The Marco Polo Problem: A Combinatorial Approach to Geometric Localization

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[2]University of Rochester

CCCG, 2025

The Marco Polo Problem I

- Point of Interest (POI) ✗
- ✗ within distance n from origin

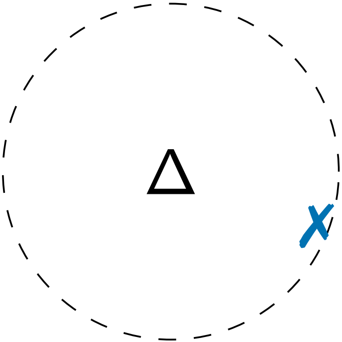

Figure 1: A search area.

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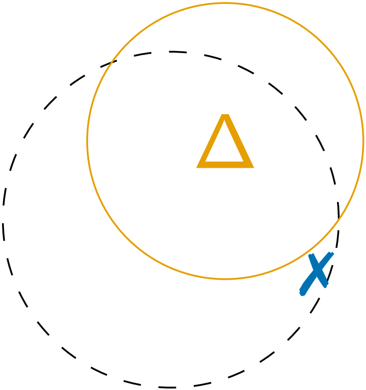

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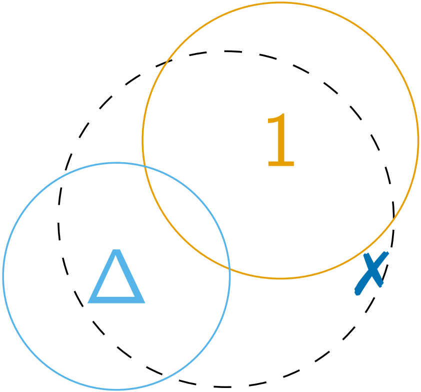
- Probe until 'finding' ✗



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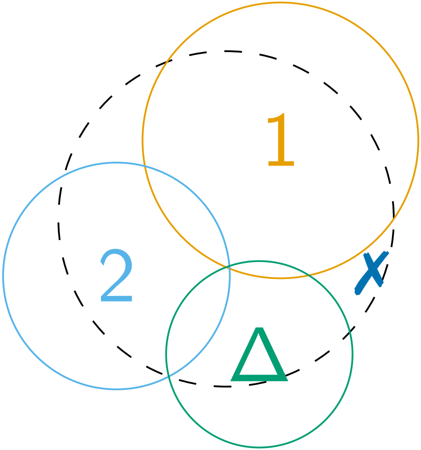

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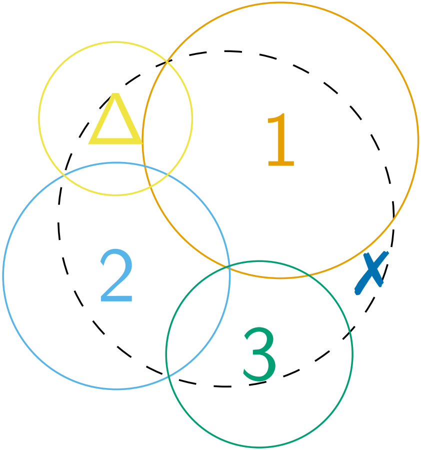

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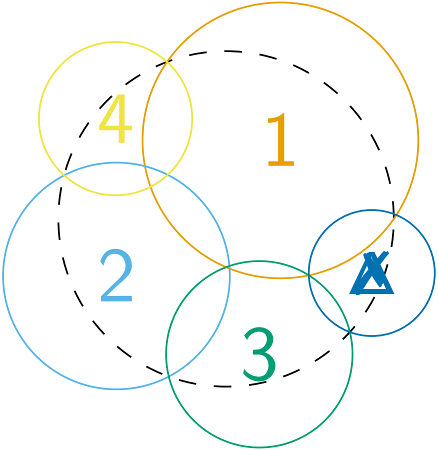
- 'finding': distance $\Delta \leftrightarrow$ ✗ ≤ 1



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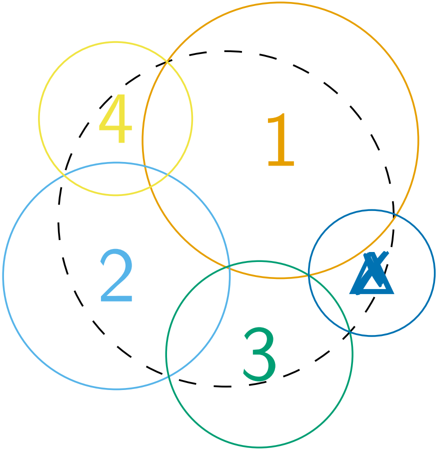

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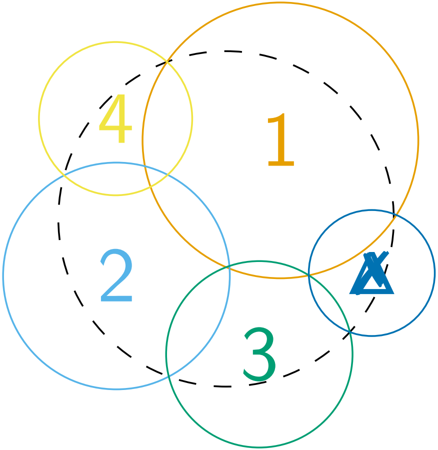

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- Variants:
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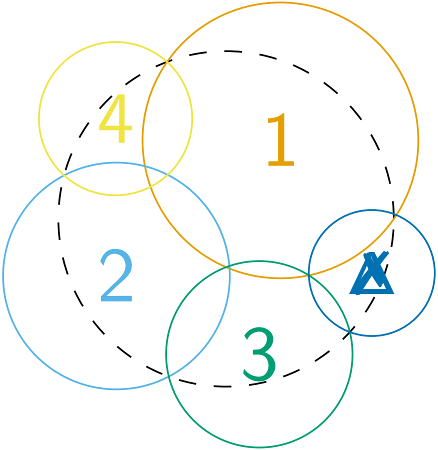
  - find all POIs



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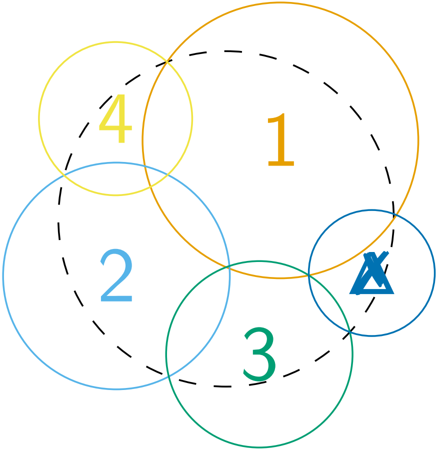

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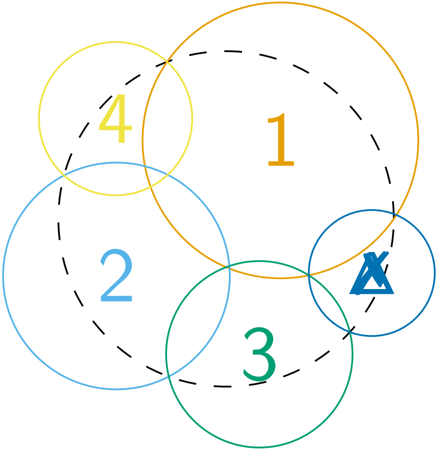

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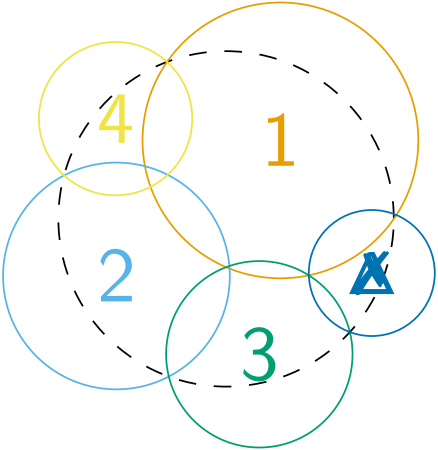

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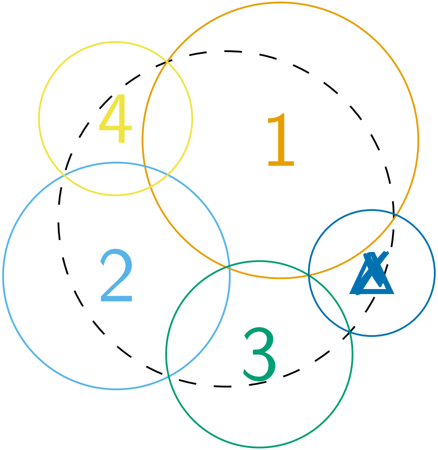

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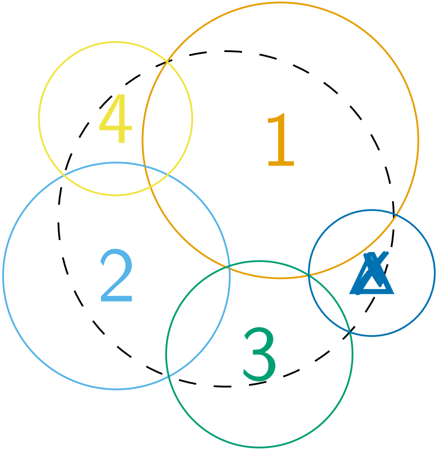

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- Variants:
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  - Distance metrics (L_1, L_∞)
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  - Probe response (T/F, d, i)
  - Δ's memory if any
  - multiple Δ, etc...
- Effectiveness metrics:

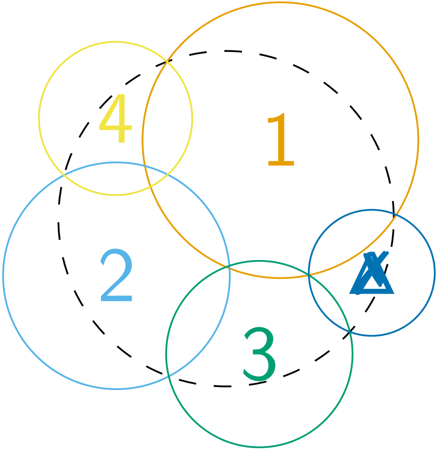

Figure 2: A search area.

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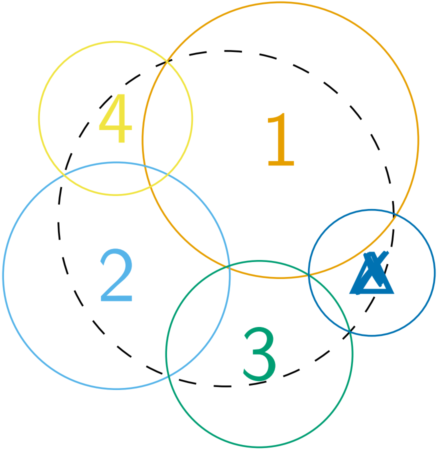
- Effectiveness metrics:
  - # of probes, $P(n)$



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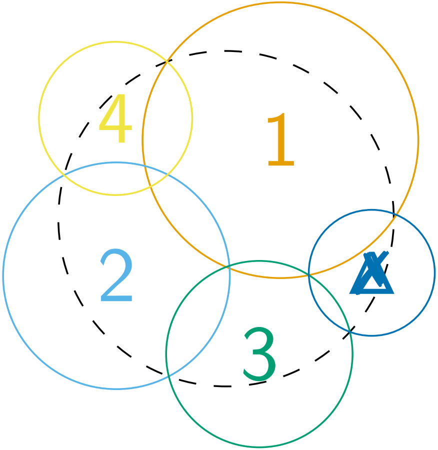
  - # of probes, $P(n)$
  - Distance traveled by Δ, $D(n)$



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- Effectiveness metrics:
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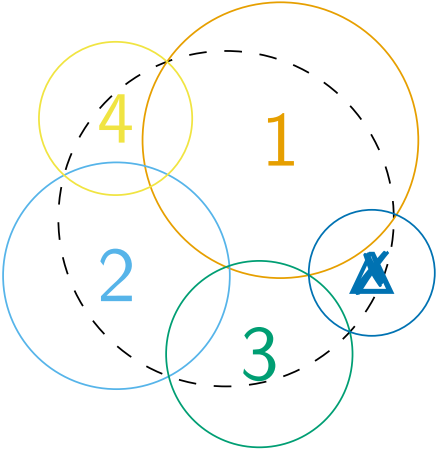
  - Distance traveled by Δ, $D(n)$
  - # of POI responses, $R(n)$



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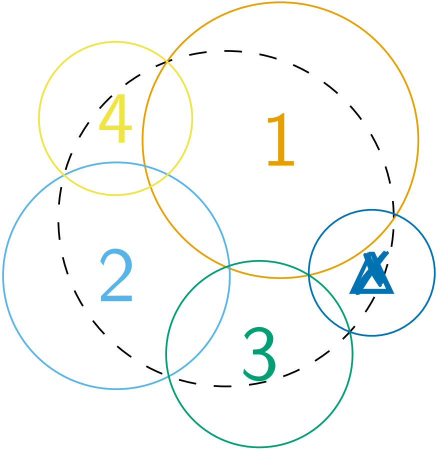
  - Input sensitivity?



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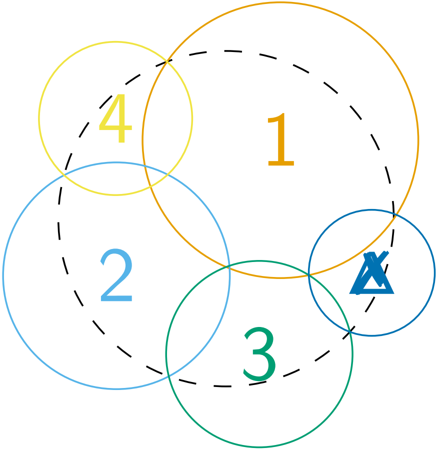
  - Input sensitivity?
    - TSP tour length, $\mathtt{OPT}$



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- Effectiveness metrics:
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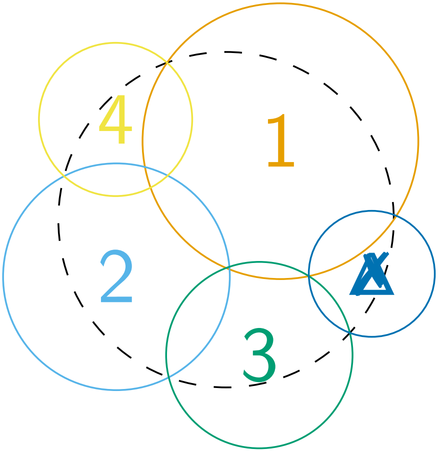
  - Distance traveled by Δ, $D(n)$
  - # of POI responses, $R(n)$
  - Input sensitivity?
    - TSP tour length, `OPT`
  - Simplicity / practicality



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- Motivation?

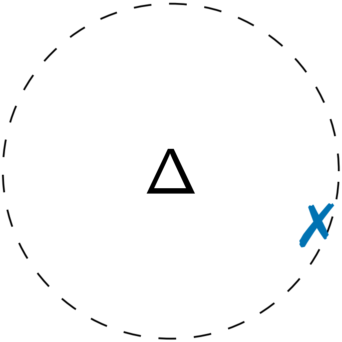


Figure 3: A search area.

Motivation

- Motivation?
- Finding gold?

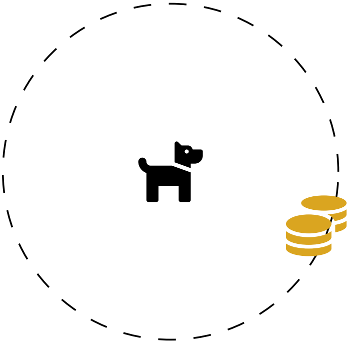
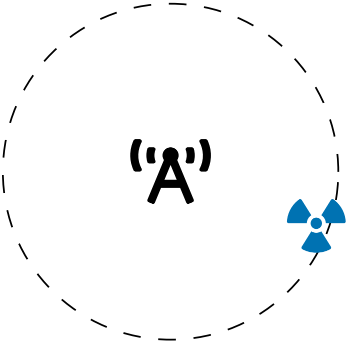


Figure 3: A search area.

Motivation

- Motivation?

- Finding gold?
- Detecting uranium?



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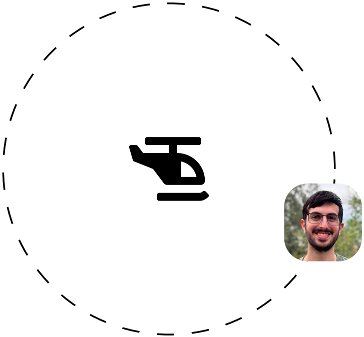
Motivation

- Motivation?

- Finding gold?
- Detecting uranium?
- Finding lost hiker / kidnap victim? 



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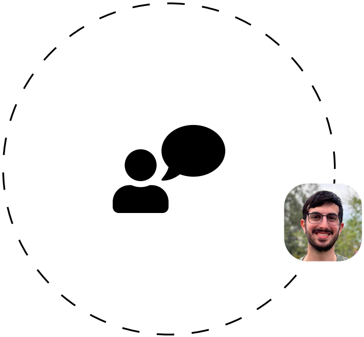
Motivation

- Motivation?

- Finding gold? 
- Detecting uranium? 
- Finding lost hiker / kidnap victim? 
- Game of Marco Polo?



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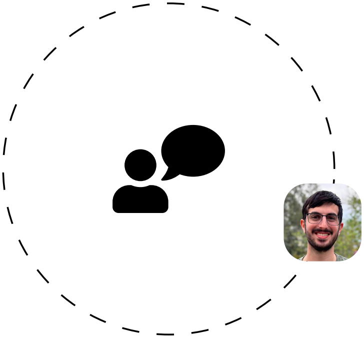
Motivation

- Motivation?

- Finding gold?
- Detecting uranium?
- Finding lost hiker / kidnap victim? 
- Game of Marco Polo?

- Whatever floats your



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Why not One?

- What if just one ✗?

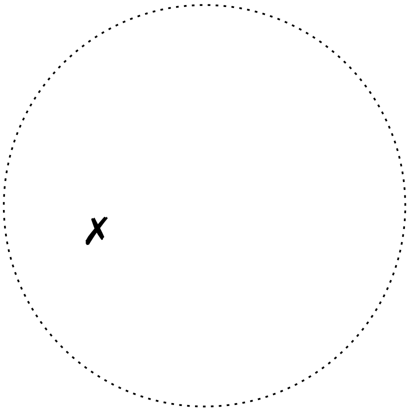
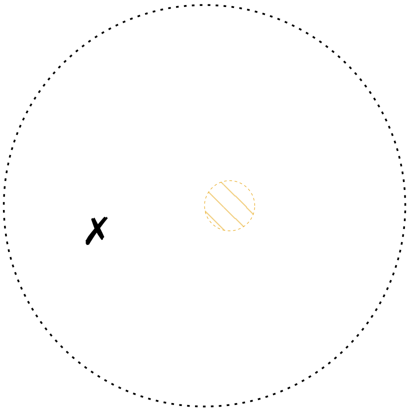


Figure 4: Trivial example w/ one ✗.

Why not One?

- What if just one ✗?
- Probe radius



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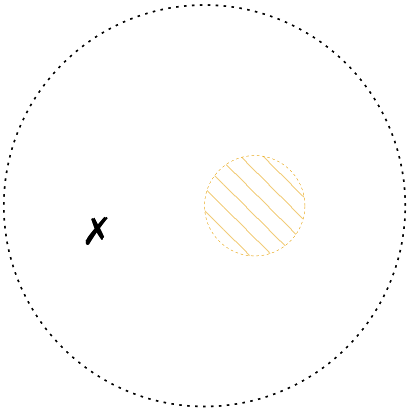

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- What if just one ✗?
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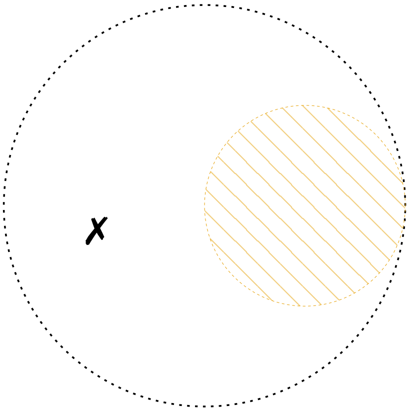

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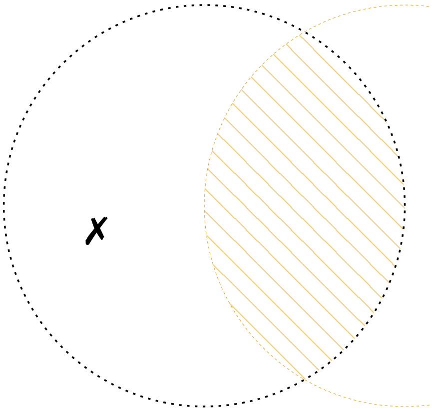

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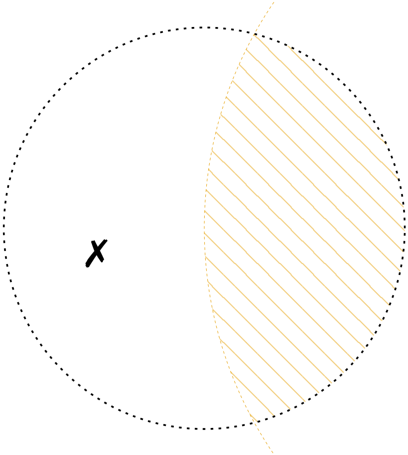

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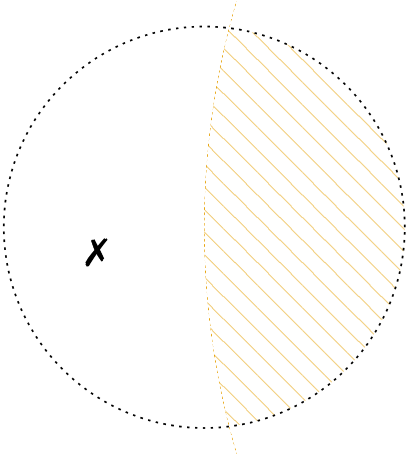

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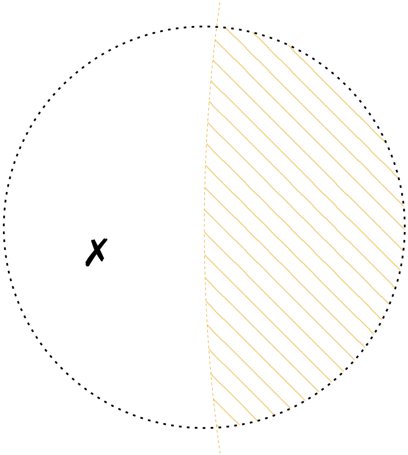

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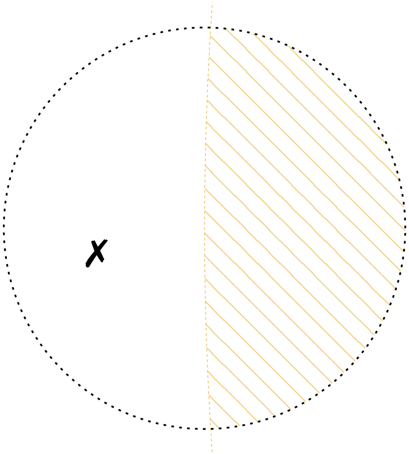

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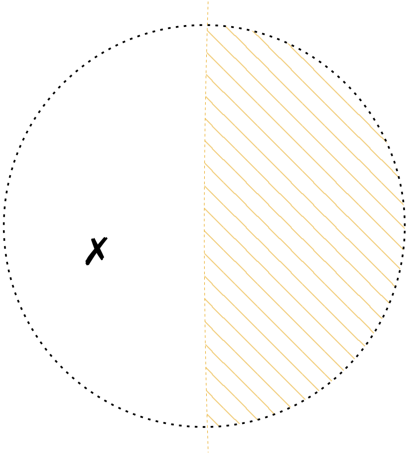
Why not One?

- What if just one ✗?
- Probe radius....... $\to \infty$



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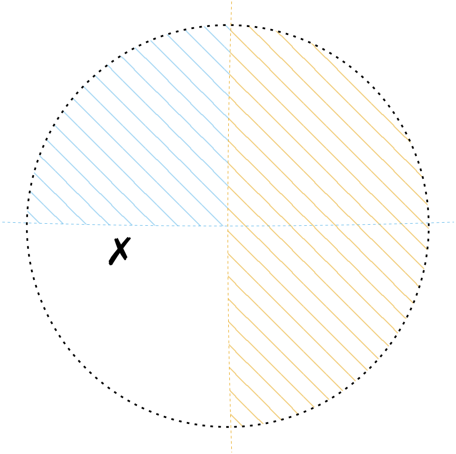
Why not One?

- What if just one ✗?
- Probe radius....... $\rightarrow \infty$
- Keep dividing



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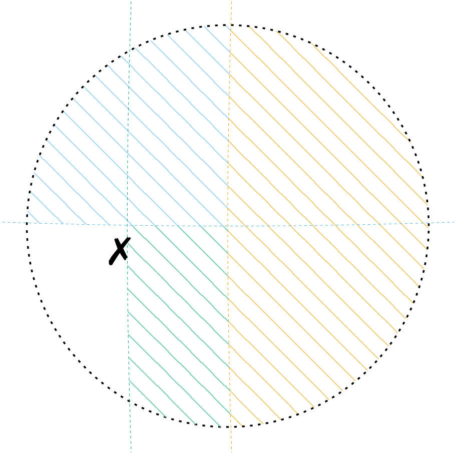

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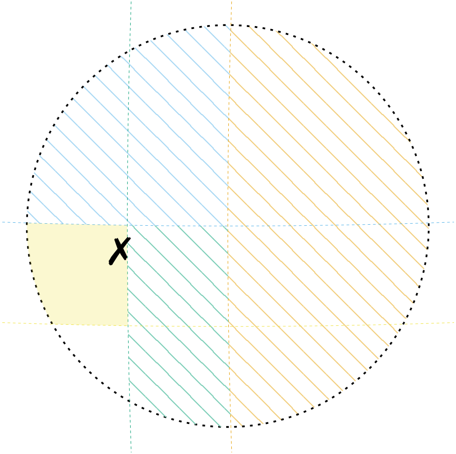

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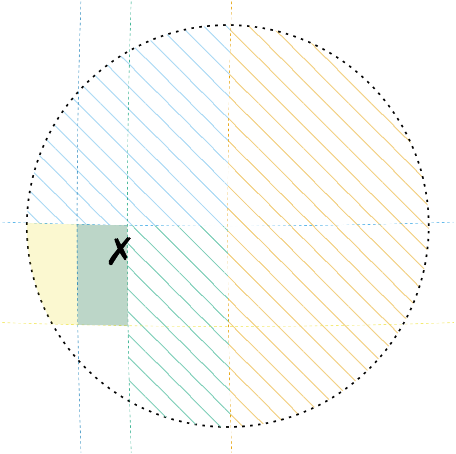

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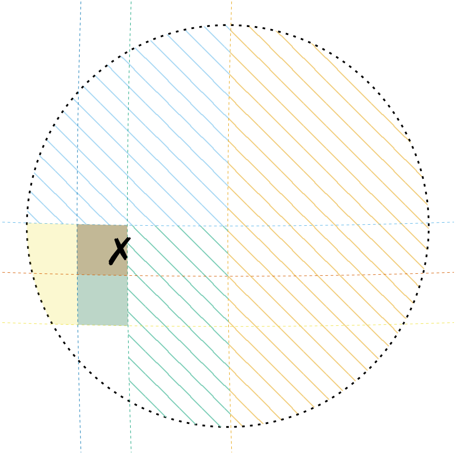

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Why not One?

- What if just one ✗?
- Probe radius....... $\rightarrow \infty$
- Keep dividing....
- Until 'found'!
  (reduced to radius 1)

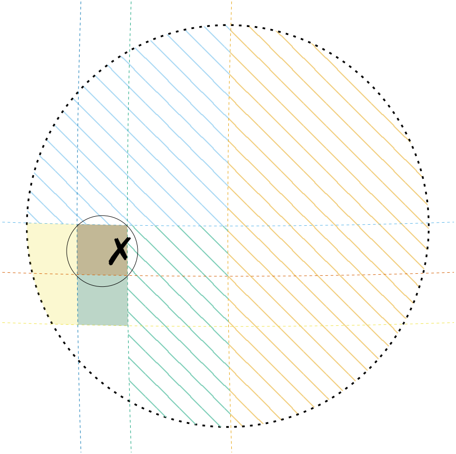
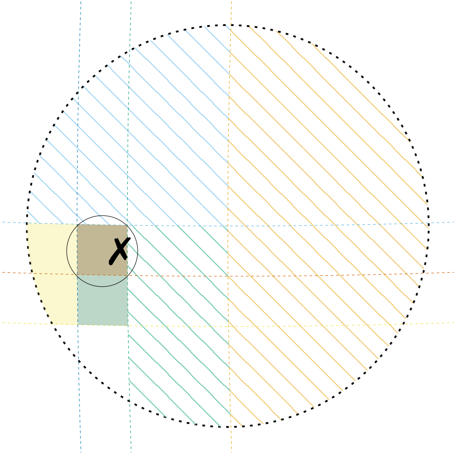


Figure 4: Trivial example w/ one ✗.

Why not One?

- What if just one ✗?
- Probe radius....... $\rightarrow \infty$
- Keep dividing....
- Until ‘found’!
  (reduced to radius 1)
- Time?



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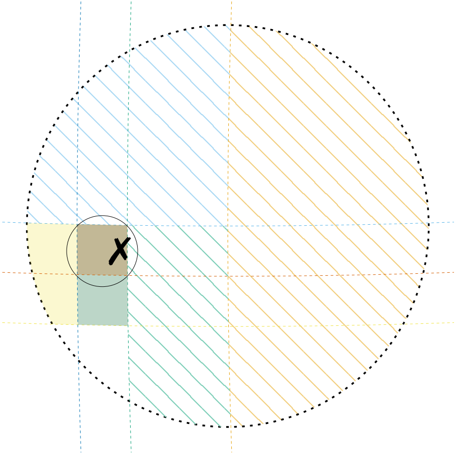
- Time?
- Initial diameter? $2n$



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- x halved $\log 2n = \lceil \log n \rceil + 1$

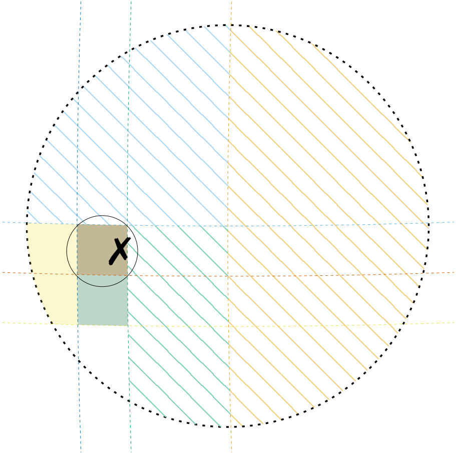

Figure 4: Trivial example w/ one ✗.

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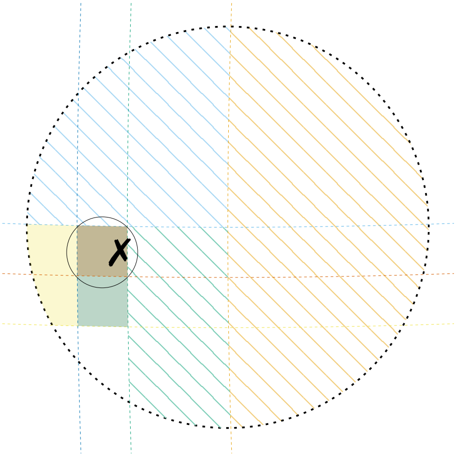
- Time? $2\lceil \log n \rceil + 2$
- Initial diameter? $2n$
- x halved $\log 2n = \lceil \log n \rceil + 1$



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- Initial diameter? $2n$
- x halved $\log 2n = \lceil \log n \rceil + 1$
- Optimal?

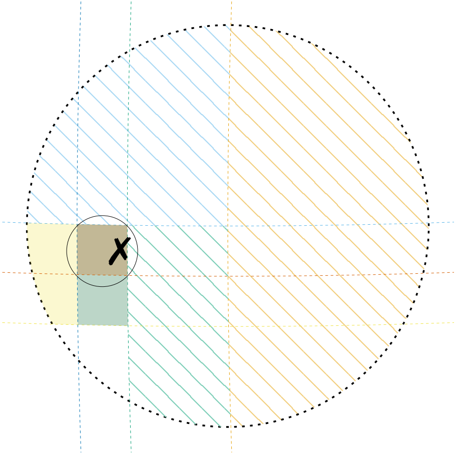

Figure 4: Trivial example w/ one ✗.

Why not One?

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- Probe radius....... $\to \infty$
- Keep dividing....
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- Time? $2\lceil \log n \rceil + 2$
- Initial diameter? $2n$
- x halved $\log 2n = \lceil \log n \rceil + 1$
- Optimal?
- Initial area: πn^2

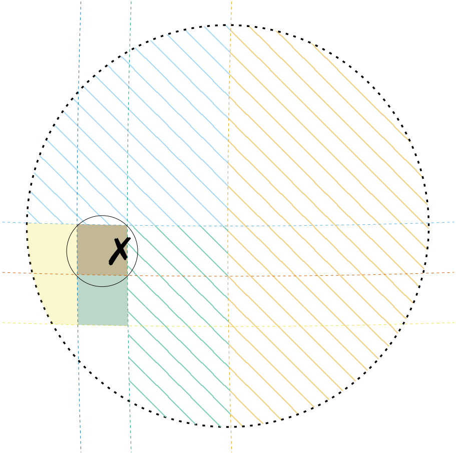

Figure 4: Trivial example w/ one ✗.

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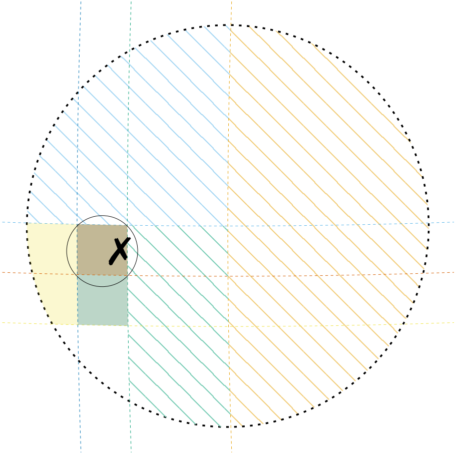
- Time? $2\lceil \log n \rceil + 2$
- Initial diameter? $2n$
- x halved $\log 2n = \lceil \log n \rceil + 1$
- Optimal?
- Initial area: πn^2
- Final area: π



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- What if just one ✗?
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- Time? $2\lceil \log n \rceil + 2$
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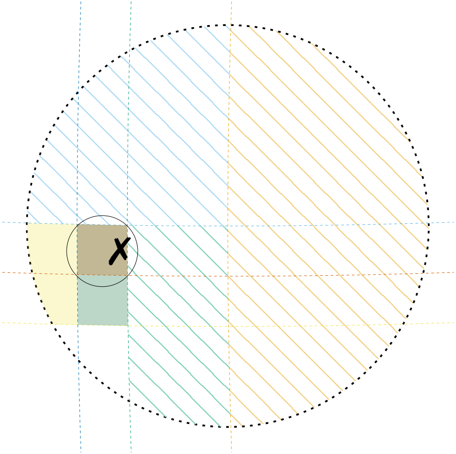
- x halved $\log 2n = \lceil \log n \rceil + 1$
- Optimal?
- Initial area: πn^2
- Final area: π
- Optimal probe $\rightarrow$ half remaining



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- Optimal?
- Initial area: πn^2
- Final area: π
- Optimal probe $\to$ half remaining
- $\lceil \log(\pi n^2/\pi) \rceil = \lceil 2 \log n \rceil$

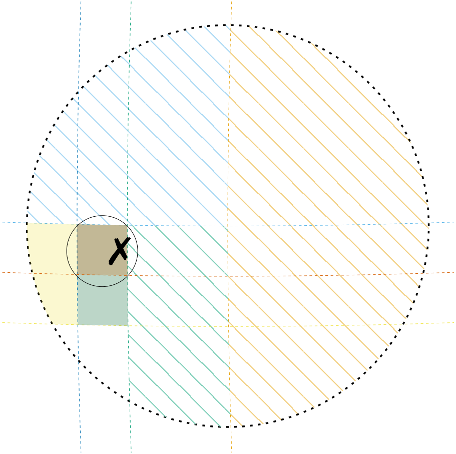

Figure 4: Trivial example w/ one ✗.

Why not One?

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- Probe radius....... $\rightarrow \infty$
- Keep dividing....
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  (reduced to radius 1)
- Time? $2\lceil \log n \rceil + 2$
- Initial diameter? $2n$
- x halved $\log 2n = \lceil \log n \rceil + 1$
- Optimal? pretty much!
- Initial area: πn^2
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- Optimal probe $\rightarrow$ half remaining
- $\lceil \log(\pi n^2 / \pi) \rceil = \lceil 2 \log n \rceil$

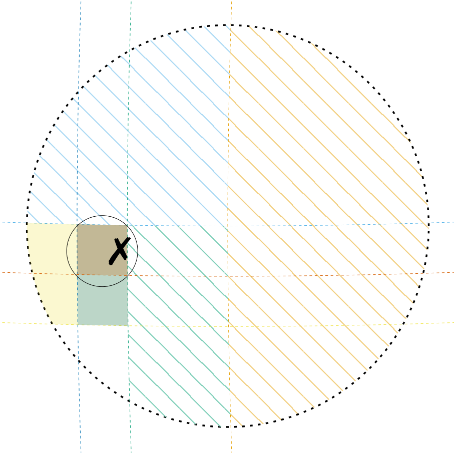

Figure 4: Trivial example w/ one ✗.

Why not One?

- What if just one ✗? too easy!
- Probe radius....... $\to \infty$
- Keep dividing....
- Until 'found'!
  (reduced to radius 1)
- Time? $2\lceil \log n \rceil + 2$
- Initial diameter? $2n$
- x halved $\log 2n = \lceil \log n \rceil + 1$
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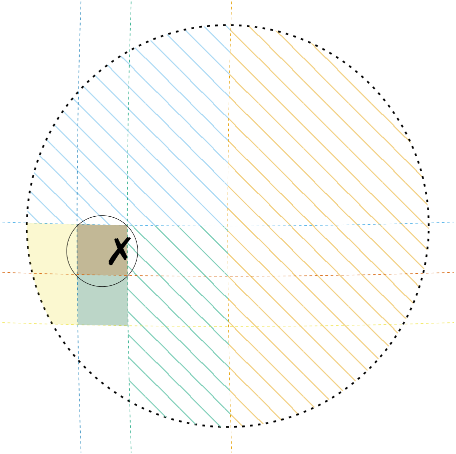

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Why not One? Restrict Radius

- Limit radius to n?

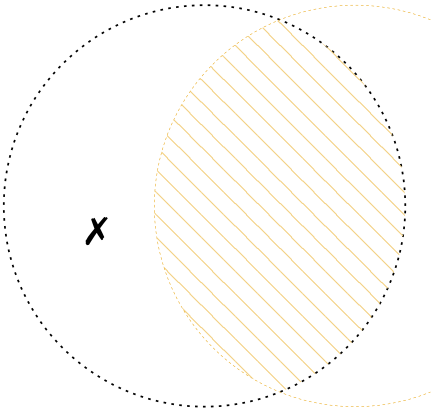
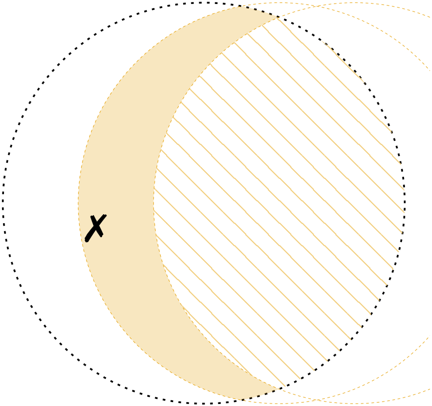


Figure 5: One ✗, probe $d \leq n$.

Why not One? Restrict Radius

- Limit radius to n?
- Restrict x to 1-wide



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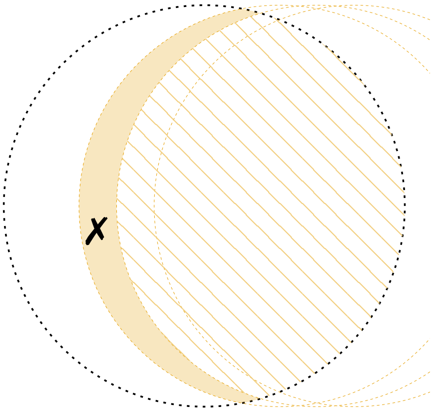
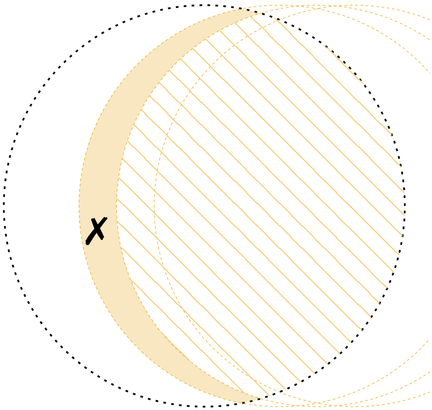


Figure 5: One ✗, probe $d \leq n$.

Why not One? Restrict Radius

- Limit radius to n?
- Restrict x to 1-wide $\lceil \log 2n \rceil$



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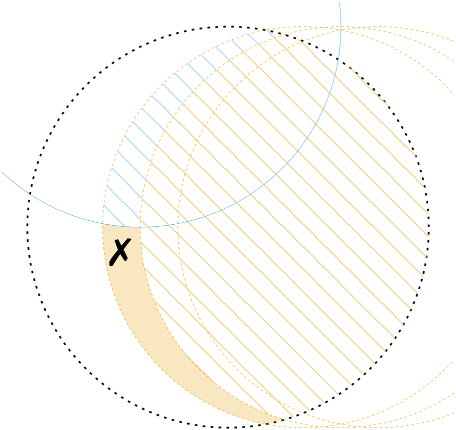
Why not One? Restrict Radius

- Limit radius to n?
- Restrict x to 1-wide $\lceil \log 2n \rceil$
- Restrict y to 1-wide



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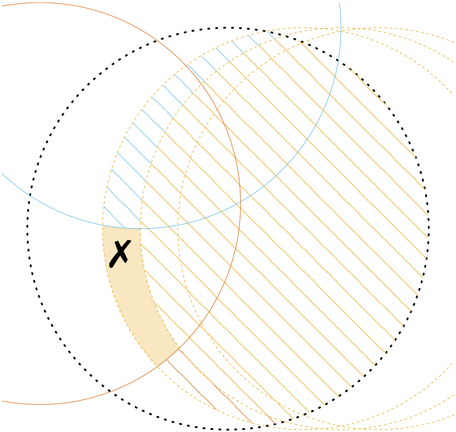

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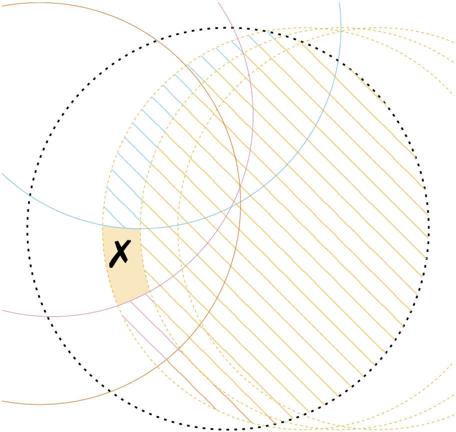

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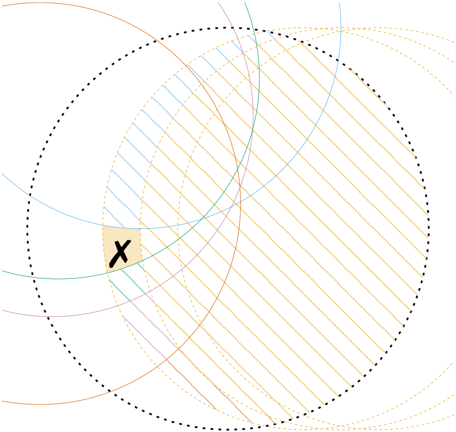

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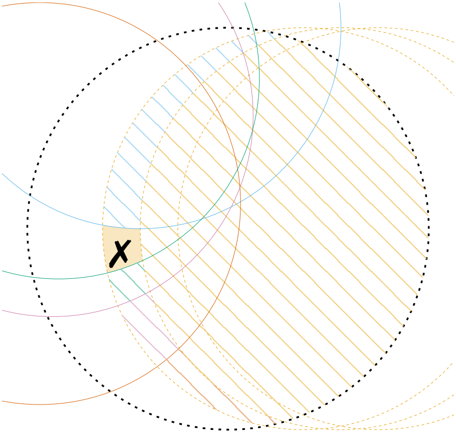
Why not One? Restrict Radius

- Limit radius to n?
- Restrict x to 1-wide $\lceil \log 2n \rceil$
- Restrict y to 1-wide $\lceil \log \pi n \rceil$
- Overall: $\leq 2\lceil \log n \rceil + 3$



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- Limit radius to n?
- Restrict x to 1-wide $\lceil \log 2n \rceil$
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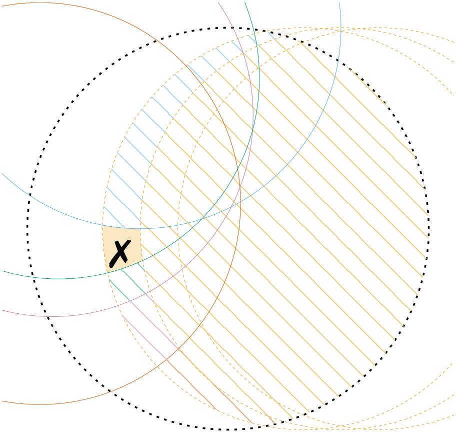
- Overall: $\leq 2\lceil \log n \rceil + 3$
- Also close enough!



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- Overall: $\leq 2\lceil \log n \rceil + 3$
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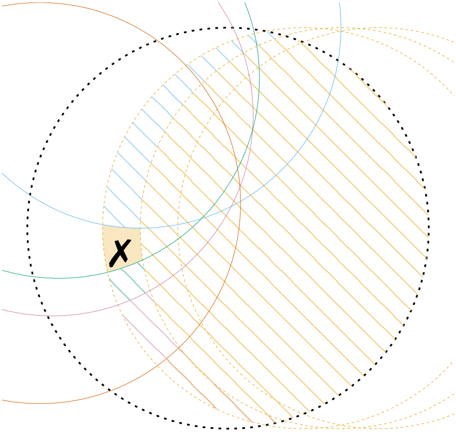
- Δ might leave initial area...



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- Limit radius to n?

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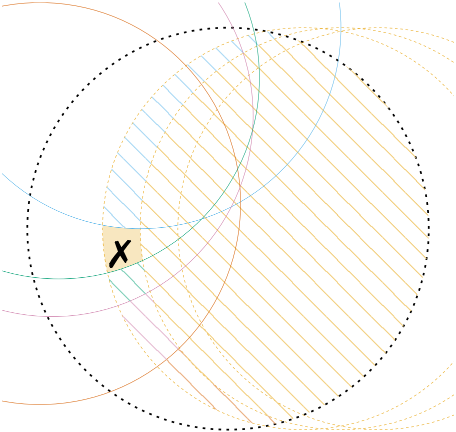
- Overall: $\leq 2\lceil \log n \rceil + 3$
- Also close enough!

- Δ might leave initial area...
- Can also solve in $2\lceil \log n \rceil + \mathcal{O}(1)$



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Why not One? Restrict Radius

- Limit radius to n?
- Restrict x to 1-wide $\lceil \log 2n \rceil$
- Restrict y to 1-wide $\lceil \log \pi n \rceil$
- Overall: $\leq 2\lceil \log n \rceil + 3$
- Also close enough!
- Δ might leave initial area...
- Can also solve in $2\lceil \log n \rceil + \mathcal{O}(1)$
- From now on...
  1. probe radius $\leq n$
  2. may be multiple ✗!

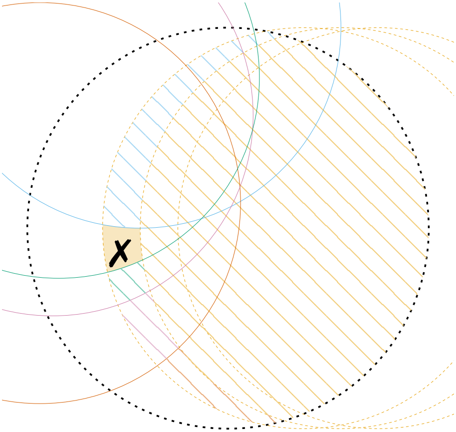


Figure 5: One ✗, probe $d \leq n$.

L_2: Hexagonal Algorithms

- Consider hexagonal lattice

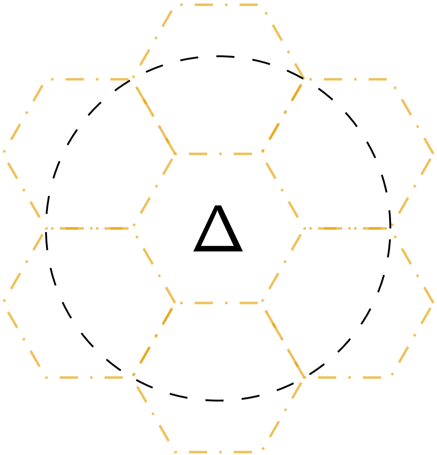


Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$

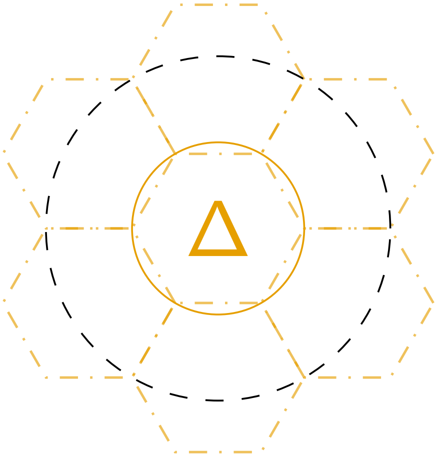
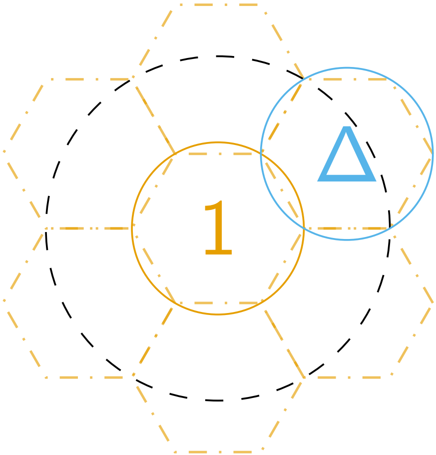


Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!



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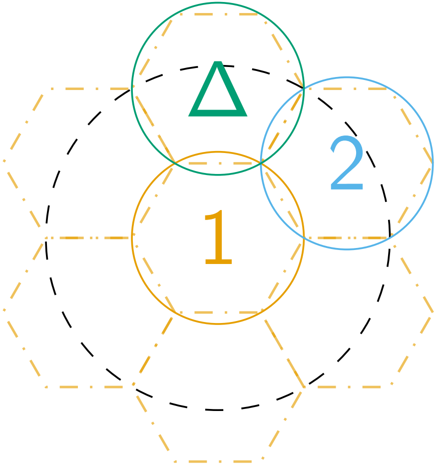

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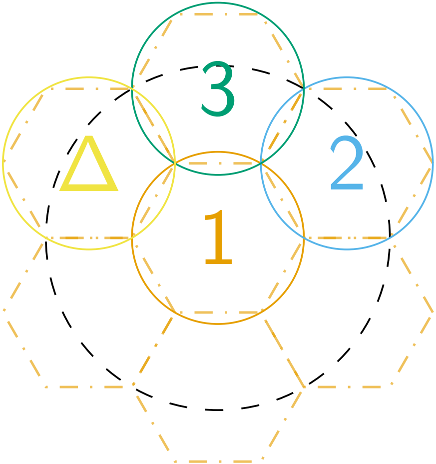

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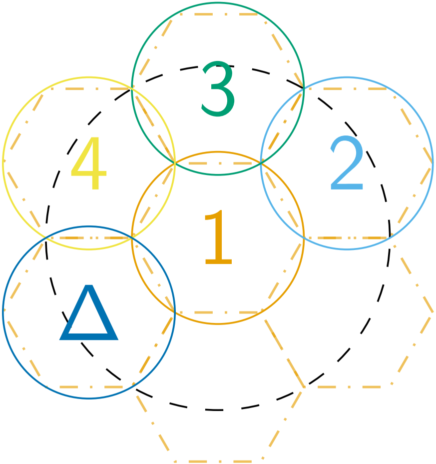

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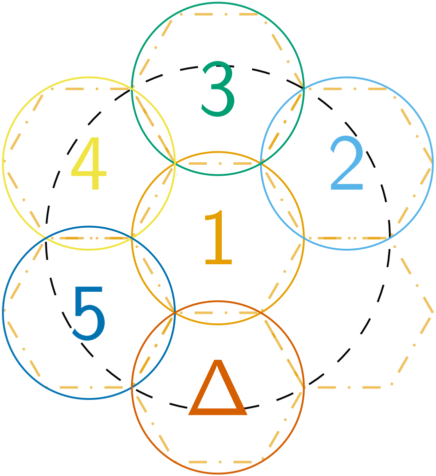


Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
- After (at most) 6 probes...

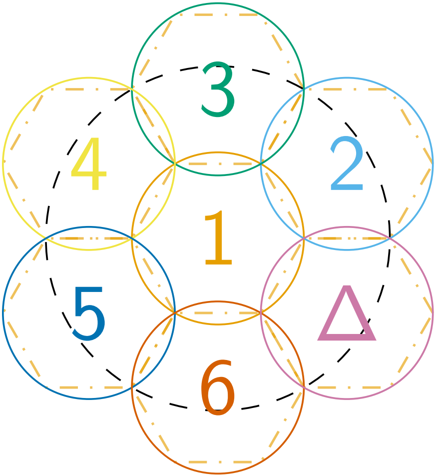


Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
- After (at most) 6 probes...
- Search area radius is halved!

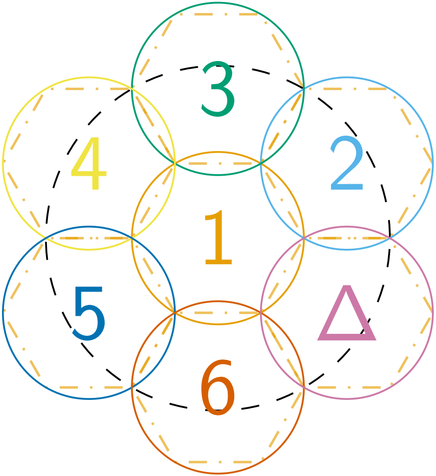
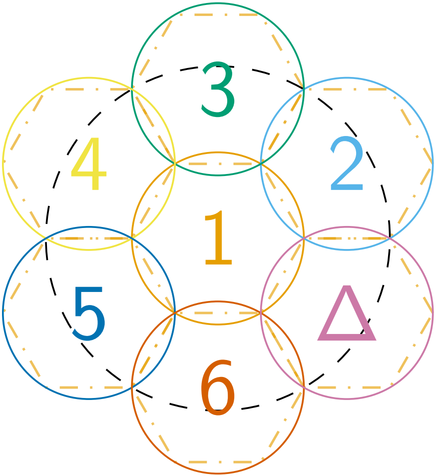


Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
- After (at most) 6 probes...
- Search area radius is halved!
- Recurse!
- $P(n) \leq 6\lceil \log n \rceil$



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- Recurse!
- $P(n) \leq 6\lceil \log n \rceil$
- Total responses?

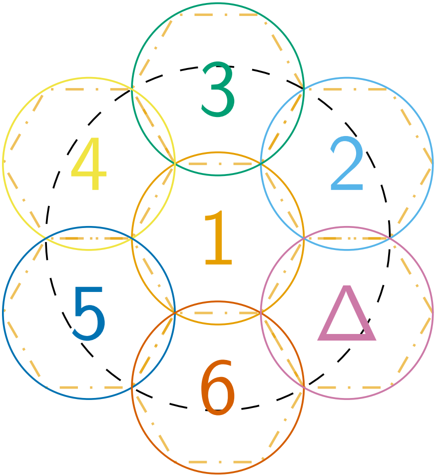

Figure 6: Algorithm 1

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- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
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- Recurse!
- $P(n) \leq 6\lceil \log n \rceil$
- Total responses?
- At most one per layer...

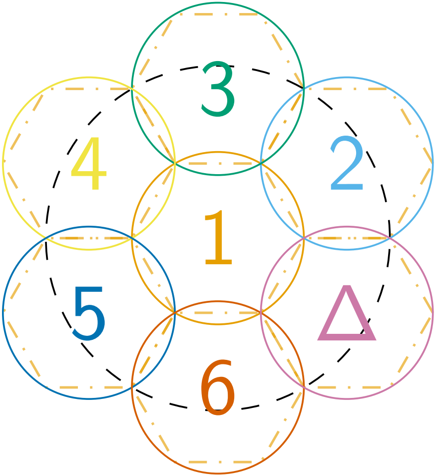

Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
- After (at most) 6 probes...
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- Recurse!
- $P(n) \leq 6\lceil \log n \rceil$
- Total responses?
- At most one per layer...
- $R(n) \leq \lceil \log n \rceil$

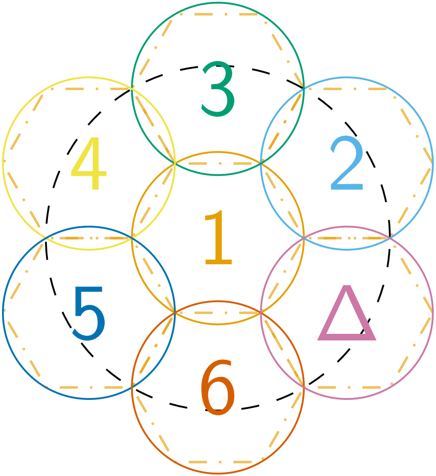

Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
- After (at most) 6 probes...
- Search area radius is halved!
- Recurse!
- $P(n) \leq 6\lceil \log n \rceil$
- Total responses?
- At most one per layer...
- $R(n) \leq \lceil \log n \rceil$
- Distance traveled?

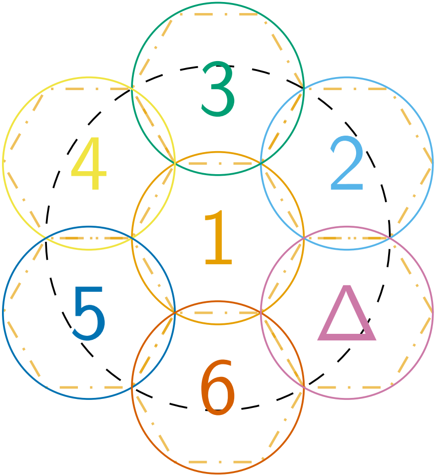

Figure 6: Algorithm 1

L_2: Hexagonal Algorithms

- Consider hexagonal lattice
- Hexagon side length: $n/2$
- Probe!
- After (at most) 6 probes...
- Search area radius is halved!
- Recurse!
- $P(n) \leq 6\lceil \log n \rceil$
- Total responses?
- At most one per layer...
- $R(n) \leq \lceil \log n \rceil$
- Distance traveled?
- $D(n) \leq 10.39n$

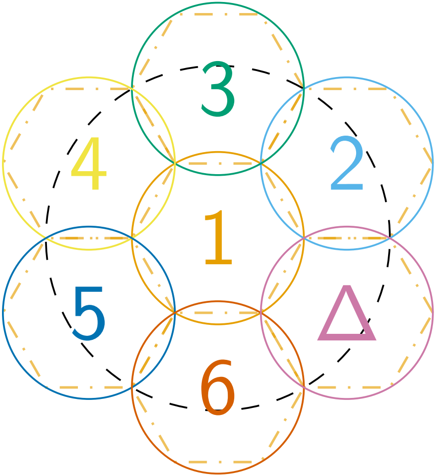

Figure 6: Algorithm 1

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice

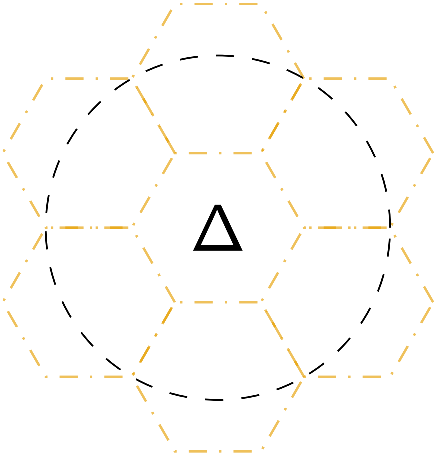

Figure 7: Algorithm 2

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice
- First probe 2 quadrants...

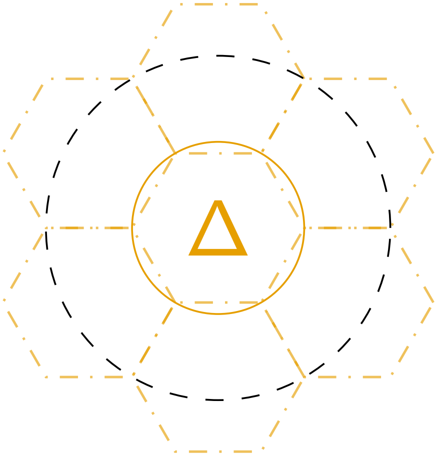

Figure 7: Algorithm 2

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice
- First probe 2 quadrants...

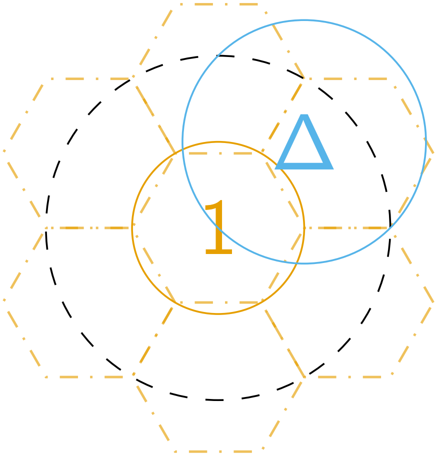

Figure 7: Algorithm 2

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice
- First probe 2 quadrants...

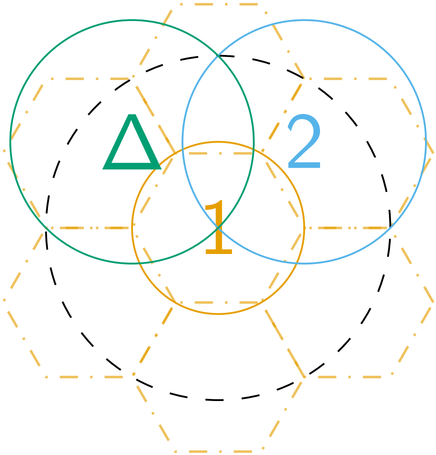

Figure 7: Algorithm 2

- Consider hexagonal lattice
- First probe 2 quadrants...

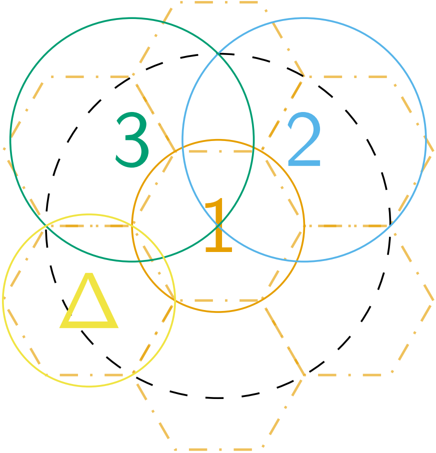

Figure 7: Algorithm 2

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice
- First probe 2 quadrants...

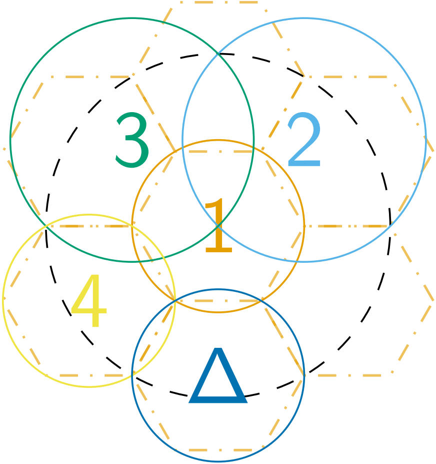

Figure 7: Algorithm 2

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice
- First probe 2 quadrants...
- After (at most) 5 probes...
- Search area radius is halved!

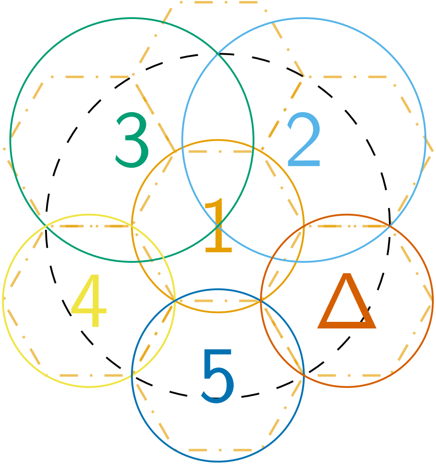

Figure 7: Algorithm 2

L_2: Hexagonal Algorithms cont.

- Consider hexagonal lattice
- First probe 2 quadrants...
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- $P(n) \leq 5\lceil \log n \rceil$

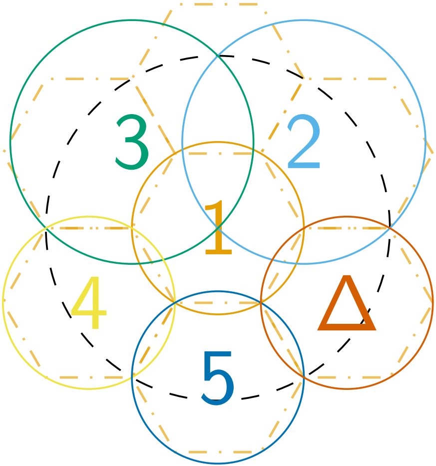

Figure 7: Algorithm 2

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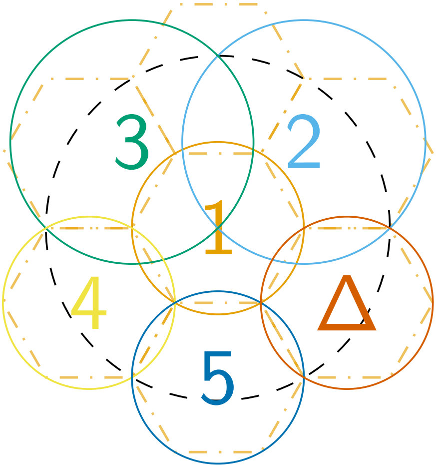

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- Total responses?
- If 3rd probe succeeds, only $1/\sqrt{2}$ reduction...

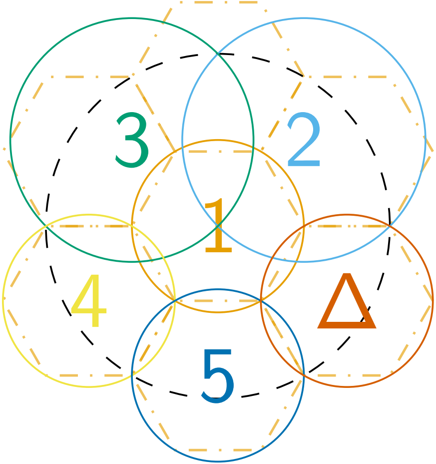

Figure 7: Algorithm 2

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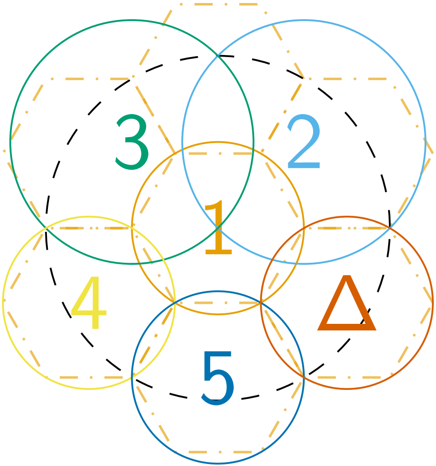

Figure 7: Algorithm 2

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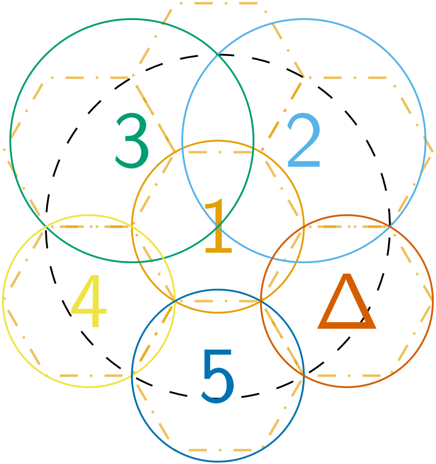

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L_2: Hexagonal Algorithms cont.

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- After (at most) 5 probes...
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- If 3rd probe succeeds, only $1/\sqrt{2}$ reduction...
- $R(n) \leq 2\lceil \log n \rceil$
- Distance traveled?
- $D(n) \leq 8.81n$

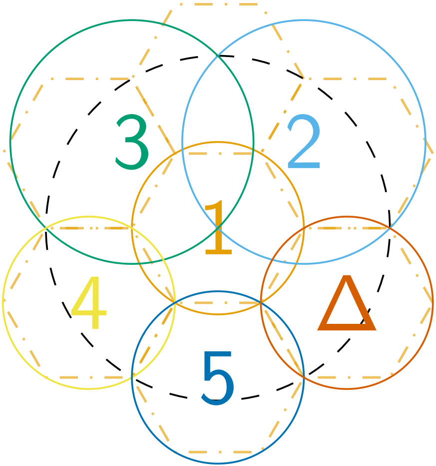

Figure 7: Algorithm 2

L_2: Progressively Shrinking Probes

- Larger probes are better?

L_2: Progressively Shrinking Probes

- Larger probes are better?
  - ☑ Reduce more when fail

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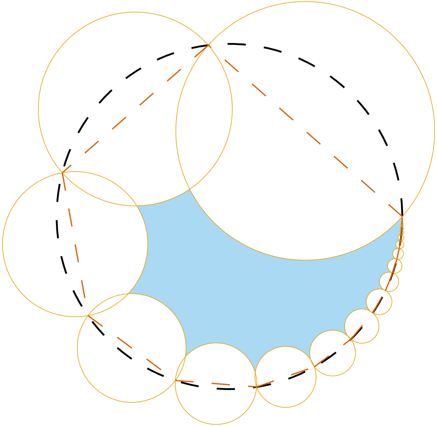

Figure 8: Must be able to cover perimeter... $\rho_1 \geq 0.74915\ldots$

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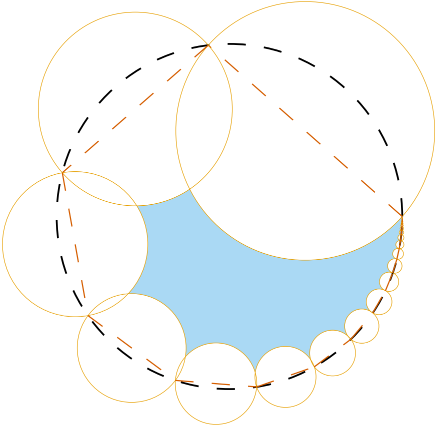


Figure 8: Must be able to cover perimeter... $\rho_1 \geq 0.74915\ldots$

Lower bound: $P(n) \geq 2.40001\lceil \log n \rceil$

L_2: Chord-Based Shrinking Algorithms

- $\rho_1 = 0.74915\ldots$ too small

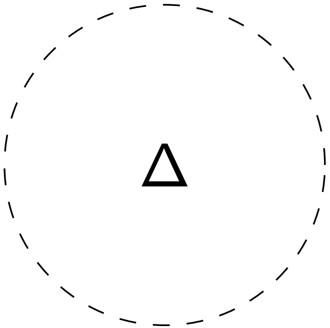
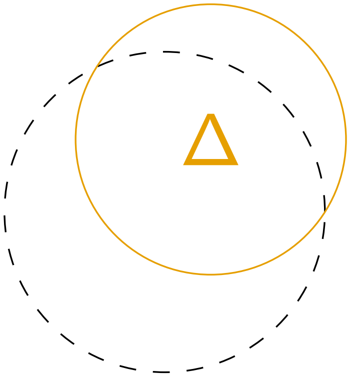


Figure 9: Algorithm 3

L_2: Chord-Based Shrinking Algorithms

- $\rho_1 = 0.74915\ldots$ too small
- How large must ρ_1 be?



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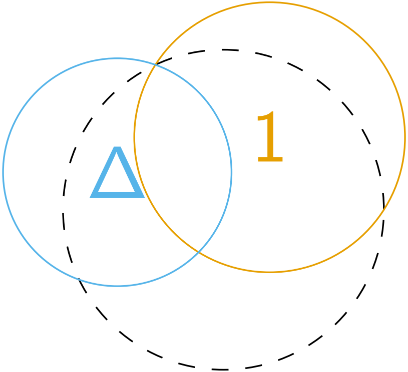

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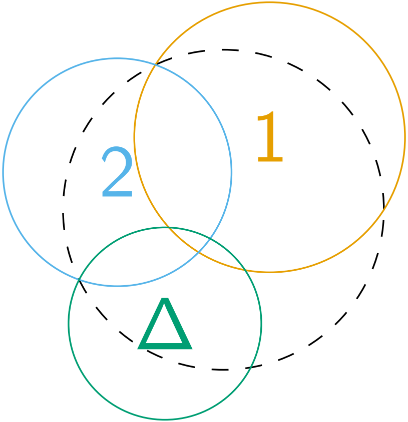

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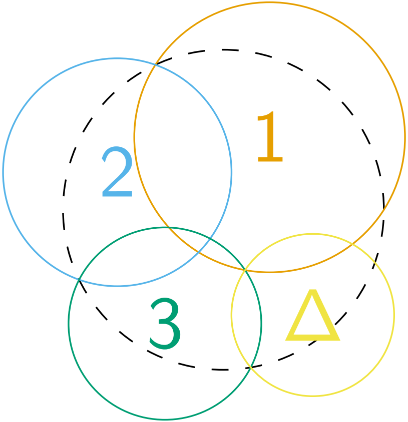

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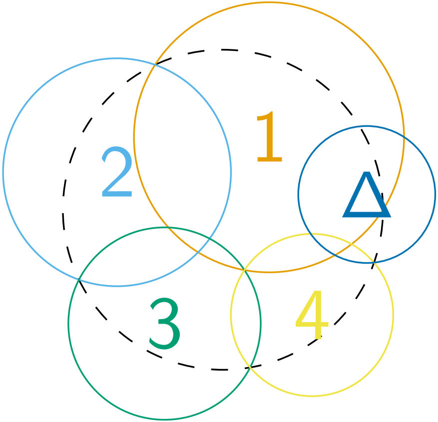

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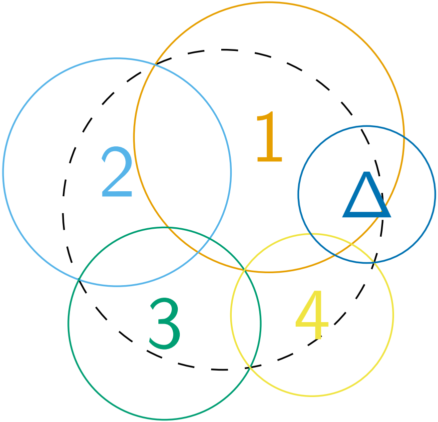
L_2: Chord-Based Shrinking Algorithms

- $\rho_1 = 0.74915\ldots$ too small
- How large must ρ_1 be?

- $\rho_1 = 0.844\ldots$



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- $\rho_1 = 0.74915\ldots$ too small
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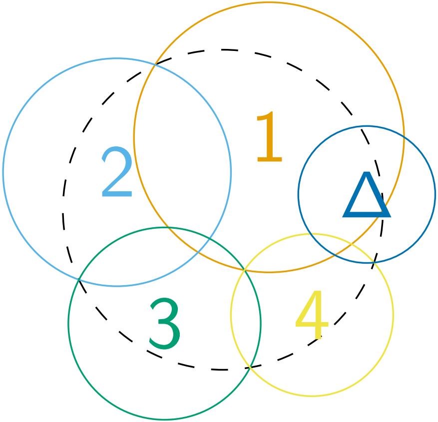
- $P(n) = R(n) \leq -\frac{1}{\log \rho_1}\lceil \log n \rceil = 4.08\lceil \log n \rceil$



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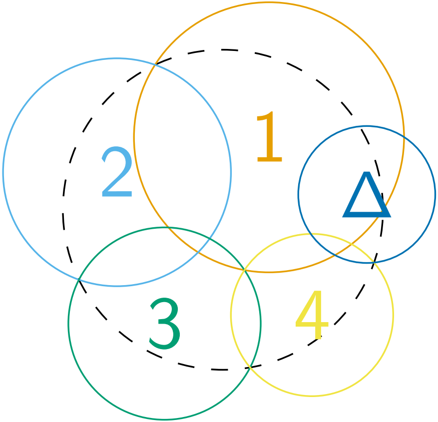

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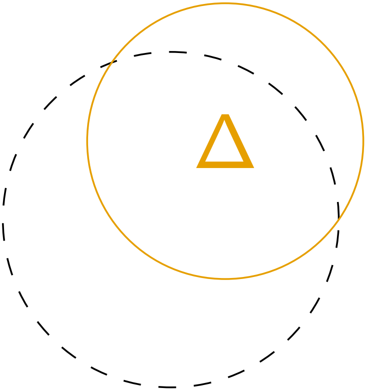

Figure 9: Algorithm 4

L_2: Chord-Based Shrinking Algorithms

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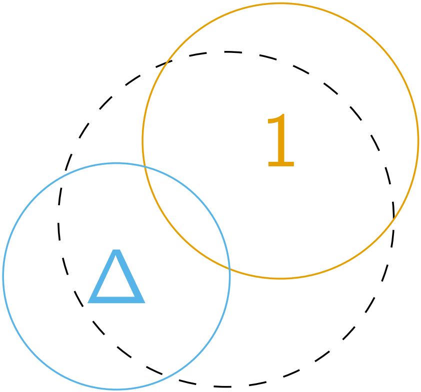

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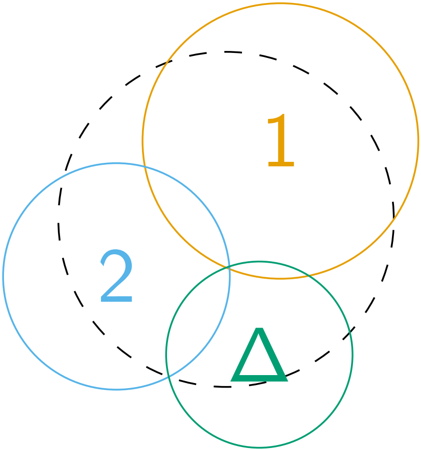

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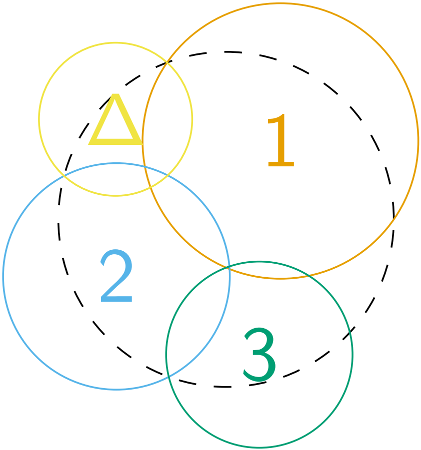

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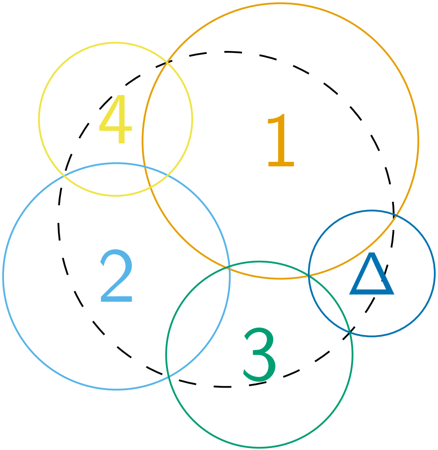
- What if we rearrange?
- Can get $\rho_1 = 0.822\ldots$



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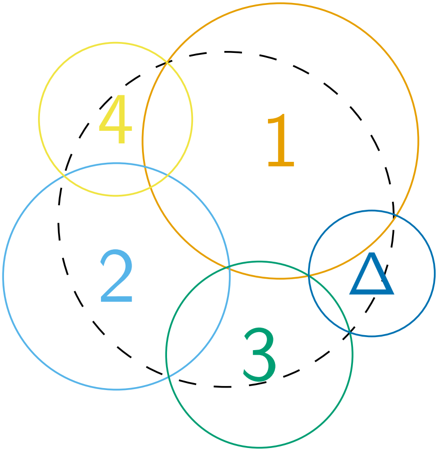
- What if we rearrange?
- Can get $\rho_1 = 0.822\ldots$
- $P(n) = R(n) \leq 3.54\lceil \log n \rceil$



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- What if we rearrange?
- Can get $\rho_1 = 0.822\ldots$
- $P(n) = R(n) \leq 3.54\lceil \log n \rceil$
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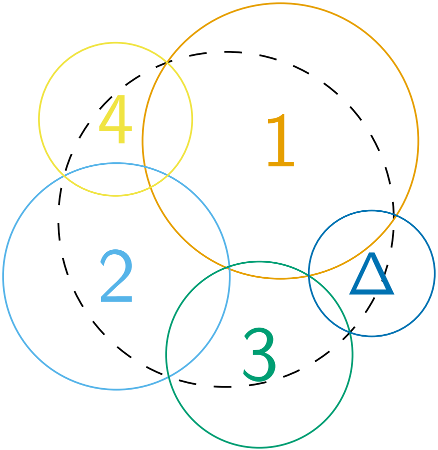

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- Why only 5 probes?

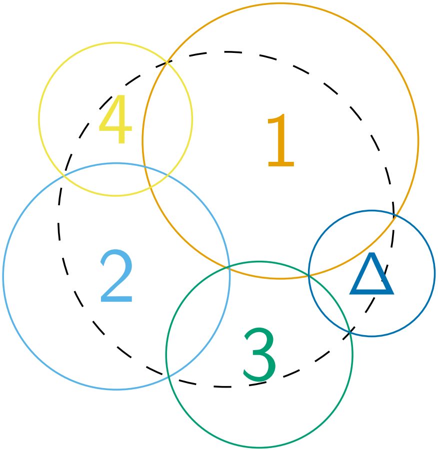

Figure 9: Algorithm 4

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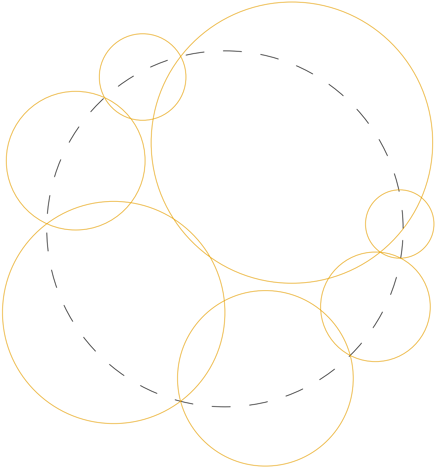
- Why only 5 probes? Smaller probes $\rightarrow$ uncovered center



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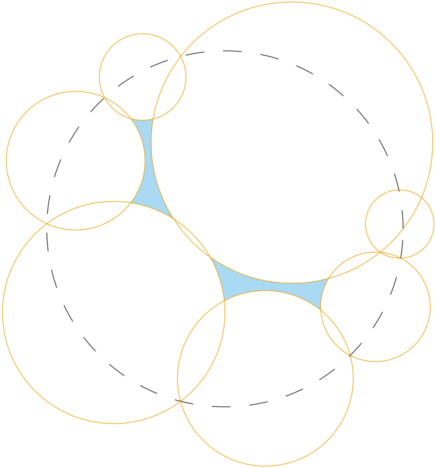

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...

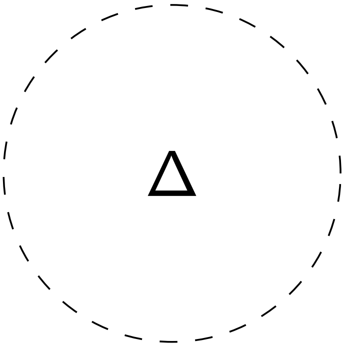

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...

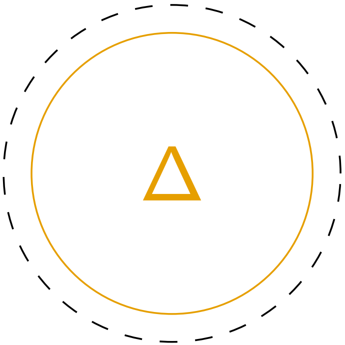

Figure 9: Algorithm 5

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.

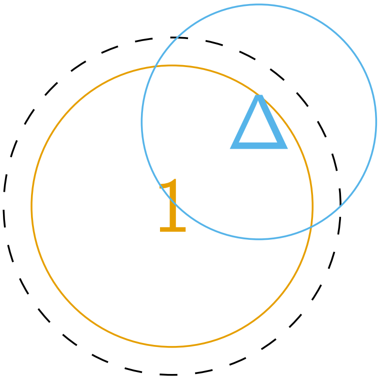

Figure 9: Algorithm 5

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- Let's start in center...
- Then perimeter..

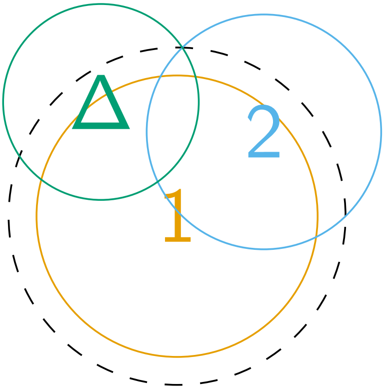

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter...

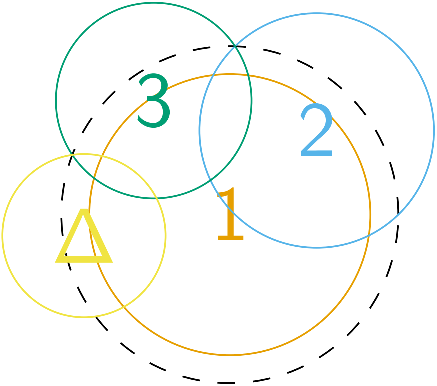

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center. . .
- Let's start in center. . .
- Then perimeter....

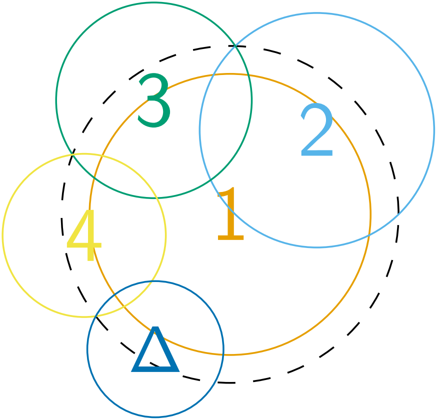

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.....

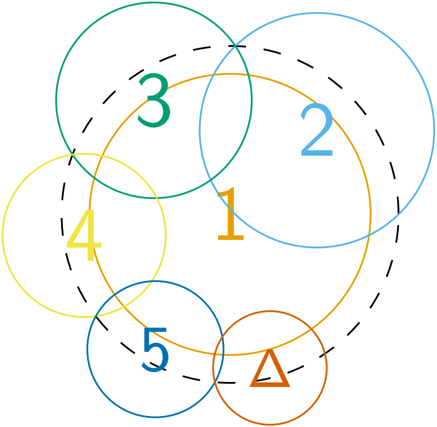

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter......

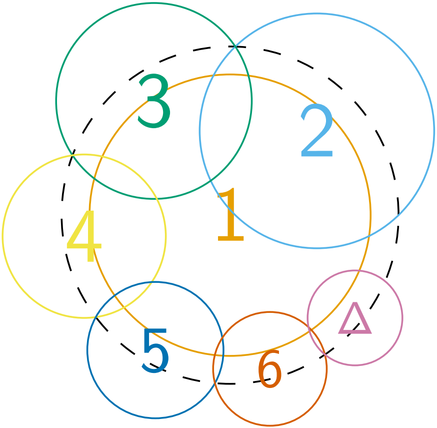

Figure 9: Algorithm 5

- Avoid uncovered center. . .
- Let's start in center. . .
- Then perimeter.......
- 7 probes!

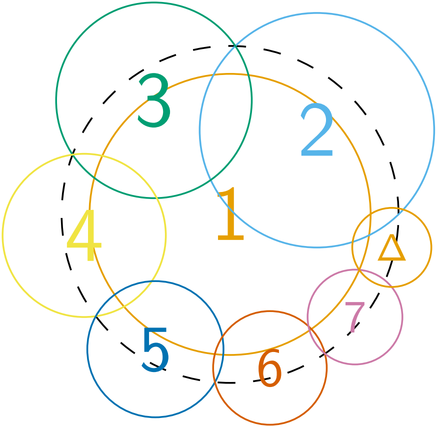

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.......

- 7 probes! but inefficient...
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$

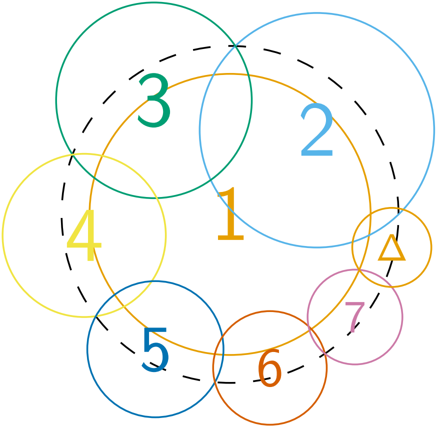

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.......

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- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$

- Problem: Outer circumference covered faster than inner

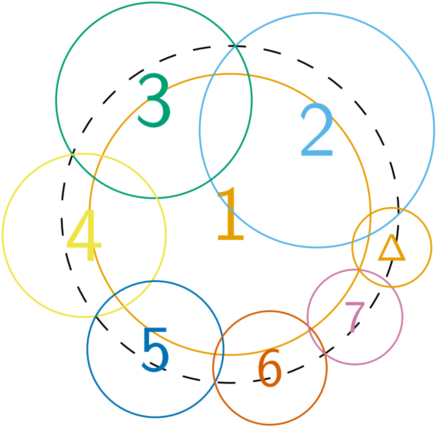

Figure 9: Algorithm 5

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.......
- 7 probes! but inefficient...
- $P(n) \leq 3.83 \lceil \log n \rceil$
- $D(n) \leq 6.72n$
- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate

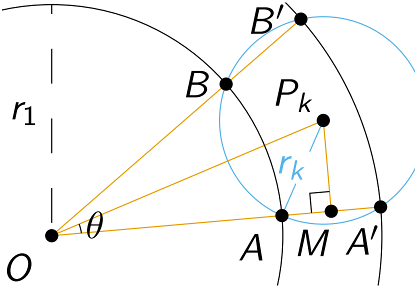

Figure 9: Some geometry

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center. . .
- Let's start in center. . .
- Then perimeter.......
- 7 probes! but inefficient. . .
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$
- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate
- Result:

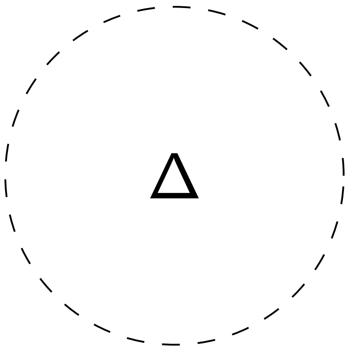

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.......
- 7 probes! but inefficient...
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$
- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate
- Result:

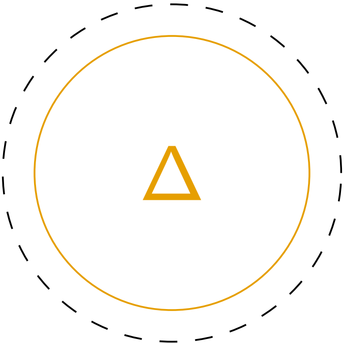

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- $D(n) \leq 6.72n$
- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate
- Result:

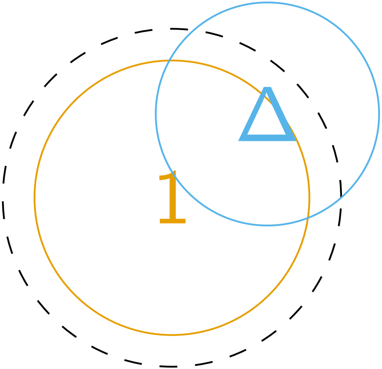

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- Problem: Outer circumference covered faster than inner
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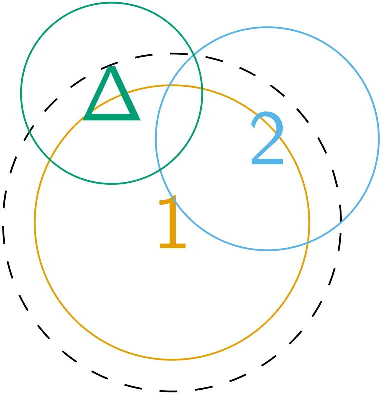

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- Solution: Cover at same rate
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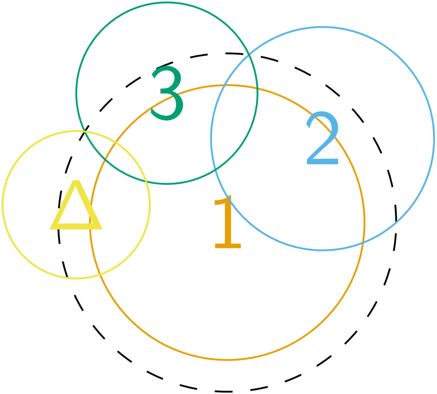

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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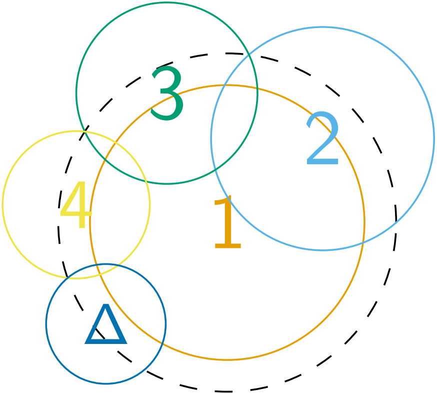

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- Result:

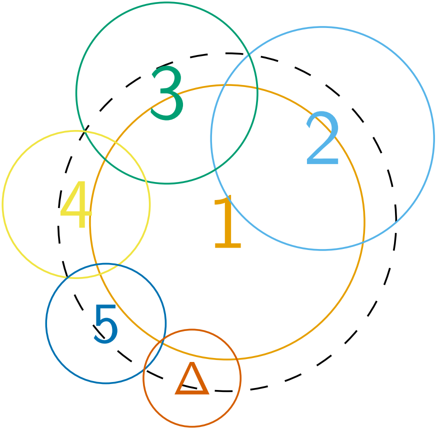

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- Solution: Cover at same rate

- Result:

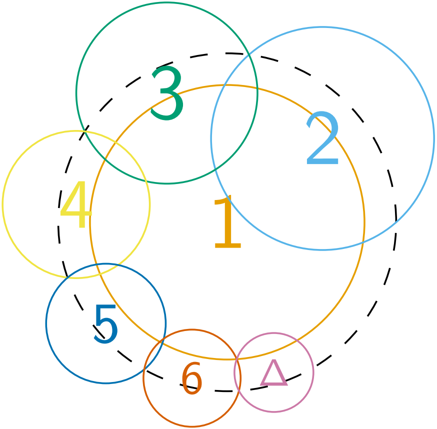

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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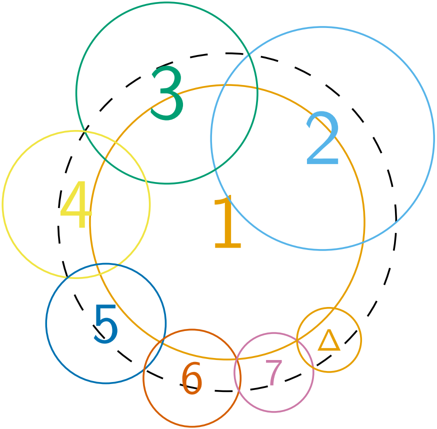

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- $D(n) \leq 6.72 n$

- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate

- Result:

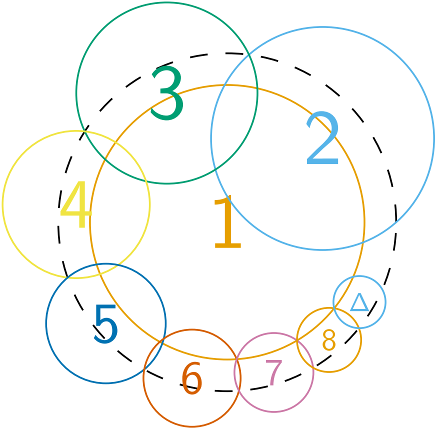

Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

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- Let's start in center...
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- Result:

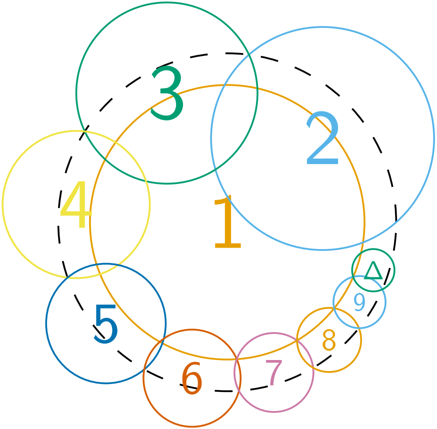

Figure 9: Algorithm 6

- Avoid uncovered center...
- Let's start in center...
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- 7 probes! but inefficient...
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$
- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate
- Result:

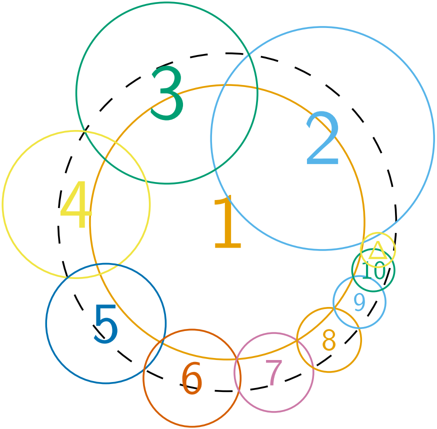


Figure 9: Algorithm 6

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.......
- 7 probes! but inefficient...
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$
- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate
- Result: turned up to 11!

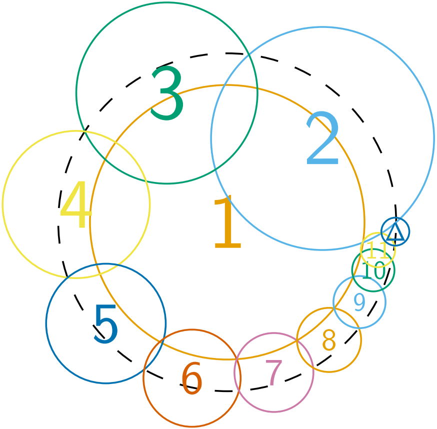


Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center...
- Let's start in center...
- Then perimeter.......

- 7 probes! but inefficient...
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$

- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate

- Result: turned up to 11!
- $P(n) \leq 3.34\lceil \log n \rceil$
- $D(n) \leq 6.02n$

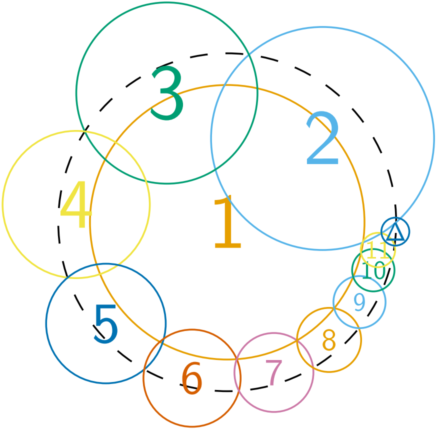


Figure 9: Algorithm 6

L_2: Higher-count Monotonic-path Algorithms

- Avoid uncovered center. . .
- Let's start in center. . .
- Then perimeter.......

- 7 probes! but inefficient. . .
- $P(n) \leq 3.83\lceil \log n \rceil$
- $D(n) \leq 6.72n$

- Problem: Outer circumference covered faster than inner
- Solution: Cover at same rate

- Result: turned up to 11!
- $P(n) \leq 3.34\lceil \log n \rceil$
- $D(n) \leq 6.02n$ – our best result!

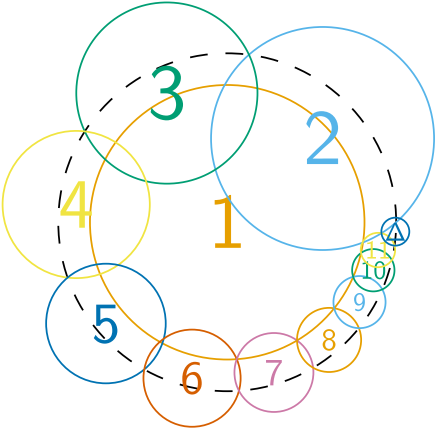
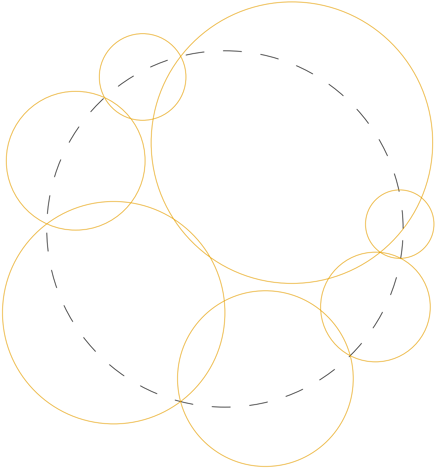


Figure 9: Algorithm 6

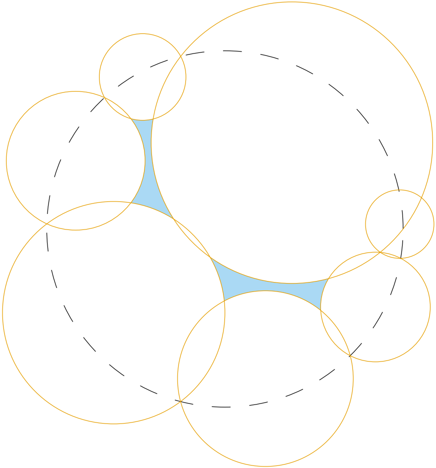
L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...



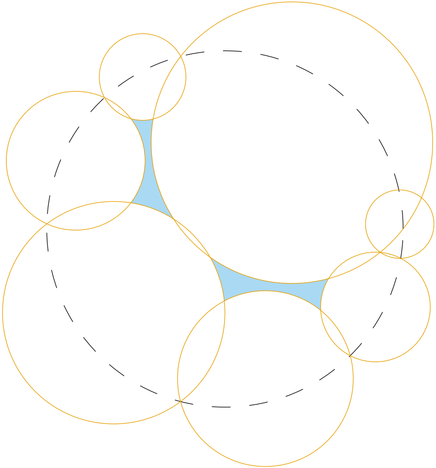
L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗



L_2: Darting Non-Monotonic Algorithms

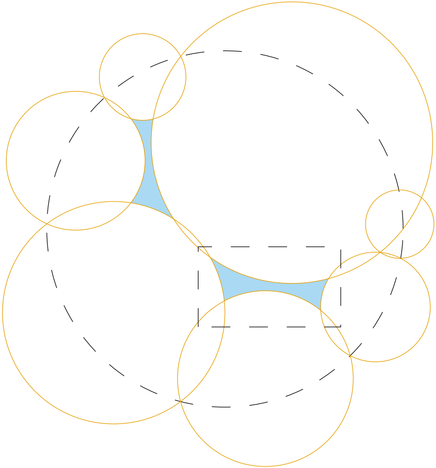
- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...



L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small…
- Uncovered internal area ✗
- Can add more probes…

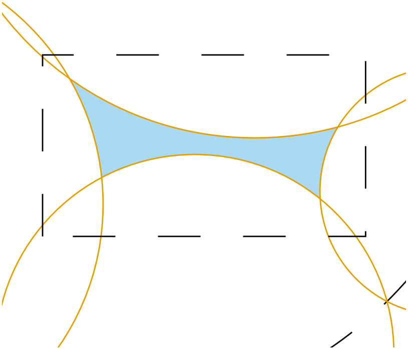
- How?



L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

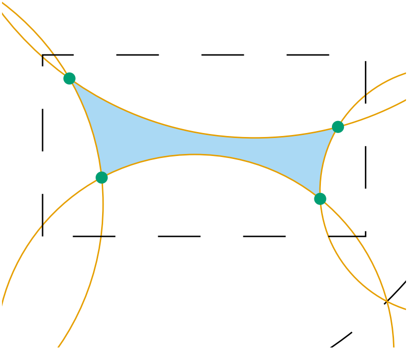
- How?



L_2: Darting Non-Monotonic Algorithms

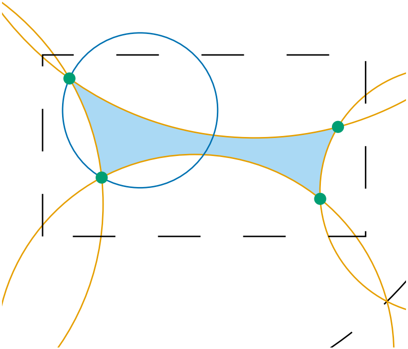
- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...



- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage.



L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...
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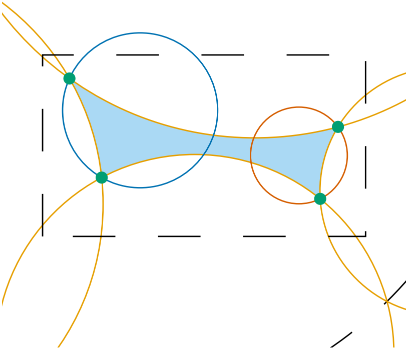
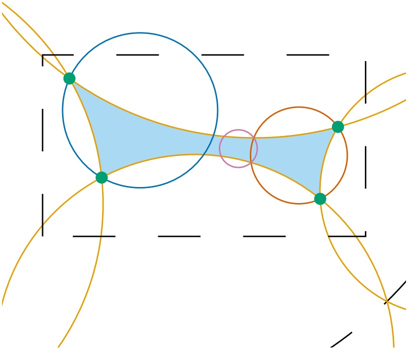


L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage...

- Fill in holes programmatically!



L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...
- How?
  1. Identify corners...
  2. Maximize coverage...
- Fill in holes programmatically!
- Start w/ Algorithm 4 ...

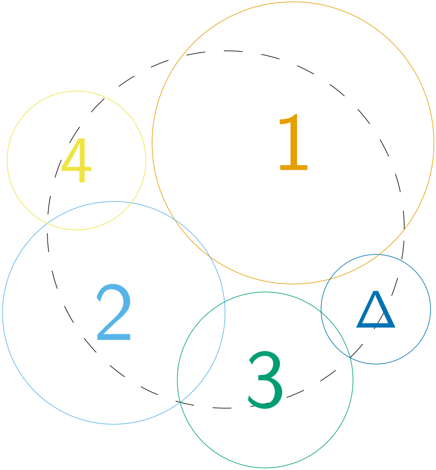


Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage...

- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically.

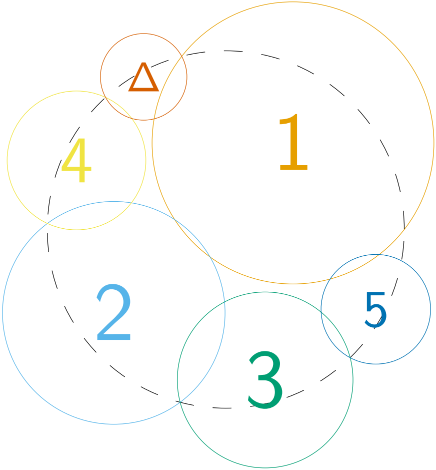


Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
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- How?
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- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically..

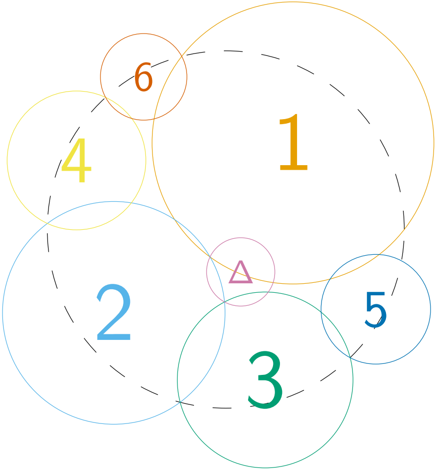

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

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- Start w/ Algorithm 4 ...
- Finish programmatically...

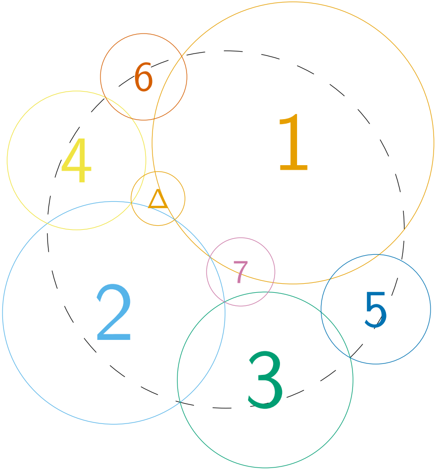

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

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- Start w/ Algorithm 4 ...
- Finish programmatically....

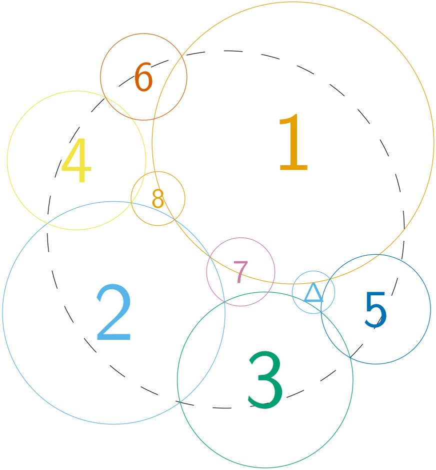

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

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- Start w/ Algorithm 4 ...
- Finish programmatically.....

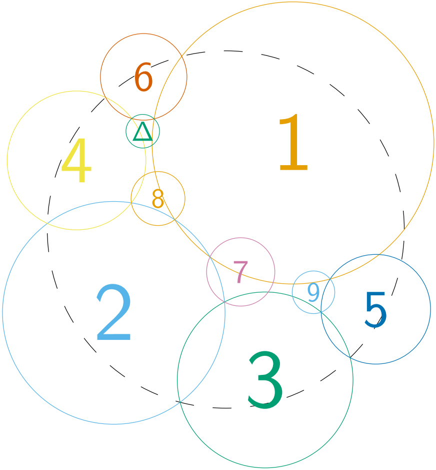

Figure 10: Algorithm 7

- Algorithm 4 w/ ρ_1 too small...
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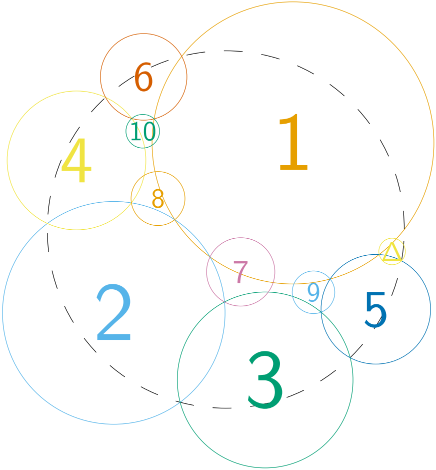

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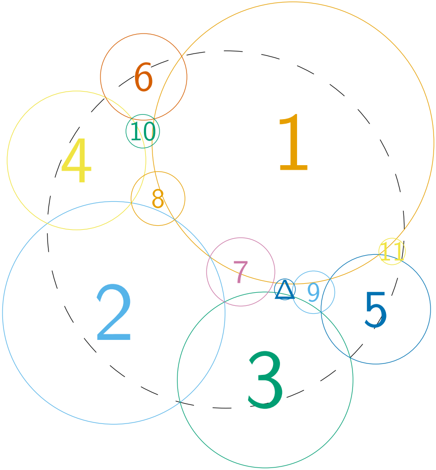

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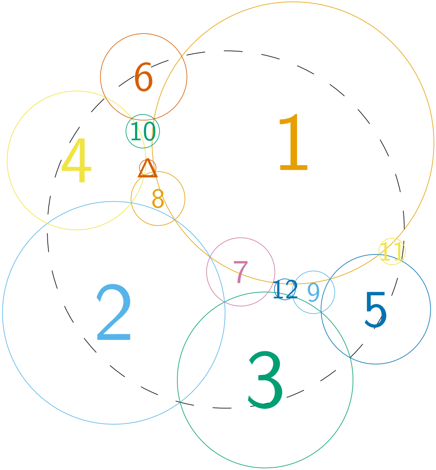

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage...

- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically........

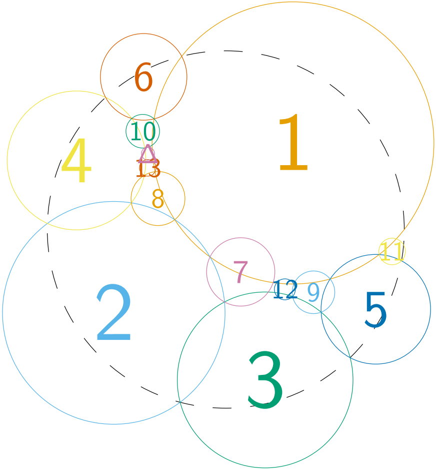

Figure 10: Algorithm 7

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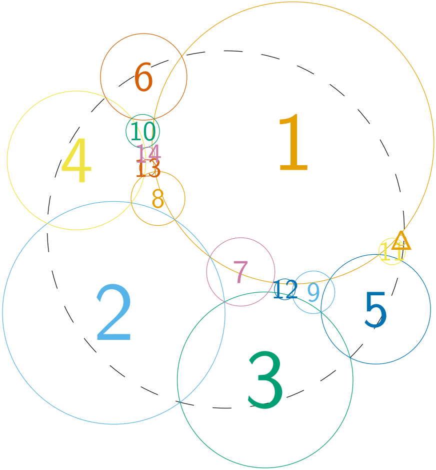

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

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- Can add more probes. . .

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- Start w/ Algorithm 4 . . .
- Finish programmatically..........

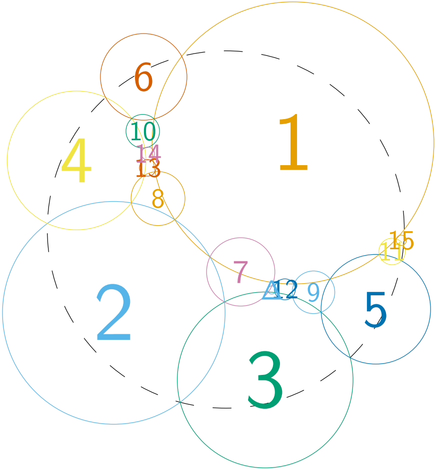

Figure 10: Algorithm 7

- Algorithm 4 w/ ρ_1 too small...
- Uncovered internal area ✗
- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage...

- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically...........
- 25 probes!

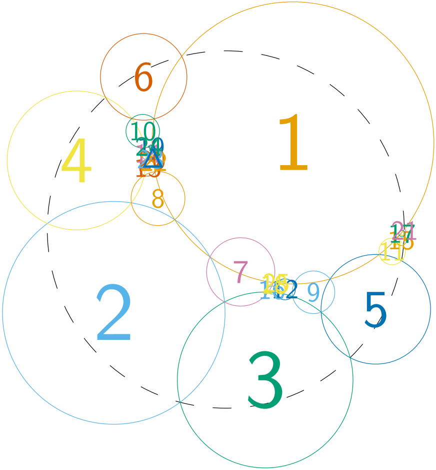

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

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  1. Identify corners...
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- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically...........
- 25 probes!
- $P(n) \leq 2.93\lceil \log n \rceil$

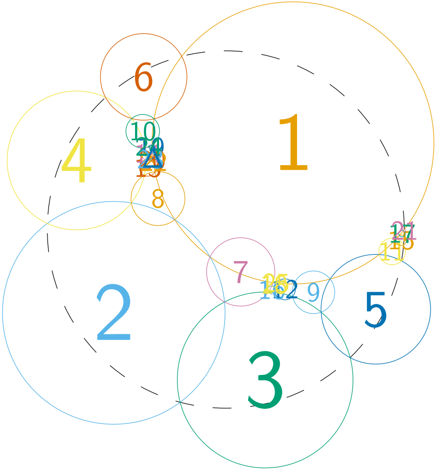

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
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- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage...

- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically...........
- 25 probes!
- $P(n) \leq$ 2.93$\lceil \log n \rceil$
- $D(n) \leq$ 25.8n

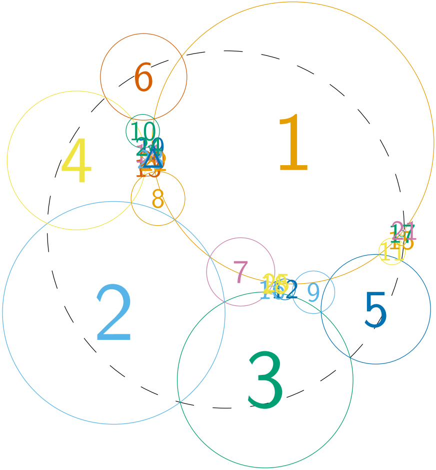

Figure 10: Algorithm 7

L_2: Darting Non-Monotonic Algorithms

- Algorithm 4 w/ ρ_1 too small...
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- Can add more probes...

- How?
  1. Identify corners...
  2. Maximize coverage...

- Fill in holes programmatically!

- Start w/ Algorithm 4 ...
- Finish programmatically...........
- 25 probes!
- $P(n) \leq 2.93\lceil \log n \rceil$
- $D(n) \leq 25.8n$ – terrible!

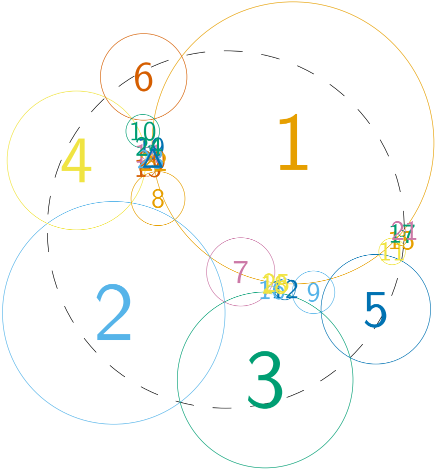
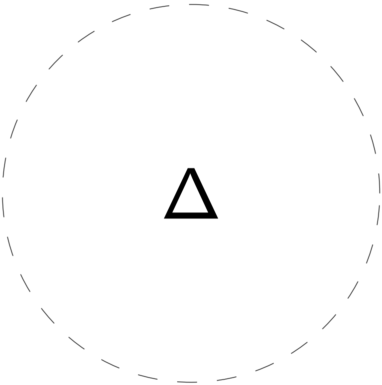


Figure 10: Algorithm 7

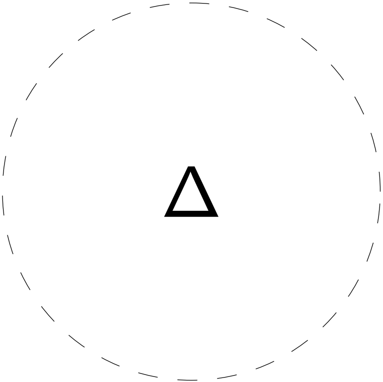
L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?



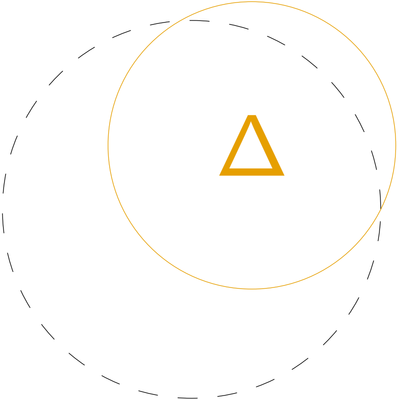
L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop



L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop
- Differential evolution.



L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop
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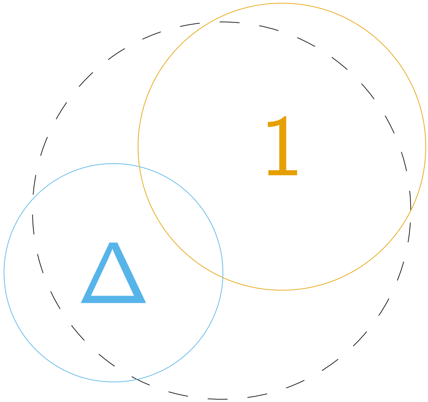

L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
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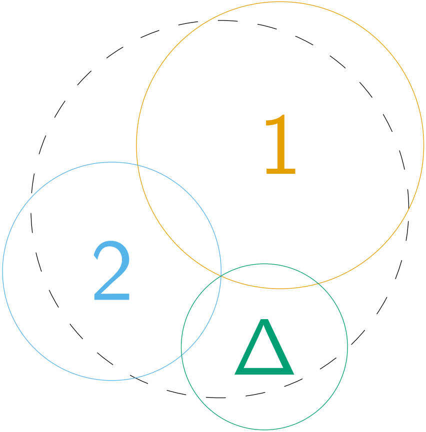

L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
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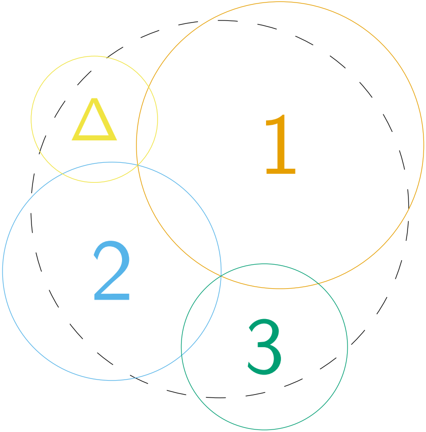
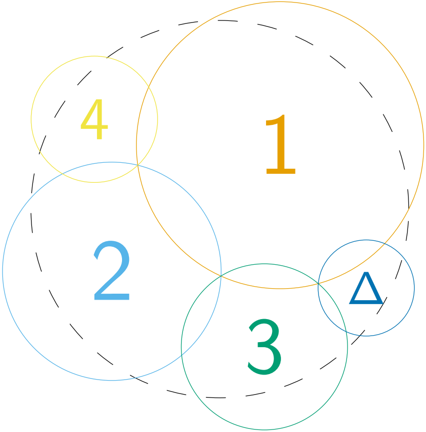
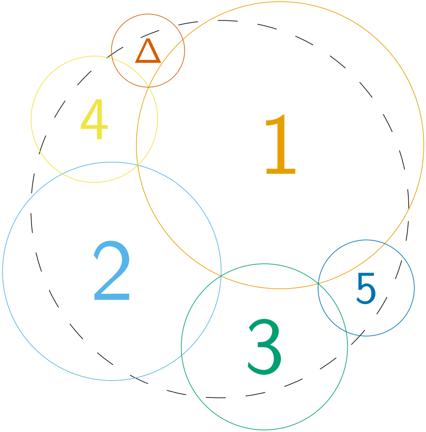


- How far can we push this?
- Take human out of the loop
- Differential evolution.....



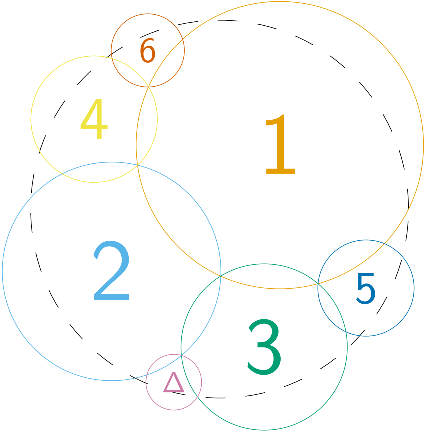
L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop
- Differential evolution......



L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic.



L_2: Darting Non-Monotonic Algorithms – cont'd

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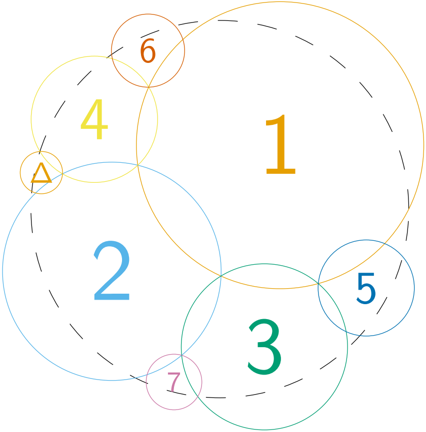

L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
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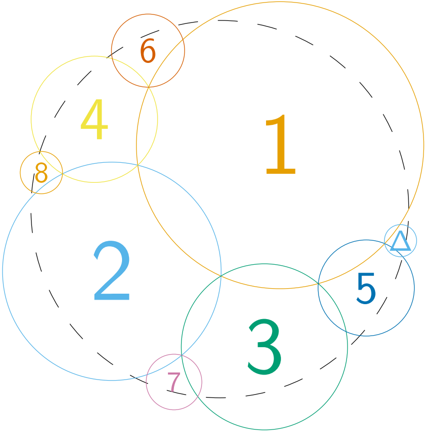
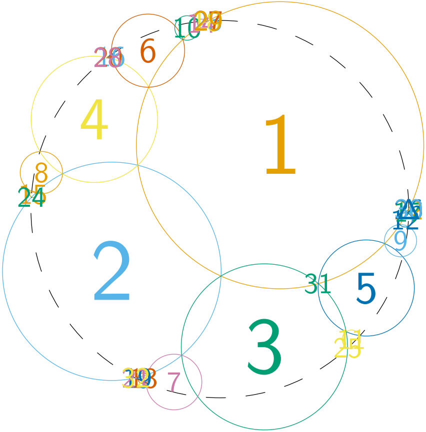


L_2: Darting Non-Monotonic Algorithms – cont'd

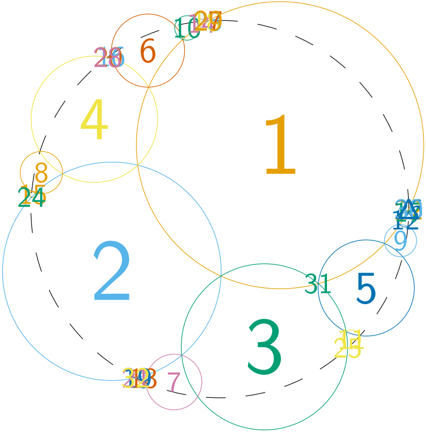
- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic............
- 32 probes!!!



L_2: Darting Non-Monotonic Algorithms – cont'd

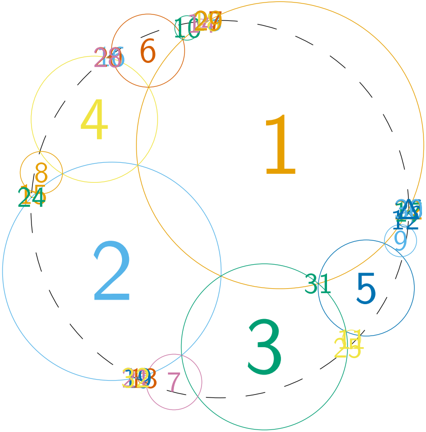
- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic............
- 32 probes!!!

- How good is it?



- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic............
- 32 probes!!!

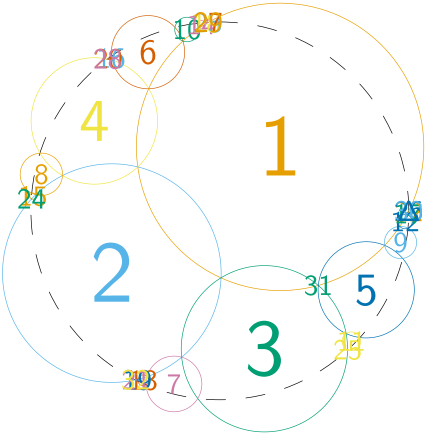
- How good is it?
- $P(n) \leq 2.53\lceil \log n \rceil$



L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic............
- 32 probes!!!

- How good is it?
- $P(n) \leq$ 2.53$\lceil \log n \rceil$
- Recall: 2.4$\lceil \log n \rceil$ – lower bound

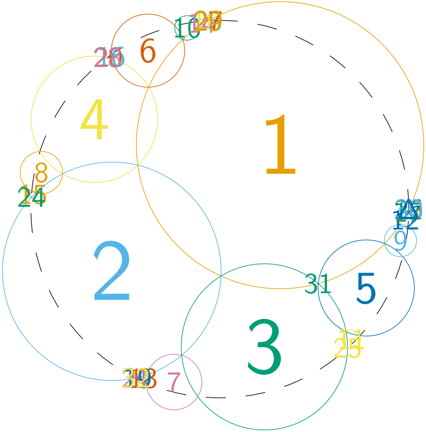


L_2: Darting Non-Monotonic Algorithms – cont'd

- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic............
- 32 probes!!!

- How good is it?
- $P(n) \leq {\color{red}2.53}\lceil \log n \rceil$
- Recall: ${\color{red}2.4}\lceil \log n \rceil$ – lower bound

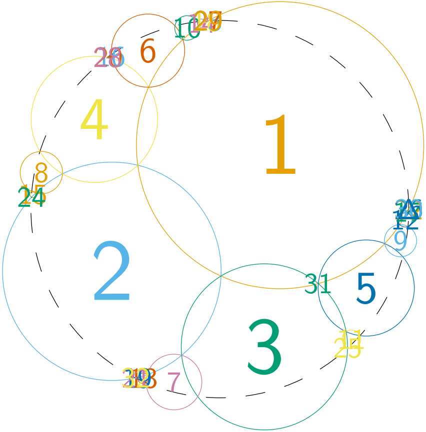
- Distance?



- How far can we push this?
- Take human out of the loop
- Differential evolution......
- Then programmatic............
- 32 probes!!!

- How good is it?
- $P(n) \leq 2.53\lceil \log n \rceil$
- Recall: $2.4\lceil \log n \rceil$ – lower bound

- Distance? abysmal
- $D(n) \leq 45.4n$



L_2: Comparing # Probes

Category	Alg. #	Probes ($P/\lceil\log n\rceil$) Min	Avg	Max	Bound
Hexagonal	Alg. 1	**1.00**	3.24	5.70	6.00
	Alg. 2	**1.00**	2.93	4.80	5.00
Chord-Based	Alg. 3	3.85	4.13	4.25	4.08
	Alg. 4	3.10	3.52	3.70	3.54
Monotonic	Alg. 5	3.55	3.87	4.15	3.83
	Alg. 6	3.25	3.41	3.85	3.34
Darting	Alg. 7	2.90	2.99	3.65	2.93
	Alg. 8	2.55	**2.59**	**3.20**	**2.53**

Table 1: A numerical comparison of simulation results for our 8 algorithms on the number of probes made (P). The best values are highlighted in bold.

L_2: Comparing Distance Traveled

Category	Alg. #	Total Distance (D/n) Min	Avg	Max	Bound
Hexagonal	Alg. 1	**0.00**	3.35	10.39	10.39
	Alg. 2	**0.00**	2.65	8.81	8.81
Chord-Based	Alg. 3	4.69	5.46	6.56	6.95
	Alg. 4	4.30	5.38	9.00	9.31
Monotonic	Alg. 5	**0.00**	**1.92**	6.72	6.72
	Alg. 6	**0.00**	1.96	**6.01**	**6.02**
Darting	Alg. 7	3.86	5.97	25.74	25.80
	Alg. 8	2.44	4.05	42.58	45.40

Table 2: A numerical comparison of simulation results for our 8 algorithms on the total distance traveled by Δ (D). The best values are highlighted in bold.

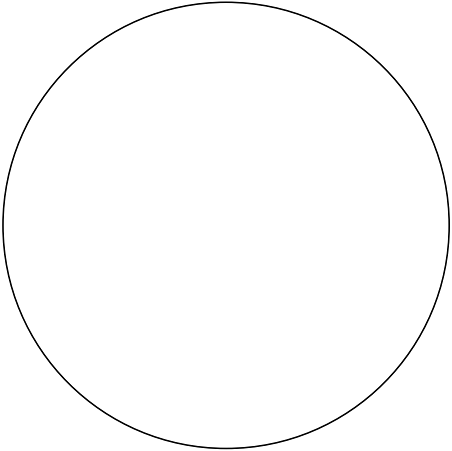
L_2: Comparing # Responses

Category	Alg. #	Responses ($R/\lceil \log n \rceil$)			
		Min	Avg	Max	Bound
Hexagonal	Alg. 1	**0.20**	**0.89**	**1.00**	**1.00**
	Alg. 2	0.35	1.11	1.45	2.00
Chord-Based	Alg. 3	1.40	1.99	2.40	4.08
	Alg. 4	0.80	1.94	2.50	3.54
Monotonic	Alg. 5	0.80	2.49	3.85	3.83
	Alg. 6	0.60	1.96	3.35	3.34
Darting	Alg. 7	0.30	1.39	2.15	2.93
	Alg. 8	0.45	1.31	1.85	2.53

Table 3: A numerical comparison of simulation results for our 8 algorithms on the number of POI responses made (R). The best values are highlighted in bold.

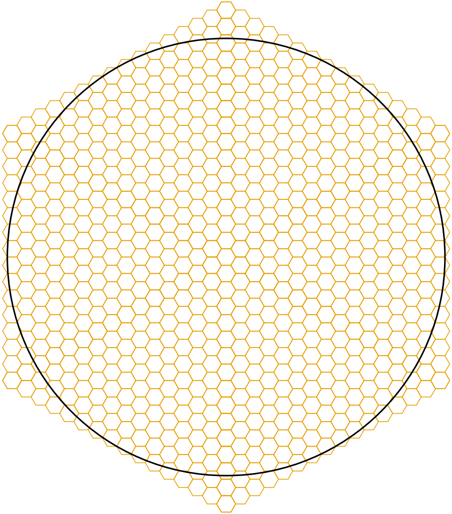
L_2: Reducing Responses

- If POI only allowed 1 response?



L_2: Reducing Responses

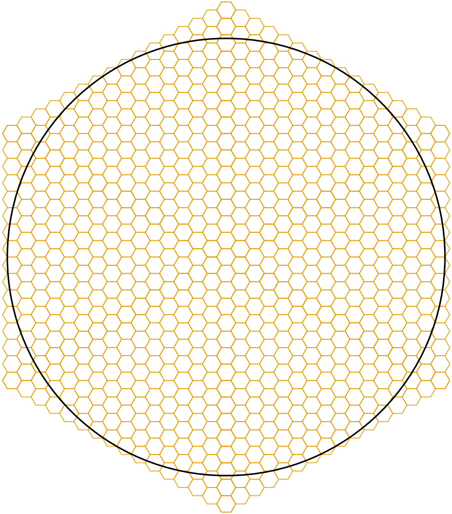
- If POI only allowed 1 response?
- Large hexagonal lattice!



L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

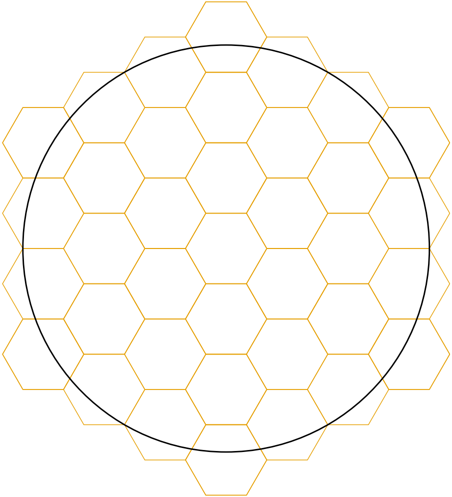
- If POI allowed 2 responses?



L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

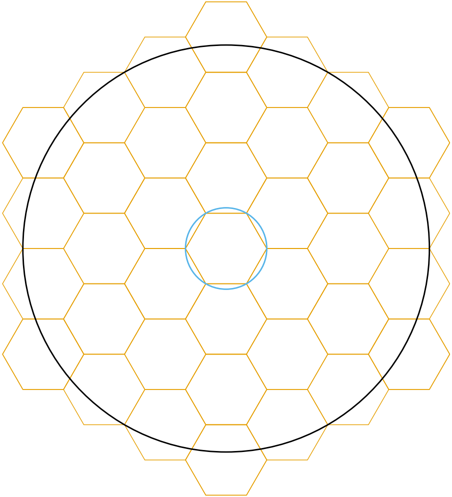
- If POI allowed 2 responses?
- Medium hexagonal lattice...



L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

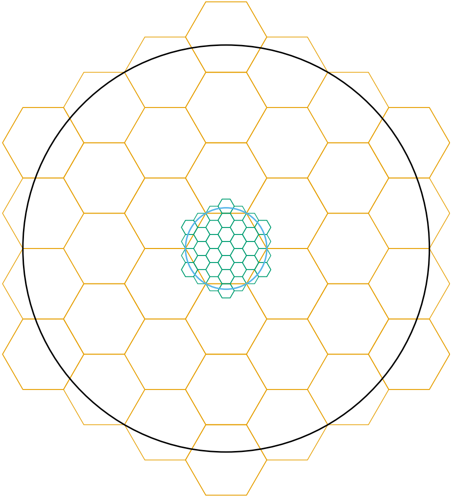
- If POI allowed 2 responses?
- Medium hexagonal lattice...
- After one response...



L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

- If POI allowed 2 responses?
- Medium hexagonal lattice...
- After one response... recurse!

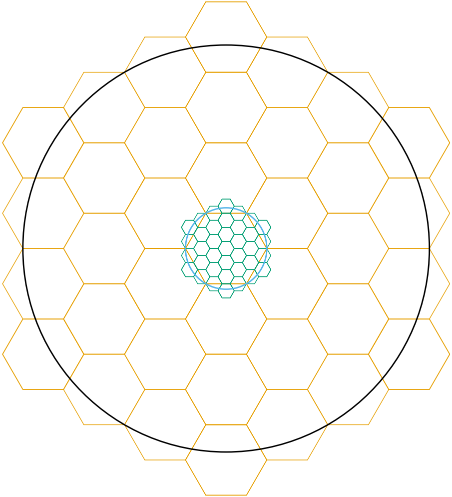


L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

- If POI allowed 2 responses?
- Medium hexagonal lattice...
- After one response... recurse!

- If POI allowed $R_{\max}$ responses?



L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

- If POI allowed 2 responses?
- Medium hexagonal lattice...
- After one response... recurse!

- If POI allowed R_{max} responses?
- R_{max} recursions!

Theorem

If a POI is allowed R_{max} responses,

$$P(n) \leq 6R_{max}\binom{\left\lceil \frac{2n^{\frac{1}{R_{max}}}+2}{3} \right\rceil}{2} \tag{1}$$

$$L = \left\lceil \frac{2n^{\frac{1}{R_{max}}}+2}{3} \right\rceil \text{ rings.} \tag{2}$$

L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

- If POI allowed 2 responses?
- Medium hexagonal lattice...
- After one response... recurse!

- If POI allowed $R_{\max}$ responses?
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Theorem

If a POI is allowed R_{max} responses,

$$P(n) \leq 6R_{max}\binom{\lceil \frac{2n^{\frac{1}{R_{max}}}+2}{3} \rceil}{2} \quad (1)$$

$$L = \lceil \frac{2n^{\frac{1}{R_{max}}}+2}{3} \rceil \text{ rings.} \quad (2)$$

Corollary

1. *If $R_{max} = 1$, $P(n) \leq \mathcal{O}(n^2)$.*
2. *If $R_{max} = 2$, $P(n) \leq \mathcal{O}(n)$.*
3. *If $R_{max} = \lceil \log n \rceil$, then $P(n) \leq 6\lceil \log n \rceil$.*

L_2: Reducing Responses

- If POI only allowed 1 response?
- Large hexagonal lattice!

- If POI allowed 2 responses?
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- After one response... recurse!

- If POI allowed $R_{\max}$ responses?
- $R_{\max}$ recursions!

- Corollary 3 is Algorithm 1!

Theorem

If a POI is allowed R_{max} responses,

$$P(n) \leq 6R_{max}\binom{\lceil \frac{2n^{\frac{1}{R_{max}}}+2}{3} \rceil}{2} \tag{1}$$

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Corollary

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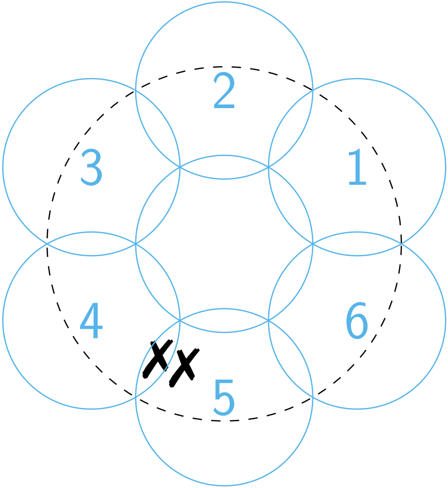
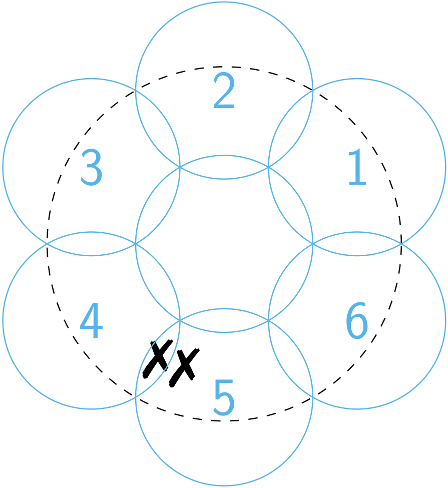
- Finding all k POIs?



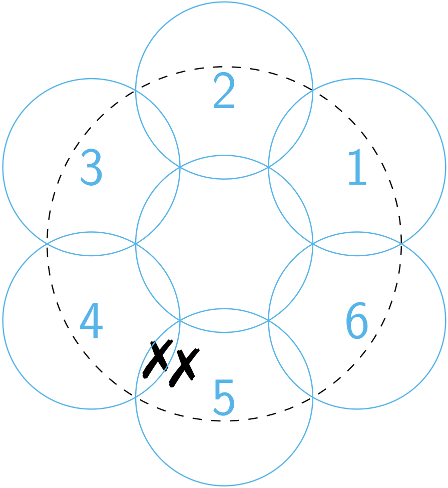
L_2: Finding All POIs

- Finding all k POIs?
- Using algorithm $\mathcal{A}(n)$



L_2: Finding All POIs

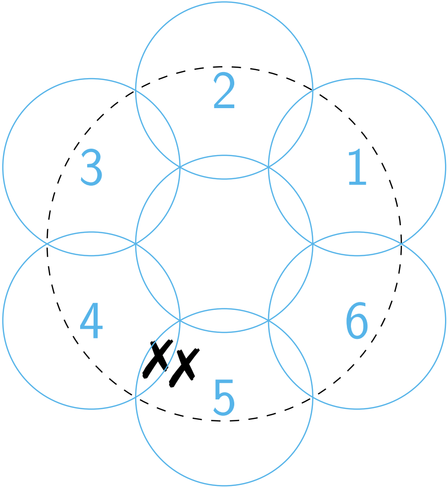
- Finding all k POIs?
- Using algorithm $\mathcal{A}(n)$
- One POI: $P(n) \leq {\color{red}c}\lceil \log n \rceil$



L_2: Finding All POIs

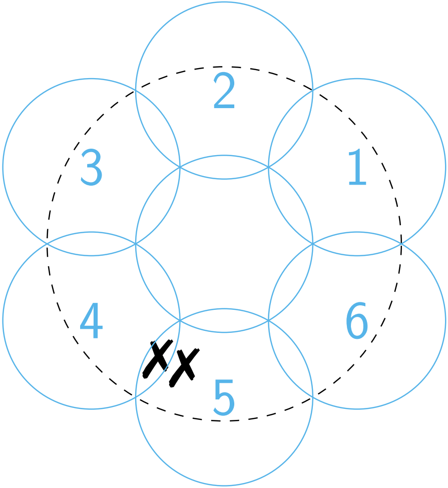
- Finding all k POIs?

- Using algorithm $\mathcal{A}(n)$
- One POI: $P(n) \leq c\lceil \log n \rceil$
- Traveling $D(n) \leq dn$



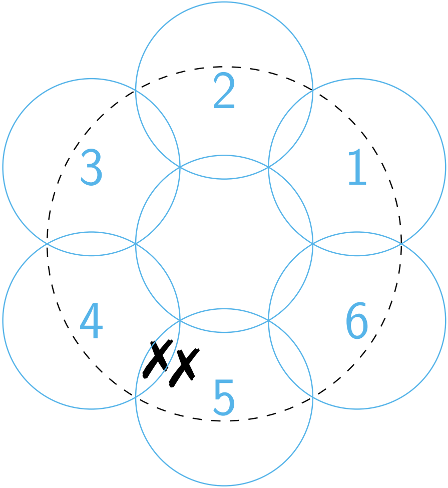
L_2: Finding All POIs

- Finding all k POIs?
- Using algorithm $\mathcal{A}(n)$
- One POI: $P(n) \leq c\lceil \log n \rceil$
- Traveling $D(n) \leq dn$
- Trivial: Call $\mathcal{A}(n)$ k times



L_2: Finding All POIs

- Finding all k POIs?
- Using algorithm $\mathcal{A}(n)$
- One POI: $P(n) \leq c\lceil \log n \rceil$
- Traveling $D(n) \leq dn$
- Trivial: Call $\mathcal{A}(n)$ k times
- $P_{\text{tot}} \leq ck\lceil \log n \rceil$
- $D_{\text{tot}} \leq dkn$



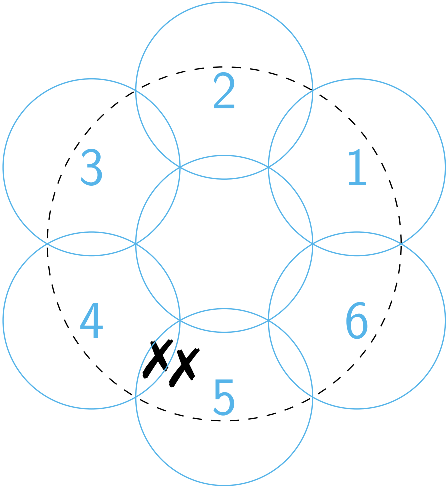
L_2: Finding All POIs

- Finding all k POIs?

- Using algorithm $\mathcal{A}(n)$
- One POI: $P(n) \leq c\lceil \log n \rceil$
- Traveling $D(n) \leq dn$

- Trivial: Call $\mathcal{A}(n)$ k times
- $P_{\text{tot}} \leq ck\lceil \log n \rceil$
- $D_{\text{tot}} \leq dkn$

- Can we do better?



L_2: Finding All POIs

- Finding all k POIs?
- Using algorithm $\mathcal{A}(n)$
- One POI: $P(n) \leq c\lceil \log n \rceil$
- Traveling $D(n) \leq dn$
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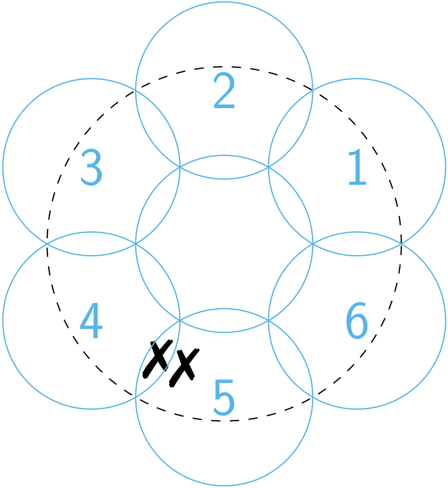
- $P_{\text{tot}} \leq ck\lceil \log n \rceil$
- $D_{\text{tot}} \leq dkn$
- Can we do better?
- Recall: Probes return boolean



L_2: Finding All POIs

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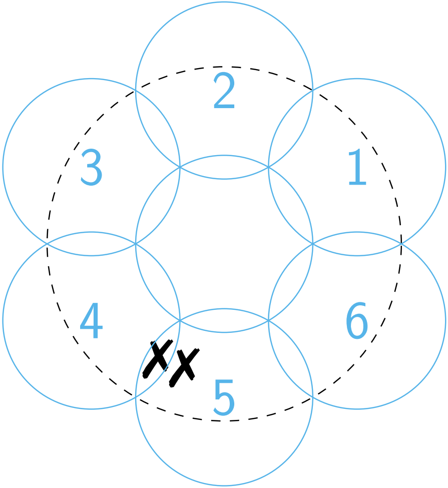
- Recall: Probes return boolean
- Even returning quantity...?



L_2: Finding All POIs – cont'd

- Trivial: Call $\mathcal{A}(n)$ k times
- $P_{\text{tot}} \leq ck\lceil \log n \rceil$
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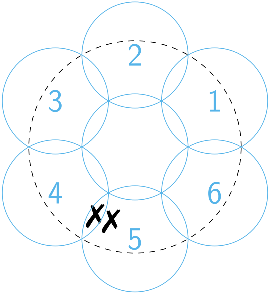
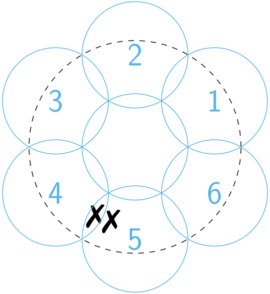


- Trivial: Call $\mathcal{A}(n)$ k times
- $P_{\text{tot}} \leq ck\lceil\log n\rceil$
- $D_{\text{tot}} \leq dkn$

- Simple idea – once one found,



L_2: Finding All POIs – cont'd

- Trivial: Call $\mathcal{A}(n)$ k times
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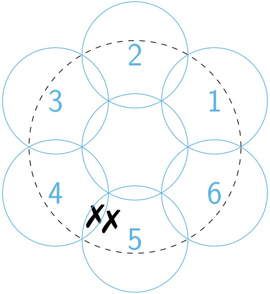
- Simple idea – once one found,
- exponential search



L_2: Finding All POIs – cont'd

- Trivial: Call $\mathcal{A}(n)$ k times
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- Simple idea – once one found,
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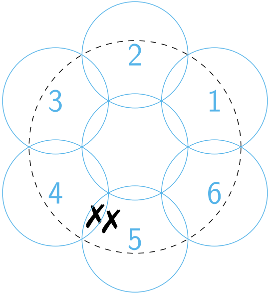

Theorem

We can perform a memoryless search for all k POIs in

$$P_{tot} \leq c\lceil \log n \rceil + (c+1)(k-1)\lceil \log \overline{e} \rceil,$$
$$D_{tot} \leq dn + 2dE,$$

where $E < OPT(\lceil \log k \rceil + 1)$, $\overline{e} = \frac{E}{k-1}$, *and OPT is the optimal tour length for the traveling salesperson problem (TSP) on the k POIs.*

Open Problems

- $P(n)$: Progressive probe LB: 2.40001, Alg. 6 achieves 2.53
  Can we tighten?
- Take advantage of known empty regions?
- Alg. 6 $D(n) \leq 6.02n$ – can we improve?
- Higher dimensions?
- Better find-all strategy?
- Other distance metrics (L_1, L_∞)?
- Instance optimal w.r.t. δ_{min}?