Lecture_1
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Lecture_1

Prereqs: Mathematics and Classical Computing

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Course Overview:

- Lecture 1: Pre-requisite overview
- Lecture 2: Qubits, Quantum Logic Gates and Quantum Circuits
- Lecture 3: Teleportation, No Cloning Theorem, Superdense Coding, and BB84
- Lecture 4: Review on Quantum Circuits, Oracle, Deutsch
- Lecture 5: Practical Workshop
- Weekend: Hackathon!

Goals of this lecture

- Get you comfortable with the basic tools of mathematics commonly used in quantum computing
- Get you comfortable with Dirac notation, used in quantum mechanics and therefore in quantum computing
- Look at the inner works of classical computers so we can compare them to those of quantum computers later

Contents

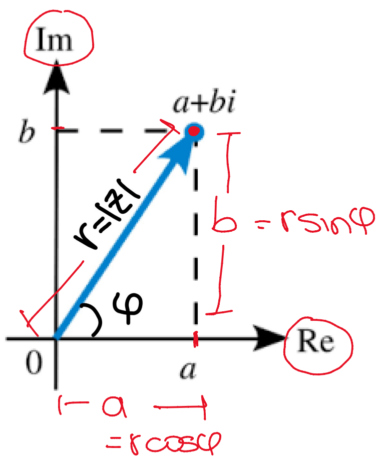
- Complex Numbers
  - Basic operations
- Linear algebra
  - Vector spaces
  - Operators
  - Eigenvalues, eigenvectors
- Classical computing
  - Gates and universality

Complex numbers

- $\underline{z} = \underline{a} + \underline{bi}$, where $\underline{i^2 = -1}$: Cartesian
- $z = \underbrace{r\cos(\varphi)}_{a} + i\,\underbrace{r\sin(\varphi)}_{b}$

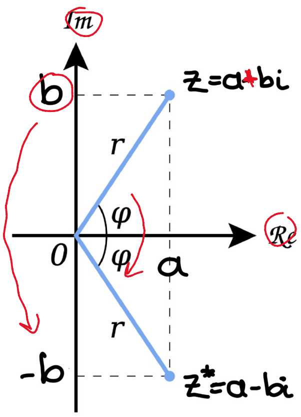
  $= |z|\,(\underbrace{\cos\varphi + i\sin\varphi}_{e^{i\varphi}}) = re^{i\varphi}$
- Magnitude of z is $|z| = \sqrt{a^2+b^2}$
- Addition: $z = a+bi$, $w = c+di$ $z+w = a+bi+c+di = (a+c)+(b+d)i$
- Subtraction: $z-w = (a-c)+(b-d)i$
- Multiplication: $zw = (a+bi)(c+di) = ac + bci + adi + bd\underbrace{i^2}_{-1}$ $-bd$

  $= \underline{(ac-bd)} + \underline{(bc+ad)i}$



Complex conjugation

- All it does is turn i into $-i$
- If we have a complex number $z = a + bi$, its complex conjugate is $z^* = a - bi$



- z times its conjugate:

$$zz^* = (a+bi)(a-bi) = a^2 - \cancel{ab}i + \cancel{ab}i - b^2 \underbrace{i^2}_{-1}$$
$$= a^2 + b^2 = |z|^2 = r^2$$

$w = c + di$

- Product of conjugates: $z^* w^* = (a-bi)(c-di)$

$$= ac - adi - bci + bdi^2$$
$$= (ac - bd) - (ad + bc)i$$
$$= (zw)^*$$

Complex conjugate

- Sum of conjugates: $z^* + w^* = a - bi + c - di$

$$= (a+c) - (b+d)i$$

- Conjugate of conjugate: $i \longrightarrow -i \longrightarrow -(-i) = i$

$$(z^*)^* = z$$

- Magnitude of z: $zz^* = |z|^2$

$$|z| = \sqrt{zz^*}$$

Linear algebra

- Since quantum states can be represented as vectors, we need to get into linear algebra
- Linear algebra allows us to study vectors, how they interact with each other, and how they are transformed about matrices
- The field is much broader than this, but we only need some basic concepts to get started

Vectors

- A vector has the form $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ or $\vec{a}^T = (a_1 \cdots a_n)$

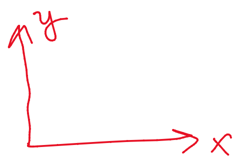
  column   row

- The dimension of a vector is its number of entries   n

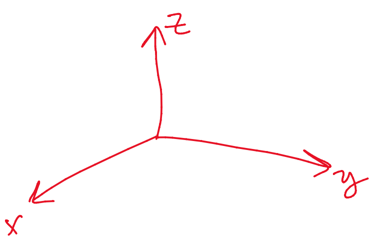
  real numbers.

  - $n = 1$

  - $n = 2$



  - $n = 3$



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Vectors

- Vector addition

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad \vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \quad (a_1 + b_1 \cdots a_n + b_n)$$

- Scalar multiplication

k

$$k\vec{a} = \begin{pmatrix} ka_1 \\ \vdots \\ ka_n \end{pmatrix}$$

$\dagger$: dagger

- Conjugate transpose:

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \vec{a}^{*T} = \vec{a}^{\dagger} = (a_1^* \cdots\cdots a_n^*)$$

- Magnitude:

$$\|\vec{a}\| = |\vec{a}| = \sqrt{|a_1|^2 + \cdots + |a_n|^2} = \left[(a_1^* \cdots a_n^*) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right]^{1/2}$$

Vector space

$\{\vec{u}_1, \vec{u}_2, \ldots\}$

- A vector space is a collection of vectors, a field of scalars, and two operations: vector addition and scalar multiplication
- There are real vector spaces and complex vector spaces
- In quantum computing, we work with **complex vector spaces**

Vector space

- Imagine a vector space V and two vectors $\vec{u}$ and $\vec{v}$ in it. The following are true:

  V: $\vec{u}$ $\vec{0}$ $\vec{u}+\vec{v}$ $\vec{v}$ $\vec{w}$

  - $\vec{u} + \vec{v}$ is in V
  - $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{w} + \vec{v})$, where $\vec{w}$ is also in V
  - There is a $\vec{0}$ vector in V, where $\vec{u} + \vec{0} = \vec{u}$
  - For every $\vec{u}$ there is a $\boxed{-\vec{u}}$ such that $\vec{u} - \vec{u} = \vec{0}$

    inverse

Vector space

- Imagine a vector space V with scalar field $\mathbb{C}$, two vectors $\vec{u}$ and $\vec{v}$ in V, and scalars c and d in $\mathbb{C}$. The following are true:

  V: $\vec{u}$ $\vec{v}$ $c\vec{u}$

  $c, d \in \mathbb{C}$

  - $c\vec{u}$ is in V
  - $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
  - $c(d\vec{u}) = d(c\vec{u})$
  - $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
  - $1(\vec{u}) = \vec{u}$

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Dot (Inner) product

Op. b/. vectors.

- Vector space → Hilbert Space

$\mathbb{C}^n$ $\quad \vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$

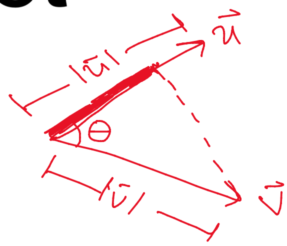
- The dot product of two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$ is denoted by: $u_1, \dots, u_n \in \mathbb{C}$

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i^* v_i = u_1^* v_1 + \dots + u_n^* v_n$$

- We can write the magnitude of a vector in terms of an inner product:

$$|\vec{u}| = \sqrt{\underbrace{\sum_{i=1}^{n} u_i^* u_i}_{\vec{u} \cdot \vec{u}}} = \underbrace{\sqrt{\sum_{i=1}^{n} |u_i|^2}}_{[|u_1|^2 + \dots + |u_n|^2]^{1/2}}$$

Dot (Inner) product



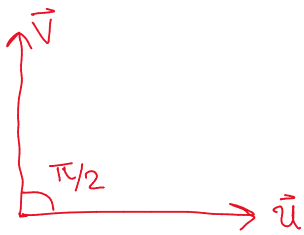
- Geometrically, the dot product is defined as

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \, cos\theta$$

- If we are working with unit vectors,

$$|\vec{u}| = |\vec{v}| = 1$$

- If the vectors are orthogonal, $\theta = \pi/2$ and thus $\boxed{\vec{u} \cdot \vec{v} = 0}$



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Orthonormal basis

Orthogonal unit

- Vectors can be represented as a sum of other vectors, i.e. a linear combination

V

$\vec{u}$ B $\vec{v}_1 \ldots$

- Particularly, we are interested in representing a vector as a sum of linearly independent vectors of unit magnitude that are orthogonal to each other

Orthonormal Basis.

$a, b \in \mathbb{C}$

- $\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{v}_2} \right\}$ $|\vec{v}_1| = |\vec{v}_2| = 1$ $\vec{v}_1 \cdot \vec{v}_2 = 0$

- $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_2}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_3} \right\}$

Orthonormal basis

- Given a set of basis vectors $\vec{u}_i$, we can write any vector as a linear combination of them

$\frac{1}{2} + \frac{1}{2} = 1$

$$\vec{a} = \sum_i \alpha_i \vec{u}_i$$

$= \alpha_1 \vec{u}_1 + \ldots + \alpha_n \vec{u}_n$

V

$\vec{a}$ B $\vec{u}_1, \ldots, \vec{u}_n$

$\left\{ \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\vec{u}_1}, \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{u}_2} \right\}$ $|\vec{u}_1| = |\vec{u}_2| = 1$ $\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{2} - \frac{1}{2} = 0$

$\alpha_1 = \alpha_2 = \frac{\sqrt{2}}{2}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 = \frac{\alpha_1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\alpha_2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Dirac notation

- Makes working with the concepts presented easier
- Uses bras and kets: $\underbrace{\langle \cdot \,|}_{\text{Bra}}$ and $\underbrace{|\, \cdot \rangle}_{\text{Ket}}$
- $|v\rangle = \underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}_{\text{column}}$ and $\langle v| = |v\rangle^\dagger = (v_1^* \quad \cdots \quad v_n^*)$

$$\vec{u} \cdot \vec{v} = \underbrace{(u_1^* \cdots u_n^*)}_{\langle u|} \underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}_{|v\rangle}$$

- A bra-ket takes an inner product!

$$\langle v|u\rangle = (v_1^* \quad \cdots \quad v_n^*) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_{i=1}^{n} v_i^* u_i$$

$\langle v|\,|u\rangle = \langle v|u\rangle$

Braket

Qubits as quantum states

$\alpha, \beta \in \mathbb{C}$

- We can represent a quantum state as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- $|0\rangle$ and $|1\rangle$ are distinguishable quantum states
- $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Therefore, we can write $|\psi\rangle = \alpha \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|0\rangle} + \beta \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|1\rangle} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

V = Quantum States

$|0\rangle$
$|1\rangle$
$\alpha|0\rangle + \beta|1\rangle$

Qubit representation

- A qubit is represented as $\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$
- Values α and β are called probability amplitudes. Furthermore, $|\alpha|^2$ is the probability of finding the qubit in state $|0\rangle$ and $|\beta|^2$ is the same but for state $|1\rangle$
- Therefore, $\boxed{|\alpha|^2 + |\beta|^2 = 1}$ (Statevector) In other words, $\boxed{||\psi\rangle| = 1}$ $Pr(|0\rangle) + Pr(|1\rangle) = 1$

Qubit representation

- What if $|\alpha|^2 + |\beta|^2 \neq 1$? We need to normalize our state!

$$|\hat{\psi}\rangle = \frac{|\psi\rangle}{||\psi\rangle|}$$

$$|\psi\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 2$$

$\alpha = 1 \qquad \beta = i$

$|\beta| = 1 = i^* i = -1 \cdot i^2 = 1$

$||\psi\rangle| = \sqrt{|\alpha|^2 + |\beta|^2}$

$$|\hat{\psi}\rangle = \frac{|\psi\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}} = \frac{|\psi\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

Multi-qubit representation

- Most of the times, we will be working with more than 1 qubit
- Suppose you have two qubits $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \delta|0\rangle + \gamma|1\rangle$. The combined state is $\boxed{|\psi\rangle \otimes |\phi\rangle}$ Tensor product.

$(\alpha|0\rangle + \beta|1\rangle) \otimes (\delta|0\rangle + \gamma|1\rangle)$

$\alpha\delta|0\rangle \otimes |0\rangle + \alpha\gamma|0\rangle \otimes |1\rangle + \beta\delta|1\rangle \otimes |0\rangle + \beta\gamma|1\rangle \otimes |1\rangle$

$\alpha\delta|00\rangle + \alpha\gamma|01\rangle + \beta\delta|10\rangle + \beta\gamma|11\rangle$

$0 \quad 1 \quad 2 \quad 3$

1 q → 2 st.
2 q. → 4 states
3 q → 8 state
⋮
n q → 2^n st..

- Dimension of vector representing n qubits = 2^n.
- A statevector in a Hilbert space $\mathcal{H}$ of dimension 2^n describes an n qubit system

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Operators

$|v\rangle$

- Operators act on qubits to transform them. $A|v\rangle = |w\rangle$
- Operators in quantum computing need to be linear and unitary to preserve probability:
  - $A(|v\rangle + |w\rangle) = A|v\rangle + A|w\rangle$
  - $A(c|v\rangle) = c(A|v\rangle)$, where c is a complex scalar
  - $||v\rangle| = |A|v\rangle|$, i.e., magnitude is conserved

Linearity

Operators

$\langle a|b\rangle = \langle a|\,|b\rangle$

$|b\rangle\langle a|$

- What is the outer product ($|a\rangle\langle b|$)?
  ket bra

$|a\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\langle b| = (b_1^* \cdots b_n^*)$

- Pauli Operators:
  - $I = |0\rangle\langle 0| + |1\rangle\langle 1|$
  - $X = |1\rangle\langle 0| + |0\rangle\langle 1|$
  - $Y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$
  - $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Identity

Pauli

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Operators

- Let's apply the Pauli operators on a qubit defined by
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

- $I|\psi\rangle = \left(\boxed{|0\rangle\langle 0| + |1\rangle\langle 1|}\right)(\alpha|0\rangle + \beta|1\rangle)$

$= \alpha|0\rangle\cancel{\langle 0|0\rangle}^{1} + \beta|0\rangle\cancel{\langle 0|1\rangle}^{0} + \alpha|1\rangle\cancel{\langle 1|0\rangle}^{0} + \beta|1\rangle\cancel{\langle 1|1\rangle}^{1}$

$= \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$

- $X|\psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle)$

$= \alpha|0\rangle\cancel{\langle 1|0\rangle}^{0} + \beta|0\rangle\cancel{\langle 1|1\rangle}^{1} + \alpha|1\rangle\cancel{\langle 0|0\rangle}^{1} + \beta|1\rangle\cancel{\langle 0|1\rangle}^{0}$

$= \beta|0\rangle + \alpha|1\rangle$

$= \alpha|1\rangle + \beta|0\rangle$ $\qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Y, Z

Operators as matrices

- To convert from outer product to matrix, just perform the operation they encode

$$|a\rangle\langle b| = \begin{pmatrix}a_1\\ \vdots \\ a_n\end{pmatrix}(b_1^* \quad \dots \quad b_n^*) = \begin{pmatrix} a_1 \cdot b_1^* & \cdots & a_1 \cdot b_n^* \\ \vdots & \ddots & \vdots \\ a_n \cdot b_1^* & \cdots & a_n \cdot b_n^*\end{pmatrix}$$

- Now, we can write the Pauli operators as matrices
- $|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1 \ \ 0) = \begin{pmatrix}1\cdot 1 & 1\cdot 0\\ 0\cdot 1 & 0\cdot 0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$
- $|0\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}\underbrace{(0 \ \ 1)}_{\langle 1|} = \begin{pmatrix}1\cdot 0 & 1\cdot 1\\ 0\cdot 0 & 0\cdot 1\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$
- $|1\rangle\langle 0| = \begin{pmatrix}0&0\\1&0\end{pmatrix}$
- $|1\rangle\langle 1| = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

Operators as matrices

- Let's apply multiple Pauli matrices to the same qubit

$$I = (\underline{|0\rangle\langle 0| + |1\rangle\langle 1|} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} X &= |0\rangle\langle 1| + |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ Y &= i|0\rangle\langle 1| - i|1\rangle\langle 0| = i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\} \text{Pauli Matrices}$$

Operators as matrices

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$X \cdot |\psi\rangle = \begin{pmatrix} \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} & \begin{matrix} 1 \\ 0 \end{matrix} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$Z|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

$$= \alpha|0\rangle \underbrace{-}_{\text{Phase}} \beta|1\rangle$$

Multi-qubit operators

Controlled NOT

- The CNOT **always** acts on two qubits. It has a control qubit and a target qubit. If the control is set to $|1\rangle$, it'll apply the X operator to the target qubit.

$|\psi\rangle$: control If $|\psi\rangle = |1\rangle$ If $|\psi\rangle = |0\rangle$

$|\phi\rangle$: Target. then $|\phi\rangle \rightarrow X|\phi\rangle$ & $|\phi\rangle \rightarrow I|\phi\rangle$

- Outer product representation:

$$CNOT = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$CNOT = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

I X

- We can define it as $CNOT|x, y\rangle = |x, x \oplus y\rangle$, where $\oplus$ denotes addition modulo 2

if $x=0$ $y \rightarrow 0 \oplus y = y$

if $x=1$ $y \rightarrow 1 \oplus y = \neg y$

Unitary matrices

- In quantum computing, all operators **need** to be unitary matrices

Unitary $\rightarrow$ Prob. preserved $\rightarrow$ $||\psi\rangle|$

- This means that $U^\dagger U = UU^\dagger = I$

$$\begin{pmatrix}u_1\\ \vdots \\ u_n\end{pmatrix} \rightarrow \begin{pmatrix}u_1^* & \cdots & u_n^*\end{pmatrix}$$

Hermitian conjugate Adjoint.

$$U = \begin{pmatrix}u_{11} & \cdots & u_{1n}\\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn}\end{pmatrix} \qquad U^\dagger = \begin{pmatrix}u_{11}^* & \cdots & u_{n1}^*\\ \vdots & \ddots & \vdots \\ u_{1n}^* & \cdots & u_{nn}^*\end{pmatrix}$$

$YY^\dagger = I$ Y^2

$Y^\dagger Y = I$

$u_{n1} \rightarrow u_{1n}$ $u_{ij} \rightarrow u_{ji}$

X Z

$$Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad Y^* = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \quad Y^\dagger = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\boxed{YY^\dagger} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ +i & 0\end{pmatrix} = \begin{pmatrix}0\cdot 0 - i\cdot i & 0\cdot(-i) - i\cdot 0\\ i\cdot 0 + 0\cdot i & i\cdot(-i) + 0\cdot 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

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Eigenvalues and eigenvectors

$\vec{u}$ $\qquad A\vec{u} = \lambda\vec{u}$

- The eigenvectors of a $\overset{A}{\text{matrix}}$ are those that remain unchanged, up to a constant factor, when the matrix acts on them

- $A|u_\lambda\rangle = \lambda|u_\lambda\rangle$, A is an operator, $|u_\lambda\rangle$ is an eigenvector, and λ is the corresponding eigenvalue

- Characteristic equation: $\qquad f(\lambda) = 0$

$$\boxed{\det(A - \lambda I) = 0}$$
$$f(\lambda)$$

$$A|u_\lambda\rangle = \lambda|u_\lambda\rangle$$
$$A|u_\lambda\rangle - \lambda|u_\lambda\rangle = A|u_\lambda\rangle - \lambda I|u_\lambda\rangle$$
$$= \underline{(A - \lambda I)|u_\lambda\rangle = 0}$$

Eigenvalues and eigenvectors

- Let's get the eigenvalues and eigenvectors of some of the Pauli matrices

Trace

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det A = ad - bc \qquad \mathrm{Tr}A = a + d$$

$$\det(A - \lambda I) = \det\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$
$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$
$$\underbrace{(ad - bc)}_{\det A} - \underbrace{(a + d)}_{\mathrm{Tr}A}\lambda + \lambda^2 = 0$$

$$\boxed{\lambda^2 - \mathrm{Tr}A \cdot \lambda + \det A = 0}$$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \mathrm{Tr}\, X = 0 \quad \det X = -1$

$\lambda^2 - 1 = 0 \implies \lambda = \pm 1$

- Eigenvectors $\quad X|u_1\rangle = |u_1\rangle \quad X|u_{-1}\rangle = -|u_{-1}\rangle$

$|u_1\rangle = \begin{pmatrix} u_{1(1)} \\ u_{1(2)} \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u_{1(1)} \\ u_{1(2)} \end{pmatrix} = \begin{pmatrix} u_{1(1)} \\ u_{1(2)} \end{pmatrix}$

$|u_{-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} u_{1(2)} \\ u_{1(1)} \end{pmatrix} = \begin{pmatrix} u_{1(1)} \\ u_{1(2)} \end{pmatrix} \quad u_{1(1)} = u_{1(2)}$

$|u_1\rangle = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \end{pmatrix}$

$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvalues and eigenvectors

- The spectral decomposition of an operator is

$$A = \sum_i \lambda_i |u_{\lambda_i}\rangle\langle u_{\lambda_i}| = \lambda_1 |u_{\lambda_1}\rangle\langle u_{\lambda_1}| + \dots + \lambda_n |u_{\lambda_n}\rangle\langle u_{\lambda_n}|$$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \lambda = 1 \quad |u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \lambda = -1 \quad |u_{-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$1 \cdot |u_1\rangle\langle u_1| - |u_{-1}\rangle\langle u_{-1}| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \cdot 1 & 1 \cdot (-1) \\ -1 \cdot 1 & -1 \cdot (-1) \end{pmatrix}$

Y, Z

$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

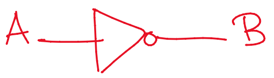
$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

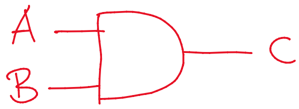
Classical computing

- In classical computing, we also use gates!

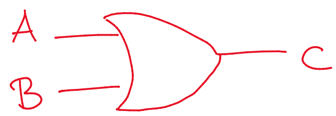
True - False
High - low
1 - 0

  - NOT: 

A	B
0	1
1	0

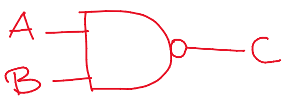
  - AND: 

A	B	$C = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

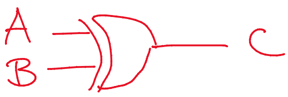
  - OR: 

A	B	C
0	0	0
0	1	1
1	0	1
1	1	1

Classical computing

- NAND: 

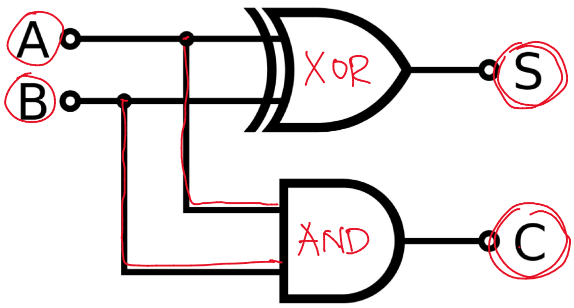
A	B	C
0	0	1
0	1	1
1	0	1
1	1	0

- XOR: 

A	B	$C = A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

Classical computing

- Let's build a half adder



$S = A \oplus B$ sum

$C = A \cdot B$ carrier

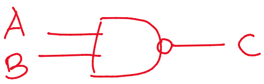
$A + B$

Universality of NAND

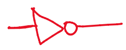
- In classical computing, you can build any gate out of NAND gates. Therefore, any computation can be done with only NAND gates

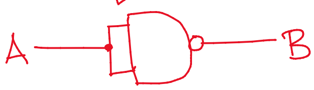
3

- Let's build the AND, OR, and NOT from NAND gates



A	B	C
0	0	1
0	1	1
1	0	1
1	1	0



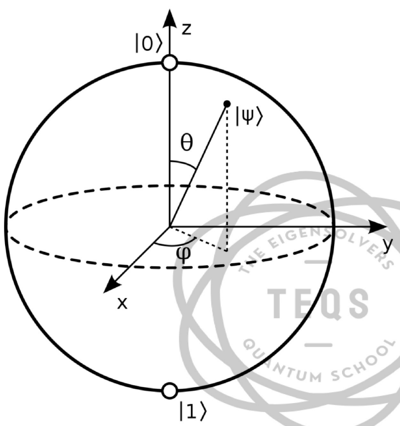


A	B
0	1
1	0

Universal quantum gates

- Like in classical computing, we also have a set of universal quantum gates that can compute any function

- These are the rotation gates $R_{i\in\{x,y,z\}}$, phase shift gate $P(\theta)$, and the $CNOT$ gate



Thanks for listening!

Good luck on Lecture 2!