Fast Geographic Routing in Fixed-Growth Graphs

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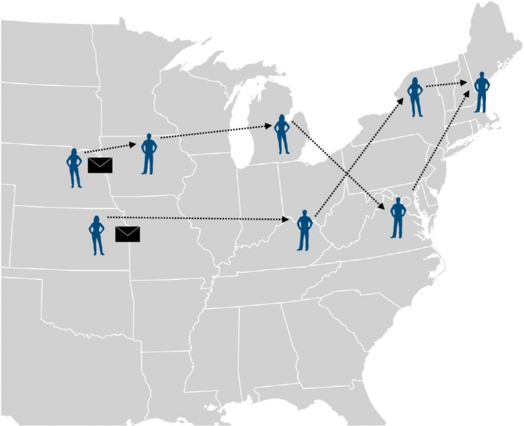
Special thanks to Evrim Ozel

University of California, Irvine

CIAC, 2025

History – Milgram's Small-World Experiments [5]

- Kansas and Nebraska → Massachusetts



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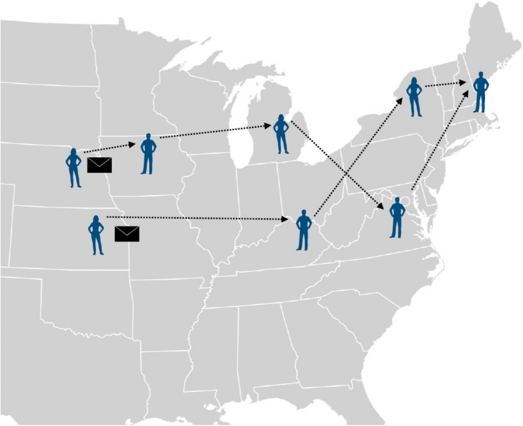

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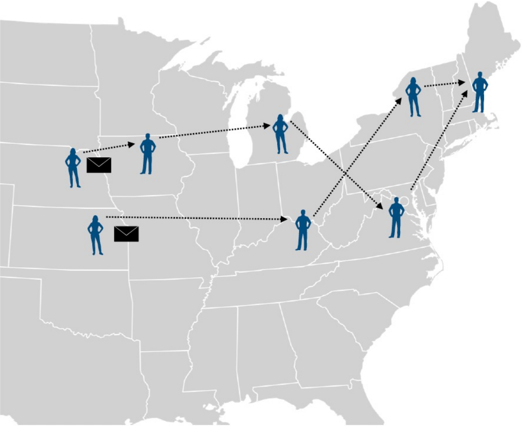

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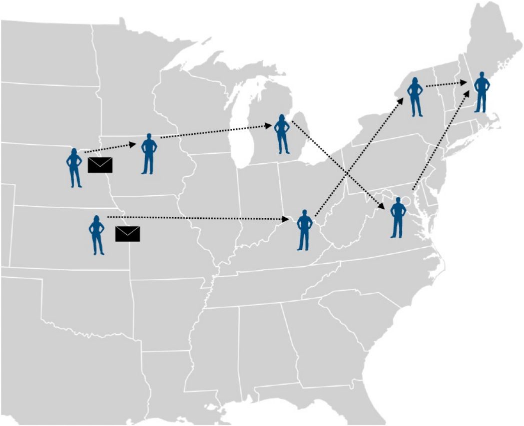

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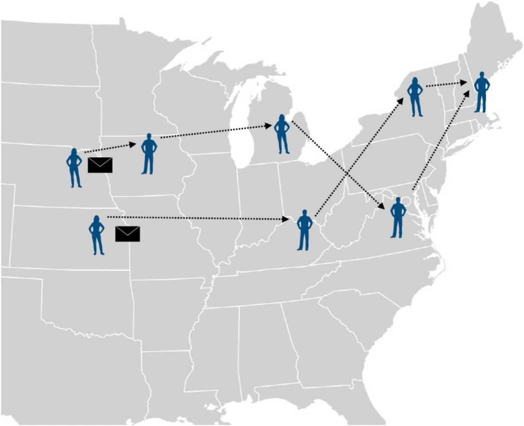

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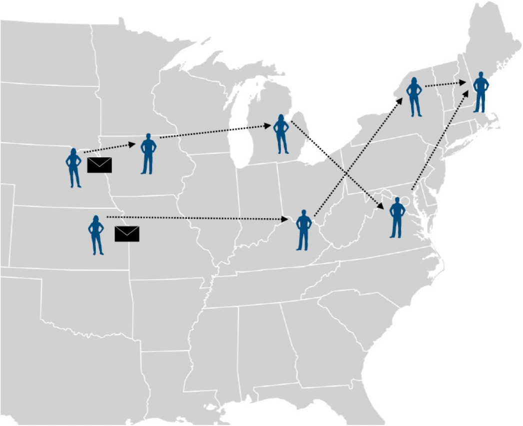
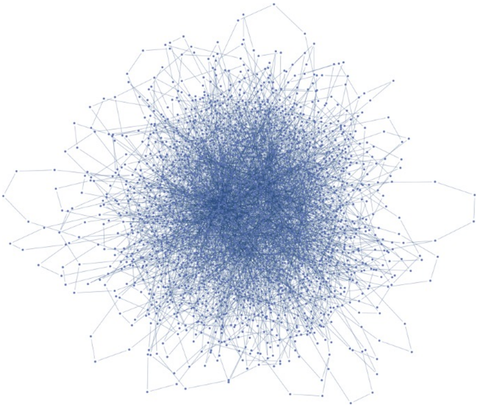
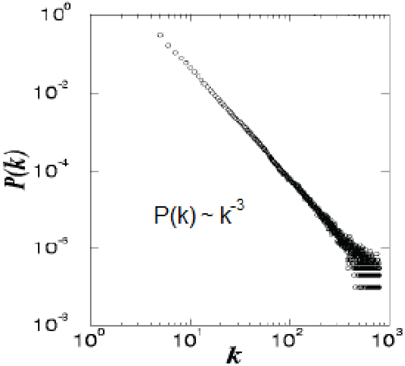
- How to model?



Preferential Attachment Models

- Rich get richer
- $P(u \to v) \propto d_v$





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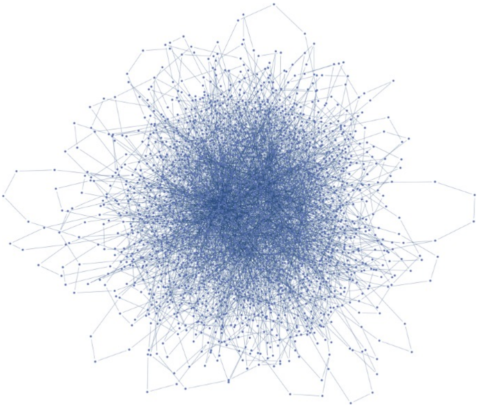
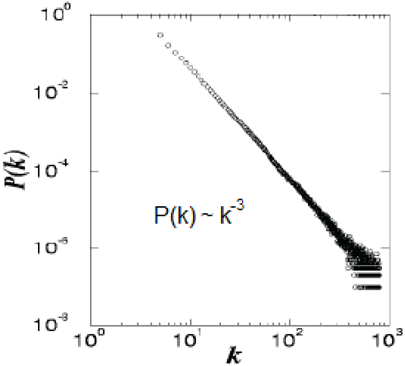
☑ Low $\mathcal{O}(\log n)$ diameter





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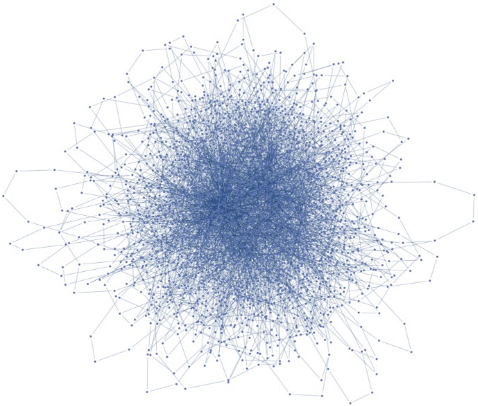
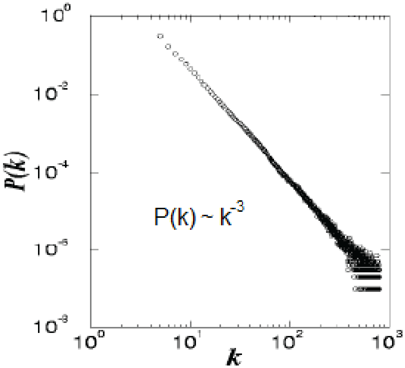
☒ No geography $\to$ no greedy routing





Kleinberg's Model $\mathcal{K}(n, p, q)$ [3]

- 2-D $n \times n$ lattice $\mathcal{L}$

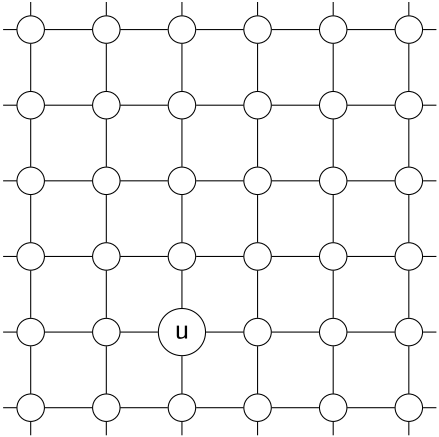


Figure 2: Kleinberg's Model $\mathcal{K}^*$

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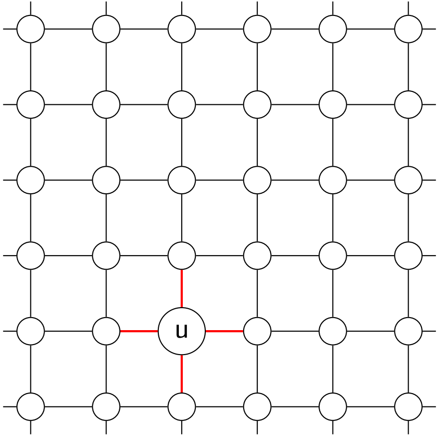

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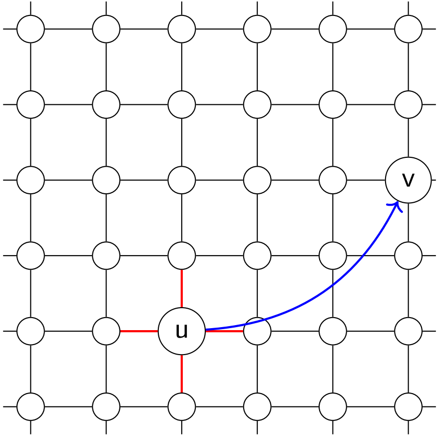

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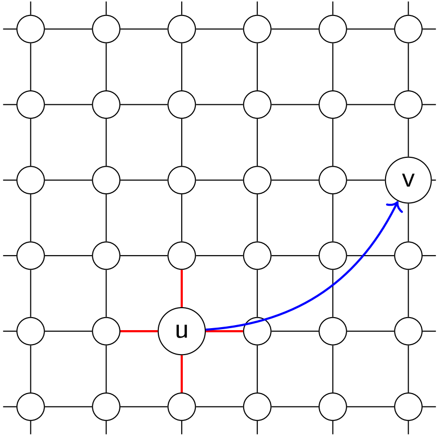

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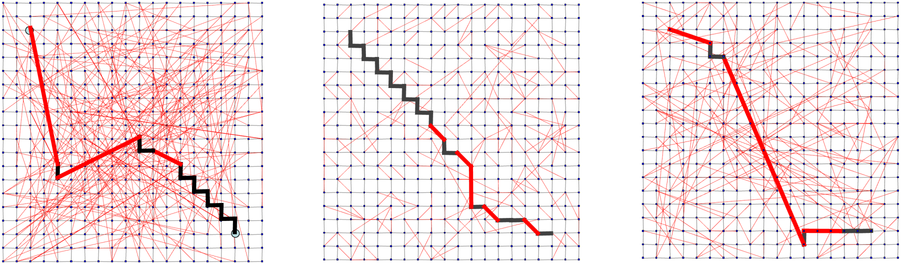

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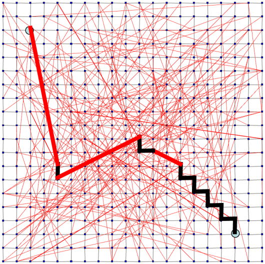

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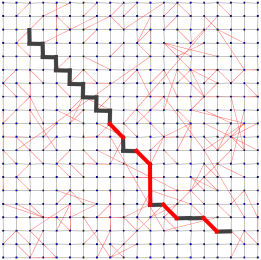

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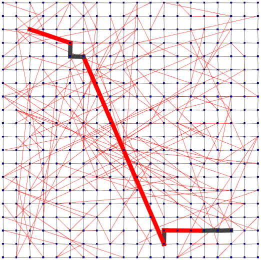

(c) $s = 2$

- Big impact, but... not 6!

Neighborhood Preferential Attachment (NPA) [2]

- Idea: Combine Kleinberg w/ Preferential Attachment
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- Experimentally good, but no theory

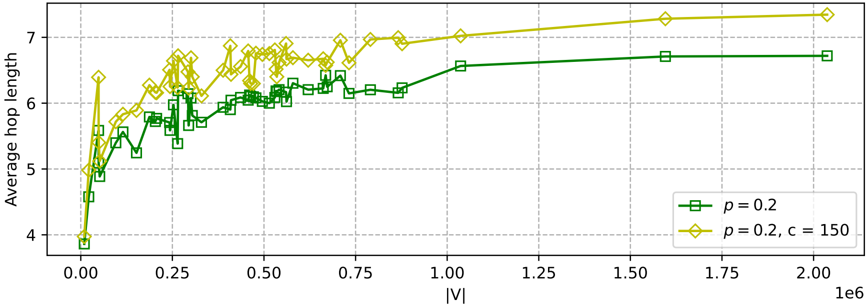

Figure 4: Average hop length when $q = 30$ [2]

Randomized Kleinberg Highway [1]

- $P(u \to v) \propto d_v / \delta(u, v)^s$
- Problem: Dependent probabilities

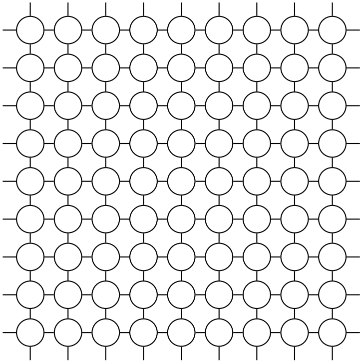

Figure 5: Randomized KH Graph
$n = 9$

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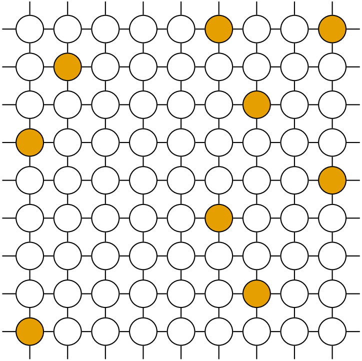

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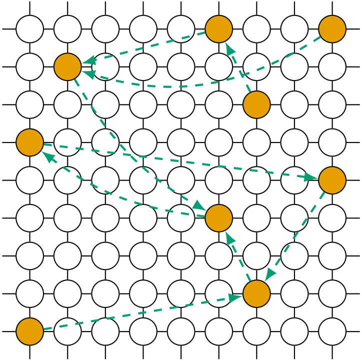

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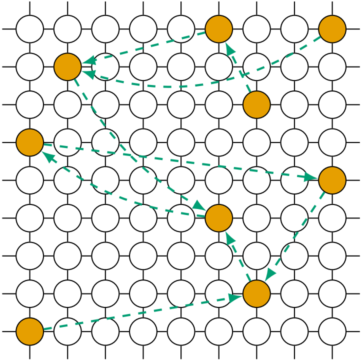
- *Only to other highway nodes!*



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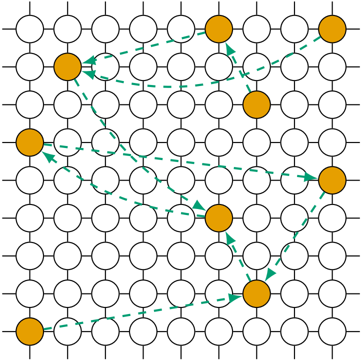
☑ Constant average degree
☑ Independent probabilities



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Randomized Kleinberg Highway – Analysis Sketch

1. Reach highway
2. Traverse highway
3. Reach destination

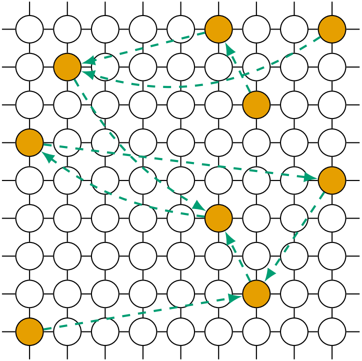
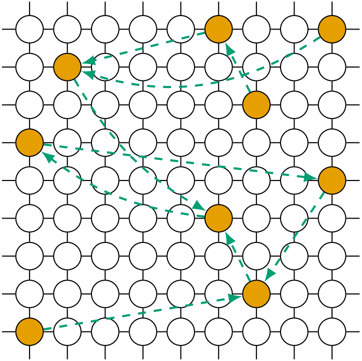


Figure 6: Randomized Kleinberg Highway Graph

Randomized Kleinberg Highway – Analysis Sketch

1. Reach highway – $\mathcal{O}(k)$
2. Traverse highway
3. Reach destination



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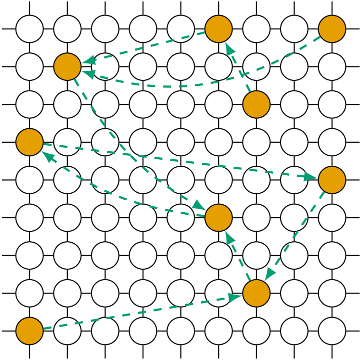

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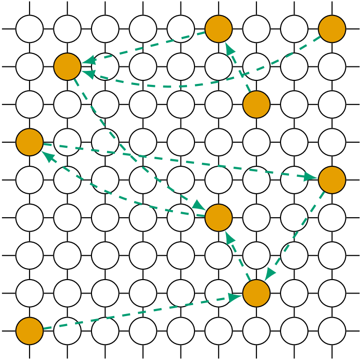
Randomized Kleinberg Highway – Analysis Sketch

1. Reach highway – $\mathcal{O}(k)$
2. Traverse highway – $\mathcal{O}(\log^2(n))$?
3. Reach destination – $\mathcal{O}(k)$



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1. Reach highway – $\mathcal{O}(k)$
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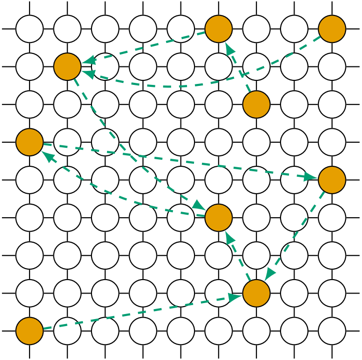
3. Reach destination – $\mathcal{O}(k)$



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1. Reach highway – $\mathcal{O}(k)$* in expectation
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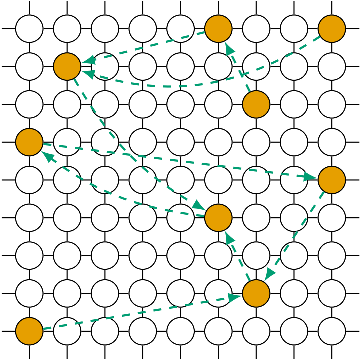
3. Reach destination – $\mathcal{O}(k)$*



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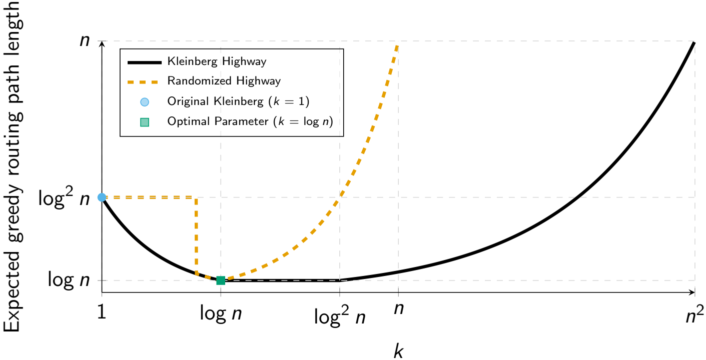
Kleinberg Highway – Results

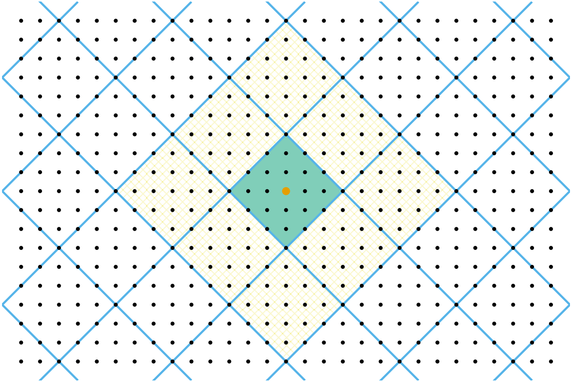
Theorem

For $k \in o\left(\frac{\log n}{\log\log\log n}\right)$, the expected decentralized greedy routing path length is $\mathcal{O}(\log^2 n)$, while for $\Theta\left(\frac{\log n}{\log\log\log n}\right) \leq k < \Theta(\log n)$, it is $\mathcal{O}(\log^2(n)/k)$, for $\Theta(\log n) \leq k \leq \Theta(n)$, it is $\mathcal{O}(k)$ and finally, for $k \in \Omega(n)$, it is $\mathcal{O}(n)$.



Nested Lattice Construction

- Nested lattices w/ radius ℓ



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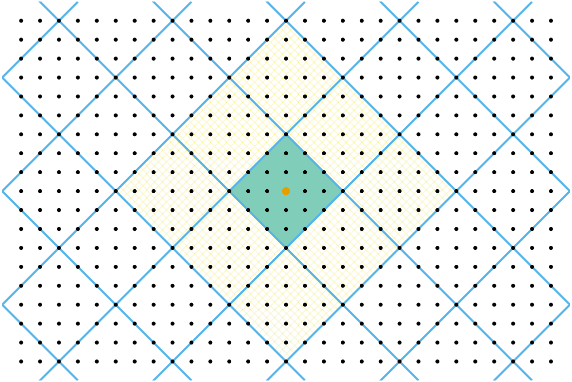

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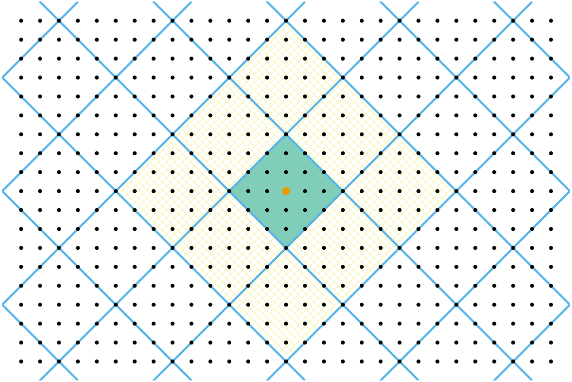

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Radius ℓ	Lower	Upper
$3\sqrt{k \log n}$	$9 \log n$	$41 \log n$
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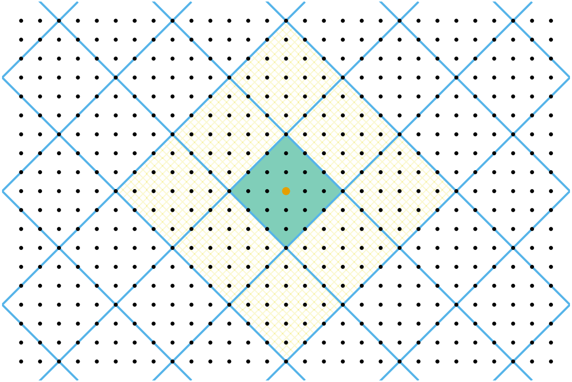

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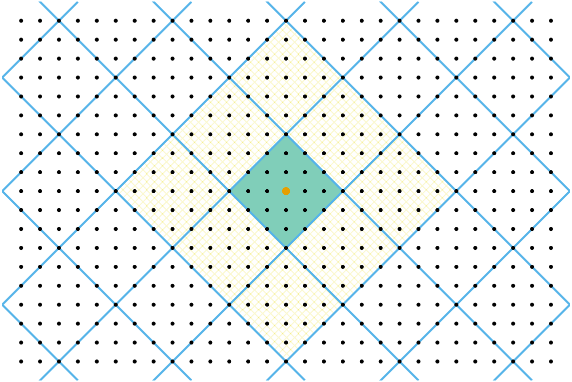
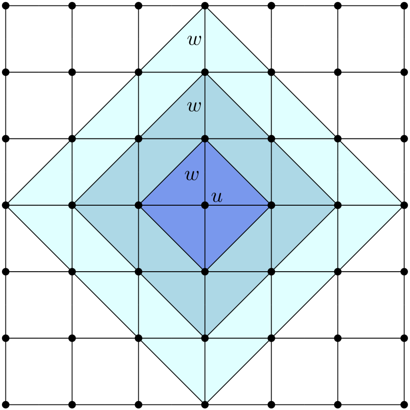


Fixed Growth Graphs

- Previous analyses: underlying lattice $\mathcal{L}$



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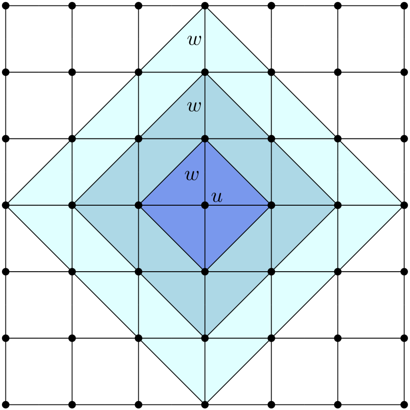

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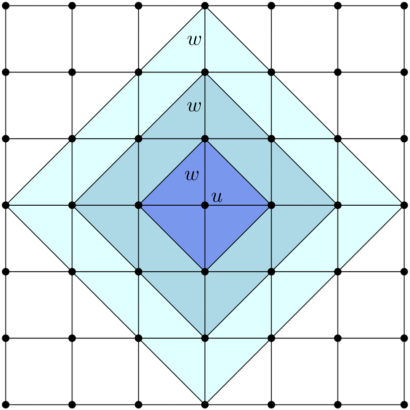

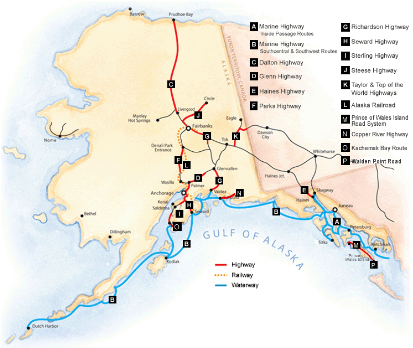

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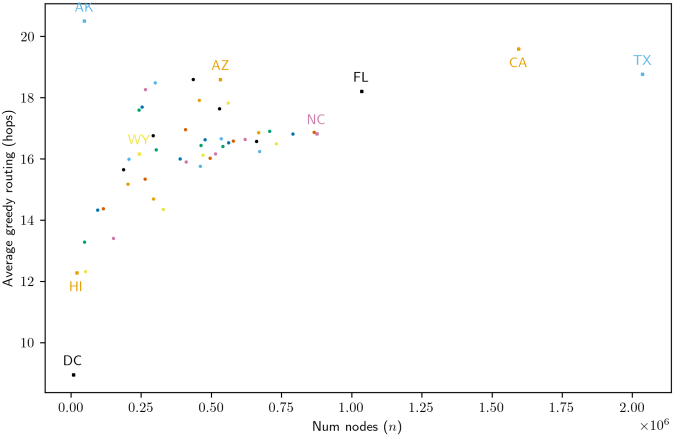

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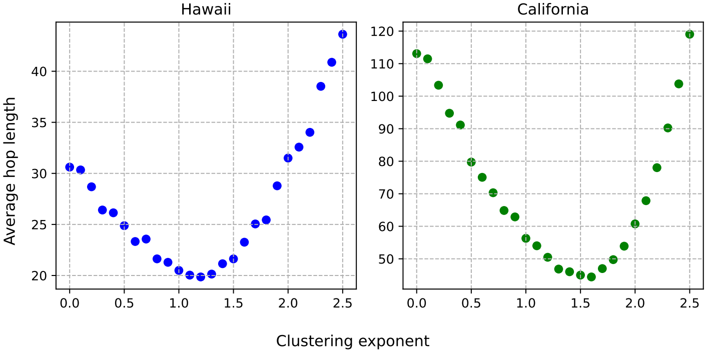

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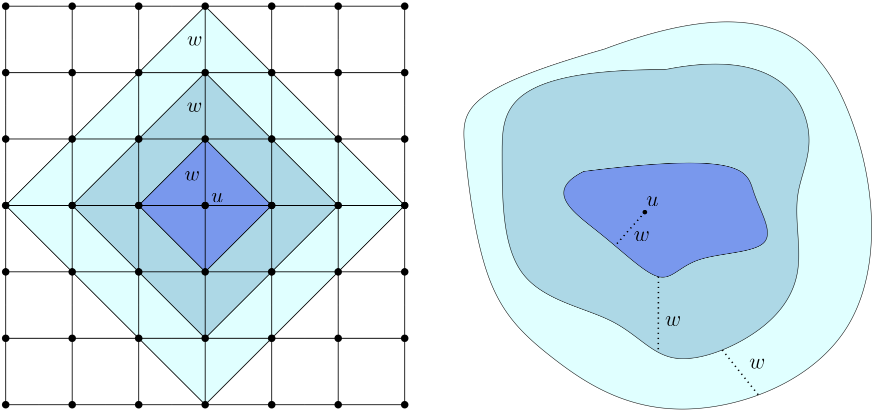
Graph family $\mathcal{F}$ has *fixed-growth* (FG) dimensionality α if $|\mathcal{B}_\ell(u)| = \Theta(\ell^\alpha)$.



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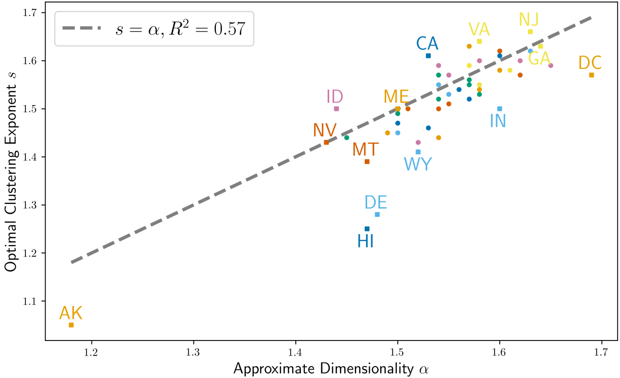

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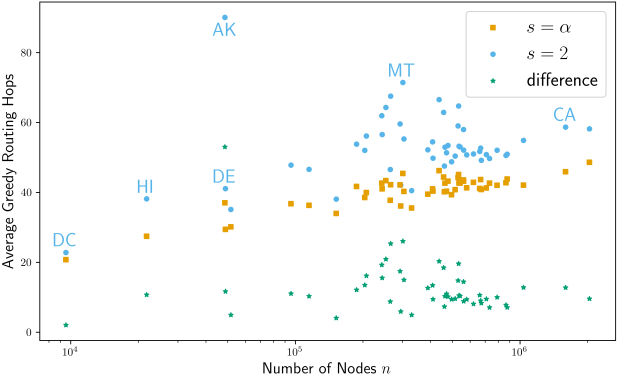
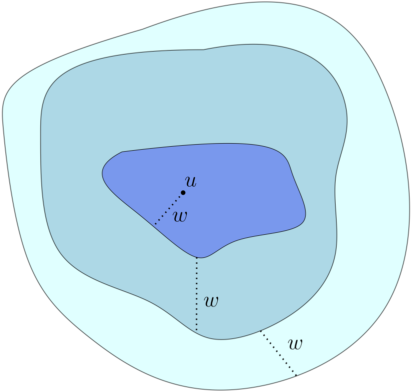
Graph family $\mathcal{F}$ has *fixed-growth* (FG) dimensionality α if $|\mathcal{B}_\ell(u)| = \Theta(\ell^\alpha)$.

- Constants in $\Theta(\ell^\alpha)$ can be extreme – infinite graphs?
- $s = 2$... only for infinite graphs?
- Experimentally 'estimate' α for finite graphs... – better routing!



FG Nested 'Lattice' Construction?

- Balls w/ radius ℓ



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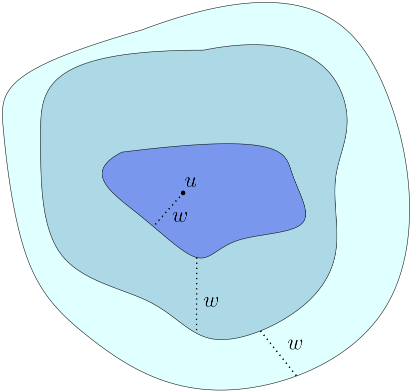

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- Balls w/ radius ℓ
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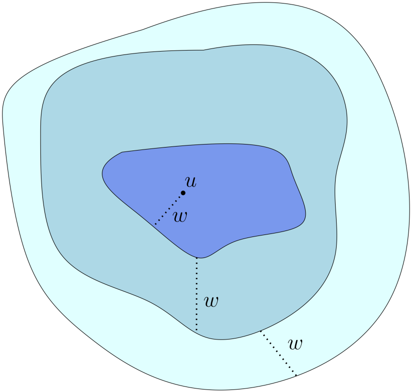
- Expected $\Theta(\ell^\alpha / k)$ highway



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- Balls w/ radius ℓ
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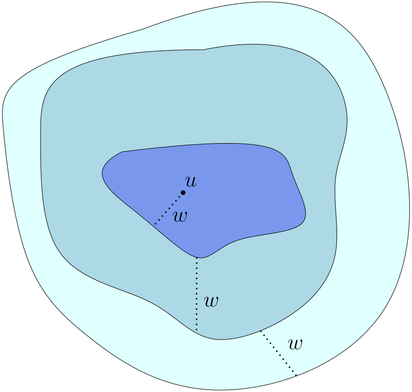
- Chernoff bounds!

Radius ℓ	# Highway
$\Omega(\sqrt[\alpha]{k \log n})$	$\Theta(\ell^\alpha/k)$
$\Theta(\sqrt[\alpha]{k \log\log n})$	$\Theta(\log\log n)$
$\Theta(\sqrt[\alpha]{k})$	$\Theta(1)$

- Not all w.h.p.



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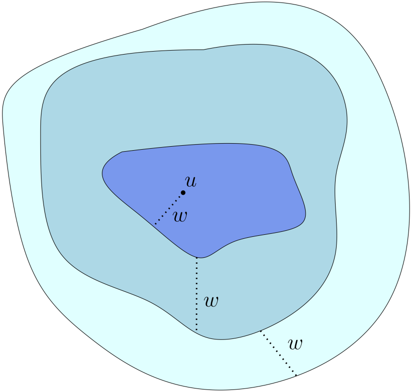

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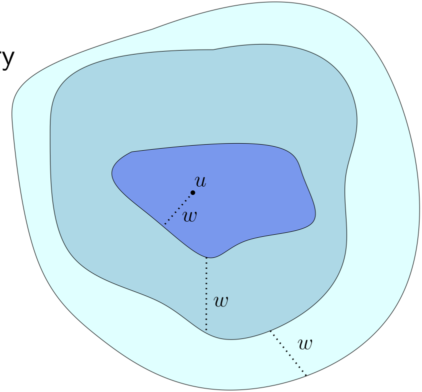

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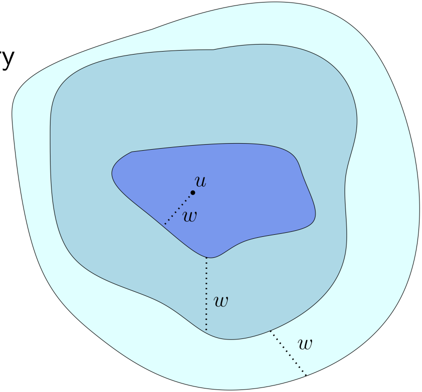
- Shells $\mathcal{S}_b^{(w)}$
- $w = \Theta(\sqrt[\alpha]{k \log n})$



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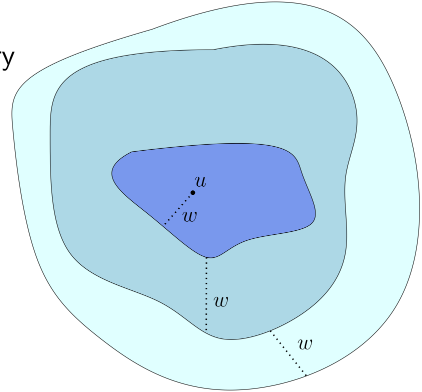
- Shells $\mathcal{S}_b^{(w)}$
- $w = \Theta(\sqrt[\alpha]{k \log n})$
- $\Theta(b^{\alpha-1} \log n)$ highway nodes



Distance to the Highway

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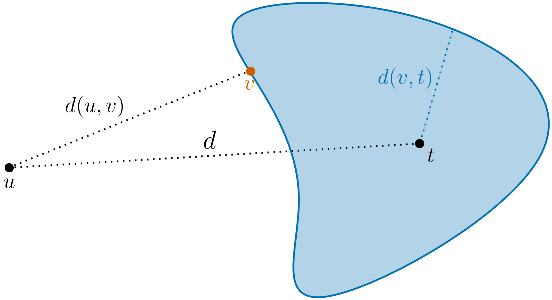
- $\exists u \ \ d(u, \mathcal{H}) = \Omega(\sqrt[\alpha]{k \log n})$ w.h.p.
- Limits greedy routing and diameter w.h.p.

Distance Halving

- Idea: Improve distance by constant factor



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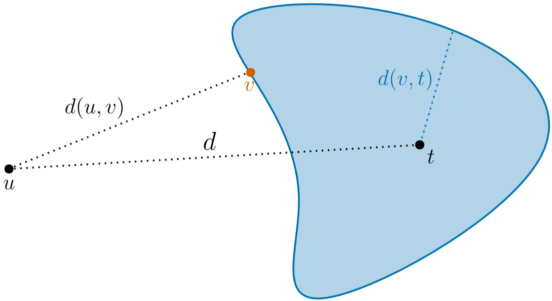

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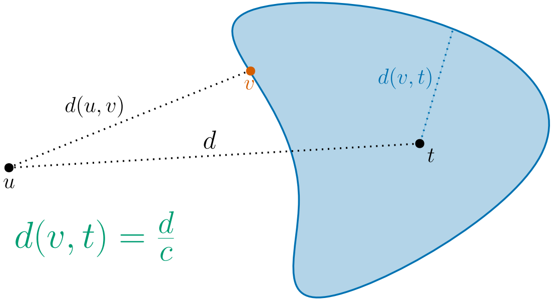

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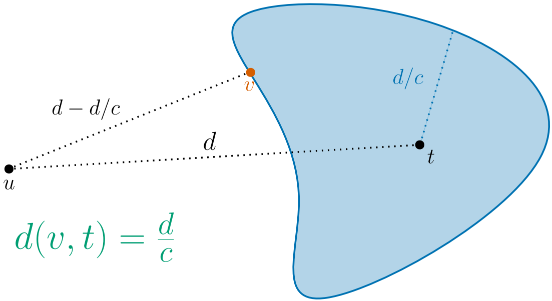

$$d(v,t) = \frac{d}{c}$$

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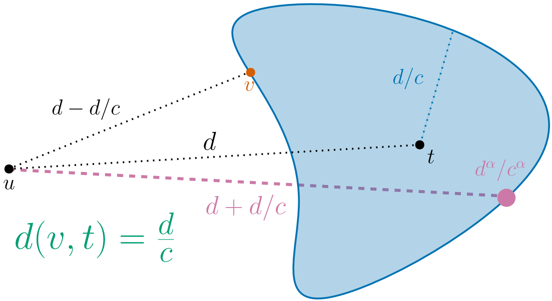
- Improvement by factor c
- Worst case: all as far away $(d + d/c)$



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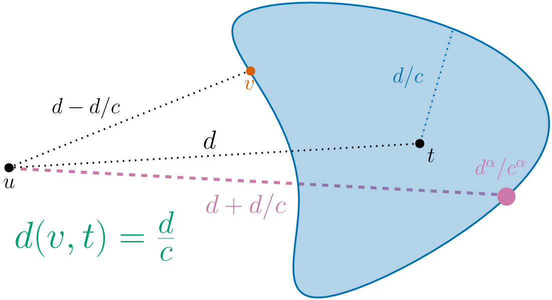
Lemma

$\frac{1}{(c+1)^\alpha} <$ *probability of factor c improvement* $< \frac{1}{(c-1)^\alpha}$



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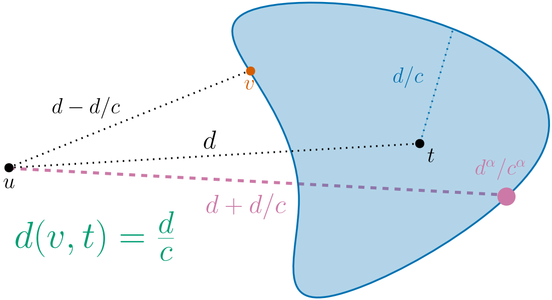

Staying on the Highway – Lattice

- Traversing Highway:
  - If can halve the distance, take it

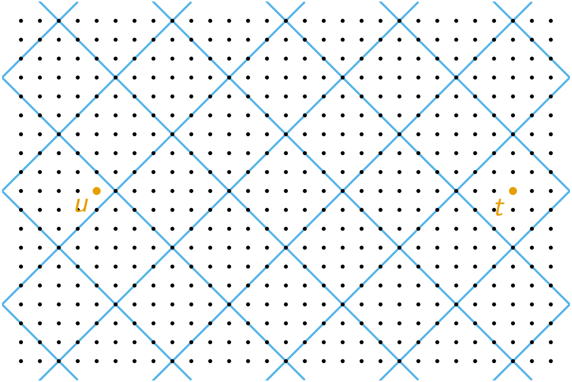

Figure 10: A highway node u routing to the destination t

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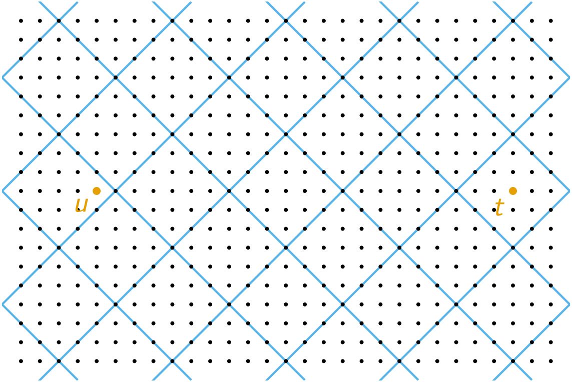

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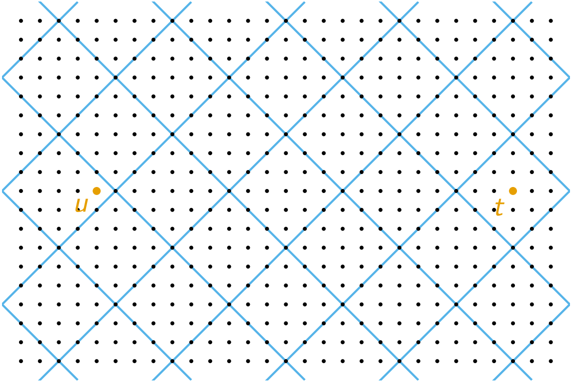

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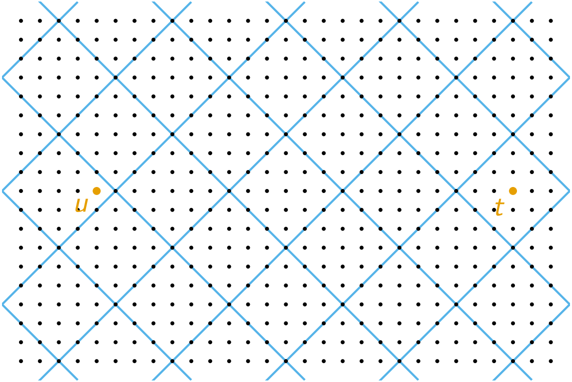

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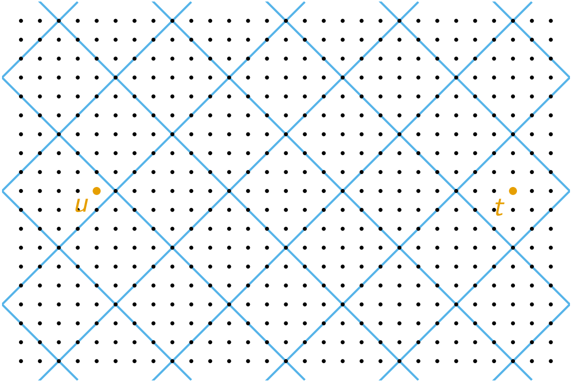

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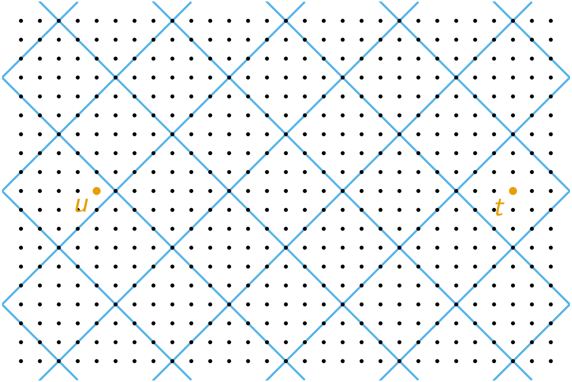

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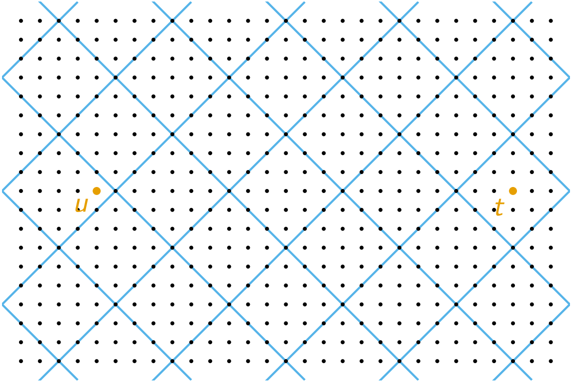

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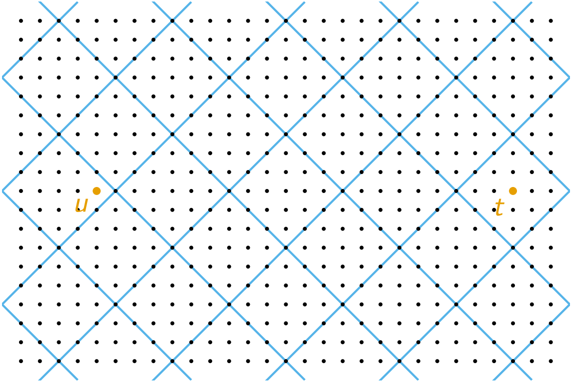

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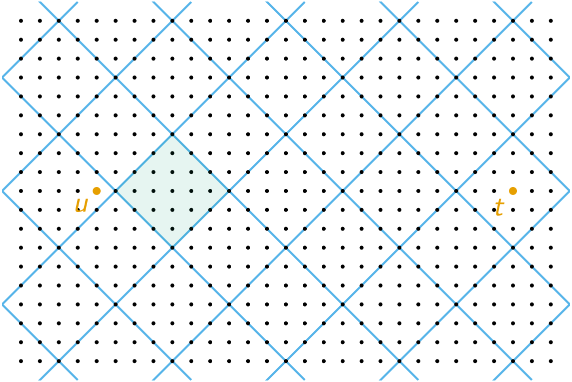

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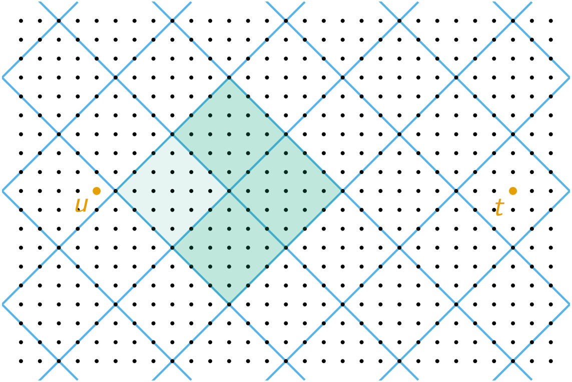

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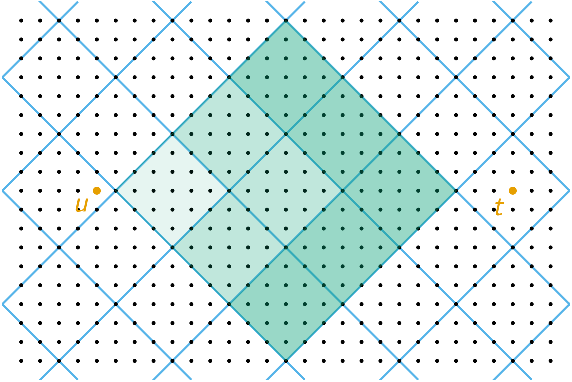

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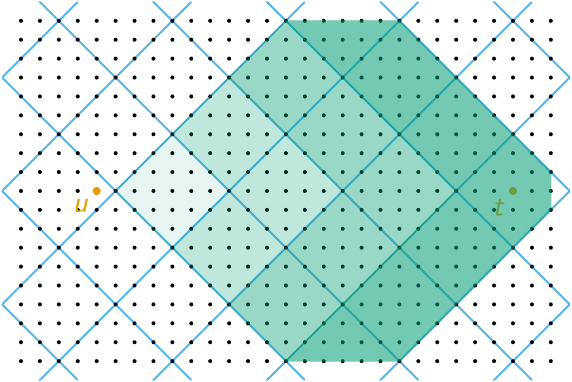

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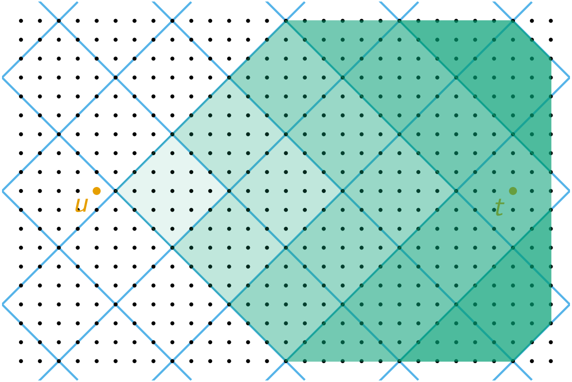

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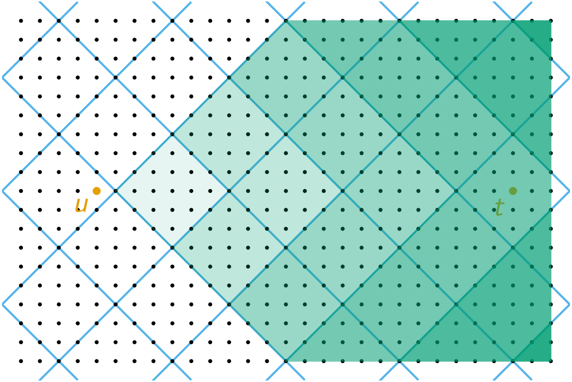

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Staying on the Highway – Fixed Growth

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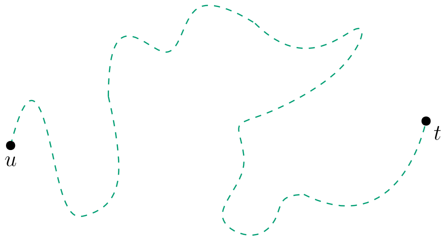

Staying on the Highway – Fixed Growth

- Claim: Closer long-range contacts exist w.h.p. – no lattice structure!?
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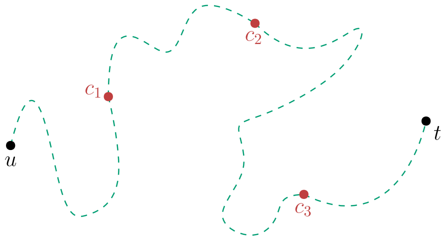

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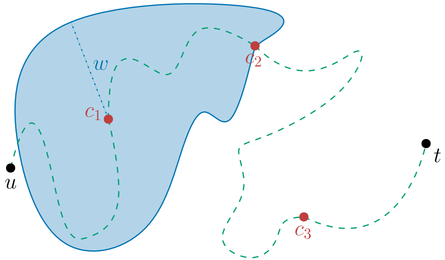

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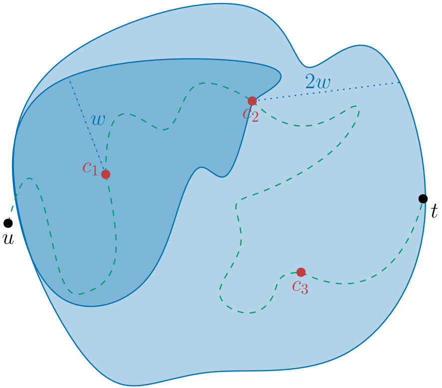

Staying on the Highway – Fixed Growth

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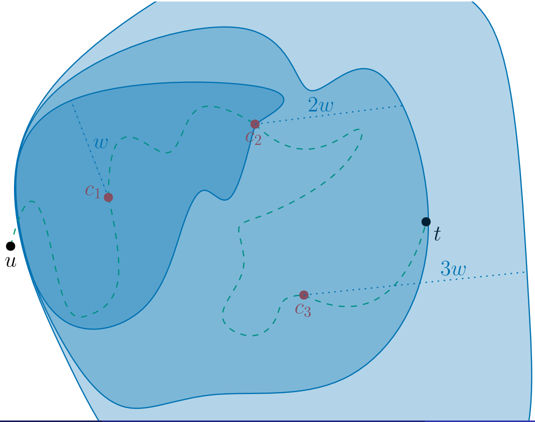
- How many nodes at which distance?
- Consider a shortest path from $u \rightarrow t$
- Pick points along path with distance w between them
- Consider balls of radius $b \times w$ around point c_b – just like shells



Greedy Routing – Lower Bounds

☑ Constant distance-halving probability

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- ☑ $\mathcal{O}(\log n)$ upper bound

Greedy Routing – Lower Bounds

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- ? Can we do better?

Greedy Routing – Lower Bounds

- ☑ Constant distance-halving probability
- ☑ Stay on highway if not w.h.p.
- ☑ $\mathcal{O}(\log n)$ upper bound
- ? Can we do better? – no

- Let x_i be factor improvement of i-th hop
- $d_f \times x_f \times x_{f-1} \times \cdots \times x_0 = d_0$
- $\prod_i x_i = d_0 / d_f$
- $\sum_i \log x_i = \log d_0 - \log d_f \approx \log n$
- $\mathbb{E}[\sum_i \log x_i] = f\mathbb{E}[\log x]$
- $\mathbb{E}_{x>c_0}[\log x] \leq \int_{c_0}^{\infty} \log c \cdot \Pr(x = c)\ dc \leq \int_{c_0}^{\infty} \log c \cdot \frac{1}{(c-1)^{\alpha}} dc = \Theta(1)$
- $\mathbb{E}[\sum_i \log x_i] = \Theta(f) \implies f = \Theta(\log n)$

Greedy Routing – Results

- In expectation:

Theorem

In any fixed-growth graph $\mathcal{G}$ with FG dimensionality α and highway constant $k \in \Theta(\log n)$, greedy routing between two arbitrary nodes s and t can expect to take $\Theta(\log n)$ hops, if $d(s,t) = \Theta(\sqrt[\alpha]{n})$.

- With high probability:

Theorem

Let $\mathcal{G}$ be a randomized highway graph with FG dimensionality α and highway constant $k \in \Theta(\log n)$. If $d(s,t) = \Theta(\sqrt[\alpha]{n})$, then greedy routing between any two nodes s and t succeeds with high probability in $\Theta(\log n)$ hops if $\alpha \geq 2$, and in $\Theta(\sqrt[\alpha]{\log^2 n})$ hops if $\alpha \leq 2$.

Diameter – Results

Theorem

Let $\mathcal{G}$ be a randomized highway graph with FG dimensionality α and highway constant $k \in \Theta(\log n)$. The diameter of $\mathcal{G}$ is $\Theta(\frac{\log n}{\log \log n})$ if $\alpha > 2$, and $\Theta(\sqrt[\alpha]{\log^2 n})$ if $\alpha \leq 2$.

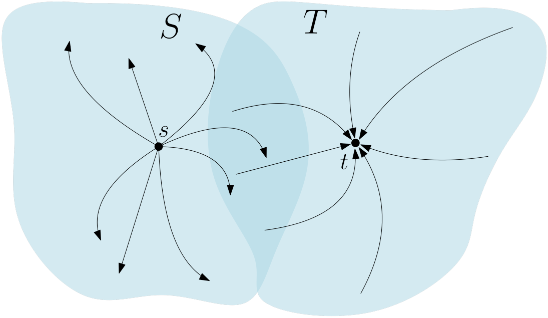
- Gap for $\alpha > 2$

Diameter – Results

Theorem

Let $\mathcal{G}$ be a randomized highway graph with FG dimensionality α and highway constant $k \in \Theta(\log n)$. The diameter of $\mathcal{G}$ is $\Theta(\frac{\log n}{\log \log n})$ if $\alpha > 2$, and $\Theta(\sqrt[\alpha]{\log^2 n})$ if $\alpha \leq 2$.

- Gap for $\alpha > 2$



Open Problems

- Is $k = \Theta(\log n)$ always optimal?
- Non-uniform α ($\alpha(\ell)$? $\alpha(n)$?)
- Other greedy routing algorithms?
- Imprecise distances?

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