MULTIPLICATIVE ANALOGUE TO METHOD OF DIFFERENCES AND EULER'S DIFFERENCE FORMULAS

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Abstract. Euler introduced a method for interpolating a function from a uniformly spaced set of known values using his celebrated difference formula:

$$f(a+x)=\sum_{k=0}^{\infty}\binom{x}{k}\Delta^k f(a),$$

where the operator $\Delta^k f(a)$ denotes the k-th forward difference of the function evaluated at the point a.

This formula is profoundly effective in discrete algebraic settings, offering a systematic way to interpolate functions from finite, equally spaced data. However, in non-algebraic analytical contexts, the expansion may exhibit irregular behavior near boundaries or at certain singular points. While such anomalies can often be mitigated, a deeper generalization is desirable to extend its utility and establish broader structural properties.

In this work, we propose a **multiplicative analogue** of Euler's interpolation method. We develop a novel product expansion suited for factorial-like functions and present a **binomial analogue of the tetration function**, derived from our formulation. This approach not only broadens the algebraic scope of the classical difference method but also opens a new direction for interpolating functions in product-based discrete structures.

Contents

1. Preliminaries

Analogous to Euler's classical forward difference expansion, we propose a multiplicative formulation of function interpolation based on a newly defined *ratio operator*. The central identity we aim to establish is:

$$f(a+x)=\prod_{k=0}^{\infty}\left(\rho^k f(a)\right)^{\binom{x}{k}}, \tag{1}$$

where $\rho^k f(a)$ denotes the k-th order ratio operator applied to the function f at the base point a. The operator is defined recursively as follows:

$$\rho f(a)=\frac{f(a+1)}{f(a)}, \tag{2}$$

$$\rho^2 f(a)=\rho\left(\rho f(a)\right)=\frac{\rho f(a+1)}{\rho f(a)}, \tag{3}$$

$$\rho^k f(a)=\rho\left(\rho^{k-1} f(a)\right)=\frac{\rho^{k-1} f(a+1)}{\rho^{k-1} f(a)}. \tag{4}$$

This operator serves as a multiplicative counterpart to the forward difference $\Delta^k f(a)$, and its structure mirrors the behavior of derivatives in a ratio-based discrete system.

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To prove the identity, we consider the discrete sequence of function values:

$$\gamma = \{f(a),\ f(a+1),\ f(a+2),\ f(a+3),\ \ldots,\ f(a+n)\}$$

From this set, we compute the successive ratio operators $\rho^k f(a)$ for all $k \in \mathbb{N}$. Let us define the index set:

$$k = \{0, 1, 2, 3, \ldots, k\}$$

and the corresponding set of operator values:

$$\rho_k = \{\rho^0 f(a),\ \rho^1 f(a),\ \rho^2 f(a),\ \rho^3 f(a),\ \ldots,\ \rho^k f(a)\}$$

By observing the structure formed through repeated ratios, and inspired by the binomial coefficients from Pascal's triangle, we find that:

$$f(a+n-1) = \rho^0 f(a)^{\binom{n-1}{0}} \cdot \rho^1 f(a)^{\binom{n-1}{1}} \cdot \rho^2 f(a)^{\binom{n-1}{2}} \cdots \rho^k f(a)^{\binom{n-1}{k}}$$

which can be compactly written as:

$$f(a+n-1) = \prod_{i=0}^{k} \left(\rho^i f(a)\right)^{\binom{n-1}{i}} \tag{5}$$

$$f(a+n) = \prod_{i=0}^{k} \left(\rho^i f(a)\right)^{\binom{n}{i}} \tag{6}$$

This forms the foundation for a multiplicative analogue of Newton's interpolation, where the exponents follow binomial growth and the base values evolve via successive multiplicative differences.

Thus our proof of (1) is completed by the derivation of (6).

2. On Transformations

Consider Euler's classical forward difference formula:

$$f(a+x) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k f(a) \tag{7}$$

Let us define the original forward difference set as:

$$\Delta = \{\Delta^0 f(a),\ \Delta^1 f(a),\ \Delta^2 f(a),\ \ldots,\ \Delta^k f(a)\}$$

Now, define a transformed set Δ', which we refer to as the **first-order Pascal transform** of Δ, denoted:

$$P_1(\Delta) = \Delta' = \left\{\Delta^0 f(a),\ \Delta^0 f(a) + \Delta^1 f(a),\ \Delta^1 f(a) + \Delta^2 f(a),\ \ldots,\ \Delta^{k-1} f(a) + \Delta^k f(a),\ \Delta^k f(a)\right\}$$

This transformation resembles the additive behavior seen in Pascal's triangle, shifting the structure of the differences to cumulative adjacent pairs.

We now aim to show the following identity: If

$$f(a+x) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k f(a), \quad \Delta^k f(a) \in \Delta$$

then the cumulative sum satisfies:

$$\sum_{x=0}^{n} f(a+x) = \sum_{k=0}^{\infty} \binom{n}{k} \Delta'^k f(a), \quad \Delta'^k f(a) \in \Delta' \tag{8}$$

This provides a discrete analogue of the integral of an interpolated function, where the difference terms are transformed via P_1. This structure hints at a broader algebra of transformations over difference sets and may lead to higher-order transforms $P_2, P_3, \ldots$, each corresponding to deeper combinatorial structures.

To prove this transformation identity, we consider the forward differences of the function $f(x)$, denoted by the set γ, defined as:

$$\gamma = \left\{ f(a),\ f(a+1) - f(a),\ f(a+2) - 2f(a+1) + f(a),\ \ldots,\ \sum_{i=0}^{n} \binom{n-1}{i} f(a+i)(-1)^{n-i-1} \right\}$$

In general, the k-th term of this sequence is given by:

$$\Delta^k f(a) = \sum_{i=0}^{k-1} \binom{k-1}{i} f(a+i)(-1)^{k-i-1} \tag{9}$$

Now, consider the cumulative sum sequence of the function:

$$\left\{ f(a),\ f(a) + f(a+1),\ f(a) + f(a+1) + f(a+2),\ \ldots,\ \sum_{i=0}^{n} f(a+i) \right\}$$

We denote the forward differences of this sequence by the set γ'. These differences can be partitioned into three subsets:

$$\gamma' = \gamma_1 \cup \gamma_2 \cup \gamma_3 \tag{10}$$

$$\gamma_1 = \{f(a)\} \tag{11}$$

$$\gamma_2 = \left\{ f(a+1),\ f(a+2) - f(a+1),\ f(a+3) - 2f(a+2) + f(a+1),\ \ldots,\ \sum_{i=0}^{n-1} \binom{n-1}{n-i-1} f(a+i+1)(-1)^{n-i-1} \right\} \tag{12}$$

$$\gamma_3 = \left\{ \sum_{i=0}^{n} \binom{n-1}{i} f(a+i)(-1)^{n-i-1} \right\} \tag{13}$$

Notably, the k-th term of the central subset γ_2 is given by the expression:

$$\Delta'^k = \sum_{i=0}^{k-1} \binom{k-1}{i} f(a+i+1)(-1)^{k-i-1} \tag{14}$$

We now demonstrate the elegant identity:

$$\Delta^k f(a) + \Delta^{k+1} f(a) = \Delta'^k f(a)$$

where $\Delta'^k f(a)$ denotes the k-th term in the forward difference sequence of the cumulative sum $S(x) = \sum_{j=0}^{x} f(j)$. Assuming $k+1 \leq n$, we expand the terms explicitly.

First, consider the standard expressions for the k-th and $(k+1)$-th forward differences of f:

$$\Delta^k f(a) = \sum_{i=0}^{k-1} \binom{k-1}{i} f(a+i)(-1)^{k-i-1} \tag{15}$$

$$\Delta^{k+1} f(a) = \sum_{i=0}^{k} \binom{k}{i} f(a+i)(-1)^{k-i} \tag{16}$$

Adding both expressions:

$$\Delta^k f(a) + \Delta^{k+1} f(a) = \sum_{i=0}^{k-1} \binom{k-1}{i} f(a+i)(-1)^{k-i-1} + \sum_{i=0}^{k} \binom{k}{i} f(a+i)(-1)^{k-i} \tag{17}$$

$$= \sum_{i=1}^{k-1} \left[\binom{k-1}{i} - \binom{k}{i} \right] f(a+i)(-1)^{k-i-1} + f(a+k) \tag{18}$$

Using the Pascal identity:

$$\binom{k-1}{i} - \binom{k}{i} = -\binom{k-1}{i-1}$$

we get:

(19)

$$\Delta^k f(a) + \Delta^{k+1} f(a) = \sum_{i=1}^{k-1} \left[-\binom{k-1}{i-1} \right] f(a+i)(-1)^{k-i-1} + f(a+k) = \sum_{i=1}^{k-1} \binom{k-1}{i-1} f(a+i)(-1)^{k-i} + f(a+k)$$

Changing the index in the summation:

$$= \sum_{i=0}^{k-2} \binom{k-1}{i} f(a+i+1)(-1)^{k-i-1} + f(a+k) = \sum_{i=0}^{k-1} \binom{k-1}{i} f(a+i+1)(-1)^{k-i-1} \tag{20}$$

This final expression is precisely the definition of $\Delta'^k f(a)$, hence:

$$\Delta^k f(a) + \Delta^{k+1} f(a) = \Delta'^k f(a)$$

This gives a relation between the central set γ_2 and the set γ, mirroring our first-order Pascal Transformation, and hence proving (8).

Thus, let

$$\underbrace{P_k\big(P_k\big(\cdots P_k(\Delta)\cdots\big)\big)}_{n \text{ times}} = \Delta_k^n$$

Then, we can write:

$$\underbrace{\sum_{x=0}^{n} \sum_{x=0}^{n} \cdots \sum_{x=0}^{n}}_{s \text{ times}} f(a+x) = \left(\sum_{x=0}^{n} \right)^s f(a+x)$$

$$= \sum_{k=0}^{\infty} \binom{x}{k} \Delta_k f(a), \quad \Delta_k f(a) \in \Delta_1^s \tag{21}$$

A similar identity for (1) can be established if we redefine the transform $P_2(\Delta)$ to perform **multiplication** instead of **addition**, as in Pascal's triangle. That is, assume a set:

$$A = \{a_0, a_1, a_2, a_3, a_4, \ldots, a_n\}$$

Then define:

$$P_2(A) = \{a_0,\ a_1 \cdot a_0,\ a_2 \cdot a_1,\ \ldots,\ a_n \cdot a_{n-1},\ a_n\}$$

Now, we define another such identity. Let:

$$\Delta_2 = \{\rho^0 f(a),\ \rho^1 f(a),\ \ldots,\ \rho^n f(a)\}$$

Let the iterative application of the transform P_2 on Δ_2, applied s times, be denoted by Δ_2^s. Then:

$$\underbrace{\prod_{x=0}^{n}\prod_{x=0}^{n}\cdots\prod_{x=0}^{n}}_{s\text{ times}} f(a+x) = \left(\prod_{x=0}^{n}\right)^{s} f(a+x)$$

$$= \prod_{k=0}^{\infty} \Delta_k f(a)^{\binom{x}{k}}, \quad \Delta_k f(a) \in \Delta_2^s \tag{22}$$

3. Factorial-like Functions and Tetrational Expansion

(7) does not expand the tetrational series and exhibits irregular behavior at certain points on the real line. To achieve a more accurate approximation of the factorial-like function, we instead utilize (1), expressing the function as an infinite product.

Using these means one can obtain a product expression for the tetration function, where if ${}^{k}a$ is the tetration of a as a base with a tower height of k, then one may find, using the product formula and methods in this paper that,

$${}^{n+1}a = \prod_{t=0}^{\infty}\prod_{i=0}^{t}\left({}^{i+1}a\right)^{(-1)^{t-i}\cdot\binom{t}{i}\cdot\binom{n}{t}} \tag{23}$$

The expression (23) is a purely algebraical expression for the tetration function analogous to the binomial expansion for exponents.

Similarly for Factorial-like functions, one may use Pascal's second order transform and the product formula (both were discussed above in this paper) to conclude that,

$$\prod_{i=0}^{n}(a+i) = a \cdot \prod_{t=0}^{\infty}\prod_{i=0}^{t}(1+a+i)^{(-1)^{t-i}\cdot\binom{t}{i}\cdot\binom{n}{t+1}} \tag{24}$$

The expression (24) is our approximation of the factorial function, however, this may misbehave at some points, as already known in interpolation theory.

References

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