

**Step 1**

$$\begin{array}{ccccc}
 S^0 & \xrightarrow{\lambda_{S^0}^{\text{Sets}_*, -1}} & S^0 \wedge S^0 & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} S^0 \\
 \downarrow [x] & & \downarrow \text{id}_{S^0} \wedge [x] & & \downarrow \text{id}_{S^0} \otimes_{\text{Sets}_*} [x] \\
 S^0 & \xrightarrow{\lambda_{S^0}^{\prime, -1}} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0 & & \\
 \downarrow [x] & & \downarrow \text{id}_{\mathbb{1}_{\text{Sets}_*}} \wedge [x] & & \\
 X & \xrightarrow{\lambda_X^{\text{Sets}_*, -1}} & S^0 \wedge X & \xrightarrow{\text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1}} & S^0 \otimes_{\text{Sets}_*} X \\
 & & \downarrow \lambda_X^{\prime, -1} & & \downarrow \text{id}_{\mathbb{1}_{\text{Sets}_*}} \wedge \text{id}_X \\
 X & \xrightarrow{\lambda_X^{\prime, -1}} & \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X & & 
 \end{array}$$

Commutative diagram illustrating the proof of the identity  $(\neq)$  in Step 1. The diagram shows the relationship between various tensor products and smash products involving  $S^0$ ,  $X$ , and the unit  $\mathbb{1}_{\text{Sets}_*}$  in the category  $\text{Sets}_*$ .

The diagram is organized into two main rows of objects, with vertical arrows labeled  $[x]$  indicating the application of a natural transformation. The objects are:

- Top row:  $S^0$ ,  $S^0 \wedge S^0$ ,  $S^0 \otimes_{\text{Sets}_*} S^0$ ,  $\mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0$
- Bottom row:  $X$ ,  $S^0 \wedge X$ ,  $S^0 \otimes_{\text{Sets}_*} X$ ,  $\mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X$

The horizontal arrows represent natural transformations:

- $S^0 \rightarrow S^0 \wedge S^0$  is  $\lambda_{S^0}^{\text{Sets}_*, -1}$ .
- $S^0 \wedge S^0 \rightarrow S^0 \otimes_{\text{Sets}_*} S^0$  is  $\text{id}_{\text{Sets}_*|S^0, S^0}^{\otimes, -1}$ .
- $S^0 \otimes_{\text{Sets}_*} S^0 \rightarrow \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0$  is  $\text{id}_{\mathbb{1}_{\text{Sets}_*}}^{\otimes, -1} \wedge \text{id}_{S^0}$ .
- $S^0 \rightarrow \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0$  is  $\lambda_{S^0}^{\prime, -1}$ .
- $S^0 \wedge X \rightarrow S^0 \otimes_{\text{Sets}_*} X$  is  $\text{id}_{\text{Sets}_*|S^0, X}^{\otimes, -1}$ .
- $S^0 \otimes_{\text{Sets}_*} X \rightarrow \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X$  is  $\text{id}_{\mathbb{1}_{\text{Sets}_*}}^{\otimes, -1} \wedge \text{id}_X$ .
- $X \rightarrow \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} X$  is  $\lambda_X^{\prime, -1}$ .
- $X \rightarrow S^0 \wedge X$  is  $\lambda_X^{\text{Sets}_*, -1}$ .

Vertical arrows are labeled  $[x]$ . The diagram is annotated with the following labels:

- $(\neq)$  above the top row.
- $(1)$  below the horizontal arrow  $S^0 \rightarrow \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0$ .
- $(2)$  below the horizontal arrow  $S^0 \wedge X \rightarrow S^0 \otimes_{\text{Sets}_*} X$ .
- $(3)$  to the left of the vertical arrow  $S^0 \rightarrow X$ .
- $(4)$  to the right of the vertical arrow  $S^0 \otimes_{\text{Sets}_*} S^0 \rightarrow \mathbb{1}_{\text{Sets}_*} \otimes_{\text{Sets}_*} S^0$ .
- $(5)$  in the center of the diagram.