. 5, 10, 685).

[Wik25] Wikipedia Contributors. *Multivalued Function — Wikipedia, The Free Encyclopedia*. 2025. url: https://en.wikipedia.org/wiki/Multivalued_function (cit. on p. 493).

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