Mathematical Morphology

Idea of MM

- Mathematical Morphology
  - Combines two ideas
- Morphology part
  - The study of shape
- Mathematical part
  - Refers to the use of "Set Theory"

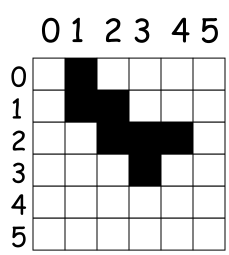
Basic Definition

A tuple is a fixed fixed-length list containing elements that need not have the same type. It can be, and often is, used as a key-value pair.

- Consider a binary image
  - This can be thought of as a *set* of tuples
  - tuples in 2-D integer space Z^2
  - each tuple is a 2-D vector whose coordinates are the (x,y) values of the black pixels
    - Black pixels are used by convention
      - Considered the "foreground"
    - You can modify this to be the white pixels

Example

Binary Image



A = { (1,0), (1,1), (2,1), (2,2), (3,2), (4,2), (3,3) }

In practice, we use the image itself to represent this set.

Basic Definitions

- Let A and B be sets in Z^2
  - with components a=(a1, a2) and b=(b1,b2), respectively

  – The *translation* of A by x = (x1,x2), denoted by $(A)_x$ is defined by:

$$(A)_x = \{c \mid c = a + x, \quad \text{for } a \in A\}$$

Basic Definitions

- The *reflection* of B, denoted by $\hat{B}$ is defined as:

$$\hat{B} = \{x \mid x = -b, \quad \text{for } b \in B\}$$

- The *complement* of set A is:

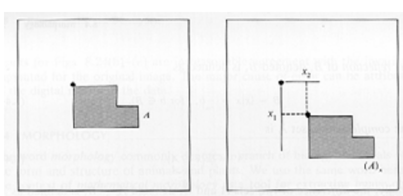
$$A^c = \{x \mid x \notin A\}$$

Basic Definitions

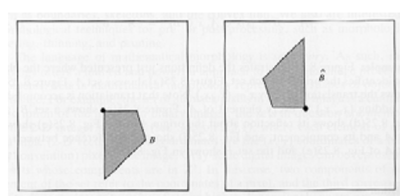
- The *difference* of two sets A and B, denoted by A-B, is defined by:

$$A - B = \{x \mid x \in A, x \notin B\} = A \cap B^c$$

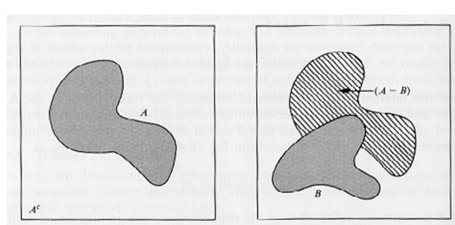
Example



Translation



Reflection



Difference

Dilation Operator

- With A and B as sets in Z^2 and $\emptyset$ denoting the empty set, the *dilation* of A by B, denoted by $A \oplus B$ is defined as:

$$A \oplus B = \{x \mid (\hat{B})_x \cap A \neq \phi\}$$

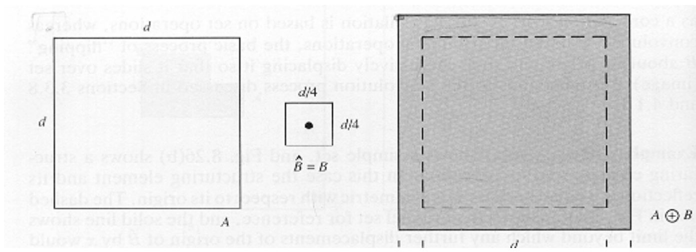
or

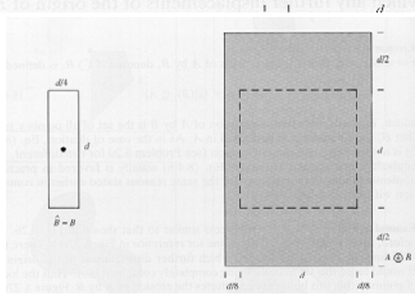
$$A \oplus B = \{x \mid (\hat{B})_x \cap A \subseteq A\}$$

Dilation

- Process consists of obtaining the reflection of B, about its origin
- Then shifting this reflection, $\hat{B}$, by *x*
- The dilation of A by B is the set of all ***x*** displacements such that $\hat{B}$ and A overlap by at least *one* element
- B is often called the "structuring element"

Example





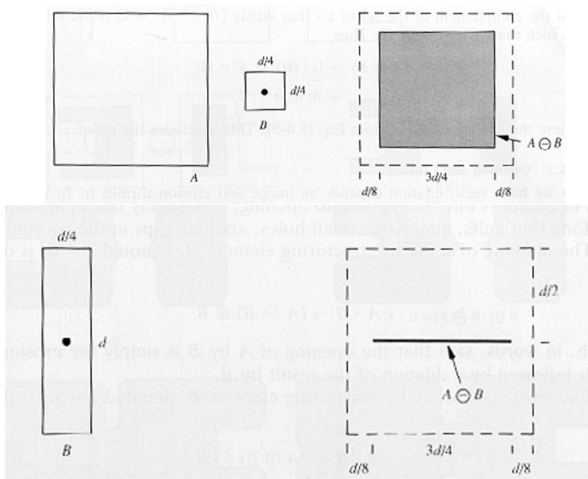
Erosion

- For sets A and B in Z^2, the erosion of A by B, denoted by $A \ominus B$, is defined by:

$$A \ominus B = \{x \mid (B)_x \subseteq A\}$$

- $A \ominus B$ is the set of all points x, such that B, translated by x, is contained in A.

Example



Dilation and Erosion Relationship

- Are duals of each other with respect to complementation and reflection, that is:

$$(A \ominus B)^c = A^c \oplus \hat{B}$$

Duality Proof

- Proof

$$(A \ominus B)^c = \{x \mid (B)_x \subseteq A\}^c$$

If set $(B)_x$ is contained in set A, then $(B)_x \cap A^c = \phi$, in which case the preceding equation becomes

$$(A \ominus B)^c = \{x \mid (B)_x \cap A^c = \phi\}^c$$

Duality Proof

But the complement of the set of x's that satisfy $(B)_x \cap A^c = \phi$, is the set of x's such that $(B)_x \cap A^c \neq \phi$

- thus

$$(A \ominus B)^c = \{x \mid (B)_x \cap A^c \neq \phi\}$$

$$= A^c \oplus \hat{B}$$

Dilation and Erosion

- Dilation
  - expands an image
- Erosion
  - shrinks an image
- From these two operators, we can construct several new operators

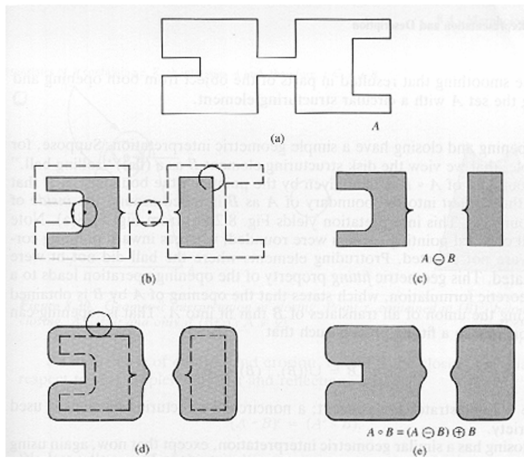
Opening

- The *opening* of set A, by structuring element B is:

$$A \circ B = (A \ominus B) \oplus B$$

- in other words, *opening* of A by B is simply the *erosion* of A by B, followed by a *dilation* by B

Example



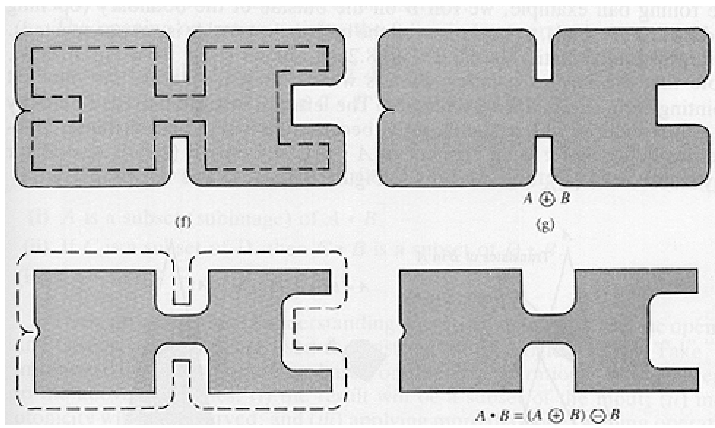
Closing

- The *closing* of set A, by structuring element B is:

$$A \bullet B = (A \oplus B) \ominus B$$

- in other words, *closing* of A by B is simply the *dilation* of A by B, followed by an *erosion* by B

Example

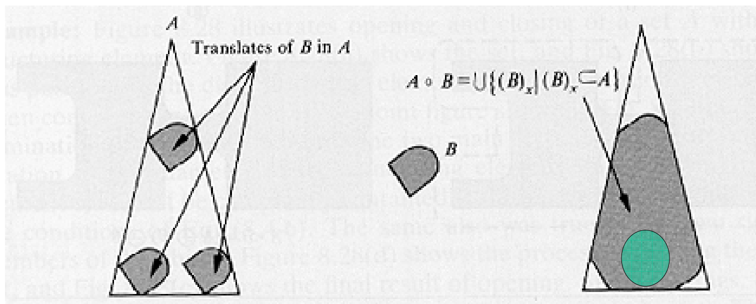


Geometric Interpretation

- Opening
  - can be considered a geometric *fitting* problem
  - it is the union of all translates of B that fit into A

$$A \circ B = \bigcup \{ (B)_x \mid (B)_x \subset A \}$$

Example



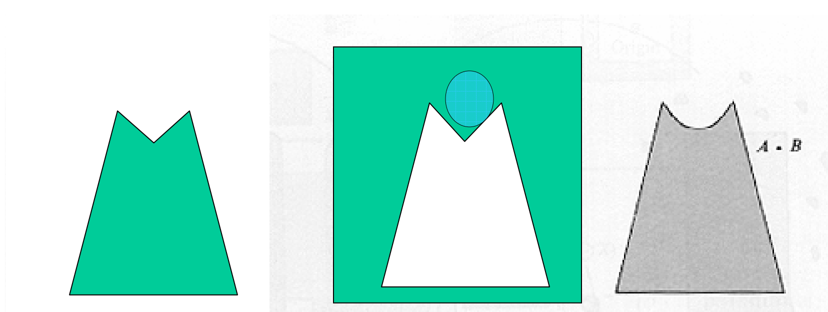
Geometric Interpretation

- Alternative Interpretation
- Opening
  - If you are painting A, using a brush shaped like B, then the opening is all the points that you can paint.

Geometric Interpretation

- *Closing*
  - If you are painting the outside of A (ie, you are paint A^c), it is all the points you cannot paint.

Example



A A^c

Example

Interesting Property

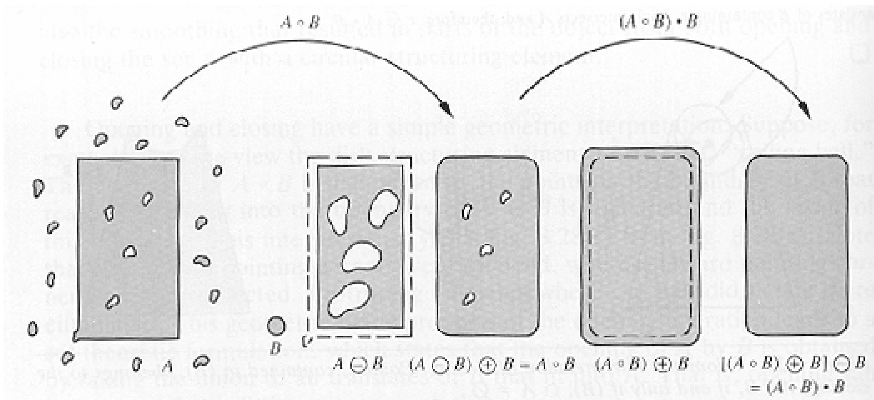
- Like dilation and erosion, *opening* and *closing* are duals with respects to set complementation and reflection

$$(A \bullet B)^c = A^c \circ \hat{B}$$

Properties of Opening and Closing

- **OPENING**
  (i) $A \circ B$ is a subset (subimage) of A
  (ii) If C is a subset of D, then $C \circ B$ is a subset of $D \circ B$
  (iii) $(A \circ B) \circ B = (A \circ B)$
- **CLOSING**
  (i) A is a subset of $A \bullet B$
  (ii) If C is a subset of D, then $C \bullet B$ is a subset of $D \bullet B$
  (iii) $(A \bullet B) \bullet B = (A \bullet B)$

Filters: Closing of the Opening

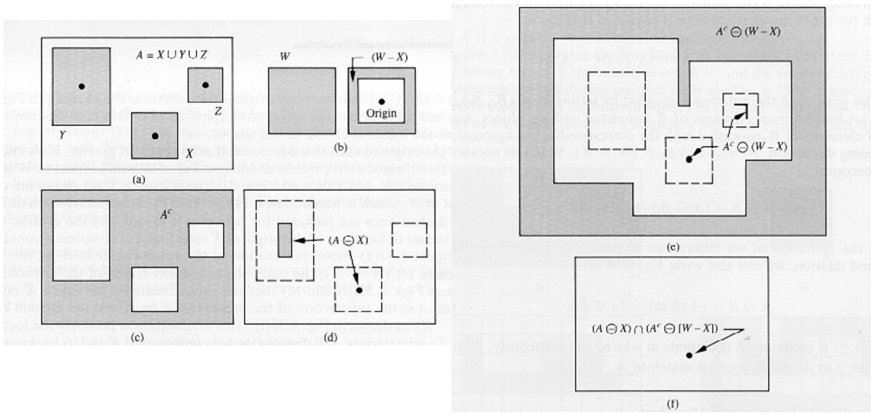


Hit-or-Miss Transform

- Basic tool for shape detection
- Construct two elements
  - *X* and *(W-X)* where *W* is window slightly larger than *X*
  - Our goal is to find *X* in the image
- Solution

$$A \circledast B = (A \ominus X) \cap (A^c \ominus [W\text{-}X])$$

Hit-or-Miss Transform



Hit-or-Miss Transform

- Original

$$A \circledast B = (A \ominus X) \cap (A^c \ominus [W\text{-}X])$$

- New notation
- Let B_1 = X and B_2= (W-X)

$$A \circledast B = (A \ominus B_1) - (A \oplus \hat{B}_2)$$

Note, B1 and B2 have to be disjoint!

Hit-or-Miss Transform

- Alternative Usage
  - B1 and B2 are two (distinct) structuring elements

$$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$$

- is therefore-
  - all the points that B1 hits in A
    - AND
  - all the points that B2 hits in A^c *(misses in A)*

Hit-or-Miss Transform

- For a normal structuring element used by dilation or erosion, zeros are ignored
  - For example:

0	1	0
1	1	1
0	1	0

Ignore these

- For the hit-or-miss structuring element
  - 0's are typically not ignored, instead we use a new notation (next slide)

Hit-or-Miss Transform

- Example

1	x	x
1	0	x
1	x	x

•This means, you have to find regions in A, where both the 1s and 0s match.

•An "x" is used to denote the areas to be ignored

•So, for the above example, you can think of B = two normal structuring elements, B1 and B2, where

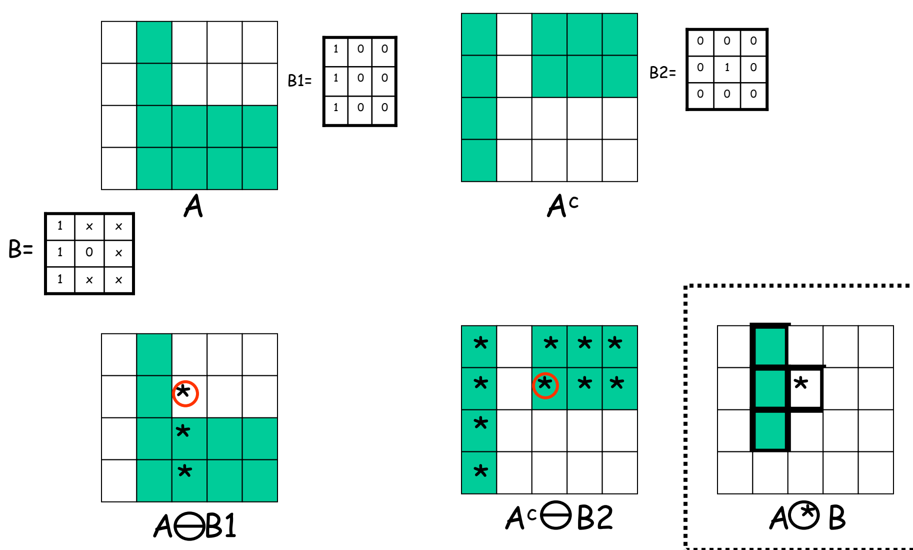
B1=

1	0	0
1	0	0
1	0	0

B2=

0	0	0
0	1	0
0	0	0

Hit or Miss Example



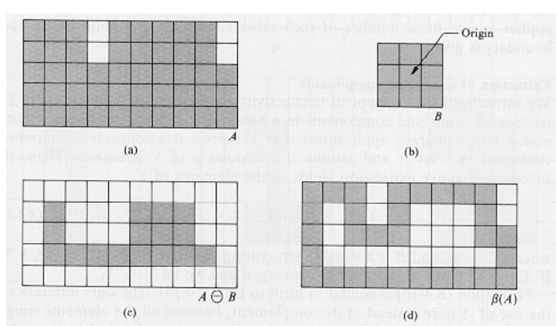
Hit-or-Miss Transform

- Using 'x', '0', and '1' is a common way of denoting hit-or-miss structuring elements
- It is a compact way of expressing two separate elements B1 and B2 as one element B

Basic Morphological Algorithms

- Boundary extraction

$$\beta(A) = A - (A \ominus B)$$



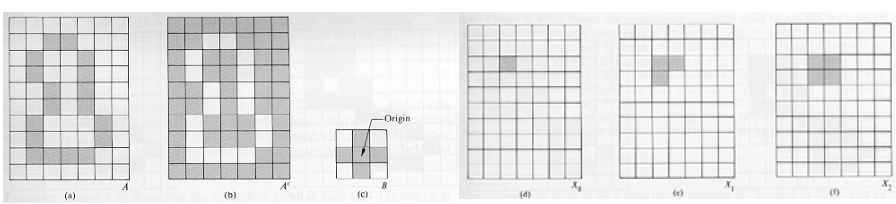
Region Filling

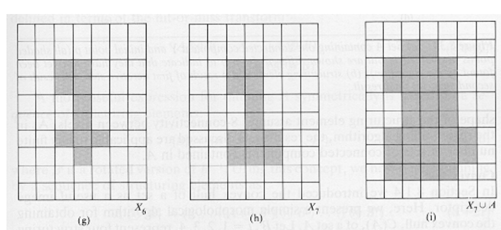
- Let A denote a set with a N_8 connected boundary
- Let X_0 be an initial point in that boundary
- Using a N_4 structuring element
- Region filling can be defined as:

$$X_k = (X_{k-1} \oplus B) \cap A^c \qquad k = 1,2,3,....$$

  – Algorithm terminates when $X_k = X_{k-1}$

Region Filling Example





Region Filling

- The repeated dilation of region filling would fill the entire image
- However, the intersection with A^c limits the results to inside the region of interest
- This "delimiting process" is sometimes called: *conditional dilation*

Connected Component Extraction

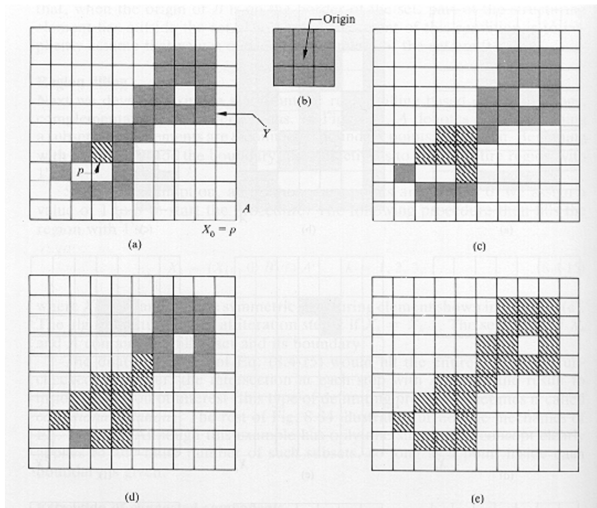
- Similar to region growing
- Let Y be a connected component
- Let X_o= p, a point on Y

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1,2,3,...$$

  - Algorithm terminates when $X_k = X_{k-1}$

Side Note: Recall that MM routines often take a variable, N, denoting the number of times to apply the operator. If N=inf, this generally means stop when $X_k = X_{k-1}$ (that is, terminate when the image stops changing)

Example



Convex Hull

- Let C(A) be the convex hull of A
- Let B^i, i=1,2,3,4, represent four structuring elements
- Let

$$X^i_k = (X \circledast B^i), \text{ for i} = 1,2,3,4 \quad \text{and k} = 1,2,....$$

- Now, let $D^i = X^i_{conv}$ where "conv" means

$$X^i_k = X^i_{k-1}$$

Convex Hull

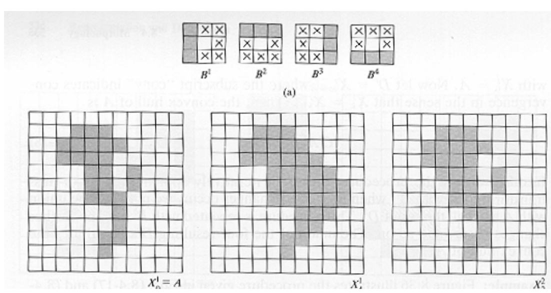
- Now, let $D^i = X^i_{conv}$ where "conv" means Xi has converged

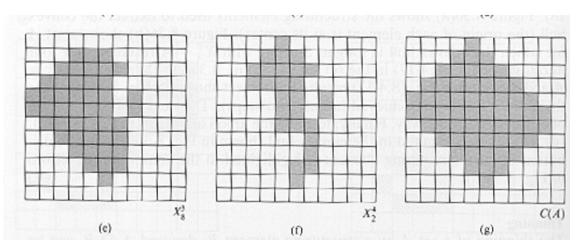
$$X^i_k = X^i_{k-1}$$

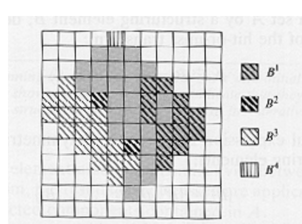
- Then, the convex hull of A is

$$C(A) = \bigcup_{i=1}^{4} D^i$$

Example







Thinning

- Thinning of A by structuring element B is defined as:

$$A \otimes B = A - (A \circledast B)$$

$$= A \cap (A \circledast B)^c$$

- Let B = {B} = {B^1, B^2, B^3, ... , B^n}

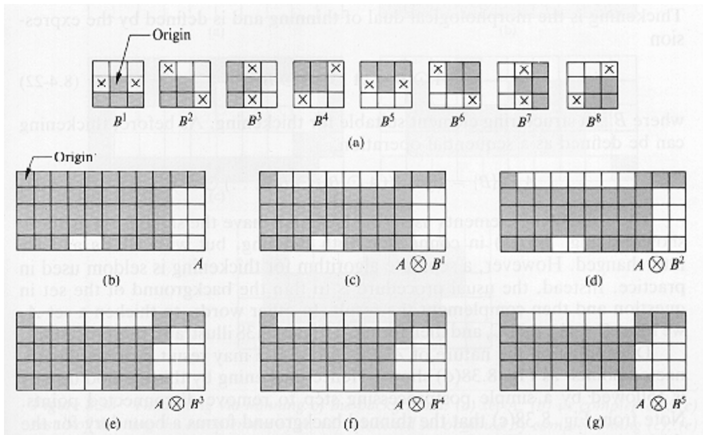
Thinning

- B is a set of structuring elements

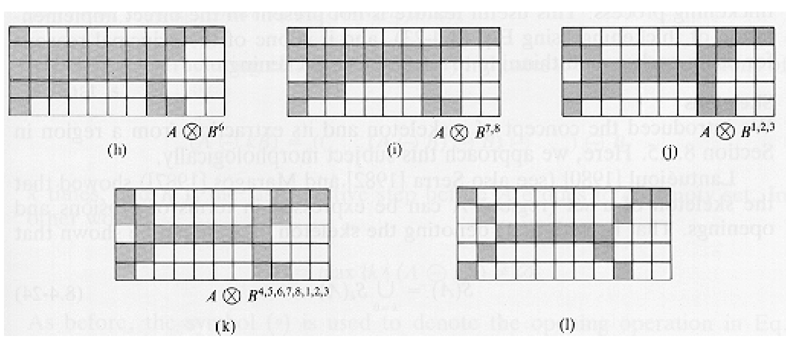
$$A \otimes \{B\} = ((\ldots((A \otimes B^1) \otimes B^2)\ldots)B^n)$$

- In other words, the process is to thin A by one pass with B1, then this the results with one pass of B2, and so on.
- Repeat the process until no further changes occur.

Example of Thinning



Example of Thinning



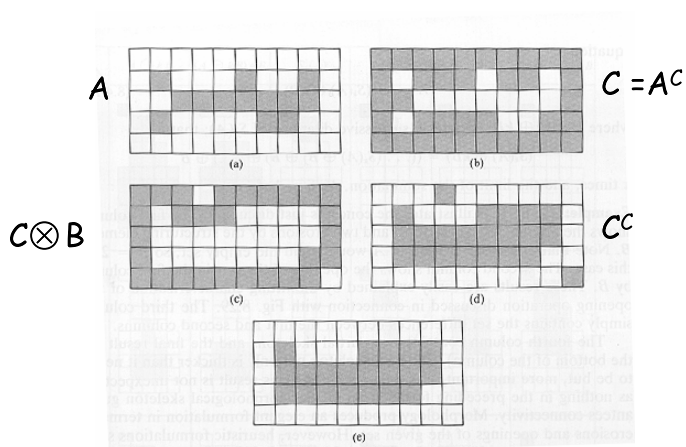
Thickening

- A thickened by B is:

$$A \odot B = A \cup (A \circledast B)$$

- Easy
  - To thickening set A,
  - Thin set C, where $C = A^c$
  - $A \odot B = C^c$

Thickening Example



Remove disconnected components

Skeletons

- Skeleton extraction ,S(A), of A, using structure element B

$$S(\mathrm{A}) = \bigcup_{\mathrm{k}=0}^{\mathrm{K}} \mathrm{S}_{\mathrm{k}}(A)$$

- where

$$S_k(A) = \bigcup_{k=0}^{K} \{(A \ominus kB) - [(A \ominus kB) \circ B]\}$$

Skeletons cont'

- B is a structuring element and ($A \ominus kB$), indicates k successive erosions of A; such that

$$(A \ominus \mathrm{k} B) = ((\ldots (A \ominus B) \ominus B) \ldots \ominus B)$$

- k times, and K is the last iterative step before A erodes to an empty set, ie:

$$K = \max\{ k \mid (A \ominus kB) \neq \phi \}$$

Skeleton

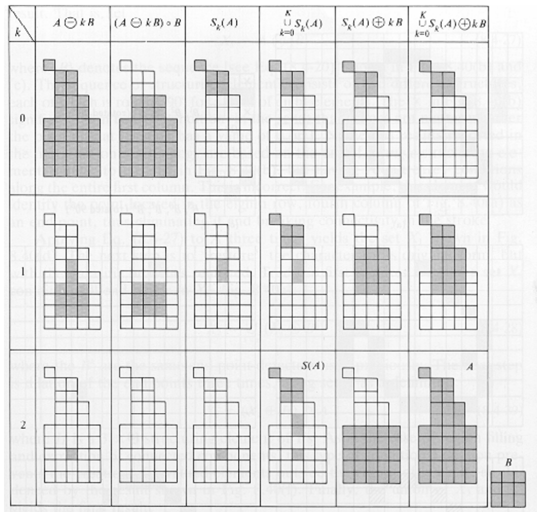
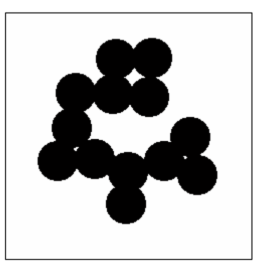
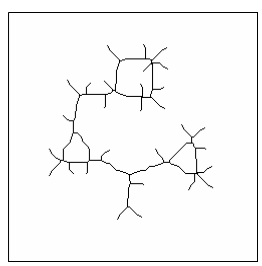


Figure 8.39 *An example of the implementation of Eqs. (8.4-24)–(8.4-26). The original set is shown at the top left, and its morphological skeleton is shown at the bottom of the fourth column. The reconstructed set is shown at the bottom of the sixth column.*

Example



A

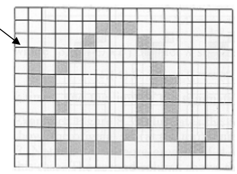


S(A)

- Restricted skeletonization . . don't allow regions to be disconnected! (Another lecture)

Pruning

- Sometimes thinning and skeletonization leaves “spurs” or “parasite” structures



- We would like to remove these structures
- This method is called “pruning”

Pruning

- 4 Steps
- First $X_1 = A \otimes \{B\}$
  - Let {B} be a sequence of structuring elements
  - Apply this K times
- Second
  - Find the end points of X1

$$X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$$

Pruning

$$X_3 = (X_2 \oplus H) \cap A$$

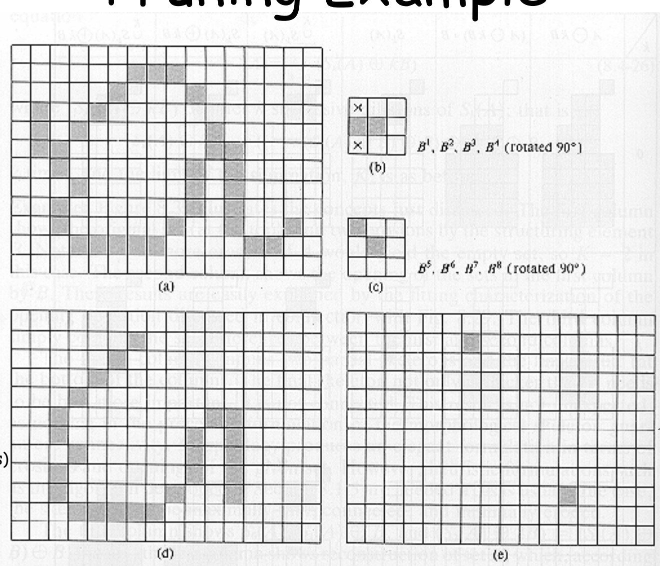
- Third
  - Use an N8 structuring element, H to grow the end-points K times
  - This is a form of conditional dilation
- Fourth
  - The union of X1 and X3 is the desired results

$$X_4 = X_1 \cap X_3$$

Pruning Example

$X_1 = A \otimes \{B\}$

(Performed 3 times)

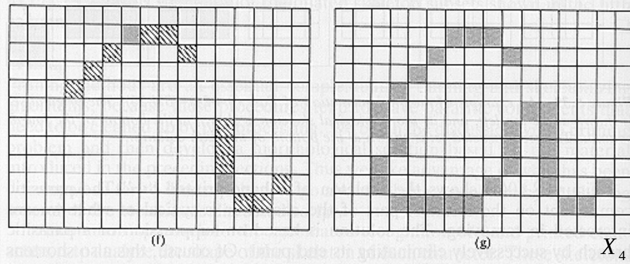


$X_2 = \bigcup_{k=1}^{8} (X_1 \otimes B^k)$

Pruning Example

$X_3 = (X_2 \oplus H) \cap A$

(performed 3 times)



$X_4 = X_1 \cap X_3$

Figure 8.40 *Example of pruning: (a) original image; (b) and (c) structuring elements used for deleting (thinning) end points; (d) result of three cycles of thinning; (e) end points of (d); (f) dilation of end points conditioned on (a); (g) pruned image.*

This technique is sometimes called "de-spurring" or "spur removal"

Modifications

- You can make modifications to MM operators
- One typical example is the "majority" dilation
  - Set a pixel to 1 if 5 or more of its N_8 neighbors are 1

Extension to Grayscale

- MM can be extended to grayscale image
  - f is a 2D discrete function (image)
  - b is a 2D discrete function (smaller image)
- Dilation

$$(f \oplus b)(s,t) = \max\{\, f(s-x,t-y)+b(x,y) \,|\, (s-x),(t-y) \in D_f; (x,y) \in D_b \}$$

- Erosion

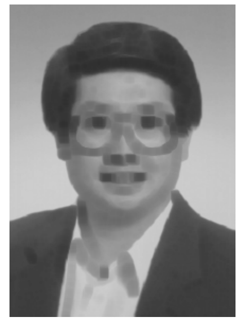
$$(f \ominus b)(s,t) = \min\{\, f(s+x,t+y)-b(x,y) \,|\, (s+x),(t+y) \in D_f; (x,y) \in D_b \}$$

Extension to Grayscale

- Grayscale MM operators are very similar to spatial domain convolution
- But instead of a summation of the "mask" coefficients w/ the image
  - you perform a *min* or *max* over the domain

Grey-Level MM Example

Dilation



Erosion

$$B=\begin{pmatrix} 1 & 2 & 2 & 3 & 2 & 2 & 1 \\ 2 & 2 & 3 & 4 & 3 & 2 & 2 \\ 2 & 3 & 4 & 4 & 4 & 3 & 2 \\ 3 & 4 & 4 & 5 & 4 & 4 & 3 \\ 2 & 3 & 4 & 4 & 4 & 3 & 2 \\ 2 & 2 & 3 & 4 & 3 & 2 & 2 \\ 1 & 2 & 2 & 3 & 2 & 2 & 1 \end{pmatrix}$$

Grey Level MM

- Dilation
  - Tends to brighten the image, remove dark regions
- Erosion
  - Tends to darken the image, remove bright regions

- You can derive *open, close*
- *Other operators . . . .*

Summary

- Mathematical Morphology
  - image processing approach based on set theory and shape (structuring elements)
  - provides a powerful framework for many operations
  - some operations can be easily expressed using MM
    - such as boundary extraction
- Often used to post-process "thresholded" images
  - Threshold
  - Perform some MM operation (clean up noise, fill in holes)
  - Result provides a nice segmented region

Active Research Areas

- Morphology Digest
  - http://www.cwi.nl/projects/morphology/
- ISMM
  - International Symposium on Mathematical Morphology
- 3D image analysis
- Scale Spaces

Active Research Areas

- Revisiting old problems
  - Using MM
  - Old examples
    - Segmentation
    - Edge Detection (Using gray-level MM)
- You'll find MM mentioned in conjunction with other techniques