University of Cape Town

Department of Statistical Sciences

STA2004F **Class Test 1** **17 March 2025**

Answer All Questions

TIME: 90 MINUTES **FULL MARKS: 48**

1. (17 marks) Consider a gambling game played as follows. You toss a *fair* coin, and if it comes up heads, you win R1, otherwise, you win an amount Z that is independently drawn from a given probability distribution with CDF F_Z. Let X be your winnings.
   (a) Show that the CDF of X is $F_X(x) = \frac{1}{2}F_Z(x) + \frac{1}{2}I_{[1,\infty)}(x)$. (3)
   (b) Find $\mathbb{E}(X)$ if $\frac{1}{2}Z \sim \text{Bernoulli}(p = 0.5)$. (3)
   (c) Now suppose $Z \sim \text{Exp}(\lambda = 1)$.
      i. Show that X is neither discrete nor continuous. (2)
      ii. Calculate $\Pr(0 < X < 1)$. (2)
      iii. Find $\mathbb{E}(X)$.
      Hint: Use Darth Vader. (3)
      iv. If $U \sim \mathcal{U}(0, 1)$, find a function $g : (0, 1) \to \mathbb{R}$ such that $g(U)$ has the same distribution as X. Be sure to prove that the given g satisfies this condition. (4)

2. (8 marks) Let $X \sim \mathcal{U}(-2, 1)$ and $Y := \lfloor |X| \rfloor$ (= floor of absolute value of X).
   (a) Find the PMF of Y. (3)
   (b) Calculate $\rho(X, Y)$. (5)

3. (12 marks) Suppose (X, Y) is uniformly distributed on the region

$$\mathcal{A} := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4, x > 0, y > 0\right\}.$$

   (a) Write down the joint PDF of (X, Y). (2)
   (b) Find the marginal PDFs of X and Y. (4)
   (c) Find the joint and marginal PDFs of $R := \sqrt{X^2 + Y^2}$ and $\Theta := \arctan \frac{Y}{X}$. Show that R and Θ are independent.
   Hint: The inverse transformations are $x = r\cos\theta$ and $y = r\sin\theta$. (6)

4. (6 marks) Let $X \sim N(0, 1)$. Prove that $\cos X$ and $\sin X$ are uncorrelated but not independent.
   Hint: Don't find the joint or marginal distributions (you can do so if you want to). For independence, consider the events $\left\{\sin^2 X < 0.5\right\}$ and $\{\cos^2 X < 0.5\}$.

5. (5 marks) Prove that the kurtosis of any random variable (with finite 4th moment) is always greater than or equal to 1. Also show that this lower bound is attained by the Bernoulli(0.5) distribution.
   Hint: Write $(X - \mu_X)^4 = ((X - \mu_X)^2)^2$ and apply Jensen's Inequality to $\varphi(x) = x^2$.