Zip-zip Trees: Making Zip Trees More Balanced, Biased, Compact, or Persistent

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[2]Princeton University

WADS, 2023

Motivation

- 2018 – Zip tree (Tarjan, Levy, Timmel)
  - $1.5 \log n$ expected average depth
    - Min key: $0.5 \log n$, Max key: $\log n$
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  - ☑ May be partially persistent
  - ☑ Supports biased keys (still $O(\log \log n)$ bits per node)

Skip List

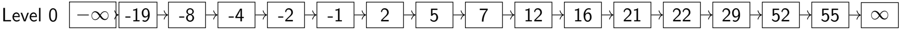
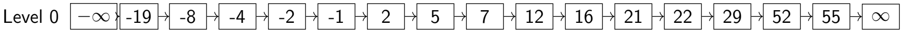


Figure 1: A sorted linked list

Skip List

- What if you add 'fast lanes'?



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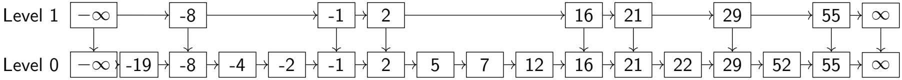

Figure 1: A skip list with one coin flip

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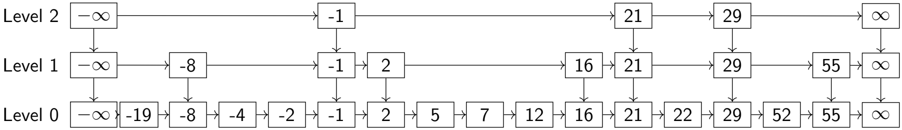

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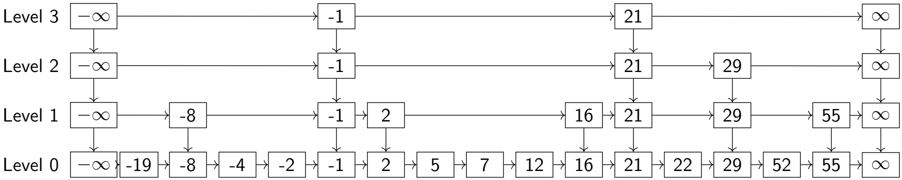

Figure 1: A skip list with three coin flips

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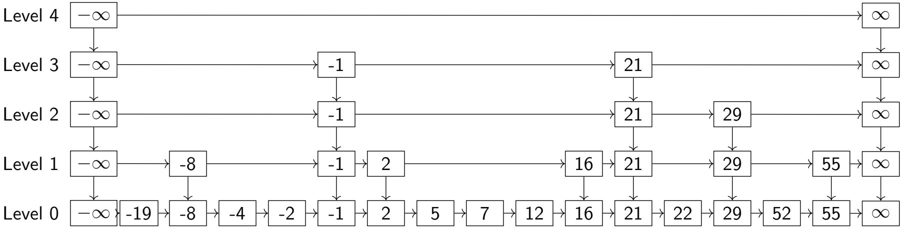

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Skip List

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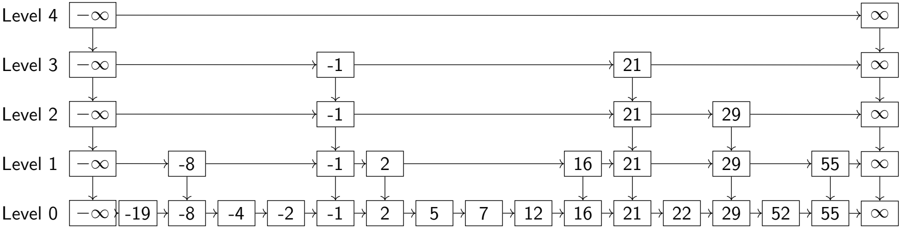
- Idea: 1 move in level $k \approx 2$ in level $k-1 \approx 2^k$ in level 0



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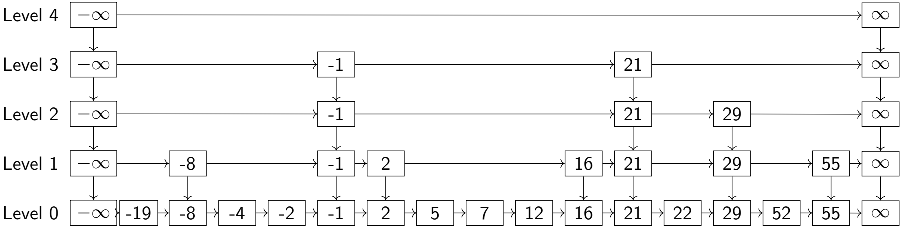

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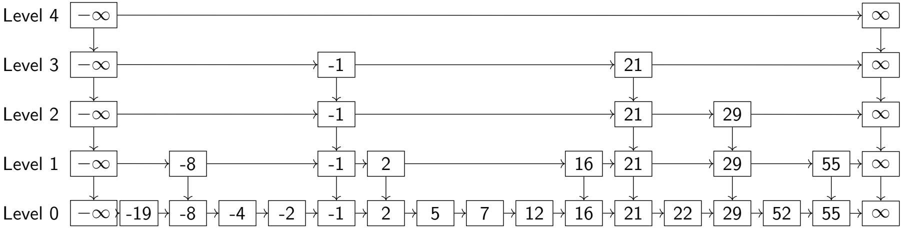

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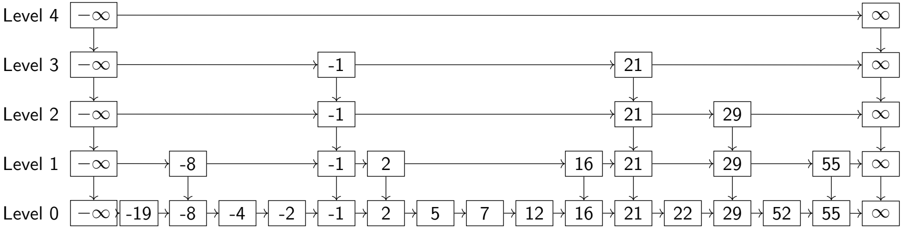

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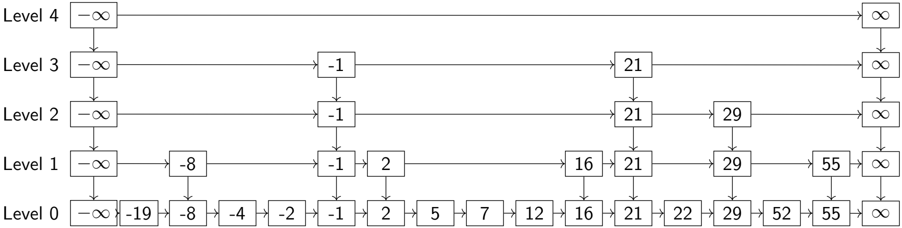
- Expected 2*n* nodes - bad



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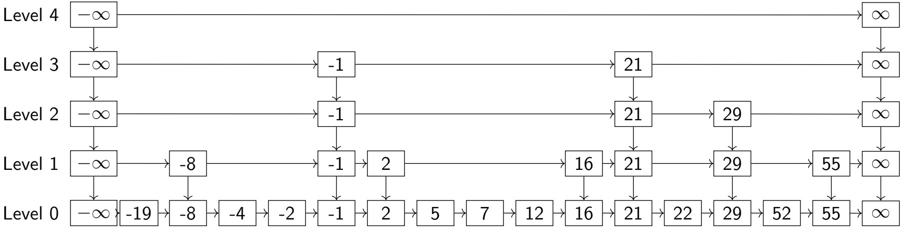

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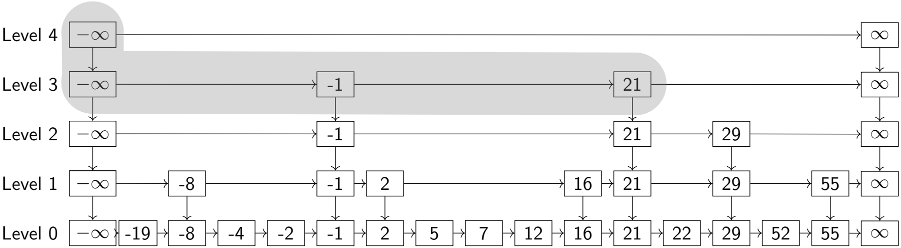

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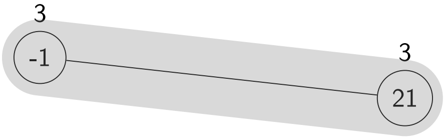

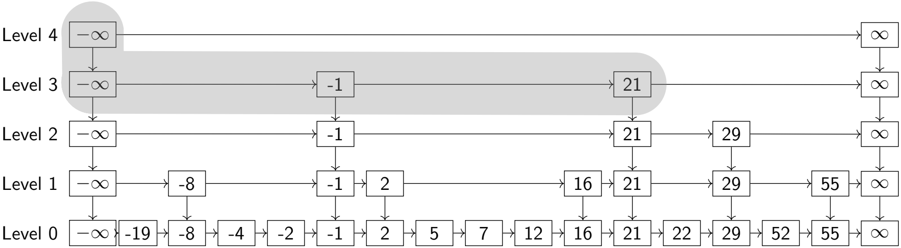

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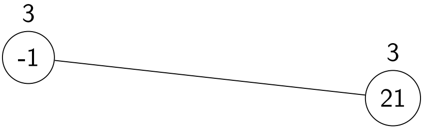

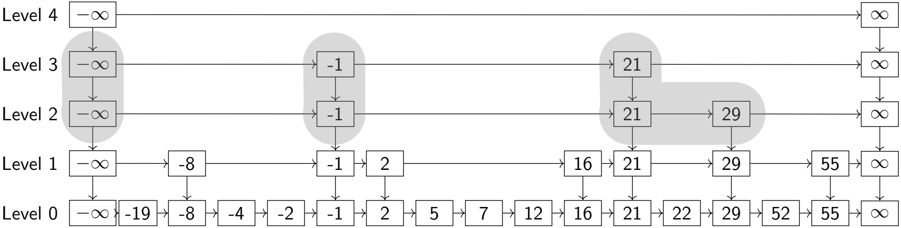

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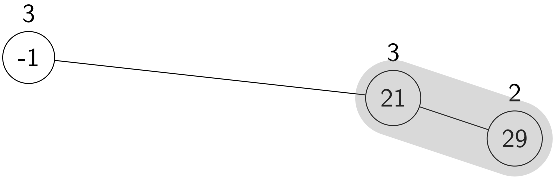

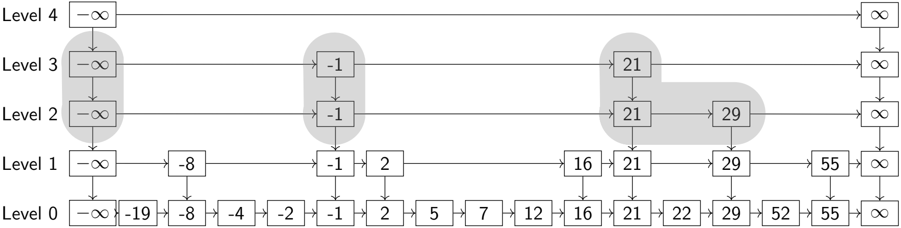

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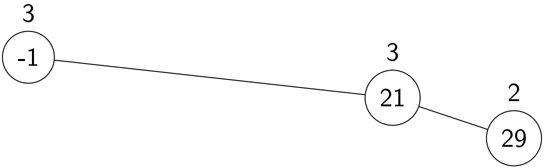

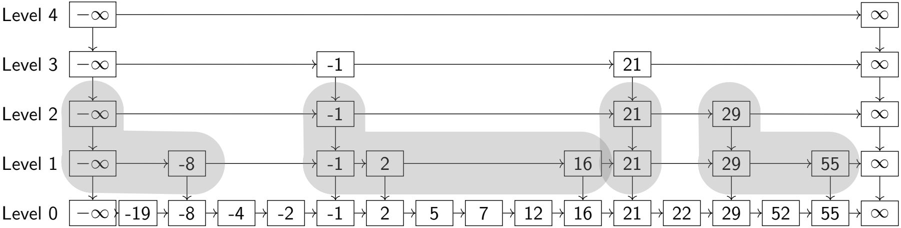

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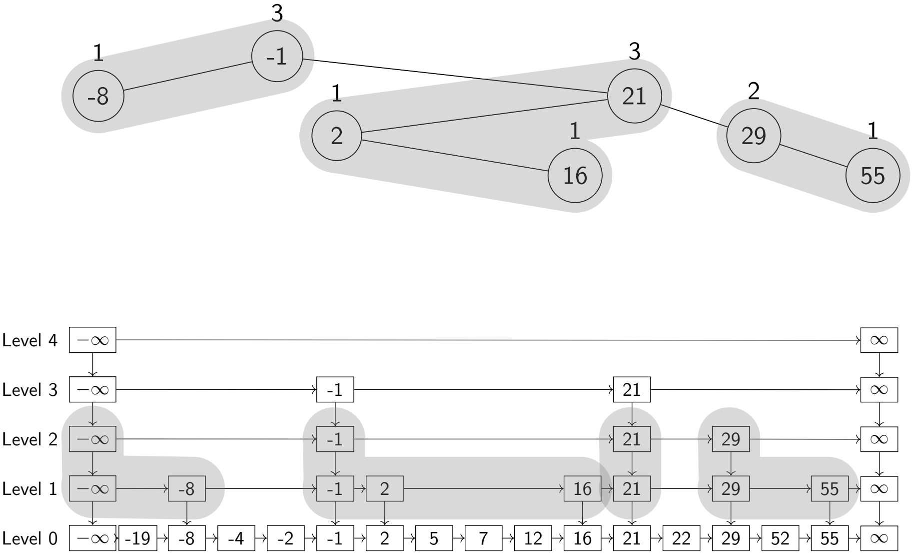

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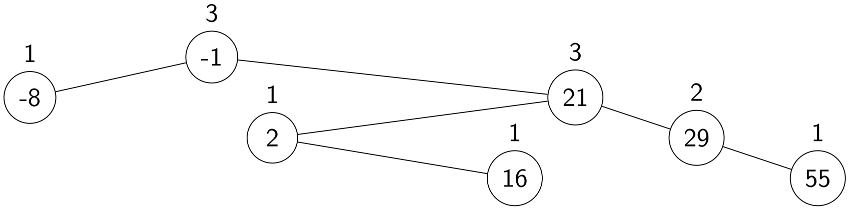

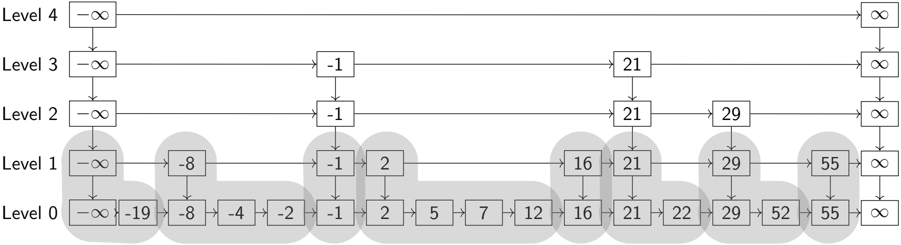

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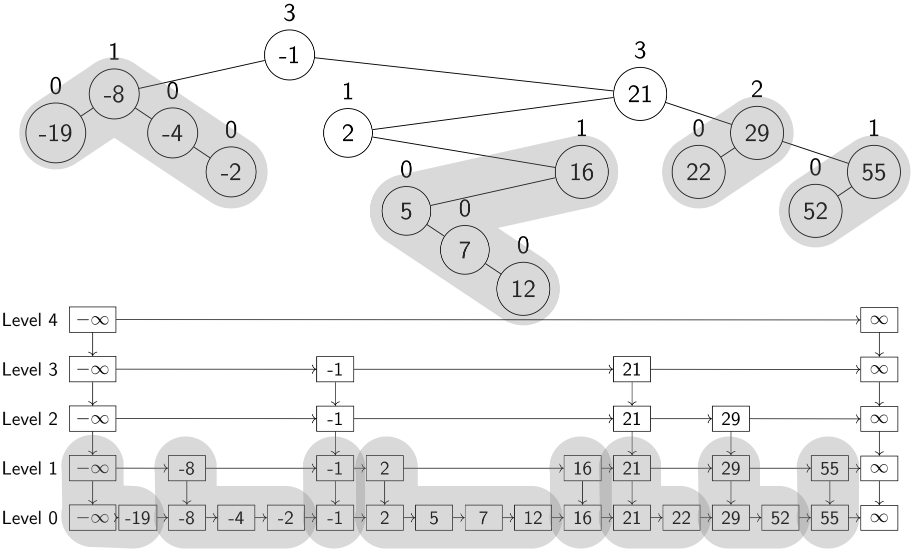

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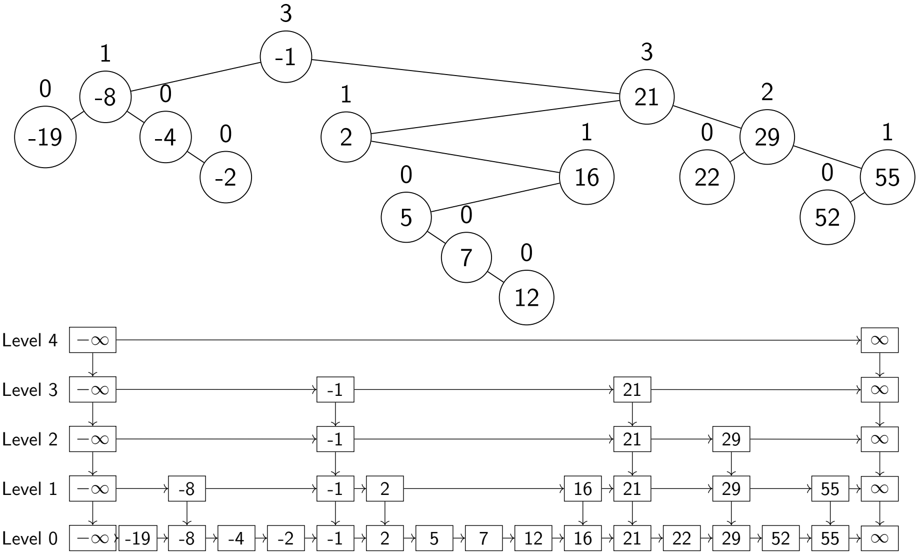

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- Can we do better?

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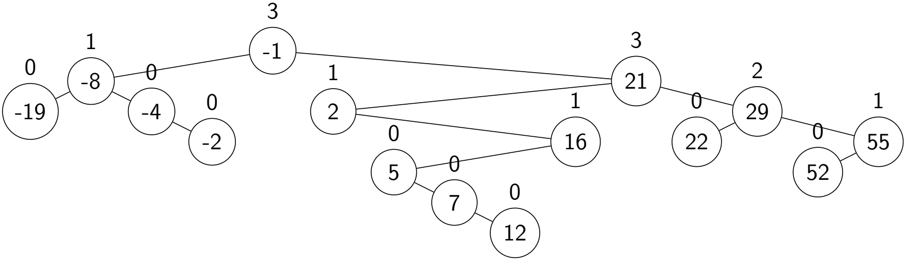
- Hope: fewer collisions, better depth?

Zip-zip Trees Example



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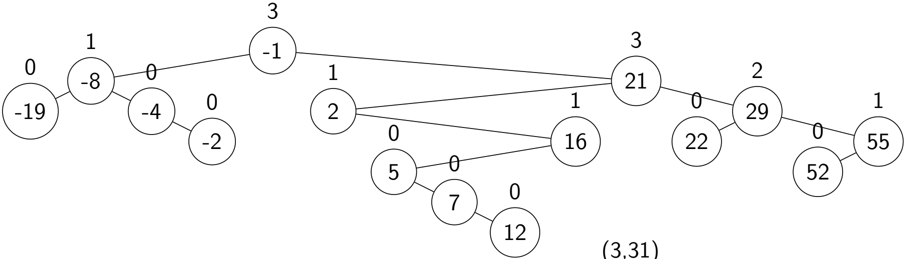

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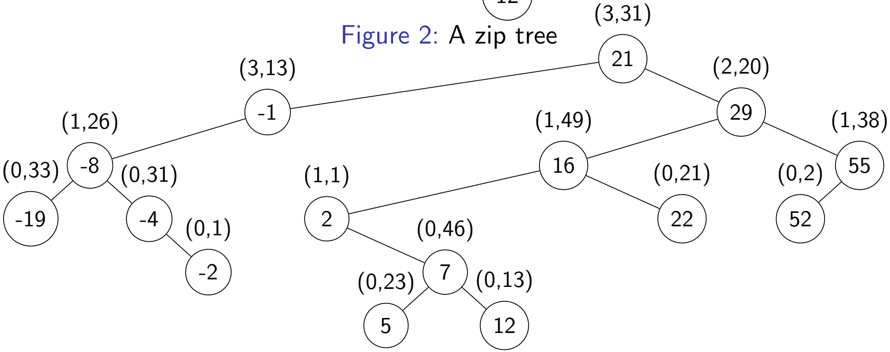

Figure 3: A random zip-zip tree generated from the above zip tree

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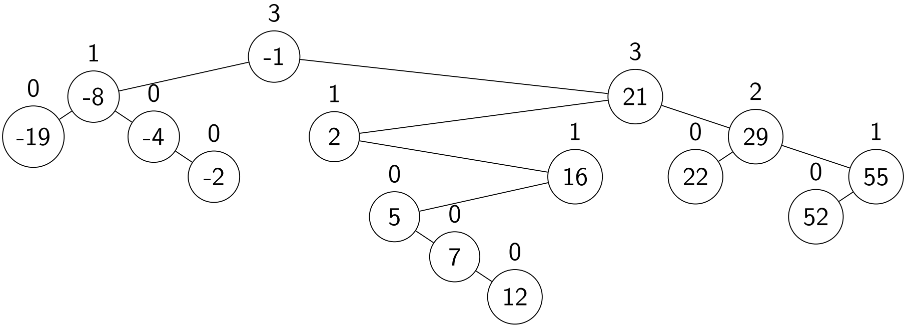
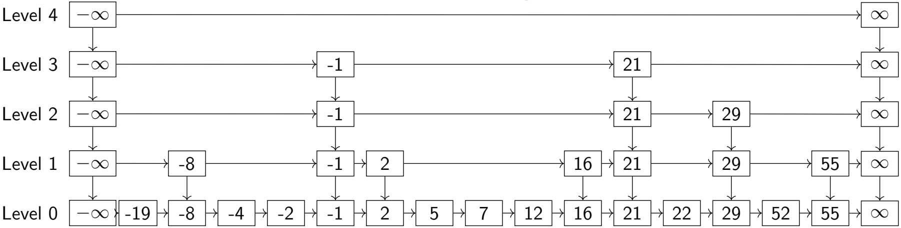
- Idea: Consider rank groups





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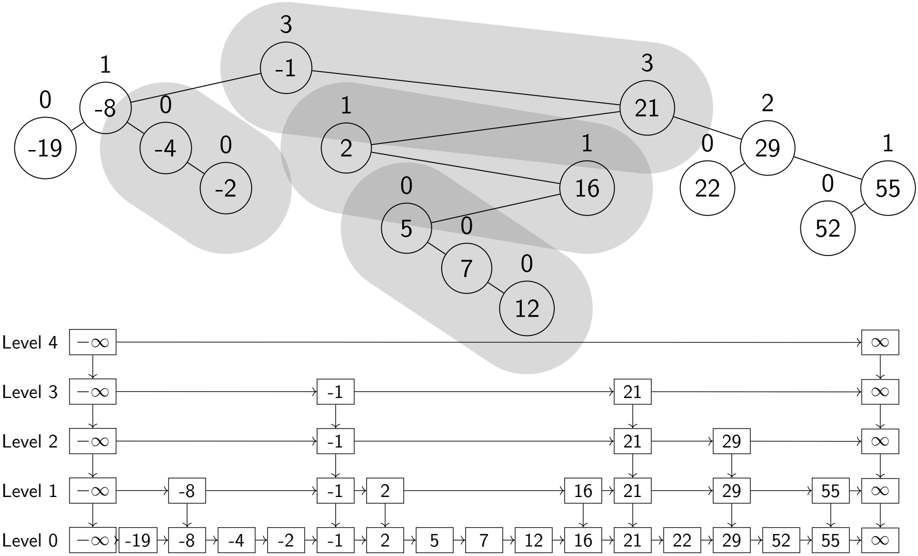

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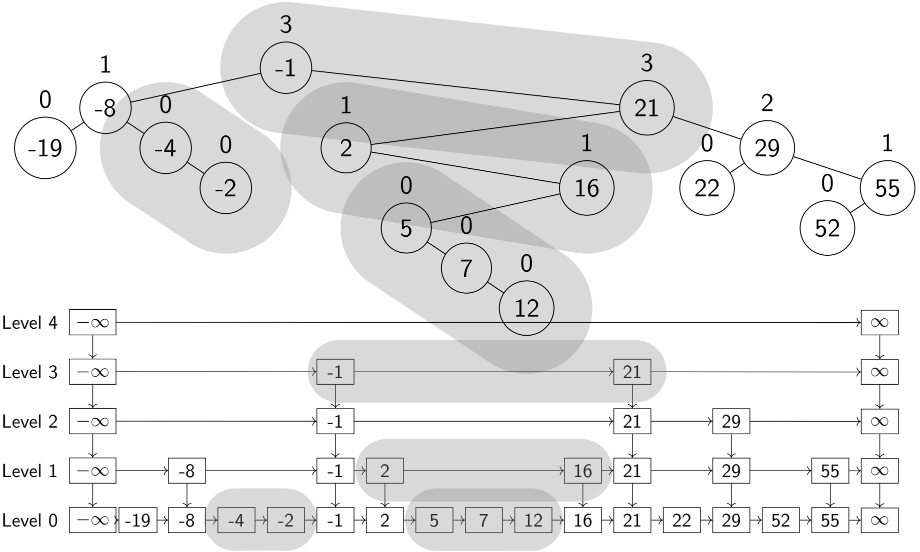

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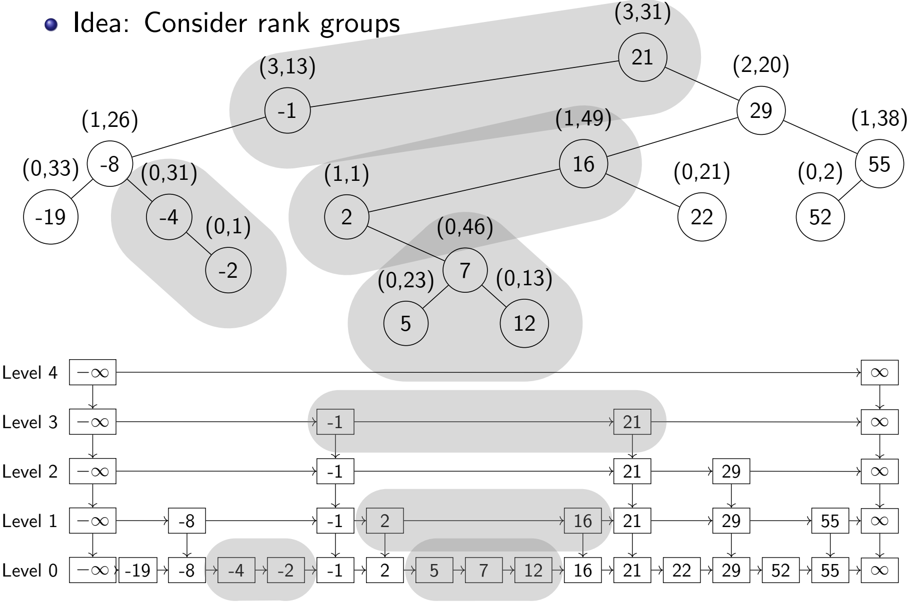


Zip-zip Trees Analysis II

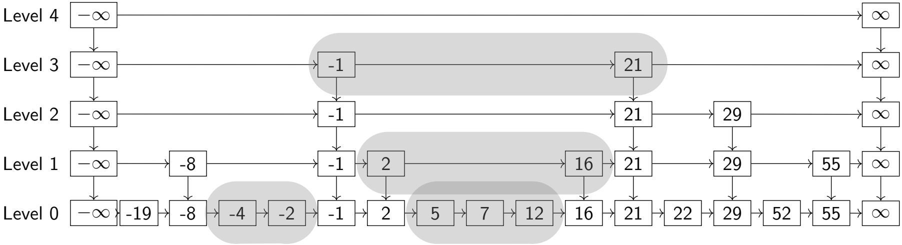
- How big are rank groups?

Lemma

The size of an r_1 rank group has expected value 2 and is $< 2\log n$ w.h.p.

Proof Sketch.

- Size is (at most) geometrically distributed $\square$



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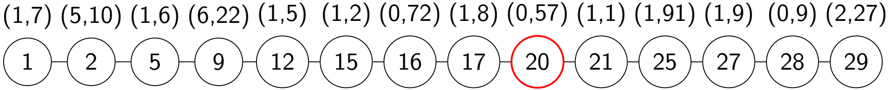

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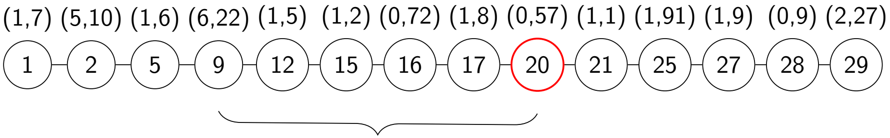

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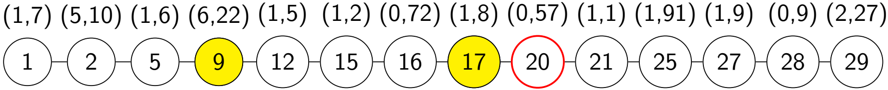

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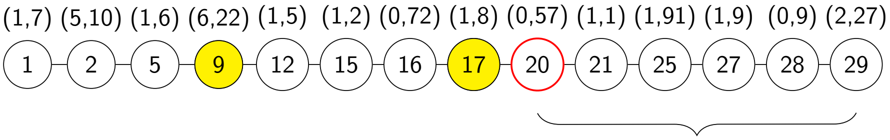

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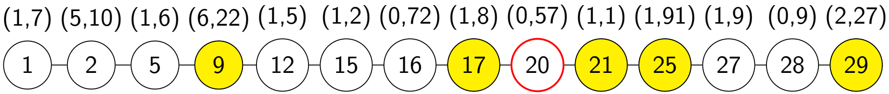

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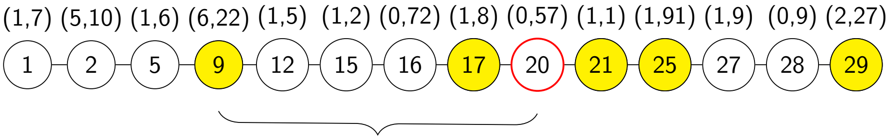

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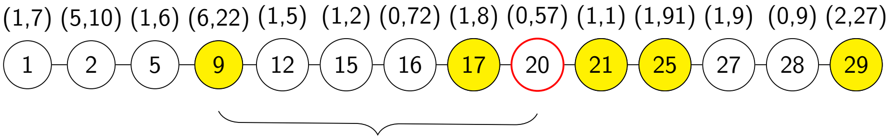
$$P(x_i \prec x_j) = \frac{1}{|i-j|+1}$$

$$\delta_j = \sum_{i=1}^{j} \frac{1}{j-i+1} + \sum_{i=j+1}^{n} \frac{1}{i-j+1} = H_j + H_{n-j+1} - 1$$

$\square$



Zip-zip Trees Analysis IIII

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Corollary

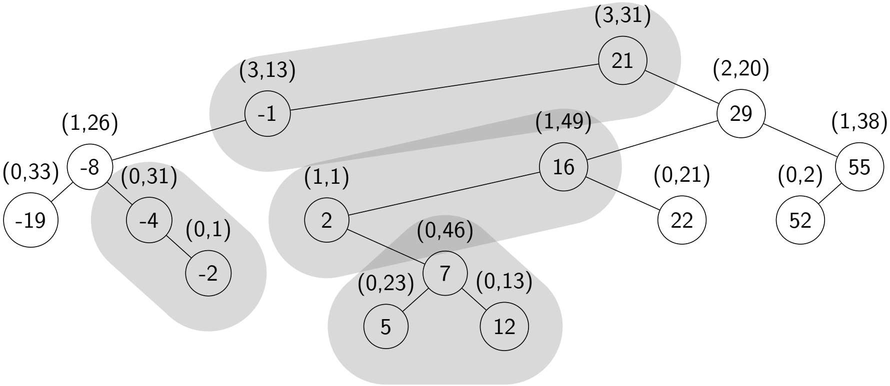
The expected depth of the min and max keys is $0.69 \log n + \gamma + o(1)$

Corollary

The expected depth of any key is at most $1.39 \log n - 1 + o(1)$

Just-in-Time (JIT) Zip-zip Trees

- Ranks can be up to $O(\log n)$, but don't differ much



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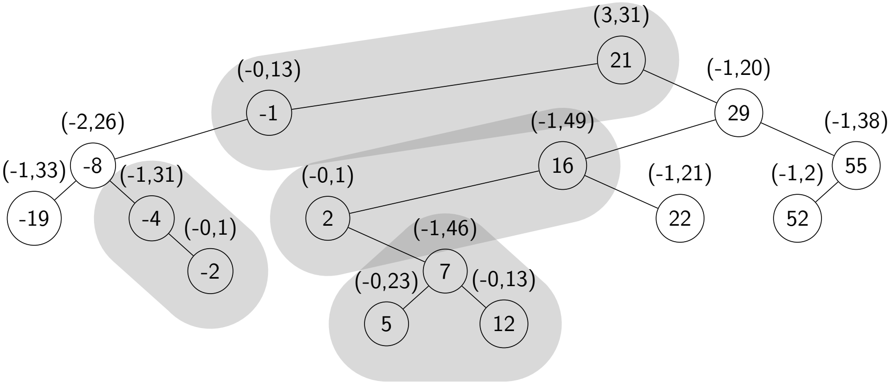
Just-in-Time (JIT) Zip-zip Trees

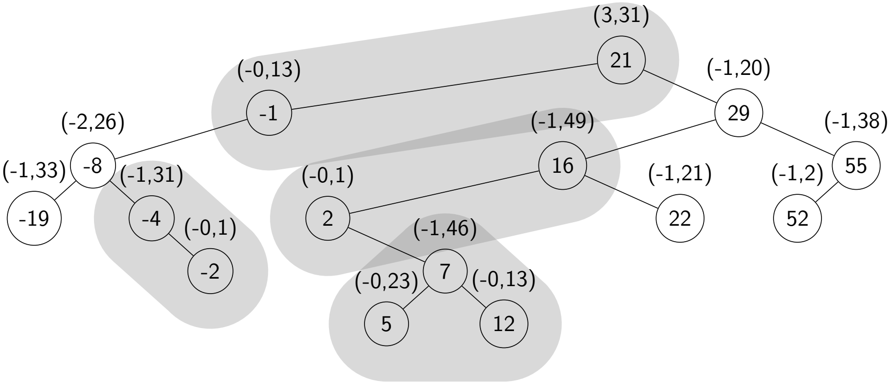
- Ranks can be up to $O(\log n)$, but don't differ much
  - Store r_1 rank differences! (Expected $O(1)$)



1

Just-in-Time (JIT) Zip-zip Trees

- Ranks can be up to $O(\log n)$, but don't differ much
  - Store r_1 rank differences! (Expected $O(1)$)
- Rank groups are small...

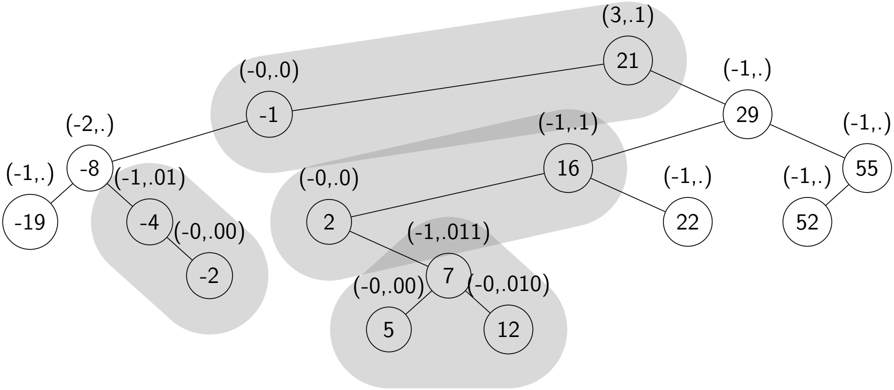


1

Just-in-Time (JIT) Zip-zip Trees

- Ranks can be up to $O(\log n)$, but don't differ much
  - Store r_1 rank differences! (Expected $O(1)$)

- Rank groups are small...
  - Generate r_2 ranks on the fly! (Expected $O(1)$)[1]



[1] r_1 differences are $O(1)$ per node, r_2 are $O(1)$ per operation

Depth Discrepancy

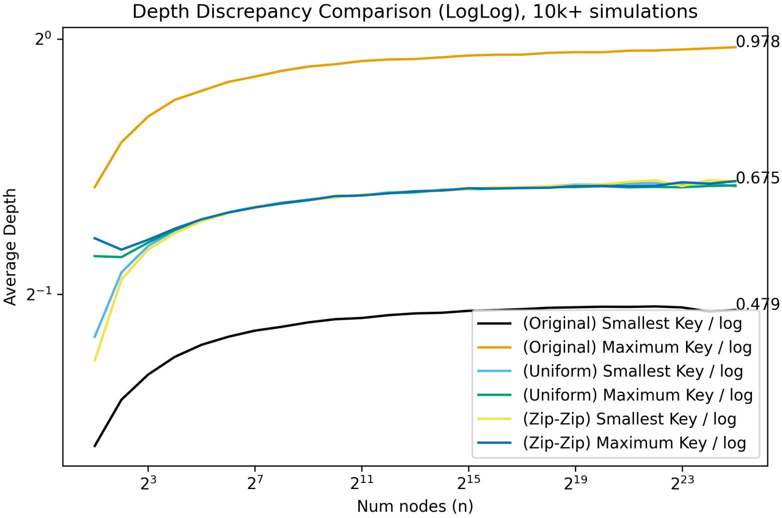


Figure 4: The depth discrepancy between the min and max keys for three variants

Average Key Depth and Tree Height

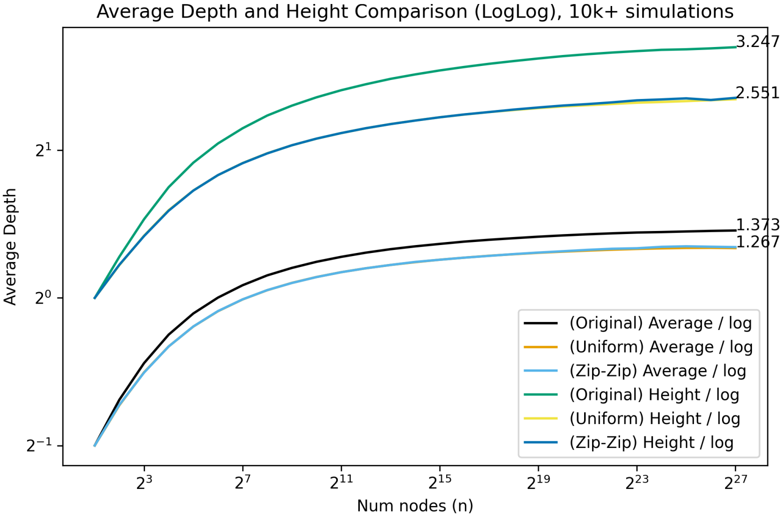


Figure 5: The average node depth and tree height for three variants

Rank Collisions

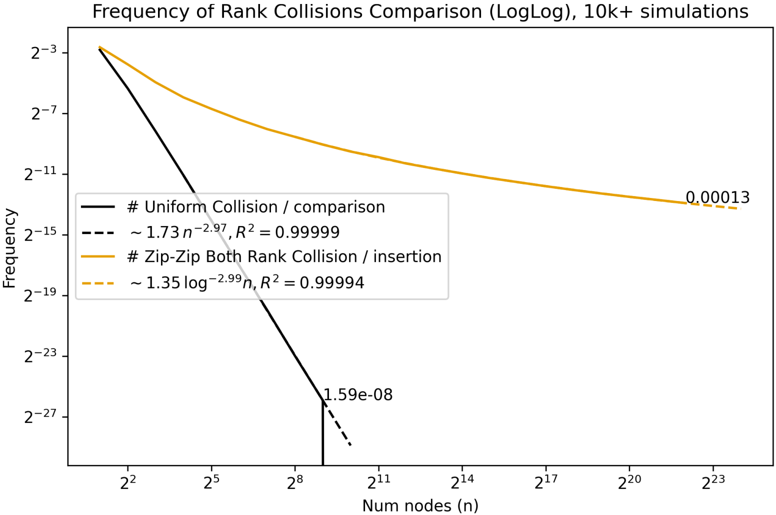


Figure 6: The frequency of encountered rank ties per rank comparison for the uniform variant and per element insertion for the zip-zip variant

Just-in-Time Zip-zip Tree Size

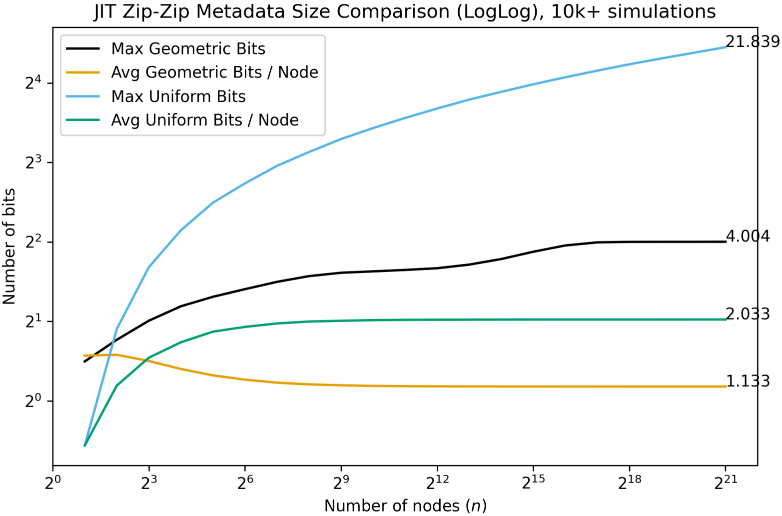


Figure 7: The metadata size for the just-in-time implementation

Summary

- 2023 – Zip-zip tree
  - ☑ $1.39 \log n$ expected average depth
  - ☑ Space cost: $\log \log n$ bits per node (or $O(1)$ bits per update w.h.p.)
  - ☑ Easy to implement
  - ☑ Strongly history independent (except JIT)
  - ☑ May be partially persistent
  - ☑ Supports biased keys (still $O(\log \log n)$ bits per node)

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- Any questions?