Highway Preferential Attachment Models for Geographic Routing

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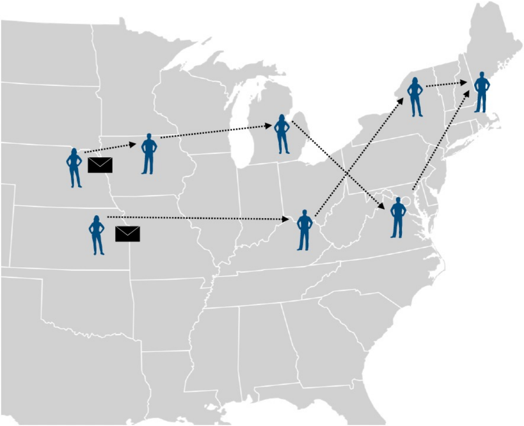
University of California, Irvine

COCOA, 2023

History – Milgram's Small-World Experiments

- Kansas and Nebraska $\rightarrow$ Massachusetts



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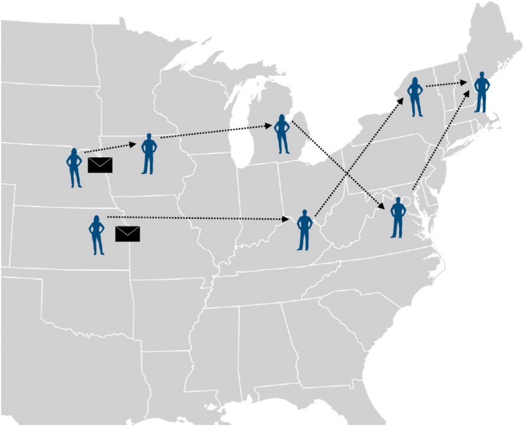

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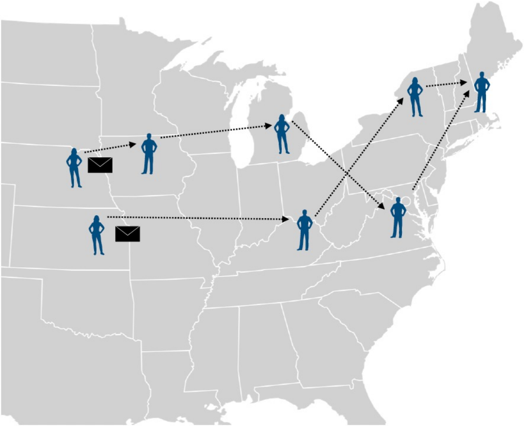

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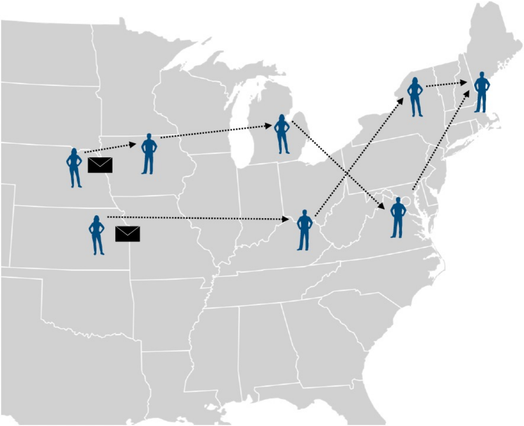

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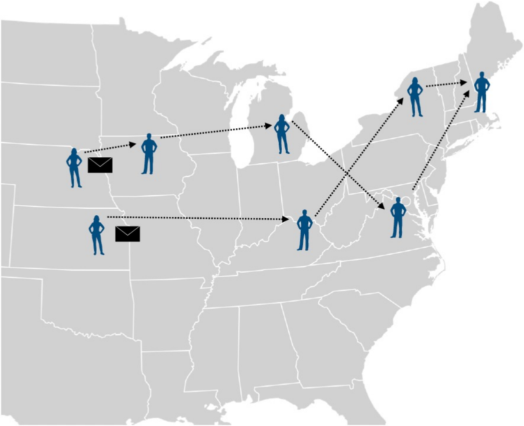

History – Milgram's Small-World Experiments

- Kansas and Nebraska → Massachusetts
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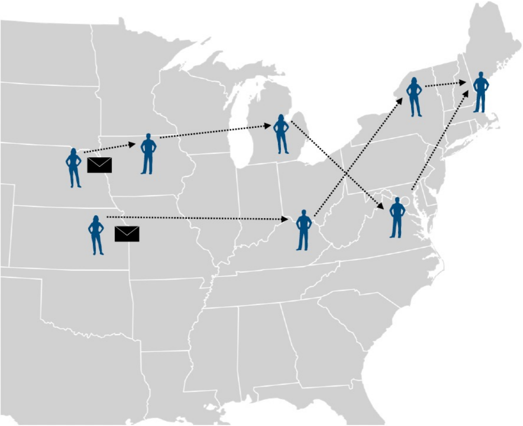
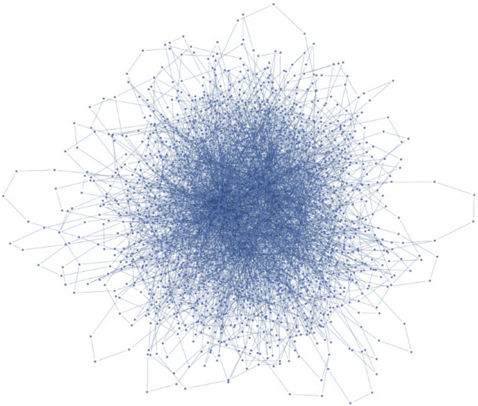
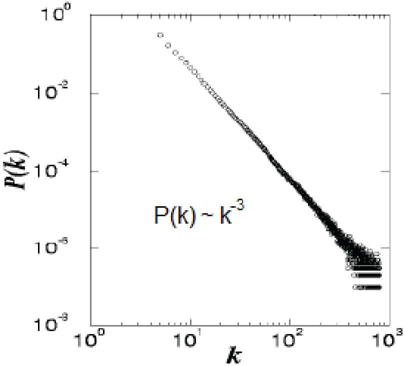
- How many 'hops'? - ~6!
- Popularized "six degrees of separation"
- How to model?



Preferential Attachment Models

- Rich get richer
- $P(u \to v) \propto d_v$





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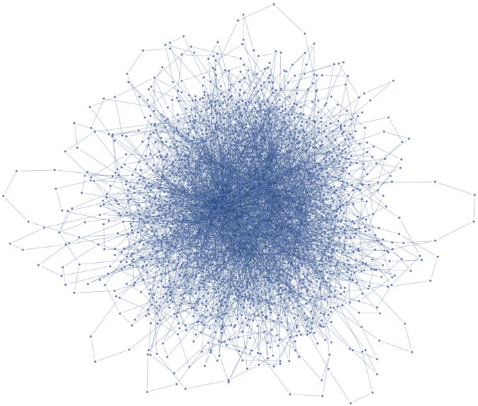
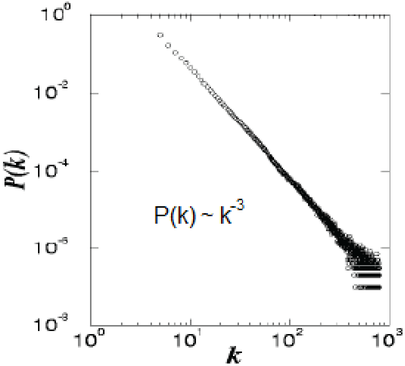
☑ Low ($\mathcal{O}(\log n)$) diameter





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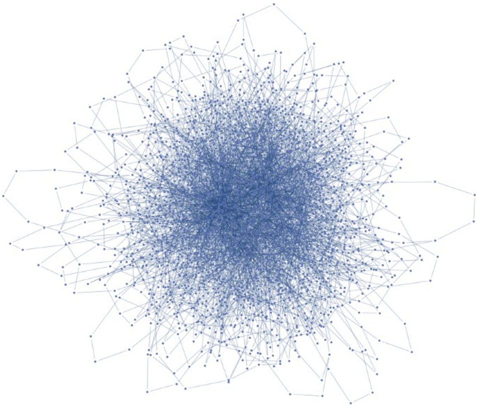
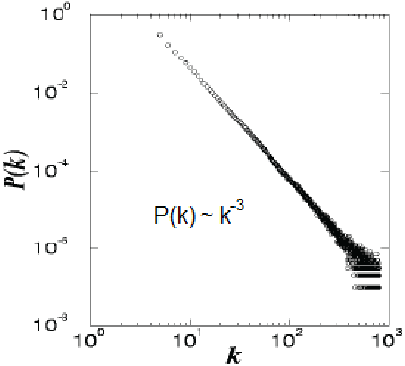
☒ No geography $\to$ no greedy routing





Kleinberg's Model $\mathcal{K}(n, p, q)$

- 2-D $n \times n$ square

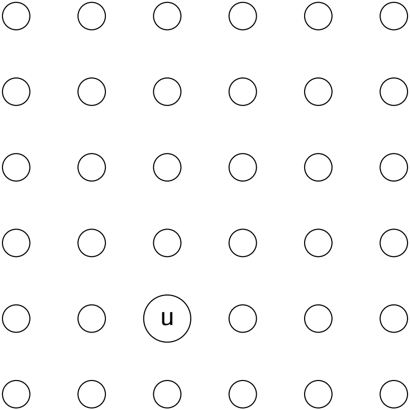
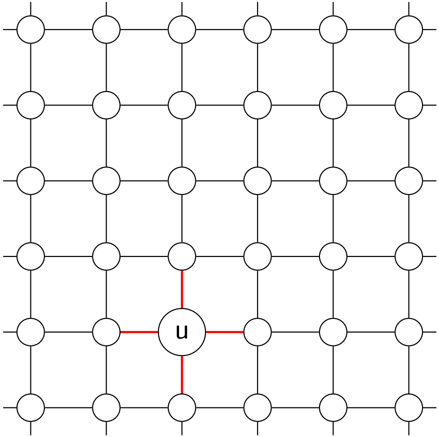


Figure 2: Kleinberg's Model $\mathcal{K}^*(6, 0, 0)$

Kleinberg's Model $\mathcal{K}(n, p, q)$

- 2-D $n \times n$ square
- Local connections p
- Wrap-around



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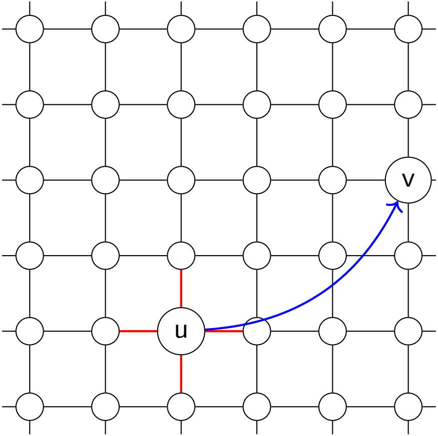

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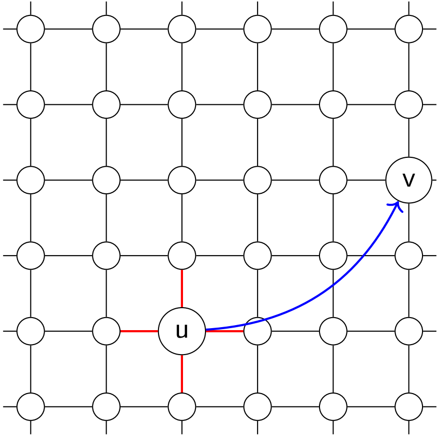
- Long-range connections q
- $P(u \to v) \propto \delta^{-s}(u, v)$



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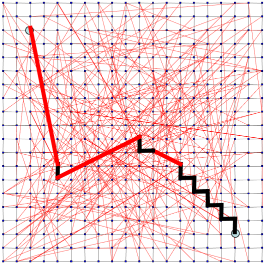
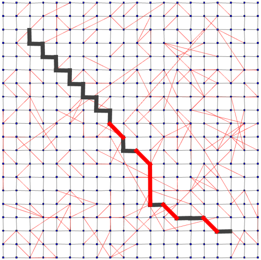
Kleinberg's Results

- $\mathcal{O}(\log^2 n)$ greedy routing when $s = 2$



(a) $s < 2$



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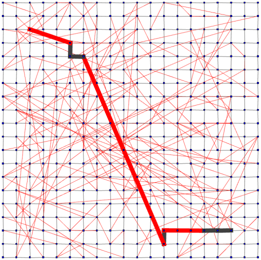

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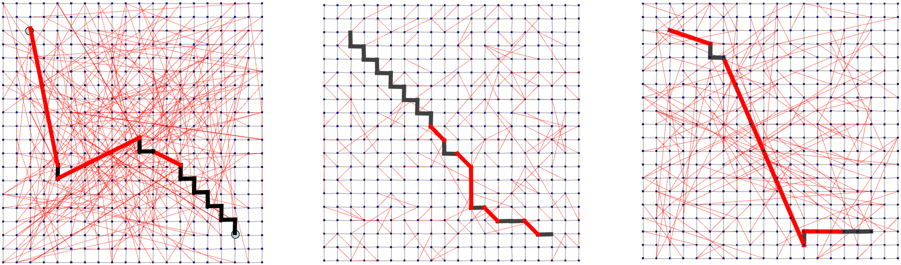

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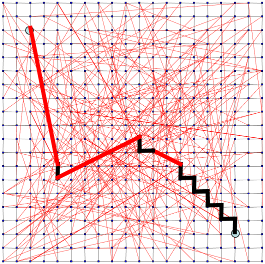

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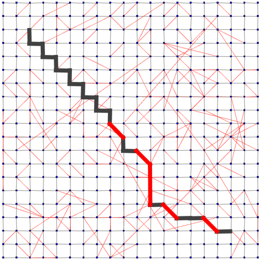

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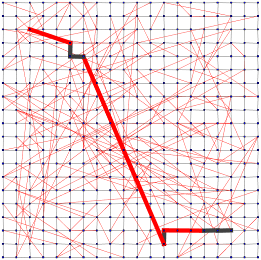

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Neighborhood Preferential Attachment (NPA)

- Idea: Combine Kleinberg w/ Preferential Attachment
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- What if $P(u \to v) \propto d_v / \delta^s(u, v)$?
- Experimentally good, but no theory

Kleinberg Highway

- Kleinberg's model: all nodes equal

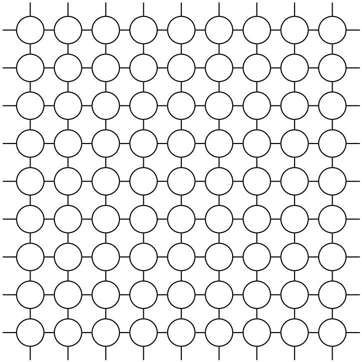
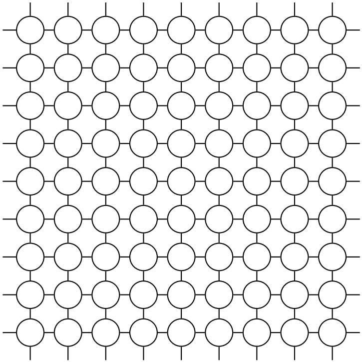


Figure 4: Kleinberg Highway Graph $n = 9$

Kleinberg Highway

- Kleinberg's model: all nodes equal
- Reality:



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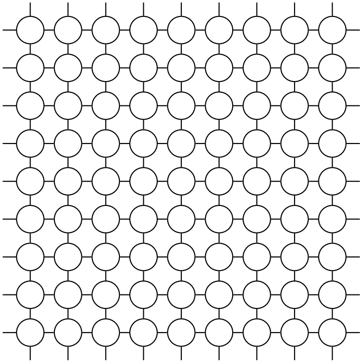

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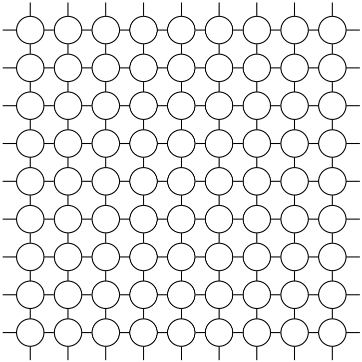

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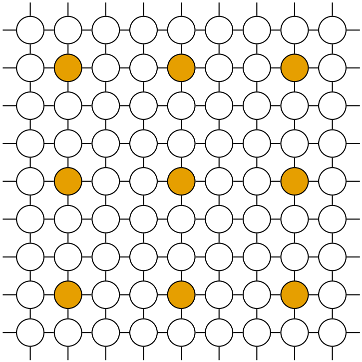

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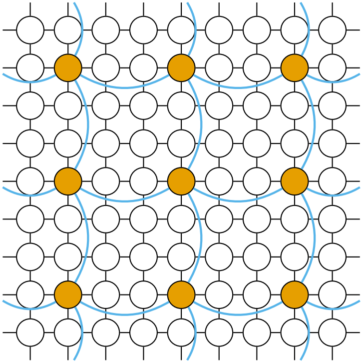

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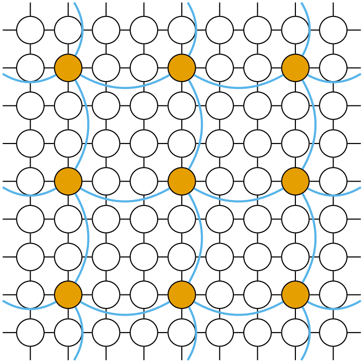

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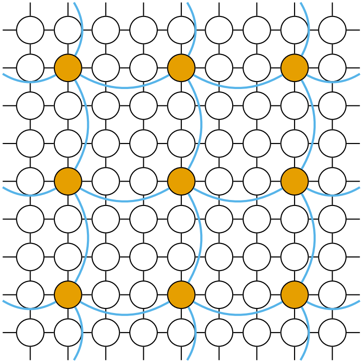
- What degree?



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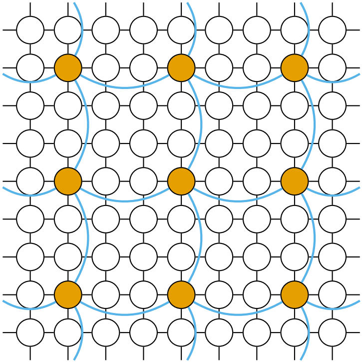
- What degree? $\Theta(k)$



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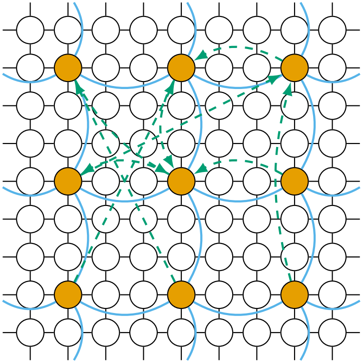

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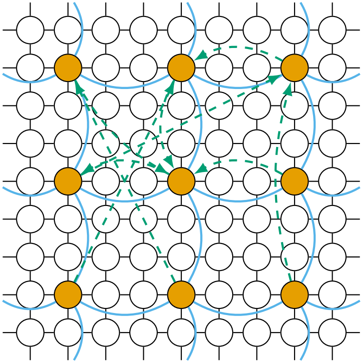
- *Only to other highway nodes!*



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Kleinberg Highway – Analysis Sketch

1. Reach highway
2. Traverse highway
3. Reach destination

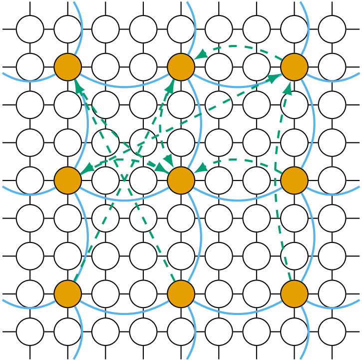
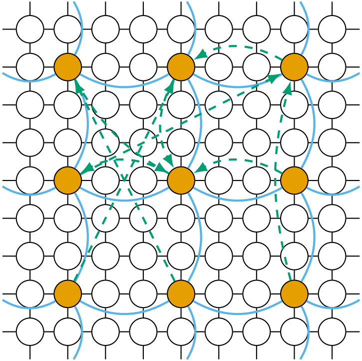


Figure 5: Kleinberg Highway Graph

Kleinberg Highway – Analysis Sketch

1. Reach highway - $\mathcal{O}(\sqrt{k})$
2. Traverse highway
3. Reach destination



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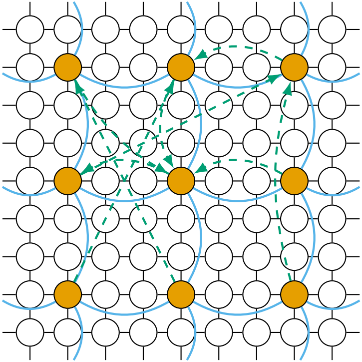

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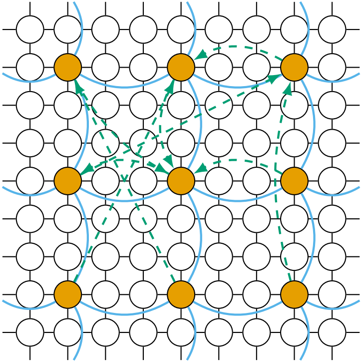
1. Reach highway - $\mathcal{O}(\sqrt{k})$
2. Traverse highway - $\mathcal{O}(\log^2(n))$?
3. Reach destination - $\mathcal{O}(\sqrt{k})$



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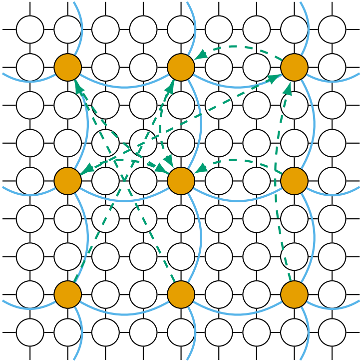

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Kleinberg Highway – Result

Theorem

The expected decentralized routing time in a **Kleinberg highway** *network is* $\mathcal{O}(\sqrt{k} + \log^2(n)/k + \log n)$ *for* $1 \leq k \leq n^2$ *when each node knows the positioning of the highway grid, and* $\mathcal{O}(k + \log^2(n)/k)$ *otherwise.*

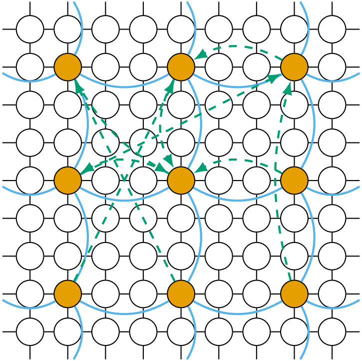


Figure 6: Kleinberg Highway Graph

- Kleinberg Highway: unrealistic

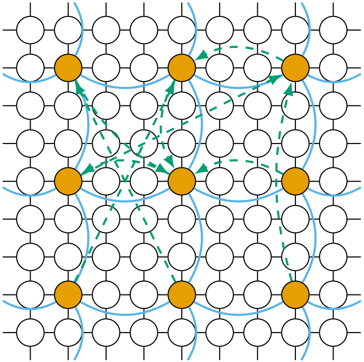
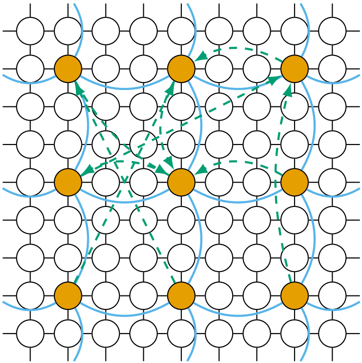


Figure 7: Kleinberg Highway Graph
$n = 9 \; k = 9 \; Q = 1/9$

Randomized Kleinberg Highway

- Kleinberg Highway: unrealistic
- Let's randomize!



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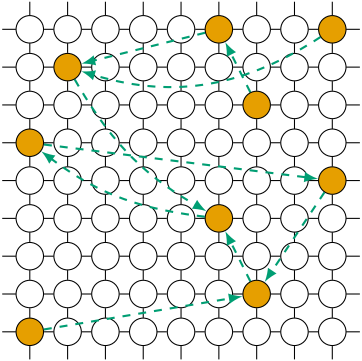
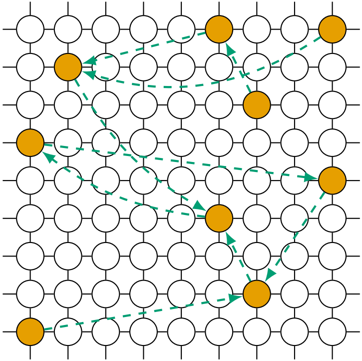


Figure 7: Randomized Kleinberg Highway Graph $n = 9$ $k = 9$ $Q = 1/9$

Randomized Kleinberg Highway

- Kleinberg Highway: unrealistic
- Let's randomize!
- Each node is in highway w/ probability $1/k$
- How to analyze?



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Randomized Kleinberg Highway – Analysis I

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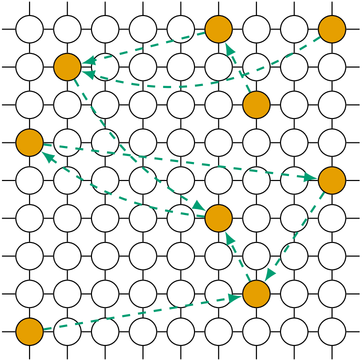

Figure 8: Randomized Kleinberg Highway Graph

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1. Reach highway - $\mathcal{O}(k)$
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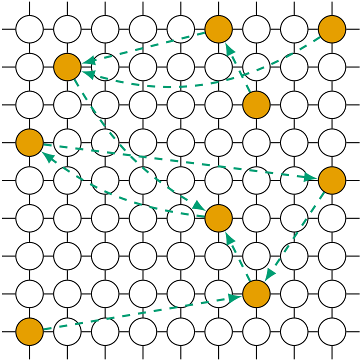

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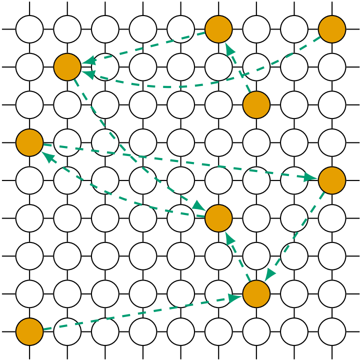

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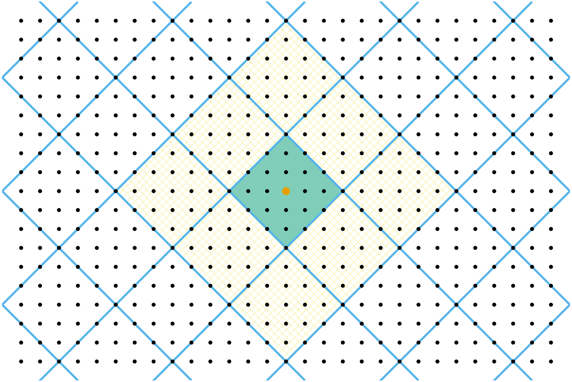
Nested Lattice Construction

- Nested lattices w/ radius r



Nested Lattice Construction

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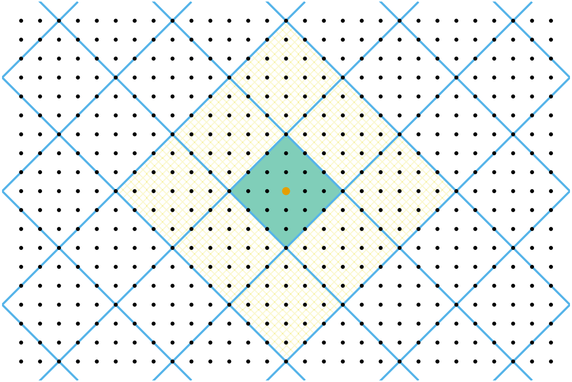
- $\Theta(r^2)$ nodes in ball



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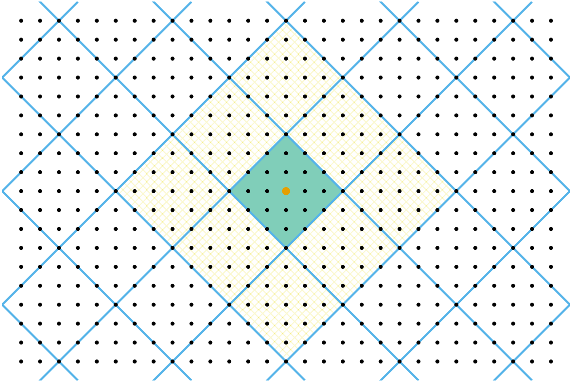

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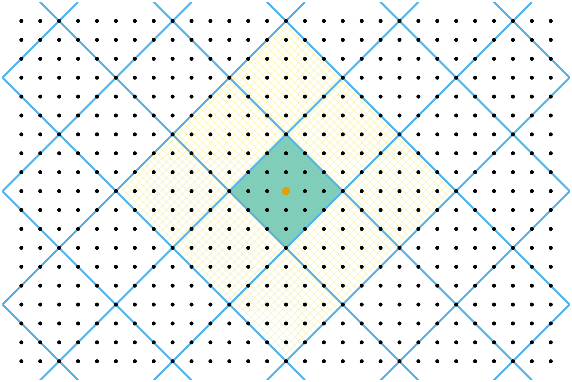
- Expected $\Theta(r^2/k)$ highway
- Chernoff bounds!



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- Nested lattices w/ radius r
- $\Theta(r^2)$ nodes in ball
- Expected $\Theta(r^2/k)$ highway

Radius r	Lower	Upper
$3\sqrt{k \log n}$	$9 \log n$	$41 \log n$
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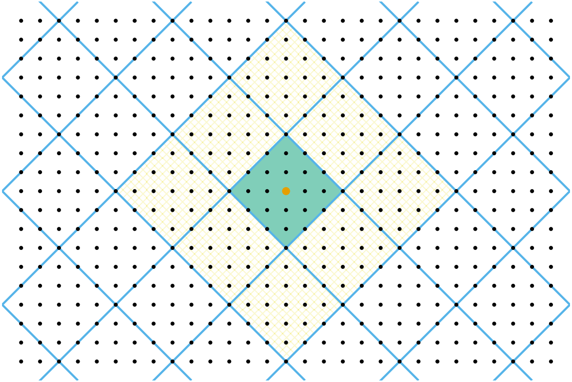
$2\sqrt{k}$		18

- Chernoff bounds!



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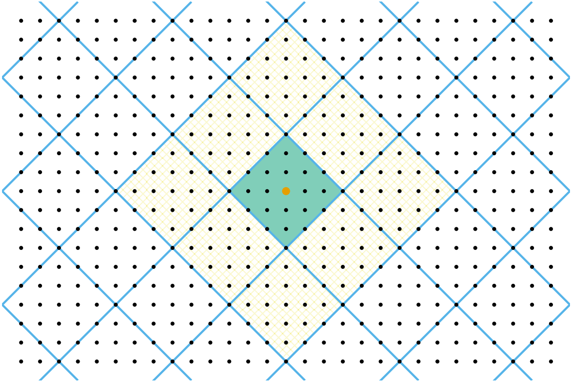
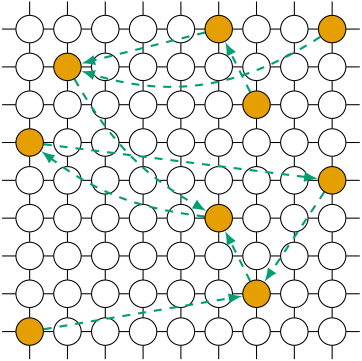
- Not all w.h.p.



Randomized Kleinberg Highway – Analysis II

Lemma

In the randomized highway model, the probability that highway node u has a long-range connection to highway node v that halves its distance to the destination is proportional to at most $k/\log n$ for $k \in \mathcal{O}(\log n)$ and is constant for $k \in \Omega(\log n)$.



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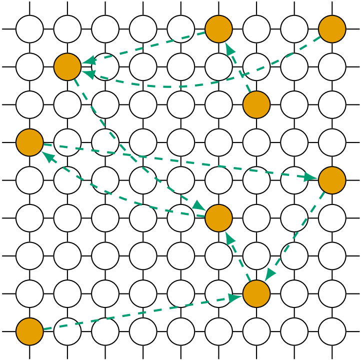

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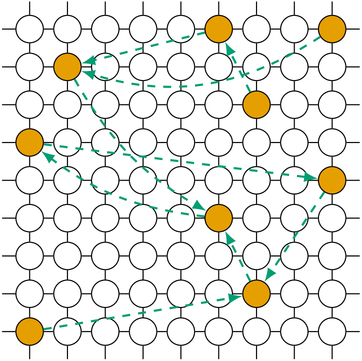

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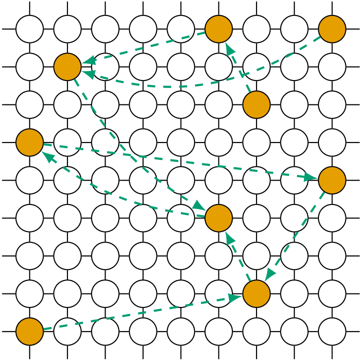

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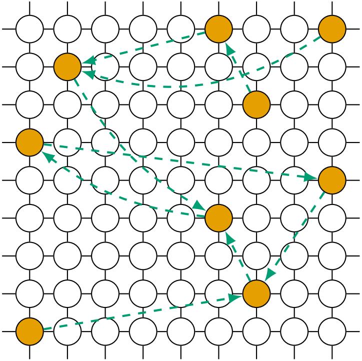

Randomized Kleinberg Highway – Variant

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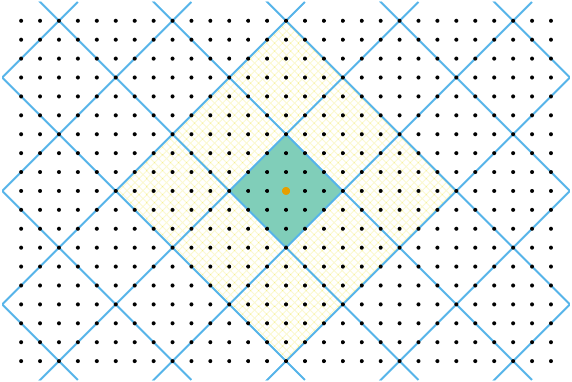

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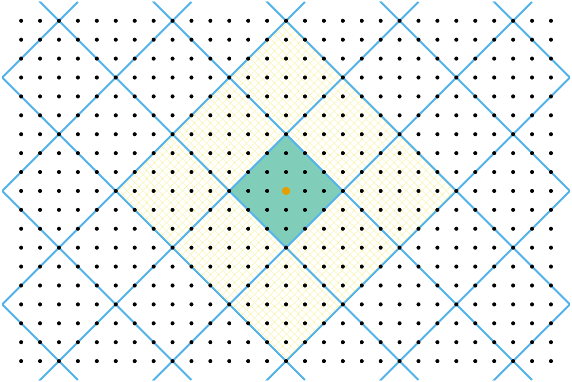

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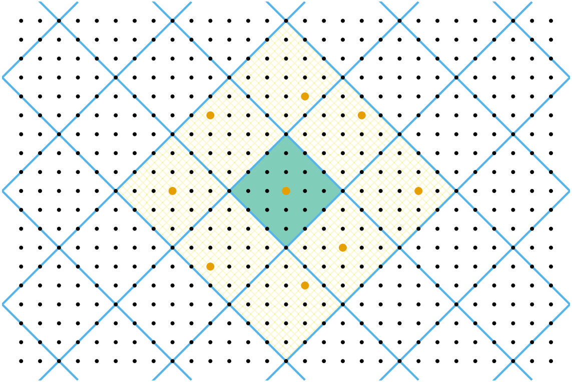

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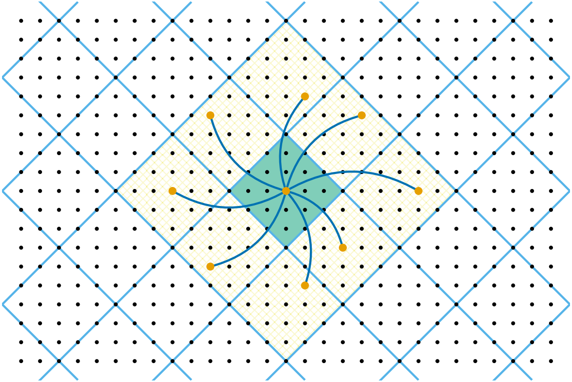

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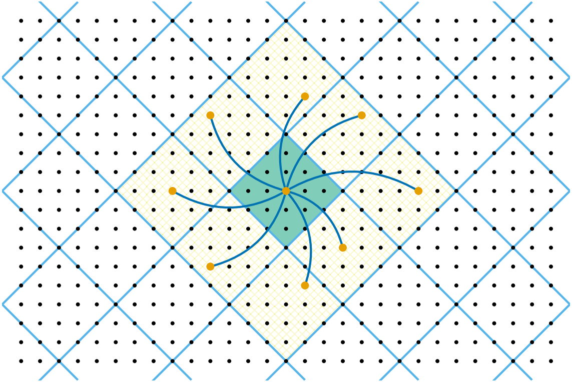

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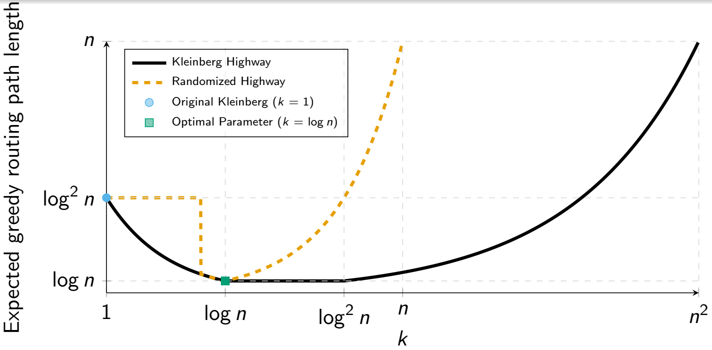
Kleinberg Highway – Result

Theorem

For $k \in o\left(\frac{\log n}{\log\log\log n}\right)$*, the expected decentralized greedy routing path length is* $\mathcal{O}(\log^2 n)$ *w.h.p., while for* $\Theta\left(\frac{\log n}{\log\log\log n}\right) \leq k < \Theta(\log n)$*, it is* $\mathcal{O}(\log^2(n)/k)$ *w.h.p., for* $\Theta(\log n) \leq k \leq \Theta(n)$*, it is* $\mathcal{O}(k)$ *and finally, for* $k \in \Omega(n)$*, it is* $\mathcal{O}(n)$*.*



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wNPA – Results

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Theorem

The windowed NPA model has a decentralized greedy algorithm that routes in $\mathcal{O}(\log^{1+\epsilon}(n))$ hops w.h.p.

wNPA — Experiments I

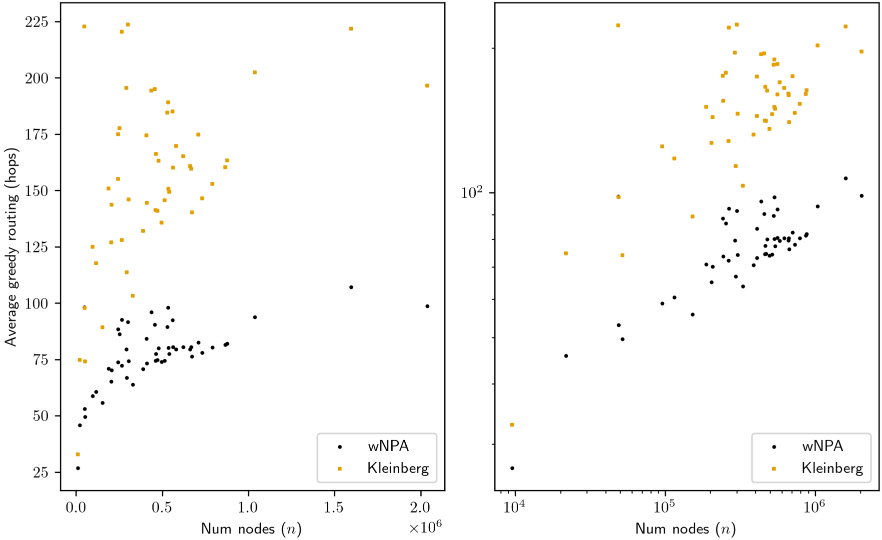

Figure 9: Comparison of greedy routing times for Kleinberg's model and the windowed NPA model when $Q = 1, \epsilon = 0.5, A = 1.01$.

wNPA — Experiments II

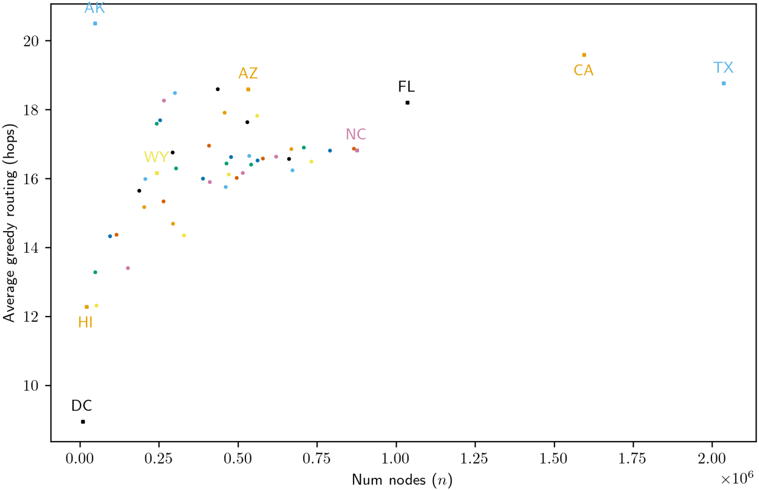

Figure 10: The greedy routing times for the windowed NPA model on the 50 US states when $Q = 32$, $\epsilon = 0.5$, and $A = 1.01$.

Open Problems

- Theory for original NPA model
- Tight greedy routing bounds for:
  - Kleinberg highway
  - Randomized Kleinberg highway
  - Windowed NPA
- Efficient (sequential) generation for:
  - Randomized Kleinberg highway
  - Windowed NPA
  - Original NPA
- Remove tight constraints from Windowed NPA
  - Each node u only connects to nodes within $[k_u/A, Ak_u]$
  - 0 probability for anyone else
  - What if decaying probability depending on k_u and k_v?