Breaking SIKE

Isogeny Club, September 13

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COSIC

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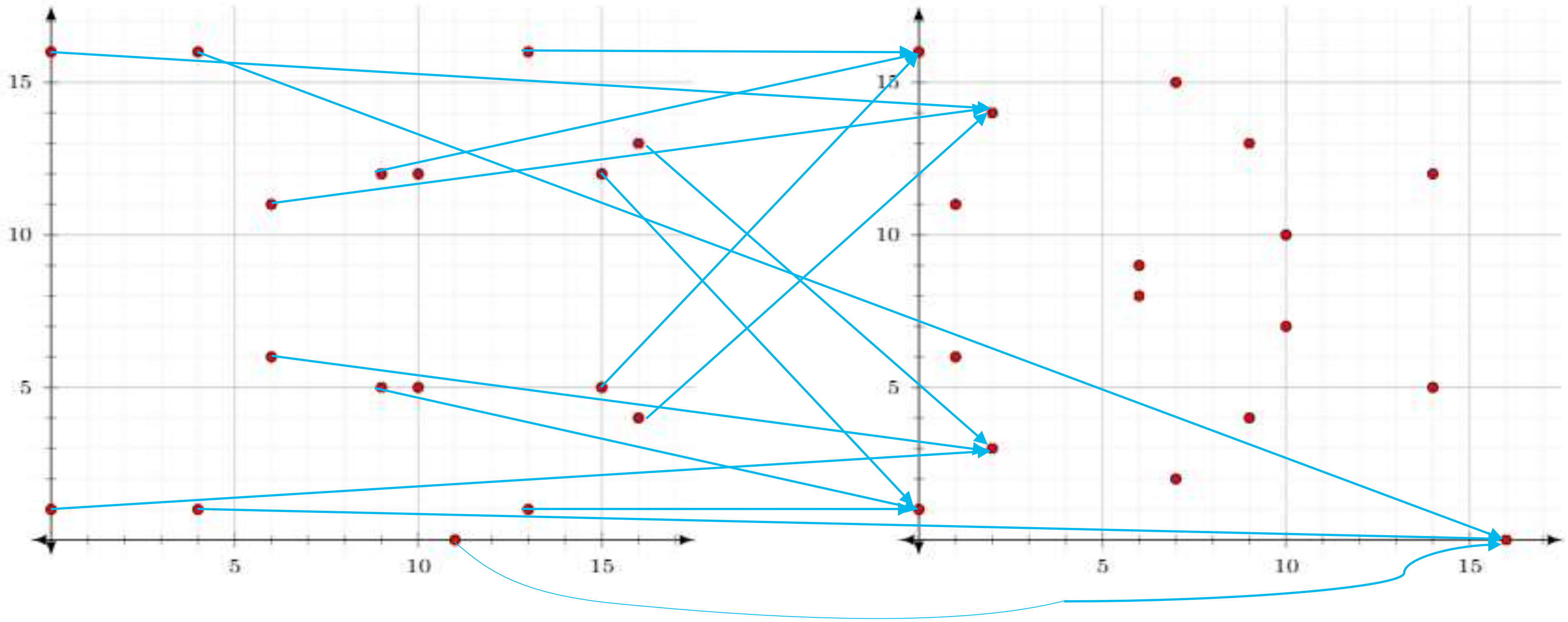
Outline

- SIDH/SIKE
- Isogenies in dimension two
- The glue-and-split attack

SIDH/SIKE

Isogenies are surjective group morphisms with finite kernel



$$\varphi: \quad E/\mathbb{F}_{17} : y^2 = x^3 + x + 1 \quad \to \quad E'/\mathbb{F}_{17} : y^2 = x^3 + 1$$

$$P = (x, y) \quad \mapsto \quad \begin{cases} \infty \text{ if } P \in \{\infty, (10,5), (10,12)\} \\ \left(\frac{x^3-3x^2+5x-4}{x^2-3x-2}, y\frac{x^3+4x^2+4x-8}{x^3+4x^2-6x+3}\right) \text{ else} \end{cases}$$

Hard problem: given two elliptic curves, it is conjecturally hard to find any isogeny between them.

- 1996/2007: CRS (Couveignes-Rostovtsev-Stolbunov)
- 2006: CGL hash function (Charles-Goren-Lauter)
- 2011: SIDH (Jao-De Feo)
- 2018: CSIDH (Castryck-Lange-Martindale-Renes-Panny)
- 2020: SQISign (De Feo-Kohel-Leroux-Petit-Wesolowski)

Supersingular Isogeny Diffie-Hellman

ALICE

BOB

Finite field $\mathbb{F}_{p^2}$ and supersingular elliptic curve E

$\varphi_A : E \to E_A$

$\varphi_B : E \to E_B$

Secret kernel
$G_A \subseteq E(\mathbb{F}_{p^2})[2^e]$

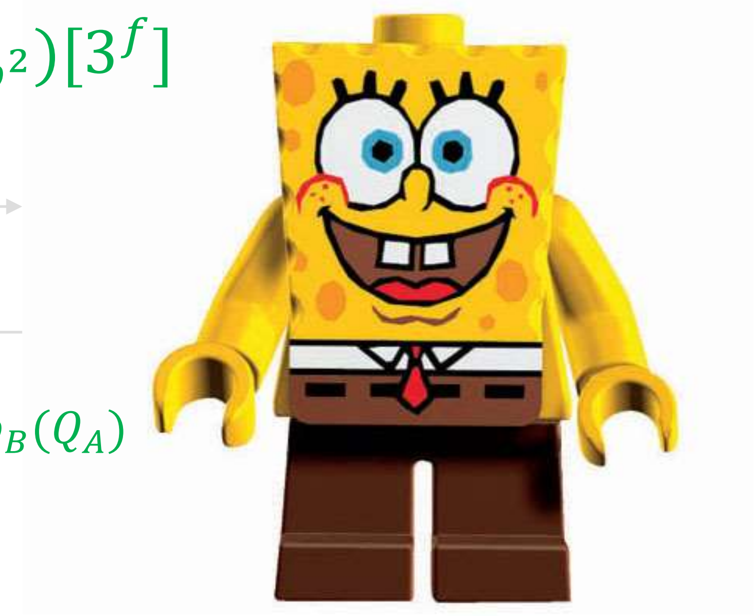
Problem! What is G_A on E_B?

Secret kernel
$G_B \subseteq E(\mathbb{F}_{p^2})[3^f]$

Supersingular Isogeny Diffie-Hellman

ALICE

BOB



Finite field $\mathbb{F}_{p^2}$, supersingular elliptic curve E,

basis P_A, Q_A of $E(\mathbb{F}_{p^2})[2^e]$, basis P_B, Q_B of $E(\mathbb{F}_{p^2})[3^f]$

$\varphi_A: E \to E_A, \varphi_A(P_B), \varphi_A(Q_B)$

$\varphi_B: E \to E_B, \varphi_B(P_A), \varphi_B(Q_A)$

Shared secret: $j(E_{AB})$ obtained from

$\varphi_A': E_B \to E_{BA} \cong E_{AB} \leftarrow E_A: \varphi_B'$

with $\ker(\varphi_A') = \langle r_A \varphi_B(P_A) + s_A \varphi_B(Q_A) \rangle$,

$\ker(\varphi_B') = \langle r_B \varphi_A(P_B) + s_B \varphi_A(Q_B) \rangle$

Secret kernel

$G_A = \langle r_A P_A + s_A Q_A \rangle$

Secret kernel

$G_B = \langle r_B P_B + s_B Q_B \rangle$

Security of SIDH

It’s complicated in part because NIST’s post-quantum security levels are vague; QRAM costs? Circuit depth? Latency? Etc.[1]

- Best generic attack is a claw-finding attack: $O\left(p^{\frac{1}{4}}\right)$ classical and $O\left(p^{\frac{1}{6}}\right)$ quantum
- 2017: torsion-point attack on unbalanced parameters $2^e, 3^f$ (Petit and follow-up work)
- Our work: heuristic polynomial time with precomputable integer factorization

- 2016: Galbraith, Petit, Shani & Ti: chosen ciphertext attack against static key SIDH
- SIKE: Supersingular Isogeny Key Encapsulation (‘key exchange with long term public key’)

[1] Good read: https://blog.cr.yp.to/20151120-batchattacks.html

SIKE parameter sets

Starting curve is always $E: y^2 = x^3 + 6x^2 + x$

$\mathbb{F}_{p^2}$ with p one of

$$2^{216} \cdot 3^{137} - 1$$
$$2^{250} \cdot 3^{159} - 1$$
$$2^{305} \cdot 3^{192} - 1$$
$$2^{372} \cdot 3^{239} - 1$$

(base points omitted)

Note: primes of this form result in $\#E(\mathbb{F}_{p^2}) = \left(2^e 3^f\right)^2$ so easy torsion/kernels/isogenies

Note: $2^e \approx 3^f$ so Alice and Bob have similar entropy

Computational versus decisional isogeny problem

Given E and E', find an isogeny of degree ℓ^k between them.

$\sim$

Given E and E', does there exist an isogeny of degree ℓ^i between them for $0 < i < k$?

$$E = E_0 \to E_1 \to E_2 \to E_3 \to \cdots \to E_{k-1} \to E_k = E'$$

$\ell + 1$ options

ℓ options

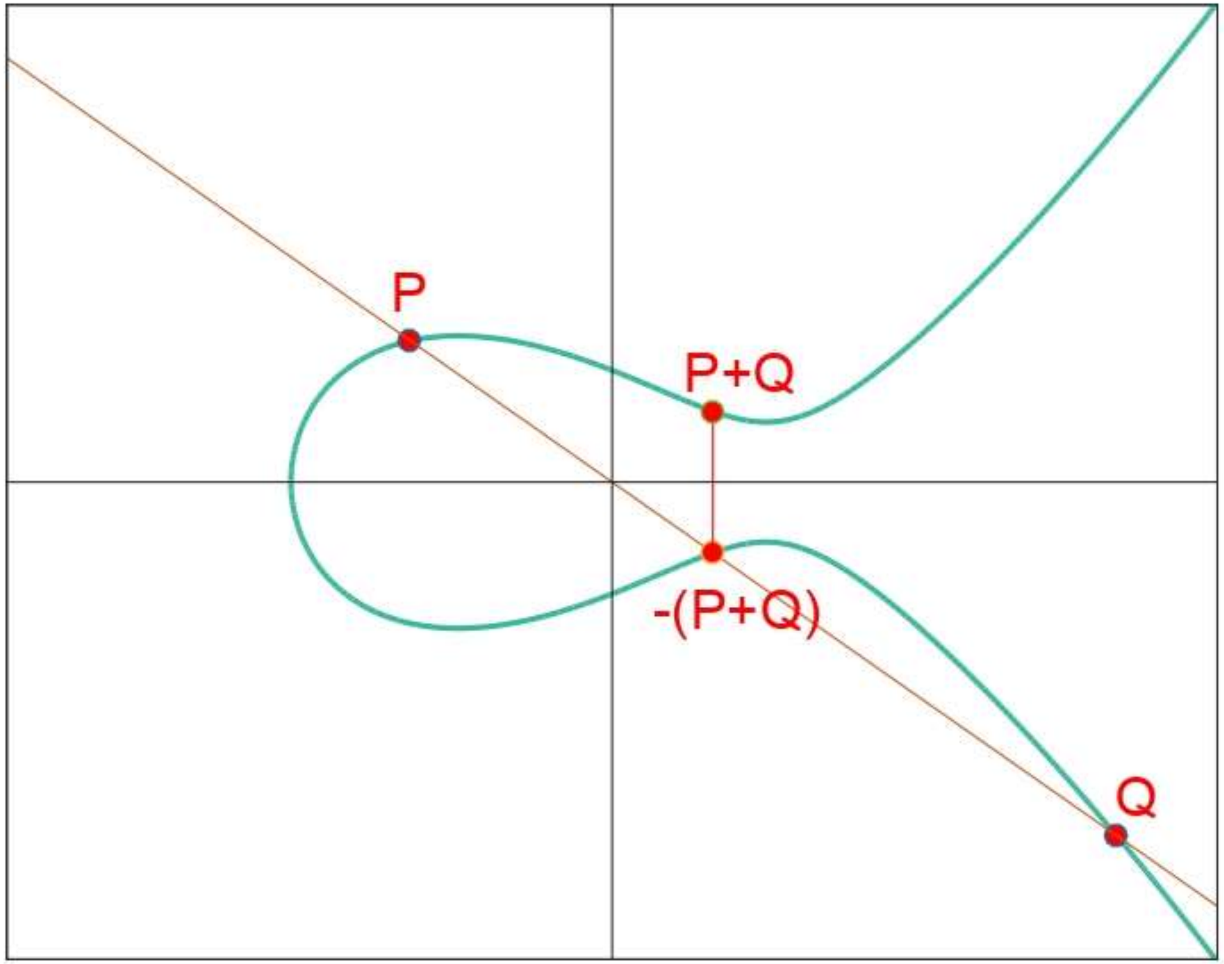
Isogenies in dimension two

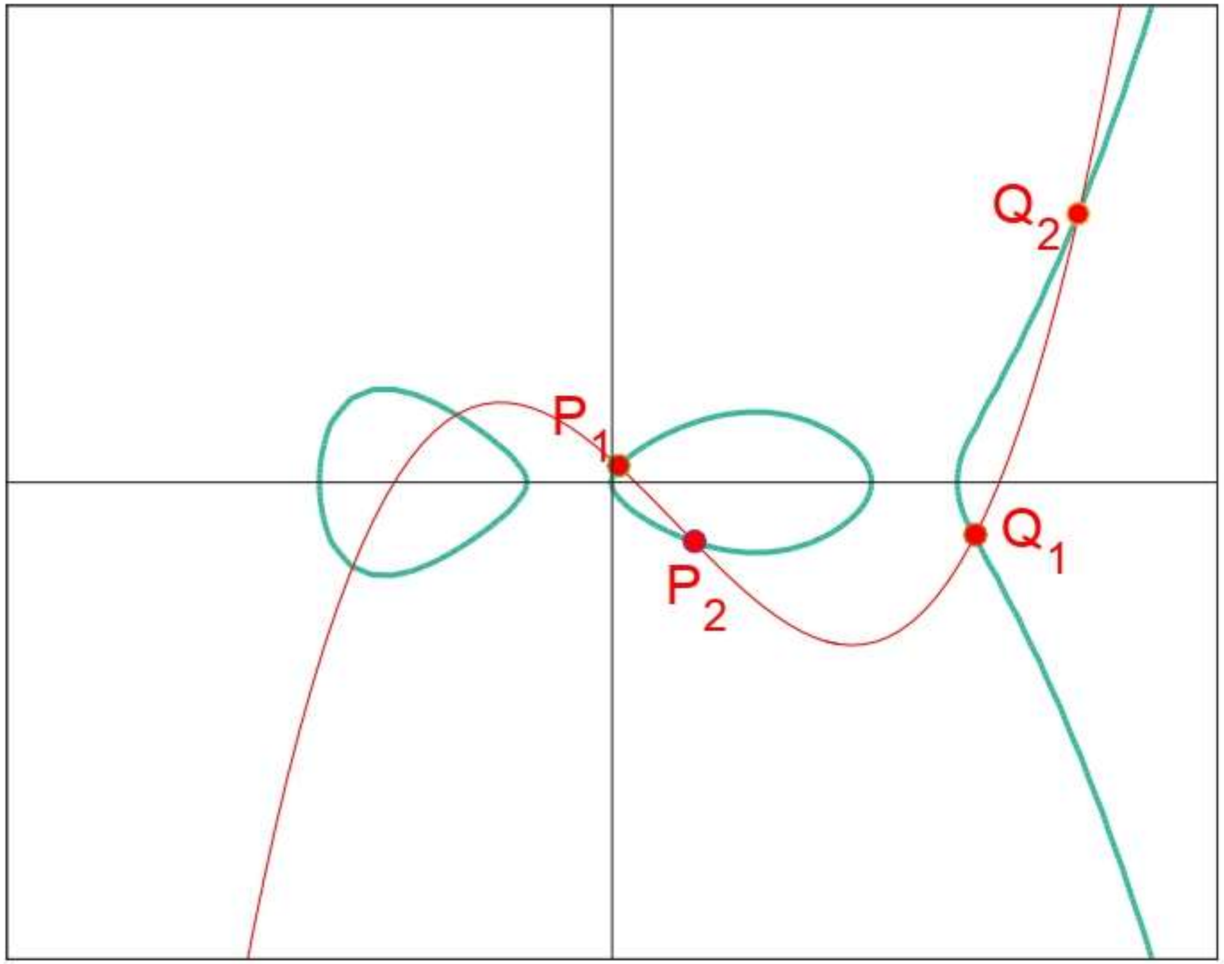
Elliptic curves → abelian varieties

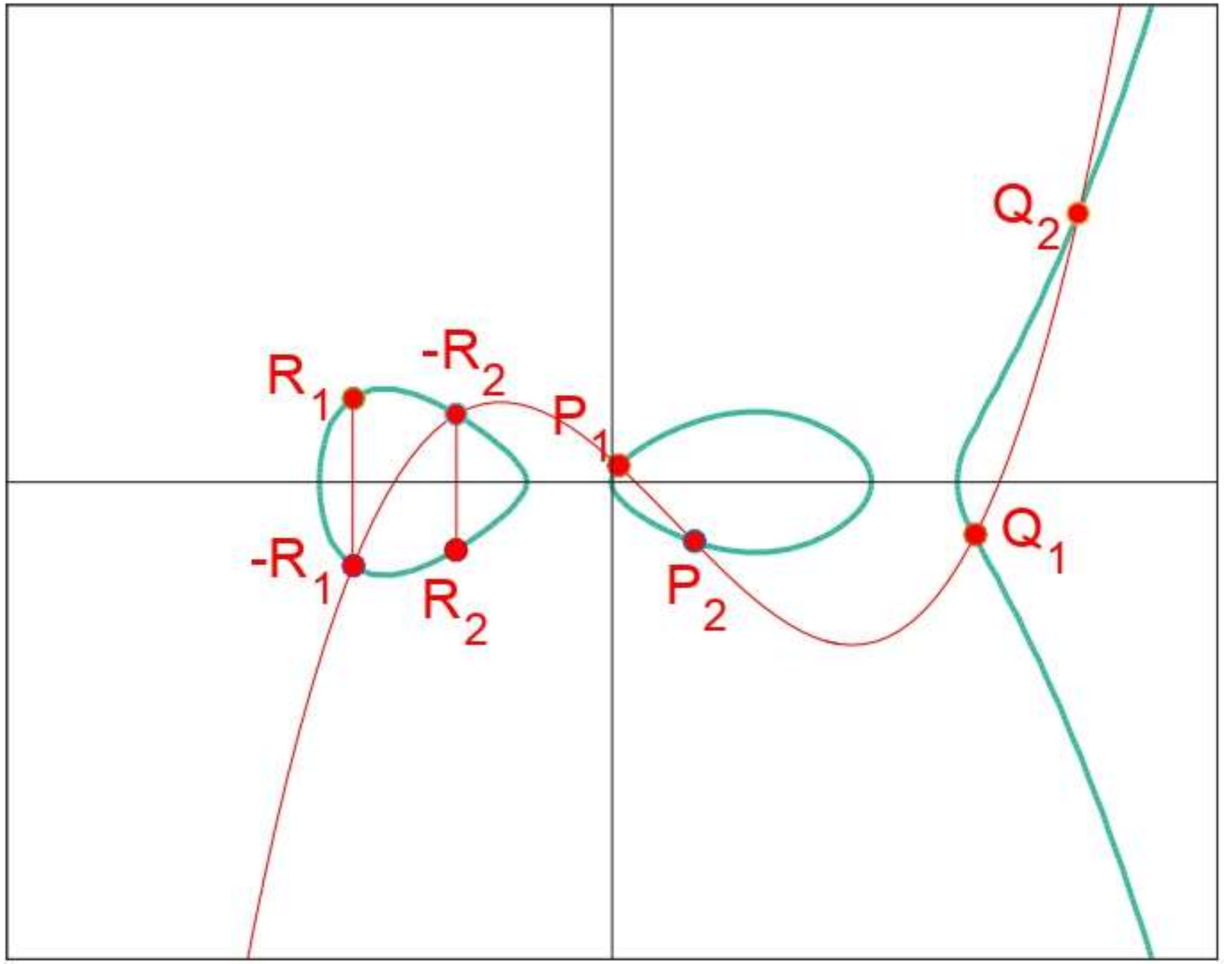
In dimension two these are called abelian surfaces:

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ \text{products of elliptic curves } E \times E' & & \text{Jacobians of genus-2 curves } C \end{tikzcd}$$

Remark: we are actually interested in principally polarized abelian surfaces!
This allows us to use equations $C: y^2 = x^5 + Ax^3 + Bx^2 + Cx + D$.


P
P+Q
-(P+Q)
Q


P1
P2
Q2
Q1



Q2
R1
-R2
P1
Q1
-R1
R2
P2

Supersingular abelian surfaces?

An elliptic curve E is supersingular if

- $E[p]$ is trivial;
- $End(E)$ is an order in a quaternion algebra;
- the trace t of Frobenius is $t \equiv 0 \ mod \ p$;
- …

We want the strongest generalization for cryptography, i.e. superspecial abelian surfaces!

Invariants in two dimensions

A genus-2 curve is defined by a triple of (absolute) Igusa invariants (i_1, i_2, i_3)

There are $\approx p^3/2880$ superspecial Jacobians of genus-2 curves

A product of elliptic curves is defined by a set of j-invariants $\{j_1, j_2\}$

$\approx p/12$ supersingular elliptic curves results in $\approx p^2/288$ superspecial products

Isogenies in dimension two

An (N, N)-isogeny $\Phi: A \to A'$ is an isogeny such that

- $\ker(\Phi) \cong \frac{\mathbb{Z}}{N\mathbb{Z}} \times \frac{\mathbb{Z}}{N\mathbb{Z}}$
- $\ker(\Phi)$ is maximal isotropic wrt to the N-Weil pairing, i.e.
$$\forall P, Q \in \ker(\Phi) : e_N(P, Q) = 1$$

Remark: the second condition ensures that A' comes equipped with a principal polarization!

Four types of isogenies!

1. $Jac(C) \to Jac(C')$ -> generic (N, N)-isogeny
2. $Jac(C) \to E_1' \times E_2'$ -> split (N, N)-Jacobian
3. $E_1 \times E_2 \to Jac(C')$ -> gluing elliptic curves along their (N, N)-torsion
4. $E_1 \times E_2 \to E_1' \times E_2'$ -> (N, N)-isogeny between products of elliptic curves

(N, N)-isogenies between products of elliptic curves

Let $\varphi_1 : E_1 \to E_1'$ and $\varphi_2 : E_2 \to E_2'$ be cyclic N-isogenies, then

$$\Phi = \varphi_1 \times \varphi_2$$

is an (N, N)-isogeny from $E_1 \times E_2$ to $E_1' \times E_2'$.

Why? Because the N-Weil pairing on products of elliptic curves equals the product of the N-Weil pairing on the respective curves.

In particular, $\ker(\Phi)$ is maximal isotropic with regards to the N-Weil pairing. It can be written as $\underbrace{\langle (P, \infty_{E_2}), (\infty_{E_1}, Q) \rangle}_{\text{this is a diagonal kernel}}$.

(N, N)-isogenies from products of elliptic curves

Let

$$\Phi: E_1 \times E_2 \to A'$$

be an (N, N)-isogeny with nondiagonal kernel

$$\ker(\Phi) = \langle (P, Q), (P', Q') \rangle.$$

When is this not an (N, N)-gluing; i.e. when is $A' \cong E_1' \times E_2'$?

Expected for superspecial abelian surfaces with probability $\approx \frac{10}{p}$.

The glue-and-split attack

Examples for failed gluings

- A $(2,2)$-isogeny $\Phi: E_1 \times E_2 \to A'$ with nondiagonal kernel *can* only have $A' \cong E_1' \times E_2'$ if $E_1 \cong E_2$.
- A $(3,3)$-isogeny $\Phi: E_1 \times E_2 \to A'$ with nondiagonal kernel *can* only have $A' \cong E_1' \times E_2'$ if there exists a 2-isogeny $\psi: E_1 \to E_2$.

- A $(5,5)$-isogeny $\Phi: E_1 \times E_2 \to A'$ with nondiagonal kernel *can* only have $A' \cong E_1' \times E_2'$ if there exists a 4- or 6-isogeny $\psi: E_1 \to E_2$.
- A $(7,7)$-isogeny $\Phi: E_1 \times E_2 \to A'$ with nondiagonal kernel *can* only have $A' \cong E_1' \times E_2'$ if there exists a 6- or 10- or 12-isogeny $\psi: E_1 \to E_2$.
- …

Kani's theorem (highly informal)

- **Theorem:** an (N, N)–gluing fails iff it comes from an isogeny diamond configuration.

  i.e. $\langle (P, x\psi(P)), (Q, x\psi(Q)) \rangle$ for some $x \in \mathbb{Z}$

- **Definition:** an isogeny diamond configuration of order N is a tuple (ψ, G_1, G_2) with
  1. $\psi: E \to E'$ an isogeny;
  2. $G_1, G_2 \subset ker(\psi)$;
  3. $G_1 \cap G_2 = \{\infty_E\}$;
  4. $deg(\psi) = \#G_1 \cdot \#G_2$;
  5. $N = \#G_1 + \#G_2$.

Attacking Bob's secret key

Given

$$(E, \underbrace{P_A, Q_A}_{\text{Alice's } 2^e\text{-torsion basis}}), (E_B, \varphi_B(P_A), \varphi_B(Q_A))$$

we want to find

$$\varphi_B. \longrightarrow \text{isogeny of degree } 3^f$$

Idea: consider

$$E = E_0 \to E_1 \to E_2 \to \cdots \to E_{f-1} \to E_f = E_B$$

Which of the 4 options is correct? (remark that we can push P_A, Q_A through easily)

Forcing an isogeny diamond configuration

Can we force E_1, E_B into Kani's theorem?

Definition: an isogeny diamond configuration of order 2^e is a tuple (ψ, G_1, G_2) with

1. $\psi: E \to E'$ an isogeny; → $\psi = \varphi_1: E_1 \to E_B$ perhaps?
2. $G_1, G_2 \subset ker(\psi)$; → $\#G_i = 3^k$ for some k
3. $G_1 \cap G_2 = \{\infty_E\}$;
4. $deg(\psi) = \#G_1 \cdot \#G_2$; → $\deg(\psi) = 3^{f-1}$ if we have correct E_1
5. $2^e = \#G_1 + \#G_2$. → $\#G_1 = 3^{f-1}$ and $\#G_2 = 1$

Forcing an isogeny diamond configuration

Construct an isogeny $\gamma: E_1 \to C$ of degree $c = 2^e - 3^{f-1}$

$c > 0$?

How? Later!

Definition: an isogeny diamond configuration of order 2^e is a tuple (ψ, G_1, G_2) with

1. $\psi = \varphi_1 \circ \hat{\gamma}: C \to E_1 \to E_B$;
2. $G_1 = \ker(\hat{\gamma})$, $G_2 = \gamma(B)$ with B Bob's secret kernel;
3. $G_1 \cap G_2 = \{\infty_E\}$;
4. $deg(\psi) = \#G_1 \cdot \#G_2 = \left(2^e - 3^{f-1}\right) \cdot 3^{f-1}$;
5. $2^e = \#G_1 + \#G_2 = \left(2^e - 3^{f-1}\right) + 3^{f-1}$.

Finishing the attack

Consider $\Phi: C \times E_B \to A'$ with kernel

$$\langle (\gamma(P_A), \varphi_B(P_A)), (\gamma(Q_A), \varphi_B(Q_A)) \rangle.$$

In practice, compute

$$C \times E_B \to Jac(C_1) \to Jac(C_2) \to \cdots \to Jac(C_{e-1}) \to A'$$

(2,2)-isogenies

If A' is a product of elliptic curves, we picked the correct E_1 with overwhelming probability!

Finding a $\gamma: E_i \to C$ of degree $c = 2^e - 3^{f-i}$

- Known endomorphism ring ($C \cong E_i$):
  - $E_i: y^2 = x^3 + x$ has endomorphism $\iota: E_i \to E_i, (x, y) \mapsto (-x, iy)$
    -> if $c = u^2 + v^2 = (u + iv)(u - iv)$ for $u, v \in \mathbb{N}$ we can find γ easily
  - $E_0: y^2 = x^3 + 6x^2 + x$ has endomorphism 2ι $\longrightarrow$ we can translate $\gamma_0: E_0 \to E_0$ to E_i
    -> similar easy trick; E_0 is actually used in SIKE as starting curve
  - E_i with small endomorphism ok too
  - In general, if $End(E_i)$ is known we can use KLPT algorithm

Finding a $\gamma: E_i \to C$ of degree $c = 2^e - 3^{f-i}$

- Unknown endomorphism ring:
  - Hope that c is smooth and work with arbitrary isogenies over extension fields
  - Add more leeway:

$$c = d \cdot 2^{e-j} - d' \cdot 3^{f-i}$$

we can guess the action of the d-torsion; in practice this means after the $(2^{e-j}, 2^{e-j})$-isogeny we check if *any* of the (d, d)-isogenies splits

probability that this happens by chance is only $O\left(\frac{d^3}{p}\right)$

if we know the action of φ_B on the 2^e-torsion, we also have it on the 2^{e-j}-torsion

we can extend φ_B with any isogeny of degree d'

we don't need all $0 < i < f$

Thanks!