Math 4MB3: Draft Group Project - Supplementary file

Group Name: The Infective Collective

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1 Construction of the model

First, consider the following coupled seasonally forced SIR model consisting of n patches:

$$\begin{aligned}
\frac{dS_i}{dt} &= \mu N - \left(\sum_{j=1}^{n} \beta(t) m_{ij} I_j + \mu\right) S_i \\
\frac{dI_i}{dt} &= \left(\sum_{j=1}^{n} \beta(t) m_{ij} I_j + \mu\right) S_i - (\mu + \gamma) I_i \\
\frac{dR_i}{dt} &= \gamma I_i - \mu R_i
\end{aligned} \tag{1}$$

where S_i, I_i and R_i are numbers of susceptible, infected and recovered individuals in patch i, respectively. N is population size in every patch and is assumed to be constant. $\beta(t)$ is the transmission rate, γ is the per capita recovery rate and μ is the per capita death rate as well as birth rate. m_{ij} is the proportion of contacts from patch j that are dispersed to patch i. The dispersal matrix is then given by $M = [m_{ij}]$ and we have $\sum_{i=1}^{n} m_{ij} = 1$. To allow for seasonal forcing, we model the transmission rate using a trigonometric function [CITE]:

$$\beta(t) = b_0(1 + b_1 \cos(2\pi t)) \tag{2}$$

Note that a susceptible individual in patch i leaves the susceptible compartment at rate $\sum_{j=1}^{n} \beta(t) m_{ij} I_j + \mu$. We can view this quantity as hazard and obtain the probability that a susceptible individual survives during a time interval $(t, t + \Delta t)$:

$$\exp\left(-\int_t^{t+\Delta t} \sum_{j=1}^{n} \beta(s) m_{ij} I_j(s) + \mu ds\right), \tag{3}$$

where for sufficiently small Δt, we can write

$$\int_t^{t+\Delta t} \sum_{j=1}^{n} \beta(s) m_{ij} I_j(s) + \mu ds \approx \left(\sum_{j=1}^{n} \beta(t) m_{ij} I_j(s) + \mu\right) \Delta t. \tag{4}$$

Rearranging, number of individuals that leave the susceptible compartment after Δ time step is given by

$$S_{i,\text{leave}}(t) = \left(1 - \exp\left(-\left(\sum_{j=1}^{n} \beta(t) m_{ij} I_j(s) + \mu\right) \Delta t\right)\right) S_i(t)$$

Assuming that the transmission rate and number of infected individuals stays constant over the interval $(t, t + \Delta)$, the probability that a susceptible individual leaves the compartment due to infection is given by

$$\frac{\sum_{j=1}^{n} \beta(t) m_{ij} I_j(s)}{\sum_{j=1}^{n} \beta(t) m_{ij} I_j(s) + \mu} \tag{5}$$

Incidence (i.e., number of suseptible individuals that become infected during $(t, t + \Delta t)$ interval) is then given by the product of above probability and $S_{i,\text{leave}}(t)$. Similarly, an infected individual and a recovered individual suffer from constant hazard of $\gamma + \mu$ and μ, respectively, and these can each be translated into survival probability, yielding

$$\begin{aligned} I_{i,\text{leave}}(t) &= (1 - \exp(-(\gamma + \mu)\Delta t)) I_i(t) \\ R_{i,\text{leave}}(t) &= (1 - \exp(-\mu \Delta t)) R_i(t) \end{aligned} \tag{6}$$

Then, the full discrete time model is given by

$$\begin{aligned} S_k(t + \Delta t) &= b_k(t) + S_k(t) - S_{k,\text{leave}}(t) \\ I_k(t + \Delta t) &= i_k(t) + I_k(t) - I_{k,\text{leave}}(t) \\ R_k(t + \Delta t) &= r_k(t) + R_k(t) - R_{k,\text{leave}}(t) \end{aligned} \tag{7}$$

where $b_k(t)$, $i_k(t)$, and $r_k(t)$ represent number of new susceptible, infected, and recovered individuals that are produced between the interval $(t, t + \Delta t)$:

$$\begin{aligned} i_k(t) &= \frac{\sum_{j=1}^{n} \beta(t) m_{ij} I_j(s)}{\sum_{j=1}^{n} \beta(t) m_{ij} I_j(s) + \mu} S_{k,\text{leave}}(t) \\ r_k(t) &= \frac{\gamma}{\gamma + \mu} R_{k,\text{leave}}(t) \\ b_k(t) &= S_{k,\text{leave}}(t) - i_k(t) + I_{k,\text{leave}}(t) - r_k(t) \end{aligned} \tag{8}$$