Ihara zeta functions of abstract isogeny graphs and modular curves

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Cryptographic motivation

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Our work gives a framework for studying expansion of non-backtracking walks in isogeny graphs with level structure through their *zeta functions*.

Zeta functions in general

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The Riemann zeta function encodes information about the distribution of prime numbers in the location of its zeros.

Zeta functions in general

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Definition

Let X be a smooth, irreducible, projective variety defined over $\mathbb{F}_\ell$. The **Hasse-Weil zeta function** for X is defined as:

$$Z(X,u) = \exp\left(\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{\ell^n})}{n} u^n\right) = \prod_{x\in[X]} \frac{1}{1-u^{\deg(x)}},$$

where the product is defined over the closed points of X.

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Again, these zeta functions have a version of the Riemann hypothesis, now known by work of Deligne (and others).

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A *prime* in a graph G is a closed walk, containing no backtracking or tail, which is not a shorter walk repeated multiple times.

But what does *no backtracking* mean?

Serre's definition of a graph

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Teaser: this definition is not appropriate for isogeny graphs, particularly isogeny graphs with level structure.

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Definition

Let G be an (undirected) graph. The *Ihara zeta function* of G is the function

$$\zeta_G(u) = \prod_{\text{primes } P} (1 - u^{|P|})^{-1}$$

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1. $u\frac{d}{du}\log\zeta_G(u) = \sum_{m\geq 1} N_m u^m$, where N_m is the number of non-backtracking cycles of length m.

Ihara zeta functions

Facts about Ihara zeta functions:

1. $u \frac{d}{du} \log \zeta_G(u) = \sum_{m \geq 1} N_m u^m$, where N_m is the number of non-backtracking cycles of length m.
2. (Bass-Ihara determinant formula): Suppose that G is a d-regular graph, and let A be the adjacency matrix of G. Then we have:

$$\zeta_G(u) = \frac{(1-u^2)^{1-\chi(G)}}{\det(I - Au + (d-1)u^2)},$$

where χ is the Euler characteristic of G.

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In these graphs, the natural notion of "backtracking" is given by the dual map on isogenies.

But recall that J needed to be a *fixed-point free involution*.

Issues with backtracking

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$$\widehat{1+i} = 1 - i = (-i)(1+i).$$

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Cryptographic Motivation
Ihara zeta functions
Graphs
Abstract Isogeny Graphs
Orientable graphs
Modular Curves
Asymptotics
References

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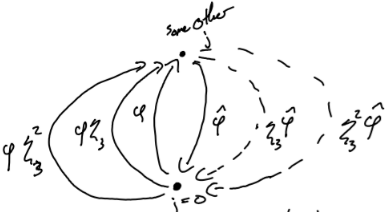

Credit: Orientations and Cycles in Supersingular ℓ-isogeny graphs

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Definition

An *abstract isogeny graph* is the following collection of data:

- A set X of vertices;
- a set Y of edges;
- functions, $s, t : Y \to X \times X$;
- a function $J : Y \to Y$; and
- a function $L : X \to X$,

such that $s(J(e)) = t(e)$ and $t(J(e)) = L(s(e))$ for all $e \in Y$.

The L function

To motivate the function L, we need to dig in to an even worse failure of the dual map in isogeny graphs with level structure. Recall:

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Definition

For $H \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, a *level H-structure* on an elliptic curve E is an isomorphism $\phi : \mathbb{Z}/N\mathbb{Z}^2 \to E[N]$, up to pre-composition by elements of H.

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Isogenies must satisfy:

$$\begin{tikzcd} (\mathbb{Z}/N\mathbb{Z})^2 \arrow[r, "id"] \arrow[d, "\phi"'] & (\mathbb{Z}/N\mathbb{Z})^2 \arrow[d, "\phi'"] \\ E[N] \arrow[r, "\psi"] & E'[N] \end{tikzcd}$$

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E[N] \arrow[r, "\psi"] & E'[N] \arrow[r, "\hat{\psi}"] & E[N]
\end{tikzcd}$$

Thus the target of the dual of $\psi : (E, [\phi]) \to (E', [\phi']])$ is $(E, [\ell\phi])$.

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$$\begin{tikzcd}
s(e) \arrow[r, "e"] \arrow[rr, bend left, "L"] & \begin{array}{c} s(Je) \\ \shortparallel \\ t(e) \end{array} \arrow[r, "Je"] & \begin{array}{c} t(Je) \\ \shortparallel \\ Ls(e) \end{array}
\end{tikzcd}$$

Motivating abstract isogeny graphs

Theorem

Choosing appropriate representatives for edges in order to define J as the dual map, we can realize $G(p, \ell, H)$ as an abstract isogeny graph for any H. The same is true for $(\ell, \ldots, \ell)$-isogeny graphs.

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Technical remark: The choice of representatives is designed to deal with the fact that the dual map is not well-defined on edges - this was already noted in [2], and our solution is equivalent to theirs.

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where the primes are non-backtracking with respect to the J function.

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where the primes are non-backtracking with respect to the J function.

Another teaser: note that primes in an abstract isogeny graph are exactly the *isogeny cycles* studied previously by [2] and [3].

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1. by combinatorial data,
2. by relation to zeta functions of modular curves.

Ihara zeta function - combinatorial formula

For a function $f : S \to S$ acting on a finite set S, we define $C_k(f)$ to be the number of k-cycles in the largest permutation induced by f.

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Theorem

Let $\Gamma = (X, Y, J, L)$ be an abstract isogeny graph with regular out degree d and adjacency matrix A. Then $\zeta_\Gamma(u)$ is given by:

$$\frac{(1-u^2)^{C_1(L)}(1+u)^{-C_1(J)} \prod_{k>1}(1-(-1)^k u^{2k})^{C_k(L)}(1-u^k)^{-C_k(J)}}{\det(1 - Au + u^2(d-1)L)}$$

Orientable graphs associated to abstract isogeny graphs

For both simplification of the previous formula, and the statement of the next, we use the *orientable graphs* associated to an abstract isogeny graph Γ.

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Definition

Let $\Gamma = (X, Y, J, L)$ be an abstract isogeny graph. We define $\sim_X$ to be the smallest equivalence relation on X such that $x \sim_X Lx$ for all $x \in X$, and $\sim_Y$ to be the smallest equivalence relation on Y such that $y \sim_Y J^2 y$ for all $y \in Y$. The *orientable graphs associated to* Γ are
$\Gamma^{+1} = (X/\sim_X, Y/\sim_Y - \{[y] : J[y] = [y]\})$ and
$\Gamma^{-1} = (X/\sim_X, Y/\sim_Y \sqcup \{[y] : J[y] = [y]\})$

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Theorem

Let Γ be a d-regular abstract isogeny graph for $d \geq 1$, and suppose that the permutation induced by J is an involution and $s(J^2 y) = s(y)$ for every edge $y \in Y$. Then we have

$$\zeta_\Gamma(u) = \frac{(1+u)^{\chi(\Gamma^{-1})}(1-u)^{\chi(\Gamma^{+1})}}{\det(\mathrm{id}_{\mathbb{C}_X} - uA + u^2 Q)}.$$

Hasse-Weil Zeta functions

Our next goal is to relate Ihara zeta functions of abstract isogeny graphs to Hasse Weil zeta functions of modular curves. This will allow us to understand asymptotics of cycles in graphs with level structure.

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Theorem

Let $G = G(p, \ell, H)$. Denote by $X_{H,\mathbb{F}_\ell}$, and $X_{H_p,\mathbb{F}_\ell}$ the associated modular curves over $\mathbb{F}_\ell$. Then

$$\frac{Z(X_{H_p,\mathbb{F}_\ell}, u)}{Z(X_{H,\mathbb{F}_\ell}, u)^2}\zeta_G(u) = \frac{(1-u^2)^{C_1(L)}}{(1+u)^{C_1(J)}} \prod_{k>1} \frac{(1-(-1)^k u^{2k})^{C_k(L)}}{(1-u^k)^{C_k(J)}}.$$

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Corollary

Let $G = (p, \ell, B_0(N))$. Let $X_0(pN)_{\mathbb{F}_\ell}$ and $X_0(N)_{\mathbb{F}_\ell}$ denote the modular curves over $\mathbb{F}_\ell$. Then we have that

$$\frac{Z(X_0(pN)_{\mathbb{F}_\ell}, u)}{Z(X_0(N)_{\mathbb{F}_\ell}, u)^2}\zeta_G(u) = (1+u)^{\chi(G^{-1})}(1-u)^{\chi(G^{+1})}.$$

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Asymptotics for graphs with level structure in arbitrary characteristic

Finally, we use this product to deduce asymptotics for the number of cycles of length r as $r \to \infty$, for arbitrary p, and in the presence of level structure.

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Theorem

Let G be the ℓ-isogeny graph with Borel level structure, and N_r be the number of non-backtracking tailless cycles of length r in G. Then we have that

$$N_r = 2\#X_0(N)(\mathbb{F}_{\ell^r}) - \#X_0(pN)(\mathbb{F}_{\ell^r}) - \chi(G^{+1}) + (-1)^{r-1}\chi(G^{-1}).$$

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Theorem

Let G be the ℓ-isogeny graph with N-level structure for an arbitrary prime p. Let N_r be the number of non-backtracking cycles of length r in G. Then N_r asymptotically approaches ℓ^r as $r \to \infty$.

Point counts on modular curves

As a final application, we can "flip the script," and use isogeny graphs to study modular curves.

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Theorem

Let p, ℓ be distinct primes and $r > 2$ such that $\ell^r < p$. Let $G := G(p, \ell)$. Then we have that

$$\begin{aligned}\#X_0(p)(\mathbb{F}_{\ell^r}) = & 2(1+\ell^r) - \chi(G^{+1}) + (-1)^{r-1}\chi(G^{-1}) \\ & - 2\sum_{n|r}\sum_{\mathcal{O}\in\mathcal{I}_n} h(\mathcal{O}),\end{aligned}$$

where $\mathcal{I}_n$ is an explicit set of imaginary quadratic orders.

The end :)

Thanks for listening!

References I

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