Mathematics Project (Section B)

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Construction of different types of conics using the concept of double cone and a plane

Acknowledgements

Firstly, I want to thank God for giving me good health and allowing me to complete this project. I am also grateful to my Maths teacher, Mrs Reena who helped me all throughout and also taught me the concepts well. I would also like to thank my family and my friends who act as my backbone and provide me with as much support as possible.

Abstract

In this project, we will start from the basics and go through the various definitions and theorems that make up conic sections and the study of ellipses, parabolas, hyperbolas and much more.

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Chapter 1
Introduction

Here we are going to go through the basics of conic sections and understand the entire coordinate geometry using the tools, a right double cone and a plane. There are also some preliminaries that we should define before starting on the project.

1.1 3 Dimensional Space

Definition 1.1.1. In mathematics, a tuple of n numbers can be understood as the Cartesian coordinates of a location in a n-dimensional Euclidean space. The set of these n-tuples is commonly denoted R^n,and can be identified to the n-dimensional Euclidean space. When n = 3, this space is called three-dimensional Euclidean space

This is the space in which our cone will be constructed and the plane will be able to intersect the plane to form all the different conic sections.

The three dimensional space has three axis, all mutually perpendicular to each other.

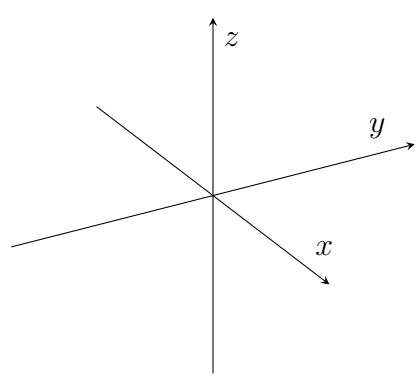

Figure 1.1: 3D Cartesian Plane

The three axes, are called X, Y and Z respectively.

Definition 1.1.2. The point where the three axes coincide is knows as the origin.

The points in the 3D Space are referred to buy using (x,y,z) which are known as the coordinates of the point in the space.

Origin is given the coordinates (0,0,0) because the point where the three axes coincide is at their respective 0s, i.e. x = 0, y = 0 and z = 0.

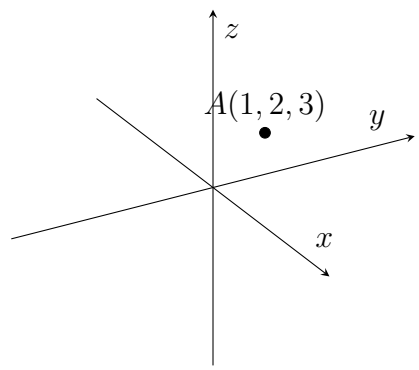

Figure 1.2: Coordinates of a point

Theorem 1.1.3. *The distance between two points (x_1,y_1,z_1) and (x_2,y_2,z_2) is given by the formula* $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

1.2 Cone

Definition 1.2.1. A cone is a three-dimensional solid geometric shape having a circular base and a pointed edge at the top called the apex. A cone has one face and a vertex. There are no edges for a cone.

Thus V = 1, F = 1 and E = 1
$\implies$ V + F - E = 1 + 1 = 2
This implies that the cone follows the Euler's formula for 3 Dimensional objects.

A cone can be formed by taking a line for example OD and moving D in a circular locus around any one axis (let it be Z axis) such that the locus of the line forms the cone.

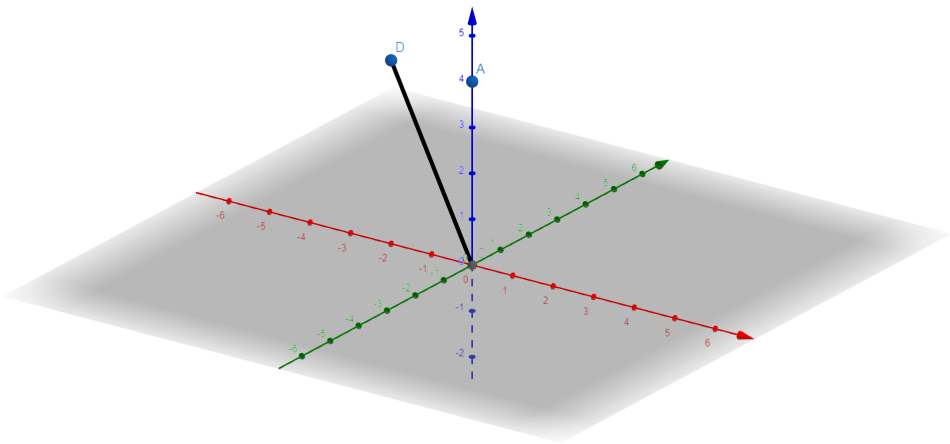

Figure 1.3: Line OD

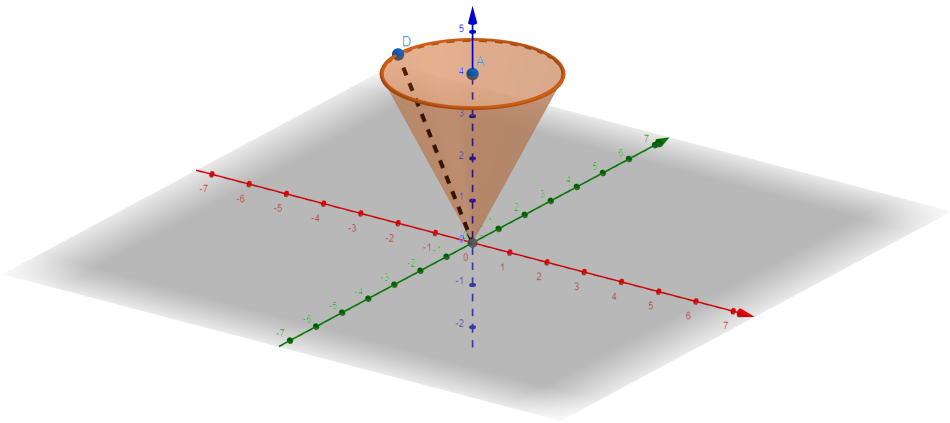

Figure 1.4: Upward cone

Thus the cone can be expressed in the form of an equation:

$$z = \pm\sqrt{Ax^2 + By^2}$$

The plus sign gives the upward facing cone and negative sign gives downward facing cone.

1.3 Double Cone

Definition 1.3.1. In the case of line segments, the cone does not extend beyond the base, while in the case of half-lines, it extends infinitely far. In the case of lines, the cone extends infinitely far in both directions from the apex, in which case it is sometimes called a double cone. Either half of a double cone on one side of the apex is called a nappe.

Here we need a line DE that passes through origin. Now we will rotate the line such that D and E trace a circle and the locus of the entire line traces the double cone. This line DE is known as generator and the Z-axis is the Axis of the cone.

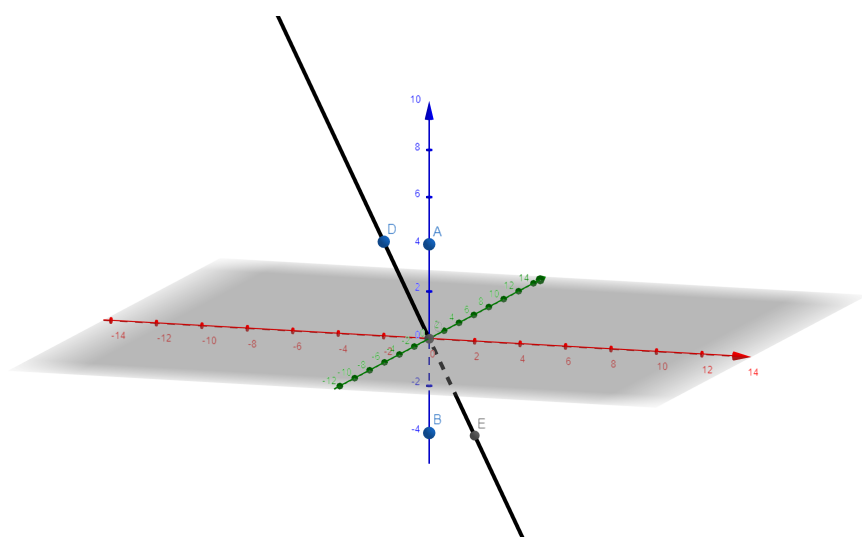

Figure 1.5: Generator

The double cone can be represented using the equation:

$$z^2 = Ax^2 + By^2$$

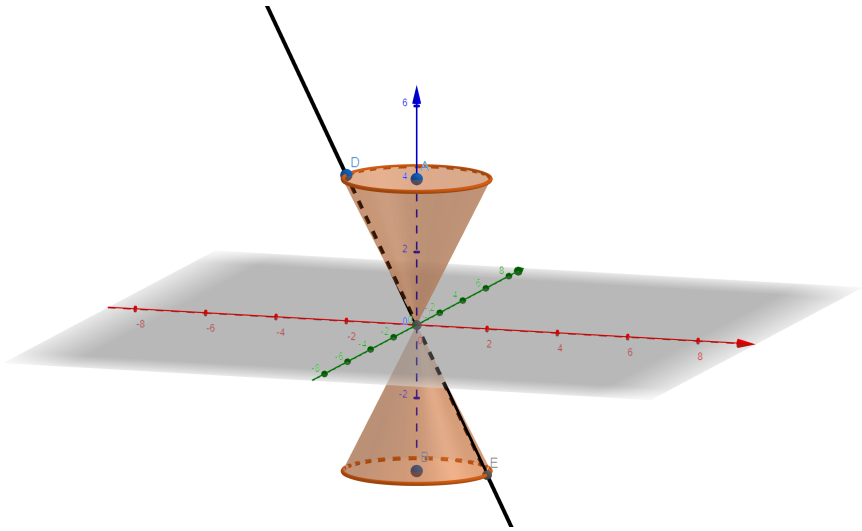

Figure 1.6: Double Cone

Chapter 2
Different Sections of a Double Cone

Here we will be using a plane to cut the cone in various places, so that we get the various sections ("conic sections"). Before continuing, we have to make some definitions.

Definition 2.0.1. A plane is a flat, two-dimensional surface that extends indefinitely.

Definition 2.0.2. The line that was rotated to make the double cone is called the generator. eg OD in the previous section.

Definition 2.0.3. The line around which the generator is rotated to make the double cone is called the axis. e.g AB(z-axis) in the previous section.

Definition 2.0.4. The vertex of the double cone is the point where the generator and the axis meets.

Definition 2.0.5. Let the angle between the generator and the axis be α

Definition 2.0.6. Let the angle between the plane and the axis be θ

2.1 Plane passes through vertex

When the plane passes through the vertex of the double cone, we get 3 cases. These conics are called degenerate conics.

2.1.1 When $90 \leq \theta < \alpha$

Here the plane will intersect with the cone to create a point in the 3D space, i.e., the plane will intersect with the vertex of the double cone only.

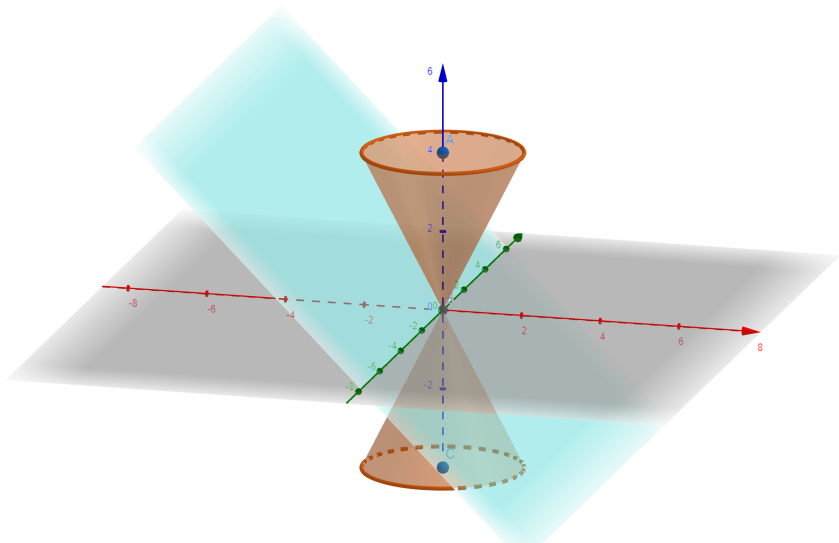

Figure 2.1: Point

2.1.2 when $\theta = \alpha$

Here the plane will intersect the edge of the double cone (which is a straight line).

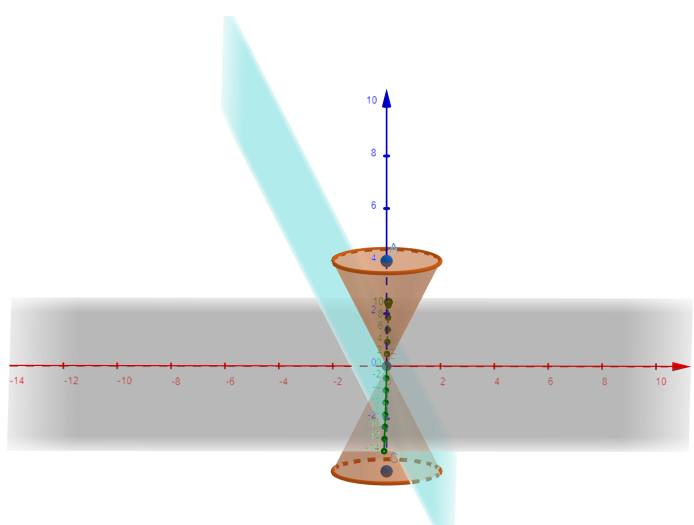

Figure 2.2: Line

2.1.3 when $\theta = 90$

Here the plane will intersect the double cone to form two straight lines or "a pair of straight lines"

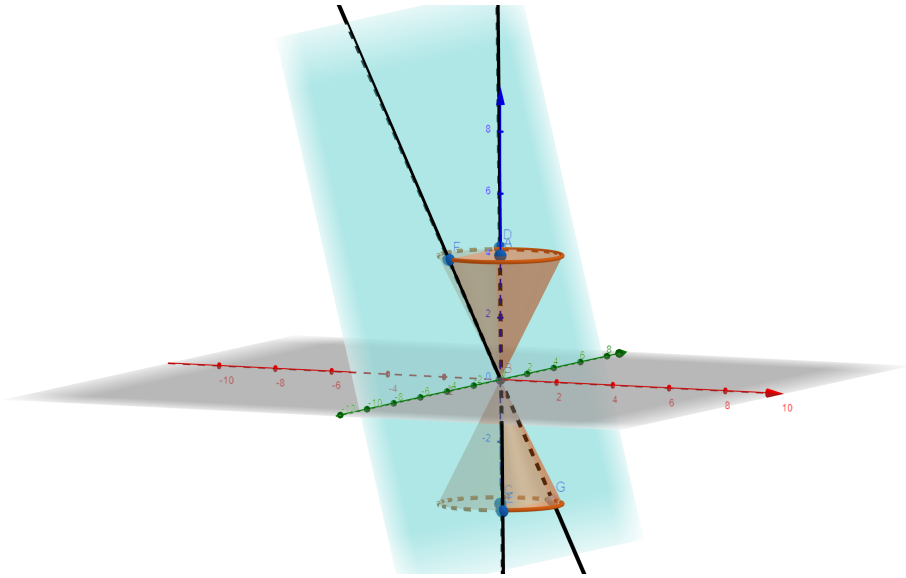

Figure 2.3: Pair of Straight Lines

2.2 Plane does not pass through vertex

Here we get 4 different conic sections.

2.2.1 when $\theta = 90$

This gives us a circle. As shown in Figure 2.4

2.2.2 When $90 \leq \theta < \alpha$

This gives us an ellipse. As shown in Figure 2.5

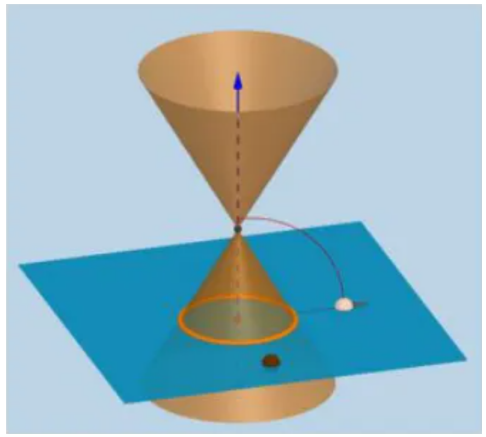

Figure 2.4: Circle

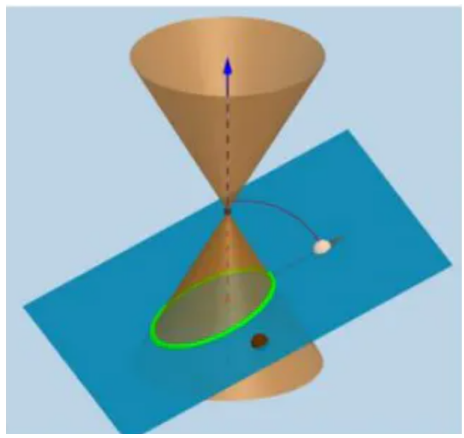

Figure 2.5: Ellipse

2.2.3 $\theta = \alpha$

This gives us a parabola. As shown in figure 2.6

2.2.4 $\theta < \alpha$

This gives us a hyperbola. As shown in figure 2.7

Also all the 4 conics can be shown in one cone as follows:

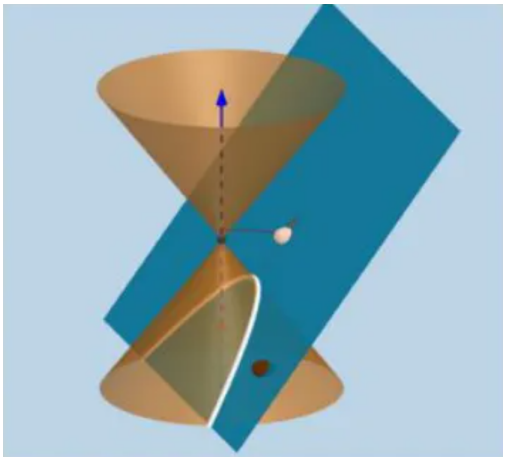

Figure 2.6: Parabola

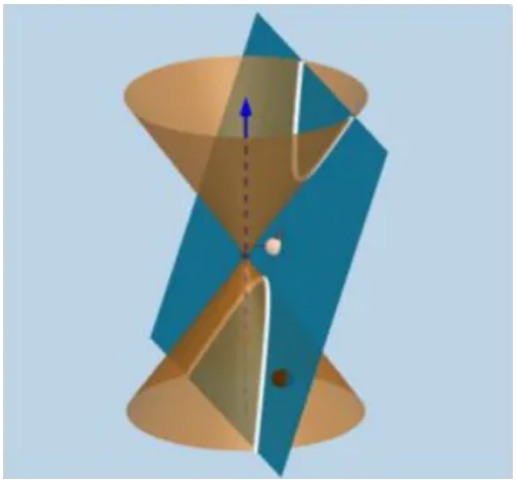

Figure 2.7: Hyperbola

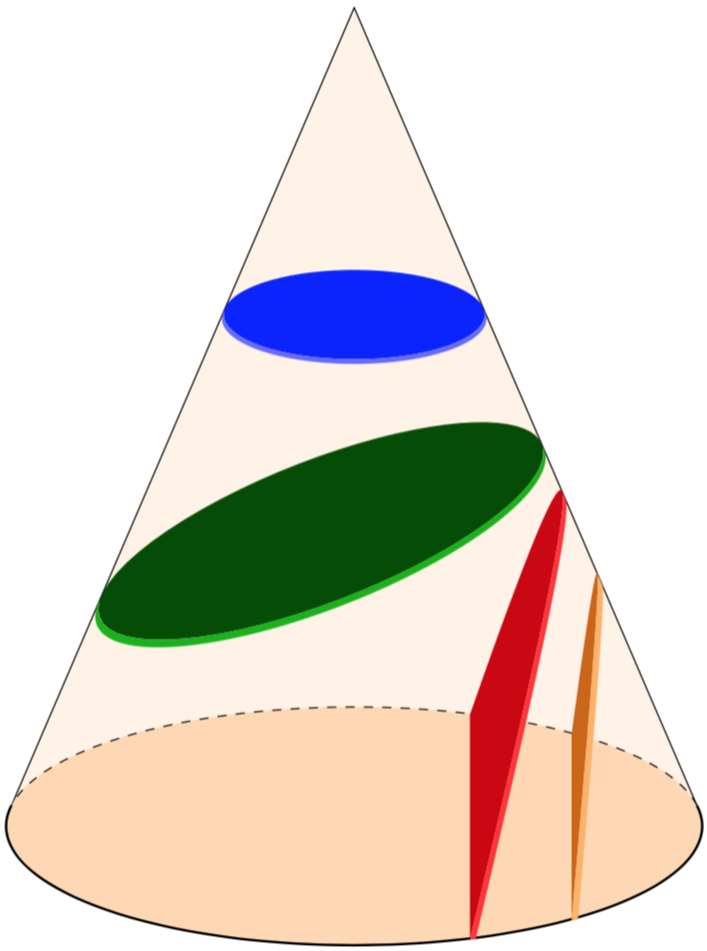

Figure 2.8: Conics

Chapter 3
Analyzing the Conics

From the previous section, we have found 7 conic sections. There were:

1. Degenerate Conics
   (a) Point
   (b) Line
   (c) Pair of Straight Lineas
2. Non-Degenrate Conics
   (a) Circle
   (b) Ellipse
   (c) Parabola
   (d) Hyperbola

As there were a lot of conics formed, mathematicians decided to make the circle as the standard.

Later they introduced a term called eccentricity.

Definition 3.0.1. Eccentricity(e) is a measure of how much the conic section deviates from being circular.

Later conic sections got a more refined definition. Before that we have to define a conic section.

Definition 3.0.2. A conic section is the locus of a point P, which moves in such a way that its distance from a fixed point S (called Focus) always bears a constant

ratio to its distance from a fixed straight line (called Directrix), all being in the same plane.

Definition 3.0.3. The eccentricity of the conic section is the constant ratio of the distance of point P from fixed point S (called Focus) to the distance of point P from a fixed straight line (called Directrix).

Also let distance from point to focus be PS and distance from point to line be PM. So e = PS/PM

3.1 Degenerate Conics

Degenerate Conics were those conics which were created when the plane passed through the vertex of the double cone. They are formed when the fixed point (focus) lies on the fixed line (directrix). More details about the degenerate conics are as

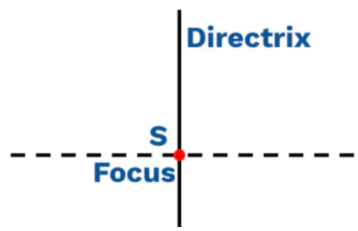

Figure 3.1: Directrix and Focus for degenerate conics

follows:

3.1.1 Point

A point is the most basic concept in all of coordinate geometry. It forms the fundamental concept on which all of the remaining conic section depends.

Generally a point is expressed as (x,y) where x represents the x coordinate of the point and y represents the y coordinate of the point.

For a point, e = 0.

3.1.2 Imaginary Line

Here $0 < e < 1$. Such a line is not possible because as e = PS/PM and if $e < 0$. PS cannot be less than PM as PM is a perpendicular and perpendicular is the shortest

distance between a line and a point.

3.1.3 Coincident Lines

Here e = 1. That is we actually get two lines but both the lines are the same. Let us take for example that the directrix is x = 0 and the focus is (0,0).

$$PM = \mod x$$
$$PS = \sqrt{x^2 + y^2}$$
$$\frac{PS}{PM} = e = 1$$
$$PS = PM$$
$$\sqrt{x^2 + y^2} = \mod x$$
$$x^2 + y^2 = x^2$$
$$y^2 = 0$$
$$y = 0$$

The general equation for a line is Ax + By + C = 0.

3.1.4 Pair of Straight Lines

Here e > 1. Such that we get a pair of straight lines. Taking the previous example and assuming e = 2.

$$PM = \mod x$$
$$PS = \sqrt{x^2 + y^2}$$
$$\frac{PS}{PM} = e = 2$$
$$PS = 2PM$$
$$\sqrt{x^2 + y^2} = 2 \mod x$$
$$x^2 + y^2 = 4x^2$$
$$3x^2 - y^2 = 0$$
$$(\sqrt{3}x - y)(\sqrt{3}x + y) = 0$$

So we get two lines y = $\sqrt{3}x$ and y = $-\sqrt{3}x$.

3.2 Non-Degenerate Conics

Non-degenerate conics are those conics that do not pass through the vertex of the double cone.

3.2.1 Circle

Here e = 0.
A circle has the general equation:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

It is the most basic Non-Degenerate Conic and it is the figure based on which eccentricity is defined.

3.2.2 Ellipse

Here 0 < e < 1.
An ellipse has two general equations:
horizontal ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and vertical ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Where a and b are the lengths of the semi-minor and semi-major axes respectively.

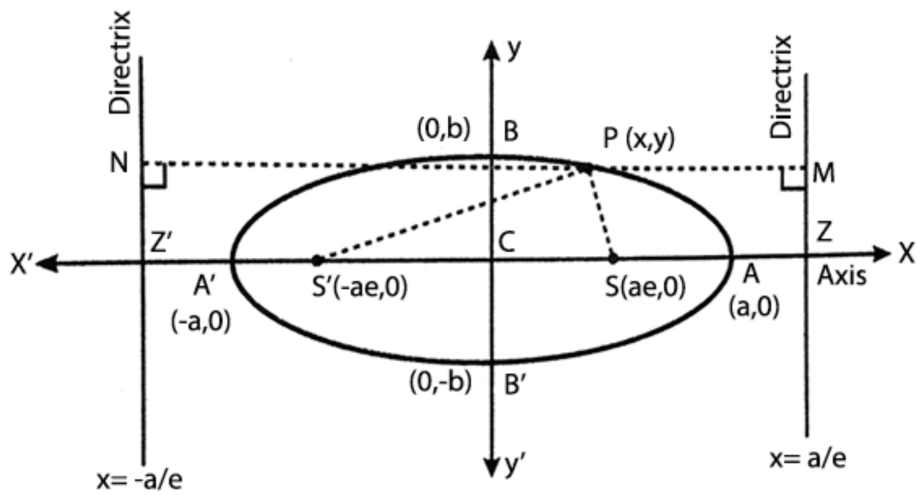

Figure 3.2: Details of Ellipse

3.2.3 Parabola

Here e = 1.
A parabola has 4 general equations:
Left Hand Parabola

$$y^2 = 4ax$$

Right Hand Parabola

$$y^2 = -4ax$$

Upward Parabola

$$x^2 = 4ay$$

Downward Parabola

$$x^2 = -4ay$$

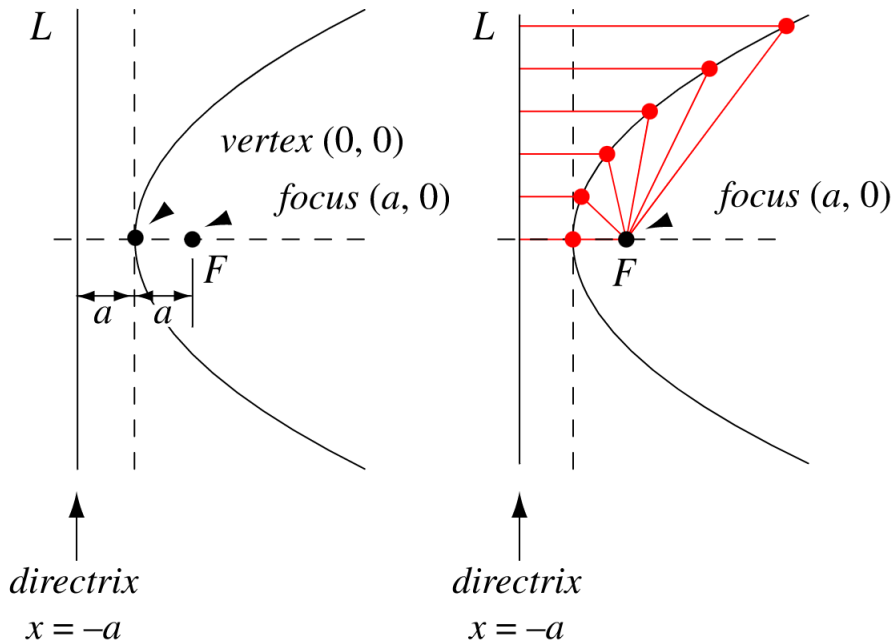

Figure 3.3: Details of Parabola

3.2.4 Hyperbola

Here e = 1.
A parabola has the general equation as:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

It has two axis - transverse and conjugate.

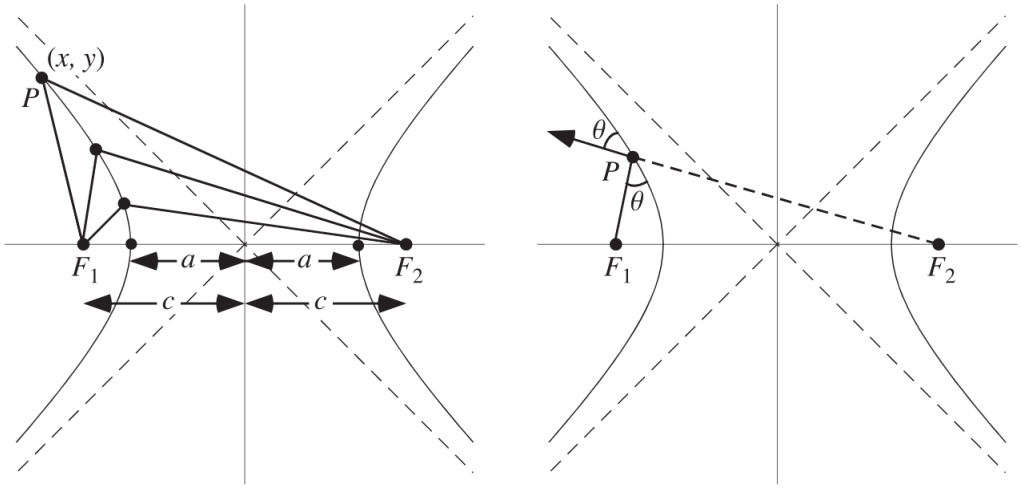

Figure 3.4: Details of Hyperbola

Chapter 4
Findings

1. Double cones are 3D objects generated using a line(generator) rotated around an axis.
2. Conic Section involves the study of the sections made on a double cone made by a plane.
3. Conic Sections includes all the important parts of coordinate geometry from a point right up to a hyperbola.
4. There are two types of sections:
   (a) Degenerate Conics: Conic section that are formed by the intersection of the plane at the vertex of the double cone are called Degenerate Conic Sections.
   (b) Non-Degenerate Conics: Conic Sections that are not formed by the intersection of the plane at the vertex of the double cone are called non-Degenerate Conic Sections.
5. There are a total of 7 different conic sections. [8 if you include 'Imaginary Lines']
6. Eccentricity of the conic section was actually created to represent how different a conic section was from a circle.
7. Later, eccentricity was defined as the ration between 'the distance between point P and focus S' and 'the distance between point P and the axis'.
8. The different conics are:
   (a) Point
      i. It forms the basics of all coordinate geometry.

ii. Here e = 0

iii. A point is represented as (x,y)

(b) Imaginary Lines

i. It is called so because we cannot actually draw the line in our plane.

ii. This is because here 0 < e < 1 which is actually not possible to draw [similar to imaginary numbers].

(c) Coincident Lines

i. In reality these are two lines which coincide.

ii. Here, e = 1.

iii. As it is a line, it can be represented as Ax + By + C = 0.

(d) Pair of Straight Lines

i. These are two different lines.

ii. Here e > 1/

iii. The two lines intersect at a point and they are not parallel.

iv. The equation for a 'Pair of Straight Lines' can actually be formed by $L_1 \times L_2 = 0$ where L_1 and L_2 are the equations of the individual lines.

(e) Circle

i. This is a locus of all points equidistant from a point.

ii. Here e = 0.

iii. Circles can be represented using the formula: $x^2 + y^2 = r^2$

(f) Ellipse

i. Here 0 < e < 1.

ii. Ellipse has two focii and two directrices.

iii. It has two axes, a major axes and a minor axes.

iv. An ellipse can be represents using the formula $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(g) Parabola

i. Here e = 1.

ii. So a parabola is the locus of all points equidistant from a line(directrix) and a point(focus).

iii. It can be represented using two equation - $y^2 = 4ax$ and $x^2 = 4ay$ for the two different kind of parabola.

iv. It only has one axis.

(h) Hyperbola

i. Here e > 1

ii. The hyperbola also has two axes, a conjugate axis and a transverse axis.

iii. A hyperbola can be represented using the equation - $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Chapter 5
References

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