ON THE SQUARE PEG PROBLEM

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Abstract. The *Square Peg Problem*, introduced by *Otto Toeplitz* in 1911, asks whether every simple closed Jordan curve C inscribes a square—that is, whether four points on C can form the vertices of a square. To approach this, we introduce the notion of 4-Δ sets and rotations, as a special case of the more general ${}^{n}\Delta_k$ configurations, defined as sets of k lines in n-dimensional space with equal mutual angles.

For $n = 2$ and $k = 4$, this configuration consists of two perpendicular line pairs (angle $\pi/2$). We prove that the continuous rotation of such a configuration over a Jordan curve causes the vertical and horizontal line pairs to swap positions. By the *Intermediate Value Theorem*, this implies the existence of a moment when vertical and horizontal lines coincide in length, thus forming the diagonals of an inscribed square.

Contents

1. Preliminaries

We begin by introducing the notion of the ${}^{n}\Delta_k$ rotation and associated sets.

Let $S \subseteq \mathbb{R}^n$ be a set of points in n-dimensional Euclidean space. The ${}^{n}\Delta_k$ configuration is defined as a system of k lines in $\mathbb{R}^n$ that are mutually equiangular and intersect at a single point $P(n, k)$, referred to as the **configuration point**.

Corresponding to this configuration, we define the ${}^{n}\Delta_k$ sets as a collection of $k + 1$ independent sets:

- k of these are termed *parametric sets*, each representing the lengths assigned to the respective k lines of the configuration.
- The remaining set is called the *angular set*, which governs the direction or orientation of the entire configuration in space.

At each instance of rotation determined by an element of the angular set, the k lines act as position vectors with respective magnitudes from the parametric sets. Together, they describe a subset of points in $\mathbb{R}^n$ relative to the configuration point $P(n, k)$.

As we rotate the configuration, at the point $P(n, k)$ by the angles in the angular set in order, we refer to this as the *Delta Rotation*.

In this paper, we primarily consider the specific configuration ${}^{2}\Delta_4$, representing two mutually perpendicular lines in the plane intersecting at a point $P(2, 4)$, which we shall simply denote by P. For ease of reference, we denote the configuration ${}^{2}\Delta_4$ by Δ throughout the paper.

Now, we formally state the central conjecture under consideration:

Conjecture I (Square Peg Problem)**.** *Every simple, closed, non-self-intersecting Jordan curve C inscribes a square; that is, there exists a square whose four vertices all lie on the boundary of C.*

Conjecture 1, widely known as the *Square Peg Problem*, was first posed by *Otto Toeplitz* in 1911 and remains one of the most intriguing open problems in elementary geometry.

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2. On Δ Rotations

Throughout this section, we assume that C denotes an arbitrary, closed, simple, and non-self-intersecting Jordan curve in the Euclidean plane $\mathbb{R}^2$. Let P be a point strictly contained in the interior of the region enclosed by C, and let P be referred to as the *configuration point.* We define a Δ-*configuration* at P as a pair of mutually perpendicular lines intersecting at P, extending in all four directions, such that each of the four resulting rays intersects the boundary of C.

Proposition I. *Let $\theta \in [0, 2\pi)$ be any arbitrary angle representing the orientation of a Δ-configuration (i.e., rotation of the perpendicular pair about P). Then, for each such θ, all four rays of the configuration touch the boundary of C in finite, non-zero lengths. That is, none of the four segments connecting P to C along the rays have zero measure.*

Proof. Since P lies strictly inside the Jordan curve C, and C is a simple, closed curve, it follows from the Jordan curve theorem that any ray emanating from P must intersect the boundary of C exactly once before escaping the interior. Given a configuration consisting of two perpendicular lines passing through P, these define four rays in distinct directions. Each of these rays, by the simplicity and closedness of C, must intersect the boundary at a unique point. Consequently, for each angle θ, the segments of the rays from P to their respective points of intersection with C are finite and non-zero in length. Hence, none of the four directional segments in the Δ-configuration can have zero measure. And hence, must have a possible measure so that it touches the boundary of C. *(end of proof)*

By Proposition 1, we have established that for every orientation θ, all four segments in the Δ-configuration are non-degenerate. We now turn to the question of *continuity* of these segment lengths as a function of rotation. Specifically, we ask: as θ varies continuously, do the corresponding segment lengths also vary continuously?

Proposition II. *Let C and P be as defined above. Then the mapping from orientation angle $\theta \in [0, 2\pi)$ to the vector of four segment lengths in the corresponding Δ-configuration is continuous. That is, small changes in orientation produce small changes in segment lengths.*

Proof. Let T_θ denote the Δ-configuration obtained by orienting the pair of perpendicular lines through P at angle θ. Suppose for the sake of contradiction that the mapping $\theta \mapsto T_\theta$ is not continuous. Then there exists some θ_0 and an $\varepsilon > 0$ such that for every $\delta > 0$, there exists θ' with $|\theta' - \theta_0| < \delta$, yet at least one of the segment lengths in $T_{\theta'}$ differs from that in T_{θ_0} by more than ε.

However, each segment length in T_θ is determined by the distance from P to the intersection point of a ray (in direction determined by θ) with the boundary of C. Since C is a compact, continuous curve and the ray varies continuously with θ, the intersection point—and hence the length—varies continuously with θ. Thus, the segment lengths form continuous functions of θ.

Therefore, our assumption of discontinuity leads to a contradiction, and we conclude that the segment lengths of the Δ-configuration vary continuously as the orientation θ varies. This establishes the desired result. *(end of proof)*

Building upon the propositions established in the previous section, we now pose a fundamental question concerning the transition between distinct Δ-configurations within a fixed Jordan curve.

Question I. *Given two distinct Δ-configurations at the same configuration point P within a Jordan curve C, does there exist a connected angular subset $\Theta \subseteq [0, 2\pi)$ and corresponding parametric subsets such that a continuous rotation through Θ induces a continuous transition between the two configurations?*

This question encapsulates the essence of rotational continuity within Δ-configurations. Specifically, we are interested in determining whether it is always possible to rotate a given configuration smoothly to another—most notably, to its *swapped* configuration, in which the original horizontal and vertical lines exchange roles.

If such a continuous transition exists, it implies the existence of a specific angle of rotation, denoted θ^*, at which the configuration is symmetric with respect to the swap—that is, the lengths of the horizontal and vertical segments (measured from P to their respective intersections with the boundary of C) are equal.

Let us define a real-valued function $f(\theta)$ representing the signed difference between the total lengths of the horizontal and vertical segments in the Δ-configuration oriented at angle θ. That is,

$$f(\theta) := \ell_H(\theta) - \ell_V(\theta),$$

where $\ell_H(\theta)$ and $\ell_V(\theta)$ denote the total lengths of the horizontal and vertical segments, respectively, in the configuration at orientation θ.

From the continuity of the segment lengths established earlier, it follows that both $\ell_H(\theta)$ and $\ell_V(\theta)$ are continuous functions of θ. Therefore, $f(\theta)$ is also continuous on $[0, 2\pi)$.

Suppose the initial configuration corresponds to an angle θ_0 with $f(\theta_0) > 0$, and the swapped configuration corresponds to an angle θ_1 with $f(\theta_1) < 0$. Then, by the Intermediate Value Theorem, there exists some $\theta^* \in (\theta_0, \theta_1)$ such that $f(\theta^*) = 0$.

This implies that at orientation θ^*, the horizontal and vertical segments are of equal length. Due to the orthogonality of the lines in a Δ-configuration, this equality implies that the configuration forms the diagonals of a square inscribed within the curve C and centered at P. And hence proves the *Conjecture 1.*

2.1. **On Δ Successors and Predecessors.** To address *Question 1*, it becomes necessary to rigorously examine the local behavior of Δ-configurations under continuous rotation. In particular, we introduce the notions of *successor* and *predecessor* configurations associated with a given configuration.

Definitions. Let C be a closed, simple, non-self-intersecting Jordan curve, and let P be a fixed configuration point in the interior of C. Consider a Δ-configuration d_0 oriented at some angle $\theta_0 \in [0, 2\pi)$. As we vary the orientation θ continuously in a neighborhood of θ_0, the corresponding Δ-configurations trace a smooth path in the configuration space.

Definition I. *A* Δ-successor *of a configuration d_0 is defined as a local configuration d_+ obtained by a small positive perturbation of the angle θ_0, i.e., $d_+ = d(\theta_0 + \varepsilon)$ for some sufficiently small $\varepsilon > 0$ (or the configuration which approaches d_0 from left-hand-side, is called successor configuration).*

Definition II. *Similarly, a* Δ-predecessor *of d_0 is the local configuration d_- obtained by a small negative perturbation of θ_0, i.e., $d_- = d(\theta_0 - \varepsilon)$ for some sufficiently small $\varepsilon > 0$ (or the configuration which approaches d_0 from the left-hand-side is called predecessor configuration).*

Note that these terms are not defined with respect to clockwise or counterclockwise rotation in the ambient space, but rather with respect to the one-dimensional topology of the angular parameter space. Since the parameter θ belongs to a connected, oriented interval (e.g., $[0, 2\pi)$ or $\mathbb{R}/\pi\mathbb{Z}$), each configuration has a well-defined left-hand and right-hand neighborhood in the sense that either the rotations needed to achieve either of the neighbourhood is different from each other fundamentally, or the configurational lengths of angles in the neighbourhoods are different, from which we derive the notions of predecessor and successor. The terms successor and predecessor configurations aren't absolute but relative to the type of rotation. If we are rotating in opposite direction, the successor and predecessor might swap themselves.

3. On the Approach

We now turn our attention to *Conjecture 1.* To investigate it, let us consider an arbitrary Jordan curve C, and a point P located in its interior. We construct two perpendicular lines passing through P, each intersecting the boundary of C. This initial setup is referred to as the *initial configuration*, denoted by c_0. By construction, both lines intersect at the point P.

As previously defined, we introduce the notion of a Δ-rotation of c_0, which refers to the simultaneous motion of the two lines along the boundary of C, while preserving both their orthogonality and intersection at P. As the configuration evolves through this Δ-rotation, each new position is called a *successor configuration*, while prior positions are *predecessor configurations.*

We now introduce a key concept, called a *restriction.* Suppose that after a certain amount of Δ-rotation, the configuration reaches a position in which it is no longer possible to continue rotating in the initial angular direction. In this case, we refer to this situation as a *restriction* in the motion of the configuration.

Proposition III. *A restriction can occur only if at least one of the four endpoints of the configuration lies tangent to a convex region of the curve C; specifically, at a local extremum of a locally upward (or downward) region.*

Proof. By the definition of a restriction, suppose that the configuration reaches a position c_1 and cannot proceed further in the direction of angular increase, say θ. Let this configuration be at angular coordinate θ_f. By continuity (as established in Proposition 2), the existence of the restriction implies that at least one of the four endpoints of c_1 has no corresponding continuation at angle $\theta_f + d\theta$ for any infinitesimal increment $d\theta > 0$.

However, since the curve C is continuous, a neighboring point must exist, and it must lie on the opposite side, at angle $\theta_f - d\theta$. Due to the closure of C, the configuration at this neighboring angular position cannot coincide with the initial configuration c_0, and hence must itself possess both a valid successor and predecessor.

This behavior characterizes a situation in which one endpoint of c_1 is tangent to a locally convex region of the curve. In such a case, the configuration reaches a local extremum—where the tangent line cannot move further in the current angular direction without violating the orthogonality or containment conditions. Thus, the predecessor configuration exists at $\theta_f - d\theta$, and any valid continuation must emerge on the opposite side.

(end of proof)

Hence, by Proposition 3, we conclude that restrictions in the Δ-rotation process can occur only at tangential points lying on convex regions of C, relative to the center of configuration P. Moreover, even at such points, a successor configuration must still exist, as dictated by the nature of convex regions in continuous curves: a local extremum must, by continuity, be flanked by regions with both positive and negative slope. Therefore, a configuration arriving from the side of a positive slope must be able to continue onto the side of a negative slope, or vice versa.

Hence for the sake of our proof, we propose another preposition:

Proposition IV. *For curve C having the properties as described above, if initiated by an initial configuration $_0$ such that c_0 has a predecessor as well as a successor configuration. Then as we Δ-Rotate the configuration c_0 it everytime may produce an unique configuration.*

Proof. For some initial configuration c_0 which has a successor as well as a predecessor configuration. Now we know that, every final configuration c_i afterwards has a predecessor configuration for certain. Now we assume that for some final configuration c_n there is a *(end of proof)*