ON AN ERDŐS PROBLEM

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ABSTRACT. The Erdős-Straus conjecture was proposed by Paul Erdős and Ernő Straus in 1948. The conjecture asks wether, for every $n \geq 2, \in \mathbb{N}$, we can express,

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Such that, x, y and z are natural numbers. In this paper, we aim to investigate the conjecture, but generalize it to any k number of fractions. That is, *for any natural number k, can we express $\frac{2(k-1)}{n}$ as the sum of reciprocals of k natural numbers.* For $k = 1$ the solution is when $x_1 = x_2 = \frac{n}{2}$, and for $k = 2$, the conjecture reduces to the famous *Erdős-Straus Conjecture*.

In this paper, by the means of *Prime Factor Anlyasis* we show that the conjecture is true, for all $k \in \mathbb{N}$.

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1. Preliminaries

We begin by formally stating the generalized form of the Erdős–Straus-type conjecture we propose:

Conjecture I (Generalized Erdős–Straus Form)**.** *For any integer $k \in \mathbb{N}$, the rational expression $\frac{2k}{n}$ can be represented as a sum of $(k+1)$ unit fractions. That is, for every pair $(k, n) \in \mathbb{N}^2$ with $n \geq 2$, there exists a set of natural numbers $\{x_0, x_1, \ldots, x_k\} \subset \mathbb{N}$ such that:*

$$\frac{2k}{n} = \frac{1}{x_0} + \frac{1}{x_1} + \cdots + \frac{1}{x_k} = \sum_{i=0}^{k} \frac{1}{x_i}.$$

We now justify the necessity of the condition $n \geq 2$ by establishing the following lemma:

Lemma I. *For every $k \in \mathbb{N}$, the equation in Conjecture 1 admits a solution when $n = 2$.*

Proof. Substituting $n = 2$ into the conjectured identity, we aim to express:

$$\frac{2k}{2} = k$$

as a sum of $k + 1$ unit fractions of natural numbers.

We construct the following multiset S of size $k + 1$:

$$S = \underbrace{\{1, 1, \ldots, 1\}}_{k-1 \text{ times}} \cup \{2, 2\}.$$

Clearly, the sum of the reciprocals of the elements in S is:

$$\left(\sum_{i=1}^{k-1} \frac{1}{1} \right) + \frac{1}{2} + \frac{1}{2} = (k-1) + \frac{1}{2} + \frac{1}{2} = k.$$

Hence, the equality

$$k = \sum_{i=0}^{k} \frac{1}{x_i}$$

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holds true for $x_i \in S$, satisfying the conjecture for $n = 2$. This completes the proof. *(end of proof)*

One may naturally ask that, *if by Lemma 1, the conjecture holds for $n = 2$, then does it also holds for any $n \geq 2, \in \mathbb{N}$?* This is specifically what builds the *Conjecture 1*.

However, one can observe that when $n = 2p$ for some $p \in \mathbb{N}$, *Conjecture 1* holds universally. To formalize this observation, we state the following lemma:

Lemma II. *Let $k \in \mathbb{N}$. If $n = 2p$ for some $p \in \mathbb{N}$, then* Conjecture 1 *holds.*

Proof. Suppose $n = 2p$. Then the conjectured equation becomes:

$$\frac{2k}{n} = \frac{k}{p} = \sum_{i=0}^{k} \frac{1}{x_i}, \quad x_i \in \mathbb{N}.$$

This equality is satisfied by the multiset:

$$X = \underbrace{\{p, p, \ldots, p\}}_{k-1 \text{ times}} \cup \{2p, 2p\},$$

since:

$$\sum_{i=1}^{k-1} \frac{1}{p} = \frac{k-1}{p}, \quad \text{and} \quad \frac{1}{2p} + \frac{1}{2p} = \frac{1}{p},$$

yielding:

$$\frac{k-1}{p} + \frac{1}{p} = \frac{k}{p}.$$

Thus, the equality holds, and the lemma is proved. *(end of proof)*

Consequently, the scope of *Conjecture 1* reduces to the case where n is an odd natural number greater than one. We therefore restate the conjecture more precisely as follows:

Conjecture II. *Let $k \in \mathbb{N}$, and let $n > 1$ be an odd natural number. Then the quantity $\frac{2k}{n}$ can be expressed as a sum of $k + 1$ distinct unit fractions of positive integers.*

The remainder of this article is dedicated to establishing a proof of *Conjecture 2*, which by *Lemma 2*, implies the validity of *Conjecture 1* in full generality.