Unit-3(Interpolation)

Polynomial forms:- The polynomial equation has in the following form:

$$a_n x^n + a_{n-1} x^{n-1} + \text{----------------} + a_0 = 0$$

where a_n=0, this equation has n roots may be real, equal or imaginary. An equation of form f(x)=0, where f(x) is the polynomial in x is called algebraic equation, if the degree of the polynomial is more than 1.

The most common form of an n^{th} order polynomial is,

$$f(x)= a_0+a_1x +a_2x^2+\text{----------------------}+a_nx^n$$

this form as known as power form is very convenient for differentiating and integrating the polynomial function.

Interpolation=> Interpolation is the technique of making an appropriate estimate of an unknown value from certain other known values under certain assumption.

Interpolation refers to the estimation of a figure or value within given limits of the available data.

Extrapolation=> Extrapolation is the technique of estimating a probable figure for the future or past under certain assumptions.

Extrapolation refers to the estimation of a figure outside the given limits of data.

Interpolation and Extrapolation can be distinguished with following examples:

Year	1951	1961	1971	1981	1991
Population (in crore)	36.11	43.92	54.82	68.3	84.6

In above example, various year 1951 to 1991 the population of the India is given but population figure of 1955 and 1996 are not available.

If we estimate population of 1955 it is termed as interpolation.

On the other hand if we make future estimate of population for the year 1996 this will be termed as extrapolation.

Linear Interpolation :- The simplest form of interpolation is to approximate two data points by a straight line. Suppose we are given two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

These two points can be constructed linearly as shown in fig(3)

Using the concept of line joining the end points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, we can write the equation of straight line as,

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

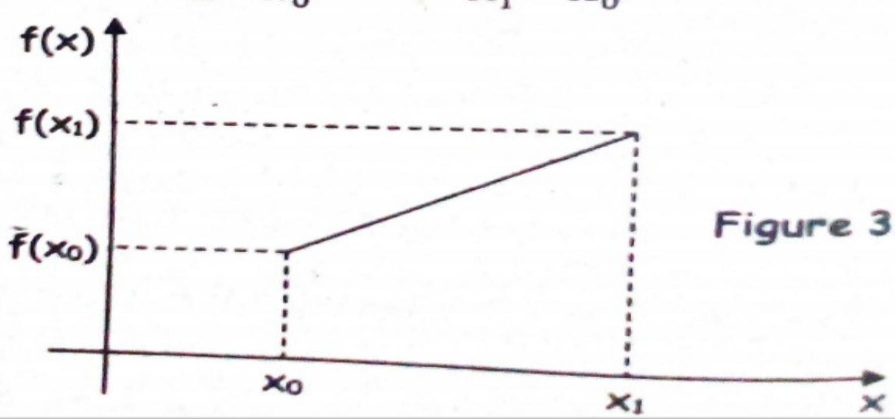

Figure 3.

For solving for f(x) we get

$$f(x) = f(x_0) + (x - x_0)\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

This formula is known as linear interpolation formula. Here the term

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

Represents the slope of the line.

Problems of Linear Interpolation

Problems of Linear Interpolation

(1) Prob ⇒ Table given below gives Square root for integer using linear interpolation formula, estimate the Square root of 2.5

x	1	2	3	4	5
$f(x)=\sqrt{x}$	1	1.4142	1.7321	2	2.2361

Sol^n ⇒ The required Value lies in between 2 and 3

$\therefore \quad x_0 = 2 \qquad f(x_0) = 1.4142$

$\quad\; x_1 = 3 \qquad f(x_1) = 1.7321$

$\Rightarrow x = 2.5, \quad f(x) = ?$

Now, By formula for linear Interpolation,

$$f(x) = f(x_0) + (x - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x) = 1.4142 + (2.5 - 2)\left(\frac{1.7321 - 1.4142}{3 - 2}\right)$$

$$f(x) = 1.4142 + (0.5)(0.3179)$$

$$f(x) = 1.57315$$

Hence the square root of 2.5 is 1.57315 by using linear interpolation formula.

(3) find the Value of $x = 9.4$ by using linear interpolation formula for the following table

x	9	10
$f(x)$	3	3.1623

Sol^n ⇒ The Required Value lies in between 9 and 10.

∴ $x_0 = 9$ $\quad f(x_0) = 3$

$x_1 = 10$ $\quad f(x_1) = 3.1623$

$x = 9.4$ $\quad f(x) = ?$

⇒ Now by formula for linear Interpolat^n

$$f(x) = f(x_0) + (x - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= 3 + (9.4 - 9)\left[\frac{3.1623 - 3}{10 - 9}\right]$$

$$= 3 + (0.4)\left[0.1623\right]$$

$$= 3 + 0.06492$$

$$f(x) = 3.06492$$

Hence the Square root of 9.4 is, 3.06492 by using linear interpolation formula.

<u>Lagrange's Interpolation Formula:-</u> If $x_0, x_1, x_2,$ ---------x_n are given set of observation, which are need not be equally spaced and let y_0, y_1, y_2--------- y_n are their corresponding value where y=f(x) be the given function then, the Lagrange Interpolation formula is given by,

<u>Proof:-</u> Let y=f(x) be a polynomial in x which takes the values y_0, y_1------, y_n corresponding to x_0, x_1,------------ x_n.

Since there are (n+1) values of f(x) so (n+1)th difference is zero, thus f(x) is a polynomial in x of degree n. let this polynomial is given as,

$Y= f(x)=a_0(x-x_1)(x-x_2)$--------------$(x-x_n)+a_1(x-x_0)(x-x_2)$--------------$(x-x_n)+a_2(x-x_0)(x-x_1)$--------------$(x-x_n)$+-------+$a_n(x-x_0)(x-x_1)$--------------$(x-x_{n-1})$ __________(1)

Where a_1, a_2,-------- a_n are constants

Substitute $x=x_0$ we get ,

$F(x_0)= a_0(x_0-x_1)(x_0-x_2)$--------------$(x_0-x_n)$

$Y_0= a_0(x_0-x_1)(x_0-x_2)$--------------$(x_0-x_n)$

$a_0 = Y_0/(x_0-x_1)(x_0-x_2)$--------------$(x_0-x_n)$

again if $x=x_1$ we get,

$a_1 = Y_1/(x_1-x_0)(x_1-x_2)$--------------$(x_1-x_n)$

hence finaly we get,

$a_n = Y_n/(x_n-x_0)(x_n-x_1)$--------------$(x_n-x_{n-1})$

put the values in eq(1)

$$f(x)= \frac{(X-X1)(X-X2)---------------(X-Xn)}{(X0-X1)(X0-X2)---------------(X0-Xn)}*y_0 +$$

$$\frac{(X-X0)(X-X2)---------------(X-Xn)}{(X1-X0)(X1-X2)---------------(X1-Xn)} * y_1+---------------------$$

$$+ \frac{(X-X0)(X-X1)---------------(X-Xn-1)}{(Xn-X0)(Xn-X1)---------------(Xn-Xn-1)} *y_n$$

Problems of Lagrange's Interpolation

Lagrange's Interpolation Examples

Problem 1

Use lagrange's interpolation formula to find the value of y when $x = 2$.

x	0	1	3	4
y	-12	0	6	12

Sol^n: here,

$x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$

$y_0 = -12$, $y_1 = 0$, $y_2 = 6$, $y_3 = 12$.

By using lagrange's formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$y = \frac{(2-1)(2-3)(2-4)}{(0-1)(0-3)(0-4)} \times -12 + \frac{(2-0)(2-3)(2-4)}{(1-0)(1-3)(1-4)} \times 0$$

$$+ \frac{(2-0)(2-1)(2-4)}{(3-0)(3-1)(3-4)} \times 6 + \frac{(2-0)(2-1)(2-3)}{(4-0)(4-1)(4-3)} \times 12$$

$$y = \frac{(1)(-1)(-2)}{(-1)(-3)(-4)} \times -12 + 0 + \left(\frac{2 \times 1 \times -2}{3 \times 2 \times (-1)}\right) \times 6$$

$$+ \frac{(2 \times 1 \times -1)}{(6 \times 3 \times 2)} \times 12$$

$$y = \frac{\cancel{24}^{2}}{\cancel{12}_{1}} + 0 + 4 + (-2)$$

$$= 2 + 4 - 2$$

$$\underline{y = 4}$$

Problem ②

Use lagrangens interpolation formula to find the value of y, where $x = 4$

x	0	1	2	5
y	2	5	7	8

Solⁿ: here

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$

$y_0 = 2$, $y_1 = 5$, $y_2 = 7$, $y_3 = 8$

By using lagrangens formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$\times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\times y_3$$

$$= \frac{(4-1)(4-2)(4-5)}{(-1)(-2)(-5)} \times 2 + \frac{(4-0)(4-2)(4-5)}{(1-0)(1-2)(1-5)} \times 5$$

$$+ \frac{(4-0)(4-1)(4-5)}{(2-0)(2-1)(2-5)} \times 7 + \frac{(4-0)(4-1)(4-2)}{(2-0)(2-1)(2-5)}$$

$$= \frac{(3)(2)(-1)}{(-1)(-2)(-5)} \times 2 + \frac{(4)(2)(-1)}{(1)(-1)(-4)} \times 5 +$$

$$\frac{(4)(3)(-1)}{(2)(1)(-3)} \times 7 + \frac{(4)(3)(2)}{(5)(4)(3)} \times 8$$

$$y = \left(\frac{3\times 2\times(-1)}{-10}\right) \times 2 + \frac{(4\times 2\times -1)}{(1\times -1\times -4)} \times 5$$

$$+ \frac{(4\times 3\times -1)}{(2\times 1\times(-3))} \times 7 + \frac{(4\times 3\times 2)}{(5\times 4\times 3)} \times 8$$

$$= \frac{\cancel{-6}^{3}}{\cancel{-10}_{5}} \times 2 + \frac{\cancel{-8}^{2}}{\cancel{-4}_{1}} \times 5 + \frac{\cancel{-12}^{2}}{\cancel{-6}_{1}} \times 7$$

$$+ \frac{\cancel{24}^{8}}{\cancel{60}_{\cancel{20}_{5}}} \times 8$$

$$= \frac{3 \times 2}{5} + -2 \times 5 + 2 \times 7 + \frac{16}{5}$$

$$= \frac{6}{5} + -10 + 2 \times 7 + \frac{16}{5}$$

$$y = 8.4$$

Problem ③

Using Lagrangens Interpolution formula Compute $y(5)$ from given data.

x	2	4	7	9
$y(x)$	10	26	65	101

Soln: Here;

$x_0 = 2$, $x_1 = 4$, $x_2 = 7$, $x_3 = 9$

$y_0 = 10$, $y_1 = 26$, $y_2 = 65$, $y_3 = 101$

By using lagrangen's formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = \frac{(5-4)(5-7)(5-9)}{(2-4)(2-7)(2-9)} \times 10 + \frac{(5-2)(5-7)(5-9)}{(4-2)(4-7)(4-9)} \times 26 + \frac{(5-2)(5-4)(5-9)}{(7-2)(7-4)(7-9)} \times 65 + \frac{(5-2)(5-4)(5-7)}{(9-2)(9-4)(9-7)} \times 101$$

$$y = \frac{(1)(-2)(-4)}{(-2)(-5)(-7)} \times 10 + \frac{(3)(-2)(-4)}{(2)(-3)(-5)} \times 26 + \frac{(3)(1)(-4)}{(5)(3)(-2)} \times 65 + \frac{(3)(1)(-2)}{(7)(5)(2)} \times 101$$

$$y = \frac{-8}{7} + 20.8 + 26 - \frac{303}{35}$$

$$\underline{y = 37}$$

Problem (4)

Using lagranges's Interpolution formula to estimate U_4 given that $U_4 = -12$, $U_3 = -3$, $U_5 = 75$, and $U_6 = 156$

x	2	3	5	6
$U(x)$	-12	-3	75	156

Soln: Here,

$x_0 = 2$, $x_1 = 3$, $x_2 = 5$, $x_3 = 6$

$y_0 = -12$, $y_1 = -3$, $y_2 = 75$, $y_3 = 156$.

By using lagrange's formula,

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$y = \frac{(4-3)(4-5)(4-6)}{(2-3)(2-5)(2-6)} \times (-12) + \frac{(4-2)(4-5)(4-6)}{(3-2)(3-5)(3-6)} \times$$
$$(-3) + \frac{(4-2)(4-3)(4-6)}{(5-2)(5-3)(5-6)} \times (75) + \frac{(4-2)(4-3)(4-5)}{(6-2)(6-3)(6-5)}$$
$$\times (156)$$

$$y = \frac{(1)(-1)(-2)}{(-1)(-3)(-4)} \times (-12) + \frac{(2)(-1)(-2)}{(1)(-2)(-3)} \times (-3) +$$
$$\frac{(2)(1)(-2)}{(3)(2)(-1)} \times (75) + \frac{(2)(1)(-1)}{(4)(3)(1)} \times 156$$

$$y = 2 + (-2) + \frac{150}{3} - \frac{156}{6}$$

$$\underline{y = 24}$$

Interpolation with Equidistance points :-

The Lagrange's Method and Newton's divided difference interpolation Method are used for non-equidistance points , so if the values of x are not equidistance than we have to use these methods. We can also use these methods for equidistance points.

There are two more popular methods for especially equidistance points,

Finite difference :-Suppose the function y=f(x) has the values y_0, y_1, y_2,------ y_n for the equally spaced values x= x_0 , x_0+h, x_0+2h,------- x_0+nh. If y=f(x) be any function then the variable 'x' is called argument and corresponding values of 'dependent' variable y is called entry.

To determine the value of y for intermediate value of x is based on the principle of difference, which requires two types of differences.

1) **Forward Differences:-** The differences y_1-y_0 , y_2-y_1 , y_3-y_2 ,--------- y_n-y_{n-1} are first forward difference of the function y=f(x) and we denote the difference as $\Delta y_0, \Delta y_1, \Delta y_2$,---------- Δy_n. respectively where Δ is called forward difference operator. In general the first forward difference is defined by

$$\Delta y_x = y_{x+1} - y_x$$

The differences of first forward difference are called second forward differences and denoted by $\Delta^2 y_0$, $\Delta^2 y_1$ etc

There fore, we have,

$$\begin{aligned}\Delta^2 y_0 &= \Delta(y_1 - y_0)\\ &= \Delta y_1 - \Delta y_0\\ &= y_2 - y_1 - (y_1 - y_0)\\ &= y_2 - y_1 - y_1 + y_0\end{aligned}$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$\Delta^2 y_1 = \Delta(y_{2\text{-}}y_1)$

$= \Delta y_2 - \Delta y_1$

$= y_3 - y_2 - (y_2 - y_1)$

$= y_3 - y_2 - y_2 + y_1$

$\Delta^2 y_1 = y_3 - 2y_2 + y_1$

So in general we have,

$\Delta^2 y_x = \Delta y_{x+1} - \Delta y_x$

Again the difference of 2nd forward difference are called as 3rd forward differences and denoted by $\Delta^3 y_0$, $\Delta^3 y_1$ etc

We have ,

$\Delta^3 y_0 = \Delta^2(y_1 - y_0)$

$= \Delta^2 y_1 - \Delta^2 y_0$

$=(y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$

$= y_3 - 2y_2 + y_1 - y_2 + 2y_1 - y_0$

$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$ and so on,

in general we have,

$$\boldsymbol{\Delta^n y_x = \Delta^{n-1} y_{x+1} - \Delta^{n-1} y_x}$$

Formation of Forward Difference Table :-

Argument (x)	Entry y=f(x)	First Difference Δy	Second Difference $\Delta^2 y$	Third Difference $\Delta^3 y$	Fourth Difference $\Delta^4 y$	Fifth Difference $\Delta^5 y$
x_0	y_0					
		$\Delta y_0 = y_1 - y_0$				
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$			
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$		
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$	$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$		
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1$	
		$\Delta y_3 = y_4 - y_3$		$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$		
x_4	y_4		$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$			
		$\Delta y_4 = y_5 - y_4$				
x_5	y_5					

(The first term in the table y_0 is called the leading term and difference which stand at the head of respective columns, namely Δy_0 , $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ are called leading difference)

2) Backward Differences:- The differences y_1-y_0 , y_2-y_1 , y_3-y_2 ,--------- y_n-y_{n-1} are first backward difference of the function y=f(x) and we denote the difference as $\nabla y_1, \nabla y_2$,---------- ∇y_n. respectively where ∇ is called backward difference operator. In general the first backward difference is defined by

$$\nabla y_x = y_x - y_{x-1}$$

The differences of first backward difference are called second backward differences and denoted by $\nabla^2 y_2$, $\nabla^2 y_3$ etc

There fore, we have,

$$\nabla^2 y_2 = \nabla(y_2 - y_1)$$
$$= \nabla y_2 - \nabla y_1$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - y_1 - y_1 + y_0$$

$$\nabla^2 y_2 = y_2 - 2y_1 + y_0$$

So in general we have,

$$\nabla^2 y_x = \nabla y_x - \nabla y_{x-1}$$

Again the difference of 2nd backward difference are called as 3rd backward differences.

$$\boldsymbol{\nabla^3 y_x = \nabla^2 y_x - \nabla^2 y_{x-1}}$$

In general nth backward differences is given by,

$$\boldsymbol{\nabla^n y_x = \nabla^{n-1} y_x - \nabla^{n-1} y_{x-1}}$$

Formation of Backward Difference Table :-

Argument	Entry $y=f(x)$	First Difference ∇y	Second Difference $\nabla^2 y$	Third Difference $\nabla^3 y$	Fourth Difference $\nabla^4 y$
$x = a$	y_0				
		$y_1 - y_0 = \nabla y_1$			
$a + h$	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$		
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	
$a + 2h$	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^3 y_4 - \nabla^3 y_3 = \nabla^4 y_4$
		$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$	
$a + 3h$	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
		$y_4 - y_3 = \nabla y_4$			
$a + 4h$	y_4				

Forward Interpolation Technique (Newton-Gregory's formula for forward interpolation**)**

The technique which is mainly used for interpolating the values of y in the beginning of the given arguments is called forward interpolation.

This formula is applied when the arguments are given with equally spaced, i.e. the difference between any two consecutive values of x is the same.

Let $y = f(x)$ be a function which takes the values $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, ………., $f(x_0 + nh)$, $f(x_n)$ respectively

Then the Newton-Gregory's Formula for forward interpolation is given by

$$y_0(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0$$

Where $u = \frac{x - x_0}{h}$

Here x is the value of the variable corresponding which y is required to be interpolated. x_0 is first value of x column and h is the difference between any two consecutive values of x.

This is Newton-Gregory's formula for forward interpolation with equal intervals.

Backward Interpolation Techniques (Newton-Gregory's formula for backward interpolation)

This formula is generally used, when our requirement for determining the values of the function y = f(x) at the end or just before the end of the given arguments. That is how, this formula is known as Backward Interpolation Formula.

Let y = f(x) be a function which takes the values $y_0, y_1, y_2, \ldots\ldots y_n$ for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots\ldots x_0 + nh$ respectively.

Then Newton's backward interpolation formula is as follows

$$y_n(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \ldots + \frac{u(u+1)(u+2)\ \ldots (u+n-1)}{n!}\nabla^n y_n$$

Where $u = \dfrac{x - x_n}{h}$

This is backward interpolation formula given by Newton-Gregory for equal intervals.

Problem of Newton –Gregory Forward Formula

(Note- if the value of y at points near to x_0 then we use Newton-Gregory forward interpolation formula or if value of x near to x_n then we use Newton-Gregory backward interpolation formula)

(2) The population of Country is given in the following table. Estimate the population for the year 1905 and 1925

Year	1891	1901	1911	1921	1931
Population	46	66	81	93	101

Sol^n ⇒ To find :- $x = 1905$, $x = 1925$

To estimate the population of the year 1905 we use, Newton's forward Interpolation Method. because 1905 near to x_0.

∴ The difference table for above value is

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0=1891$	46				
		20			
$x_1=1901$	66		-5		
		15		2	
$x_2=1911$	81		-3		-3
		12		-1	
$x_3=1921$	93		-4		
		8			
$x_4=1931$	101				

Case 1 :-

$x_0 = 1891$, $h = 10$, $x = 1905$

$$u = \frac{1905 - 1891}{10} = \frac{14}{10} = 1.4$$

Newton forward's interpolation formula is given,

$$y_0(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= 46 + 1.4 \times 20 + \frac{1.4(1.4-1)(-5)}{2!} + \frac{1.4(1.4-1)(1.4-2)(2)}{3!} + \frac{1.4(1.4-1)(1.4-2)(1.4-3)}{4!} \times (-3)$$

$$= 46 + 28 + \frac{(-2.8)}{2} + \frac{(-0.672)}{6} + \frac{(-1.6128)}{24}$$

$$= 46 + 28 - 1.4 - 0.112 - 0.0672$$

$y_0(x) = 72.4208$

$y_0(1905) = 72.4208$

Hence the population of 1905 is 72.4208

Case II

To estimate the population for the year 1925 we use Newton's backward interpolation because 1925 is near to x_n.

$\therefore$ The difference table for above formula is given by.

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
$x_0=1891$	44				
		20			
$x_1=1901$	66		-5		
		15		2	
$x_2=1911$	81		-3		-3
		12		-1	
$x_3=1921$	93		-4		
		8			
$x_4=1931$	101				

$x_n = 1931$, $h=10$, $x = 1925$

$$u = \frac{x-x_n}{h} = \frac{1925-1931}{10} = -0.6,$$

$y_n = 101$

By Newton backward interpolation formula is given by,

$$y_n(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$= 101 + (-0.6)8 + \frac{(-0.6)(-0.6+1)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) + (-0.6)(-0.6+1) \times (-0.6+2)(-0.6)+3$$

$$\Rightarrow y_n(x) = 101 + (-0.6)8 + \frac{(-0.6)(-0.6+1)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(-3)$$

$$= 101 - 4.8 + \frac{(-0.6)(-1.6)}{2} + \frac{0.336}{6} + \frac{2.4192}{24}$$

$$= 101 - 4.8 + \frac{0.96}{2} + 0.056 + 0.1008$$

$$= 101 - 4.8 + 0.48 + 0.056 + 0.1008$$

$$\Rightarrow y_n(x) = 96.8368$$

Hence population of year 1925 is 96.8368

Prob ② Find the value of the area of the circle of diameter 82 from the following given data:

x (d) diameter	80	85	90	95	100
A (area)	5026	5674	6362	7088	7854

Solution ⇒ The difference table of the above data is given as follows,

x	$y=f(x)$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
$x_0=80$	5026				
		648			
$x_1=85$	5674		40		
		688		-2	
$x_2=90$	6362		38		4
		726			
$x_3=95$	7088			2	
		766	40		
$x_4=100$	7854				

Here, $x_0 = 80$, $h=5$,
$x=82$

$$\therefore \quad u = \frac{82-80}{5} = 0.4$$

⇒ Using Newton-Gregory forward interpolation formula we, get, Page

$\Rightarrow y_0(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$

$\Rightarrow y_0(x) = 5026 + 0.4(648) + \frac{0.4(0.4-1)}{2!} \times 40 + \frac{(0.4)(0.4-1)(0.4-2)(-2)}{3!} + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!} \times 4$

$y_0(x) = 5026 + 259.2 - 4.8 - 0.128 - 0.1664$

$y_0(x) = 5280.1056$

$\Rightarrow$ Hence Area of circle at diameter 82 is 5280.1056

Prob. (2) Marks obtained by the Candidates in an examination are given

Marks less than:	20	30	40	50	60
No. of students:	42	87	126	174	193

Find the number of Candidates who obtained less than 44 marks.

Sol^n ⟹ The backward difference table is given below.

x (marks less than)	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
20	42				
		45			
30	87		-6		
		39		15	
40	126		9		-53
		48		-38	
50	174		-29		
		19			
60	193				

Here,

$x_n = 60, \quad h = 10, \quad x = 44$

$\therefore \quad u = \dfrac{x - x_n}{h} = \dfrac{44 - 60}{10} = -1.6$

By Newton's Backward Interpolation formula we get,

$$\Rightarrow y_n(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$= 193 + (-1.6)\,19 + \frac{(-1.6)(-1.6+1)}{2!}(-29) + \frac{(-1.6)(-1.6+1)(-1.6+2)}{3!}(-38) + \frac{(-1.6)(-1.6+1)(-1.6+2)(-1.6+3)}{4!}(-53)$$

$$\Rightarrow y_n(x) = 193 - 30.4 - 13.92 - 2.432 - 1.1872$$

= ~~[illegible]~~

$$\Rightarrow y_n(x) = 145.0608$$

Here $\Rightarrow$ Number of Candidates who obtained less than 44 marks is 145.0608

Inverse Interpolation

Consider the function y=f(x) where y is the dependent variable and x is the independent variable, The technique of determining y for x lying between two tabulated values of argument is known as interpolation whereas the technique of determining the values of x corresponding to the value of y is known as inverse interpolation.

We can use Lagrange's method to determine inverse interpolation. The main difference between normal interpolation and inverse interpolation is that we assume x as the dependent variable and y as the independent variable.

The general equation for inverse interpolation is as follows

$$x = \frac{(y-y_1)(y-y_2)\text{-----}(y-y_n)}{(y_0-y_1)(y_0-y_2)\text{----}(y_0-y_n)} * x_0 + \frac{(y-y_0)(y-y_2)\text{-----}(y-y_n)}{(y_1-y_0)(y_1-y_2)\text{----}(y_1-y_n)} * x_1 + \text{------------}$$

$$\text{-----}+\frac{(y-y_0)(y-y_1)\text{-----}(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\text{----}(y_n-y_{n-1})} * x_n$$

Problems of Inverse Interpolation

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Prob: For the data given below obtain the Value of x for $y=16$ by inverse interpolation method.

x	1	3	5	7	9
$Y=x^2$	1	9	25	49	81

Sol^n ⇒ By Inverse Interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)} \times x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)} \times x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)} \times x_2$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2(y_3-y_4)} \times x_3$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)} \times x_4$$

Here, $y=16$, $y_0=1$, $y_1=9$, $y_2=25$
$y_3=49$, $y_4=81$, $x_0=1$, $x_1=3$
$x_2=5$, $x_3=7$ and $x_4=9$

after substituting values we get,

$$\Rightarrow x = \frac{(16-9)(16-25)(16-49)(16-81)}{(1-9)(1-25)(1-49)(1-81)} \times 1$$

$$+ \frac{(16-1)(16-25)(16-49)(16-81)}{(9-1)(9-25)(9-49)(9-81)} \times 3$$

$$+ \frac{(16-1)(16-9)(16-49)(16-81)}{(25-1)(25-9)(25-49)(25-81)} \times 5$$

$$+ \frac{(16-1)(16-9)(16-25)(16-81)}{(49-1)(49-9)(49-25)(49-81)} \times 7$$

$$+ \frac{(16-1)(16-9)(16-25)(16-49)}{(81-1)(81-9)(81-25)(81-49)} \times 9$$

$$\Rightarrow x = -0.18328 + 2.35657 + 2.1820 - 0.29159 + 0.02719$$

$$\Rightarrow x = 4.090881348$$

Problems

Find values of x where $y = 19$ for the given data

x	0	1	2
y	0	1	20.

Let,

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$

$y_0 = 0$, $y_1 = 1$, $y_2 = 20.$

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} \times x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} \times x_1$$

$$\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} \times x_2$$

$$= \frac{(19-1)(19-20)}{(0-1)(0-20)} \times 0 + \frac{(19-0)(19-20)}{(1-0)(1-20)} \times 1$$

$$+ \frac{(19-0)(19-1)}{(20-0)(20-1)} \times 2$$

$$= 0 + 1 + 1.89$$

$$= \underline{2.89}$$