Control, Estimation, Modeling

Compiled Notes on Control Theory

June 2025

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1 Goals of Control Theory

Control theory is a branch of engineering that deals with the behavior of dynamical systems and how to influence that behavior through inputs. Typical considerations are:

Stability Ensuring the system remains bounded and predictable over time, and preventing oscillations or otherwise chaotic behavior.

Controllability Determining whether a system can be driven from an initial state to a desired final state using the available control inputs.

Observability Assessing how much of the internal state of a system can be inferred from its outputs. Crucial for systems where direct measurement of all variables isn't possible.

Optimality Achieving the desired behavior with minimal cost (energy, time). Involves balancing between speed, accuracy, and resource usage.

Robustness Maintaining performance despite uncertainties, disturbances, or modelling inaccuracies. The controller must be able to handle real world imperfections!

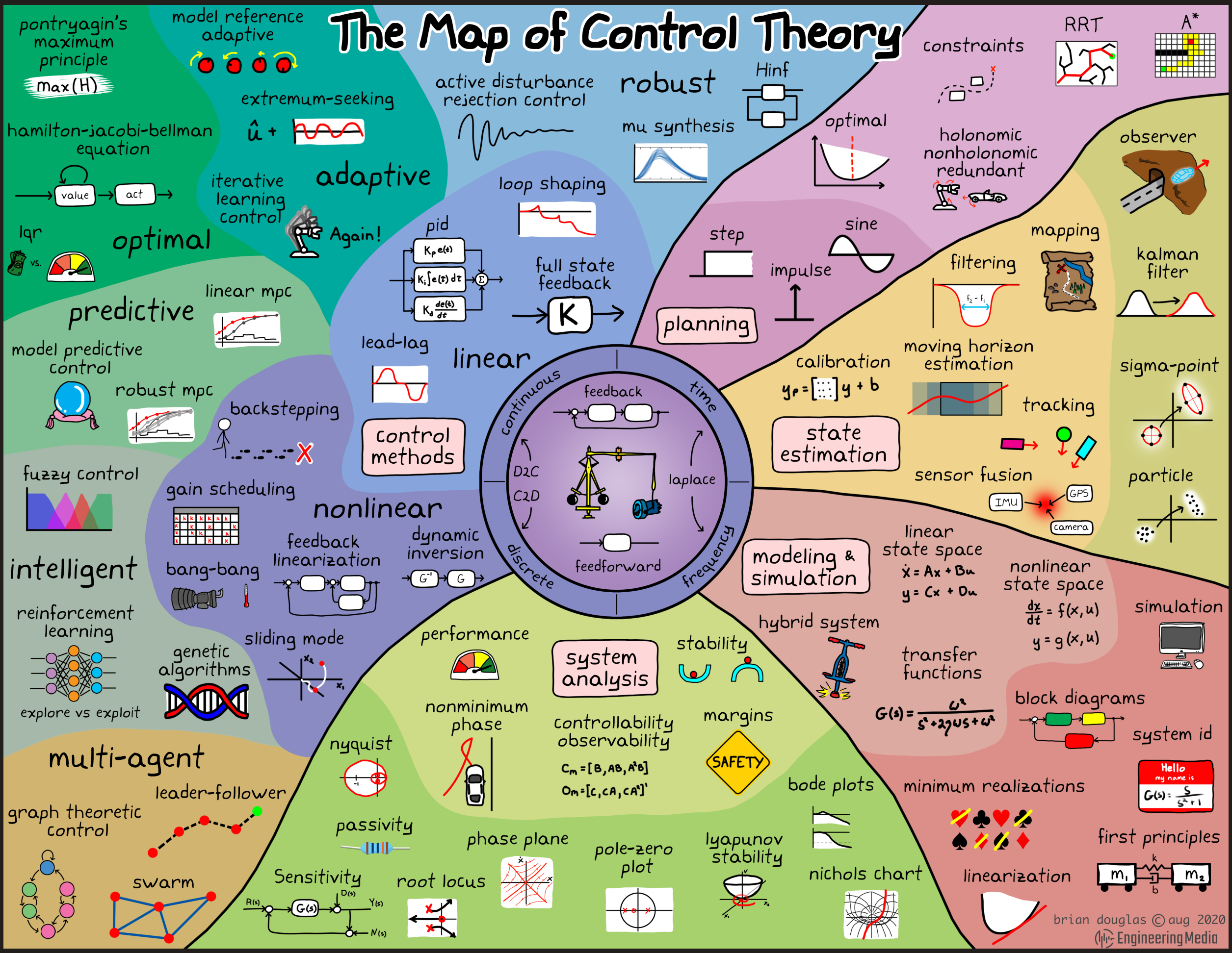


Figure 1: Brian Douglas' Concept Map of Control Theory

Control theory is a interdisciplinary topic. It can be applied to mechanical, electrical, thermal, fluid, really any system.

1.1 Applications to Aerospace

The main application of control theory in aerospace is in Guidance, Navigation, and Control (GNC). In order from most broad to least:

Navigation (Statistics) ”Given the measurements we have, where do we think we are?”
Guidance (Optimization) ”Given where we want to go and where we think we are, which path should we follow?”
Control (Differential Equations) ”What effort should we apply and how should we control actuators given where we are and the chosen path?”

All require knowledge of dynamics and linear algebra, and can be further broken down into numerous subcategories.
Control theory offers many benefits in the aerospace application.

Automation Remove the need for human intervention, preserving the attention of the operator for more important matters.
Performance Operate more effectively than a human operator.
Safeguards / Protections Prevent the craft from exceeding design limits.
Deferred Decision-making Let the human operator make certain flight critical decisions.

2 Background

Some background information must be understood before delving into control theory.

2.1 Calculus

I am not trying to rewrite a textbook here, nor do I have to knowledge to. This will just be a quick refresher on the most important topics.

2.1.1 *Differentiation*

It can be simply described as the rate of change of a quantity or function. Represented by the slope of a function.

2.1.2 *Integration*

It can be simply described as the sum of change of a quantity or function. Represented by the area under the curve of a function.

2.1.3 *Taylor Series*

2.2 Differential Equations

2.2.1 *Ordinary (ODE)*

2.2.2 *Partial (PDE)*

2.2.3 *Coupled First-order Nonlinear Differential Equations*

Coupled - variables x_i might appear in equations x_j for $i \neq j$. Control input u_i may affect multiple states x_j simultaneously. A single state may be affected by multiple control inputs at once.

First-order - Highest derivative is the first derivative.

Nonlinear - equations include terms that cannot be expressed as linear combinations (scaling, addition) of x and u. In practice, this happens when x and u have non-unity exponents or are included inside

Ordinary - Equations include a partial w.r.t. only one variable (usually t, in contrast to a PDE)

2.2.4 *Jacobian Matrix*

2.2.5 *Convolution*

Convolution integrals are useful for solving initial value problems with general forcing functions.

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau \tag{2.1}$$

2.2.6 *Fourier Transforms*

Any periodic function can be represented by a Fourier series (series of periodic functions).

For periodic functions that decay to zero, we can use the Fourier transform.

$$f(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}dt \tag{2.2}$$

This transform converts a continuous time domain signal into a continuous frequency domain signal. In other words, it converts a signal into its component frequencies.

2.2.7 *Laplace Transforms*

The Laplace transform is a more general case of the Fourier transform.

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{2.3}$$

A system's transfer function is the Laplace transform of the system's impulse response. For a given LTI system, the transfer function is unique, whereas the state space represention may not be. Like the transfer function, the feedthrough D matrix is always unique.

Convolution Theorem Convolution in the time domain is equal to multiplication in the frequency domain.

$$\mathcal{L}\{[x * y](t)\} = X(s) \cdot Y(s) \tag{2.4}$$

Transfer Functions Applying the Laplace transform to a time domain function will produce the frequency domain representation the input-output relationship of a SISO LTI system. Classical transfer functions do not exist for LTV or nonlinear systems. In the typical state space system:

$$\begin{aligned} \dot{X}(s) = sX(s) - X(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

Solving this system gives

$$Y(s) = C(sI - A)^{-1}[BU(s) + X(0)] + DU(s) \tag{2.5}$$

The quantity $(sI - A)^{-1}$ is called the resolvant of A and represents the Laplace transform of the state transistion matrix. Like the state transistion matrix, the resolvant is always invertible, regardless of the invertibility of A. If the initial condition is zero:

$$T(s) := \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \tag{2.6}$$

Geometry of Transfer Functions

2.2.8 *Delayed Differential Equations*

DDE are when the system depends on a delayed version of state.

$$\text{eg. } u(t) = x(t - \tau), \text{ with } \tau > 0 \tag{2.7}$$

These types of equations require specialized solvers.

2.2.9 *Poles & Zeros*

The poles and zeros of a transfer function provide some critical information on the behavior of a system.

Poles

Zeros

Relative Degree of Transfer Functions A transfer function with n_p poles and n_z zeros has a relative degree of $n_p - n_z$.

Strictly Proper A transfer function whose numerator has a lesser order than the denominator (relative degree greater than zero) is called strictly proper.

Semi-Proper When the relative degree is zero, the transfer function is bi-proper or semi-proper.

Improper When the numerator is a greater order than the denominator, the transfer function is improper.

The feed-through matrix D is zero when the transfer function is strictly proper, and non-zero when the transfer function is semi-proper. A state-space represention of a system only exists if the transfer function is proper (either strictly proper or semi-proper). No definitions of x, A, B, C, D can be found for an improper system. Physical systems are always proper, and practically always strictly proper.

2.3 **Linear Algebra**

2.3.1 *Basic Operations*

2.3.2 *Matrix Inversion*

2.3.3 *Eigenvectors & Eigenvalues*

The eigenvalue problem seeks to find the directions v of the square matrix A that result in a pure scaling of λ of v when A is right-multiplied by v.

$$\lambda - \text{Eigenvalues}$$
$$v - \text{Eigenvectors}$$

This must satisfy the following equation:

$$A v_i = \lambda_i v_i \tag{2.8}$$

where in general $\lambda_i \in \mathbb{C}$ is a complex scalar and $v_i \in \mathbb{C}^{n_x}$ is a complex $n_x \times 1$ vector. Since we are interested in physical systems, we restrict our attention to real squares matrices $A \in \mathbb{R}^{n_x \times n_x}$.

- λ_i and v_i can be complex when A is real.
- When A is real, λ_i and v_i will either be both real or both complex.

- When A is real, λ_i will always come in complex conjugate pairs. A real matrix A should never have an odd number of complex eigenvalues, and these eigenvalues should be symmetric about the real axis when viewed in the s-plane.

Finding Eigenvalues

$$\det(\lambda_i I - A) = 0$$

The computation forms a n_x^{th}-order polynomial in λ_i when n_x roots - each root representing possibly repeated eigenvalues. The eigenvector corresponding to an egienvalue represents the direction associated with the null-space of the matrix $(\lambda_i I - A)$. The eigenvectors associated with an eigenvalue can be used to span the null space of $\lambda_i I - A$.

Relating Eigenvalues and Poles det($sI - A$) is the characteristic polynomial, the roots of which are both the poles and eigenvalues of the system. This also means the eigenvalues and poles are not only dependent on A and are thus independent of the input or output.

2.4 Mechanics (Newtonian)

Understanding the physics that inform the behavior of a system is critical to modeling and controlling it.

Newton's 1st Law A body remains at rest, or in motion at a constant speed in a straight line, unless it is acted upon by a force

Newton's 2nd Law At any instant of time, the net force on a body is equal to the body's acceleration multiplied by its mass or, equivalently, the rate at which the body's momentum is changing with time.

Newton's 3rd Law If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

2.4.1 *Linear*

Position / Velocity / Acceleration

Momentum

2.4.2 *Rotational*

Angular Position / Velocity / Acceleration

Angular Momentum

Coriolis Force The presense of rotation adds an additional (faux) force. This is described further in (7.1.5).

2.4.3 *Energy*

2.4.4 *Electrical*

Electrical circuits can be modeled very similarly to the mechanical system. The classic example is RLC (resistor-inductor-capacitor) circuits.

2.5 Lagrangian Mechanics

Lagrangian mechanics are preferable to use for dynamical systems that have multiple components whose interaction is complicated to express (eg. multi-body systems). This is especially true if internal reactions are not of immediate interest.

$$\text{Action: } S = \int_{t_1}^{t_2} L(q, \dot{q})dt \tag{2.9}$$

Where $L(q, \dot{q})$ is the Lagrangian and $q \in \mathbb{R}^{nq}$ is a vector of generalized degrees of freedom (positions, angles). $\dot{q}$ is the vector of corresponding rates of change of the degrees of freedom (linear + angular velocities).

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q) \tag{2.10}$$

$T(q, \dot{q})$ is the kinetic energy function, $V(q)$ is the potential energy function. Thus the Lagrangian is the difference between the system's total kinetic and total potential energy.

Principle of Least Action A conservative system (one that does not gain or lose energy) starting in state $q(t_1)$ at time t_1 and ending at state $q(t_2)$ at time t_2 follows a path that makes the action S stationary (first derivative goes to zero). This is the path of least action. While the path of least action minimizes the action the vast majority of the time, the stationary condition does not guarantee minimization.

Euler-Lagrange Equations for Conservative Systems The path of least action can be thought of as a continuous steam of states $q(t)$ over the time interval. It must satisfy the Euler-Lagrange Equations.

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0, \quad j \in \{1, ..., n_q\} \tag{2.11}$$

This yields n_q equations of motion, one corresponding to each of the generalized degrees of freedom. This we obtain all the equations of motion necessary to describe a conservative (energy-preserving) dynamical system.

Euler-Lagrange Equations for Non-conservative Systems We can modify Eq. 2.11 to account for energy dissipation (eg. friction) and energy (eg. forcing).

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = Q_j - \frac{\partial R}{\partial \dot{q}_j} \tag{2.12}$$

where $Q_j := Q_j(t)$ is the forcing function and $R := R(\dot{q})$ is the energy dissipation function. If we plug in the definition for $V(q)$ from the Least Action Principle, we arrive at:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) + \frac{\partial R}{\partial \dot{q}_j} + (\frac{\partial V}{\partial q_j} - \frac{\partial T}{\partial q_j}) = Q_j \tag{2.13}$$

This is the equation of motion for each degree of freedom.

Example We can use the classic example of the mass-spring-damper system. The advantages of the Lagrangian method become clear when an inverted pendulum is added to the system.

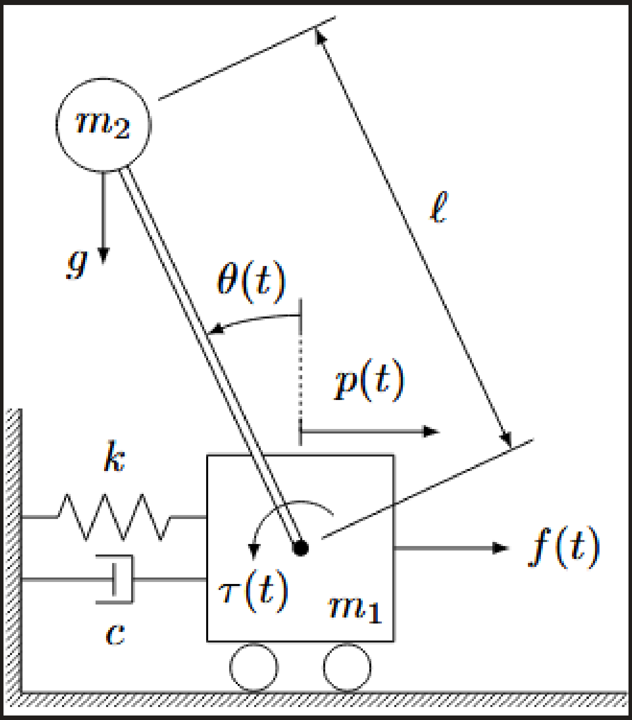


Figure 2: Inverted Pendulum on Cart System

With two bodies (masses) in the system, we can describe the degrees of freedom and the velocities of each as:

$$\begin{aligned} q_1(t) &:= p(t), & \dot{q}_1(t) &= \dot{p}(t), \\ q_2(t) &:= \theta(t), & \dot{q}_2(t) &= \dot{\theta}(t), \\ Q_1(t) &:= f(t), & Q_2(t) &:= \tau(t), \\ v_1 &= \begin{bmatrix} \dot{p} \\ 0 \end{bmatrix}, & v_2 &= \begin{bmatrix} \dot{p} - \ell\dot{\theta}\cos(\theta) \\ -\ell\dot{\theta}\sin(\theta) \end{bmatrix} \end{aligned}$$

Now, the components of the Euler-Lagrange equation (Eq. 2.13) can be found to assemble the complete equations of motion of the system.

$$T_i = \frac{1}{2} m_i v_i^T v_i$$

$$T_1 = \frac{1}{2} m_1 (\dot{p}^2 + 0^2), \quad T_2 = \frac{1}{2} m_2 [(\dot{p} - \ell\dot{\theta}\cos(\theta))^2 + (-\ell\dot{\theta}\sin(\theta))^2]$$

The sum of the kinetic energy of the system is the sum of the kinetic energy of both masses:

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}m_1\dot{p}^2 + \frac{1}{2}m_2[\dot{p}^2 - 2\ell\cos(\theta)\dot{p}\dot{\theta} + \ell^2\theta^2] \end{aligned}$$

We can then differentiate the above equation with respect to the generalized velocities $\dot{q}_i$. This gives us the following partial derivatives:

$$\frac{\partial T}{\partial \dot{q}_1} = (m_1 + m_2)\dot{p} - m_2\ell\dot{\theta}cos(\theta) \quad \text{and} \quad \frac{\partial T}{\partial \dot{q}_2} = m_2\ell[\ell\dot{\theta} - \cos(\theta)\dot{p}]$$

Differentiating again, with respect to time:

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_1}) = (m_1{+}m_2)\ddot{p}{-}(m_2\ell)(\ddot{\theta}\cos(\theta){+}\sin(\theta)\dot{\theta}^2) \quad \text{and} \quad \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_2}) = m_2\ell(\ell\ddot{\theta}{-}\ddot{p}\cos(\theta){-}\dot{p}\sin(\theta))$$

Then, we also differentiate the total kinetic enery equation with respect to position:

$$\frac{\partial T}{\partial q_1} = 0 \quad \text{and} \quad \frac{\partial T}{\partial q_2} = m_2\ell\sin(\theta)\dot{\theta}\dot{p}$$

There are two sinks/sources of potential in the system. One is the spring, and the other is the gravitational potential of m_2. Thus, the total potential energy of the system is:

$$V = \frac{1}{2}kq_1 + m_2g\ell\cos(q_2)$$

This equation is differentiated with respect to position:

$$\frac{\partial V}{\partial q_1} = kp \quad \text{and} \quad \frac{\partial V}{\partial q_2} = -m_2g\ell\sin(\theta)$$

There is one source of energy dissipation in the system: the damper. The energy dissipated by this component is given by:

$$R = \frac{1}{2}c\dot{p}^2$$

Differentiating this with respect to velocity:

$$\frac{\partial R}{\partial \dot{q}_1} = c\dot{p} \quad \text{and} \quad \frac{\partial R}{\partial \dot{q}_2} = 0$$

Now that all of the derivatives have been solved for, the nonlinear ODEs can be plugged into the Euler-Lagrange Equation, resulting in the coupled equations:

$$\begin{aligned} f &= (m_1 + m_2)\ddot{p} - m_2\ell\cos(\theta)\ddot{\theta} + m_2\ell\sin(\theta)\dot{\theta}^2 + c\dot{p} + kp \\ \tau &= m_2\ell^2\ddot{\theta} - m_2\ell\cos(\theta)\ddot{p} - m_2g\ell\sin(\theta) \end{aligned}$$

The coupled differential equations can be rearranged to isolate $\ddot{p}$ and $\ddot{\theta}$. In order to do this, the constants $\alpha(\theta)$ and $\beta(\theta)$ are defined as follows:

$$\alpha(\theta) := \frac{\cos(\theta)}{\ell} \quad \text{and} \quad \beta(\theta) := \frac{m_2 \ell \cos(\theta)}{m_1 + m_2}$$

These constants are used to eliminate the second derivative term in the coupled differential equations. Through some algebraic manipulation, it can be shown that the equations with the isolated second derivative are given by:

$$\ddot{p} = \frac{m_2 \ell \sin(\theta)[\alpha(\theta) g - \dot{\theta}^2] - c\dot{p} - kp + f + \alpha(\theta)\tau}{m_1 + m_2[1 - \alpha(\theta)\ell \cos(\theta)]}$$

$$\ddot{\theta} = \frac{m_2 \ell \sin(\theta)[g - \beta(\theta)\dot{\theta}^2] + \beta(\theta)[f - c\dot{p} - kp] + \tau}{m_2 \ell[\ell - \beta(\theta)\cos(\theta)]}$$

These are the final equations of motion, without solving for the reaction force of the pendulum on the cart!

3 System Identification & Analysis

Before we can understand how to control a system, we must first understand how that system changes in response to input. System identification is the process by which a mathematical model of the dynamical system can be constructed. Designing a controller is infinitely easier with such a model.

The system (excluding the controller) is referred to as the ”plant”.

3.1 Linear Regression

3.2 State-space Models

State space models are a useful way to describe a system. A typical LTI system can be arranged as follows:

$$x(t) - \text{state} \quad y(t) - \text{output} \quad u(t) - \text{input}$$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$$\begin{aligned} A &\in \mathbb{R}^{n_x \times n_x} && \text{zero input dynamics matrix} \\ B &\in \mathbb{R}^{n_x \times n_u} && \text{input matrix} \\ C &\in \mathbb{R}^{n_y \times n_x} && \text{output matrix} \\ D &\in \mathbb{R}^{n_y \times n_u} && \text{pass-through matrix} \end{aligned}$$

3.3 Linear, Time-Invariant Systems (LTI)

3.3.1 *Properties*

Homogeneity $\begin{cases} x(t_0), u(t) \rightarrow y(t) \\ \alpha x(t_0), \alpha u(t_0) \rightarrow \alpha y(t) \end{cases}$

Additivity $\begin{cases} x_1(t_0), u_1(t) \rightarrow y_1(t) \\ x_2(t_0), u_2(t) \rightarrow y_2(t) \end{cases} \Rightarrow x_1(t_0) + x_2(t_0), u_1(t) + u_2(t) \rightarrow y_1(t) + y_2(t)$

Time Invariance For the same input, the response at an time will be the same.

3.3.2 *Convolution*

The uniformity and superposition allows us to use the response to a known unput to comput the response to a generic input. Convolution is the mathematical operation by which we generate the response to said generic input. It's only because of the time uniformity and superposition that convolution is useful and gets its own name (unlike other mathematical constructs).

In the time domain the definition is:

$$y(t) = [h * u](t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t h(t-\tau)u(\tau)d\tau \tag{3.1}$$

Where h(t) is the impulse response of the LTI system impulse response is the response to a unit impulse $\delta(t)$.

$$\delta(t) = \begin{cases} \infty, \text{ at } t = 0 \\ 0 \text{ otherwise} \end{cases} \tag{3.2}$$

Assuming zero initial conditions:

$$h(t) = Ce^{At}B + D\delta(t)$$

The impulse response is by definition a time domain concept, as it is defined as the time domain output of an LTI system to a unit time domain impulse in the input.

Visualization of Convolution The input can be interpreted as a sequence of impulses offset in time, each setting off its own impulse response that is scaled in magnitude by the magnitude of $u(t)$ at said time. The output can be understood as the sum of the resulting scaled and time shifted impulse responses.
For any initial condition the output $y(t)$ of an LTI system due to a sinusoidal input $u(t)$ will converge to a steady state output $y_{ss}(t)$ as $t \to \infty$, assuming a steady state output exists. The output will be a sinusoid of the same frequency as $u(t)$, with a different magnitude and phase shift. The magnitude and phase shift can be gleaned from the transfer function (specifically from the Bode plot).
Each block in a control diagram represents a convolution. Thus, we need a more intuitive way to perform the operation, which can be done in the frequency domain, which can be done with the Laplace transform.

3.4 Linear, Time-Varying Systems (LTV)

For an LTV system, the system is time-varying because at least one of the four system matrices changes with time, not because x, y, u change with time! The primary shortcoming of LTV systems is their dependence on the initial time. An input and initial condition might have different responses over different time intervals. We cannot guarantee homogeneity and additivity over shifted time intervals.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

$$A(t) \in \mathbb{R}^{n_x \times n_x} \quad B(t) \in \mathbb{R}^{n_x \times n_u} \quad C(t) \in \mathbb{R}^{n_y \times n_x} \quad D(t) \in \mathbb{R}^{n_y \times n_u}$$

3.4.1 *Properties*

Homogeneity $\begin{cases} x(t_0), u(t) \to y(t) \\ \alpha x(t_0), \alpha u(t_0) \to \alpha y(t) \end{cases}$

Additivity $\begin{cases} x_1(t_0), u_1(t) \to y_1(t) \\ x_2(t_0), u_2(t) \to y_2(t) \end{cases} \Rightarrow x_1(t_0) + x_2(t_0), u_1(t) + u_2(t) \to y_1(t) + y_2(t)$

Together, these two properties are called superposition. These properties do not hold under time shifts, however.

State Transition Matrix Governs zero-input dynamics.

$$\Phi_A(t_0,t_0) = I_{n_x \times n_x}, \quad \dot{\Phi}_A(t_0,t_0) = A(t)\Phi_A(t_0,t_0)$$

3.5 Modal Analysis

Eigenvalues (2.3.3) are critical for understanding how a system reacts to input. A mode of a system is the time response associated with either a simple eigenvalue and associated eigenvector, or a complex conjugate pair of eigenvalues + eigenvectors.
Maximum number of modes is n_x, and minimum is $n_x/2$ (which is if all eigenvalues are complex conjugate pairs.)

$$\dot{x}(t) = Ax(t)$$

If we define the initial conditions as:

$$x(0) = v_i \text{ where } v_i \text{ is an eigenvector of } A$$
$$\dot{x}(0) = Ax(0) = Av_i = \lambda_i v_i$$

Since λ is a scalar, this equation implies the state of the system will evolve parallel to v_i. This means that $\dot{x}(t)$ will always remain aligned with v_i, and $x(t)$ will never come out of alignment.

Observations

- The real part of the eigenvalue λ_i dictates the exponential growth of decay of $x_{mj}(t)$.
- The imaginary part of eigenvalue λ_i dictates the frequency with which $x_{mj}(t)$ oscillates.
- The eigenvector v_i encodes the relative phase and magnitude of each state in $x(t)$. The phase information is encoded by the relative phase of the complex elements of the vector v_i.
- The scalar α_0 determines theh initial scaling and phase shift of all the states in $x(t)$.

$$X_{mj}(t) = \alpha_0 v_i e^{\lambda_i t} = \alpha_0 v_i e^{\text{Re}(\lambda_i)t}[\cos(\text{Im}(\lambda_i)t) + j\sin(\text{Im}(\lambda_i)t)]$$

Any zero input motion that a given LTI system can exhibit can be expressed as a superposition of the system modes.

Example: Conventional Aircraft Modes

Phugoid
Dutch Roll
Subsidence

Example: System Modes Take the following 2D mass-spring-damper system:

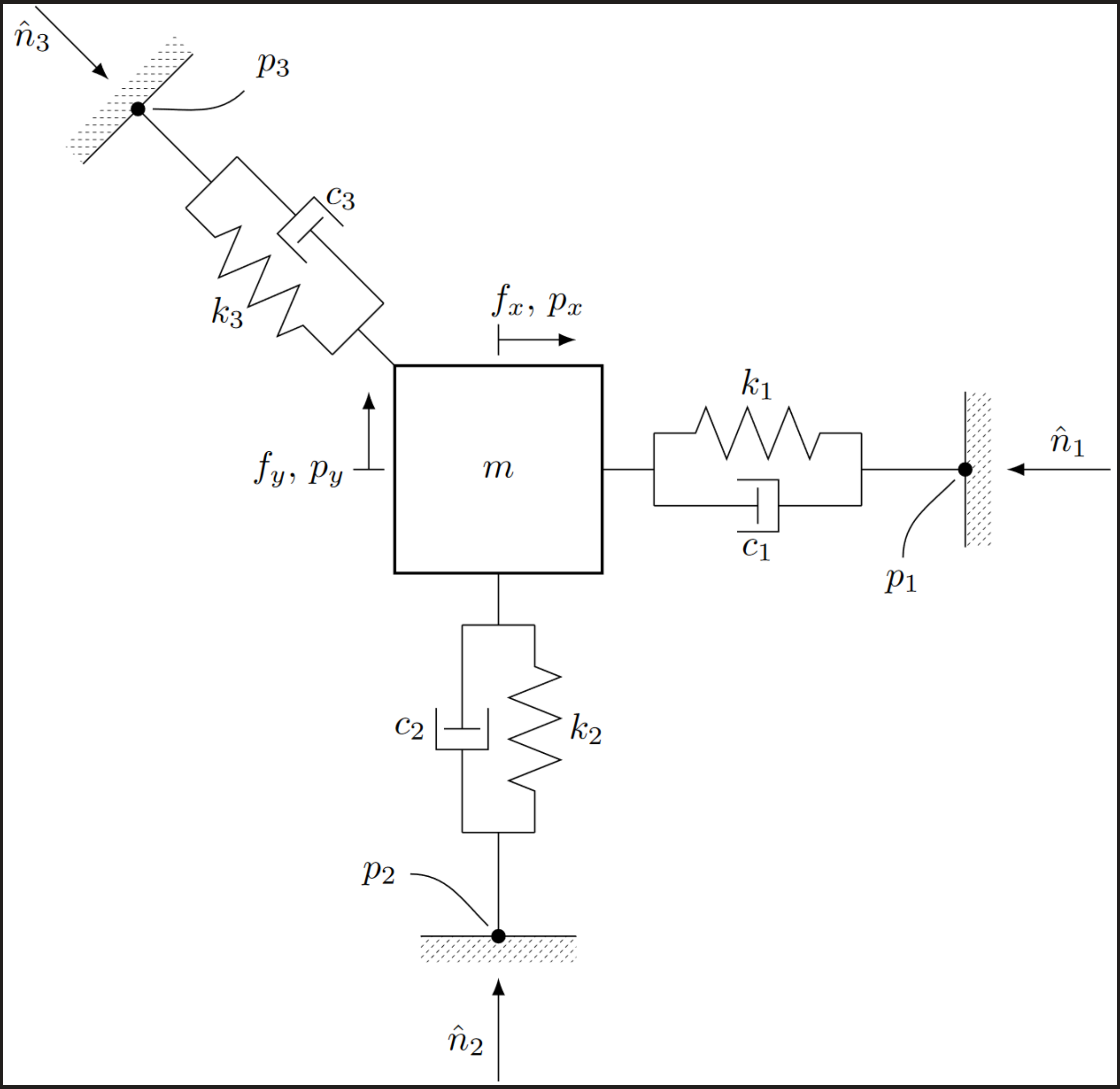


Figure 3: 2D Nonlinear Mass-Spring-Damper System

We can define the state, control input, and desired output vectors as follows:

$$x(t) = \begin{bmatrix} p_x \\ p_y \\ \dot{p_x}(t) \\ \dot{p_y}(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}, \quad y(t) = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

For this system, gravity can be neglected. We can define variables as follows: δ_i is the displacement of each mass/damper pair, $\dot{\delta}_i$, is the rate of displacement of each mass/damper pair, $\hat{n}_i$ is the unit vector pointing from the attachment point of each mass/damper pair towards the position of the mass. The variables k_i and c_i are the spring and damper constants respectively, where i represents each mass/damper grouping.

In order to find the function describing the dynamics of the system, the state vector is differentiated, because $\dot{x}(t) = f(x(t), u(t))$. The second derivative of the position of the mass, $\ddot{p}$, can be found using Newton's Laws of Motion, as seen in Eq. 3.3.

$$\dot{x}(t) = \begin{bmatrix} \dot{p}_x(t) \\ \dot{p}_y(t) \\ \ddot{p}_x(t) \\ \ddot{p}_y(t) \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{p}_x(t) \\ \dot{p}_y(t) \\ \sum[k_i \cdot \delta_i \cdot \hat{n}_{x,i}] + \sum[c_i \cdot \dot{\delta}_i \cdot \hat{n}_{x,i}] + f_x \\ \sum[k_i \cdot \delta_i \cdot \hat{n}_{y,i}] + \sum[c_i \cdot \dot{\delta}_i \cdot \hat{n}_{y,i}] + f_y \end{bmatrix} \tag{3.3}$$

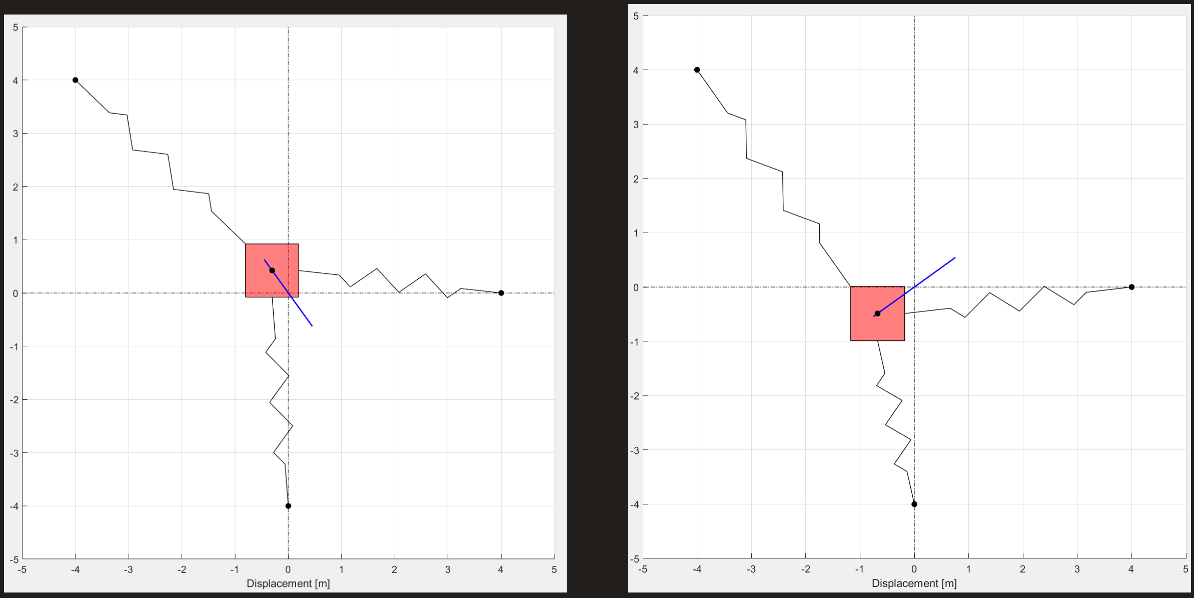
Next, the equations of the system can be derived by linearizing about the equilibrium point, represented by $\bar{x}(t)$ and $\bar{y}(t)$. The linearization is done by holding one variable constant about the equilibrium point while the derivative with respect to the other is computed. The system matrices calculated (numerical Jacobian) from the initial system variables given are:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.5 & 1.5 & 0 & 0 \\ 1.5 & -3.5 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

MATLAB's built in `eig()` function can be used to find the eigenvalues and corresponding eigenvectors. The eigenvalues and their corresponding eigenvectors are found below. Each row in the eigenvalue vector corresponds to the same indexed column in the eigenvector matrix.

$$e = \begin{bmatrix} -0 + 2.1404i \\ -0 - 2.1404i \\ 0 + 1.1912i \\ 0 - 1.1912i \end{bmatrix}, \quad V = \begin{bmatrix} 0 + 0.2475i & 0 - 0.2475i & 0 - 0.5216i & 0 + 0.5216i \\ 0 - 0.3434i & 0 + 0.3434i & 0 - 0.3760i & 0 + 0.3760i \\ -0.5297 - 0i & -0.5297 + 0i & 0.6213 + 0i & 0.6213 + 0i \\ 0.7350 + 0i & 0.7350 + 0i & 0.4478 + 0i & 0.4478 - 0i \end{bmatrix}$$

The A matrix has two complex conjugate pairs of eigenvectors, meaning that the system has two modes. The modes are found by taking the real part of the complex eigenvectors and plugging them in as the initial state. The first two eigenvectors correspond to the direction seen in Fig. 4a, and the second two correspond to Fig. 4b. Both modes oscillate, although the first mode oscillates faster. Neither exponentially grows for two reasons: there is no forcing input applied, and the absence of damping results in a purely imaginary eigenvalue, which means motion is purely oscillatory with no gain. Neither exponentially decays for the same reason. The relationship between the modes and the damping coefficients is such that when the system is undamped, it is in a state of pure oscillation, and thus the eigenvalues are purely imaginary. Adding a damping coefficient adds a real component to the eigenvalues, which represents the damping effect. Moreover, the damped natural frequency will be less than that of an undamped system.



(a) Direction of First Mode (b) Direction of Second Mode

Figure 4: System Modes

In order to excite the system along its mode, the eigenvectors corresponding to either eigenvalue pair can be plugged into the function as an initial state $x(0)$. It can be observed that the linear simulations do not match the nonlinear ones. This is due to the imprecision of the linearization operation performed; The farther the mass travels from the equilibrium point, the farther the nonlinear simulation diverges from the linear one. This is because the system matrices derived through linearization are only very accurate around the point about which they were linearized, that being the equilibrium point.

3.5.1 *System Poles*

Example: System Poles

3.5.2 *System Zeros*

The presence of a zero in an LTI system is an indication that the system can exhibit zero output even when the state and control input are nonzero. Intuitively: the presence of a zero implies that there are special combinations of non-zero initial conditions and non-zero control input that produce non-zero state time histories and zero output time histories.

$$x(0) \neq 0, u(t) \neq 0 \Rightarrow x(t) \neq 0, y(t) = 0$$

Components of the initial conditions and control input may combine and vanish, resulting in no perceptible contribution to the input. It follows that removing said components from the initial conditions and control input would generate the same output $y(t)$.
Not all LTI systems have zeros. Zeros are properties of the system at hand. Unlike poles, zeros do not influence the BIBO stability of the system. The BIBO stability is determined by the location of its poles, even in cases that even have zeros in the right hald plane.

Formal Definition of a Zero Goal: find non-zero initial conditions and control input so that the output is zero for all time.

$$\zeta(t) := \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \neq 0 \Rightarrow y(t) = 0$$

Example: System Zero To demonstrate a system zero, take the example of a triple mass-spring-damper system, in which the input is the forcing applied to m_1. Defining the output as p_2, the input that results in a zero output (motion) in m_2 can be found.

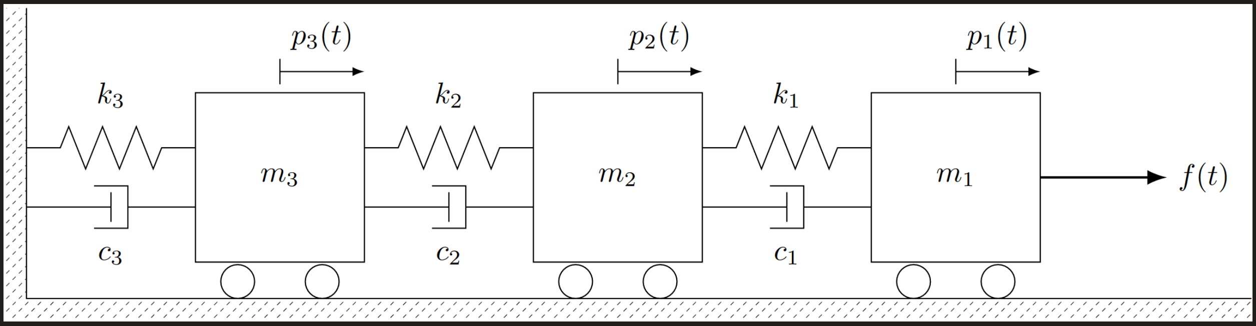


Figure 5: Triple Mass-Spring-Damper System

The equations of motion (derived from Newton's laws) are as follows:

$$\begin{aligned} m_1\ddot{p}_1(t) &= k_1(p_2(t) - p_1(t)) + c_1(\dot{p}_2(t) - \dot{p}_1(t)) + f(t) \\ m_2\ddot{p}_2(t) &= k_1(p_1(t) - p_2(t)) + k_2(p_3(t) - p_2(t)) + c_1(\dot{p}_1(t) - \dot{p}_2(t)) + c_2(\dot{p}_3(t) - \dot{p}_2(t)) \\ m_3\ddot{p}_3(t) &= k_2(p_2(t) - p_3(t)) - k_3p_3(t) + c_2(\dot{p}_2(t) - \dot{p}_3(t)] - c_3\dot{p}_3(t) \end{aligned}$$

These equations of motion can then be arranged into the LTI state-space matrix form, giving the following state matrices.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_1/m_1 & k_1/m_1 & 0 & -c_1/m_1 & c_1/m_1 & 0 \\ k_1/m_2 & -(k_1+k_2)/m_2 & k_2/m_2 & c_1/m_2 & -(c_1+c_2)/m_2 & c_2/m_2 \\ 0 & k_2/m_3 & -(k_2+k_3)/m_3 & 0 & c_2/m_3 & -(c_2+c_3)/m_3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

The system zeros can easily be found by converting the state space system into a zero-pole-gain form using MATLAB's `zpk()` function. The poles of the system are compared to the zeros to establish which pole-zero pairs cancel, and which pole will lead to the desired system behavior. None of the zeros overlap a pole, yet the first zero has only a real component, indicating a lack of oscillation. The second and third zeros are conjugate pairs with only imaginary components. The second zero $s_0 = 0 + 1.4142i$ was used for the following computations.

The initial condition vector was computed by setting up a generalized eigenvalue problem matrix using the system matrices.

$$A_z = \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix}$$

The null space of the A_z matrix corresponds to the initial state x_0 and forcing input u_0. The values calculated for these variables is as follows:

$$x_0 = \begin{bmatrix} -0.0098 \\ 0.0000 \\ 0.4964 \\ -0.0973 \\ 0.0000 \\ -0.0003 \end{bmatrix}, \quad u_0 = \begin{bmatrix} -0.4768 - 0.1378i \end{bmatrix}$$

The system is then solved using MATLAB's `ode45()` function. The initial state x_0 and input u_0 can be plugged into the system to confirm that m_2 demonstrates zero motion.

3.5.3 *System Modes for Second-order Systems*

3.5.4 *Bounded Input, Bounded Output Stability*

3.6 Frequency Domain Methods

4 Linear Control Methods

4.1 Open Loop

The system identification reveals how the inputs affect the outputs and the state of the system. We can work backwards from the desired output to find the required input.

Example In the context of a car's cruise control, we would like to achieve a desired speed with our throttle input. With a model of the car, we can determine the effect of the factors at play in the system: Throttle response, engine force, drag force, tire friction, etc. Treating the system as an equation and solving through, the throttle input required can be solved.

4.2 Closed Loop (Feedback)

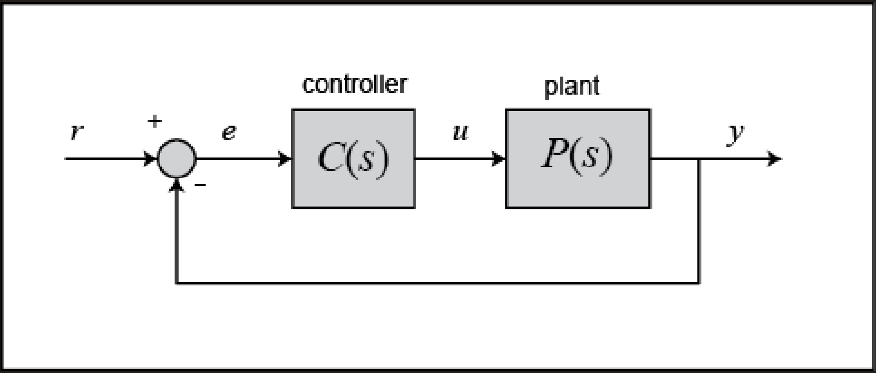


Figure 6: Block Diagram of Feedback Loop

Example Returning to the example of the cruise control, imagine that the same system (plant + controller) encounters a hill. Climbing up the slope changes the effect of the gravity force, meaning that our previous throttle input is not longer bringing the desired results. If a feedback system is implemented, the error between the desired speed and the actual speed can be calculated. With this information available, we can design our controller to account for disturbances, such as that hill.

4.2.1 *Purpose*

Note: Feedback control systems may be myopic! In many designs, they do not have the information or sophistication to understand that current decisions might make control of the vehicle impossible later on!

4.3 Bode Plots

4.4 PID Control

The PID controller is one of the simplest types of controllers.

Proportional Gain (P term) Applies a control input proportional to the amount of error.

Integral Gain (I term) Applies a control input proportional to the integral of error (sum of error over time). This term is useful for countering steady-state error.

Derivative Gain (D term) Applies a control input proportional to the rate of change of the error. This term is useful for countering overshoot.

The PID controller can also be represented as a transfer function.

$$K_p + \frac{K_i}{s} + K_d s = \frac{K_d s^2 + K_p s + K_i}{s} \tag{4.1}$$

4.4.1 *Integrator Anti-windup Term*

In certain cases where the error does not correct rapidly, the I term may ”wind up” (error over time builds up, resulting in large corrections). The large correction applied may destabilize the system. To counter this, an ”anti-windup” term can be used, to limit the sum of error the integral term sees.

Example

4.5 Higher-order Controllers

4.5.1 *Lead-Lag Control*

4.6 Feedforward

Feedforward control can address some of the drawbacks of feedback control, such as the case of myopia (4.2.1). For this reason, feedforward is sometimes referred to as guidance.

4.7 Designing the Controller

4.7.1 *Frequency Response Method*

4.7.2 *Loop-Shaping Method*

Loop-shaping is somewhat of an enhancement to the frequency response method, using the Nyquist plot in addition to the Bode plot.

4.7.3 *Root Locus Method*

4.8 Multi-Input, Multi-Output (MIMO)

5 Nonlinear Methods

A nonlinear plant is one that includes nonlinear functions in its dynamics (eg. powers, sine, cosine). Refer to (2.2.3). Many of the assumptions that allow us to solve linear systems do not apply. For example, superposition is generally not true for a nonlinear system.
A typical nonlinear dynamical system:

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), u(t))\\ y(t) &= g(t, x(t), u(t))\end{aligned}$$

where $t \in \mathbb{R}$ is monotonically increasing. f defines the dynamics of the system (nonlinear). g defines the output (measured quantities).

Example A classic example is the two dimensional mass-spring-damper system. No forces are present in the springs and dampers when $p_x = p_y = \dot{p}_x = \dot{p}_y = 0$. Gravity is neglected for this problem.

$$x = \begin{bmatrix} p_x \\ p_y \\ \dot{p}_x \\ \dot{p}_y \end{bmatrix}, \quad u = \begin{bmatrix} f_x \\ f_y \end{bmatrix}, \quad y = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

Defining δ_p as the spring displacement, the equations of motion are therefore

$$f(t, x, u) = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \sum_{i=1}^{3} \frac{k_i}{m} \delta_{p,i} \hat{n}_{x,i} + \sum_{i=1}^{3} \frac{c_i}{m} \delta_{\dot{p},i} \hat{n}_{x,i} + \frac{f_x}{m} \\ \sum_{i=1}^{3} \frac{k_i}{m} \delta_{p,i} \hat{n}_{y,i} + \sum_{i=1}^{3} \frac{c_i}{m} \delta_{\dot{p},i} \hat{n}_{y,i} + \frac{f_y}{m} \end{bmatrix}$$

Assuming that the mass has neglible dimension, and defining $p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$:

$$\begin{aligned}\delta_{p,i} &= ||p_i||_2 - ||p_i - p||_2\\ \delta_{\dot{p},i} &= -\dot{p}^\top \hat{n}_i\\ \hat{n}_i &= \frac{p - p_i}{||p - p_i||_2}\end{aligned}$$

Note that we have arbitrarily defined the outputs $y(t)$ of the system to be the x, y position of the mass. If we were measuring the position and velocity, the choice of output would be different.

5.1 Linearization

The easiest way to control a nonlinear system is by making it linear. The Jacobian matrix (2.2.4) can be calculated about the point.

Beware that the accuracy of the linearized model decreases the farther away (in terms of state) you are calculating. If compute power allows,

5.1.1 *Taylor Series Expansion*

One method to linearize is to use the Taylor series expansion and retain only the zero-th and first order term.

5.1.2 *Jacobian*

Refer to section (2.2.4).

5.2 Sliding Mode Control

6 State Estimation

6.1 Purpose

6.2 Observer

6.3 Measurement Model

6.4 Filtering

Filtering is the real-time subset of estimators.

6.4.1 *Calibration*

6.5 Complementary Filter

6.6 Alpha-Beta Filter

6.6.1 *Alpha-Beta-Gamma Filter*

6.7 Least-Squares Estimator

6.8 Linear-Quadratic Estimator (Kalman Filter)

6.8.1 *Extended Kalman Filter (EKF)*

The Kalman filter is limited to linear systems. One method to apply it to nonlinear systems is to linearize the system at the current time step, and perform the same algorithm.

6.8.2 *Unscented Kalman Filter (UKF)*

6.9 Particle Filter

6.9.1 *SIS Filter*

6.9.2 *SIR Filter*

6.9.3 *The Curse of Dimensionality*

6.10 Ensemble Kalman Filter (EnKF)

6.11 Sensor Fusion

6.11.1 *Madgwick Filter*

Note the quaternion derivative appears (Eq. 7.1).

6.11.2 *Mahoni Filter*

7 Modeling & Simulation

7.1 Reference Frames

7.1.1 *Inertial Frame*

7.1.2 *Body Frame*

7.1.3 *North-East-Down Frame*

7.1.4 *Intermediate Frames*

Transforming Between Frames

Wind Frame

Stability Frame

7.1.5 *Coriolis Forces*

7.2 Determining the State Vector

7.2.1 *Position & Derivatives*

7.2.2 *Rotation & Derivatives*

Euler Angles

Direction Cosine Matrices

Quaternions

Quaternion Derivatives

$$\dot{q} = \frac{1}{2}\Omega q, \quad \text{where } \Omega = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{7.1}$$

$$\text{Recall that } \omega^b_{b/v} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Converting Between Representations There are different methods based on which representation is being converted from/to. For ease, MATLAB has the built-in functions `eul2dcm`, `dcm2quat`, `quat2eul`. There is also the `angle2quat` function.

7.2.3 *Homogenous Transformation Matrix*

The idea behind the homogenous transformation matrix is to combine rotation and translation.

7.3 Flat Earth Equations of Motion

8 Optimal Control

8.1 Linear-Quadratic Regulator (LQR)

9 Robust Control

10 Adaptive Control

11 Predictive Control

12 Other Important Topics

12.1 Frequency Domain vs Time Domain

Computing the convolution integral is much more difficult in the time domain compared to the frequency domain. Using the frequency domain to analyze a system is means that the Laplace transform can be used, simplifying the convolution step significantly. See (2.2.7)

12.2 Continuous Time vs Discrete Time

When the controller is calculated by a computer, the system and feedback cannot be evaluated in continuous time. The system can be thought of as proceeding in time-steps.

12.2.1 *Z Transform*

The Z transform is the corollary to the Laplace Transform, only for discrete time where Laplace is for continuous.

12.3 Optimization

12.3.1 *Gradient Descent*

12.4 Singular Value Decomposition

13 Appendix: Quick Reference

14 Appendix: MATLAB / Simulink Reference

This chapter contains practical methods / tutorials to accomplish tasks in MATLAB and Simulink

14.1 Linearization

14.1.1 *MATLAB*

14.1.2 *Simulink*

14.2 Animation

14.2.1 *MATLAB*

14.2.2 *Simulink*

15 Appendix: Python Reference

16 References