NNDL:

Problem Set #4

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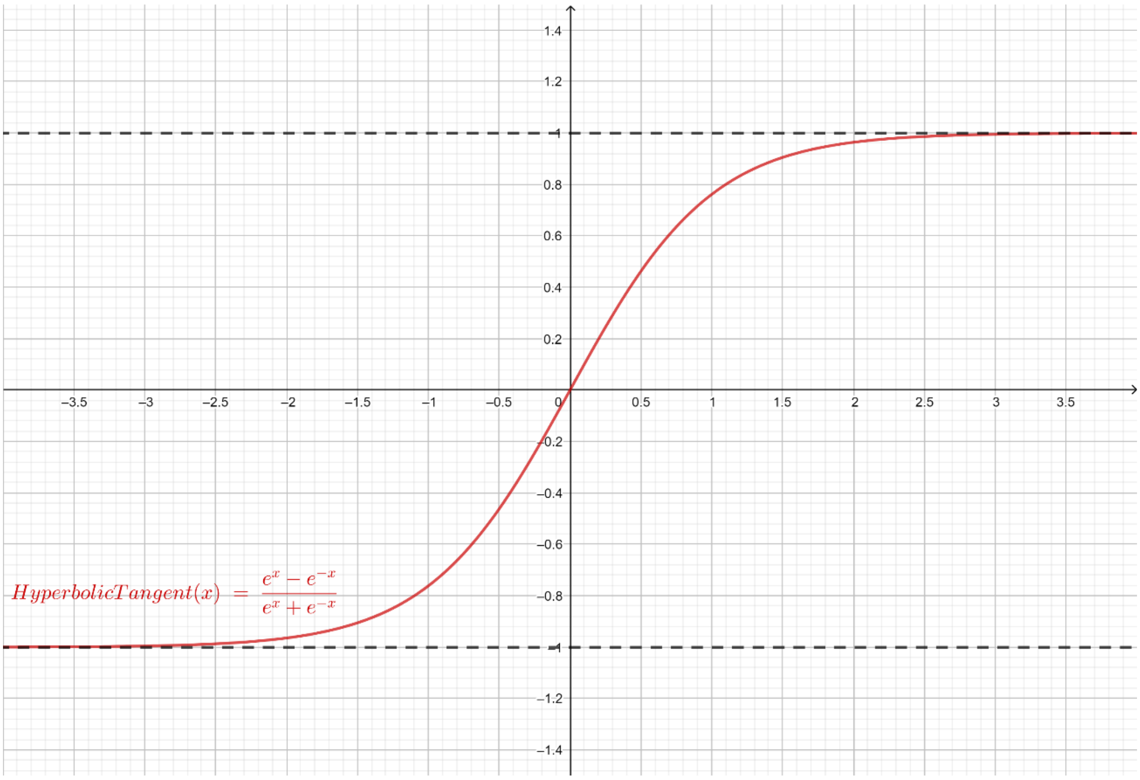
Problem 1: Hyperbolic Tangent

Sketches done using: [GeoGebra]

1. Sketch the function



2. Differentiate the function (give the formula for df/dx)

$$f(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d\,f}{d\,x} = \frac{d\ \tanh(x)}{d\,x} = \frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Use the Quotient Rule to derivate:

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\frac{d}{d\,x}(e^x - e^{-x}) * (e^x + e^{-x}) - (e^x - e^{-x}) * \frac{d}{d\,x}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

Now we need to differentiate two smaller functions:

$$\frac{d}{d\,x}(e^x - e^{-x})$$

$$\frac{d}{d\,x}(e^x + e^{-x})$$

Use the Sum and Difference Rule:

$$\frac{d}{d\,x}(e^x - e^{-x}) = \frac{d}{d\,x}e^x - \frac{d}{d\,x}e^{-x}$$

$$\frac{d}{d\,x}(e^x + e^{-x}) = \frac{d}{d\,x}e^x + \frac{d}{d\,x}e^{-x}$$

We now need to derivate 4 Expontials, that are repeated, so only 2 are needed:

$$\frac{d}{d\,x}e^x$$

$$\frac{d}{d\,x}e^{-x}$$

Now we derivate the Exponentials, Using the Exponential Rule:

$$\frac{d}{d\,x}e^{g(x)} = e^{g(x)} * g'(x)$$

$$\frac{d}{d\,x}e^x = e^x * 1 = e^x$$

$$\frac{d}{d\,x}e^{-x} = e^{-x} * (-1) = -e^{-x}$$

Now we need to go back and combine everything:

$$\frac{d}{d\,x}(e^x - e^{-x}) = \frac{d}{d\,x}e^x - \frac{d}{d\,x}e^{-x} = e^x - (-e^{-x}) = e^x + e^{-x}$$

$$\frac{d}{d\,x}(e^x + e^{-x}) = \frac{d}{d\,x}e^x + \frac{d}{d\,x}e^{-x} = e^x + (-e^{-x}) = e^x - e^{-x}$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\frac{d}{d\,x}(e^x - e^{-x}) * (e^x + e^{-x}) - (e^x - e^{-x}) * \frac{d}{d\,x}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x}) * (e^x + e^{-x}) - (e^x - e^{-x}) * (e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

Since:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{[(e^x + e^{-x})^2 - (e^x - e^{-x})^2] * \frac{1}{4}}{(e^x + e^{-x})^2 * \frac{1}{4}} = \frac{\frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}}{\frac{(e^x + e^{-x})^2}{4}}$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\frac{(e^x + e^{-x})^2}{2^2} - \frac{(e^x - e^{-x})^2}{2^2}}{\frac{(e^x + e^{-x})^2}{2^2}} = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}$$

And:

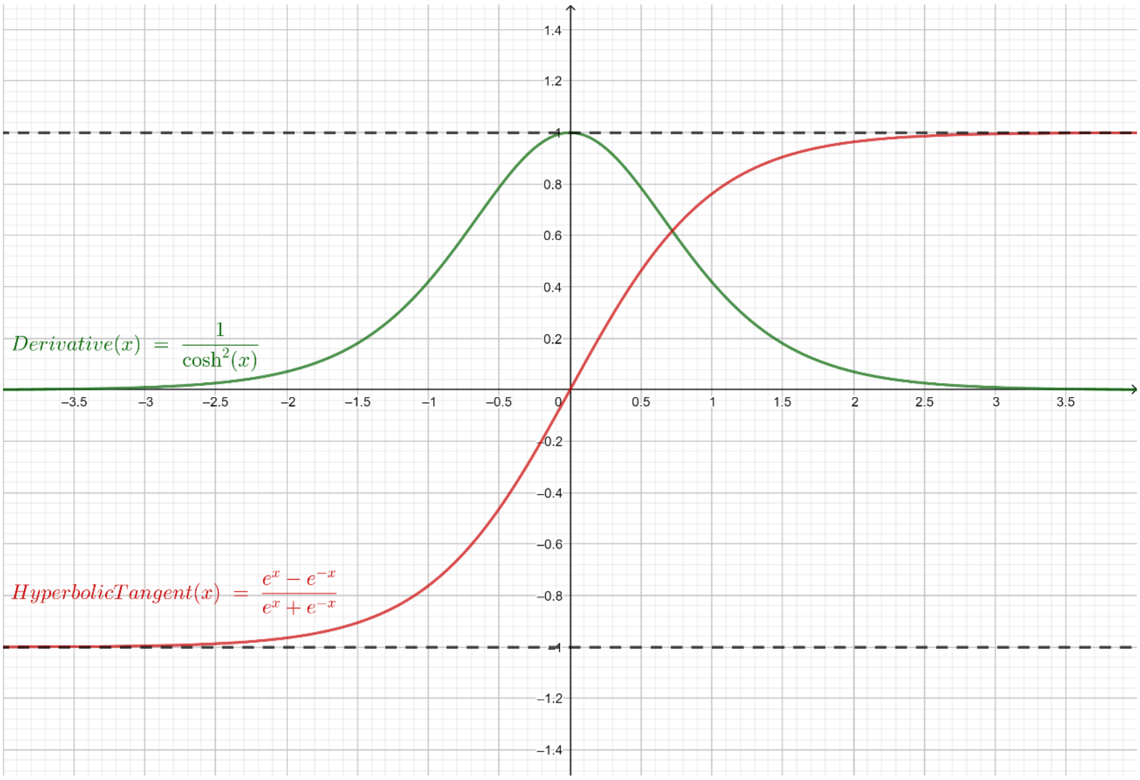
$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\frac{d}{d\,x}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{\cosh^2(x)}$$

3. Identify any tricky points in taking the derivative and propose a reasonable solution

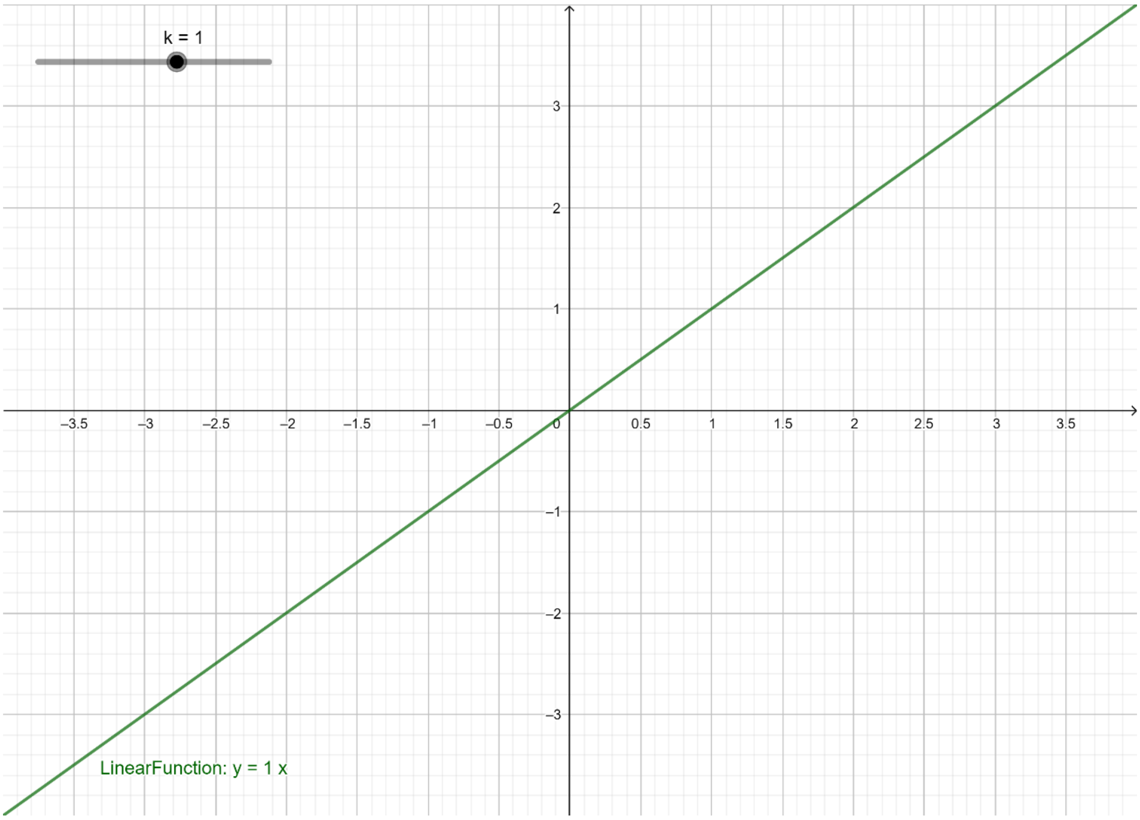
In this Activation function there are no Tricky Points, and thus no solution for them is needed

4. Sketch the derivative



Problem 2: Linear Function

1. Sketch the function



2. Differentiate the function (give the formula for df/dx)

$$f(x) = k * x$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x} k * x$$

Using:

$$\frac{d}{d\,x} k * x^n = k * n * x^{n-1}$$

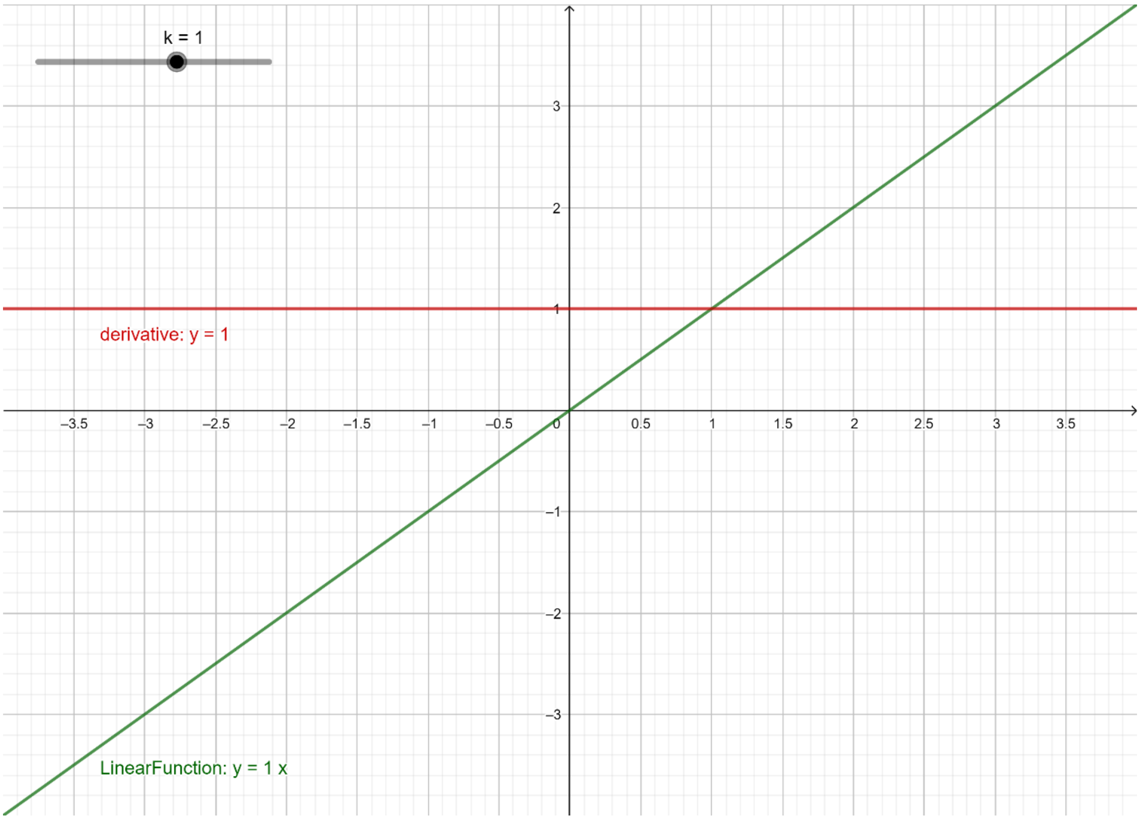
$$\frac{d}{d\,x} k * x = k * 1 * x^0 = k * 1 * 1 = k$$

$$\frac{d}{d\,x} k * x = k$$

3. Identify any tricky points in taking the derivative and propose a reasonable solution

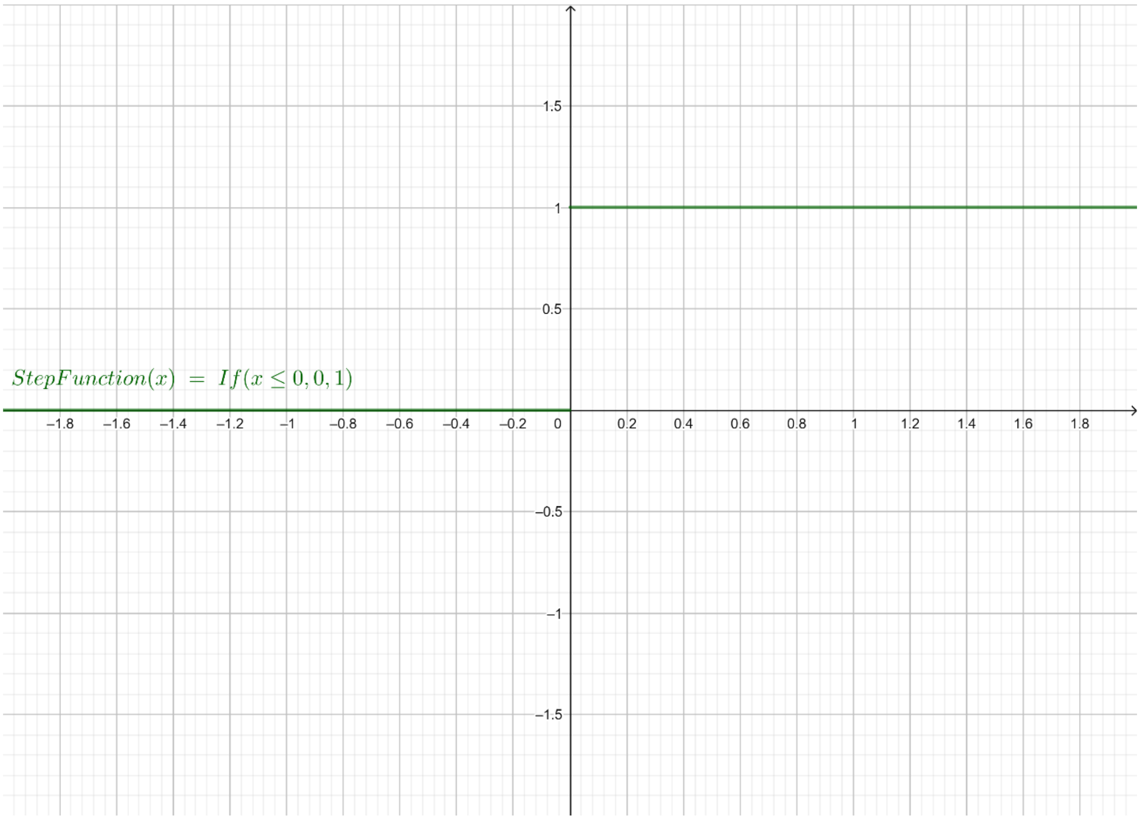
In this Activation function there are no Tricky Points, and thus no solution for them is needed

4. Sketch the derivative



Problem 3: Step Function

1. Sketch the function



2. Differentiate the function (give the formula for df/dx)

$$f(x) = \begin{cases} 0 \ if \ x \leq 0 \\ 1 \ if \ x > 0 \end{cases}$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\begin{cases} 0 \ if \ x \leq 0 \\ 1 \ if \ x > 0 \end{cases}$$

In order to do the derivative, we need to do it by its parts. Since both cases are a constant the derivative is 0 in both cases:

$$\frac{d}{d\,x}c = 0$$

$$\frac{d}{d\,x}\begin{cases} 0 \ if \ x \leq 0 \\ 1 \ if \ x > 0 \end{cases} = \begin{cases} 0 \ if \ x < 0 \\ 0 \ if \ x > 0 \end{cases}$$

Since the step function is discontinue in x=0 we can’t derivate in that point

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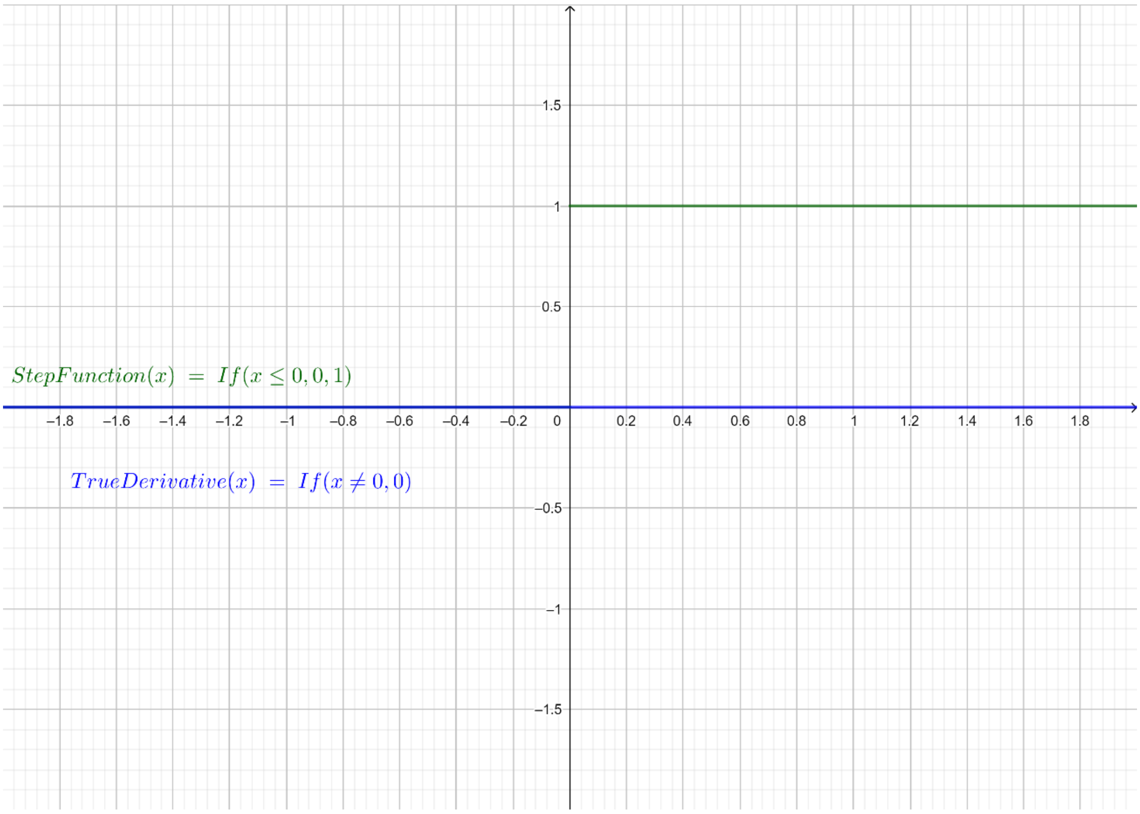
3. Identify any tricky points in taking the derivative and propose a reasonable solution

There is a discontinuity in x=0. Since the derivative is 0 in all points except for x=0, we can interpolate a 0 into that point in order to keep the derivative continuous.

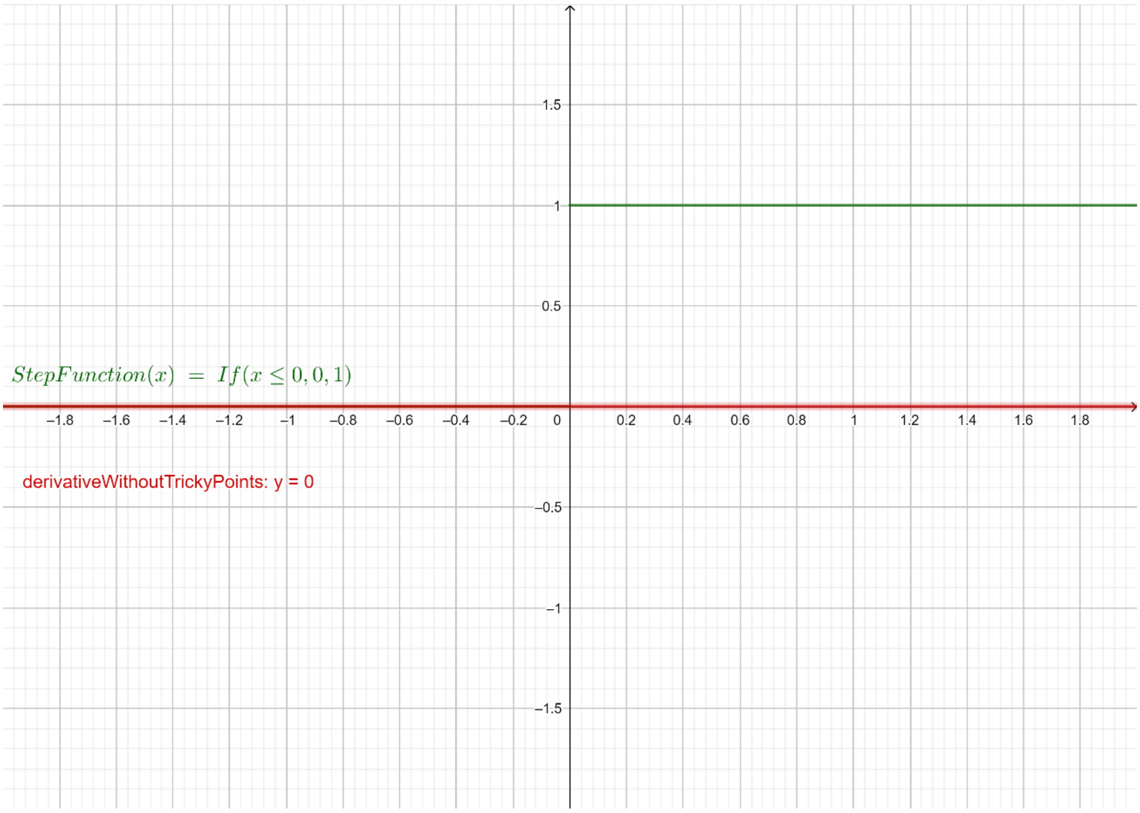
Another option is to put a 1 in x=0, that way the derivative indicates a change in original function in x=0. The problem with this approach is that the derivative is not continuous.

4. Sketch the derivative

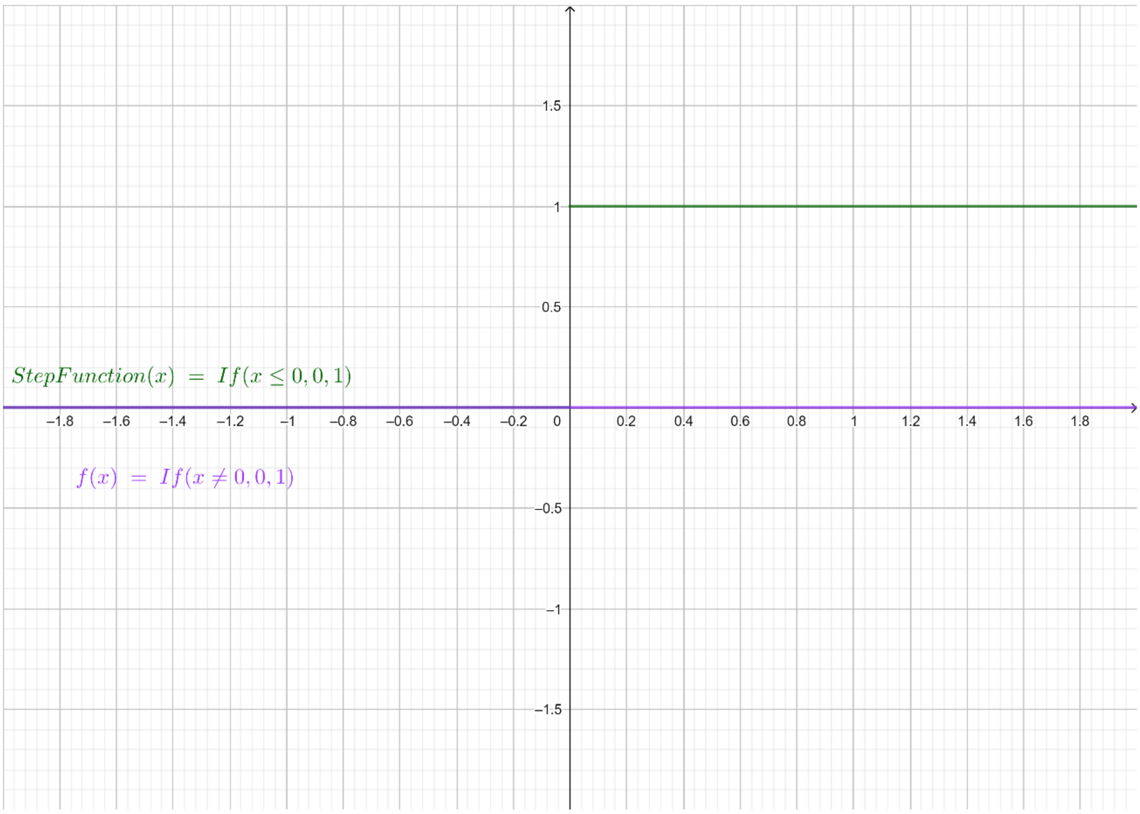
True Derivative: (Due to 0 being an infinitesimaly small point, we can't see the gap)



Derivative with a 0 in x=0, in relation with point 3

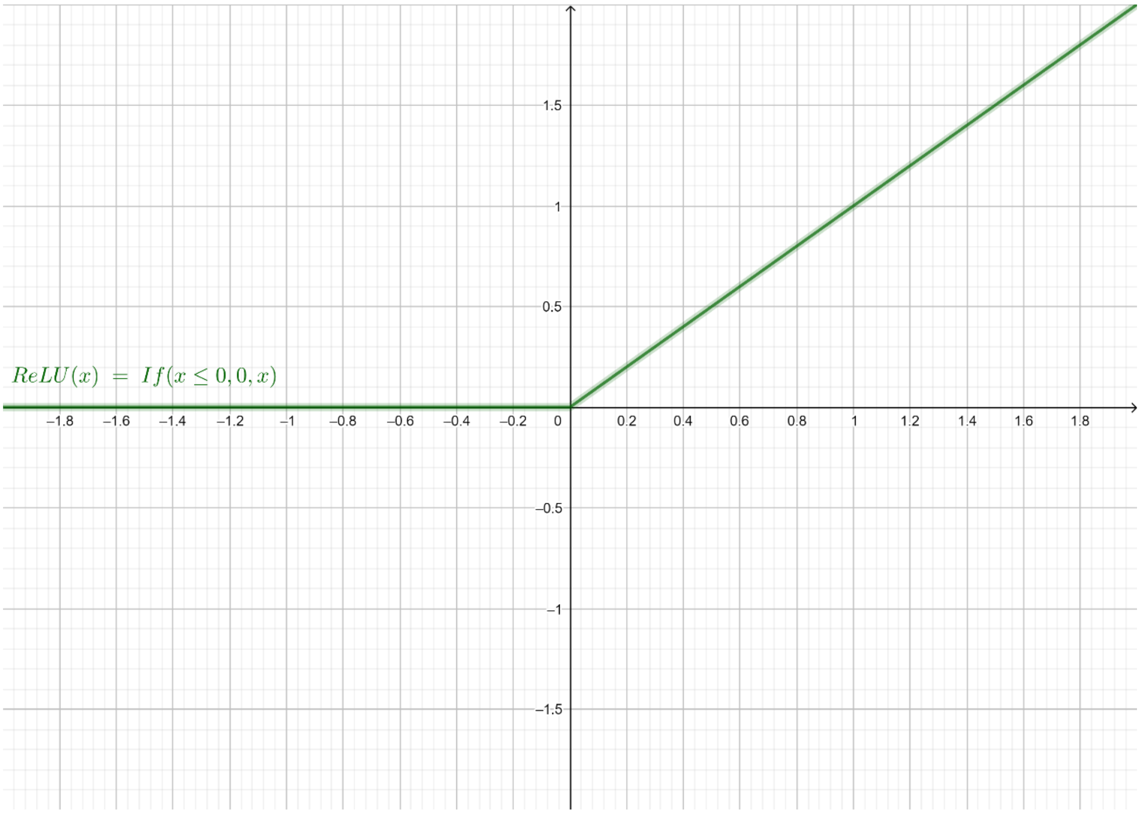


Derivative with a 1 in x=0, in relation with point 3 (We can't see that point, because is infinitesimaly small)



Problem 4: Rectified Linear Unit (ReLU)

1. Sketch the function



2. Differentiate the function (give the formula for df/dx)

$$f(x) = \begin{cases} 0 \; if \; x \leq 0 \\ x \; if \; x > 0 \end{cases}$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x} \begin{cases} 0 \; if \; x \leq 0 \\ x \; if \; x > 0 \end{cases}$$

In order to do the derivative, we need to do it by its parts.

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\, 0 \; if \; x \leq 0$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\, x \; if \; x > 0$$

The derivative of a constant is 0 (Problem 3), and the derivative of a line is the slope (Problem 2):

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\, 0 \; if \; x \leq 0 = 0 \; if \; x < 0$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\, x \; if \; x > 0 = 1 \; if \; x > 0$$

$$\frac{d}{d\,x}\begin{cases} 0\ if\ x \leq 0 \\ x\ if\ x > 0 \end{cases} = \begin{cases} 0\ if\ x < 0 \\ 1\ if\ x > 0 \end{cases}$$

The function is continuous in x=0 but it's not derivable since:

$$\lim_{x \to 0^-} f'(x) = 0$$

$$\lim_{x \to 0^+} f'(x) = 1$$

3. Identify any tricky points in taking the derivative and propose a reasonable solution

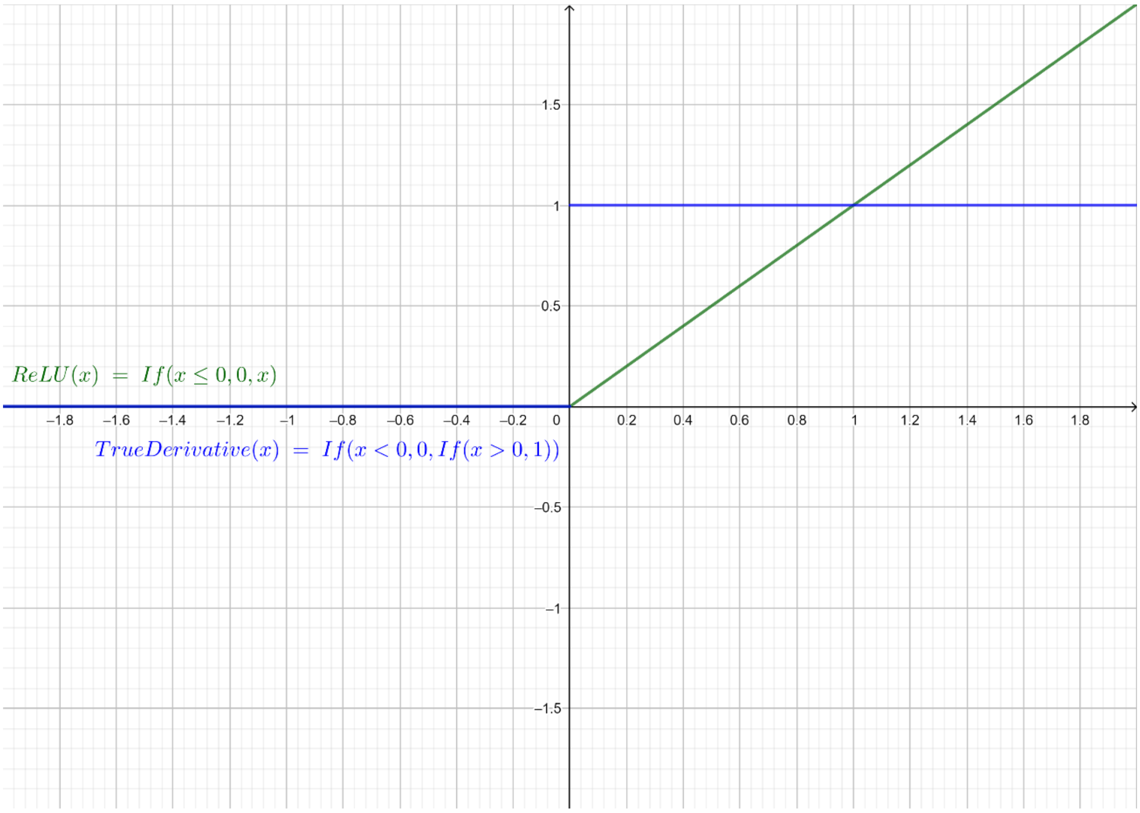
There is a tricky point in x=0, since the function is continuous but it's not derivable.

One option is to put one of the border cases. 0 or 1 for x=0

Another option will be to use and intermediate value, like 0.5, but this is not a good option, since it will split the derivative into 3 segments, adding unnecessary complexity.

4. Sketch the derivative

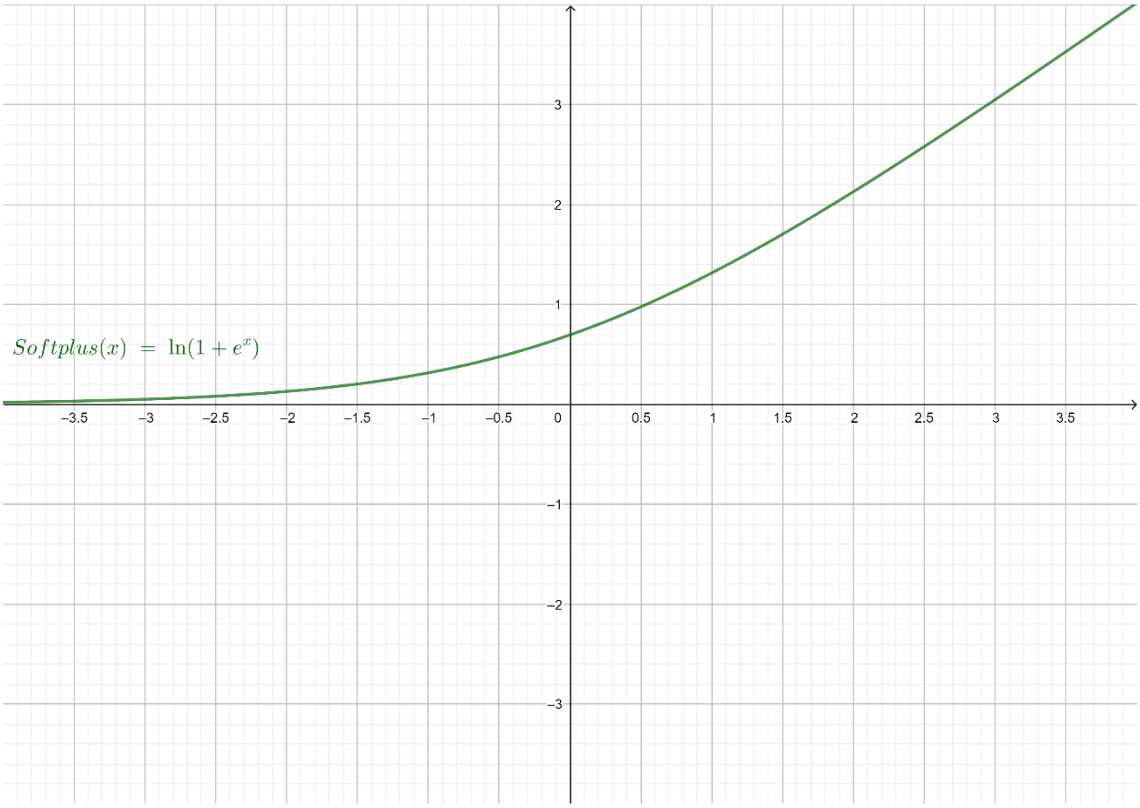
True Derivative:



Since changing the point of x=0, to 0, 1 or 0.5 won't be visible on the graph, I'm not going to draw them

Problem 5: Softplus

1. Sketch the function



2. Differentiate the function (give the formula for df/dx)

$$f(x) = \ln(1 + e^x)$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\ln(1 + e^x)$$

Since it's a logarithm we need to apply the rule for logarithms:

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\ln(g(x)) = \frac{1}{g(x)} * g'(x)$$

$$\frac{d\,f}{d\,x} = \frac{d}{d\,x}\ln(1 + e^x) = \frac{1}{1 + e^x} * \frac{d}{d\,x}(1 + e^x)$$

$$\frac{d}{d\,x}(1 + e^x) = \frac{d}{d\,x}1 + \frac{d}{d\,x}e^x$$

$$\frac{d}{d\,x}1 + \frac{d}{d\,x}e^x = 0 + e^x = e^x$$

We can conclude that:

$$\frac{d}{d\,x}\ln(1 + e^x) = \frac{1}{1 + e^x} * e^x = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}$$

3. Identify any tricky points in taking the derivative and propose a reasonable solution

There are no tricky points, as the function is continuous and derivable in all its points

4. Sketch the derivative