# Unit - 4

## Curve Fitting ⇒

Usually a mathematical equation is fitted to experimental data by plotting the data on the graph paper and then passing a straight line through the data points. This method has the obvious drawback in the straight line drawn may not be unique. The method of least-squares regression is probably the most systematic procedure to fit a unique curve through given data points and is widely used in practical computations. It can also be easily implemented on a digital computer.

## Linear Regression ⇒

The simplest example of a least-squares approximation is fitting a straight line to set of paired observations $(x_1, y_1)$, $(x_2, y_2)$, ----, $(x_n, y_n)$. The mathematical expression for the straight line is.

$$y = a + bx + e \quad \text{—— (1)}$$

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Where a and b are coefficients representing the intercept and the slope, respectively and e is the error or residual between the model and the observations, which can be represented by rearranging equation 1 as,

$$e = y - a - bx$$

Thus, the error or residual, is the difference between the true value of y and the approximate value a+bx, predicate by linear equation.

The strategy for best fit is to minimize the sum of squares of the residuals between the measured y and the y calculated with the linear model:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2 \quad (2)$$

To determine the value of a and b equation 2 is differentiated with respect to each coefficient a and b,

$$\frac{\partial S_r}{\partial a} = -2\sum(y_i - a - bx_i) \quad \text{--- (3)}$$

$$\frac{\partial S_r}{\partial b} = -2\sum[(y_i - a - bx_i)x_i] \quad \text{--- (4)}$$

To minimize the error we have,

$$\frac{\partial S_r}{\partial a} = 0 \text{ and } \frac{\partial S_r}{\partial b} = 0,$$

Hence we get

using (3)

$$-2\sum(y_i - a - bx_i) = 0$$

$$\Rightarrow \sum(y_i - a - bx_i) = 0$$

$$\sum y_i = \sum a + \sum bx_i$$

$$\Rightarrow \sum y_i = na + b\sum x_i \quad \text{--- (5)}$$

using (4)

$$-2\sum[(y_i - a - bx_i)x_i] = 0$$

$$\Rightarrow \sum[(y_i - a - bx_i)x_i] = 0$$

$$\Rightarrow \quad \Sigma x_i y_i = a \Sigma x_i + b \Sigma x_i^2 \quad \text{—— (6)}$$

By Solving equation (5) and (6)

Simultaneously we will get the value of a and b, Hence we will get least square straight line, by using equation (1) when $e = 0$ we have,

$$\boxed{y = a + bx}$$

* Rules for Solving linear Regression Problems

Step 1 ⇒ Take the equation of straight line i.e. $y = a + bx$

Step 2 → By the given data find the values of $\Sigma x$, $\Sigma y$, and $\Sigma xy$ and $\Sigma x^2$

Step 3 → After, Calculating, Substitute the above value in the least square regression equation for straight line i.e,

$$\Sigma y_i = na + b\Sigma x_i$$
$$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2$$

Step 4 ⇒ Then find the Values of a & b.

Step 5 ⇒ now Substitude the Value of a and b in the equation of straight line $y = a + bx$

Prob. 1 ⇒ Fit a straight line to a given data below using linear Regression Method

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | 2 | 3 | 4 | 7 | 8 | 9 | 5 | 5 |
| y | 9 | 6 | 5 | 10 | 9 | 11 | 2 | 3 |

Sol$^n$ ⇒ let $y = a + bx$ is required straight line,

we have the least square straight line equations as,

$$\Sigma y_i = na + b\Sigma x_i \quad -① $$
and $$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad -②$$

| | $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|---|
| | 2 | 9 | 18 | 4 |
| | 3 | 6 | 18 | 9 |
| | 4 | 5 | 20 | 16 |
| | 7 | 10 | 70 | 49 |
| | 8 | 9 | 72 | 64 |
| | 9 | 11 | 99 | 81 |
| | 5 | 2 | 10 | 25 |
| | 5 | 3 | 15 | 25 |
| | $\Sigma x =$ | $\Sigma y =$ | $\Sigma xy =$ | $\Sigma x^2 =$ |
| Total | 43 | 55 | 322 | 273 |

By Substituting these values in the above two equations, (2) We get

$$8a + 43b = 55$$
$$43a + 273b = 322$$

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By Cramer's Rule,

let

$$D = \begin{vmatrix} 8 & 43 \\ 43 & 273 \end{vmatrix}$$

$$D = (8 \times 273) - (43 \times 43)$$

$$D = 335$$

Now where,

$$a = \frac{\begin{vmatrix} 55 & 43 \\ 322 & 273 \end{vmatrix}}{D}$$

$$a = \frac{(55 \times 273) - (43 \times 322)}{335}$$

$$\therefore a = \frac{1169}{335} = 3.489$$

and

$$b = \frac{\begin{vmatrix} 8 & 55 \\ 43 & 322 \end{vmatrix}}{D}$$

$$b = (8 \times 322) - (55 \times 43) / 335$$

$$\therefore b = \frac{211}{335} = 0.6298$$

By solving these equation we get,

$a = 3.489$, $b = 0.6298$

Hence required straight line is,

$$\boxed{y = 3.489 + 0.6298x}$$

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Prob 2 ⇒ Fit a straight line to the data given below

| $x$ | 1 | 3 | 4 | 6 | 8 | 9 | 11 |
|---|---|---|---|---|---|---|---|
| $y$ | 1 | 2 | 4 | 4 | 5 | 7 | 8 |

Sol$^n$ ⇒ let $y = a + bx$ is the required straight line.

we have the least square straight line equations as,

$$\Sigma y_i = na + b\Sigma x_i \quad — (1)$$

and $$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 \quad — (2)$$

| | $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|---|
| | 1 | 1 | 1 | 1 |
| | 3 | 2 | 6 | 9 |
| | 4 | 4 | 16 | 16 |
| | 6 | 4 | 24 | 36 |
| | 8 | 5 | 40 | 64 |
| | 9 | 7 | 63 | 81 |
| | 11 | 8 | 88 | 121 |
| | $\Sigma x =$ | $\Sigma y =$ | $\Sigma xy =$ | $\Sigma x^2 =$ |
| Total | 42 | 31 | 238 | 328 |

By substituting these values in equation (1) and (2) we get,

$$7a + 42b = 31$$
$$42a + 328b = 238$$

By Cramer's Rule,

$$D = \begin{vmatrix} 7 & 42 \\ 42 & 328 \end{vmatrix} = (7 \times 328) - (42 \times 42)$$

$$= 532$$

where, $a = \dfrac{\begin{vmatrix} 31 & 42 \\ 238 & 328 \end{vmatrix}}{D} = \dfrac{172}{532}$

$\Rightarrow a = 0.3233$

and $b = \begin{vmatrix} 7 & 31 \\ 42 & 238 \end{vmatrix} = \dfrac{364}{532}$

$\Rightarrow b = 0.6842$

Solving these equation we get,

$a = 0.3233$ and $b = 0.6842$

Hence Required Straight line is,

$$\boxed{y = 0.3233 + 0.6842x}$$

# Solution of ODE



## Ordinary Differential Equation ⇒

When a differential equation contains one independent variables and one or more dependent variables and the derivatives of that single independant variable equation are called ODE (ordinary Differential Equation)

(1) $\frac{dy}{dx} + x^2 y = \sin x$

(2) $\frac{dy}{dt} + \frac{dx}{dt} = \sin t$

## Order and Degree of Diffe eqⁿ ⇒

(1) **Order**: order of DE is the order of highest order derivative involved in equation.

(2) **Degree**:- The degree of DE ~~of~~ which ~~the~~ differential Coefficient ~~are~~ is the power of its highest Derivaties.

Ex (1) $\frac{d^2x}{dy^2} - b^2 x = 0$ order $= 2$

degree $= 1$

$$\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$$ order 2, degree 1

$$\sqrt{\left(\frac{dy}{dx}\right)^2 + 1} = \frac{d^2y}{dx^2}$$ order = 2, degree = 2

# Various Method to Solve ODE ⇒

## ① Euler's Method

Consider the $I^{st}$ order ODE,

$$\frac{dy}{dx} = f(x, y) \quad \text{--- (1)}$$

with initial condition,

$$y(x_0) = y_0 \quad \text{--- (2)}$$

Rewrite eqⁿ (1)

$$dy = f(x, y)\, dx$$

Integrating the above equation in the interval $(x_0, x_1)$ and $(y_0, y_1)$

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx \quad \text{--- (3)}$$

we have to solve eqⁿ (1) as,

$x_n = x_0 + nh$ (where $n = 1, 2, 3, 4 \ldots$)

$$\therefore y_1 - y_0 = f(x, y)(x_1 - x_0) \quad \text{--- (4)}$$

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In the interval $(x_0, x_1)$ we assume that

$$f(x, y) \simeq f(x_0, y_0)$$

from eq$^n$ (4)

$$y_1 - y_0 = f(x_0, y_0)(x_1 - x_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$\therefore \quad y_1 = y_0 + f(x_0, y_0) h$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$\left(\text{where } h = (x - x_0)\right)$$

Similarly in the interval $(x_1, x_2)$ we have

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_3 = y_2 + h f(x_2, y_2)$$

Continuing in this way we obtain formula,

$$\Rightarrow \quad y(x_n) = y_{n+1} = y_n + h f(x_n, y_n)$$

This formula is Euler's formula used to solve ODE

Prob $\Rightarrow$ Solve $\frac{dy}{dx} = 1 + y^2$ with $y(0) = 0$ by Euler's Method also find, $y(0.1)$, $y(0.2)$ and $y(0.3)$

Sol$^n$ $\Rightarrow$

Given that $y(0) = 0$, $y(x_0) = y_0$

$$\frac{dy}{dx} = 1 + y^2 \doteq f(x, y)$$

Also we have;

$$y(x_0) = y_0 \rightarrow \text{(Initial Condition of eq}^n\text{)}$$

$$y(0) = 0 \Rightarrow \text{(given)}$$

Initial Value of

$$x = x_0 = 0$$

$$y = y_0 = 0$$

Euler's formula

$$y(x_n) = y_{n+1} = y_n + h f(x_n, y_n)$$

Iteration 1

$x_0 = 0, \quad y_0 = 0$

We know that

$$x_1 = x_0 + h$$

for $y(0.1)$,

$$x_1 = 0.1, \quad x_0 = 0$$

$$\Rightarrow h = x_1 - x_0 = 0.1 - 0$$

$$\boxed{h = 0.1}$$

Also $f(x_0, y_0) = 1 + y_0^2$
$= 1 + 0$

$$\Rightarrow \boxed{f(x_0, y_0) = 1}$$

The approximate value of $y$ at $x_1 = 0.1$ is given by

$$\cancel{y_0 = y_0 +}$$

$$y_{n+1} = y_0 + h f(x_0, y_0)$$

$$y_1 = y_0 + (0.1)(1)$$

$$y_1 = 0 + 0.1$$

$y_1 = 0.1$

$\therefore$ At $x_1 = 0.1$ we get
$y_1 = 0.1$ — (1)

Iteration (2)

$x_0$ is replace by $x_1$ and

$y_0$ is replace by $y_1$.

$x_1 = 0.1$ and $y_1 = 0.1$

$x_2 = x_1 + h$

for $y(0.2)$

$x_2 = 0.2,\ x_1 = 0.1$

$\Rightarrow h = 0.2 - 0.1$

$h = 0.1$

$f(x_1, y_1) = 1 + y_1^2 = 1 + 0.1^2$

$= 1 + 0.01$

$= 1.01$

$\therefore$ The approximate value of $y$ at $x = 0.2$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_2 = 0.1 + 0.1 (1.01)$$

$$= 0.1 + 0.101$$

$$y_2 = 0.201 \quad \text{—— } 2$$

$\therefore$ At $x_2 = 0.2$, we get $y_2 = 0.201$

Iteration 3 $\Rightarrow$ $x_1$ is replace by $x_2$ and $y_1$ is replace by $y_2$

$x_2 = 0.2$, $y_2 = 0.201$

$\therefore$ for $y(0.3)$

$x_3 = 0.3$, $h = 0.1$

$$f(x_2, y_2) = 1 + y_2^2 = 1 + (0.201)^2$$

$$\Rightarrow f(x_2, y_2) = 1.040401$$

$\therefore$ Approximate Value of $y$ at $x = 0.3$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 0.201 + 0.1 (1.040401)$$

$$y_3 = 0.3050401 \quad \text{—— } 3$$

$\therefore$ from 1, 2 & 3

$y(0.1) = 0.1$, $y(0.2) = 0.201$, $y(0.3) = 0.305$

# (II) Runge-Kutta Method ⇒

Euler's method is less efficient to solve the problems, it requires $h$ to be small for obtaining reasonable accuracy. The Runge-Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points on the subinterval.

If we substitute,

$$y_1 = y_0 + h f(x_0, y_0)$$

in the trapezoidal rule

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + hf_0)\right]$$

where $f_0 = f(x_0, y_0)$; if we now set,

$$K_1 = hf_0, \quad K_2 = hf(x_0 + h, y_0 + K_1)$$

Then the above equation becomes,

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

Which is the second order Runge Kutta formula, The error in this formula Can be shown to be of order $h^3$. after solving it,

Hence we get the fourth - order Runge Kutta formula ~~a~~ is,

Higher-order Runge-Kutta formula :-

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Where

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1)$$

$$K_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2)$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

in which the error is of order $h^5$.

RangeKutta Methods are advantag in accuracy and Stability perspective, but the Computation cost is more so it is very expensive Method. compare to other Method.

Prob ⇒ Solve $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ correct to four decimal places by Runge-Kutta Second order Method.

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Sol$^n$ ⇒ 1) for $y(0.1)$

With $h = 0.1$, $f_0 = y_0 - x_0$; (Where $y_0 = 2$, $x_0 = 0$)

$$f_0 = 2 - 0 = 2$$

$$\Rightarrow \quad K_1 = hf_0 = 0.1(2) = 0.2$$

$$K_2 = hf(x_0 + h, y_0 + K_1)$$

$$f(x_0+h, y_0+K_1) = f(0+0.1, 2+0.2)$$
$$= f(0.1, 2.2)$$

$$\Rightarrow f(0.1, 2.2) = y - x$$
$$= 2.2 - 0.1$$
$$= 2.1$$

$$\therefore K_2 = h(2.1) = 0.1(2.1) = 0.21$$

By Runge-Kutta 2$^{nd}$ order formula

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$= 2 + \frac{1}{2}(0.2 + 0.21)$$

$$\boxed{y(0.1) = 2.2050}$$

(2) for $y(0.2)$ with $x_0 = 0.1$,
$y_0 = 2.2050$

$\Rightarrow f_0 = (y_0 - x_0) = 2.105$

Hence $k_1 = h f_0 = (0.1)(2.105)$
$= 0.2105$

$\Rightarrow K_2 = h f(x_0 + h, y_0 + K_1)$

Hence
$f(x_0 + h, y_0 + k_1) = f(0.1 + 0.1, 2.2050 + 0.2105)$

$= f(0.2, 2.4155)$

$\Rightarrow f(0.2, 2.4155) = y_0 - x_0$

$= (2.4155 - 0.2)$
$= 2.2155$

$\Rightarrow K_2 = h(2.2155)$
$= 0.1(2.2155)$
$= 0.22155$

$\Rightarrow y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$

$= 2.2050 + \frac{1}{2}(0.2105 + 0.22155)$

$\boxed{y(0.2) = 2.4210}$

**(2) Crammer's Rule:**

Crammer's rule is another solution technique that is best suited to small number of equations. This rule says that, each unknown in a system of a linear algebraic may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of coefficient of the unknown in equation by the constants $b_1$, $b_2$, ---------, $b_n$.

For example, if we have three equations.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

then,

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D}, \text{ and } x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

example

$$3x_1 + 2x_2 = 18$$
$$-x_1 + 2x_2 = 2$$

$$D = \begin{vmatrix} 3 & 2 \\ -1 & 2 \end{vmatrix} = 6+2=8$$

$$x_1 = \frac{\begin{vmatrix} 18 & 2 \\ 2 & 2 \end{vmatrix}}{D} = \frac{36-4}{8} = 4$$

$$x_2 = \frac{\begin{vmatrix} 3 & 18 \\ -1 & 2 \end{vmatrix}}{D} \cdot \frac{6+18}{8} = 3$$

For more than three equations, crammer's rule becomes impractical because, as the number of equations increases, the determinants are time consuming to evaluate by hand or by computer.