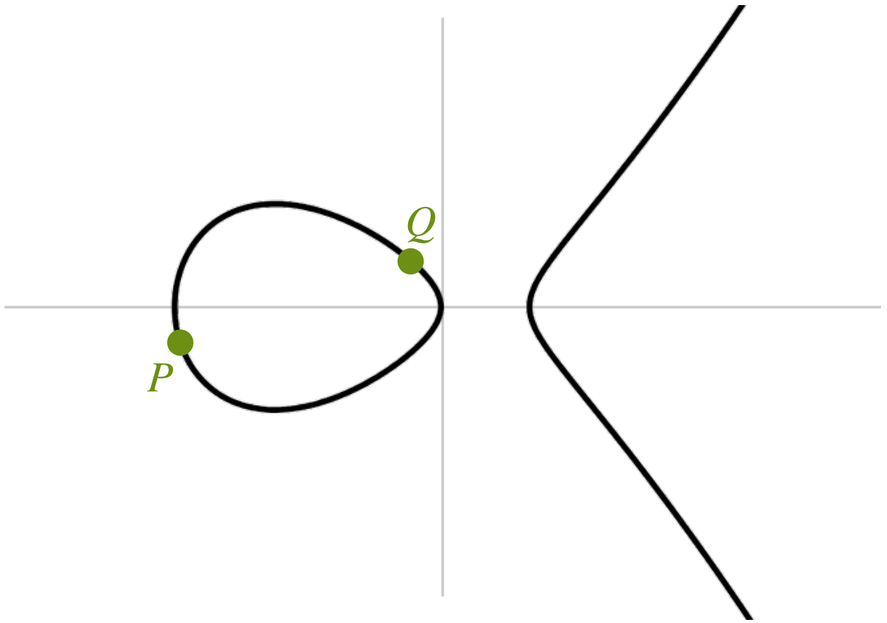
Return of the Kummer

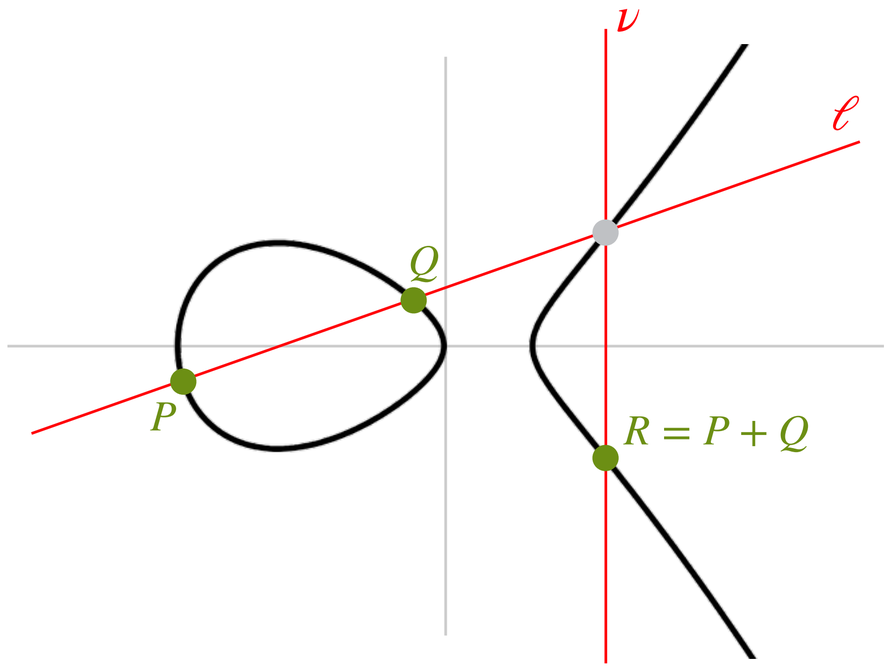
from curves to Kummers and back

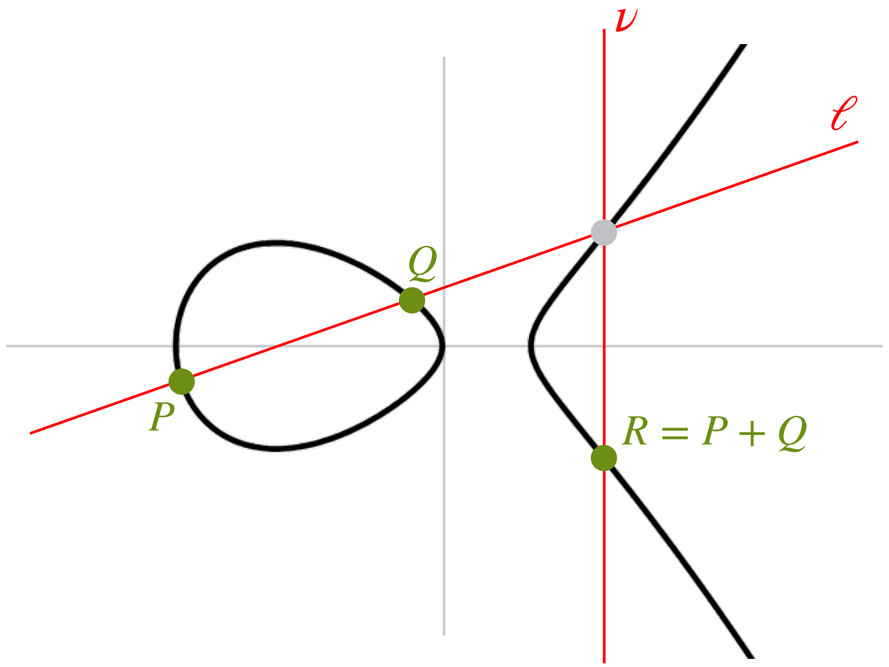
joint work with Maria Corte-Real Santos

Krijn Reijnders
KULB - March 15, 2024

Radboud University

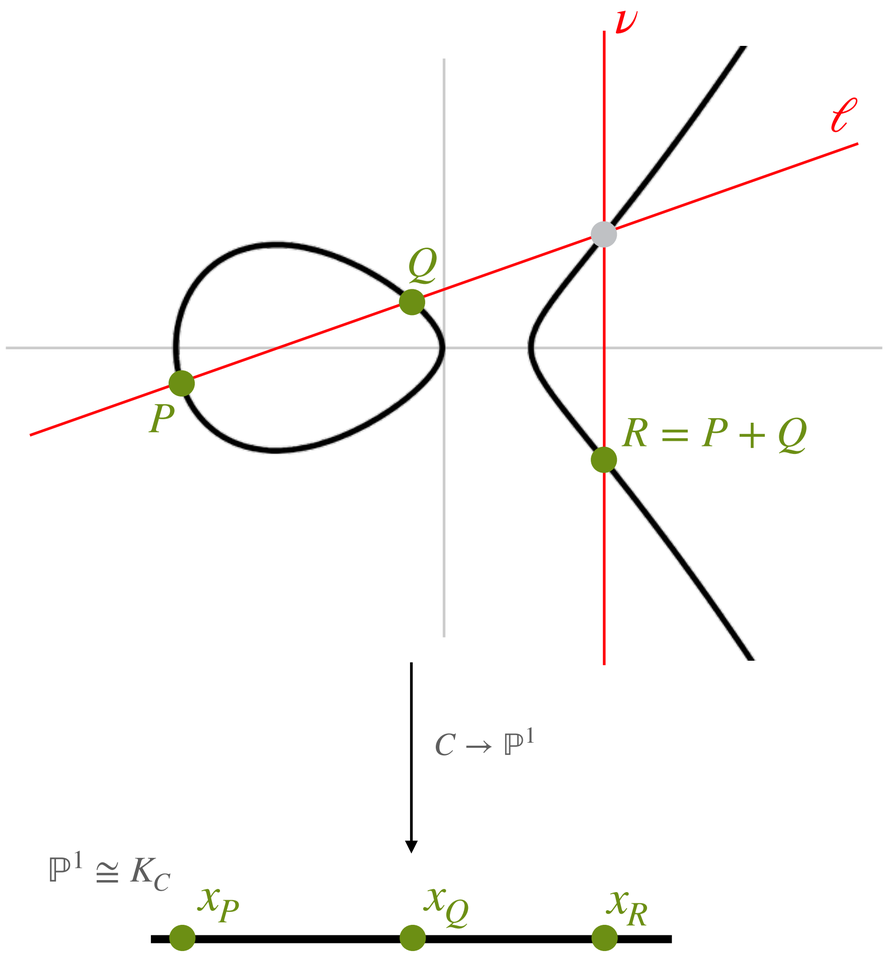

Q
P


v
ℓ
Q
P
R = P + Q



1. Points on the curve $P, Q, R \in C(k)$, i.e. satisfying some (geometric) curve equation $C : y^2 = f(x)$

2. Addition of divisors using lines from the function field $\ell, \nu \in k(C)$ for reduction, i.e. arithmetic on the Jacobian $J_C = \mathrm{Pic}^0(C)$



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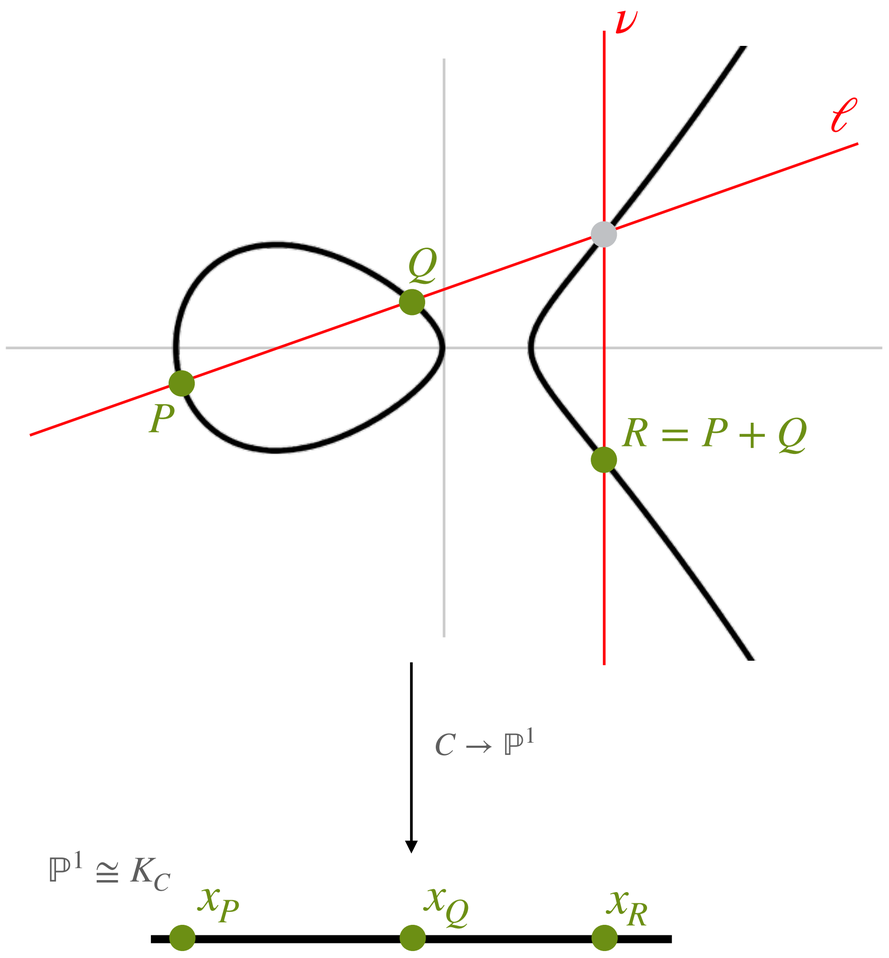
(in cryptography)

3. x-only arithmetic, i.e. working on the *Kummer line* K_C



1. Points on the curve $P, Q, R \in C(k)$, i.e. satisfying some (geometric) curve equation $C : y^2 = f(x)$

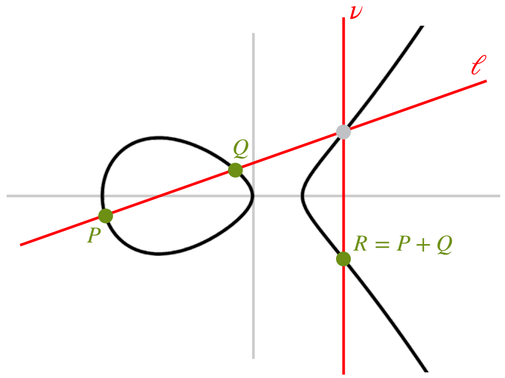
in genus 1, C is isomorphic to J_C

2. Addition of divisors using lines from the function field $\ell, \nu \in k(C)$ for reduction, i.e. arithmetic on the Jacobian $J_C = \mathrm{Pic}^0(C)$

(in cryptography)

3. x-only arithmetic, i.e. working on the *Kummer line* K_C

1. The set of points $C(k)$ of a **curve** $C : y^2 = f(x)$



2. The **Jacobian** $J_C = \mathrm{Pic}^0_k(C)$

$$J_C = \{\ (P) - (\mathcal{O})\ :\ P \in C(k)\ \}$$

3. The **Kummer line** $K_C = J_C/\pm$

$\mathbb{P}^1 \cong K_C$

1. The set of points $C(k)$ of a **curve** $C : y^2 = f(x)$

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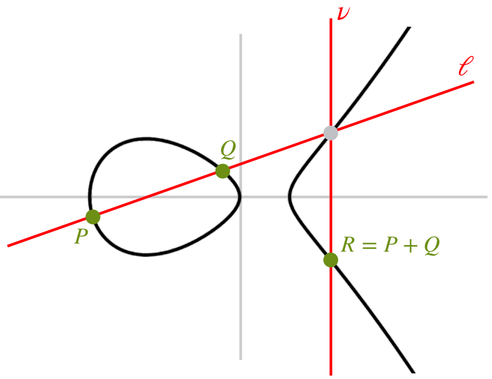
surface

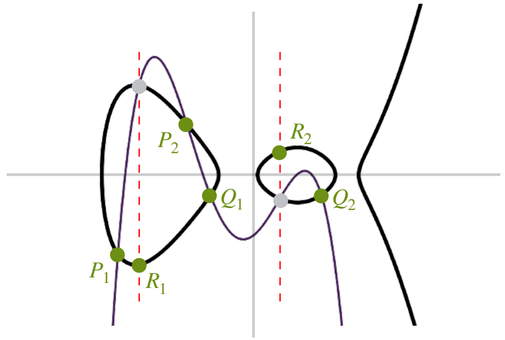
Genus 1



$$J_C = \{\ (P) - (\mathcal{O})\ :\ P \in C(k)\ \}$$

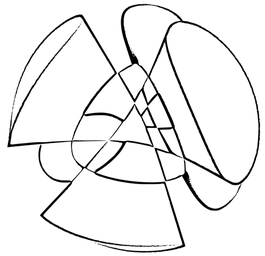
$\mathbb{P}^1 \cong K_C$

Genus 2



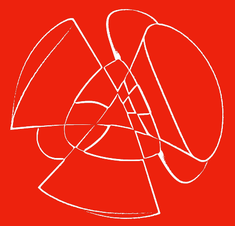
$$J_C = \{\ (P) + (Q) - D_\infty\ :\ P, Q \in C\ \}$$

$K_C \hookrightarrow \mathbb{P}^3$



Return of the Kummer

1



Kummer Surfaces

2

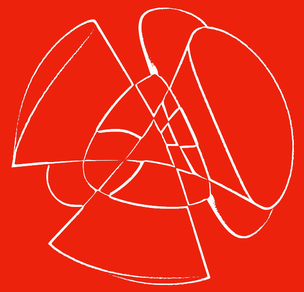
$$\begin{array}{ccc} \mathcal{J}_\alpha & \xrightarrow{\kappa} & \mathcal{J}_{\lambda,\mu,\nu} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}_\alpha & \xrightarrow{\kappa^*} & \mathcal{K}_{\lambda,\mu,\nu} \end{array}$$

Kummers In Cryptography

3

$$\begin{array}{ccccc} 0 & \longrightarrow & E[n] & \longrightarrow & E \\ & & & & \downarrow{\scriptstyle [n]} \\ E & \longrightarrow & E/[n]E & \longrightarrow & 0 \end{array}$$

Pairings on Kummers



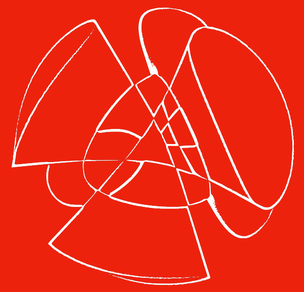
Kummer Surfaces

1

Starting point
genus 2 *hyperelliptic curve*

$$C : y^2 = f(x)$$

(general form)



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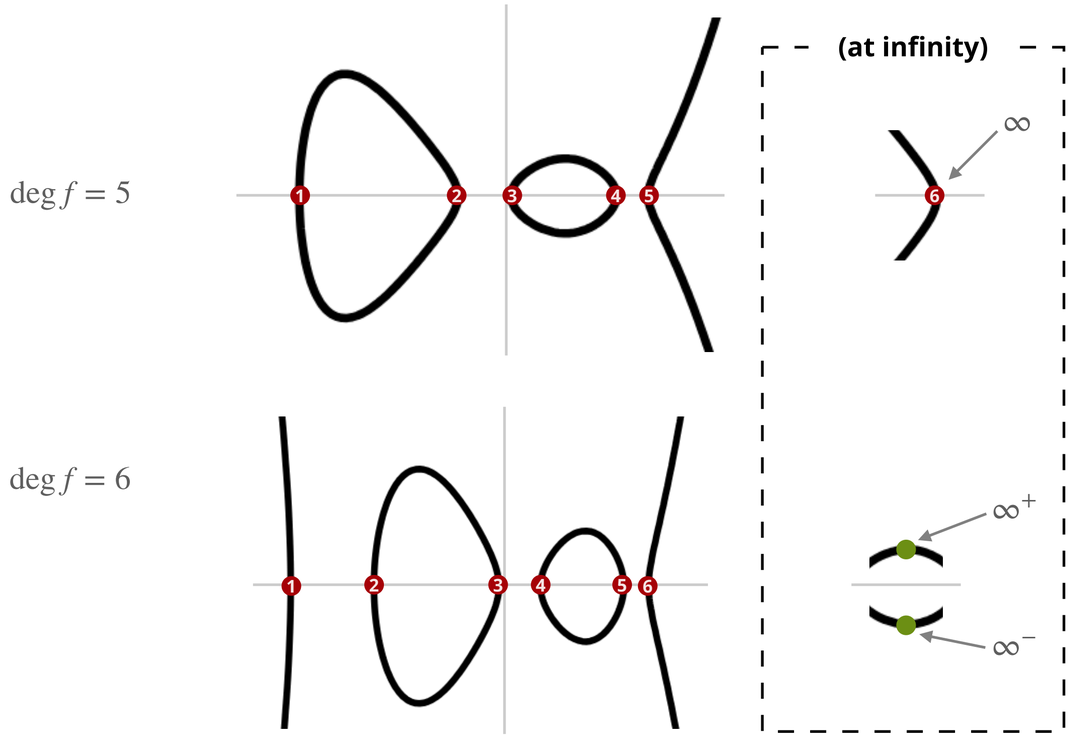
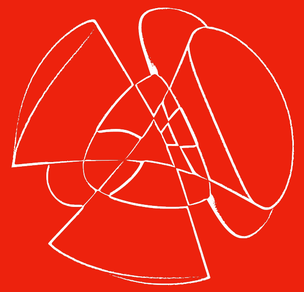
Interpretation of curves





Kummer Surfaces

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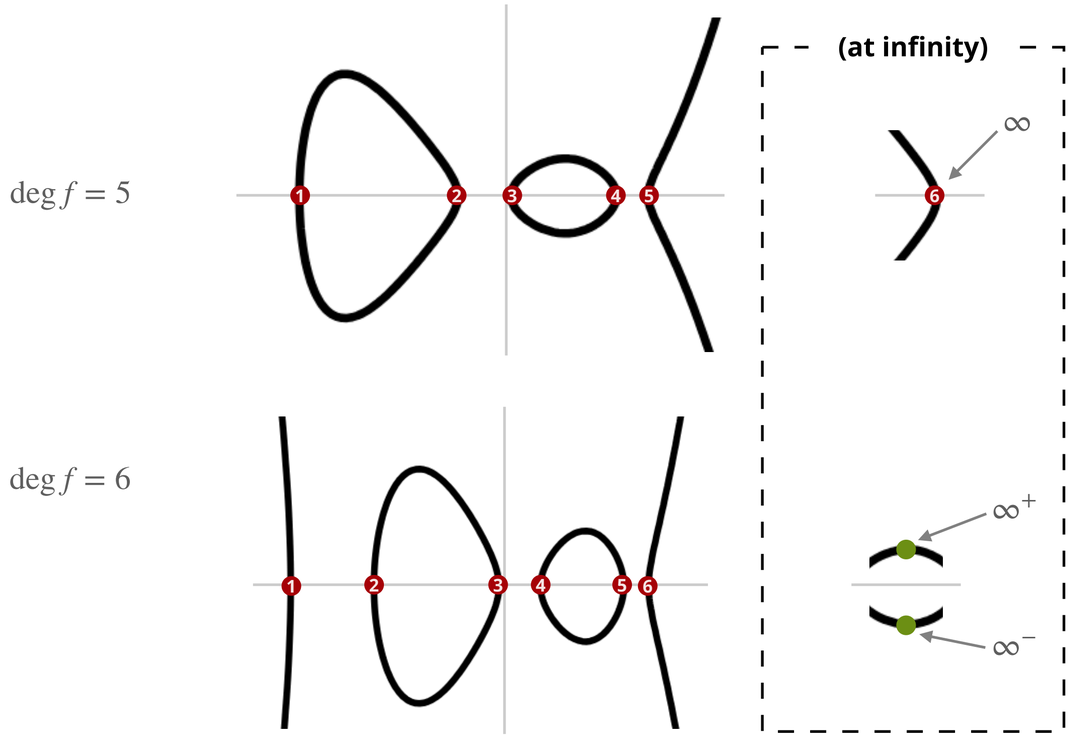
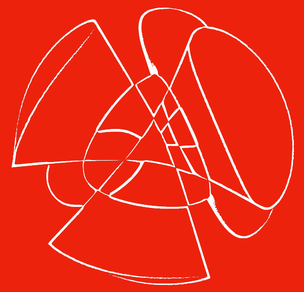
Rosenhain form

$$C_{\lambda,\mu,\nu} : y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\nu)$$

$\lambda, \mu, \nu \in k$ are Rosenhain invariants

Interpretation of curves





Kummer Surfaces

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$$C : y^2 = f(x)$$

(general form)

$$\kappa \downarrow \quad (x, y) \in C \quad \mapsto \quad \left(\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^3}\right) \in C_{\lambda,\mu,\nu}$$

! when 2-torsion is rational

2

Rosenhain form

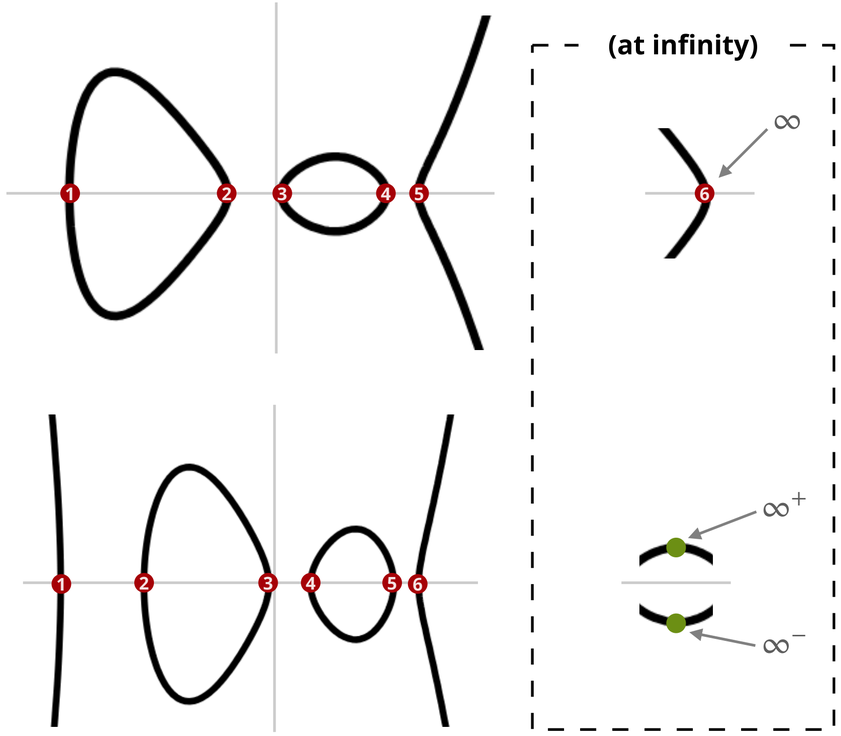
$$C_{\lambda,\mu,\nu} : y^2 = x(x-1)(x-\lambda)(x-\mu)(x-\nu)$$

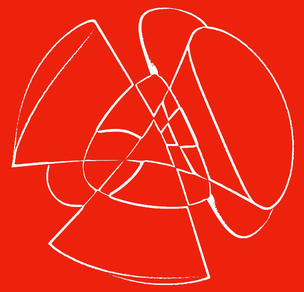
$\lambda, \mu, \nu \in k$ are Rosenhain invariants

Interpretation of curves

$\deg f = 5$

$\deg f = 6$





Kummer Surfaces

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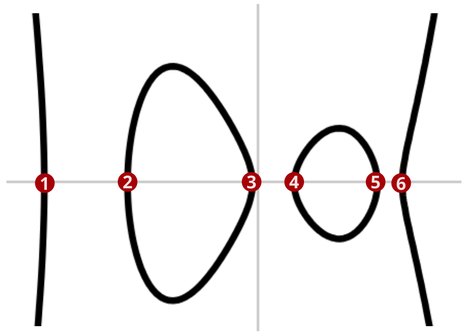
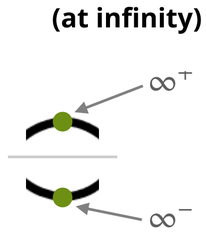
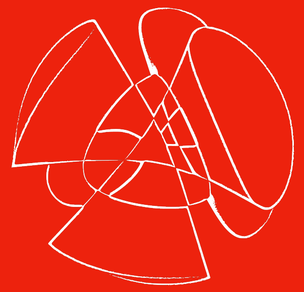
Jacobian
start with *two* points on C,
map to divisors

$$(P_1, P_2) \mapsto (P_1) + (P_2) - D_\infty$$







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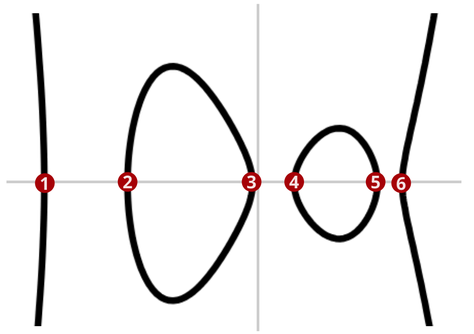
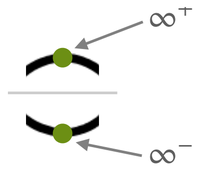
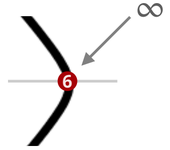


(at infinity)

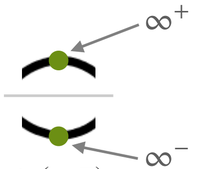


$\deg f = 5$

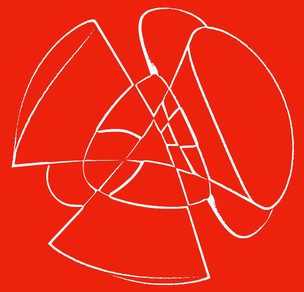


$D_\infty = 2(\infty)$

$\deg f = 6$



$D_\infty = (\infty^+) + (\infty^-)$



Kummer Surfaces

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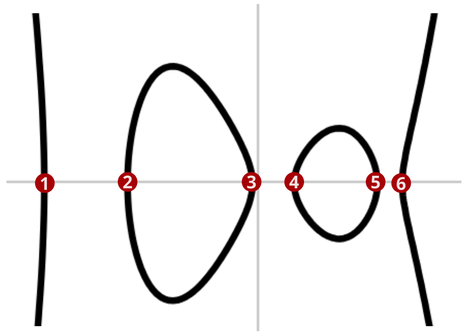
(general form)



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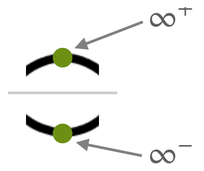
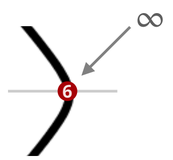


2

Jacobian
start with *two* points on C,
map to divisors

$$(P_1, P_2) \mapsto (P_1) + (P_2) - D_\infty$$

$\deg f = 5$



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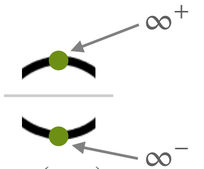


$D_\infty = (\infty^+) + (\infty^-)$

Mumford representation

We will represent an element of the Jacobian

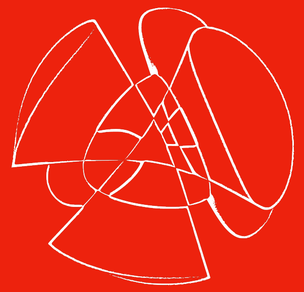
$$D_P = (P_1) + (P_2) \in J_C \qquad P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2)$$

using the *Mumford representation* $\langle a(x), b(x) \rangle$

$$a(x) = (x - x_1)(x - x_2)\ , \qquad b(x_i) = y_i\ .$$

With D_∞ as $\langle 1,0 \rangle$

1



Kummer Surfaces

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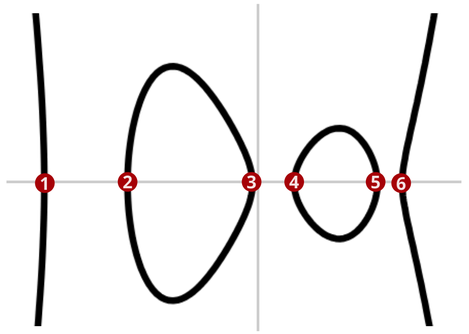
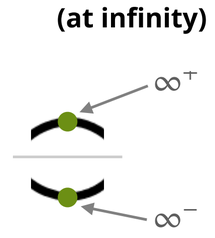
genus 2 *hyperelliptic curve*

$$C : y^2 = f(x)$$

(general form)





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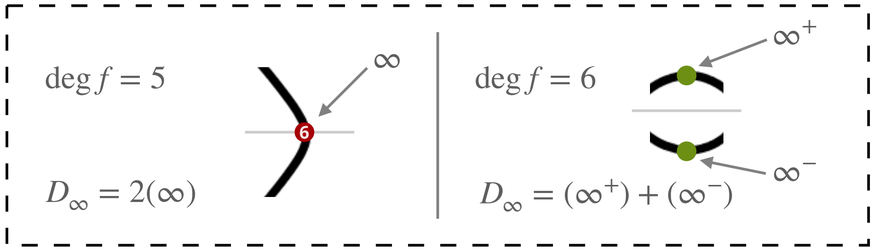

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Galois invariance

Elements of J_C are not just pairs of points from $C(k)$, but can come from larger fields $C(K)$, as long as they are Galois invariant under $\mathrm{Gal}(K/k)$!

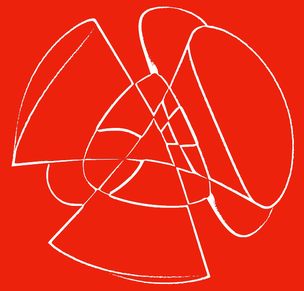
For us, in practice with $k = \mathbb{F}_p$, this means also elements

$$D_P = \big((i, 3 - i)\big) + \big((-i, 3 + i)\big)$$

Convenient: Mumford rep. is then defined over k:

$$D_P = \big(x^2 + 1, -x + 3\big)$$

1



Kummer Surfaces

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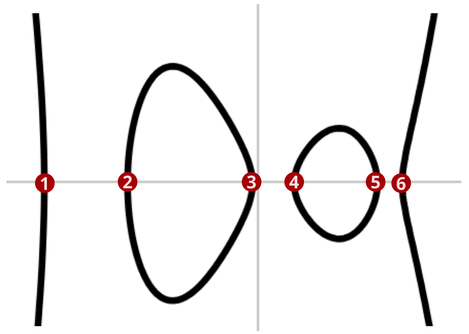

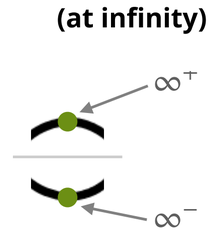

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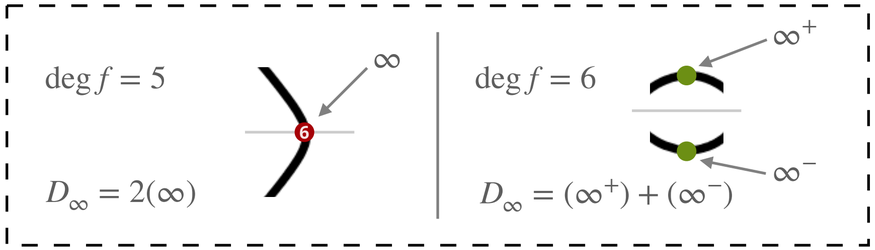

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With D_∞ as $\langle 1,0 \rangle$

Galois invariance

Elements of J_C are not just pairs of points from $C(k)$, but can come from larger fields $C(K)$, as long as they are Galois invariant under $\mathrm{Gal}(K/k)$!

For us, in practice with $k = \mathbb{F}_p$, this means also elements

$$D_P = \big((i, 3 - i)\big) + \big((-i, 3 + i)\big)$$

Convenient: Mumford rep. is then defined over k:

$$D_P = \big(x^2 + 1, -x + 3\big)$$

!

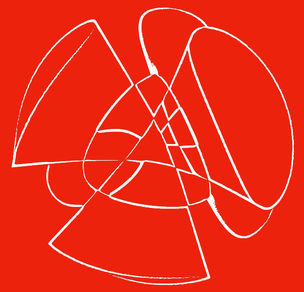
genus 1

This should remind you of how $\langle K \rangle$ can be the kernel of an isogeny over $\mathbb{F}_p$, even if K is a point in $C(\mathbb{F}_{p^2})$:

$$f(x) = \prod (x - x_{[i]K})$$

is defined over $\mathbb{F}_p$!

Radboud University



Kummer Surfaces

1

Starting point
genus 2 *hyperelliptic curve*

$$C : y^2 = f(x)$$

(general form)

2

Jacobian
start with *two* points on C,
map to divisors

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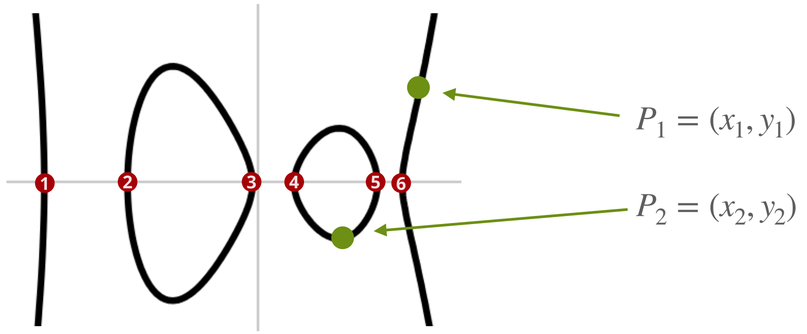

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Given two elements D_P, D_Q in Mumford rep.
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Roughly all we need to do arithmetic on the Jacobian, e.g. scalar mults and so on!
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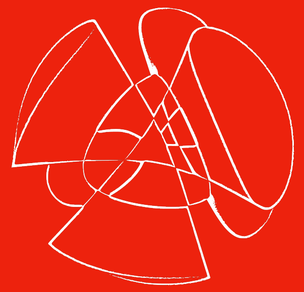

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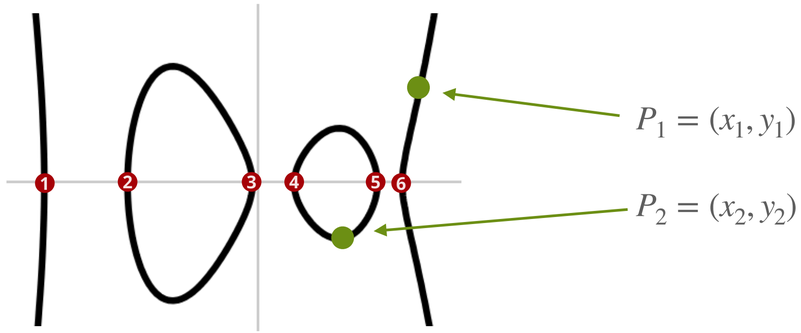

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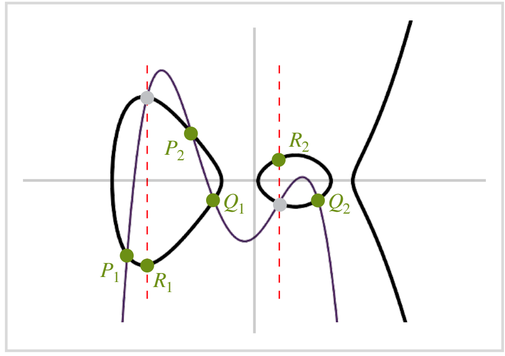

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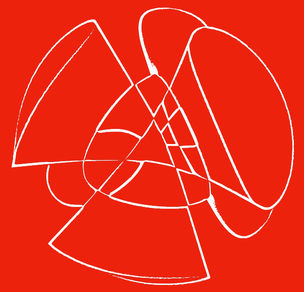

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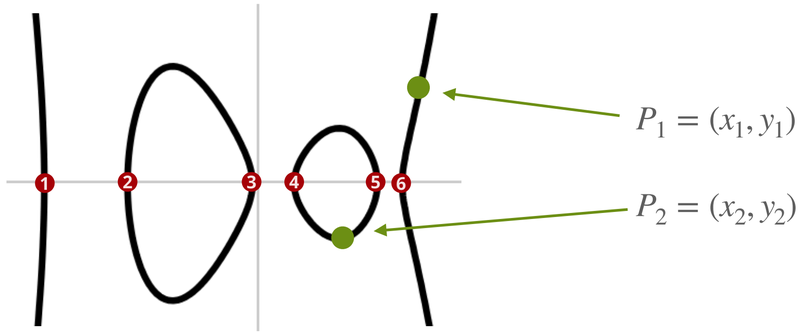

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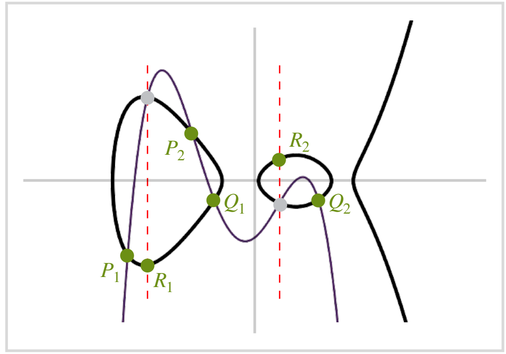
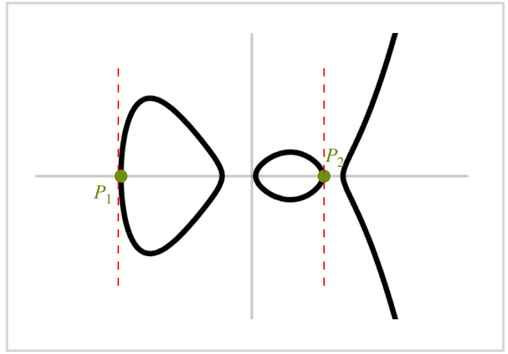


order 2? →



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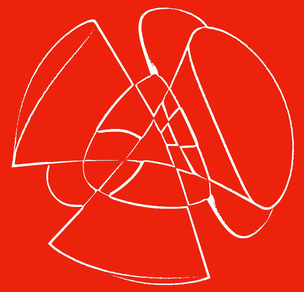

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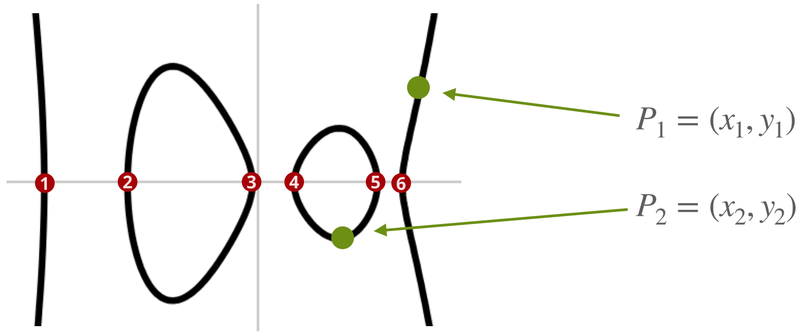

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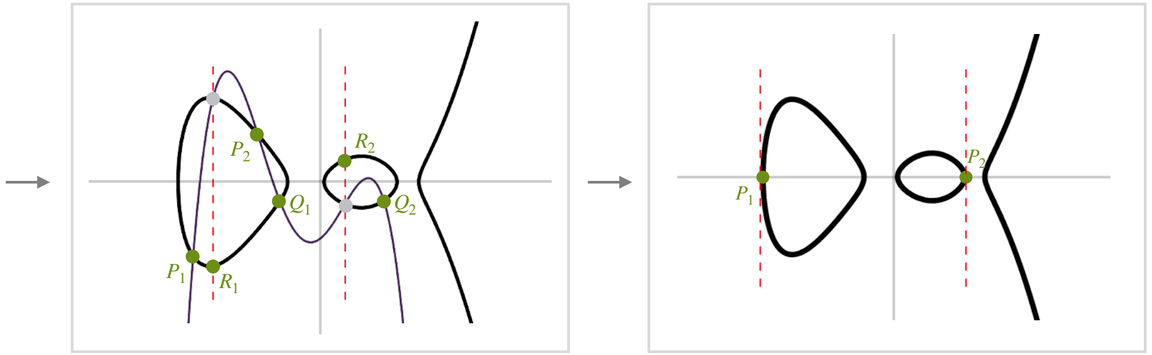

Points of order 2

Precisely the elements given by pairs of *Weierstrass points* $(w_i,0)$, for $1 \le i \le 6$.

In Mumford representation

$$L_{i,j} = \langle (x - w_i)(x - w_j),\ 0 \rangle\ ,$$

and we easily count $\binom{6}{2} = 15$ of them:

$$J_C[2] \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

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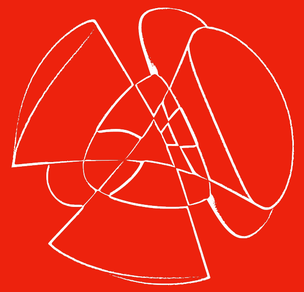

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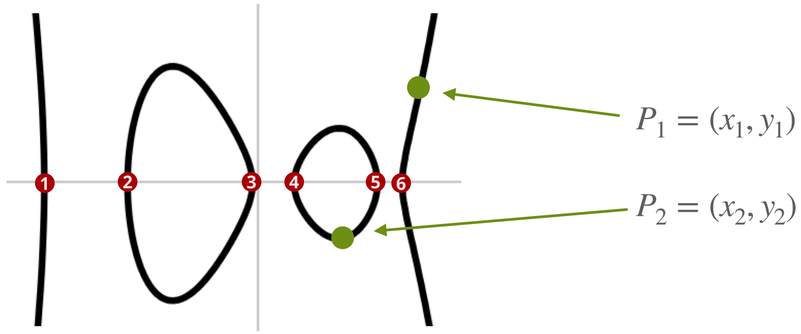

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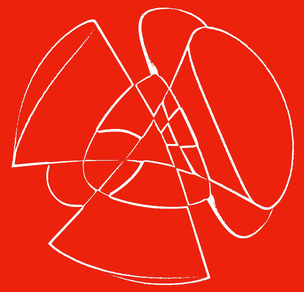

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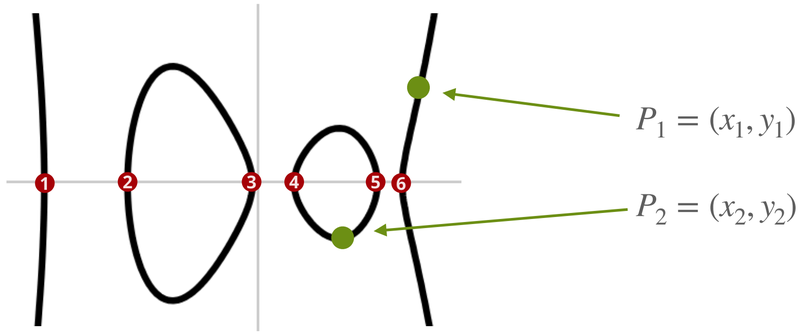

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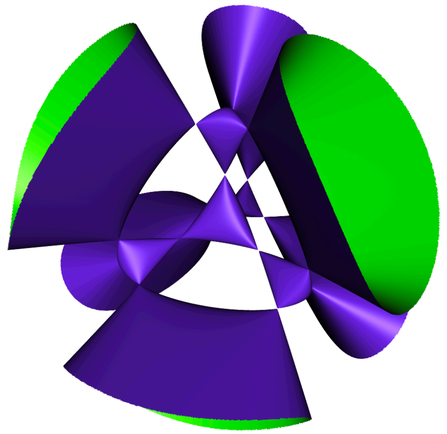

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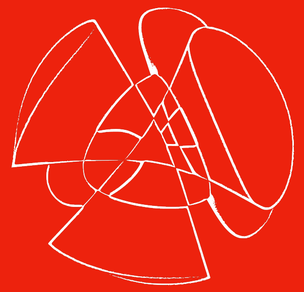

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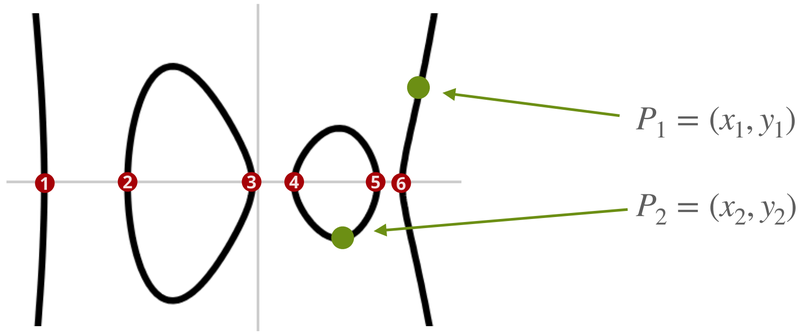

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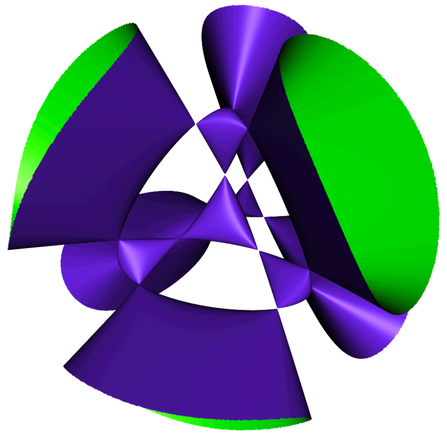

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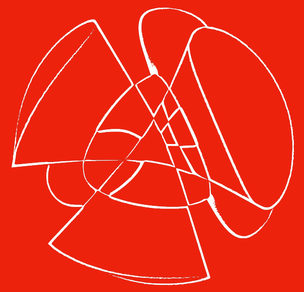

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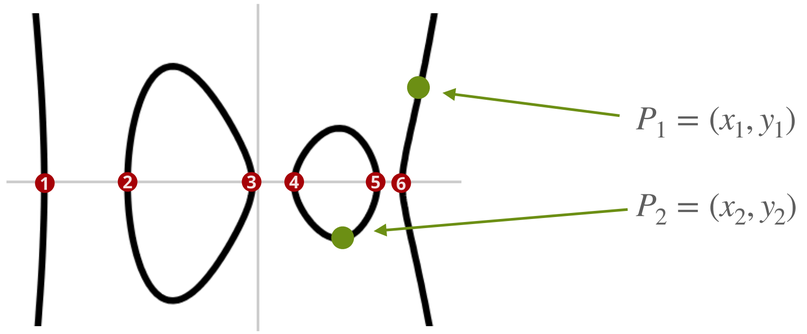

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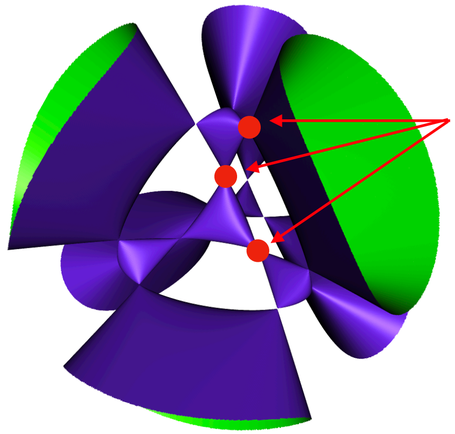

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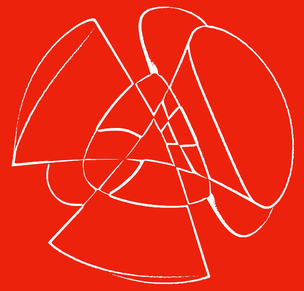

Kummer Surfaces

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Mumford Reps. $D_P = \langle a(x), b(x) \rangle$ of pairs of points of C

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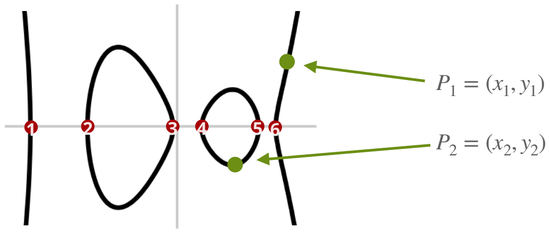

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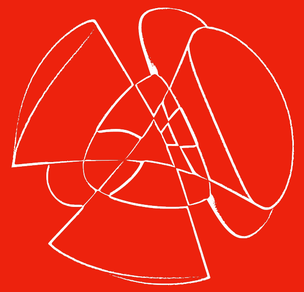

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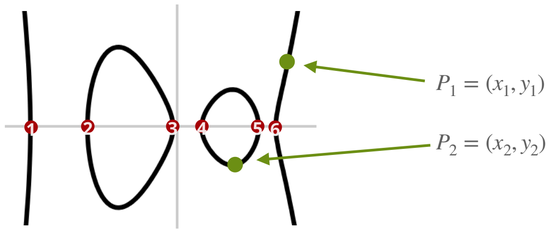

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warning!

The construction *so far* assumes a **general** curve C and constructs the **general** Kummer surface K_C

Advantage:
- mathematically elegant
- always exists
- well-described by Cassels & Flynn

Disadvantage:
- not fast enough

[1] J.W.S. Cassels and E.V. Flynn, *Prolegomena to a Middlebrow Arithmetic of Curves of Genus 2,* Cambridge University Press, 1996.
[2] D.V Chudnovsky, G.V Chudnovsky, *Sequences of numbers generated by addition in formal groups and new primality and factorization tests,* Advances in Applied Mathematics, Volume 7, Issue 4, 1986.
[3] P. Gaudry. *Fast genus 2 arithmetic based on Theta functions.* J. Mathematical Cryptology, 1(3):243– 265, 2007.
[4] D. J. Bernstein. *Elliptic vs. Hyperelliptic, part I.* Talk at ECC (slides at http://cr.yp.to/talks/ 2006.09.20/slides.pdf), September 2006.

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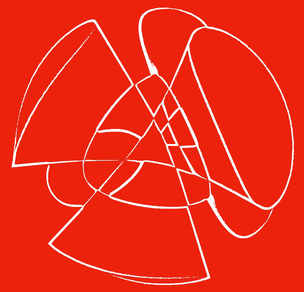

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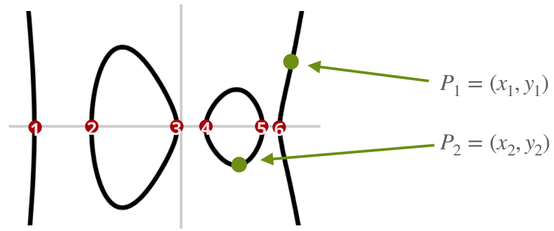

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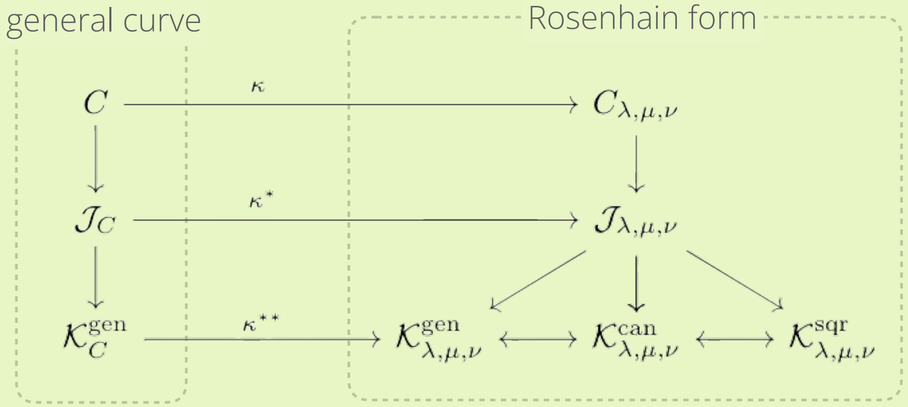

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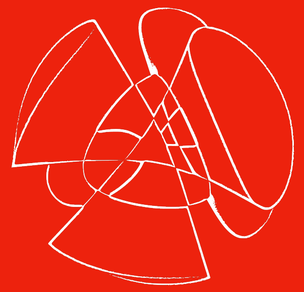

Kummer Surfaces

1

Curve

$$C : y^2 = x^6 + c_5x^5 + c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$$

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Jacobian

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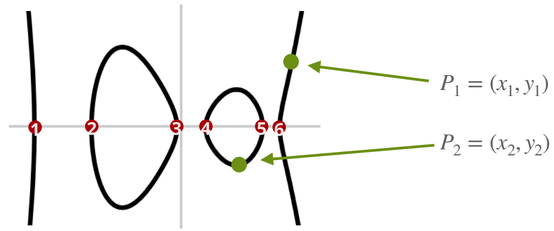

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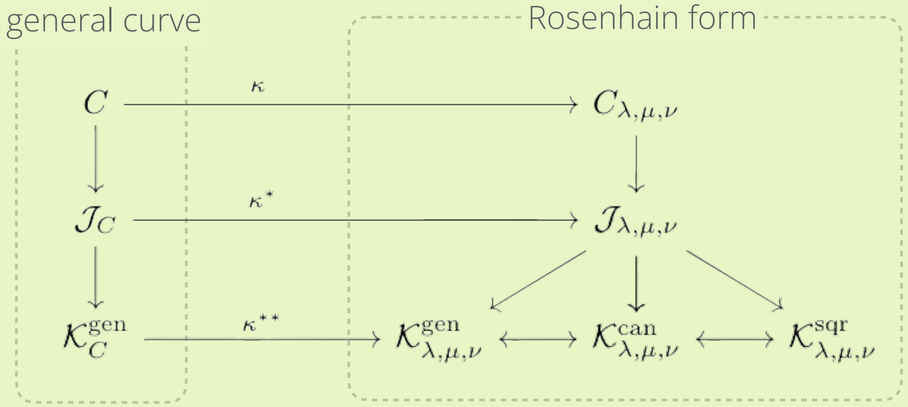

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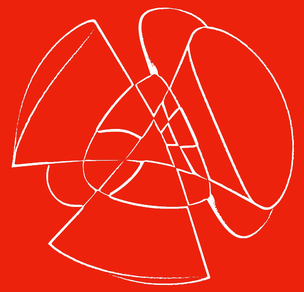

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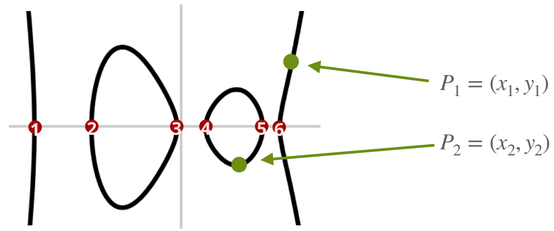

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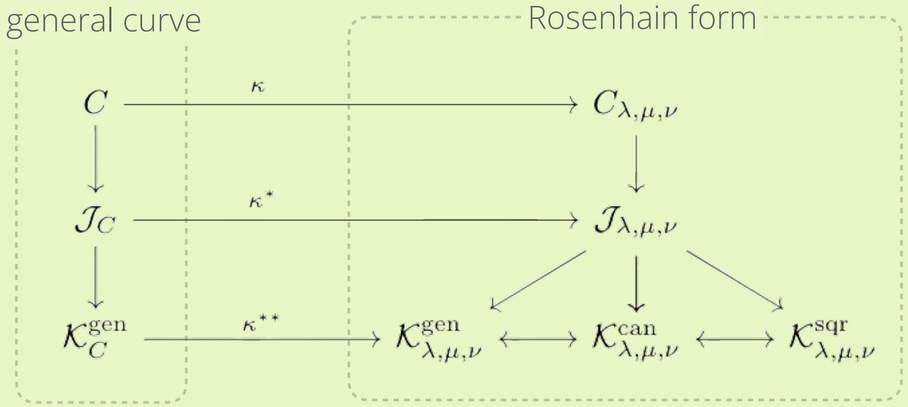

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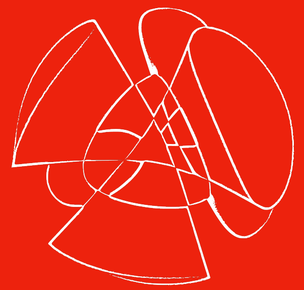

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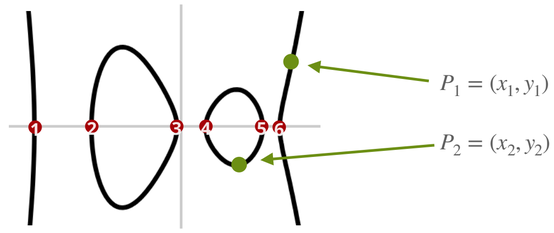

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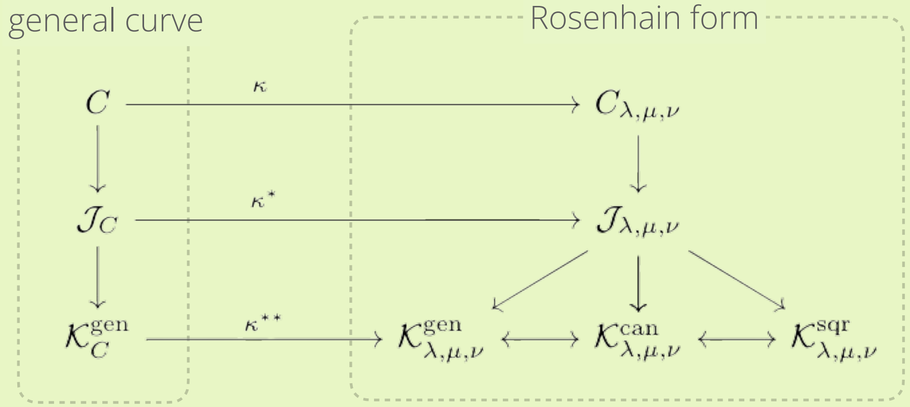

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Squared Kummer, adapted Canonical, by Bernstein for high-speed crypto, with

$$\mathcal{K}_{\lambda,\mu,\nu}^{\mathrm{sqr}} \ : \ F \cdot X_1X_2X_3X_4 = \\ (X_1^2 + X_2^2 + X_3^2 + X_4^2 - G \cdot (X_1 + X_2)(X_3 + X_4) - H \cdot (X_1X_2 + X_3X_4))^2$$

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Radboud University

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(Bernstein, Chuengsatiansup, Lange, Schwabe)

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Isogeny-based with Kummers

Works by Cosset, Lubicz, Robert since 2010, expanded by Costello in 2018. Recently, interest from HD-perspective, see e.g. Dartois, Maino, Pope, and Robert.

Goal: do cryptography on higher-dimensional abelian varieties, for example computing $(2^n,2^n)$-isogeny, either for products of curves (DMPR)

$$E \times E' \to E'' \times E'''$$

or simulate SIDH on Kummers (Costello)

$$E_\alpha \to K_\alpha \xrightarrow{\varphi} K_\beta \to E_\beta$$

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Advanced protocols on Kummers

The tools from Kummer arithmetic (1) and Kummer isogenies (2) allow us to do a single long $(2^n,2^n)$-isogeny. Can we build more tools, achieve SQIsign verification?

Goal: develop tools for point sampling, three point ladder, faster isogenies, to be able to do the full SQIsign verification only on Kummer surfaces.

$$K_0 \xrightarrow{\varphi_1} K_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} K_n$$

Hope: use parallelisation from Kummer arithmetic to achieve fast(er?) verification.

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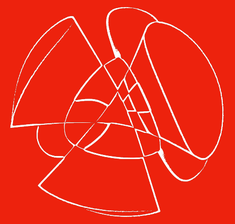
Return of the Kummer

✓



Kummer Surfaces

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$$\begin{array}{ccc} \mathcal{J}_\alpha & \xrightarrow{\kappa} & \mathcal{J}_{\lambda,\mu,\nu} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}_\alpha & \xrightarrow{\kappa^*} & \mathcal{K}_{\lambda,\mu,\nu} \end{array}$$

Kummers In Cryptography

3

$$\begin{array}{ccccc} 0 & \longrightarrow & E[n] & \longrightarrow & E \\ & & \downarrow{\scriptstyle [n]} & & \\ E & \longrightarrow & E/[n]E & \longrightarrow & 0 \end{array}$$

Pairings on Kummers

Running example: SQIsign verification using Kummers

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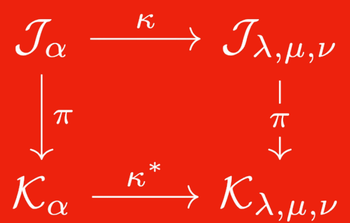
Kummers in Cryptography

Scholten's construction

Any elliptic curve $E_\alpha/\mathbb{F}_{p^2}$ has a genus-2 friend $J_\alpha/\mathbb{F}_p$ which is $(2,2)$-isogenous, where $E_\alpha : y^2 = x(x-\alpha)(x-1/\alpha)$ with $\alpha = \alpha_0 + i \cdot \alpha_1 \in \mathbb{F}_{p^2}$



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ORIGINAL POINT-OF-VIEW

1. Take the *Weil restriction* $W_\alpha := W_{\mathbb{F}_p}^{\mathbb{F}_{p^2}}(E_\alpha)$
2. Construct a specific hyper ell. curve $C_\alpha : y^2 = f_0(x) \cdot f_1(x) \cdot f_2(x)$, with $f_i(x)$ derived from α_0, α_1
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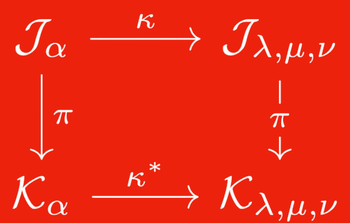

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REPHRASED POINT-OF-VIEW

1. The Weil restriction is a specific *glueing*
2. Glue $E_\alpha \times E_\alpha^{(p)}$ along the right 2-torsion
3. This gives us $\eta' : E_\alpha \times E_\alpha^{(p)} \to J_\alpha$

$$\begin{array}{ccc} \mathcal{J}_\alpha & \xrightarrow{\kappa} & \mathcal{J}_{\lambda,\mu,\nu} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}_\alpha & \xrightarrow{\kappa^*} & \mathcal{K}_{\lambda,\mu,\nu} \end{array}$$

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2. Glue $E_\alpha \times E_\alpha^{(p)}$ along the right 2-torsion
3. This gives us $\eta' : E_\alpha \times E_\alpha^{(p)} \to J_\alpha$

1 Take ell curve over $\mathbb{F}_{p^2}$ → E_α

$$\begin{array}{ccc} \mathcal{J}_\alpha & \xrightarrow{\kappa} & \mathcal{J}_{\lambda,\mu,\nu} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}_\alpha & \xrightarrow{\kappa^*} & \mathcal{K}_{\lambda,\mu,\nu} \end{array}$$

Kummers in Cryptography

Scholten's construction

Any elliptic curve $E_\alpha/\mathbb{F}_{p^2}$ has a genus-2 friend $J_\alpha/\mathbb{F}_p$ which is $(2,2)$-isogenous, where $E_\alpha : y^2 = x(x-\alpha)(x-1/\alpha)$ with $\alpha = \alpha_0 + i \cdot \alpha_1 \in \mathbb{F}_{p^2}$

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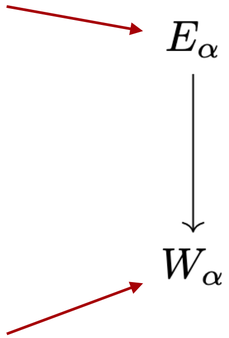
1 Take ell curve over $\mathbb{F}_{p^2}$

2 Take Weil restriction over $\mathbb{F}_p$



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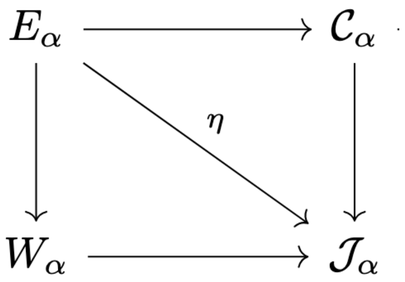
2 Take Weil restriction over $\mathbb{F}_p$

3 Re-order restricted roots



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4 Map to Rosenhain

$$\begin{array}{ccccc} E_\alpha & & \longrightarrow & \mathcal{C}_\alpha & \xrightarrow{\kappa} & \mathcal{C}_{\lambda,\mu,\nu} \\ \downarrow & \searrow^{\eta} & & \downarrow & & \downarrow \\ W_\alpha & & \longrightarrow & \mathcal{J}_\alpha & \xrightarrow{\kappa^*} & \mathcal{J}_{\lambda,\mu,\nu} \end{array}$$

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5 Go to squared Kummer

$$\begin{array}{ccccccc} E_\alpha & & \longrightarrow & & \mathcal{C}_\alpha & \xrightarrow{\kappa} & \mathcal{C}_{\lambda,\mu,\nu} \\ & \searrow^{\eta} & & & & & \\ W_\alpha & & \longrightarrow & & \mathcal{J}_\alpha & \xrightarrow{\kappa^*} & \mathcal{J}_{\lambda,\mu,\nu} \\ & & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ & & & & \mathcal{K}_\alpha^{\mathrm{gen}} & \xrightarrow{\kappa^{**}} & \mathcal{K}_{\lambda,\mu,\nu}^{\mathrm{gen}} \xrightarrow{M} \mathcal{K}_{\lambda,\mu,\nu}^{\mathrm{Sqr}} \end{array}$$

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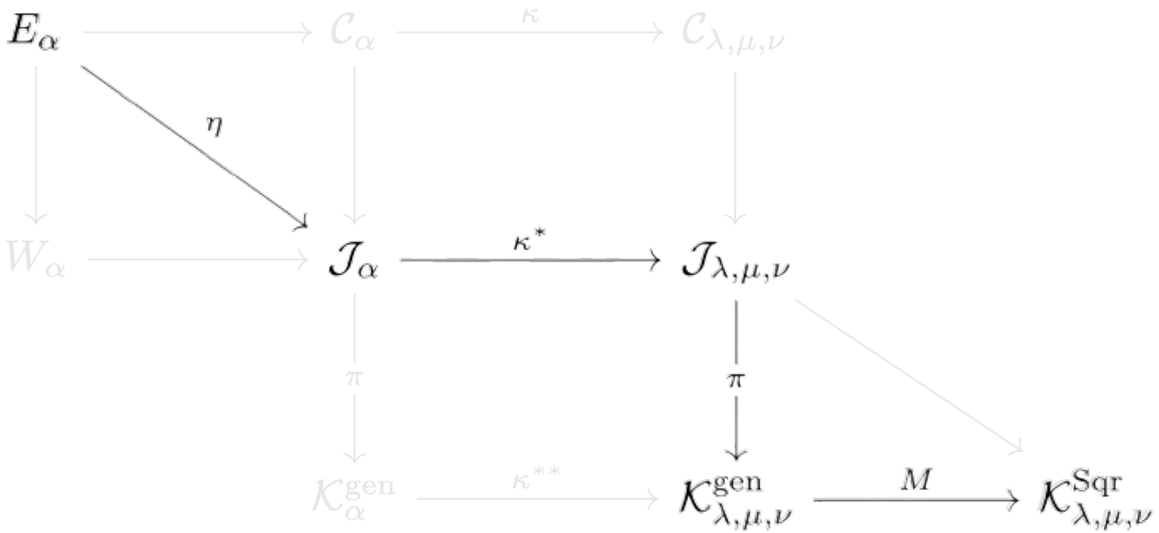

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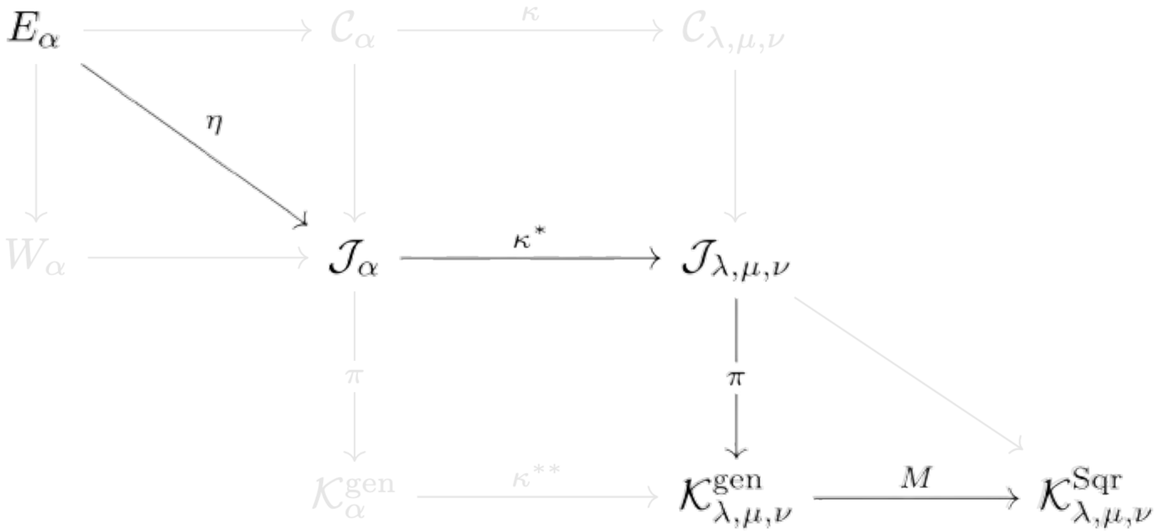
For remaining slides, ignore most of this



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!

Just think of $E_\alpha \to K_\alpha$

$$E_\alpha \xrightarrow{\zeta := M \circ \pi \circ \kappa^* \circ \eta} \mathcal{K}_\alpha := \mathcal{K}_{\lambda,\mu,\nu}^{\mathrm{Sqr}}$$

!

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Kummers in Cryptography

Costello's SIDH on Kummer

Any elliptic curve $E_\alpha/\mathbb{F}_{p^2}$ has a $(2,2)$-isogenous Kummer friend $K_\alpha/\mathbb{F}_p$

- The curve E_α has three 2-isogenies, defined either by $P_0, P_\alpha, P_{1/\alpha}$
- The Kummer K_α has corresponding $(2,2)$-isogenies
- By magic, these $(2,2)$'s can be described by single points $P'_0, P'_\alpha, P'_{1/\alpha}$ instead of subgroups!
- Thus, given a point of order 2^n on $K_{\alpha'}$ we can do a $(2^n,2^n)$-isogeny on the Kummers

1 Point P of order 2^{n+2}

E_α, P

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2 maps to $\zeta(P)$ of order 2^{n+1}

$$\begin{array}{c} E_\alpha, P \\ \downarrow{\scriptstyle \zeta} \\ \mathcal{K}_\alpha, \zeta(P) \end{array}$$

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$$\begin{array}{ccc} E_\alpha, P & & \\ \downarrow{\scriptstyle \zeta} & & \\ \mathcal{K}_\alpha, \zeta(P) & \xrightarrow{\varphi'} & \mathcal{K}_\alpha/\langle 2\zeta(P)\rangle \end{array}$$

3 gives a $(2^n,2^n)$-isogeny on Kummers

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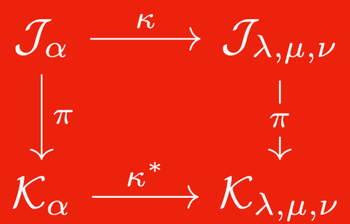
$$\begin{array}{ccc} E_\alpha, P & \xrightarrow{\varphi} & E_\alpha/\langle\langle 4P\rangle\rangle \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta'} \\ \mathcal{K}_\alpha, \zeta(P) & \xrightarrow{\varphi'} & \mathcal{K}_\alpha/\langle 2\zeta(P)\rangle \end{array}$$

3 gives a $(2^n,2^n)$-isogeny on Kummers

4 which matches up with isogeny generated by $4P$



Kummers in Cryptography

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- Costello gives us the tools to translate $\varphi : E_\alpha \to E'_\alpha$ to Kummer isogeny $\varphi' : K_\alpha \to K'_\alpha$
- Can we then translate SQIsign isogenies $\sigma : E_A \to E_2$ of degree 2^{1000} to Kummers?
- First, try *uncompressed* signatures
- Then, see if we have enough tools to compress signatures

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1 SQIsign signature

$$E_0 \xrightarrow{\sigma_1} E_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} E_n$$

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1 SQIsign signature

2 Map down starting curve

$$\begin{array}{ccccccc} E_0 & \xrightarrow{\sigma_1} & E_1 & \xrightarrow{\sigma_2} & \cdots & \xrightarrow{\sigma_n} & E_n \\ \downarrow{\scriptstyle \zeta_0} & & & & & & \\ K_0^{\mathrm{Sqr}} & & & & & & \end{array}$$

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3 Generate the right kernel points on K_i

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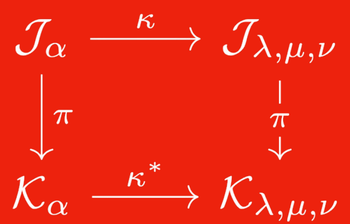
4 SQIsign on Kummers!



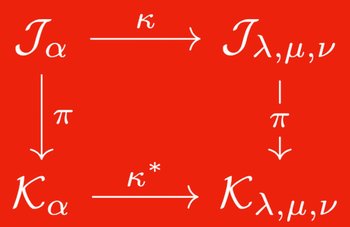
Kummers in Cryptography

Compressed SQIsign

Elliptic Curves

- instead of $\varphi : E \to E'$ given by kernel $R \in E[2^f]$
- find deterministic basis P, Q of $E[2^f]$
- then find $s \in [1..2^f]$ such that $\langle R \rangle = \langle P + sQ \rangle$
- then φ is compressed to s, which takes f bits

  from R, which takes $2 \log p$ bits



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Kummer Surfaces

- instead of $\varphi : K \to K'$ given by kernel $R \in K[2^f]$
- find deterministic basis of K ?
- can we express R simply in these terms?
- we are missing many **essential** tools to do these protocols on Kummer surfaces!

2

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Compressed SQIsign

Elliptic Curves

- instead of $\varphi : E \to E'$ given by kernel $R \in E[2^f]$
- find deterministic basis P, Q of $E[2^f]$
- then find $s \in [1..2^f]$ such that $\langle R \rangle = \langle P + sQ \rangle$
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Kummer Surfaces

- instead of $\varphi : K \to K'$ given by kernel $R \in K[2^f]$
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- can we express R simply in these terms?
- we are missing many **essential** tools to do these protocols on Kummer surfaces!

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given $P \in K$, is origin $J(\mathbb{F}_p)$ or $J^T(\mathbb{F}_p)$?

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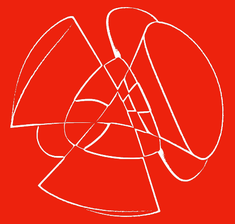
we need to talk about **pairings**

Return of the Kummer

✓



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Kummers In Cryptography

3

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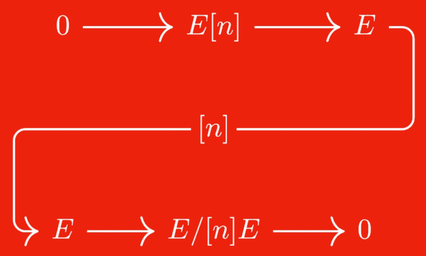
Pairings on Kummers

Pairings (on Kummers)

3



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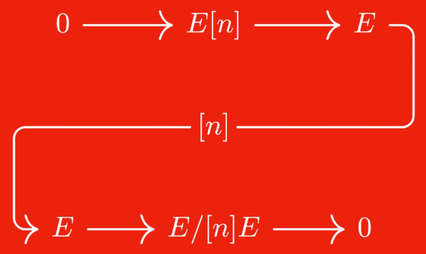

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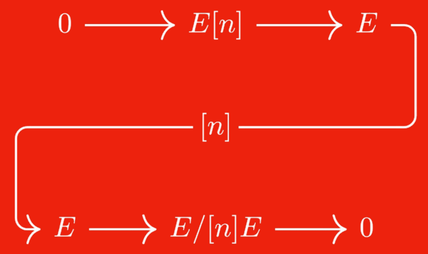

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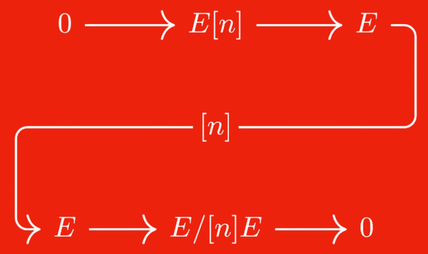

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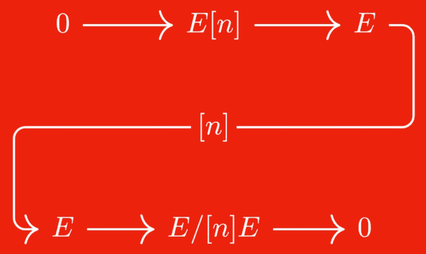

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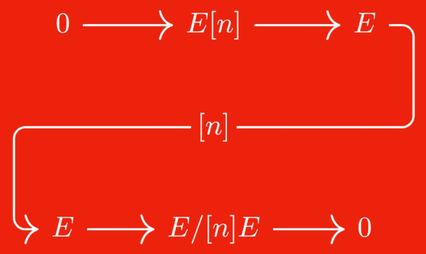

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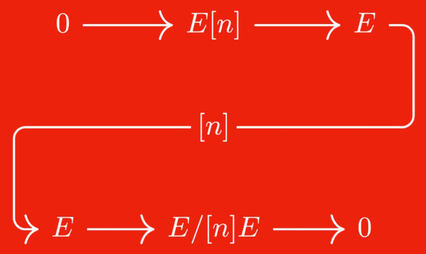

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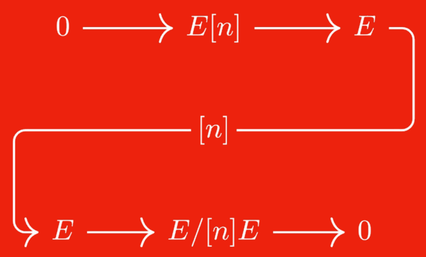

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$P \in [2]E \Leftrightarrow t_2(L_i, P) = 1$ for $L_i \in E[2]$

MISTAKE 1

DON'T: think of this as $E(\mathbb{F}_{p^2})$ or $E(\mathbb{F}_{p^2})[2]$

DO: think of this as cosets, $P + [2]E(\mathbb{F}_{p^2})$

$$t_2 : \quad E(\mathbb{F}_{p^2})[2] \quad \times \quad E(\mathbb{F}_{p^2})/[2]E(\mathbb{F}_{p^2}) \longrightarrow \mu_2$$

In SQIsign, SIDH/SIKE, we have E supersingular over $\mathbb{F}_{p^2}$ with $2^f \mid p + 1$, think of

$$E(\mathbb{F}_{p^2}) \cong \mathbb{Z}_{2^f} \times \mathbb{Z}_{2^f} \times \mathbb{Z}_h \times \mathbb{Z}_h$$

For P, Q basis of $E[2^f]$, these cosets are essentially

$$\mathcal{O} + [2]E(\mathbb{F}_{p^2}), \quad P + [2]E(\mathbb{F}_{p^2}), \quad Q + [2]E(\mathbb{F}_{p^2}), \quad (P + Q) + [2]E(\mathbb{F}_{p^2})$$

So $R \in [2]E(\mathbb{F}_{p^2})$ whenever $t_2(K, R) = 1$ for all points $K \in E(\mathbb{F}_{p^2})[2]$

profile

For $R \in E(\mathbb{F}_{p^2})$, the **profile** of R is the array

$(t_2(K_1, R), t_2(K_2, R), \ldots, t_2(K_n, R))$

for a fixed basis K_i of $E(\mathbb{F}_{p^2})[2]$

! $R \in [2]E$ if and only if profile of R is trivial: (1,1)

! profile of R determines coset $R \in P + [2]E$

1. Fun facts for elliptic curve friends
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$$0 \longrightarrow E[n] \longrightarrow E \xrightarrow{[n]} E \longrightarrow E/[n]E \longrightarrow 0$$

Pairings on Kummers

genus 1 example

Let $E : y^2 = x^3 + Ax^2 + x$ be a supersingular curve over $\mathbb{F}_{p^2}$, with $E(\mathbb{F}_{p^2}) \cong \mathbb{Z}_{p+1} \times \mathbb{Z}_{p+1}$ for $p = 2^a \cdot 3^b - 1$.

fun fact

To find $P \in E(\mathbb{F}_{p^2})$ with ``full 2^a-torsion'':

It is enough to find $P \in E(\mathbb{F}_{p^2})$ with x_P **non-square**

why?

theorem

The image $[2]E$ is given by all points $(x, y) \in E(\mathbb{F}_{p^2})$ with

$x, x - \alpha, x - \frac{1}{\alpha}$ square

with α root of $x^2 + Ax + 1$

why?

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MISTAKE 2

DON'T: think of the Tate-Lichtenbaum pairing as single value

DO: think of an array of values, e.g. evaluated on *all* of $\ker[2]$

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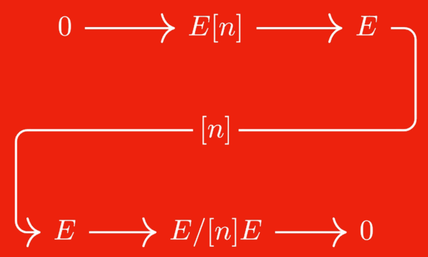

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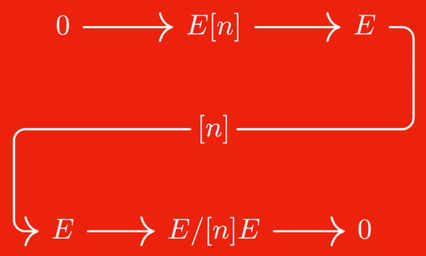

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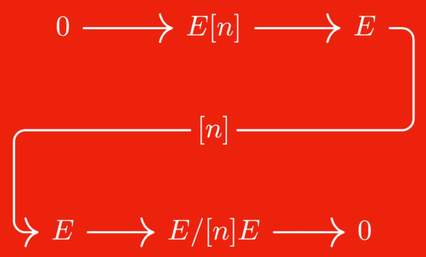

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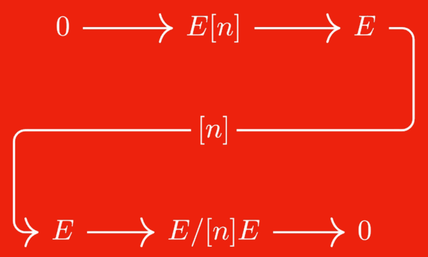

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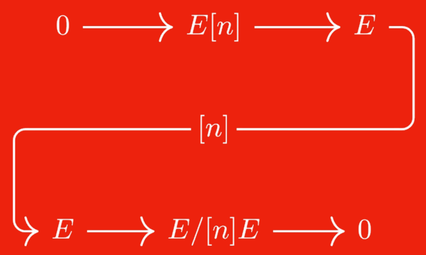

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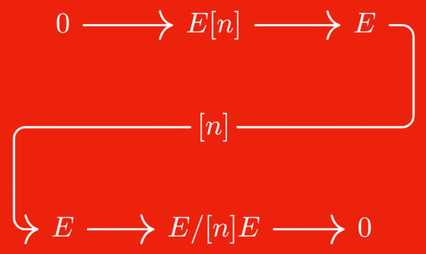

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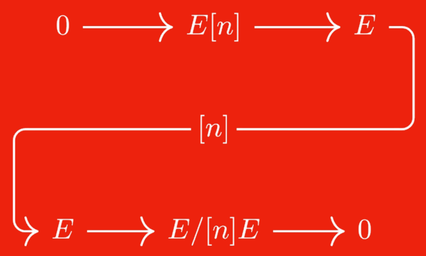

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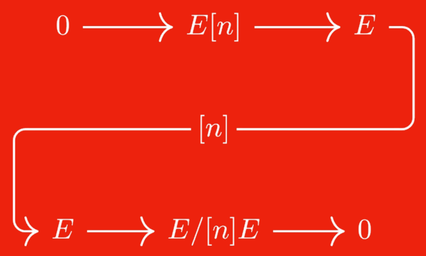

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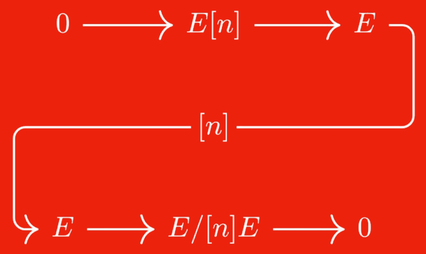

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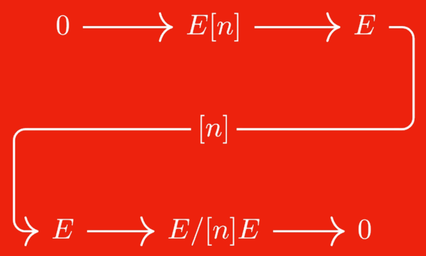

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Radboud University

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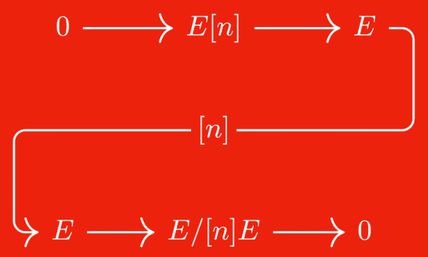

Pairings on Kummers

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2 Decompose cokernel using pairings

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generalized Tate pairing (Bruin, 2011)

Let $\varphi : J \to J'$ be an isogeny* between Jacobians** of curves, then there exists a perfect pairing

$$t_\varphi : \ker \varphi(k) \times \operatorname{coker} \hat{\varphi}(k) \to k^*$$

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decompose J

Again, profile **determines** coset:

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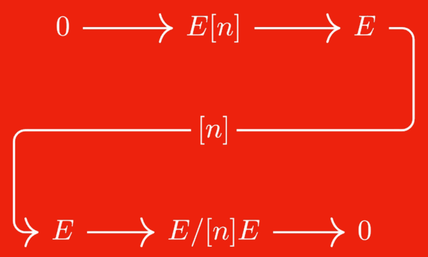

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! TAKE AWAYS

1. for φ dividing $[n]$, the profile t_φ is essentially a "sub-profile" of $t_{[n]}$
2. the profile $t_\varphi(R)$ allows us to identify a coset $P_i + \operatorname{im}\hat{\varphi}(E'(\mathbb{F}_{p^2}))$
3. in particular, easy to identify $R \notin \operatorname{im}\hat{\varphi}$, we just need one non-trivial $t_\varphi(K_i, R)$

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$$0 \longrightarrow E[n] \longrightarrow E \xrightarrow{[n]} E \longrightarrow E/[n]E \longrightarrow 0$$

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genus 2 example

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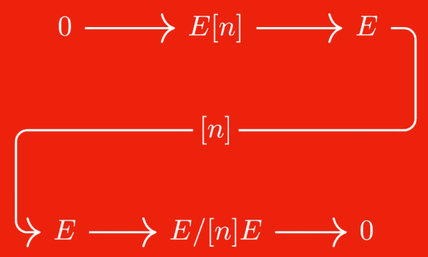

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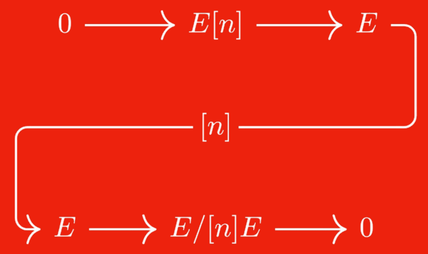

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profiles

In genus 1:
find $R \in E \backslash [2]E$
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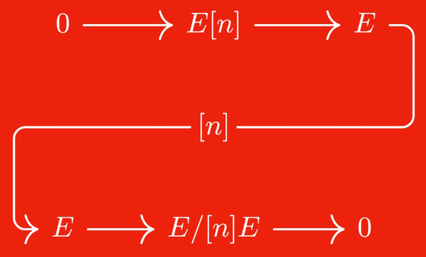

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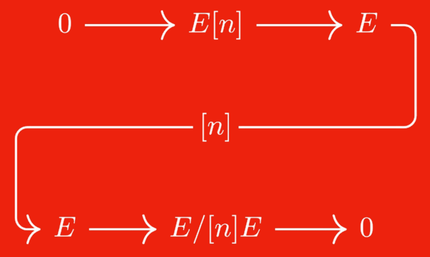

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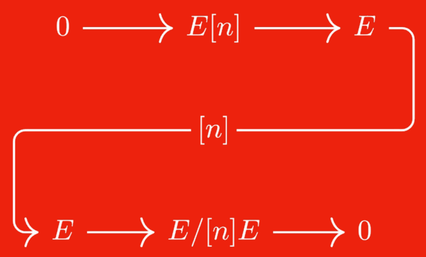

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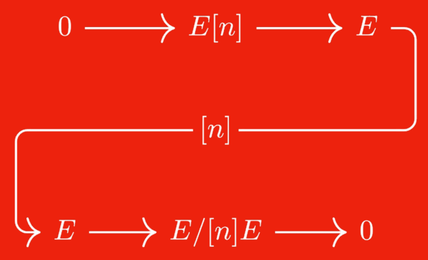

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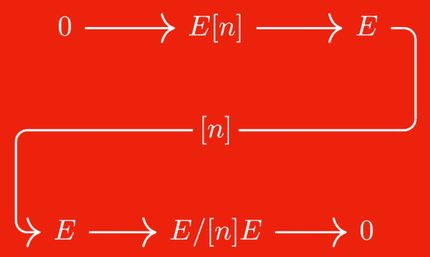

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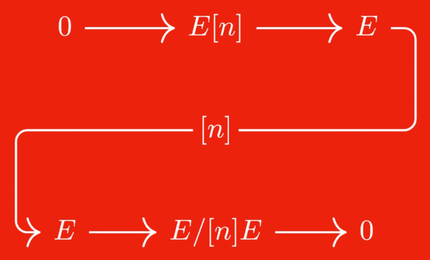
practice

Want to compute profiles, e.g. 2-Tate pairings, on Kummer K_α

3



Pairings on Kummers

1. Fun facts for elliptic curve friends
2. Decompose cokernel using pairings
3. Higher genus fun facts for Jacobian friends

genus 2 example

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how? →

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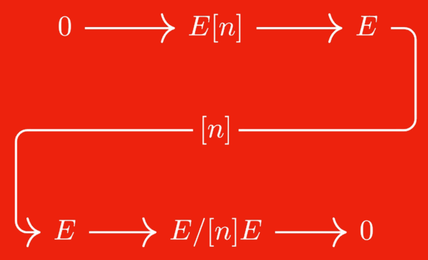

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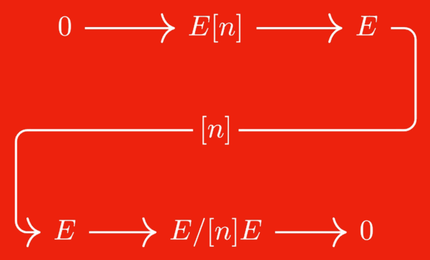

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2. Use elegant monodromy approach of Robert (2024)!
3. Develop more efficient methods for partial map $K_\alpha \to J_\alpha$ which allows very efficient profiling 😎😎😎

Concluding

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In general, can do many things on the Kummer that we can do on ell curve.

Allows for isogeny-based crypto on the Kummer over $\mathbb{F}_p$!

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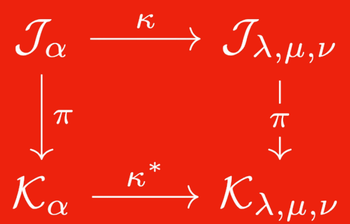
Doing compressed SQIsign requires fun but complex techniques

Pairings are great to decompose cokernels!
Applies for efficient sampling.



Kummers in Cryptography

Costello's SIDH on Kummer

Any elliptic curve $E_\alpha/\mathbb{F}_{p^2}$ has a $(2,2)$-isogenous Kummer friend $K_\alpha/\mathbb{F}_p$

- The curve E_α has three 2-isogenies, defined either by $P_0, P_\alpha, P_{1/\alpha}$
- The Kummer K_α has corresponding $(2,2)$-isogenies
- By magic, these $(2,2)$'s can be described by single points $P'_0, P'_\alpha, P'_{1/\alpha}$ instead of subgroups!
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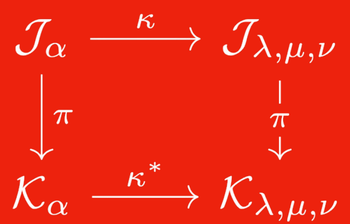

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Each elliptic curve E_α over $\mathbb{F}_{p^2}$ has a (2,2)-isogenous Kummer K_α over $\mathbb{F}_p$ whenever $\alpha = \alpha_0 + \alpha_1 i$ has $\alpha_0, \alpha_1 \neq 0$.

K_α is derived through a curve $C_{\lambda,\mu,\nu}$ with $\lambda, \mu, \nu \in \mathbb{F}_p$ and $\lambda \cdot \mu = \nu$.

Categorically, we map **objects**
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$$\begin{array}{ccc} \mathcal{J}_\alpha & \xrightarrow{\kappa} & \mathcal{J}_{\lambda,\mu,\nu} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{K}_\alpha & \xrightarrow{\kappa^*} & \mathcal{K}_{\lambda,\mu,\nu} \end{array}$$

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These elliptic curves E_α naturally have 2-isogenies, given by $E_\alpha[2]$.

Similarly, Kummer surface K_α has associated (2,2)-isogenies, that are identified by point $P \in K_\alpha[2]$.

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Corte-Real Santos & R.

Given $\lambda, \mu, \nu \in \mathbb{F}_p$ with $\lambda \cdot \mu = \nu$, the Kummer surface K over $\mathbb{F}_p$ derived from $C_{\lambda,\mu,\nu}$ has a (2,2)-isogenous elliptic curve friend E_α over $\mathbb{F}_{p^2}$.

The 2-isogeny graph of elliptic curves over $\mathbb{F}_{p^2}$ **reappears** as a subgraph of (2,2)-isogeny graph!