Attacking the General Isogeny Problem

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University College London

Based on joint work with Craig Costello, Sam Frengley and Jia Shi

The Isogeny Club
October 11, 2022

Outline

- The General Isogeny Problem in Dimensions 1 and 2

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- Results

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Definition

Let $E : y^2 = x^3 + ax + b$. Then, the j-invariant of E is

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}.$$

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We consider N-isogenies between them: isogenies with kernel $\mathbb{Z}/N\mathbb{Z}$ generated by a point $P \in E(\bar{\mathbb{F}}_p)$.

The General Isogeny Problem in One Dimension

In its most general form, the *supersingular isogeny problem* asks to find an isogeny

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between two given supersingular elliptic curves $E_1/\mathbb{F}_{p^2}$ and $E_2/\mathbb{F}_{p^2}$.

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Note, we do not assume knowledge of:

- torsion point information
- degree of the isogeny

Generalising to Two Dimensions

To generalise supersingular elliptic curves over $\bar{\mathbb{F}}_p$ ($p > 3$) to dimension 2, we consider *superspecial (principally polarised) abelian surfaces* over $\bar{\mathbb{F}}_p$.

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Every superspecial p.p. abelian surface admits a model over $\mathbb{F}_{p^2}$ and its invariants lie in $\mathbb{F}_{p^2}$.

(N, N)-Isogenies

An (N, N)-isogeny is an isogeny[1] $\phi\colon A \longrightarrow A'$, between p.p. abelian surfaces A, A' where:

- $\ker \phi \cong (\mathbb{Z}/N\mathbb{Z})^2$; and
- $\ker \phi$ is maximal isotropic with respect to the N-Weil Pairing, i.e.

$$\forall P, Q \in \ker \phi : \ e_N(P, Q) = 1.$$

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There are

$$N^3 \prod_{\substack{\text{primes} \\ \ell | N}} \frac{1}{\ell^3}(\ell + 1)(\ell^2 + 1)$$

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Again, we do not assume any extra information.

Reframing as a Path Finding Problem

The general isogeny problem in both dimensions can be viewed as finding a path between two nodes in a related graph.

Reframing as a Path Finding Problem

The N-Isogeny Graph $\Gamma_1(N; p)$ in Dimension One

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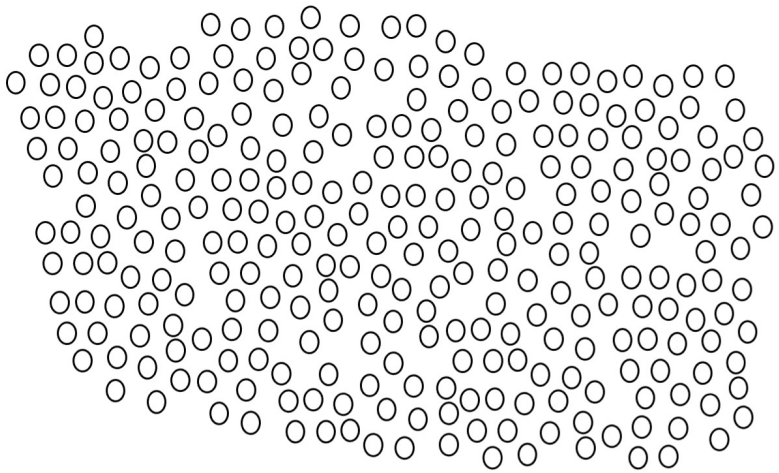

Vertices: $\bar{\mathbb{F}}_p$-isomorphism classes of supersingular E, represented by the j-invariant
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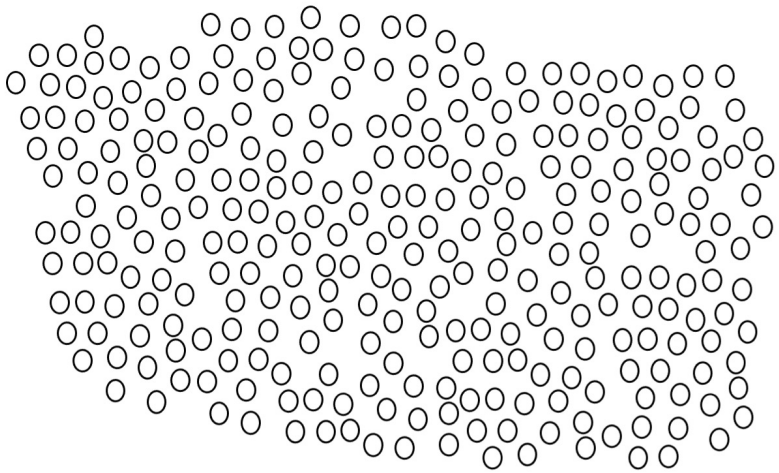

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Properties:

- There are $\approx \frac{p}{12}$ vertices
- A random walk of $\log(p)$ steps is almost as good as uniformly sampling vertices (Ramanujan property)
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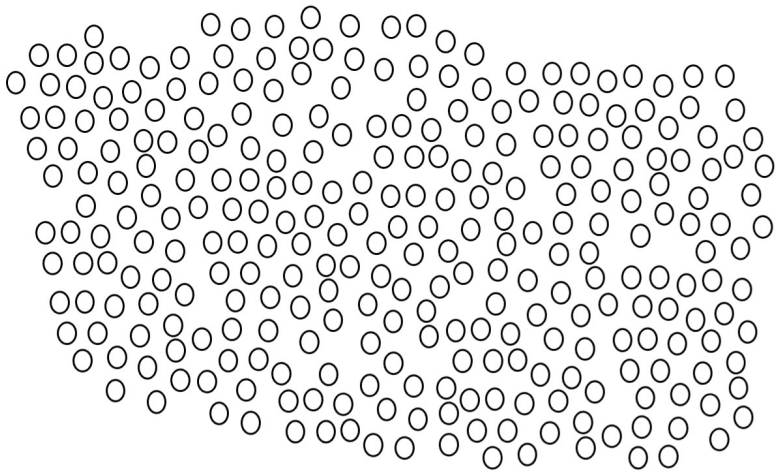

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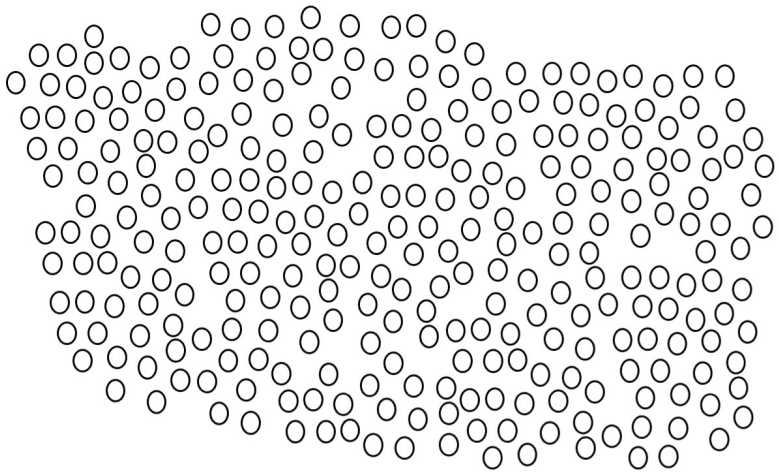

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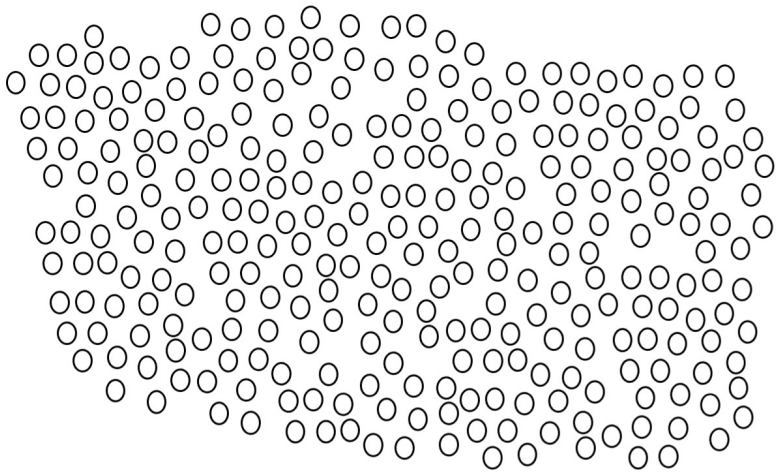

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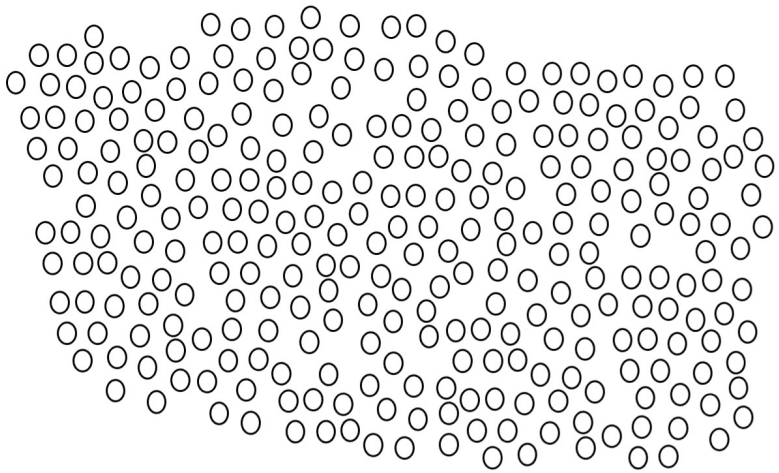

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Attacking Path Finding Problems

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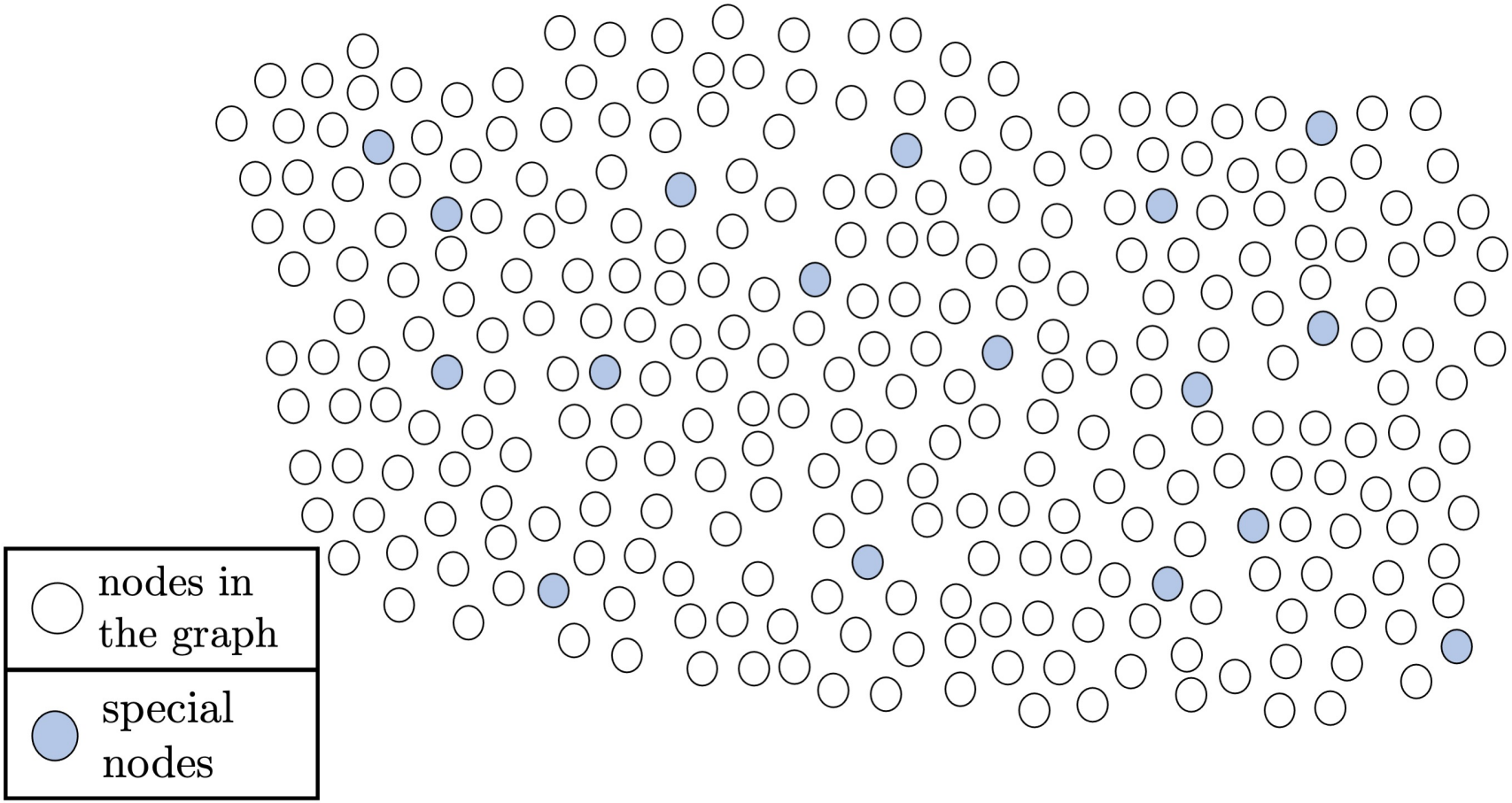
Can we find a better method (excluding the MitM approach)?

Reducing to Easier Problems using Special Subsets



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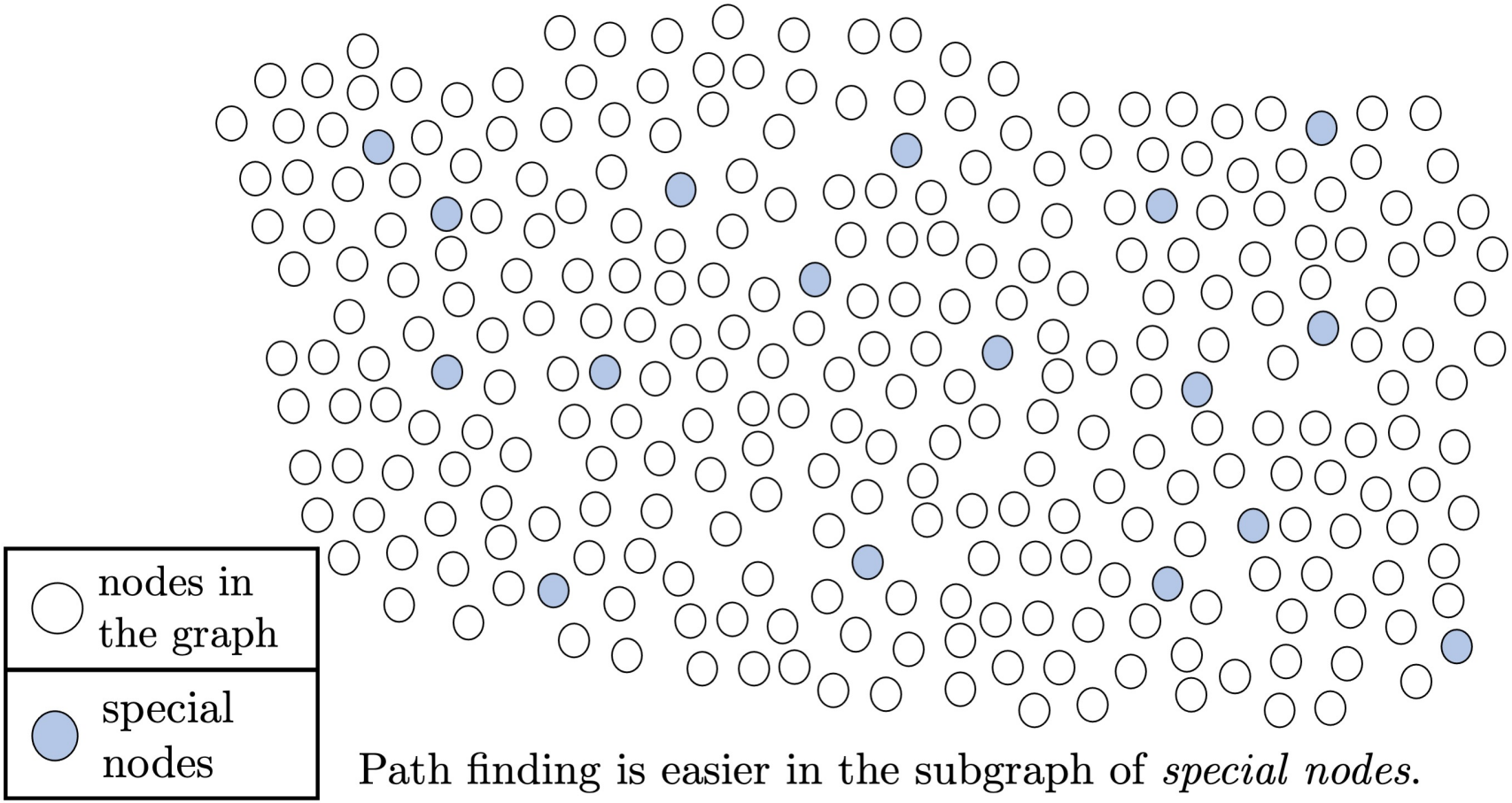

Path finding is easier in the subgraph of *special nodes*.

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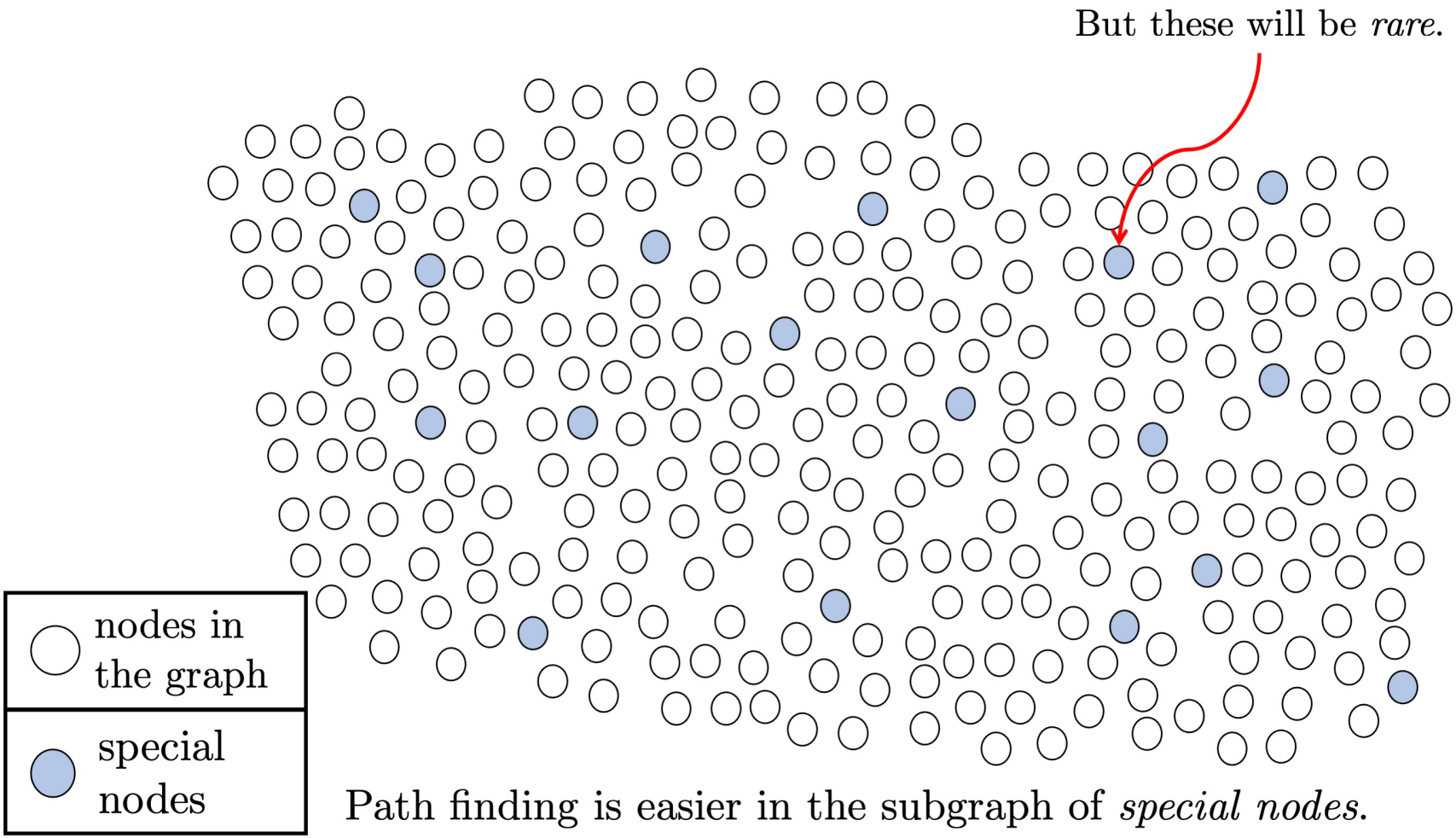

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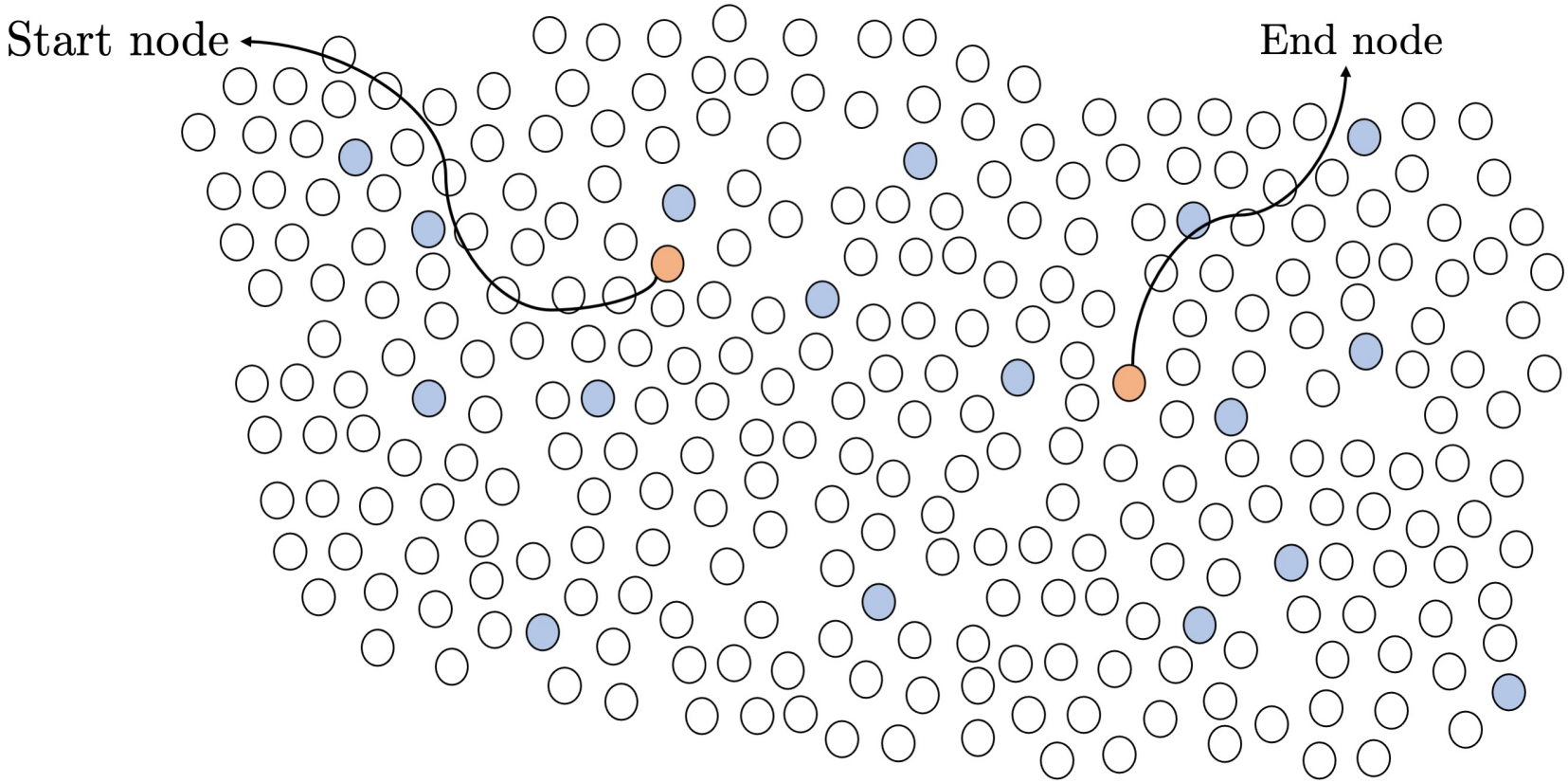

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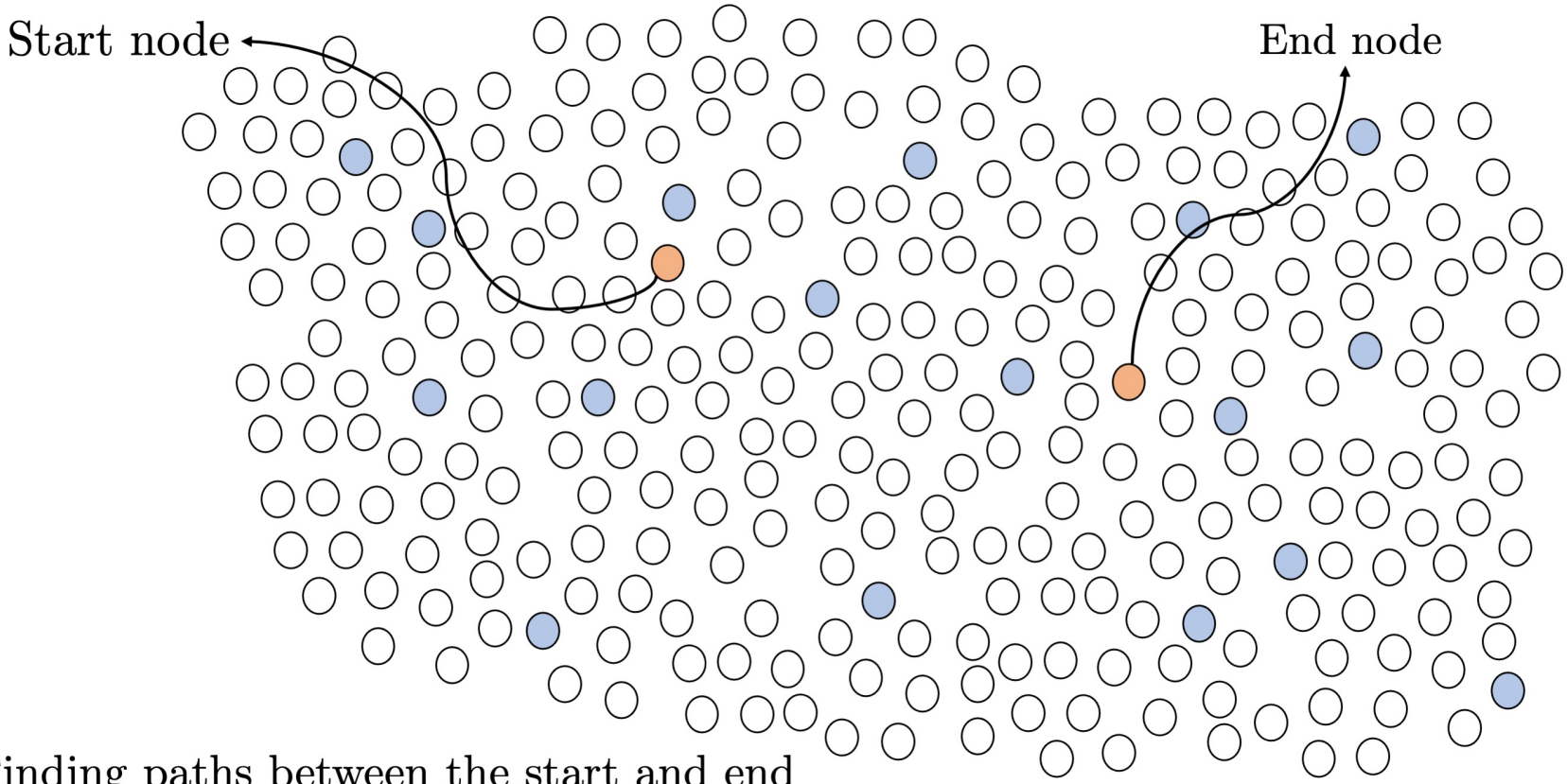

Finding paths between the start and end node can be split up into two steps.

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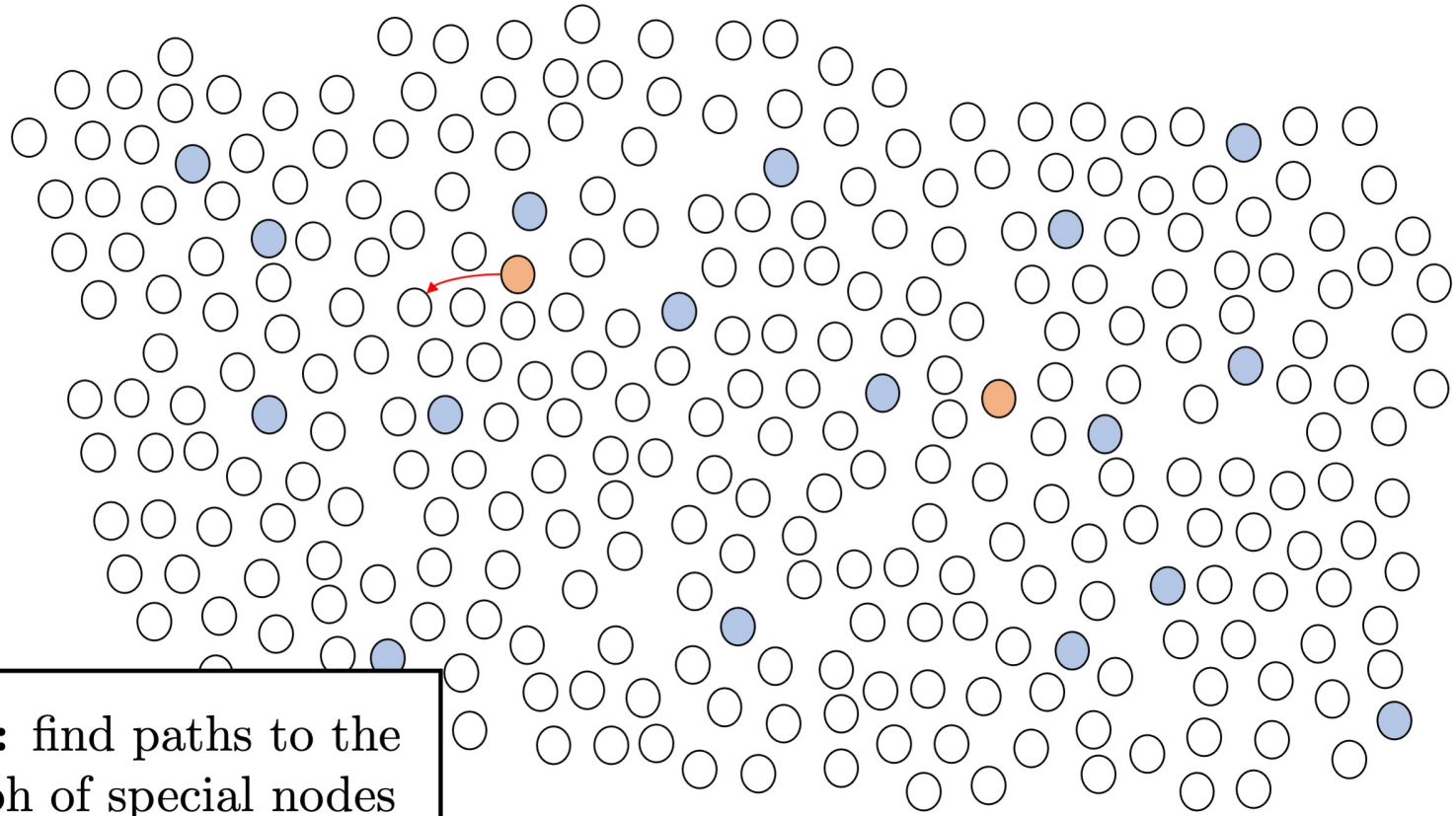

Step 1: find paths to the subgraph of special nodes

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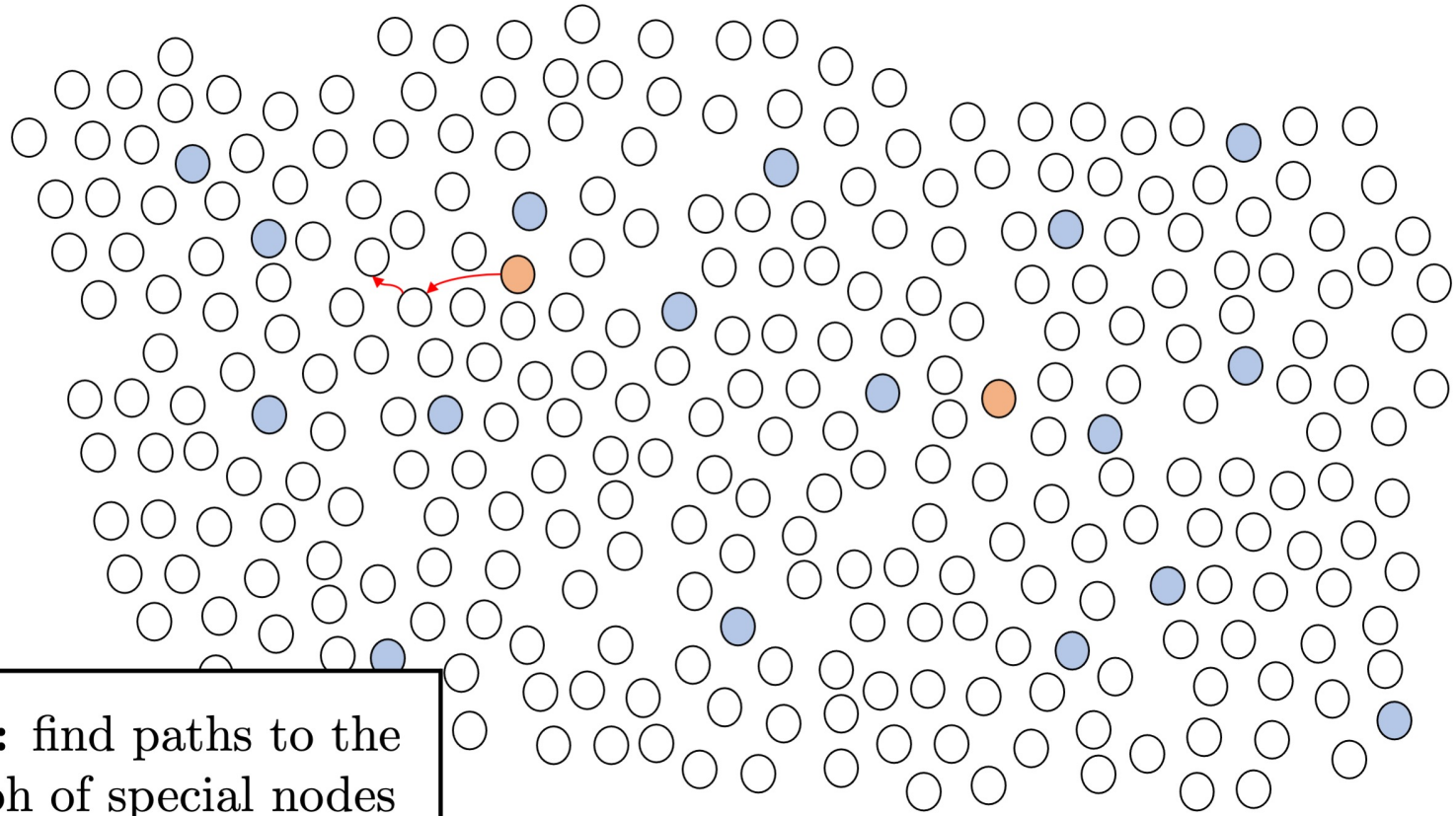

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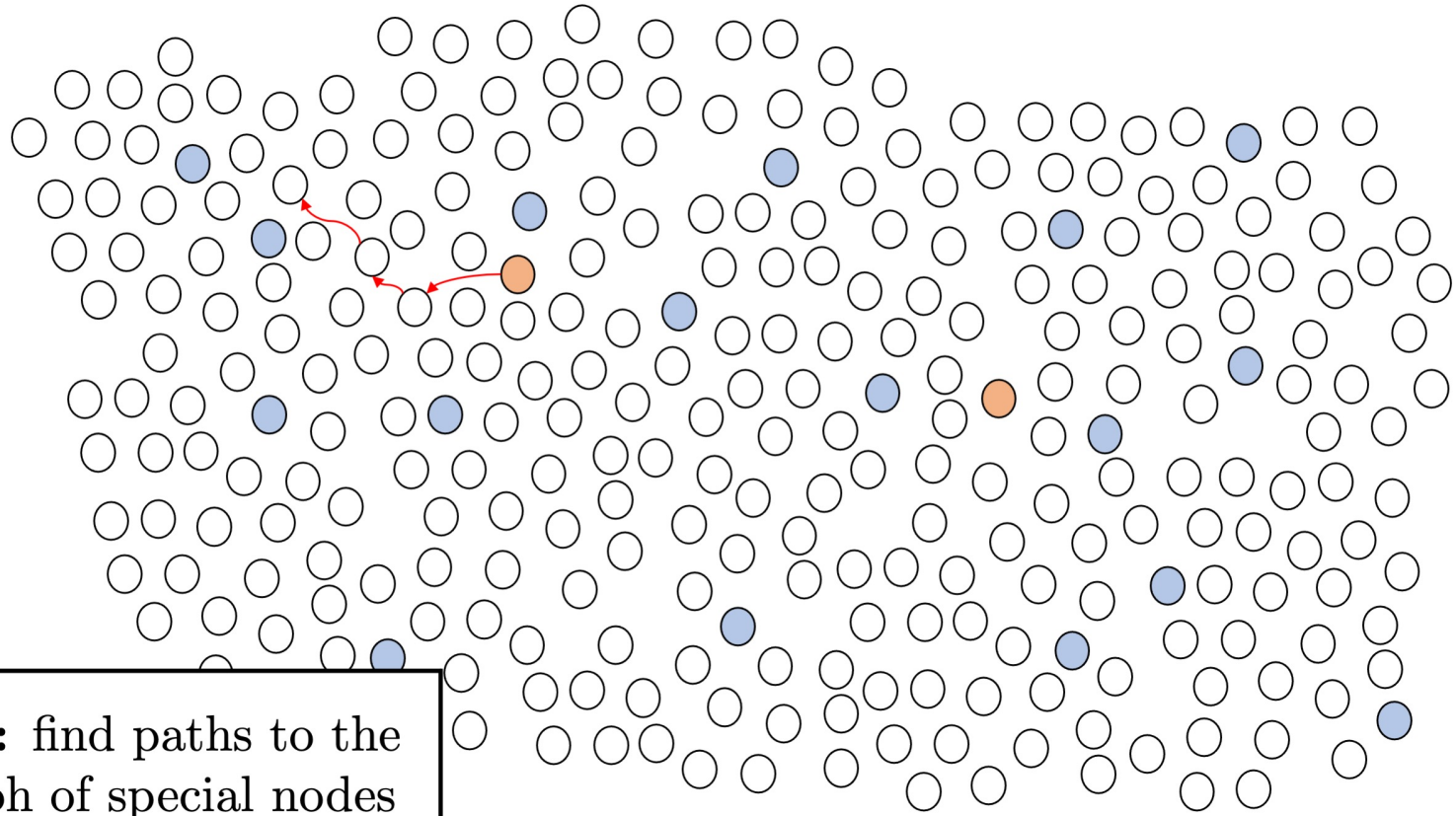

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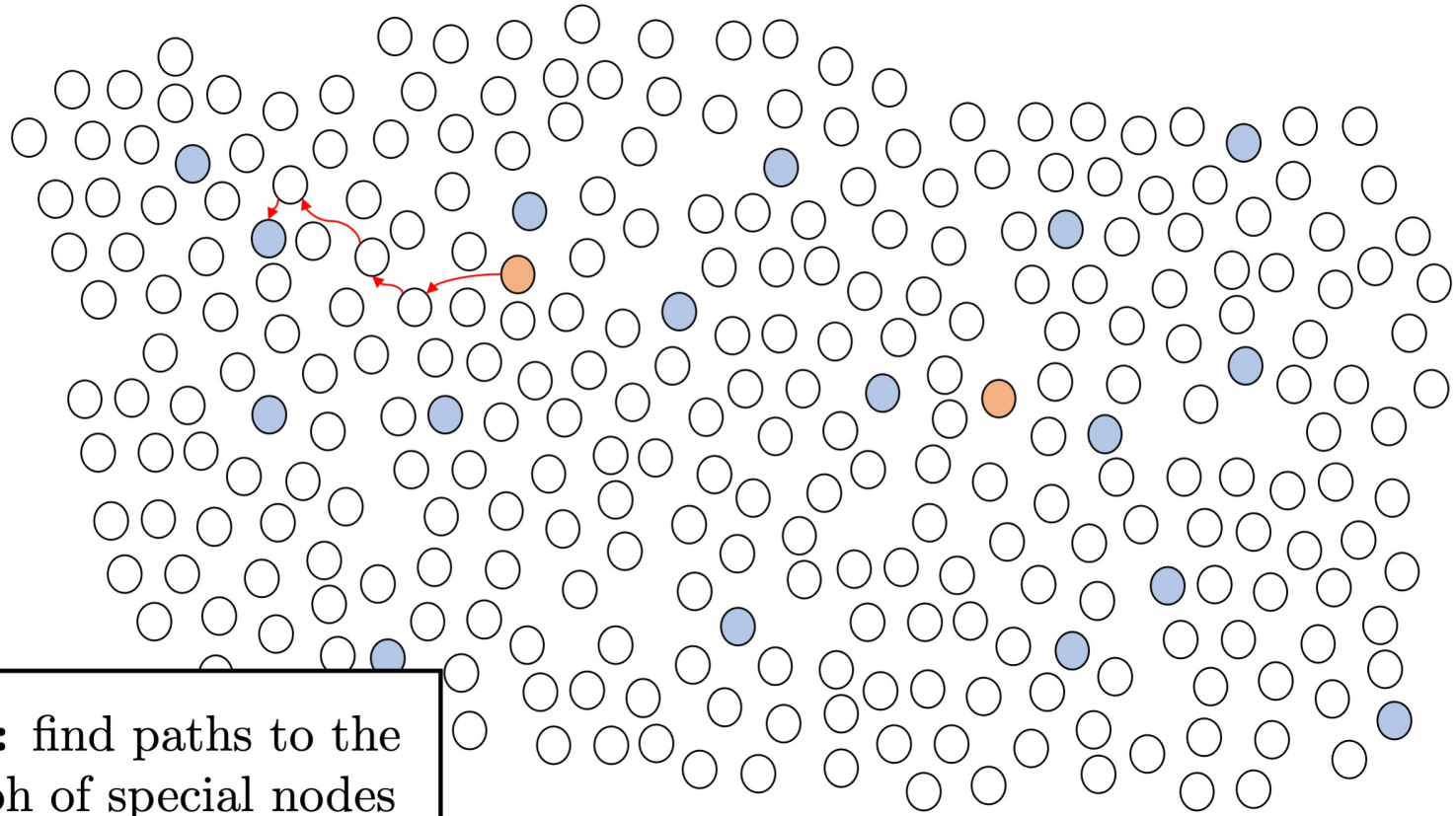

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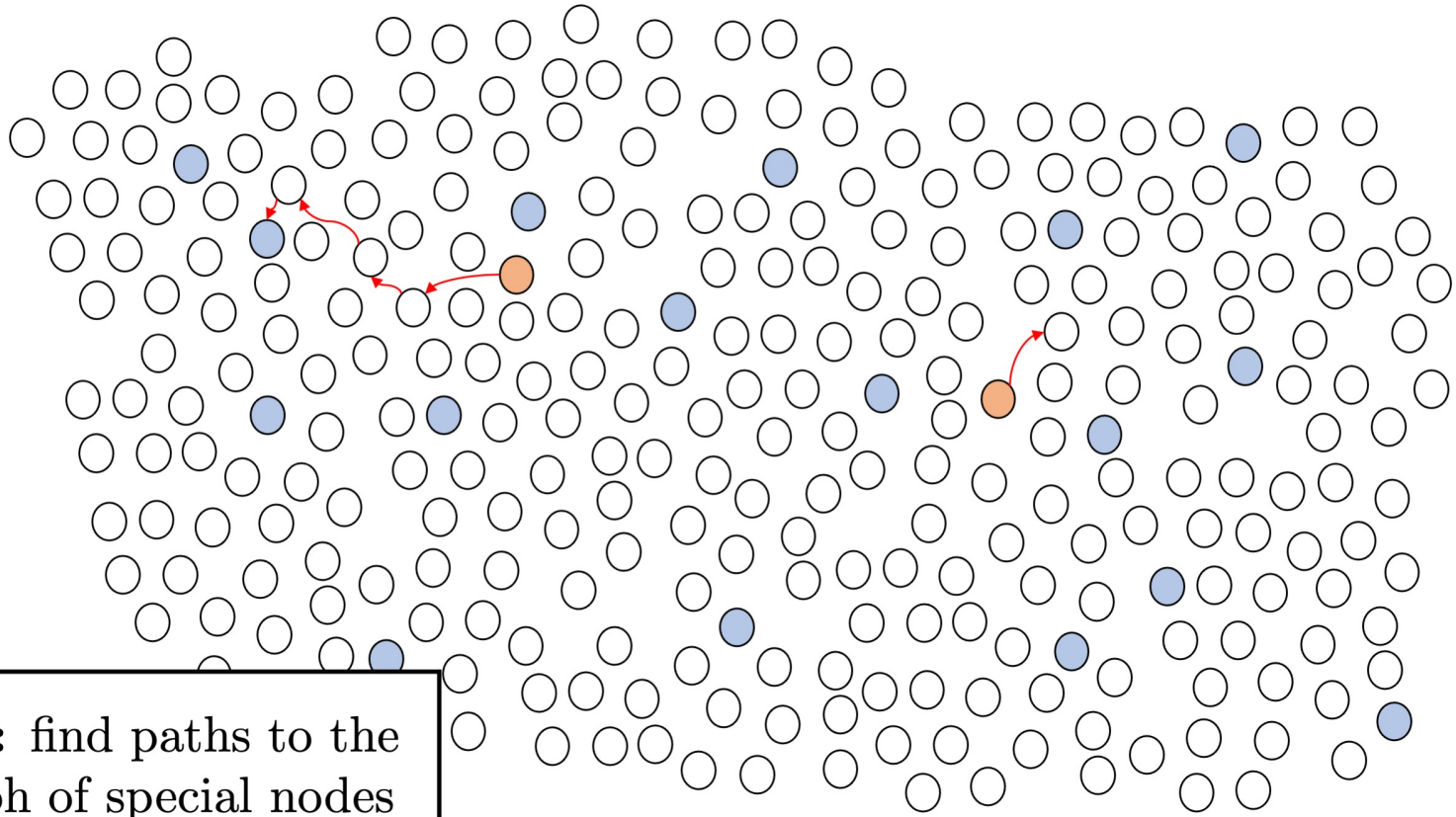

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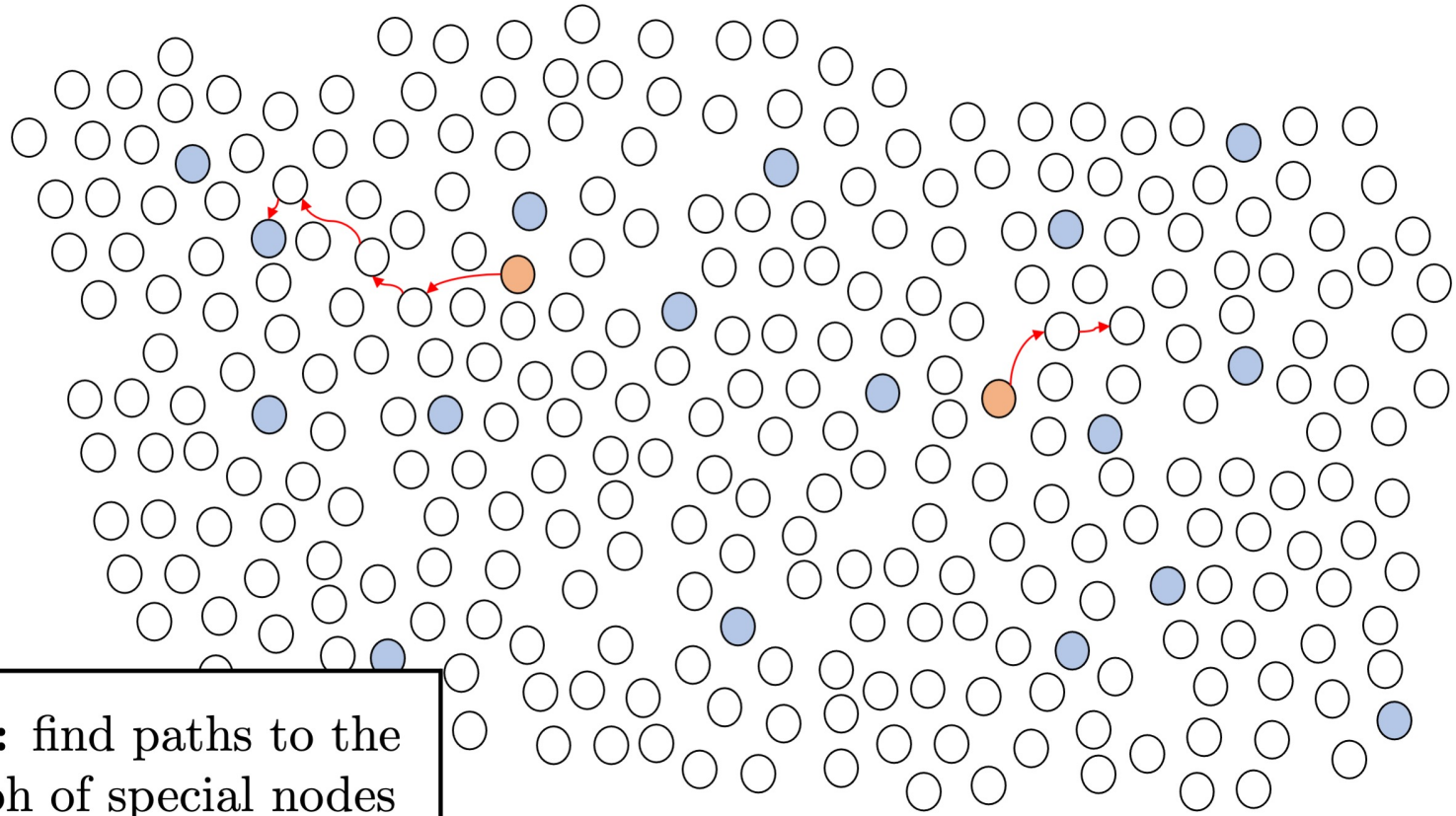

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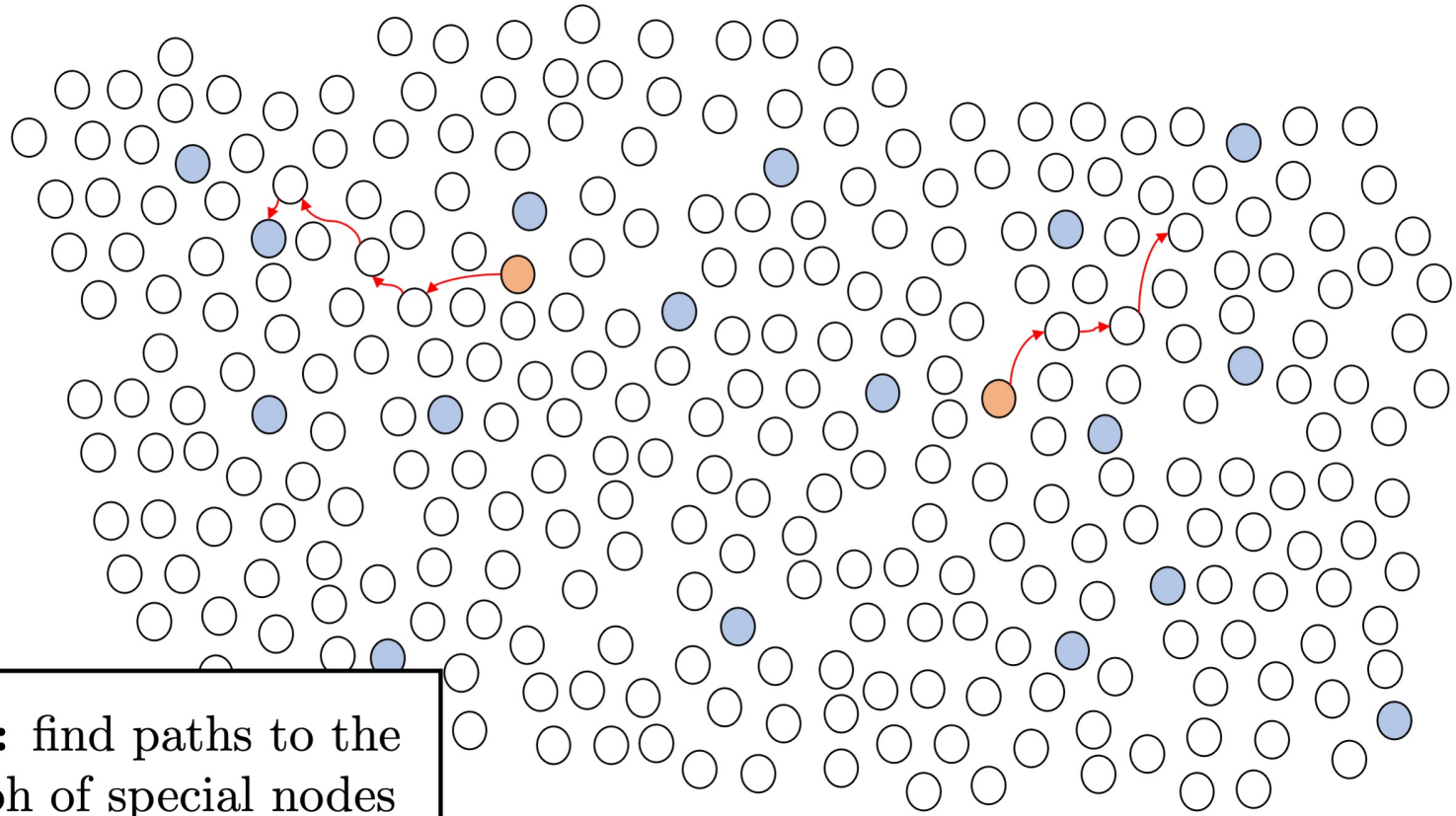

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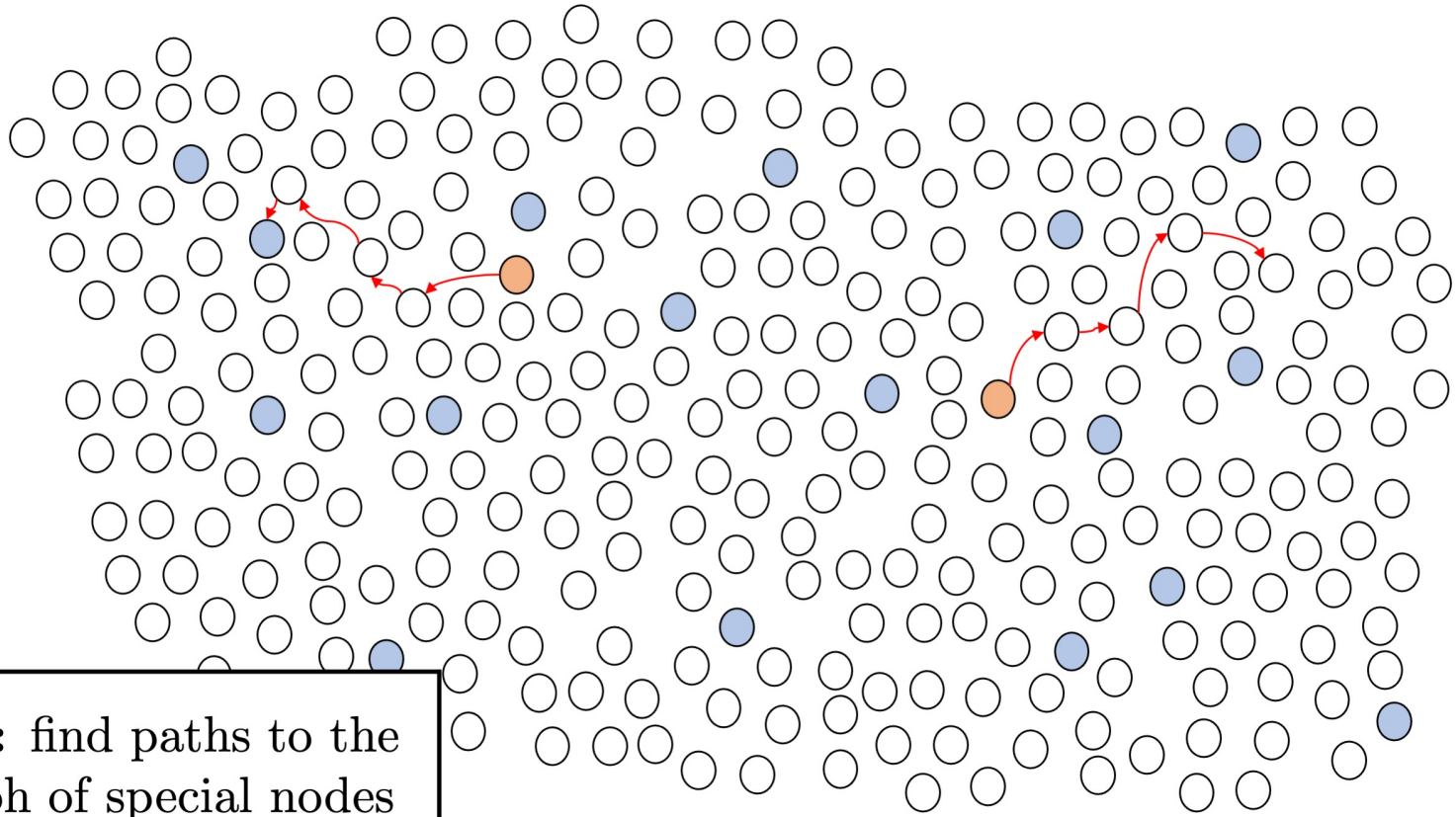

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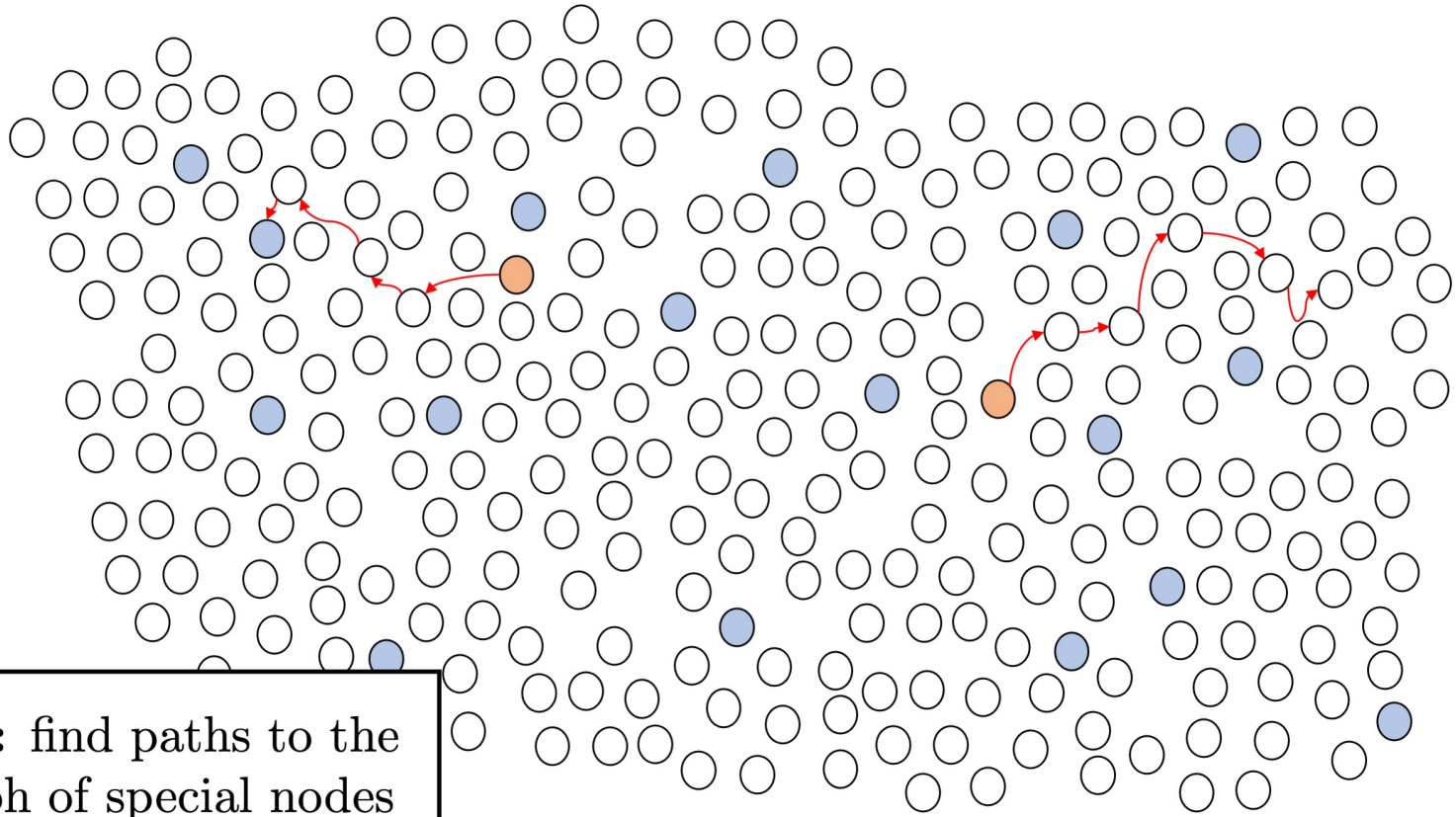

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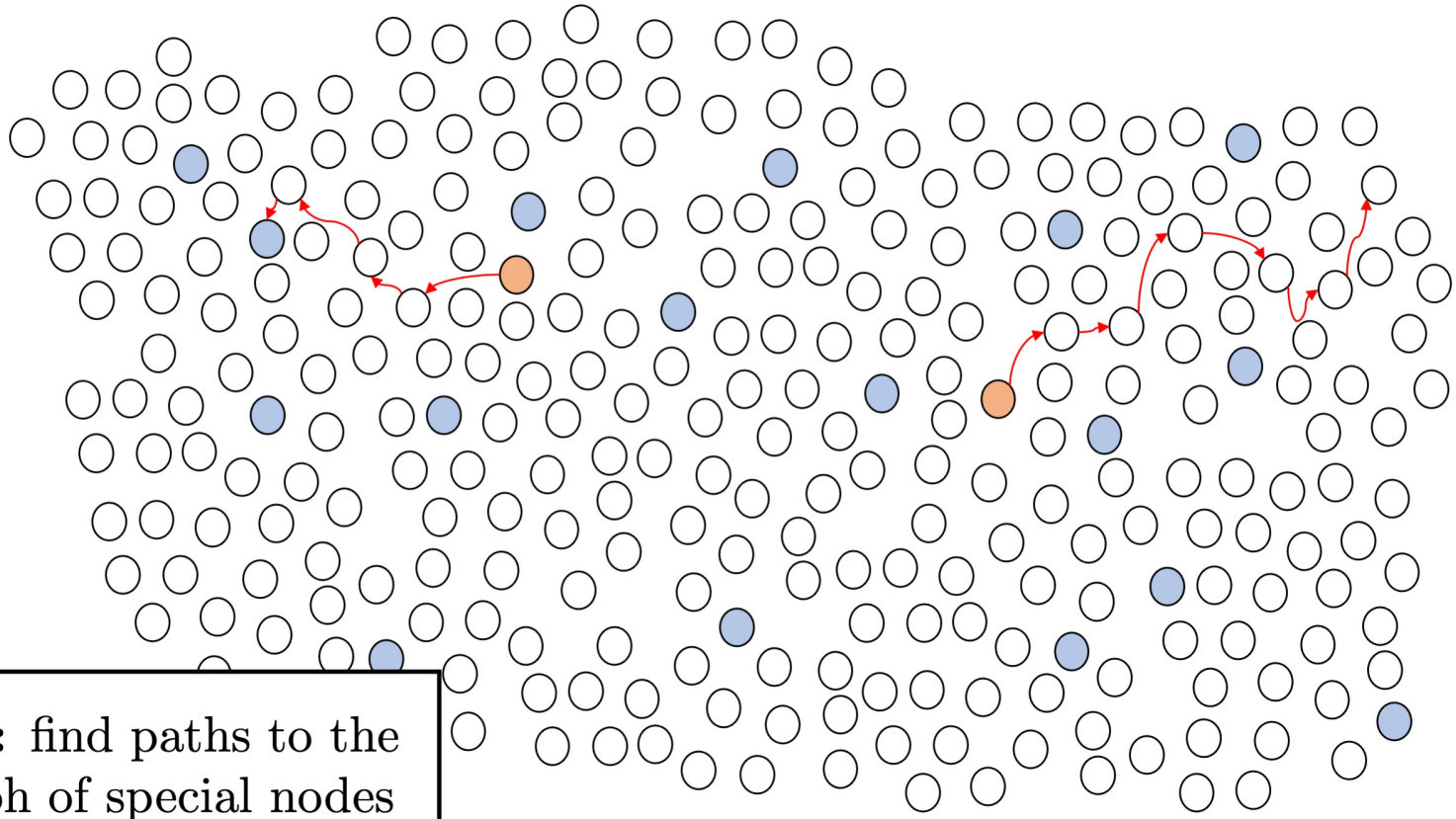

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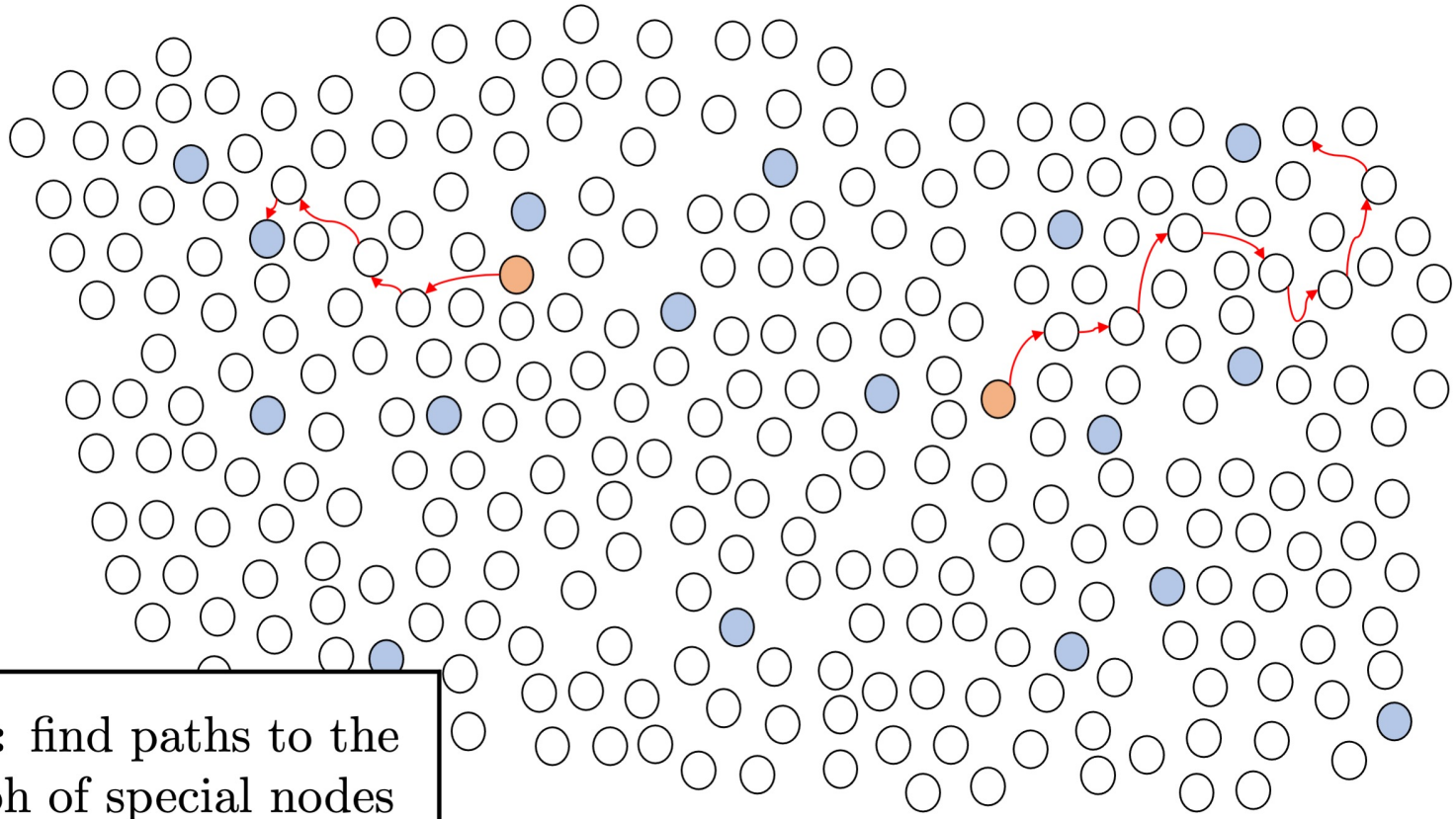

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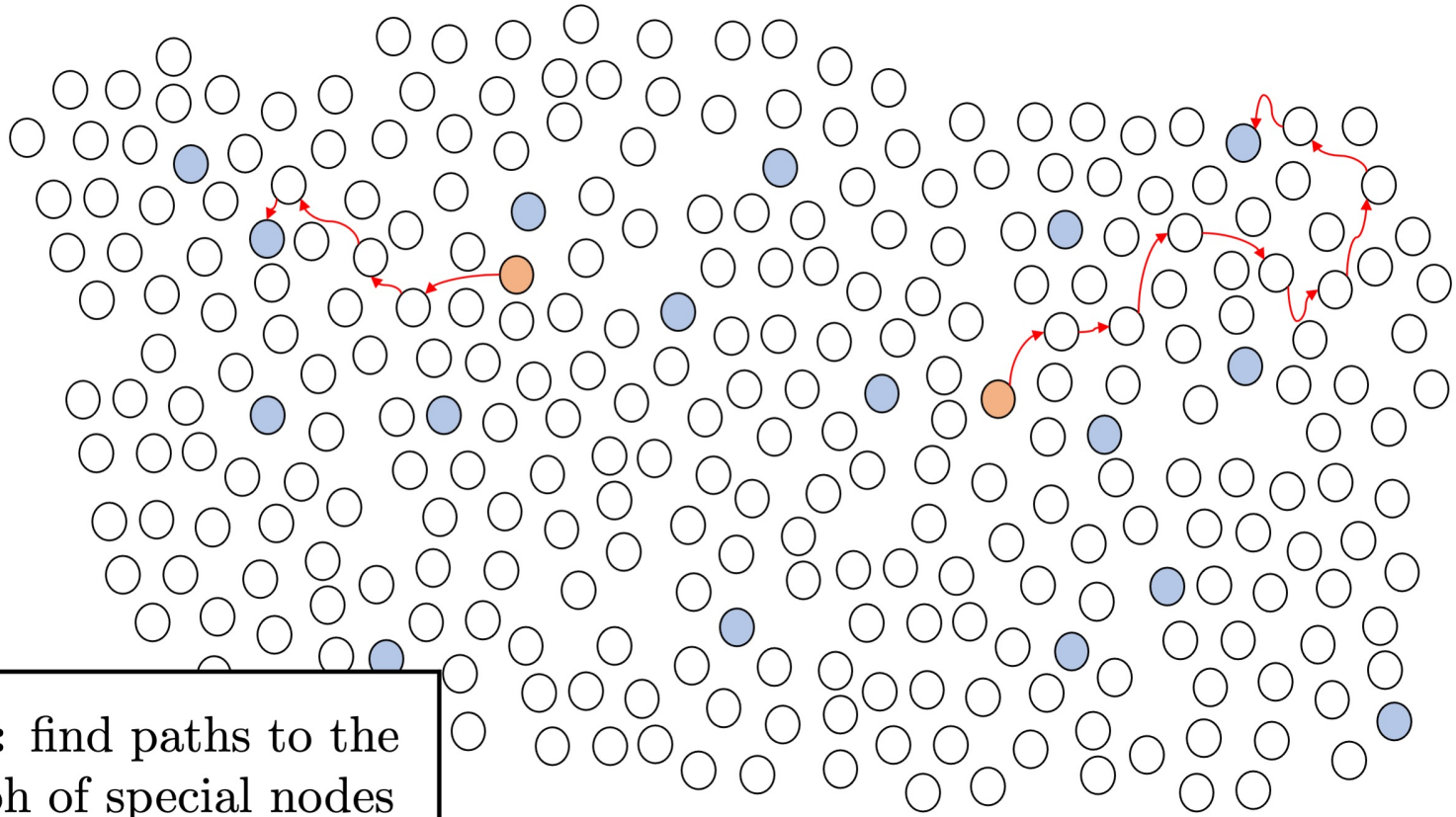

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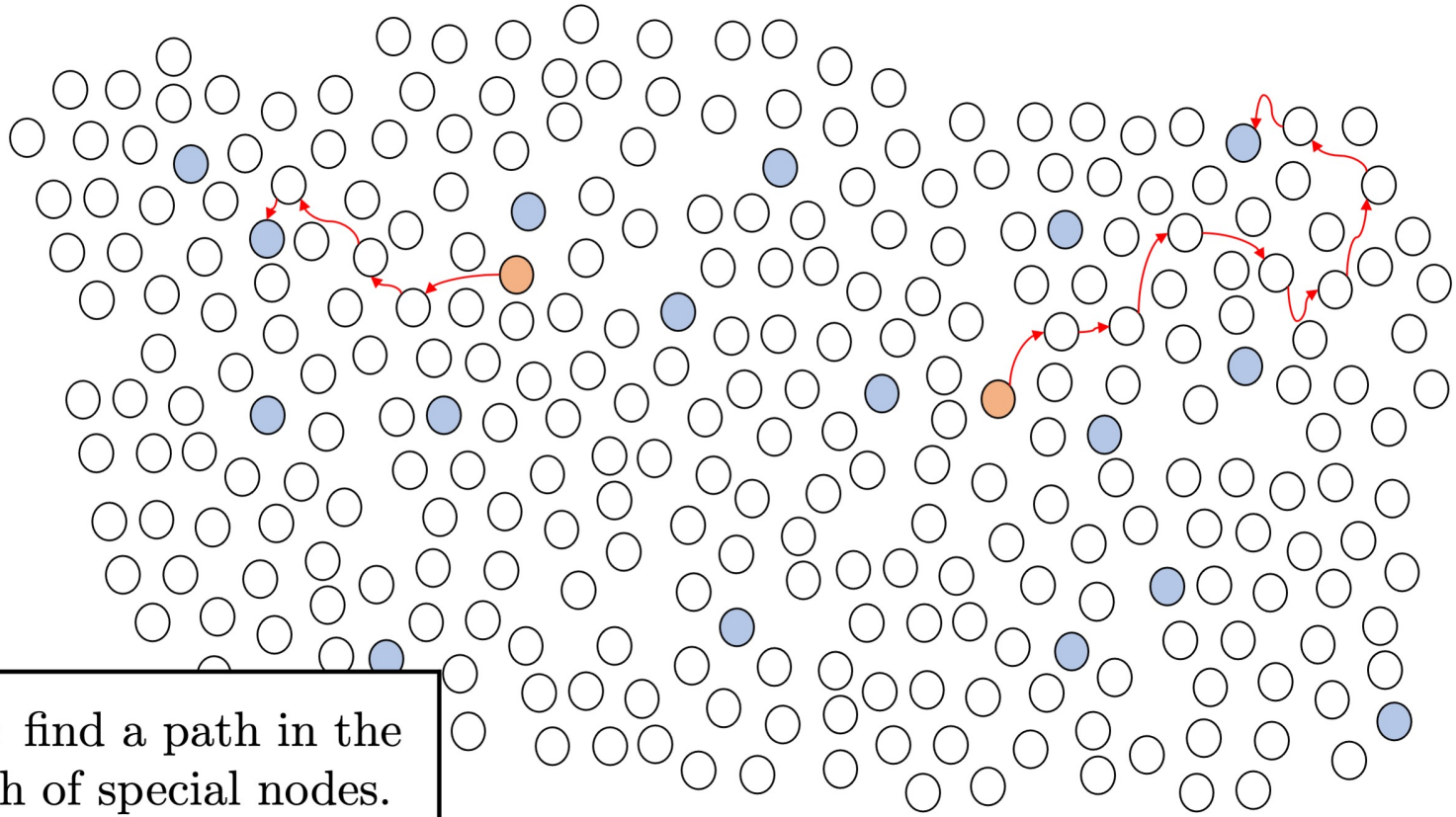

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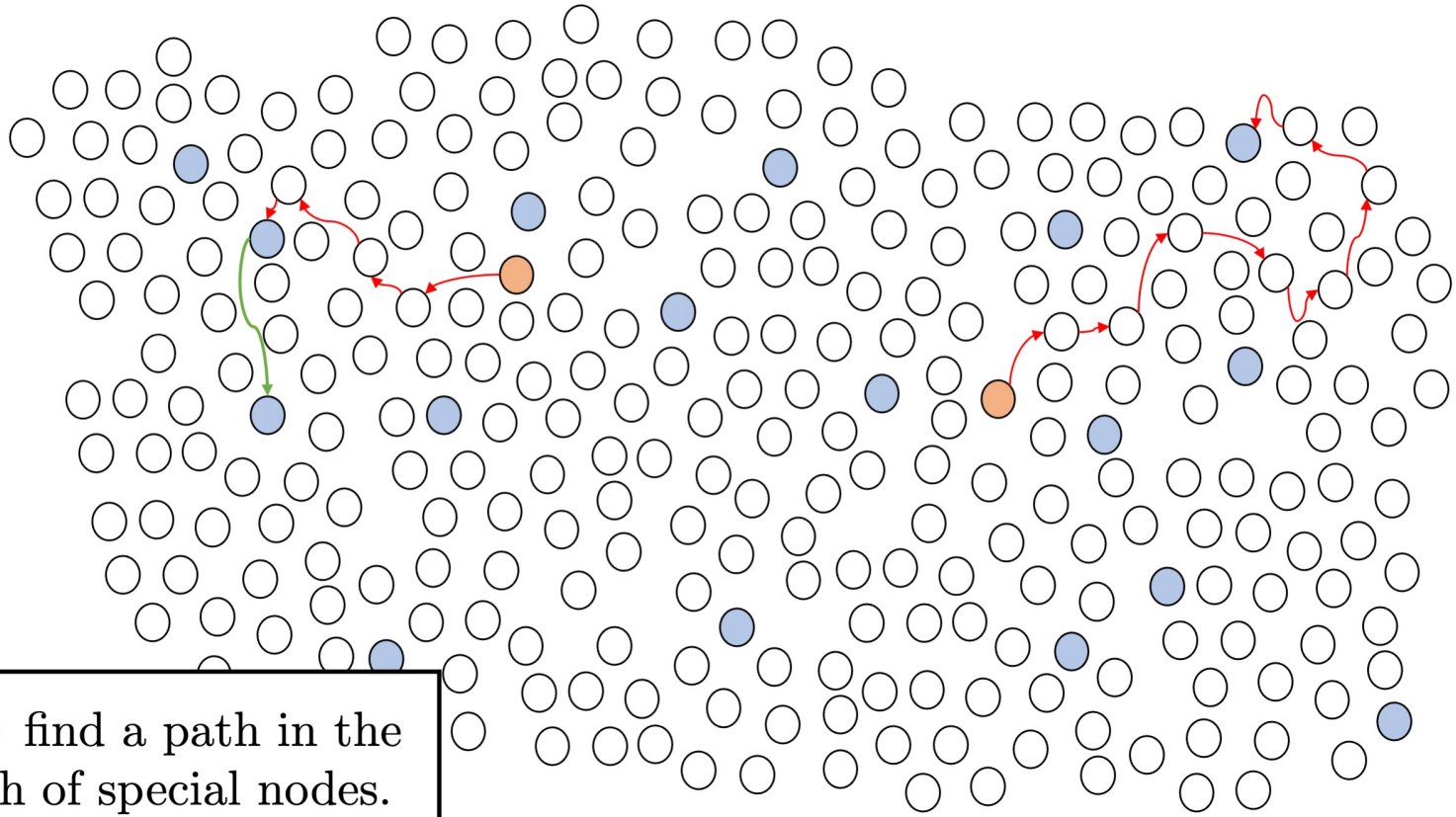

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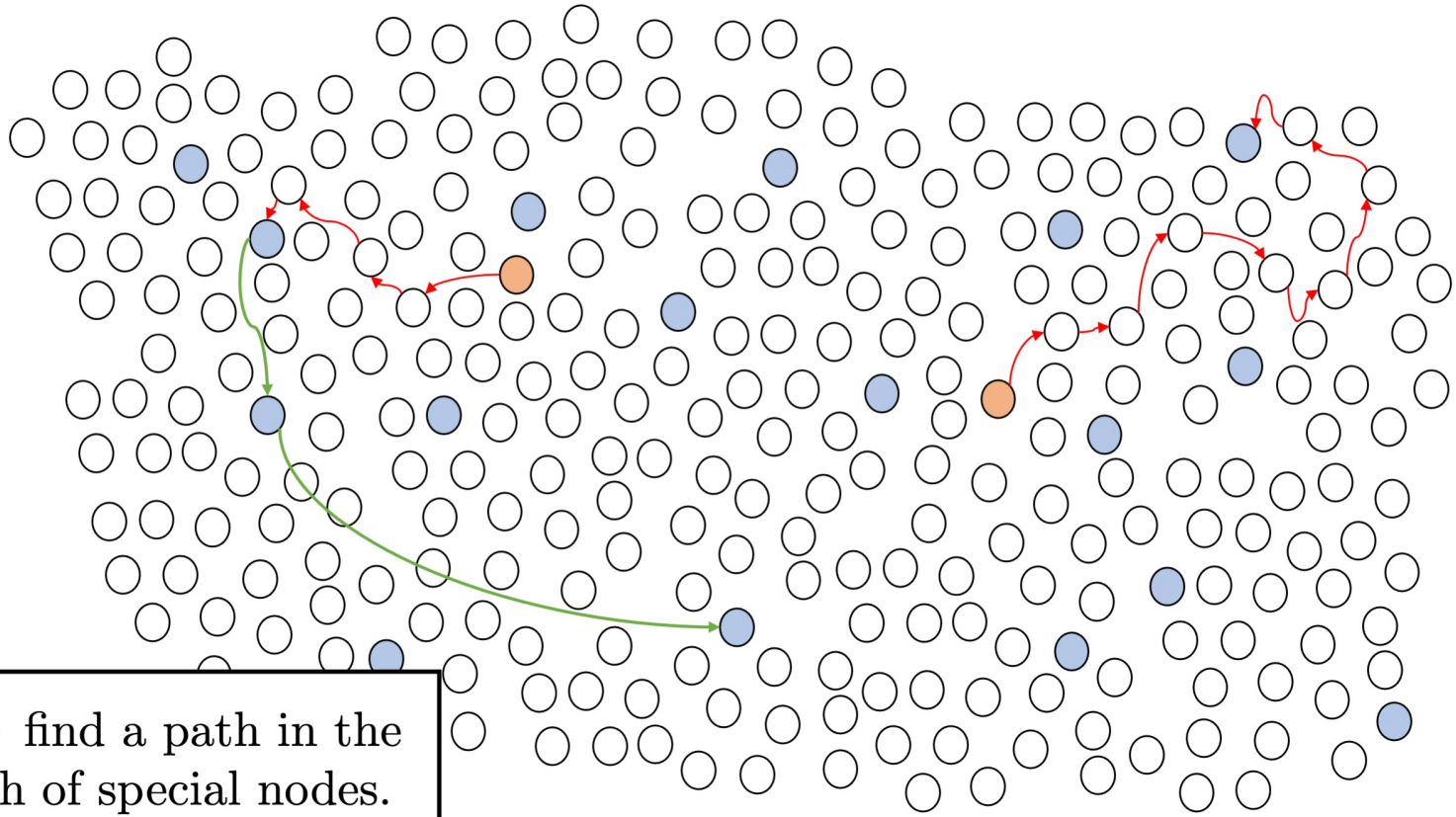

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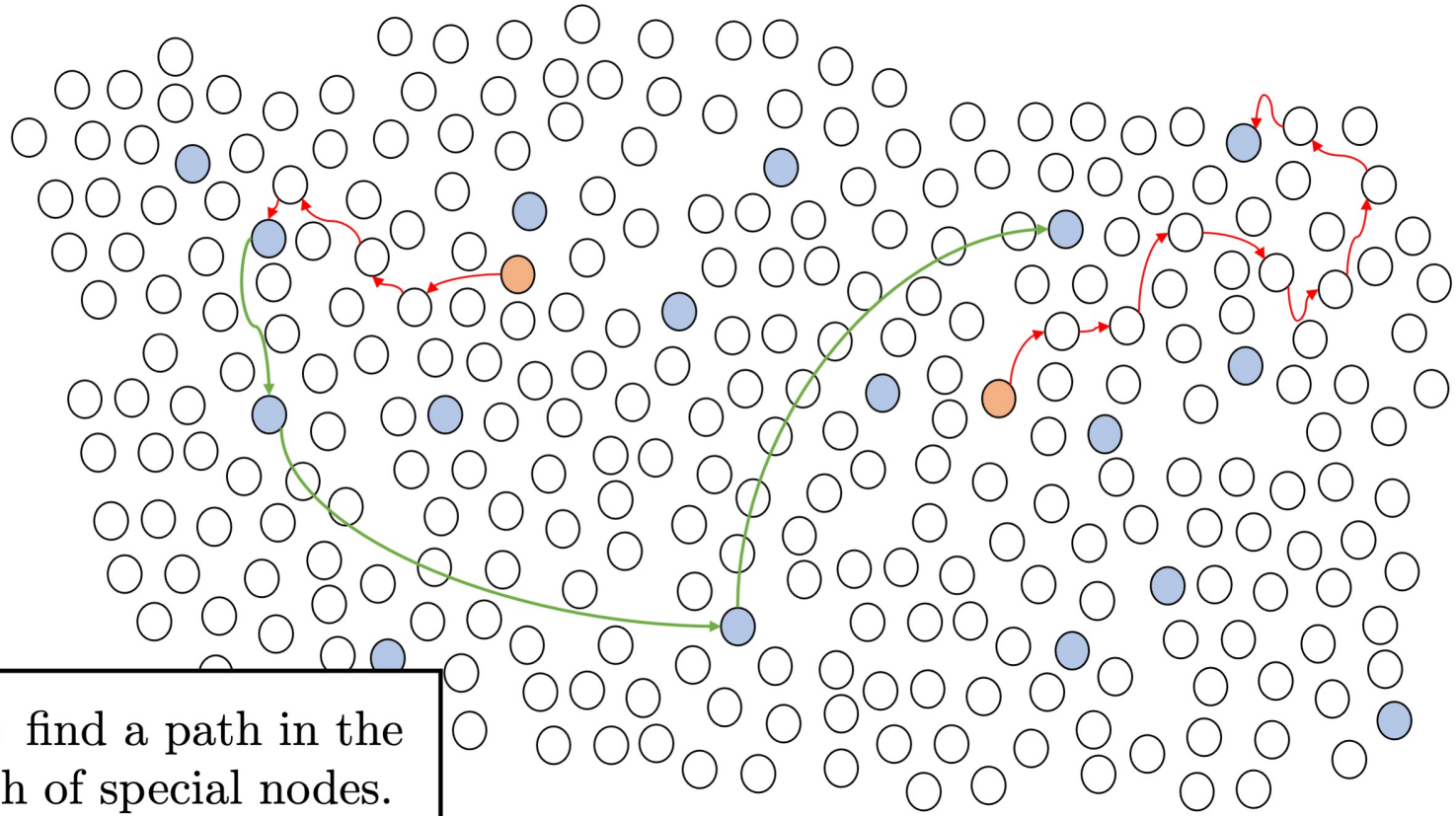

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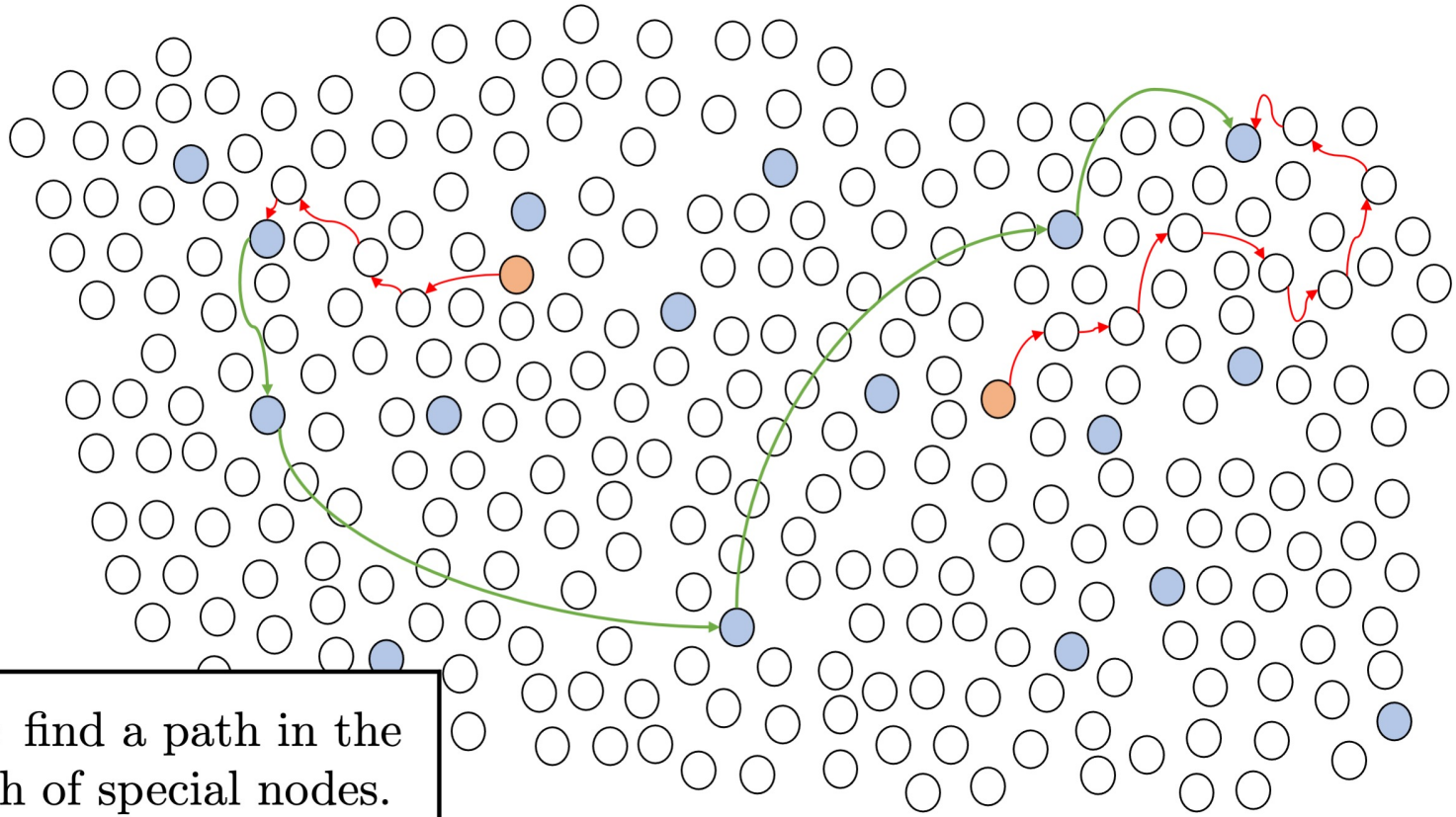

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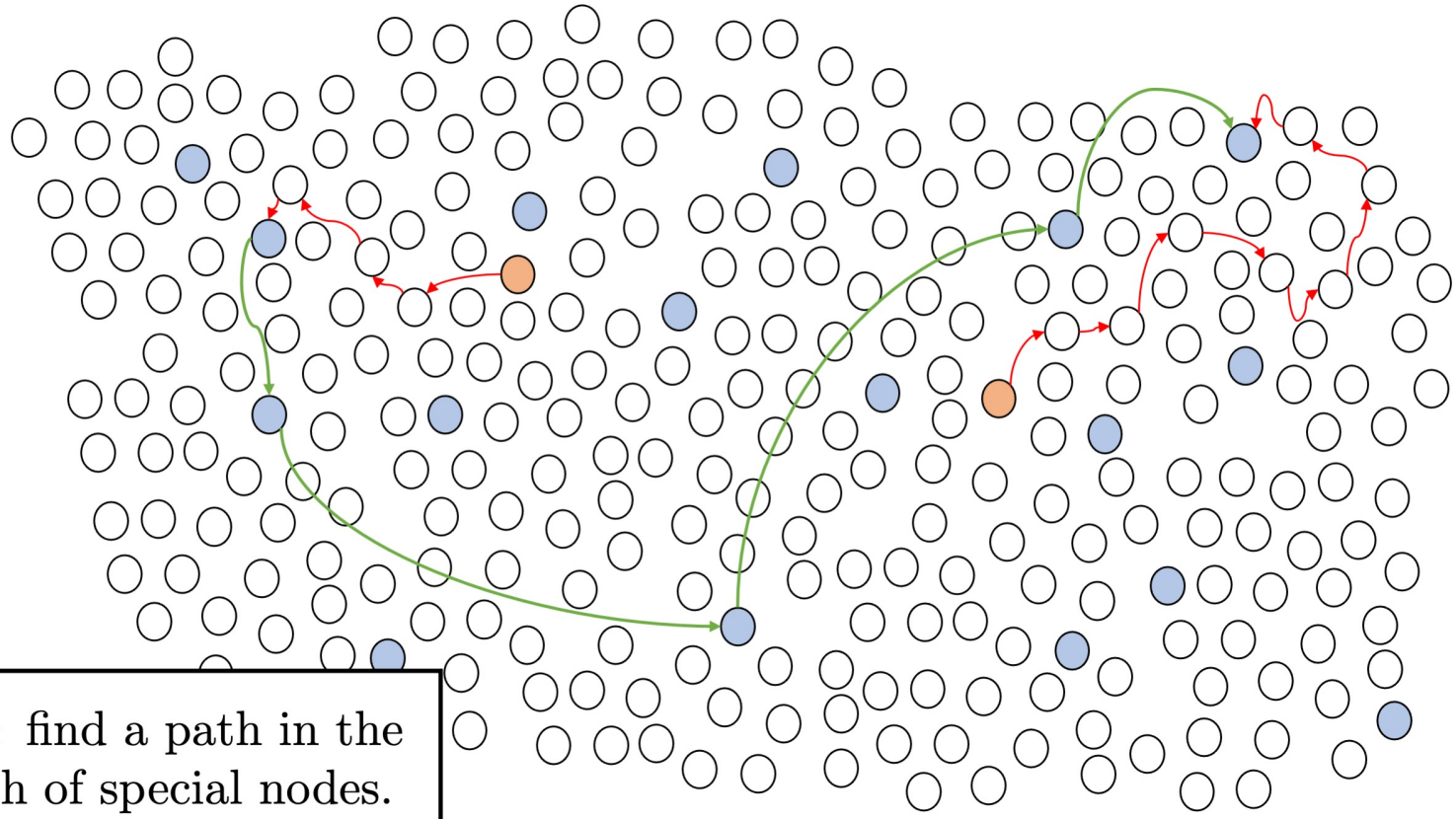
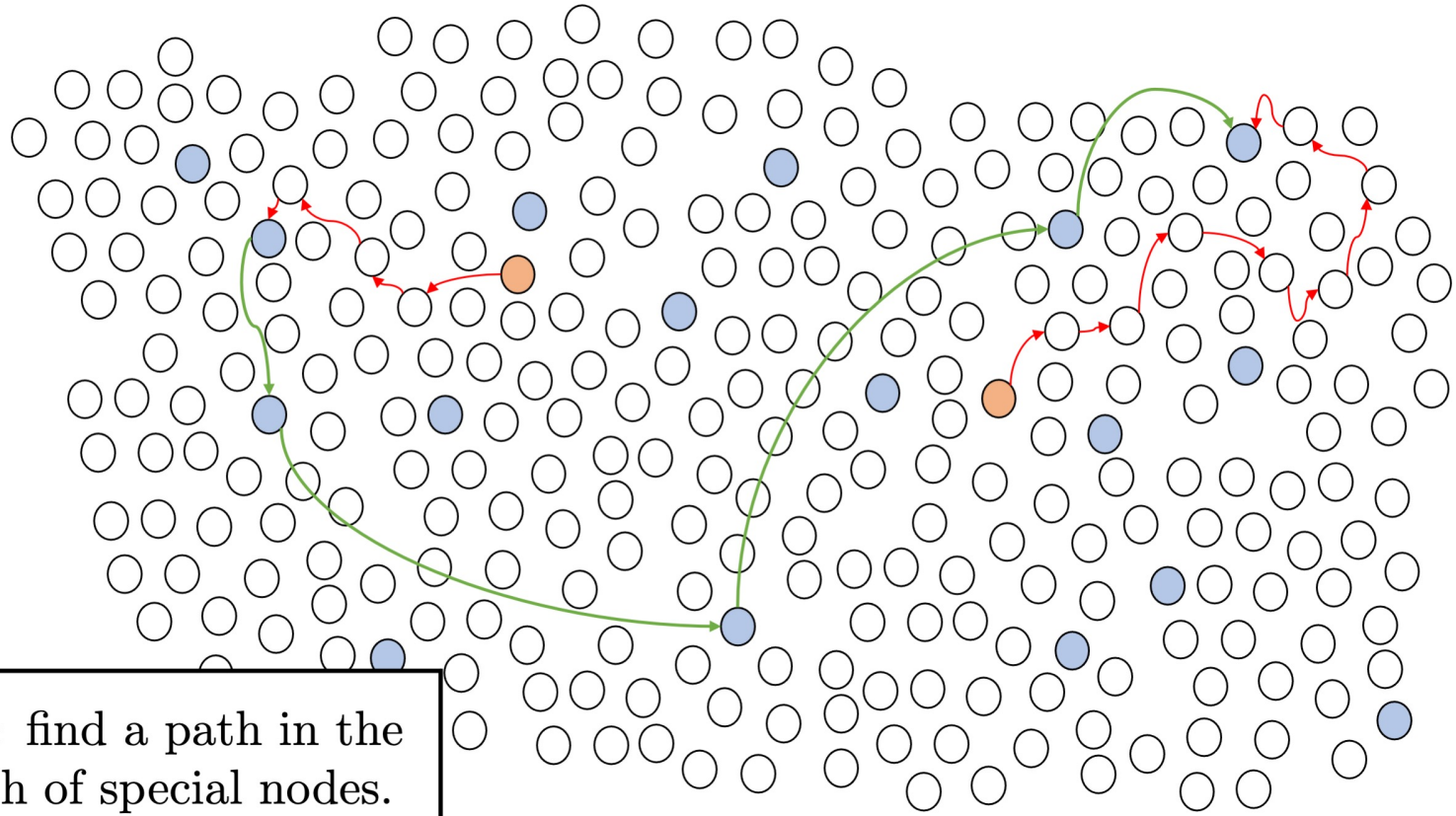
Reducing to Easier Problems using Special Subsets



Step 2: find a path in the subgraph of special nodes.

This should be *easier* so **Step 1** is bottleneck

Reducing to Easier Problems using Special Subsets



Step 2: find a path in the subgraph of special nodes.

Concatenate the paths to get the full path.

Improving Detection of the Special Subset

As Step 1 is the bottleneck, we focus on improving this part, i.e., walking in the graph until we reach a node in the special subgraph.

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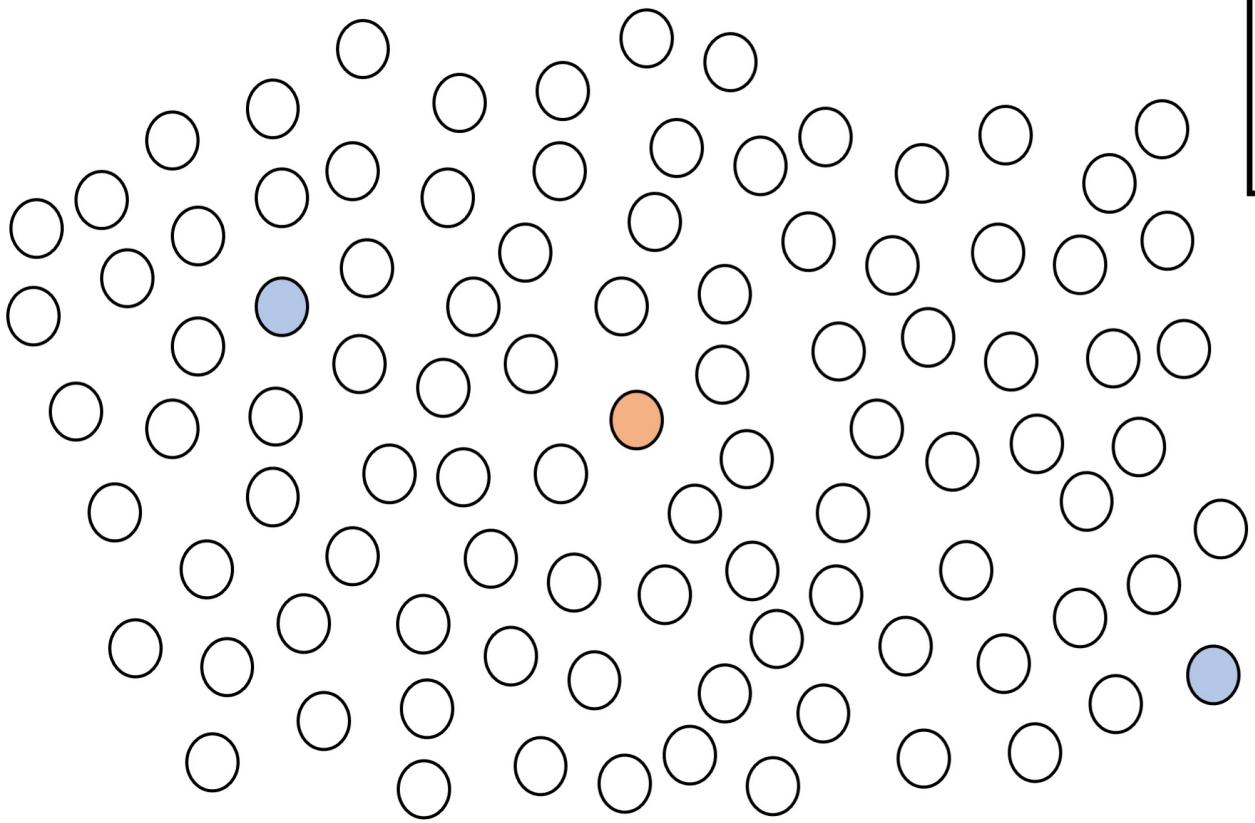
At each step of the walk, we reveal **one** node. Can we detect more nodes at each step in an efficient way?

Improving Detection of the Special Subset

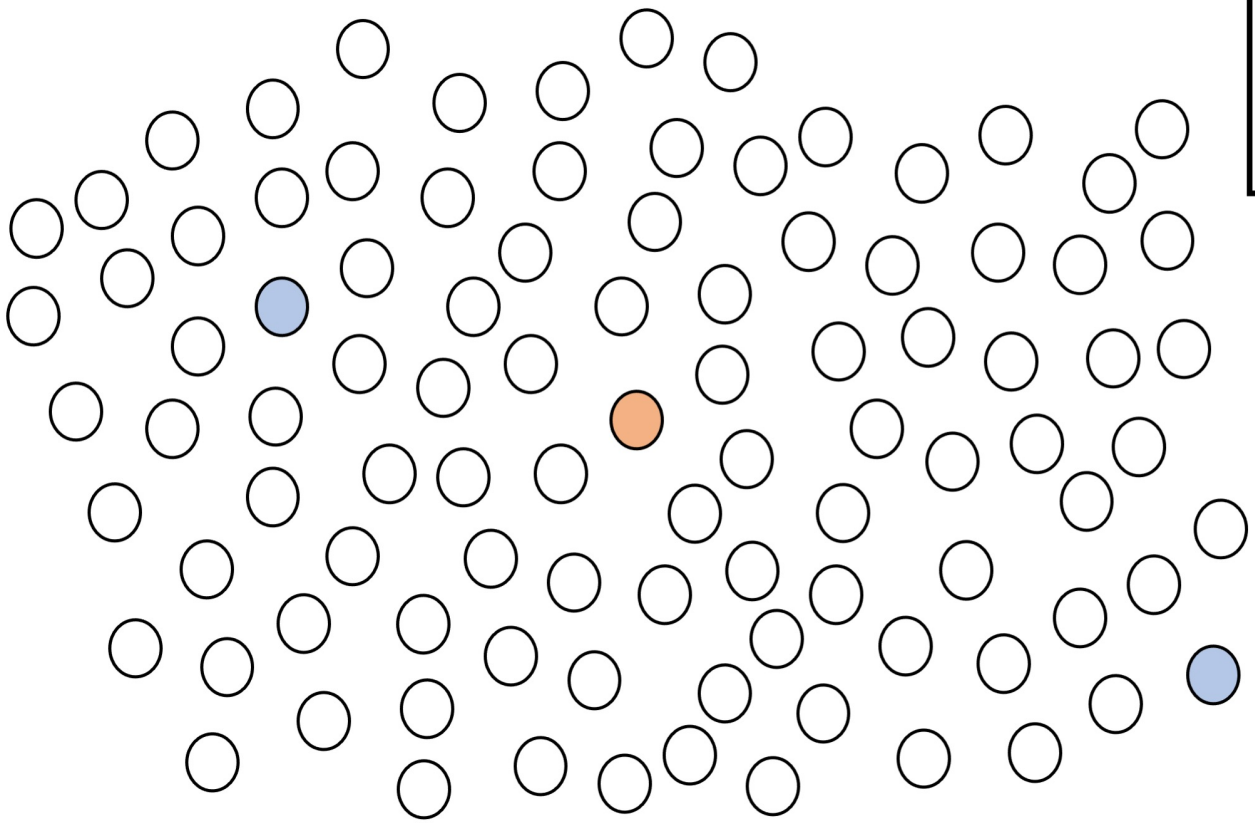


Step 1: Choose a detection set $\mathcal{D} = \{d_1, \ldots, d_n\}^*$

special nodes

*more on this later

Improving Detection of the Special Subset



Step 2: Perform detection for d_1, $d_2, d_3 \ldots$

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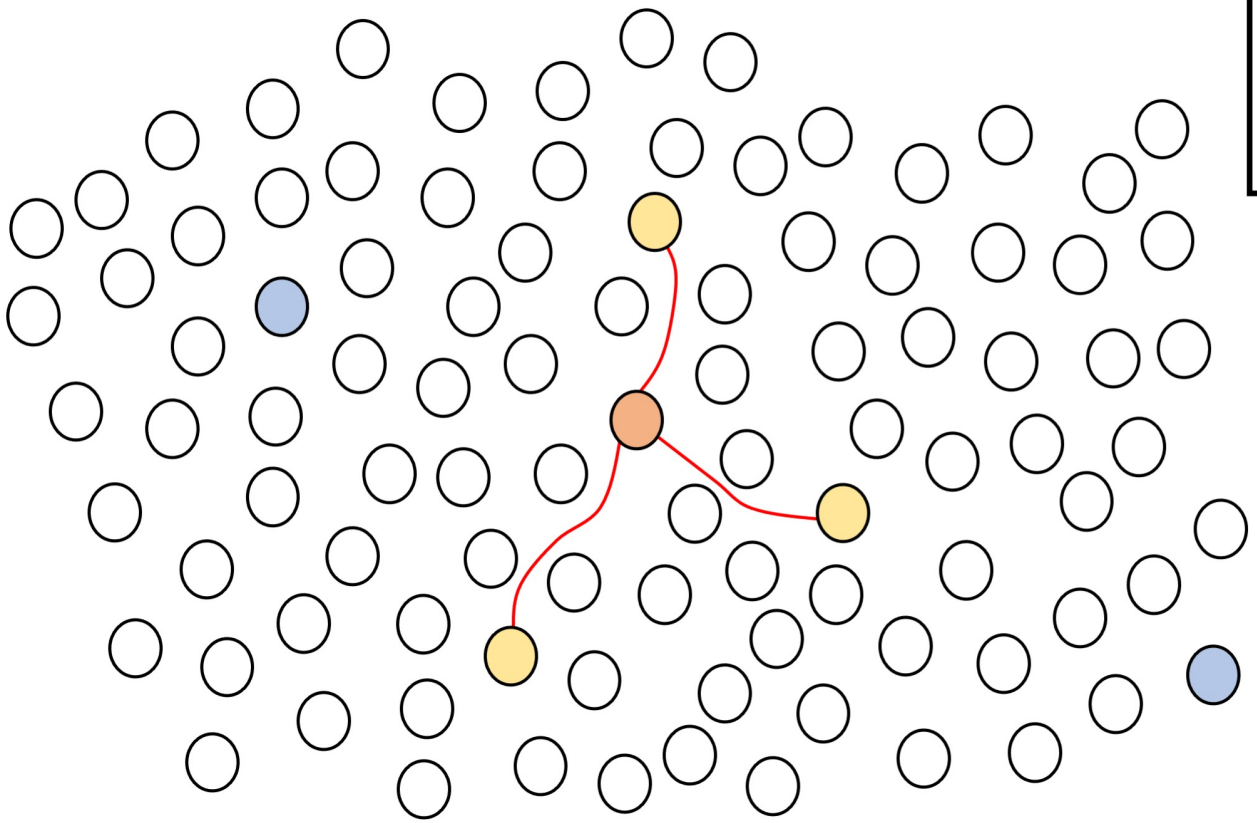

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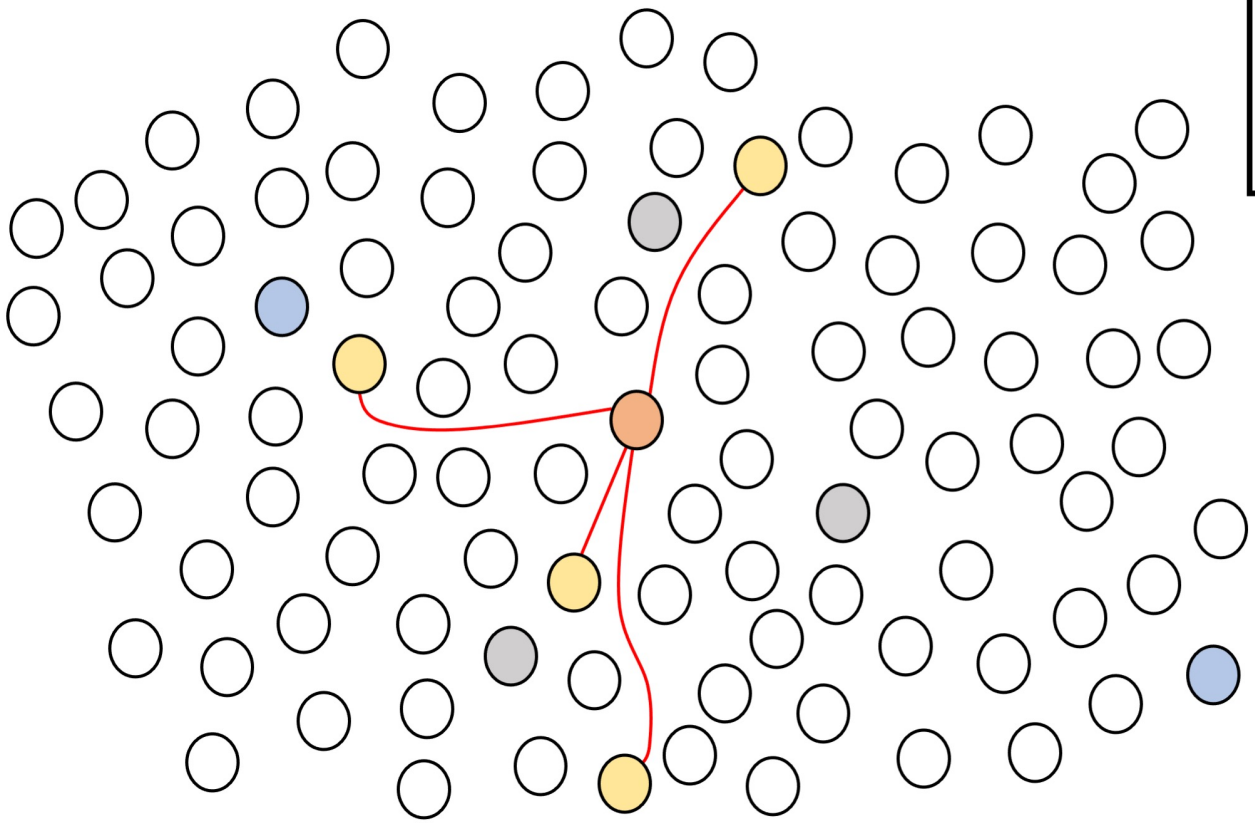

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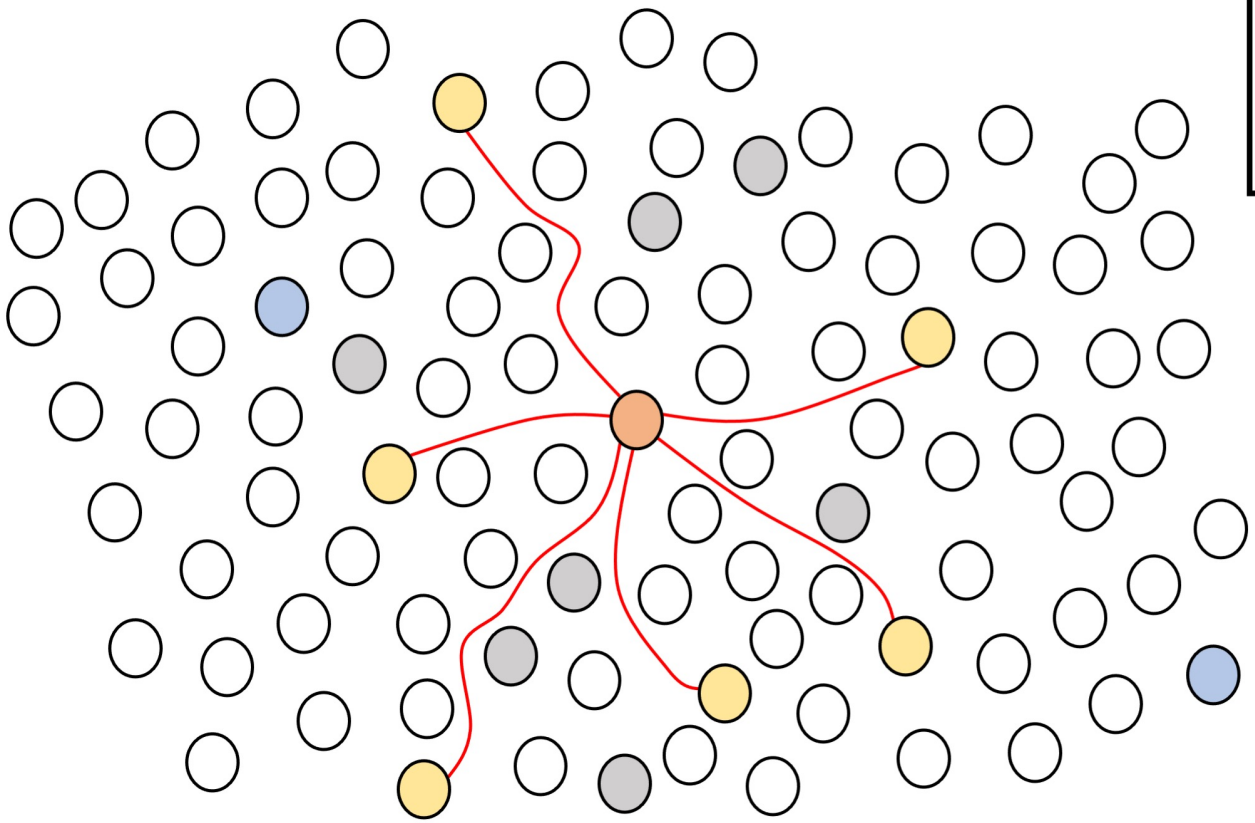

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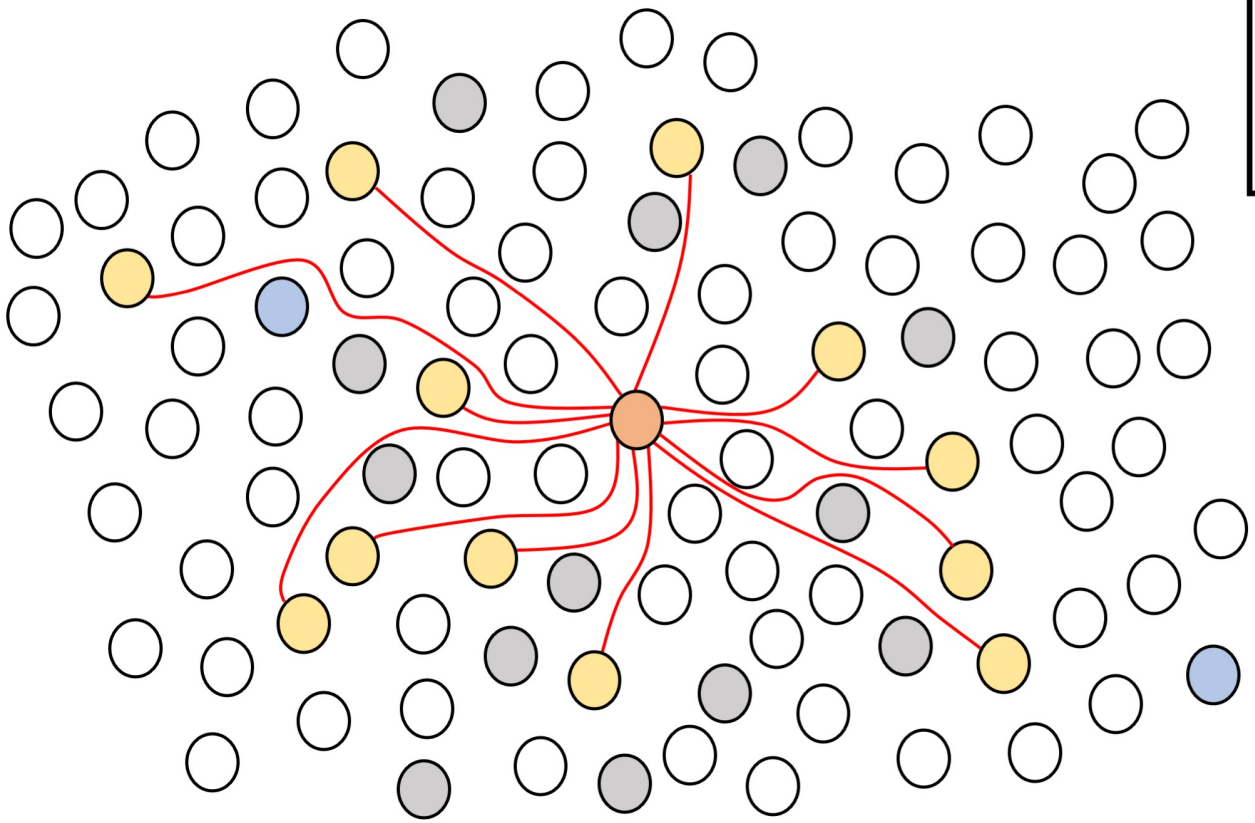

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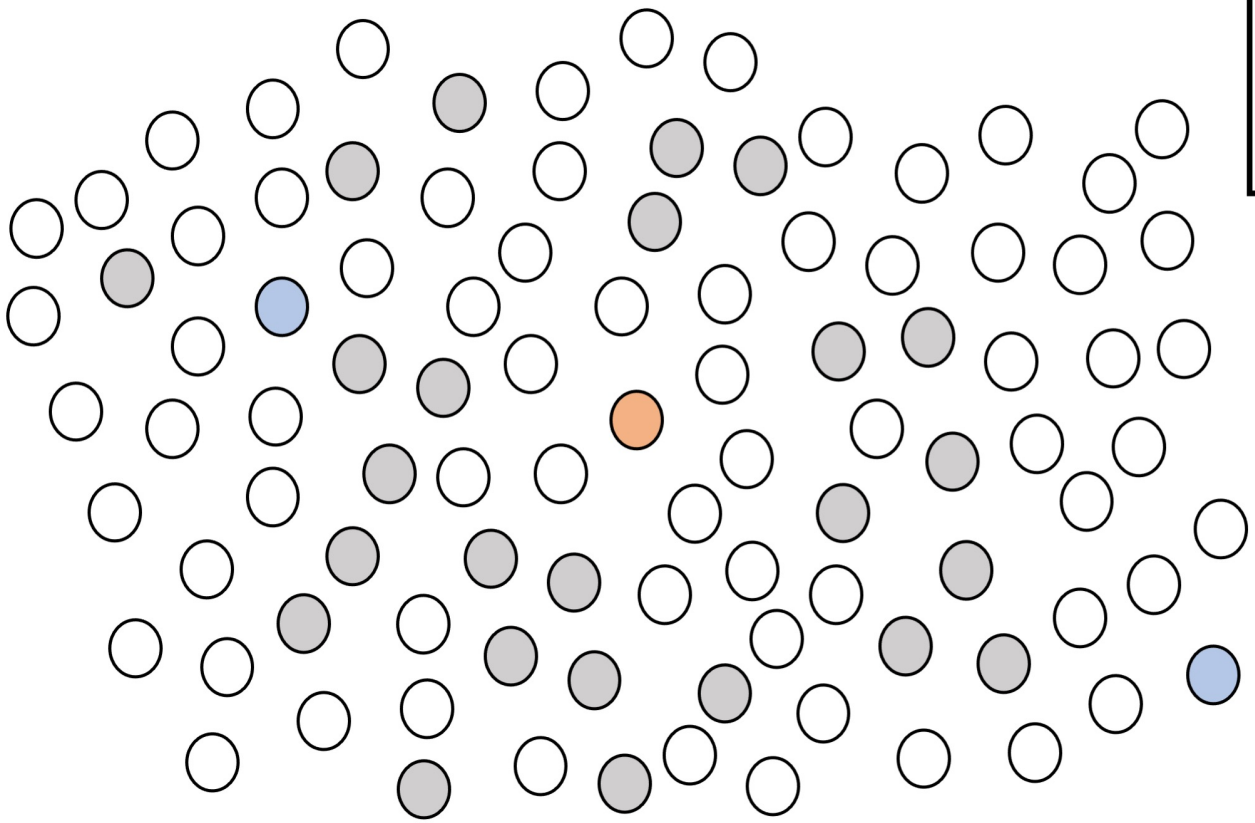

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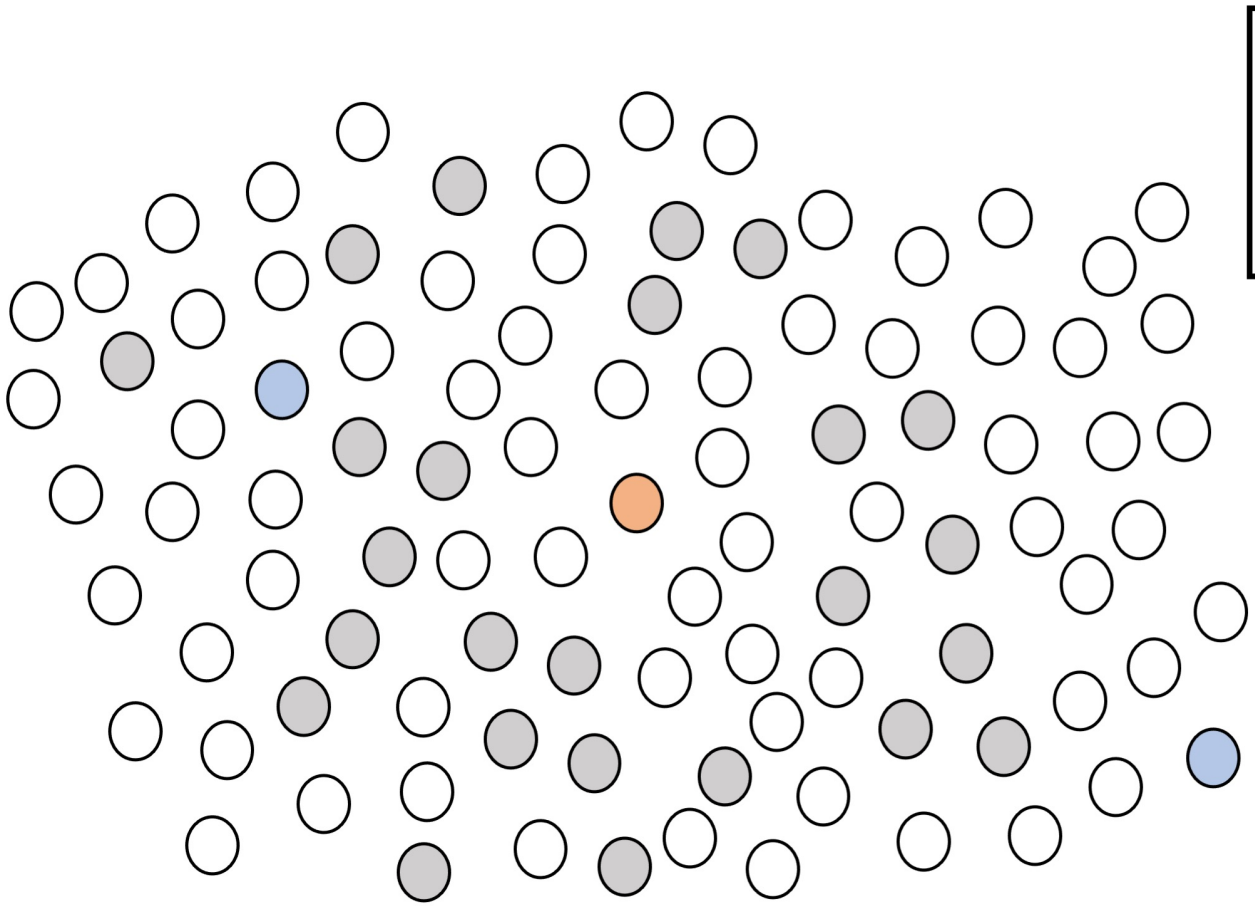

Step 3: If no detected move to another node

special nodes

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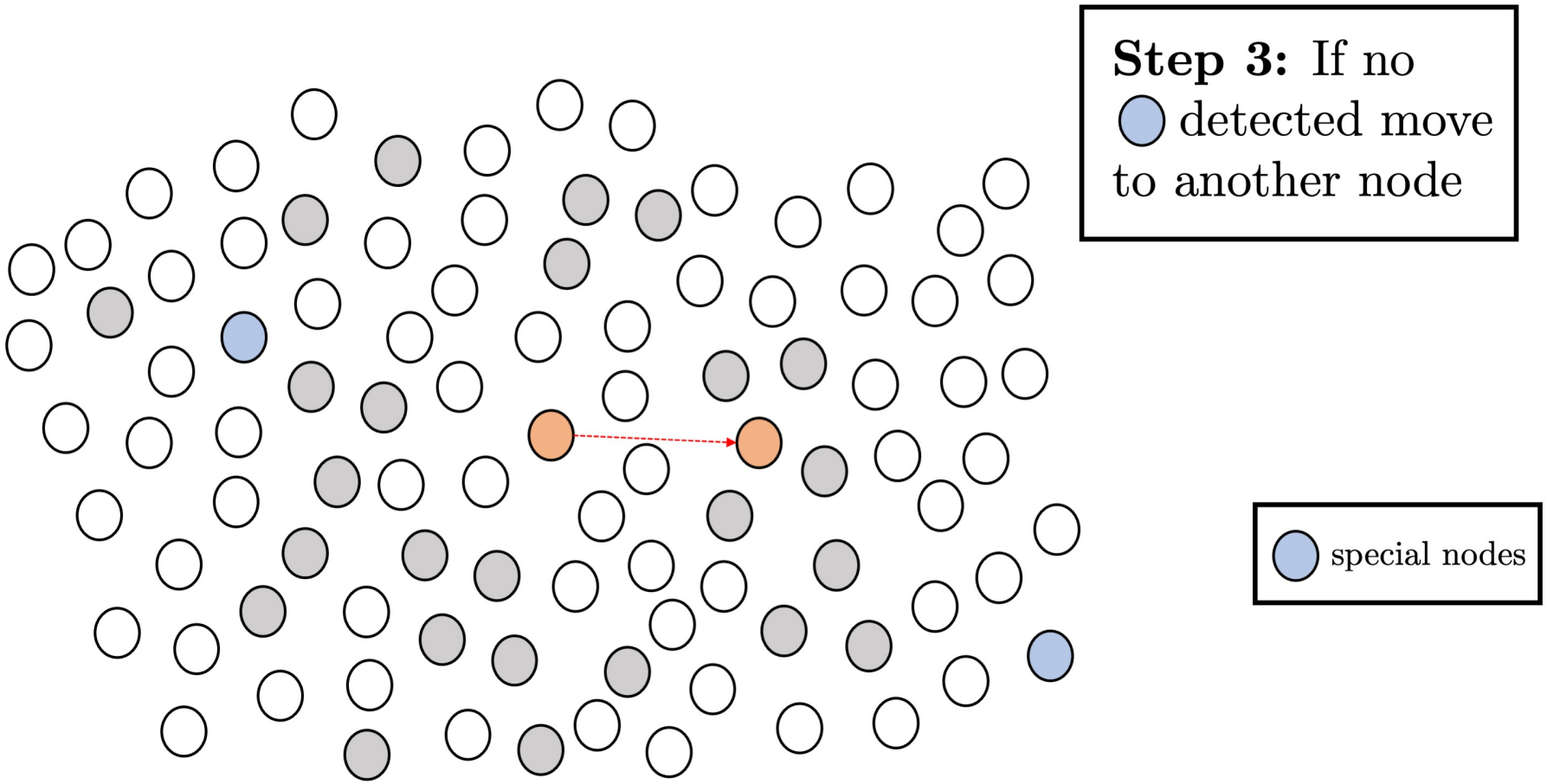

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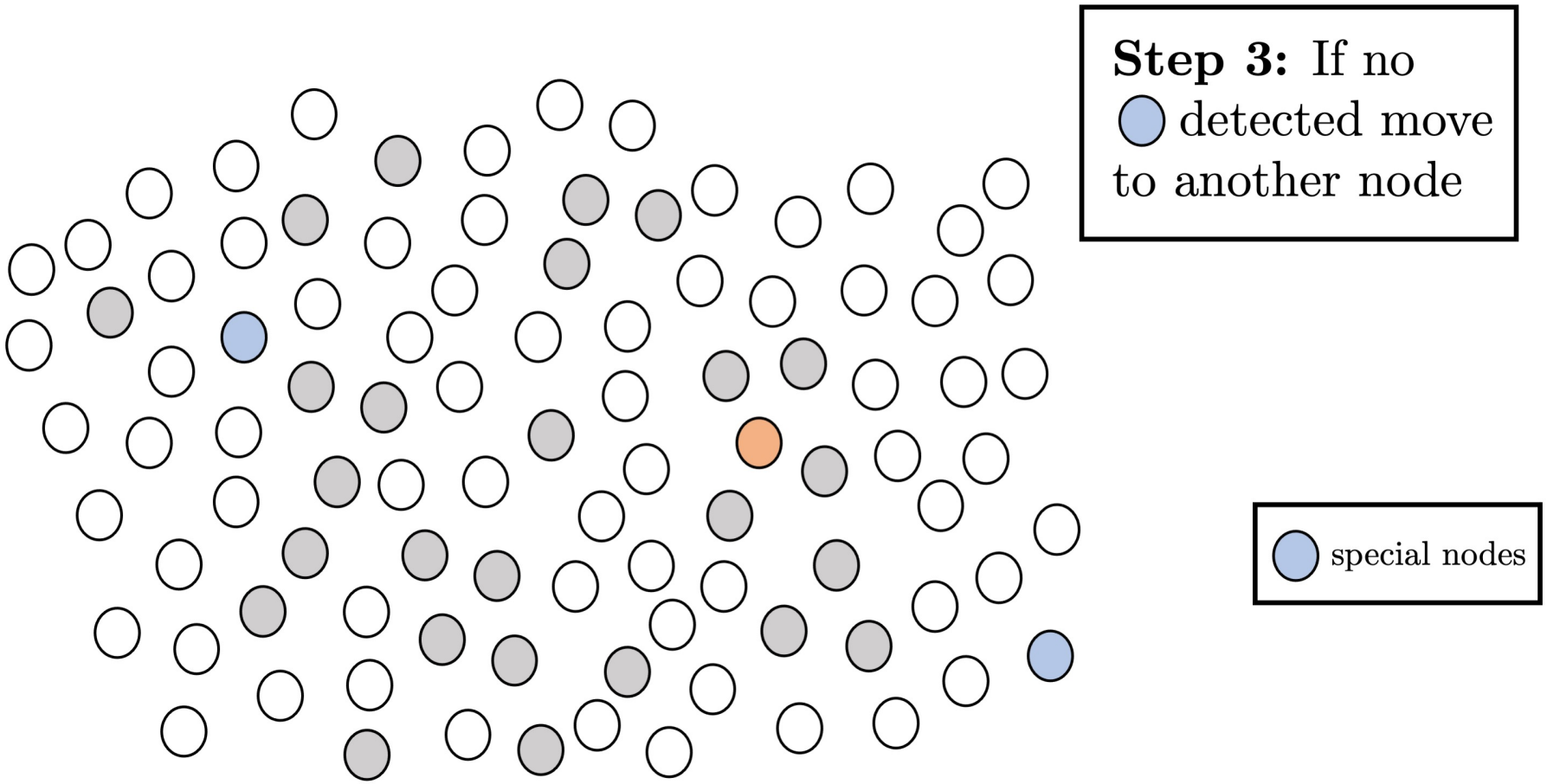

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If detection cost is *small* compared to nodes revealed, then we could hope to improve the concrete complexity of the attack.

Application to Dimension One

Based on joint work with Craig Costello, Jia Shi (eprint 2021/1488): SuperSolver

How can we apply these ideas to solving the general isogeny problem in dimension 1: given supersingular, $E_1, E_2/\mathbb{F}_{p^2}$ find an isogeny $\phi : E_1 \to E_2$?

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First, how do we take steps in this graph?

Application to Dimension One

The N-Isogeny Graph $\Gamma_1(N; p)$ in Dimension One: how to take steps

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[2] Reducing coefficients mod p we can work with $\Phi_{N,p}(X, Y) \in \mathbb{F}_p[X, Y]$.

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$$D_N := \prod_{i=1}^{n} (\ell_i + 1)\ell_i^{e_i - 1}, \text{ for prime decomposition } \prod_{i=1}^{n} \ell_i^{e_i} \text{ of } N.$$

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This tells us that the roots of $\Phi_{N,p}(X, j)$ are neighbours of j in $\Gamma_1(N; p)$.

[2] Reducing coefficients mod p we can work with $\Phi_{N,p}(X, Y) \in \mathbb{F}_p[X, Y]$.

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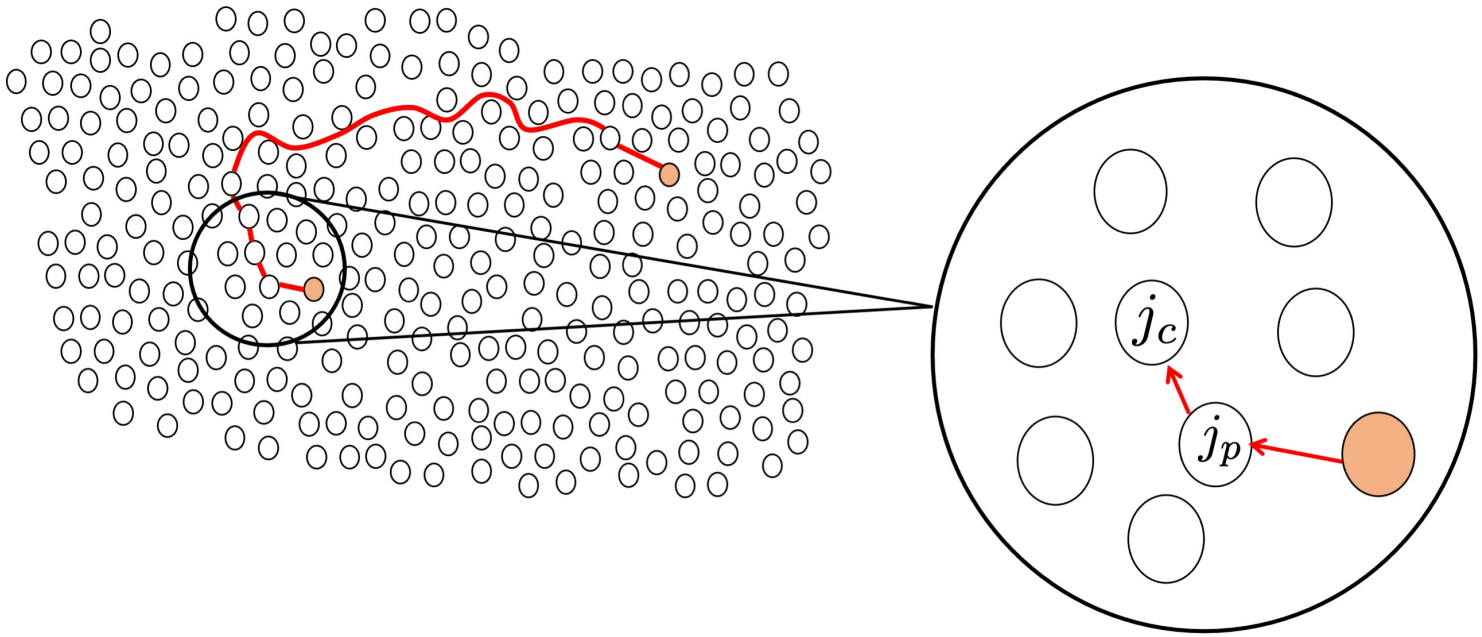

1. Store the current and previous j-invariants j_c and j_p.

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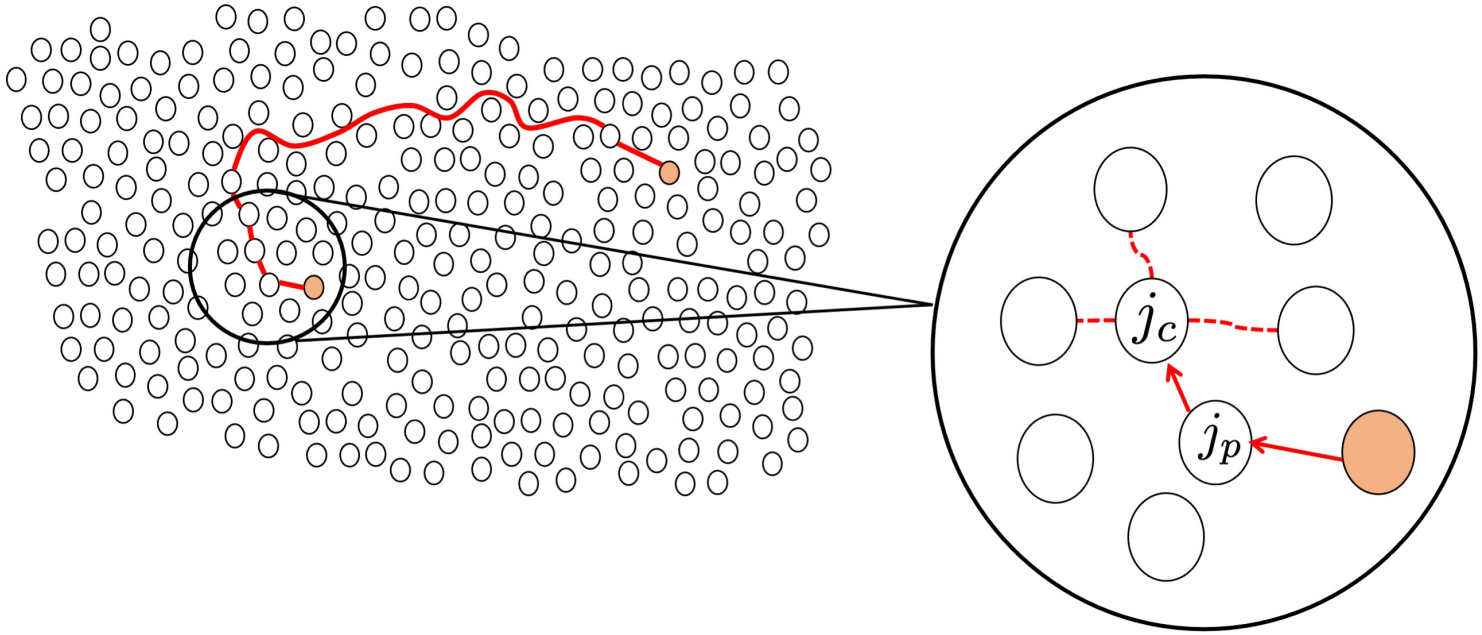

2. Find the $D_N - 1$ roots of $\Phi_{N,p}(X, j_c)/(X - j_p)$.

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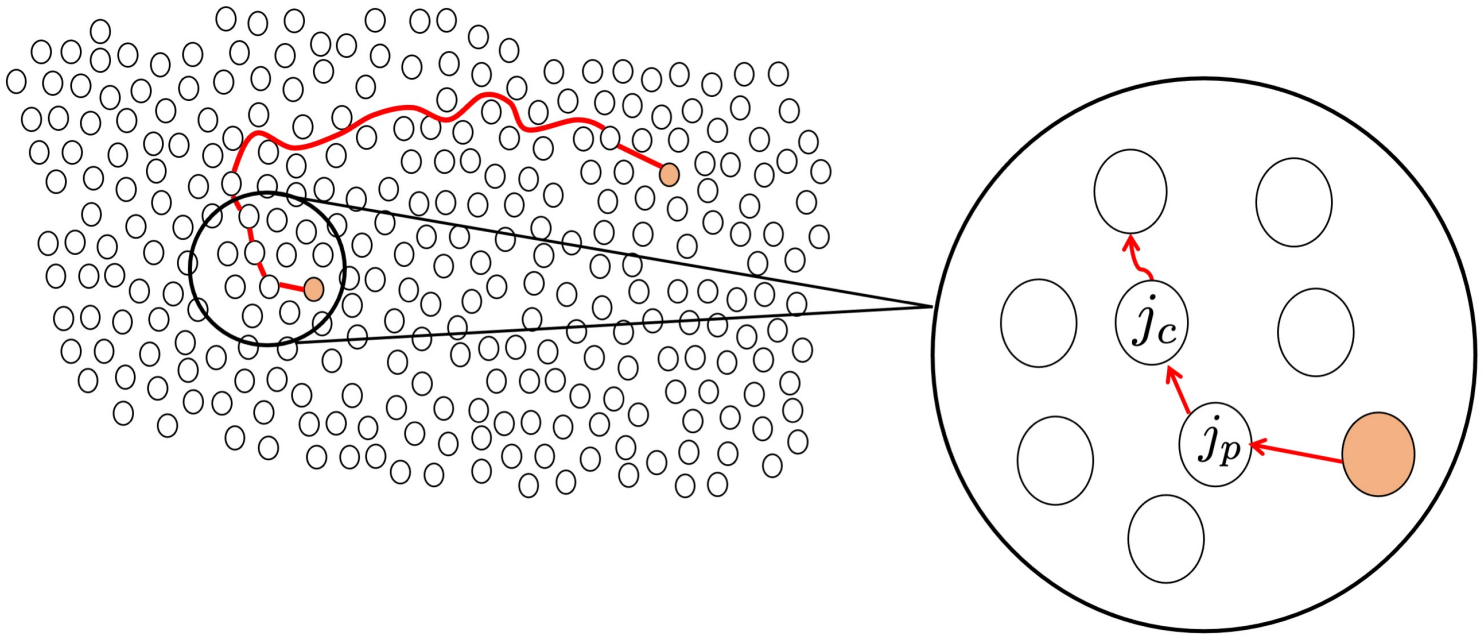

3. Choose one of these and walk to the corresponding node.

Application to Dimension One

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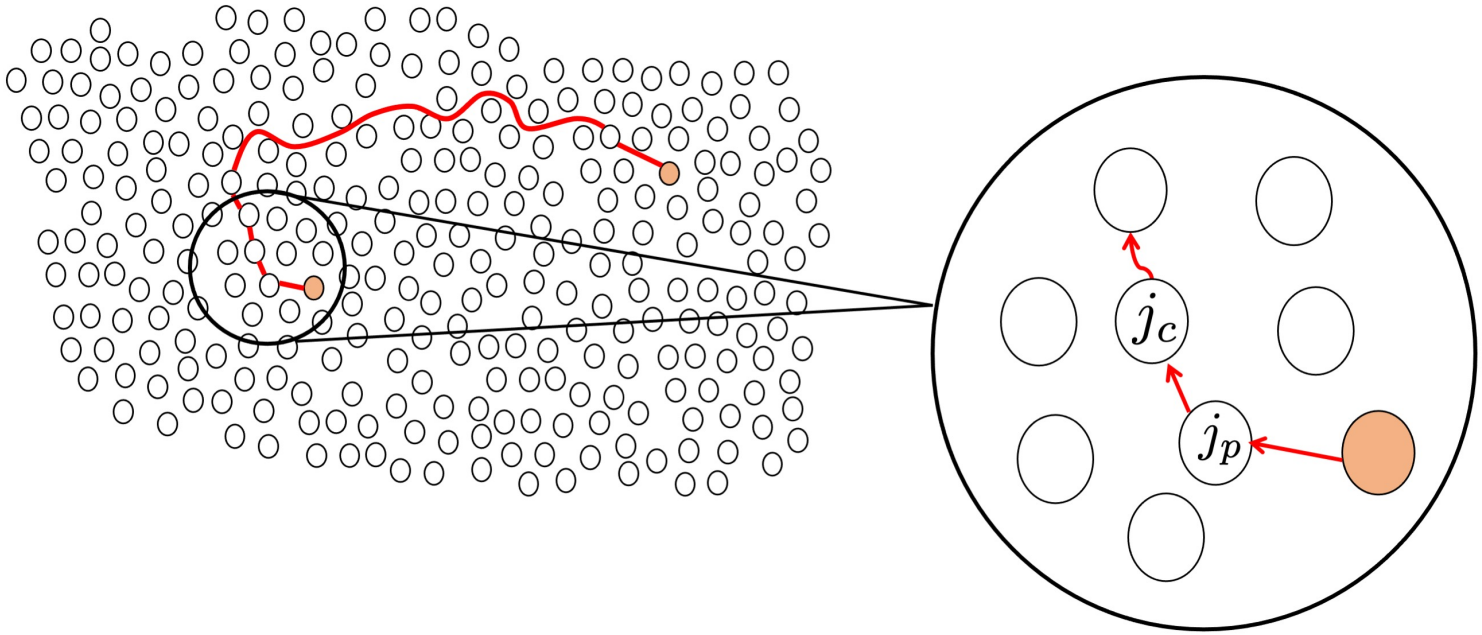
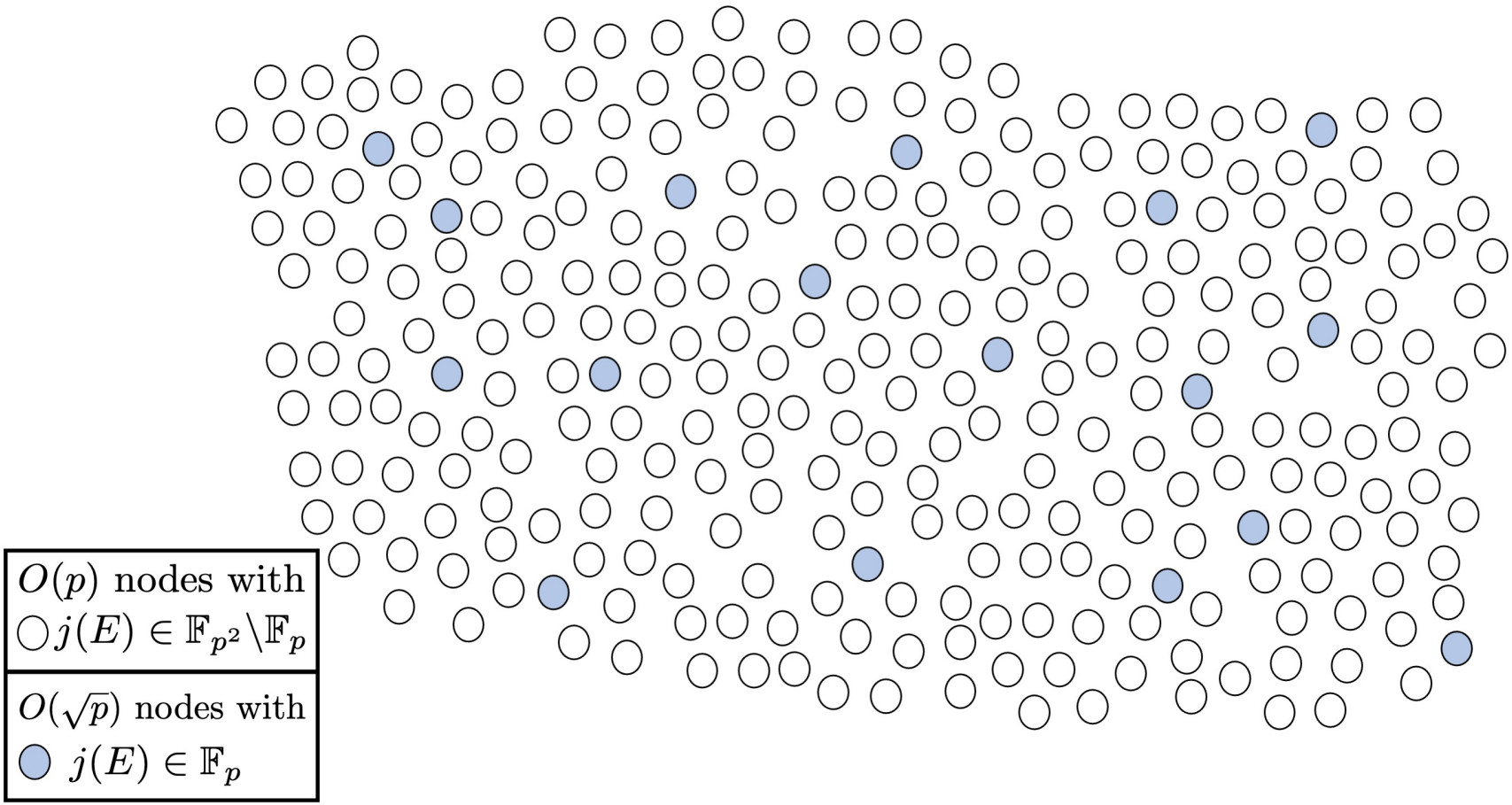


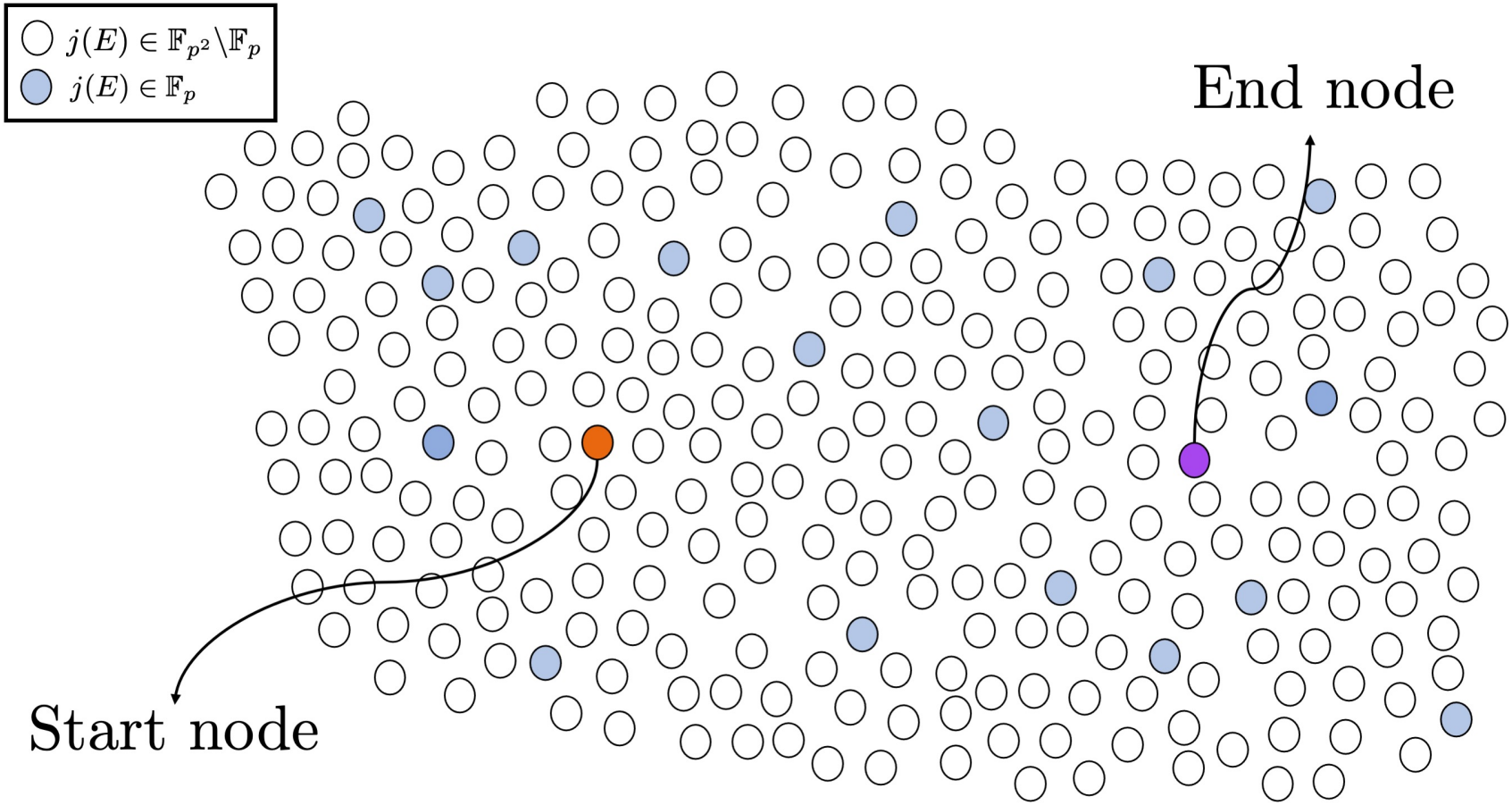
3. Choose one of these and walk to the corresponding node.

We choose to take *non-backtracking* walks in $\Gamma_1(2; p)$ as this corresponds to factoring a quadratic. So, each step reveals 2 nodes.

Special Subset: $\mathbb{F}_p$ nodes



The Delfs–Galbraith Algorithm



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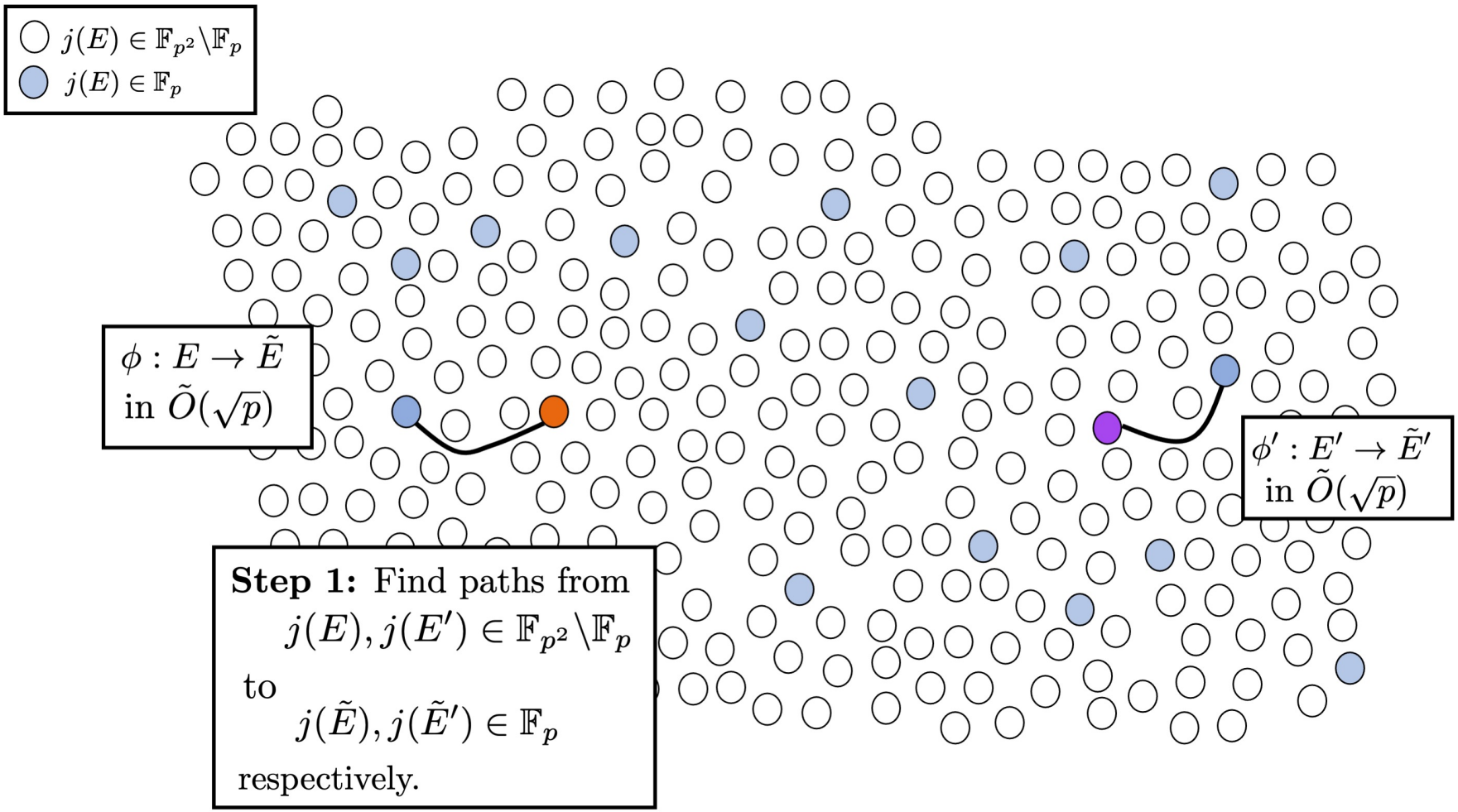

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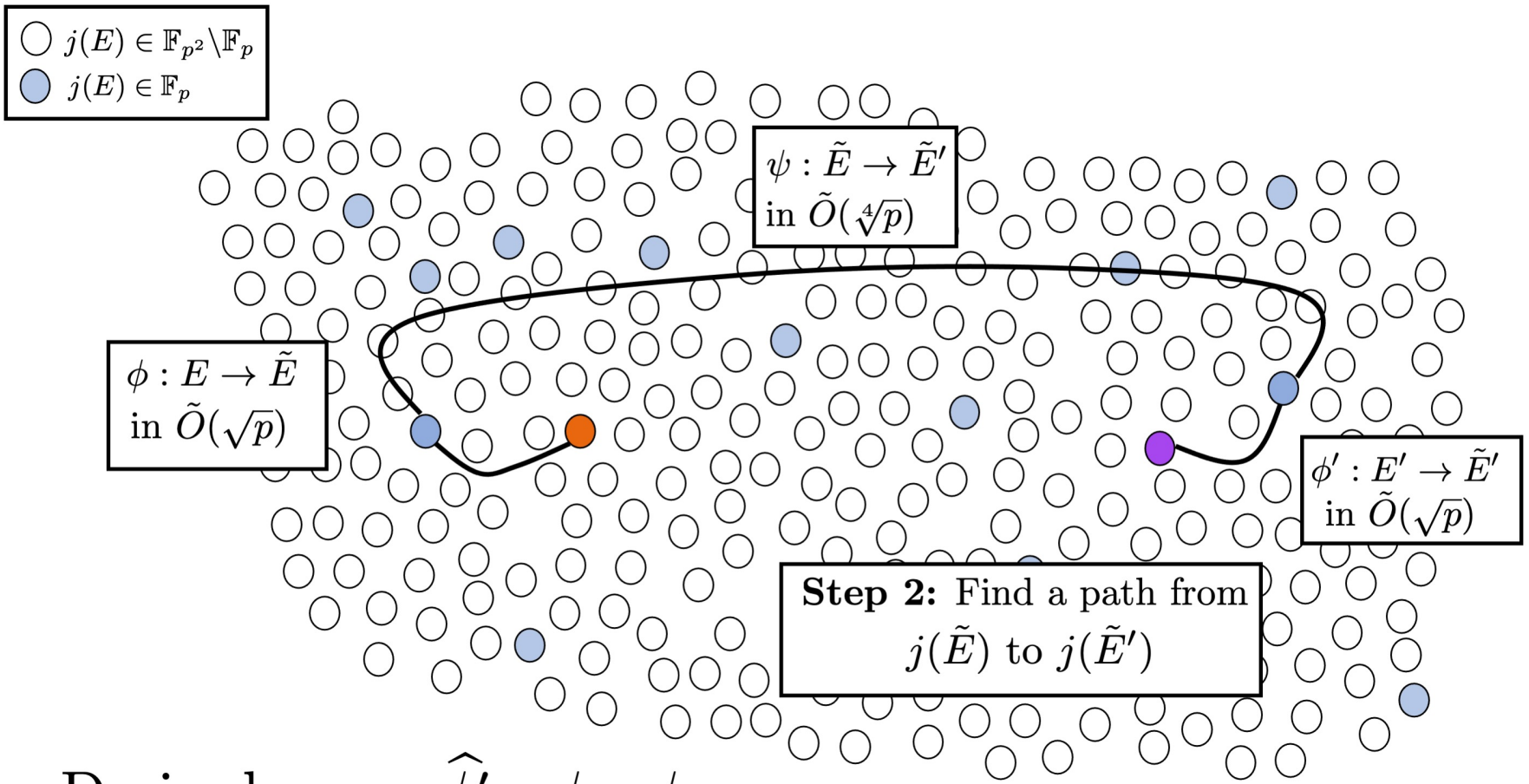

Desired map: $\widehat{\phi'} \circ \psi \circ \phi$

Improved Detection of $\mathbb{F}_p$-Subgraph

For $N > 2$, we want to determine whether the current node we are on $j_c \in \mathbb{F}_{p^2}$ is N-isogenous to node in $\mathbb{F}_p$, i.e., we ask:

$$\text{Does } \Phi_{N,p}(X, j_c) \text{ have a root in } \mathbb{F}_p?$$

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At each step, the precise values roots do not need to be known, only whether one lies in $\mathbb{F}_p$.

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We also show how to transform $f, \pi(f) \in \mathbb{F}_{p^2}[X]$ to give $g_1, g_2 \in \mathbb{F}_p[X]$ with the same gcd and avoid *all* costly multiplications in $\mathbb{F}_{p^2}$.

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$$\frac{\text{cost of step in graph} + \text{cost of (1) \& (2)}}{\text{nodes revealed by walk and detection}}.$$

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Application to Dimension Two

Based on joint with Craig Costello, Sam Frengley (eprint soon): SplitSearcher

We now consider the general isogeny problem in dimension 2, which can be viewed as finding a path between two nodes A_1, A_2 in the superspecial isogeny graph.

Application to Dimension Two

The (N, N)-Isogeny Graph $\Gamma_2(N; p)$ in Dimension Two: taking a steps

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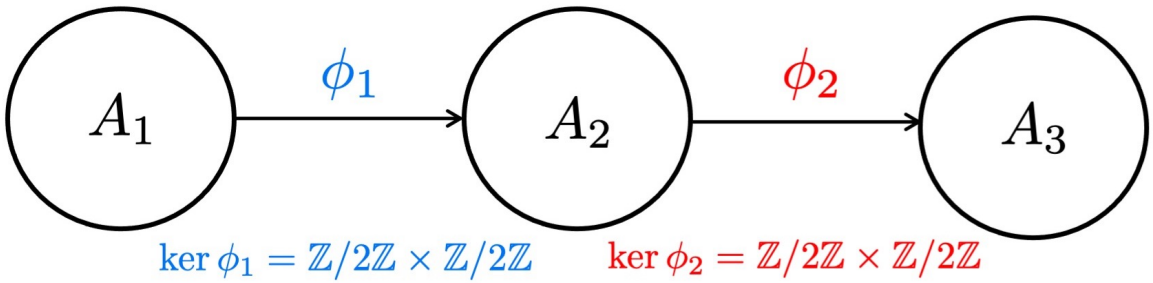

$$\ker \hat{\phi}_1 \cap \ker \phi_2 = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \phi_2 \text{ is dual to } \phi_1 \\ \mathbb{Z}/2\mathbb{Z} & \phi_2 \text{ is a bad extension of } \phi_1 \\ \emptyset & \phi_2 \text{ is a good extension of } \phi_1 \end{cases}$$

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$$\mathcal{J}(p) := \{[A] \in \mathcal{S}(p) \ : \ A \cong \mathrm{Jac}(C)\} \text{ and}$$
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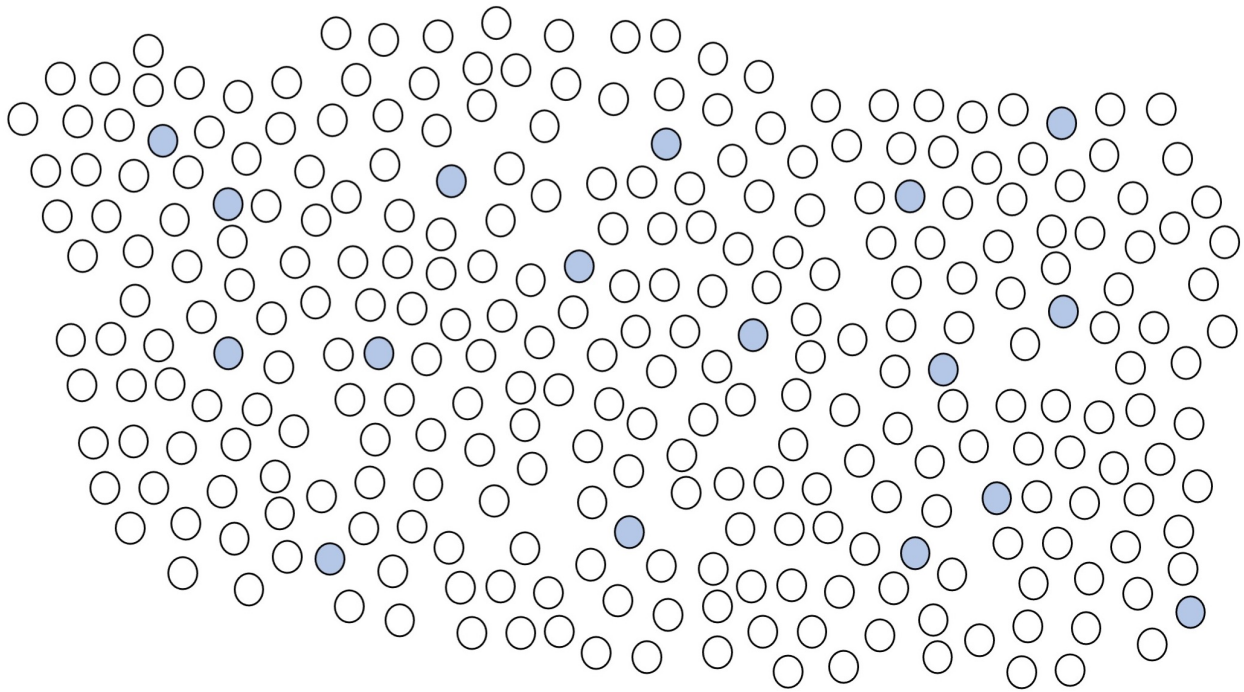
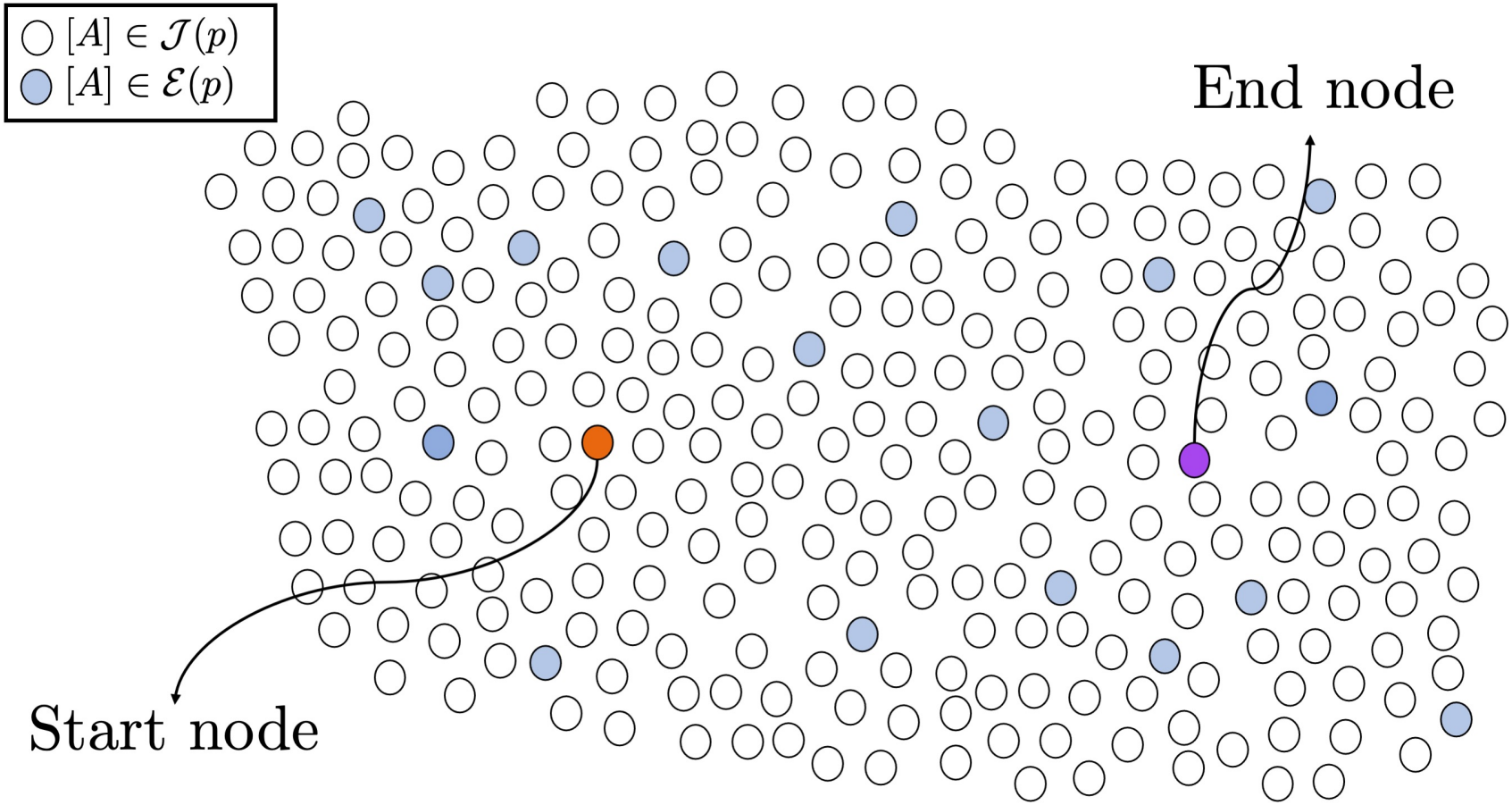


The Costello–Smith Algorithm



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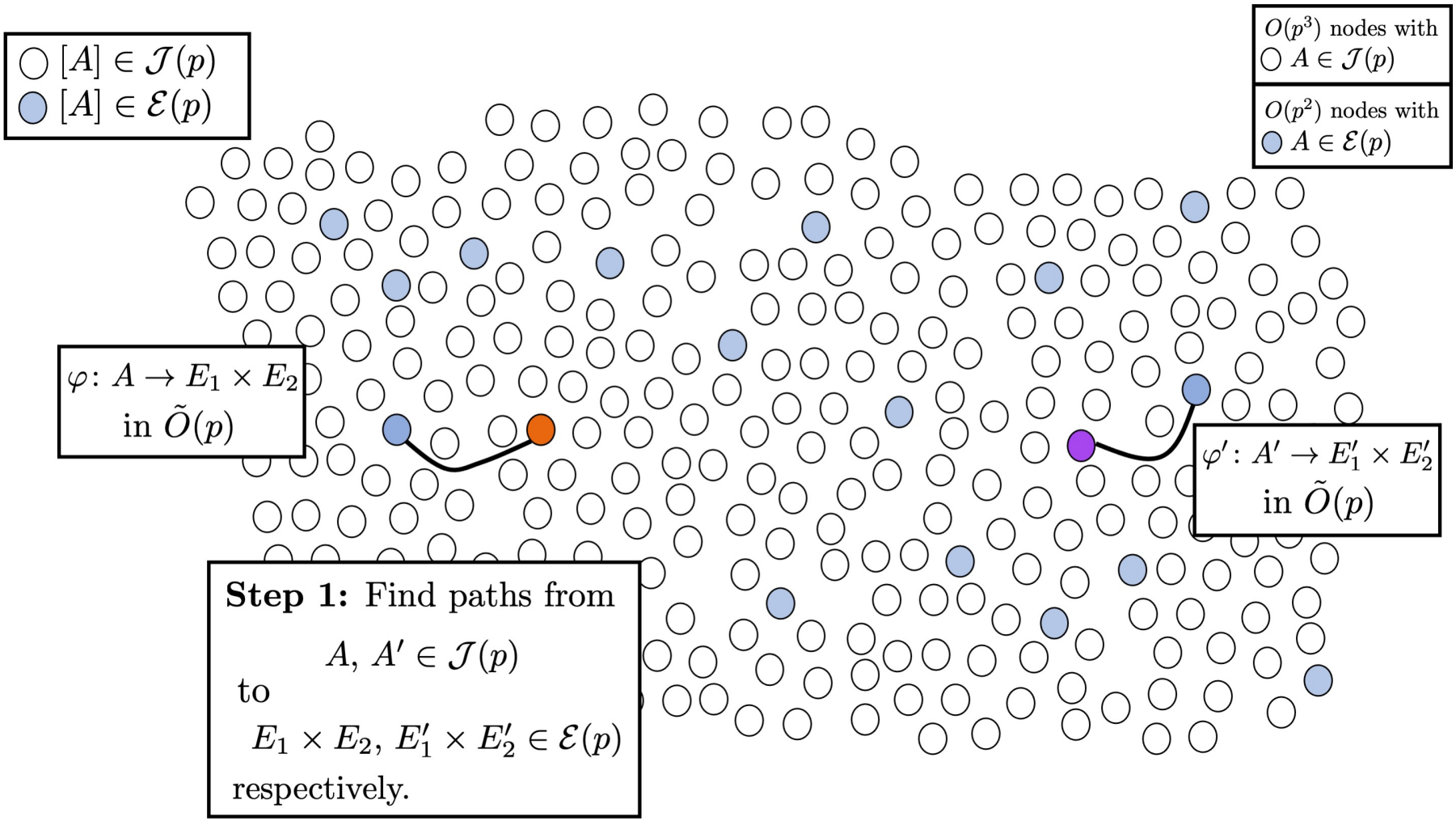

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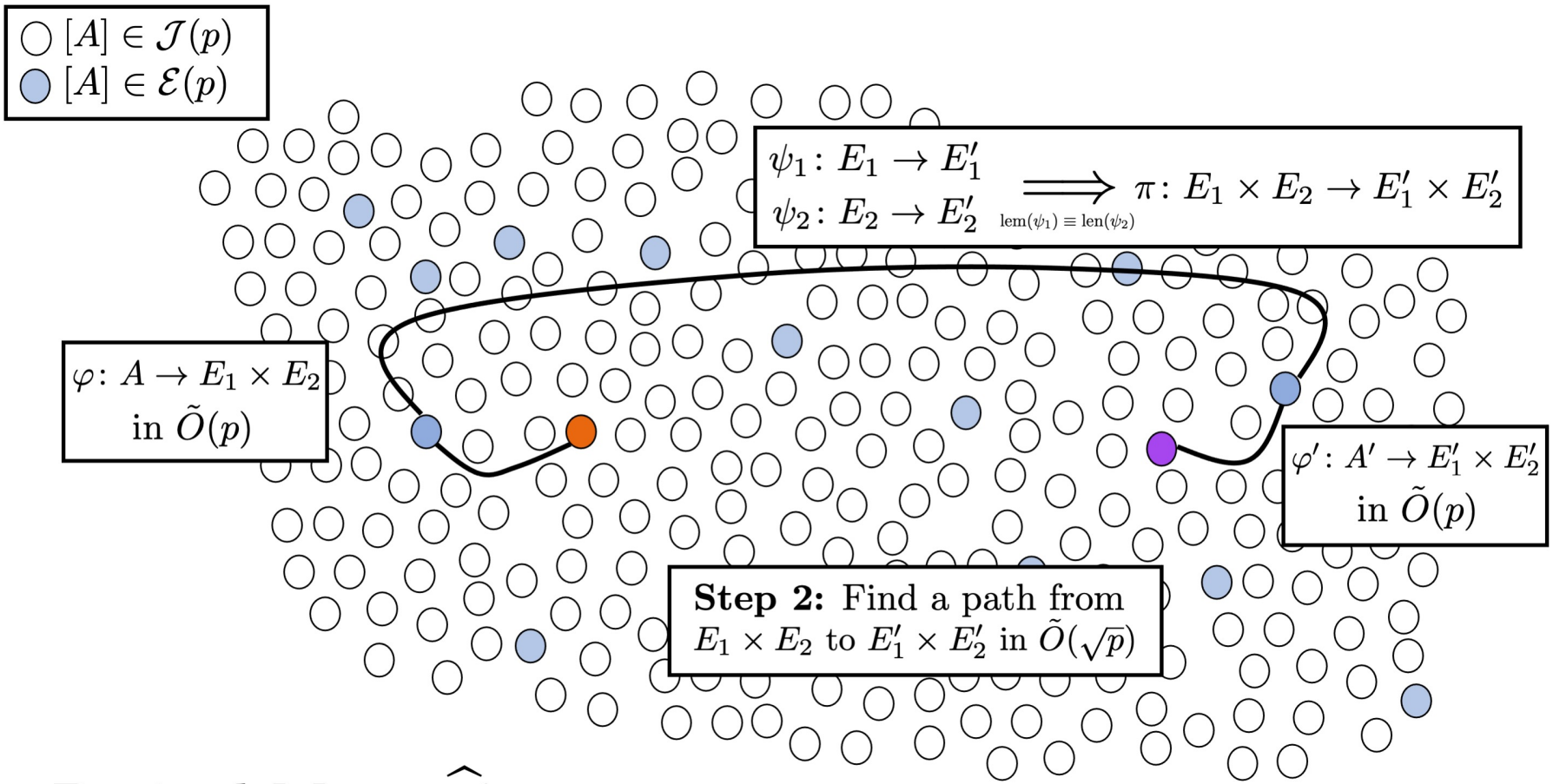

Desired Map: $\widehat{\varphi'} \circ \pi \circ \varphi$

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Kumar [Kum15] gives us the map

$$\varphi_N = \Big(\mathcal{I}_2(r,s) : \mathcal{I}_4(r,s) : \mathcal{I}_6(r,s) : \mathcal{I}_{10}(r,s)\Big),$$

such that

$$\mathrm{Jac}(C) \text{ is } (N,N)\text{-split (over } \bar{\mathbb{F}}_p) \iff [C] \text{ in } \overline{\mathrm{Im}\ \varphi_N}$$

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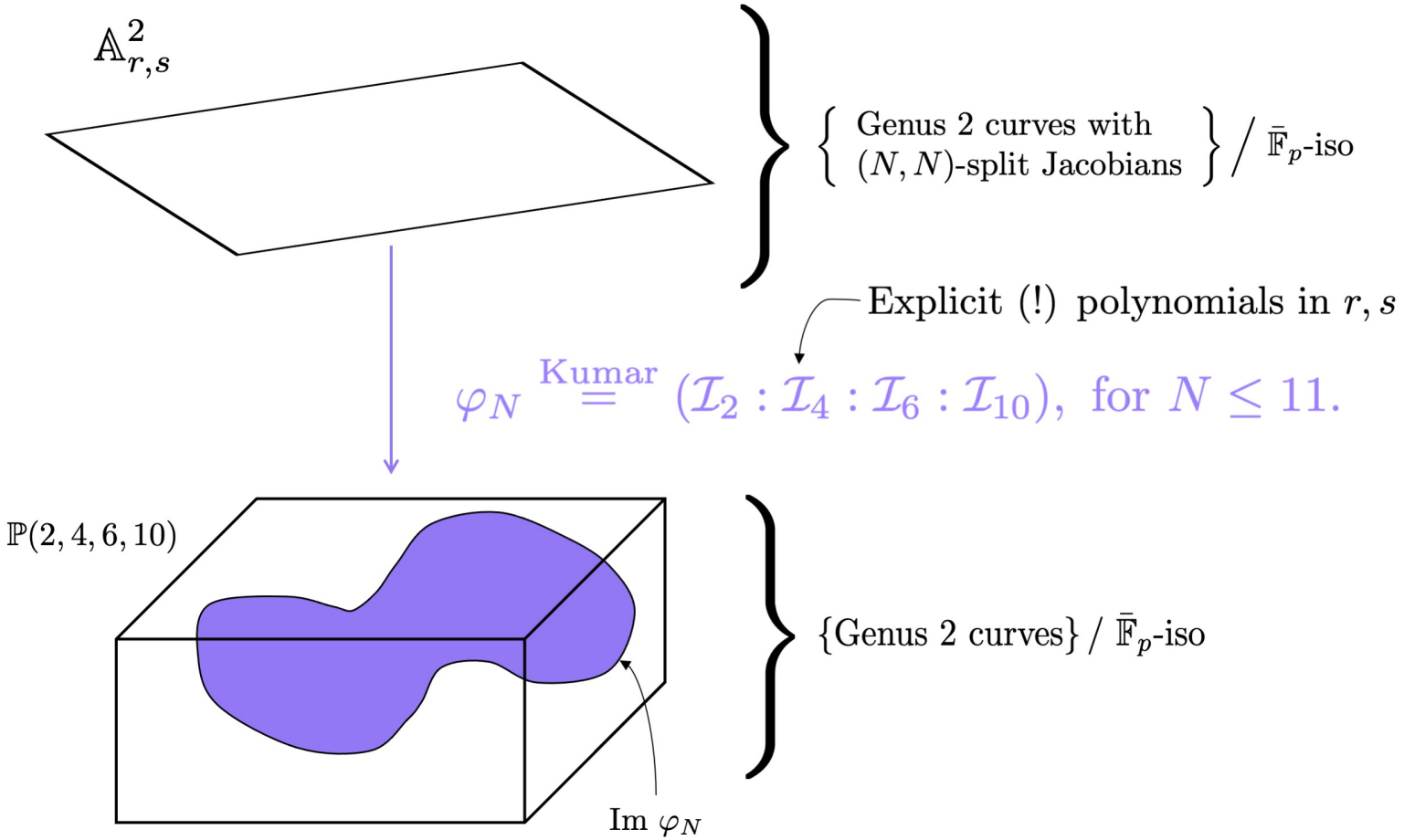

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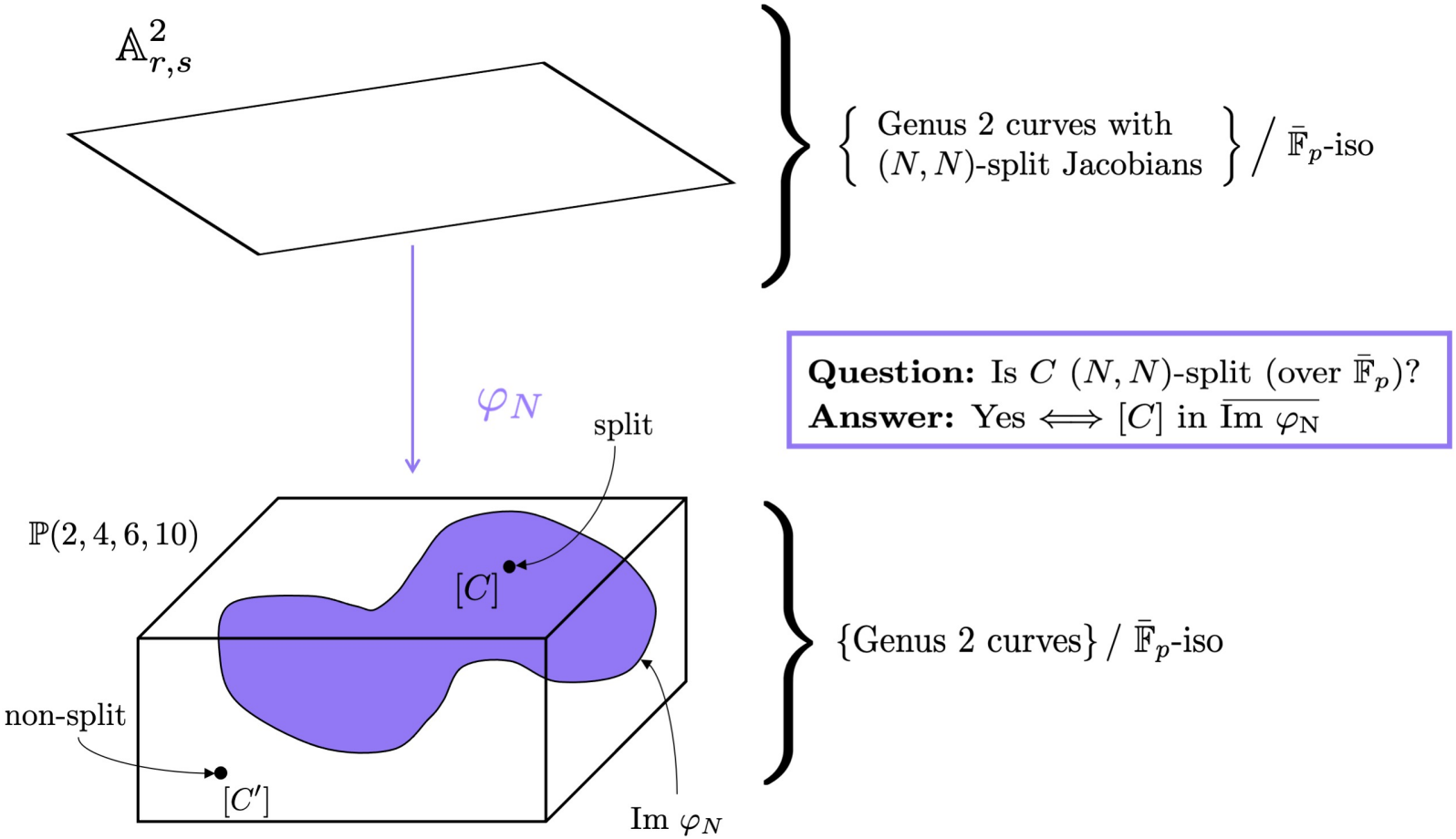

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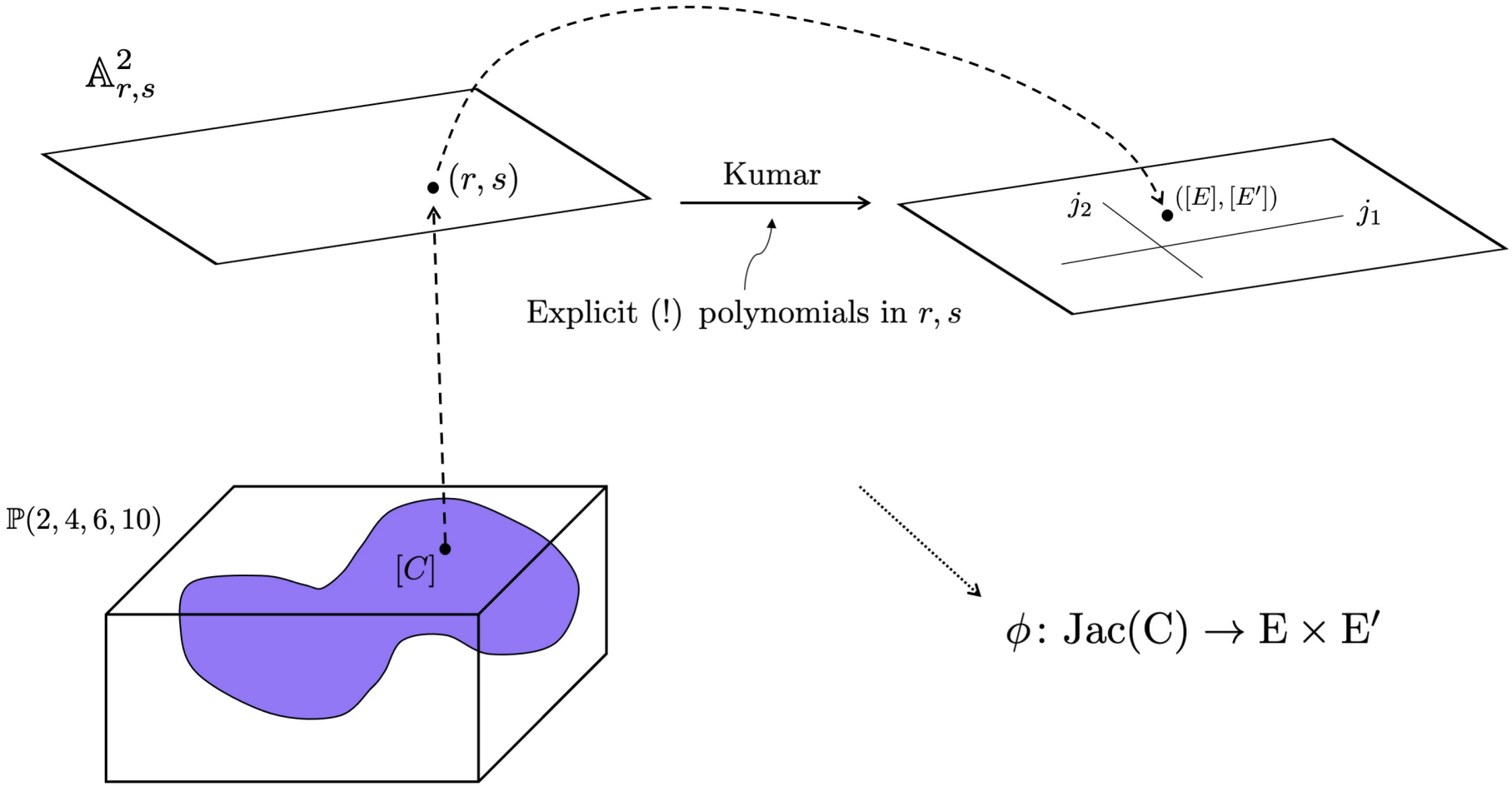

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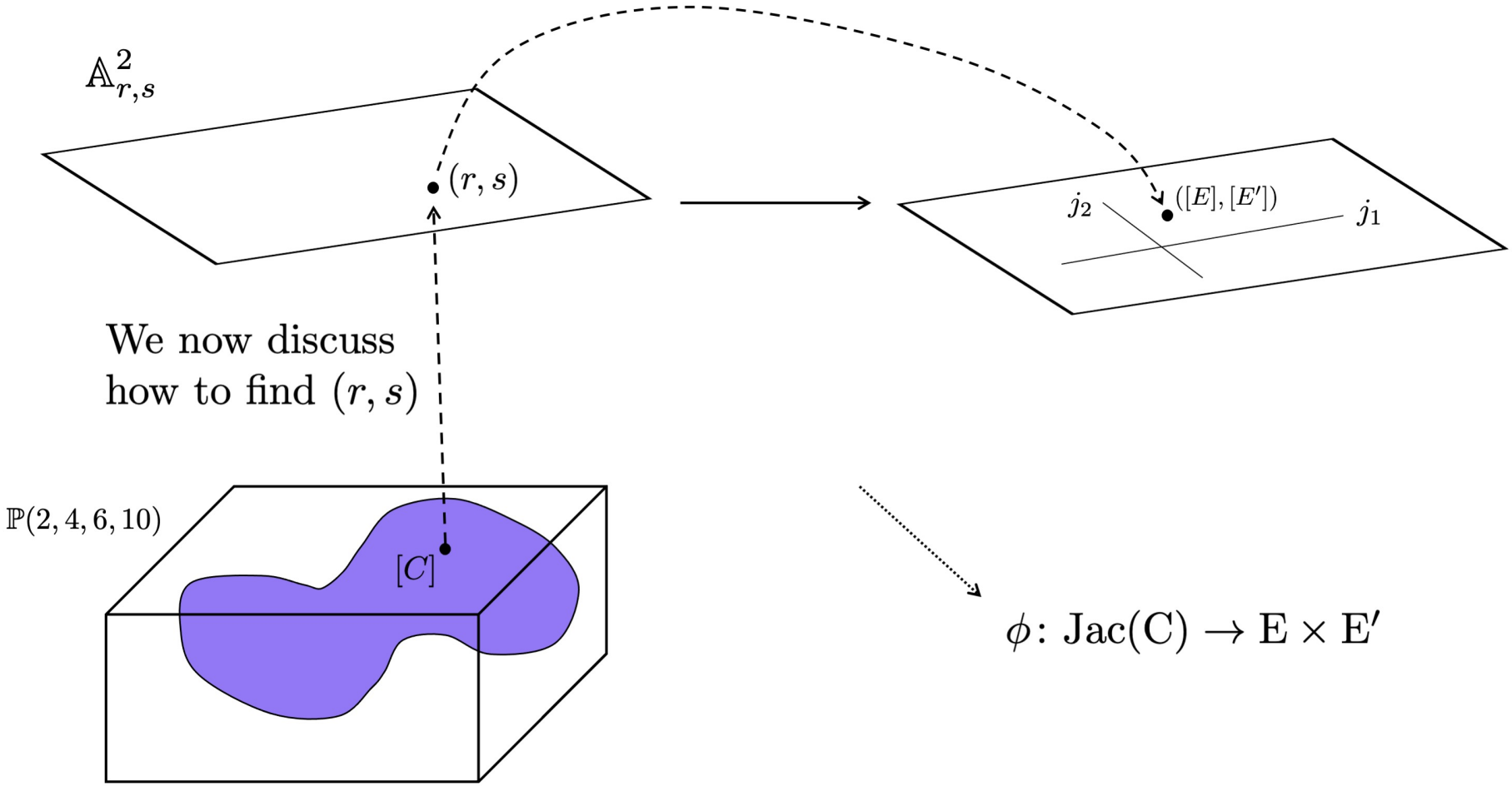

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Using the same normalisation for coordinates of Kumar's map $\mathcal{I}_k(r, s)$, we get $i_1(r, s)$, $i_2(r, s)$ and $i_3(r, s)$.

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		Walks in $\Gamma_g(2;p)$ without additional detection		**Walks in $\Gamma_g(2;p)$ with additional detection in $\Gamma_g(N;p)$**			
$\log p$	**Dimension** g	**nodes per 10^8 muls**	**$\mathbb{F}_p$ muls per node**	**Detection Set** $N \in \mathcal{D}$	**nodes per 10^8 muls**	**$\mathbb{F}_p$ muls per node**	**Improvement Factor**
100	1	443,951	223	$\{3,5,7\}$	2,165,681	46	4.9×
	2	85,034	1176	$\{2,3\}$	2,076,517	48	24.5×
250	1	191,115	526	$\{3,5,7,9,11,13\}$	1,607,145	62	8.4×
	2	34,083	2934	$\{4,6\}$	1,771,608	56	52.4×
500	1	97,510	1032	$\{3,5,7,9,11,13,17\}$	1,260,243	79	12.9×
	2	20,239	4941	$\{4,6\}$	1,667,360	60	82.4×
800	1	62,191	1609	$\{3,4,5,6,7,8,9,11,13,17,19\}$	1,096,056	86	17.6×
	2	13,228	7560	$\{4,6\}$	1,548,504	65	116.3×

Results

We implemented and optimised walks in $\Gamma_g(2; p)$ for the first step of attack in dimension $g = 1, 2$ with *and* without detection in $\Gamma_g(N; p)$. We ran these for different primes until reaching 10^8 $\mathbb{F}_p$ multiplications, counting number of nodes revealed and $\mathbb{F}_p$ multiplications per node revealed.

		Walks in $\Gamma_g(2; p)$ without additional detection		**Walks in $\Gamma_g(2; p)$ with additional detection in $\Gamma_g(N; p)$**			
$\log p$	**Dimension** g	**nodes per 10^8 muls**	**$\mathbb{F}_p$ muls per node**	**Detection Set** $N \in \mathcal{D}$	**nodes per 10^8 muls**	**$\mathbb{F}_p$ muls per node**	**Improvement Factor**
100	1	443,951	223	$\{3, 5, 7\}$	2,165,681	46	4.9×
	2	85,034	1176	$\{2, 3\}$	2,076,517	48	24.5×
250	1	191,115	526	$\{3, 5, 7, 9, 11, 13\}$	1,607,145	62	8.4×
	2	34,083	2934	$\{4, 6\}$	1,771,608	56	52.4×
500	1	97,510	1032	$\{3, 5, 7, 9, 11, 13, 17\}$	1,260,243	79	12.9×
	2	20,239	4941	$\{4, 6\}$	1,667,360	60	82.4×
800	1	62,191	1609	$\{3, 4, 5, 6, 7, 8, 9, 11, 13, 17, 19\}$	1,096,056	86	17.6×
	2	13,228	7560	$\{4, 6\}$	1,548,504	65	116.3×

Dimension One – SuperSolver: eprint 2021/1488
Dimension Two – SplitSearcher (*UltraSolver?*): coming soon

Results

We implemented and optimised walks in $\Gamma_g(2;p)$ for the first step of attack in dimension $g = 1, 2$ with *and* without detection in $\Gamma_g(N;p)$. We ran these for different primes until reaching 10^8 $\mathbb{F}_p$ multiplications, counting number of nodes revealed and $\mathbb{F}_p$ multiplications per node revealed.

		Walks in $\Gamma_g(2;p)$ without additional detection		**Walks in $\Gamma_g(2;p)$ with additional detection in $\Gamma_g(N;p)$**			
$\log p$	**Dimension** g	**nodes per 10^8 muls**	**$\mathbb{F}_p$ muls per node**	**Detection Set** $N \in \mathcal{D}$	**nodes per 10^8 muls**	**$\mathbb{F}_p$ muls per node**	**Improvement Factor**
100	1	443,951	223	$\{3,5,7\}$	2,165,681	46	4.9×
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	2	13,228	7560	$\{4,6\}$	1,548,504	65	116.3×

Dimension One – SuperSolver: eprint 2021/1488
Dimension Two – SplitSearcher (*UltraSolver?*): coming soon
Any questions?