Quantum Field Theory I
Notes

The_Reto

HS 2021

Introduction

On this document

This document are my personal notes on the lecture *Lecture-No. - Quantum Field Theory I* at ETH Zürich in the fall semester 2021. I type part of this document during the lectures (while the Professor speaks), and another part as a write up when I personally read the associated literature. All of this happens with minimal editing - there are *A LOT* of spelling mistakes, typos and errors of various severity.

Information about this Course

The course is taught by Professor Graf.
The lecture is hel in person as well as a remote lecture. Lecture Notes and the exercises are published on moodle.

Exercises

Exercises start in week two

Exercises

The exercises are posted on moodle on mondays. It's heavily recommended to solve them before going to the exercise classes. The exam is, in part, based on the exericses.

Introduction - Why QFT?

The goal of this lecture is to describe particles and interactions.

Reqirements on a theory we want to find:

1. Relativistically covariant
2. Quantum Mechanical
3. describe interactions between different particles

Picking just two of these requirements is relatively easy: Electrodynamics is relativistically covariant and describes interactions but is not quantum mechanical, regular quantum mechanics describes the world quantum mechanically and allows interactions but is not relativistic, in this lecture we will see a theory

Answer *QFT!*

1. Locality principle:
   causality travels from one point to the direct neigbouring point in time and space. The way to ensure sech a locality principle is to describe an underlying field instead of particles.
2. By imposing quantum mechanical relations on the field particles emerge automatically. Particles that emerge that way may or may not be what we thought particles are.
3. If you consider interacting fields we can automatically describe interacting particles. We will consider two interacting field theories; QED and ϕ^4-theory.
   We will describe these interacting fieldtheories with Feynman diagrams.
   We will then see that divergencies appear frem these computations and will subsequentnly learn to renormalize the theories to get rid of these divergencies.

Contents

Relativistic quantum mechanics

We now want to find a relativistically covariant description of quantum mechanics (in essence we want to unite the first two requirements from section 0.1).

1.1 Symmetries of Spacetime

Events happen at coordinates in inertal reference frames (IF):

$$x = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} ct \\ \vec{x} \end{bmatrix}$$

In addition we have a metric on this spacetime

$$x \cdot y = x^T g y = g_{\mu\nu} x^\mu y^\nu$$

with g given by

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbb{I} \end{bmatrix}$$

We also know the "raising" and "lowering" of indecies

$$x_\mu = g_{\mu\nu} x^\nu$$

We are also already familiar with Lorenz transformations (LT) that occure when changing inertial frames

$$x' = \Lambda x + a \text{ with } \Lambda^T g \Lambda = g, \, a \in \mathbb{R}^4$$

The lorenz trasformations preserve the metric on the space time and thus it's symmetries. We call the set of all possible Λ matricies the Lorenzgroup

$$L = \{\Lambda\}$$

The Lorenz group has 4 connected componets. One of them is denoted

$$L_+^\uparrow = \{\Lambda \in L | \Lambda_0^0 \geq 1, det\,(\Lambda) = +1\}$$

This connected componet is a subgroup of all lorenztransformations. The three remaining components are optained by multiplication of this first group with certain operations:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -\mathbb{I}_3 \end{bmatrix}, T = \begin{bmatrix} -1 & 0 \\ 0 & \mathbb{I}_3 \end{bmatrix}, PT = -\mathbb{I}_4$$

In the following we only consider $(\Lambda, a) \in L_+^\uparrow \times \mathbb{R}^4$ as transformations between IF's, we call this group the Poincare group P. In this notation we got a composition law:

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1 \Lambda_2, a_1 + \Lambda_1 a_2)$$

and the inverse law

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} a)$$

We now consider a family of LT's

$$x_1 = \Lambda(\lambda) x + a(\lambda) \, \lambda \in \mathbb{R}$$

$$(\Lambda(0), a(0)) = (1, 0)$$

Such a family defines an ininitesimal LT

$$x' = \frac{d}{d\lambda}\big|_{\lambda=0} (\Lambda(\lambda) + a(\lambda)) = \epsilon x + \tau$$

$$x' = x + \lambda (\epsilon x + tau) + O(\lambda^2)$$

These inifitessimal elements are no longer part of the poincare group, but are elements of the Lie Algebra of P, ie. $Lie(P)$. There are certain constraints on what values the τ or ϵ can take:

$$\tau^\mu \in \mathbb{R}$$

$$\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$$

As for any Lie group, here $Lie(P)$, is closed with respect to some operations.

1. it's a real vectorspace

2. it's closed with respect to lie brakets

Given an element of P

$$\text{Ad}_{(\Lambda,a)} : Lie(P) \to Lie(P)$$

$$(e, \tau) \to \text{Ad}_{(\Lambda,a)} (\epsilon, \tau)$$

Lie Brackets

$$[(\epsilon_1, \tau_1), (\epsilon_2, \tau_2)] := \frac{\partial}{\partial\lambda_1} \frac{\partial}{\partial\lambda_2}\big|_{\lambda_1=\lambda_2=0} (\Lambda_1, a_1)(\Lambda_2, a_2)(\Lambda_1, a_1)^{-1}(\Lambda_2, a_2)^{-1} \in Lie(P)$$

Important Formulae:

- $Ad_{(\Lambda,a)} (\epsilon, \tau) = (\Lambda\epsilon\Lambda^{-1}, \Lambda\tau - \Lambda\epsilon\Lambda^{-1})$

- $[(_1, \tau_1), (\epsilon_2, \tau_2)] = ([\epsilon_{1,2}], \epsilon_1\tau_2 - \epsilon_2\tau_1)$

To find the generators of $Lie(P) : M^{\mu\nu}, P^\sigma$ we expand $(\epsilon, \tau) = (\epsilon, 0) + (0, \tau)$ with $\epsilon = \epsilon_{\mu\nu}M^{\mu\nu}$ with $M^{\mu\nu} + M^{\nu\mu} = 0$ we get

$$M^{0,1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M^{1,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the commutators:

$$[M^{\mu\nu}, M^{\rho,\sigma}] = (g^{\nu\rho}M^{\mu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma)$$

$$[M^{\mu\nu}, P^\sigma] = g^{\mu\sigma}P^\mu - (\mu \leftrightarrow \nu)$$

$$[P^\mu, P^\nu] = 0$$

In total we find 10 generators of the symmetries of spacetime: (in the following $i = 1, 2, 3$):

Rotations about the axis i: $J^i = M^{i+2}_{i+1}$ (3 generators)

Boosts in direction i: $K^i = M^i_0$ (3 generators)

Spacial translations: P^i (3 generators)

Temporal translation: P^0 (1 generator)

1.2 What is a Particle?

Wigner 1939 tried to explain what a particle is, from pure theory. So what's a free elementary particle? A free elementary particle has only kinematic degrees of freedom: these are those dof that result from the relation of the particle to the inertial frame. Wiegner postulates: *The Hilbertspace $\mathcal{H}$ carries an irreducible projective representation $U(\Lambda, a)$ of the Poincare group $\mathcal{P}$.*

Remarks on this postulate:

Representation $\mathcal{P} \to \{$ unitaries in $\}\mathcal{H}$ homomorphically. Each element of the group can be mapped to a unitary.

Projective "up to a phase": It's important to keep in mind that the unitaries are not acting on vectors in the Hilbertspace, but on states. The unitaries correspond to ordinary representations of the universal cover of $\mathcal{P}$. (SO(3) compared to SU(2))

Irreducible the only invariant subspaces in $\mathcal{H}$ that stay invariant under $U(\Lambda, a) \forall \Lambda, a$ are the whole hilbert space or the empty group.

meaning $U(\Lambda, a)$ is a passive transformation of states $\Psi \in \mathcal{H}$ when changing the inertial frame. There is also the active interpretation, switching between the two can offer different perspectives.

Formulation this entire formulation is given in a Heisenberg picture.

Elementary What now makes a particle elementary? It's the word irreducible (no longer as a point particle). An elementary particle is thus a particle whose representation cannot be further reduced.

This representation is a one of the poincare group that induces the Lie algebra of infinitessimal Lorenz transformations $(\epsilon,) = \frac{d}{d\lambda}(\Lambda, a)$ at $\lambda = 0$. We now use this to define our unitaries

$$U(\epsilon, \tau) := \frac{d}{d\lambda} U(\Lambda, a) \text{ at } \lambda = 0$$

$$\implies U(\epsilon,)^* + U(\epsilon, \tau) = 0$$

$$-iU(\epsilon, \tau) \text{ is self adjoint}$$

Thus, with slight abuse of notation

$$U(P^\mu) = \frac{i}{\hbar} P^\mu$$

$$U(M^{\mu\nu}) = \frac{i}{\hbar} M^{\mu\nu}$$

If we only consider spacetime translations:

$$\to U(1, a) = e^{\frac{1}{\hbar} P^\mu a_\mu} = e^{\frac{i}{\hbar} P_\mu a^\mu} = e^{\frac{i}{\hbar}(P^0 a^0 - \vec{P}\vec{a})}$$

conforming with establishod use and sign conventions we find that P is the generator of translations.

$$\frac{\partial}{\partial a^\mu} U(1, a) = \frac{i}{\hbar} P_\mu U(1, a)$$

Example: before continuing we'll consider the example of the familiar Schroedinger Equation

$$i\hbar \frac{\partial}{\partial t} \psi(t) = H\psi(t)$$

This is a special case of the equation above with $\mu = 0$ and $\psi(t) U(1'a) \psi_0$ and $a = (-ct, 0)$ and $H = cP^0$. Interpretation: Whot one observer O sees at time $t = t_0 + \Delta t$ is what O' sees at time $t = t_0$, if $t' = t - \Delta t$. We can take the point of view that there is an active change of view in which the world changes, or the passive picture at which different observers at different times observe the same, unchanging, state differently.
Let's go back to the analysis of this irreducible representation.

$$\mathcal{P} = \Lambda_+^\uparrow \times \mathbb{R}^4$$

We now consider

$$(1, a)(\Lambda, 0) = (\Lambda, a) = (\Lambda, 0)(1, \Lambda^{-1}a)$$

now we go to the representation

$$\rightarrow e^{\frac{i}{\hbar}P_\mu a^\mu} U(\Lambda, a) = U(\Lambda, a) e^{\frac{i}{\hbar}P_\mu (\Lambda^{-1}a)^\mu} = U(\Lambda a) e^{\frac{i}{\hbar}(\Lambda P)_\mu a^\mu}$$

Differentiating both sides yields

$$P_\mu U(\Lambda, 0) = U(\Lambda, 0)(\Lambda P)_\mu = U(\Lambda, 0)\Lambda_\mu^\nu P_\nu$$

$$\rightarrow U(\Lambda, 0)^* P_\mu U(\Lambda, 0) = (\Lambda P)_\mu = \Lambda_\mu^\nu P_\nu$$

Which yields the result that P^μ transforms like a vector under Lorenztransformations.
We now consider Spacetime translators $U(1, a)$ which all commute with each other. This means that the generatorn P_μ also all commute, which then in turn means that they can be diagonalized jointly. We can therefor eigenvectors $|\psi\rangle$ of all at once

$$P^\mu |\psi\rangle = p^\mu |\psi\rangle$$

Let $\mathcal{H}_P$ be the eigenspace of P. We now consider our previous point

$$U(\Lambda, 0) : \mathcal{H}_P \rightarrow \mathcal{H}_{\Lambda P}$$

Lorenztransformations thus move one eigenspace to another one. We now consider the space

$$V = \{p \in \mathcal{R}^4 | \mathcal{H}_P \neq \{0\}\}$$

which is a lorenz invariant set.
We now use the property of irreducibility: V is not the union of two such subsets with the same property.

$$\implies V = \{\Lambda p | \Lambda \in L_+^\uparrow\} \text{ for some } p \in \mathbb{R}^4$$

We call the set of points reachable via lorenztransformations from a point p, Λp the orbit of p. How many orbits are there in minkowsky space? There are 6 types of orbits

1. $V_m^+ = \{p | p^2 = m^2 c^2, p^0 > 0\}$ (mass shell)

2. $V_m^- = \{p | p^2 = m^2 c^2, p^0 < 0\} = -V_m^+$

3. $V_0^+ = \{p | p^2 = 0, p^0 > 0\}$

4. $V_0^- = \{p | p^2 = 0, p < 0\} = -V_0^+$
   It's worth noting that there's no restframe for orbits of type 3 and 4.

5. $V_\rho = \{p | p^2 = -\rho\}, \rho > 0$

6. $V_0^0 = \{0\}$

Let's discuss these cases in detail:

1. fits tho notion of a massive relativistic particle: classically this would be $p^\mu = \left(\frac{E}{c}, \vec{p}\right) = m\frac{dx^\mu}{d\tau} = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \implies$
   $p^2 = m^2 c^2$.

2. also fits the notion of a massive particle, but they have $E = cp^0 < 0$. This contradicts the convention for the proper time $\frac{dx^0}{dt} > 0$.

3. Is like the first case, but with massless particles.

4. also fits the notion of a massless particle, but they have $E = cp^0 < 0$. This contradicts the convention for the proper time $\frac{dx^0}{dt} > 0$.

5. describe tachyons (particles at speed larger than the speed of light). This we dismiss as being unphysical.

6. Corresponds to the trivial representation of P $(dim\mathcal{H} = 1, U(\Lambda, a) = 1)$ which fittingly describes the vacuum.

A few remarks

- The orbits in minkovski space are not just lorenz invariant sets' but also lorenz invariant measures.

$$d\mu\left(\Lambda p\right) = d\mu(p) \ ; \text{ is unique up to a multiple:}$$

$$d\mu(p) = \delta\left(p^2 - m^2 c^2\right) \Theta\left(\gamma^0\right) d^4 p$$

because $p^2, d^4 p, \Theta(p^0)$ are all invariant. Also $d\mu\left(p\right) = \frac{d^3 p}{2 p^0}$

- Recall that $\mathcal{H}_p$ is the Hilbert space after elimination of the translational DoF.
  We can now think of this as the space of inner degrees of freedom and $dim\left(\mathcal{H}_p\right) < \infty$ (it turns out that this is true in some cases, but has to be enforced in others).
  We introduce the following notation for transformations without translations

$$\hat{U}\left(\Lambda\right) = U\left(\Lambda, 0\right) : \mathcal{H}_p \to \mathcal{H}_{\Lambda p}$$

The Hilbertspce $\mathcal{H}$ has $dim\left(\mathcal{H}\right) = \infty$ if $V?p$ has ∞-many elements.

1.2.1 Method of the little group (or stabilizer)

$$H_p := \{\Lambda \in L_+^\uparrow | \Lambda p = p\}$$

If we now consider H_p and $H_{\Lambda p}$ are isomorphic via $\hat{U}\left(\Lambda\right)$.
The whole irreducible representation $U\left(\Lambda, a\right)$ of $\mathcal{P}$ on $\mathcal{H}$ with orbit Vp are one to one with irreps. $\hat{U}\left(\Lambda\right)$ of $H_p \Lambda$ onn $\mathcal{H}_p$
The method of the little group is possible because $\mathcal{P} = \Lambda_+^\uparrow \times \mathbb{R}^4$.
A word of caution: With projective representations, some of this analysis is a bit more tricky.
What we now archieved is that we just need to analyze one $p \in V$, since we can transform it to any other $p' \in V$.

Massive case: $(m > 0)$ Pick $p\left(m, \vec{0}\right)$ (rest frame) $\implies H_p = SO(3)$ only rotations remain. H_p is thereofro generated by $\vec{J} = (J_1, J_2, J_3)$. The projective representations are ordinary reps of covering group (double) $SU(2)$: which is labeled *SPIN* j

$$\mathcal{D}_j\left(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots\right)$$

$dim\left(\mathcal{D}_j = 2j + 1\right)$.
Spin is intrinsic angular momentum in the rest frame.

Examples:

- Electron: $j = \frac{1}{2}$
- pion $j = 0$
- C-atom $j = 1$

Massless case: $(m = 0)$ Pick $p = \left(p^0, 0, 0 p^3\right), p^0 > 0$. We are looking for Lorentz transformations $\Lambda \in H_p$. One generator of H_p is given by

$$M_1^2 = J^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

another is given by

$$\Theta_1 = K^1 + J^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Which is a boost in the 1 direction and a rotation in the 2 direction. There's also

$$\Theta_2 = K^2 - J^1$$

The last two generators commute

$$[\Theta_1, \Theta_2] = 0$$

What subgroups do they generate?

- $\Lambda(w)$ is the subgroup generated by J_3 : it corresponds to a rotation by angle, or phase, $w = w_1 + iw_2, |w| = 1$.
- $\Lambda(t)$ is the subgroup generated by Θ_1 and Θ_2. With the paramenter $t = t_1 + it_2$.

All in all H_p consists of $\Lambda(w, t) = \Lambda(t)\Lambda(w)$. The group multiplication rule is given by

$$\Lambda(w', t')\Lambda(w, t) = \Lambda(w'w, t' - w't)$$

H_p is isomorphic to Euclidean groupt of rigid transformation in the plane $\mathbb{C}$.

$$\Lambda(w, t) : z \to wz + t$$

The irreducible representations of H_p by little group method. We diagonalize the abelian part Θ_1, Θ_2 and find the Eigenvalues $\xi = \xi_1 + i\xi_2$ of $\Theta_1 + i\Theta_2$ that transform as $\xi \to w\xi$ under $w \in U(1)$. The minimal invariant subsets are either $\{\xi \in \mathbb{C} || \xi | = R\}$ with $R > 0$, or $\{\xi = 0\}$. We can reject the first option, because it leads to $dim(\mathcal{H}_p) = \infty$ (which we said we'd impose to not be the case). So we are left with $H_p = U(1)$. We can think of that as the covering group of $U(1)$ which is $\mathbb{R} \implies \hat{U}(w) = w^s, s \in \mathbb{R}$. It turns out however that this is not quite correct. In truth $U(1)$ is generated by J_3. Only transformations compatible with the double cover should be retrieved. Therefore only $s = \frac{n}{2}, n \in \mathbb{N}$ are allowed. This property is called the *Helicity*.

Helicity is the intrinsic angle of momentum about the propagation direction of the massless particle.

Examples:

- Right circularized photon: $s = 1$
- Left circularized photon: $s = -1$

It's important to be precise: a Photon does not have spin $= \pm 1$, it has helicity $s = \pm 1$.

We claim that if we include parity P but not T; we get that:

- In the $m > 0$ case the D_j are as before

- In the $m = 0$ case the $\pm s$ merge

Spin and Helicity are unified in th Pauli-Lubouski vector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}p^\nu M^{\rho\sigma}$$

$$W^0 = \vec{P} \cdot \vec{J} , \; \vec{W} = P^0\vec{J} + \vec{P}?\vec{K}$$

- For $m > 0$ we choose $p = \left(m, \vec{0}\right)$ and we get

$$W^0 = 0 , \; \vec{W} = P^0\vec{J} = m\vec{J}$$

- For $m = 0$ we again take $p = \left(p^0, 0, 0, p^3\right)$ and find

$$W^\mu = p^0 \left(J^3, \underbrace{J^1 - K^2}_{-\Theta_2}, \underbrace{J_2 + K^1}_{\Theta_1}, J_3 \right) = p^0 \left(J^3, 0, 0, J^3\right) = J^3 p^\mu$$

so we find

$$W^0 = J^3 p^0 = sp^0 , \; \vec{W} = s\vec{p}$$

From which we can deduce

$$\vec{P} \cdot \vec{J} = sp^0$$

So we get that the helicity is $s > 0$ for $\vec{J}$ and $\vec{P}$ parallel and $s < 0$ for $\vec{J}$, $\vec{P}$ antiparallel.

We know that under parity P

$$\vec{p} \to -\vec{p}$$
$$p^0 \to p^0$$
$$\vec{J} \to \vec{J}\vec{K} \qquad\qquad\qquad \to -\vec{K}$$

And we find therefore

- $m > 0$: D_j are invariant

- $m = 0$: $\vec{p}\vec{J} \to -\vec{p}\vec{J}$, which means $s \to -s$.

1.3 The Klein-Gordon Equation

Staes of free particles at a fixed 4-momentum is given by

$$\Psi(x) = e^{i\vec{k}\vec{x} - \omega t} = e^{-ik_\mu x^\mu} = e^{-ik\cdot x}$$

with $k^\mu = \left(\frac{\omega}{c}, \vec{k}\right)$ describing

$$p = \hbar k, \text{i.e.} \begin{cases} E & = \hbar\omega \\ \vec{p} & = \hbar\vec{k} \end{cases}$$

By using $p_\mu p^\mu = m^2 c^2$ ie $p^0 = +\sqrt{\vec{p}^2 + m^2 c^2} > 0$ we get the dispersion law

$$\omega = c\sqrt{\vec{k}^2 + \frac{m^2 c^2}{\hbar^2}}$$

and we find the goup velocity

$$\frac{\partial\omega}{\partial\vec{k}} = \frac{\partial E}{\partial\vec{p}} = c\frac{\partial p^0}{\partial\vec{p}} = c\frac{\vec{p}}{p^0} = \vec{v}$$

Which agrees with classical velocity.
We now consider general states we get

$$\Psi(x) = (2\pi\hbar)^{-\frac{3}{2}} \int \underbrace{d^4 p\,\theta\left(p^0\right)\delta\left(p^2 - m^2 c^2\right)}_{d\mu(p)} \hat{\Psi}(p)\, e^{-\frac{i}{\hbar}p\cdot x}$$

Which can also be written as

$$\Psi(x) = (2\pi\hbar)^{-\frac{3}{2}} \int \frac{d^3\vec{p}}{2p^0} \hat{\Psi}(\vec{p})\, e^{-\frac{i}{\hbar}p\cdot x}$$

This equation then describes a so called positive energy solution of the Klein-Gordan equation

$$\left(\Box + \frac{m^2 c^2}{\hbar^2}\right)\Psi(x) = 0$$

With $\Box = \partial_\mu\partial^\mu$ the d'alembersian operator. We will now discuss the state $\Psi(x)$, $x = \left(x^0, \vec{x}\right)$ a bit further. It is a solution over all of time (instead of at one time like in the schroedinger equation), non the less the entire wave function is determined by $\Psi(0, \vec{x})$. By taking

$$\Psi(0, \vec{x}) = (2\pi\hbar)^{-\frac{3}{2}} \int \frac{d^3 p}{2p^0} \hat{\Psi}(\vec{p})\, e^{i\frac{\vec{p}\vec{x}}{\hbar}}$$

and by fourier transform

$$\Psi(0, \vec{x}) \to^{\mathcal{F}} \frac{\hat{\Psi}(\vec{p})}{2p^0} \to \Psi(x)$$

Insertirng this into the equation given above we find

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

with H the multiplication operator in momentum space by p^0.

$$H : \hat{\Psi}(\vec{p}) \to cp^0\hat{\Psi}(\vec{p}) = c\sqrt{\vec{p}^2 + m^2 c^2}\hat{\Psi}(\vec{p})$$

But: H is not a differential operator in $\vec{x}$, since $\sqrt{\vec{p}^2 + m^2 c^2}$ is not a polynomial in $\vec{p}$. This description is non-local in $\vec{x}$ at the lengthscale $\frac{\hbar}{mc}$ (the compton wave-length: for the electron we find $\lambda_c = 3.8 \cdot 10^{-18}$m).
We now phrase our schroedingerlike equation to

$$\frac{\partial\Psi}{\partial x^0} = -\frac{i}{2\hbar}(2\pi\hbar)^{-\frac{3}{2}} \int d^3 p\,\hat{\Psi}(p)\, e^{-\frac{i}{\hbar}p\cdot x}$$

Remember: The general solution of the Klein-Gordan equation is the some integral as above but without the $\Theta\left(p^0\right)$. i.e. not only positive energies are allowed. This means that for the general solution we have to initially define $\Psi(0, \vec{x})$ and $\frac{\partial\Psi}{\partial x^0}(0, \vec{x})$ independently. (which follows from the fact that the K-G equation is second order in t, only by restricting ourself to positive energy solution can we recover the Schroedinger Equation.)

1.3.1 Lorenz Invariance

We still have to show that the K-G equation is invariant under Lorentz transformations.
Given a LT $x' = \Lambda x + a$ let Ψ transform to $\Psi' = U(\Lambda, a)\Psi$ as $\Psi'(x') = \Psi(x)$ which means that

$$\Psi'(x') = \Psi\left(\Lambda^{-1}(x'-a)\right) \forall x' \in \mathbb{R}^4$$

We now check that form a positive energy solution we again get a positive energy solution. By $p^{\Lambda^{-1}(x-a)} = (\Lambda p) \cdot (x - a)$ and $d\mu(p) = d\mu\left(\Lambda^{-1}p\right)$ we get

$$\Psi'(x) = (2\pi\hbar)^{-\frac{3}{2}} \int d\mu(p)\hat{\Psi}(p) e^{-\frac{i}{\hbar}(\Lambda p)(x-a)}$$

$$\underbrace{=}_{\Lambda p \to p} (2\pi\hbar)^{-\frac{3}{2}} \int d\mu(p)\hat{\Psi}\left(\Lambda^{-1}p\right) e^{-\frac{i}{\hbar}px} e^{-\frac{i}{\hbar}pa}$$

which yields again a positive energy solution

$$U(\Lambda, a) : \hat{\Psi}(p) \to e^{\frac{i}{\hbar}pa}\hat{\Psi}\left(\Lambda^{-1}p\right)$$

To find a hilbert space we need to defien an inner product (a Lorentz invariant one) we propose

$$(\psi, \phi) := \int d\mu(p)\overline{\hat{\psi}}(p)\hat{\phi}(p) = \int \frac{d^3p}{2p^0}\overline{\hat{\psi}(\vec{p})}\hat{\phi}(\vec{p})$$

$$(U(\Lambda, a)\Psi, U(\Lambda, a)\phi) = \int d\mu(p)\overline{e^{\frac{i}{\hbar}pa}\Psi\left(\Lambda^{-1}p\right)}e^{\frac{i}{\hbar}pa}\phi\left(\Lambda^{-1}p\right) = \int d\mu(p)\overline{\Psi\left(\Lambda^{-1}p\right)}\phi\left(\Lambda^{-1}p\right) = \int \underbrace{d\mu(\lambda p')}_{=d\mu(p')}\overline{\Psi(p')}\phi(p')$$

Which means

$$(U(\Lambda, a)\Psi, U(\Lambda, a)\phi) = (\Psi, \phi)$$

which is what we wanted.
We therefore find the Hilbert-space

$$\mathcal{H} = \{\psi | \psi \text{ is a positive energy solution and } (\psi, \psi) \leq +\infty\}$$

It's worth noting that this means that there's a different Hilbertspace for particles of different mass.
Recall the generators of momentum P_μ, and of angular momentum $M_{\mu\nu}$ of a representation. We now try to find their representations

Momentum

$$P_\mu = \frac{\hbar}{i}\frac{\partial}{\partial a^\mu}U(1, a)$$

Hence we find

$$P_\mu : \Psi(x) \to i\hbar\frac{\partial\Psi}{\partial x^\mu}$$

and in momentum space

$$P_\mu : \hat{\Psi}(p) = p_\mu\hat{\Psi}(p)$$

In particular see $cP^0 = H$ and the eigenspace of joint eigenvalues p^μ $\mathcal{H}_P = \mathbb{C}$ if $p \in V_m^+$, otherwise the eigenspace is $\mathcal{H}_P = \{0\}$.
We therefore see that $U(\Lambda, a)$ is an irreducible representation of $\mathcal{P}$.
Spin of the particle we're describing $s = 0$.

Angular Momentum For $\epsilon = \frac{d\Lambda}{d\lambda}|_{\lambda=0}$ we get

$$\frac{1}{2}\overset{\mu\nu}{}M_{\mu\nu} = \frac{1}{2}\underset{\mu\nu}{}M^{\mu\nu} = \frac{\hbar}{i}\frac{d}{d\lambda}U(\Lambda, 0)|_{\lambda=0}$$

Letting this act on Ψ we get

$$\frac{\hbar}{i}\frac{d}{d\lambda}\Psi\left(\Lambda^{-1}x\right)|_{\lambda=0} = \frac{\hbar}{i}\frac{\partial\Psi}{\partial x^\mu}\underbrace{\frac{d}{d\lambda}\left(\Lambda^{-1}x\right)^\mu|_{\lambda=0}}_{=-(\epsilon x)^\mu}$$

$$= -\frac{\hbar}{i}\epsilon^{\mu\nu}x_\nu\frac{\partial\Psi}{\partial x^\mu}$$

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We conclude

$$M_{\mu\nu} : \Psi(x) \to \frac{\hbar}{i}\left(x_\mu \frac{\partial \Psi}{\partial x^\nu} - x_\nu \frac{\partial \Psi}{\partial x^\mu}\right)$$

$$M_{\mu\nu} : \hat{\Psi}(p) \to \frac{\hbar}{i}\left(p_\mu \frac{\partial \hat{\Psi}}{\partial p^\nu} - p_\nu \frac{\partial \hat{\Psi}}{\partial p^\mu}\right)$$

This means that if $\mu, \nu = i, j = \{1, 2, 3\}$ these are angular monenta, if $\mu = 0, \nu = i$ these are boosts.
For $p = \left(m, \vec{0}\right) : M_{ij}\hat{\Psi}(0) = 0$ which means that intrinsic angular momentum $s = 0$.

More over

- $[H, P_\mu] = 0 \forall \mu$

- $[H, M_{i,j}] = 0 \forall i, j \in \{1, 2, 3\}$
  These two points correspond to conservation laws for P_μ and M_{ij}.

- P_μ $M_{\mu\nu}$ transform under LT like classical quatities (including expectation values).

How can we represent the inner product given above in the position space

$$(\psi, \phi) = \int_{x^0 = ct} d^3x \, i\hbar\left(\overline{\Psi}\frac{\partial \phi}{\partial x^0} - \frac{\partial \overline{\psi}}{\partial x^0}\phi\right)$$

Which is invariant of t. Wich can be seen when we choose $t = 0$ we get $\Psi(0, \vec{x}) \to \frac{\hat{\Psi}(\vec{p})}{2p^0}$ and $\frac{\partial \Psi}{\partial x^0}(0, \vec{x}) \to -\frac{i}{2\hbar}\hat{\psi}(\vec{p})$ under fourier transformation

$$\int_{x^0=0} d^3x \, \overline{\Psi(0, \vec{x})}\frac{\partial \phi}{\partial x^0}(0, \vec{x}) = -\frac{i}{2\hbar}\int \frac{d^3p}{2p^0}\overline{\hat{\psi}(\vec{p})}\phi(\vec{p}) = -\frac{i}{2\hbar}(\psi, \phi)$$

for the second term we we similarly we get $\frac{i}{2\hbar}(\psi, \phi)$ (the same just with opposite sign). This proof also generalizes to all time since introducing time just results in a phase factor which canceles out as we multiply with a complex conjugate.
In particular we get

$$||\Psi||^2 = (\Psi, \Psi) = \int d^3x \, j_0(x)$$

With j_0 the 0-component of a 4-current

$$j_\mu(x) = i\hbar\left(\overline{\Psi}\frac{\partial \Psi}{\partial x^\mu} - \frac{\partial \overline{\Psi}}{\partial x^\mu}\Psi\right)$$

Which is divergence free $\partial^\mu j_\mu(x) = 0$.
Sofar we found the QM of a single relativistic particle, we now list a few simmilarities & differences to the non-relativistic quantum mechanics (schroedinger)

- We recovered the schroedinger equation but with an H that is not a differential operator. i.e. the new Schroedinger-esqe equation we found is not a PDE.

- Formal agreement with non-relativistic case: where we get a probability density and a probability current density. As

$$\rho(x) = |\psi(x)|^2$$

$$\vec{j}(x) = \frac{\hbar}{2im}\left(\overline{\psi}\vec{\nabla}\psi - \left(\vec{\nabla}\psi\right)\psi\right)$$

which fullfill the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div}\vec{j} = 0$$

In the relativistic case we have the simmilar equations

$$(\psi, \psi) = \int_{x^0=ct} d^3x \, j^0(x)$$

$$j_\mu(x) = i\hbar\left(\overline{\psi}\frac{\partial \Psi}{\partial x^\mu} - \frac{\partial \overline{\psi}}{\partial x^\mu}\psi\right)$$

which fullfills

$$\partial^\mu j_\mu = 0$$

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However we have the problem that

$$j^0 = 2\mathrm{Re}\left(i\hbar \frac{\partial \psi}{\partial x^0}\overline{\psi}\right)$$

is not neccesarily positive in x, which means that we cannot interpret $j^0(x)$ as probability density.
We now prove that j^0 is not neccesarily positive. We consider $x = 0$:

$$2i\hbar \frac{\partial \psi}{\partial x^0} = (2\pi\hbar)^{-\frac{3}{2}} \int d^3p \hat{\psi}(p)$$

$$\psi(0) = (2\pi\hbar)^{-\frac{3}{2}} \int \frac{d^3p}{2p^0} \hat{\psi}(\vec{p})$$

Two functions, one of them $\vec{p} \to 1$ the other $\vec{p} \to \left(2p^0\right)^{-1}$, one is constant, one isn't. These two functions are linearly independent. This means it's possible to choose the real $\hat{\psi}(\vec{p})$ such that the two integrals have opposite sign. $\implies j^0(0) < 0$.
If we forego the notion of a point particle then the differences to the non-relativistic case are acceptable

  – The non-locality in the schroedinger eq was at a lengthscale $\frac{\hbar}{mc} = \lambda_c$
  – On length scales much larger than that $\Delta x \gg \lambda_c = \frac{\hbar}{mc}$ we have copatability with $\Delta p \gg mc$ we have

$$\Delta \frac{1}{2p^0} \leq \frac{1}{2}\frac{1}{2}\left(\vec{p}^2 + m^2c^2\right)^{-\frac{3}{2}} 2|\vec{p}|\Delta p \leq \frac{1}{2p^0}|\Delta p| \ll 1$$

  Hence $\frac{\left(2p^0\right)^{-1}}{1} \approx const.$ and therefore the function is essentially linearly dependent. Meaning $j^0 > 0$. Meaning that the j^0 becomes positive as soon as we smear it over the compton wavelength.

Yet: the non-localities were still considered deficiencies which then lead to Dirac finding a truly local hamiltonian.
Before we look at the Dirac-eq we now give another interpretation of the KG eq:

(Anti-)particle interpretation of the KG Equation We call the positive/negative energy solution to the KG equation $\psi_\pm(x)$. The interpretation is then that $\psi_-(x)$ describes an anti-particle with energy $+E$ (we swich the sign of the energy - which seem ad hoc).
A general solution to the KG eq would then be given as $\psi(x) = \psi_+(x) + \psi_-(x)$. In the interpretation up to now specifying $\psi(0, \vec{x})$ is not enough for a unique solution, we also have to specify $\frac{\partial \psi}{\partial x^0}(0, \vec{x})$, in this interpretation however we can then uniquely identify a solution by spcifying $\psi_+(0, \vec{x})$ and $\psi_-(0, \vec{x})$.
In this new Interpretation the evolution of ψ is now given by a PDE (our first problem is solved). What about the problem of non-positive probability densities? We could take the inner product on ψ as:

1. $(\psi_+, \psi_+) + (\psi_-, \psi_-)$
   Which is positive definite, but it still has no pointwise positive density.

2. $\langle \psi, \psi \rangle = i\hbar \int_{x^0=ct} d^3x \left(\overline{\psi}\frac{\partial \psi}{\partial x^0} - \frac{\partial \overline{\psi}}{\partial x^0}\psi\right) = (\psi_+, \psi_+) - (\psi_-, \psi_-)$
   Which is even worse, as it's not even positive definite as a whole as $\psi, \frac{\partial \psi}{\partial x^0}$ can be chosen independently.

Conclusion: neither candiadte for an inner product has a strictly local quantum mechanical interpretation. The second one suggests j^0 is some sort of charge density, we'd then get $\langle \psi, \psi \rangle$ a total charge, and we get that particles and anti-particles have opposite charge.
To couple the theory to an external electromagnetic field $A^\mu(x)$ we use so called "minimal coupling": $i\hbar\partial_\mu \to i\hbar\partial_\mu - \frac{e}{c}A_\mu$, with $A^\mu = \left(\phi, \vec{A}\right)$ the four-vector potential. The KG equation then is given by

$$\left(\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\left(i\hbar\partial^\mu - \frac{e}{c}A^\mu\right) - m^2c^2\right)\psi = 0$$

And the current as

$$j_\mu = e\left(\overline{\psi}\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\psi + \psi\left(-i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\overline{\psi}\right)$$

$$\implies \partial_\mu j^\mu = 0$$

1.4 The Dirac Equation

By Dirac (1925): The Idea arose from the deficiencies of the KG equation, and was to replace the Klein-Gordan equation

$$\left(\hat{p}^2 - m^2 c^2\right)\psi = 0$$

$\hat{p} = i\hbar\partial_\mu$, by a new first order PDE in t and $\vec{x}$, in analogy to how the wave equations arrise from the free Maxwell equations $\Box\vec{E} = \Box\vec{B} = 0$. For particles of $m > 0$ we get

$$\left(\gamma^\mu p_\mu - mc\right)\psi = 0$$

With $\psi : \mathbb{R}^4 \to \mathbb{C}^n$ and $x \to \psi(x)$ called a Dirac spinor. γ^μ are then complex $n \times n$ matricies, which we find by

$$\left(\gamma^\nu \hat{p}_\nu + mc\right)\left(\gamma^\mu \hat{p}_\mu - mc\right) = \gamma^\nu \gamma^\mu \hat{p}_\nu \hat{p}_\mu - m^2 c^2$$

The Dirac equation implies the klein Gordan equation provided

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \; \forall \mu, \nu \in \{0, 1, 2, 3\}$$

i.e.

$$\left(\gamma^0\right)^2 = 1$$
$$\left(\gamma^k\right)^2 = -1$$
$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \; \forall \mu \neq \nu$$

Which is called a Dirac- or Clifford-Algebra.

Theorem: The Dirac-Algebra has exactly one irreducible representation of dimension $n = 4$. The proof of this theorem is in the exercises.
In particular, any two representations γ^μ, $\hat{\gamma}^\mu$ are equivalent, meaning

$$\hat{\gamma}^\mu = S\gamma^\mu S^{-1}$$

for all μ at once, with a complex matrix S. In any representation of the γ matricies have $tr\gamma^\mu = 0$. One special representation is the so called standard representation:

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\gamma^k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}$$

With σ_k the palul matricies.
In this special representation we have that $\gamma^{0*} = \gamma^0$ with $*$ the hermitian conjugate, and $\gamma^{k*} = -\gamma_k$. Later we'll see other representations (such as the Majorana or the chiral representation), where these relations also hold (despite them not strictly being a universal feature of all representations).
We consider plain wave solutions to the dirac equation (i.e. with fixed 4-momentum p^μ):

$$\psi(x) = u e^{-i\frac{p \cdot x}{\hbar}} \; , \; u \in \mathbb{R}^4$$

is a solution if

$$\left(p_\mu \gamma^\mu - mc\right)u = 0$$

we summarize $p_\mu \gamma^\mu$ as $\not{p}$.

$$\left(\not{p} - mc\right)u = 0$$
$$\implies \not{p}u = mcu$$

Note: $\not{p}^2 = p^2 \mathbb{I}^4$ and $tr\not{p} = 0$. Meaning that the eigenvalues of $\not{p}$ are $\pm\sqrt{p^2}$, two for each sign (to fullfill the trace). This then implies

$$\implies p \in V_m^+ \text{ or } p \in V_m^-$$

We find a two dimensional space of u. By applying Wiegners analysis we find that the dirac equation contains at least two irreducibles. Or two particles with two hilbertspaces $\mathcal{H} = \{u \in \mathbb{C}^4 | \not{p}u = mcu\}$. We find $2j + \frac{1}{2}$ and see that we describe a spin $\frac{1}{2}$ particle. To show this formally we'd need to consider in more detail how the lorenzgroup acts on these spaces.

We can use the dirac equation to rediscover the schroedinger equation. This reformulation is optained by multiplying γ^0 to the dirac equation from the left:

$$\gamma^0 \left(\gamma^\mu p_\mu - mc\right) \psi = 0$$

$$\implies H = c\gamma^0 \left(-\gamma^k p_k + mc\right) = c\vec{\alpha} \cdot \vec{p} + \beta mc^2$$

with $\alpha_k = \gamma^0 \gamma^k$ and $\vec{p} = -i\hbar\vec{\nabla}$ and $\beta = \gamma^0$. For which we have

- $\alpha_k^2 = 1$

- $\beta^2 = 1$

- $\alpha_k \alpha_l + \alpha_l \alpha_k = 0$

- $\beta \alpha_k + \alpha_k \beta = 0$

- are all Hermition: $\alpha_k^* = \left(\gamma^k\right)^* \left(\gamma^0\right)^* = -\gamma^k \gamma^0 = \gamma^0 \gamma^k = \alpha_k$

Which leads us to $H = H^*$ is self-adjoint on $L^2 \left(\mathbb{R}^3, \mathbb{C}^4\right)$ with the natural inner product

$$(\psi, \phi) = \int d^3 x \psi^* (x) \phi (x)$$

Which induces a norm which is conserved:

$$||\psi||^2 = (\psi, \psi) = const.$$

It's worth noting that, contrary to the KG case, there's no reason to prefer the forward mass shell in order to get a schroedinger type formulation.

The inner density

$$j^0 (x) = \psi^* (x) \psi (x) \geq 0$$

Which is suitable for an interpretation of a probability density. This acctually comes as a 4-current

$$j^\mu (x) = \overline{\psi} (x) \gamma^\mu \psi (x)$$

Where $\overline{\psi}$ is not just the complex conjugate, but instead the dirac conjugate $\overline{\psi} (x) = \psi (x)^* \gamma^0$.

We now make several claims

- Conjugate Dirac equation: $\overline{\psi} (x) \left(i\hbar\gamma^\mu \overleftarrow{\partial}_\mu + mc\right) = 0$, with $\overleftarrow{\partial}_\mu$ a deriviative that acts to the left.

- $\partial_\mu j^\mu = 0$

The proofs to these claims are as follows:

- We take the hermitian conjugate of the Dirac equation

$$\psi^* (x) \left(-i\hbar\gamma^{\mu *} \overleftarrow{\partial}_\mu - mc\right) = 0$$

If we now multiply with γ^0 from the right we get

$$\gamma^{\mu *} \gamma^0 = \begin{cases} \gamma^0 \gamma^0 & \forall \mu = 0 \\ -\gamma^k \gamma^0 & \forall \mu = 1, 2, 3 \end{cases} = \gamma^0 \gamma^k$$

$$\overline{\psi} (x) \left(-i\hbar\gamma^{\mu *} \overleftarrow{\partial}_\mu + mc\right)$$

-

$$\partial_\mu j^\mu = \overline{\psi}\gamma^\mu \overleftarrow{\partial}_\mu \psi + \overline{\psi}\gamma^\mu \partial_\mu \psi$$

$$i\hbar\partial_\mu j^\mu = i\hbar \left(\overline{\psi}\gamma^\mu \overleftarrow{\partial}_\mu \psi + \overline{\psi}\gamma^\mu \partial_\mu + mc (1 - 1) \overline{\psi}\psi\right)$$

$$= 0 + 0 = 0$$

A few remarks

- In the KG equations we discussed two deficiencies, one relating to the fact that it wasn't a PDE and the other one the problem of having no probabilistic interpretation. Both of these are fixed in the dirac equation.

- However there is a price to that: The energy spectrum of H is not bounded below

$$\sigma(H) = (-\infty, -mc^2], [mc^2, +\infty)$$

This can be seen from the EV equation we get

$$H\phi = E\phi \rightarrow \psi(t, \vec{x}) = \phi(\vec{x}) e^{-\frac{iEt}{\hbar}}$$

Which has the same form as

$$= u e^{-i\frac{px}{\hbar}}$$

Yielding

$$E = \pm c\sqrt{\vec{p}^2 + m^2c^2}$$

We now want to consider the physical interpretation of all this (Dirac only formulated this interpretation four years after the Dirac equation).

- States of energy $-E < 0$ is interpreted as an antiparticle of energy $+E > 0$, (as seen for the KG equation). However we have a new reason, as it's only corresponds to fermions, meaning we have the Pauli principle. In the vacuum all states of energy $-E < 0$ are all filled (Pauli principle $\rightarrow$ filled exactly once), and all states $+E > 0$ are empty - in such a interpretation we have $\epsilon = 0$. In the case where one state of energy $-E < 0$ is not filled (a hole punched into the Dirac sea), such a whole is an anti-particle, for the total energy we get

$$\epsilon = 0 \qquad \underbrace{-(-E)}_{\text{an anti particle is missing}} \qquad = +E$$

. States of positive Energy $+E > 0$ are stable because all the states in the dirac sea are already full, it can't fall indefinitelly. The creation of a particle-hole (anti-particle) then corresponds to the process of one state jumping from a negative to a positive state - the energy cost for such a process is then at least $\geq 2mc^2$.

- The Dirac Sea is stable by isotropy.

- just two years after this interpretation the anti-particle of the electron (the positron) was discovered.

- This is still not the final picture, QFT, will allow us to explain the positive energy of an antiparticle without going to an ad hoc explanation.

Lorenz Covariance The full discussion of lorenz covariance will be delayed to a later chapter. We will now just give the generators of $\mathcal{P}$.

$$P_\mu = i\hbar\partial_\mu$$
$$M_{\mu\nu} = -i\hbar(x_\mu\partial_\nu - x_\nu\partial_\mu) - \hbar\sigma_{\mu\nu}$$

with $\sigma_{\mu\nu} = \frac{1}{\hbar}[\gamma_\mu, \gamma_\nu]$. The generators of rotations are given

$$J^i = M^{i+2}_{i+1} = i\hbar(x_{i+1}\partial_{i+2} - x_{i+2}\partial_{i+1}) + \hbar\sigma_{i+1,i+2}$$

In the standard represesntation we have

$$\vec{J} = \vec{x} \wedge \vec{p} + \frac{\hbar}{2}\begin{bmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{bmatrix}$$

$$\vec{J} = \underbrace{\vec{L}}_{\text{orbital angular momentum}} + \underbrace{\vec{S}}_{\text{spin angular momentum}}$$

We find

$$\vec{S}^2 = \frac{\hbar^2}{4}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \frac{3}{4}\hbar^2\mathbb{I}$$

Which leads to $s = \frac{1}{2}$ as we would expect.

We now consider the minimal coupling to an external magnetic field $A^\mu(x)$. We get

$$\left(\gamma^\mu\left(p_\mu - \frac{e}{c}A_\mu\right) - mc\right)\psi = 0$$

This is invariant under gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\chi$, $\psi- : e^{i\frac{e\chi}{\hbar c}}\psi$. The hamiltonian is then given as

$$H = c\vec{\alpha} \cdot \underbrace{\left(\vec{p} - \frac{e}{c}\vec{A}\right)}_{\vec{\pi}} + \beta mc^2 + e\phi$$

Which is still hermitian.

Lecture 14.10 Missing

2
Non-Relativistic Quantum Electro Dynamics

We now take a step back and describe a non-relativistic theory, that non-the-less models a good deal of the world correctly. In addition it's a good "playground" to learn some of the relevant techniques and see some concepts that will later come up again in a relativistic theory.

2.1 Maxwell's Equations

Electromagnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$ that obey

$$div\vec{B} = 0$$

$$rot\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

Which are equivalent to

$$\vec{B} = rot\vec{A}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Where the potentials $\phi, \vec{A}$ are determined by $\vec{E}, \vec{B}$ up to gauge transformations

$$\phi \rightarrow \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}, \vec{A} \rightarrow \vec{A} + \vec{\nabla}\chi$$

With $\chi(\vec{x}, t)$ an arbitrary generating function of the gauge transformation.
$\phi, \vec{A}$ are almost unique given $\vec{E}, \vec{B}$ by imposing a gauge condition.
The inhomogenious Maxwell's equations are given by

$$div\vec{E} = \rho$$

$$rot\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{\vec{j}}{c}$$

with ρ the charge density and $\vec{j}$ the current density.
This leads te the integrability condition of the continuity equation

$$\frac{\partial \rho}{\partial t} + div\vec{j} = 0$$

Or in the converse we can ask: how many of the inhomogenious equations can we recover from the continuity equation? What we find is that we cannot retrieve the Gauss equation from the continuity eq and the ampere maxwell equation.

$$\frac{\partial}{\partial t}\left(div\vec{E} - \rho\right) = div\left(c \cdot rot\vec{B} - \vec{j}\right) - \frac{\partial \rho}{\partial t} = 0$$

Using the homogenious maxwell equations we can write, using the dalambertian operator ($\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$)

$$\Box\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t} - div\vec{A}\right) = 0$$

$$\Box\vec{A} + \vec{\nabla}\left(\frac{1}{c}\frac{\partial\phi}{\partial t} + div\vec{A}\right) = \frac{\vec{j}}{c}$$

A few remarks

- The solution should be determined by ϕ and $\partial_t\phi|_{t=0}$ and $\vec{A}$ and $\partial_t\vec{A}|_{t=0}$. This however is not the case because of the gauge freedom. If we choose $\chi(t,\vec{x}) = 0$ near $t = 0$, we get two solutions that start identically but later diverge.

- Looking at the second inhomogenious Maxwell Eq can be regarded as a eq. of motion of $\vec{A}$, but the first inhomogenious equation is not an eq. of motion for ϕ, as $\frac{\partial^2\phi}{\partial t^2}$ drops out! Recall: the gauss law is a propagating constraint, not itself a propagator.

To impose unique solutions we have to impose a particular gauge.

2.1.1 Coulomb gauge

The Coulomb gauge is given by

$$div\vec{A} = 0$$

Which still allows a limited set of residual gauge transformations $\Delta\chi = 0$. The inhomogeneous Maxwell Eqs are now given

$$-\Delta\phi = \rho$$

$$\Box\vec{A} + \vec{\nabla}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{\vec{j}}{c}$$

The general solution is then given as

$$\varphi(\vec{x},t) = -\frac{1}{4\pi}\int d^3y\,\frac{\rho(\vec{y},t)}{|\vec{x}-\vec{y}|} + \hat{\phi}(\vec{x},t)$$

with $\Delta\hat{\varphi} = 0$ which can be set to 0 without loss of generality by choosing $\chi = -\int^t \hat{\varphi}dt'$. And we found a unique solution determined by $\vec{A}$ and $\partial_t\vec{A}|_{t=0}$

2.2 The free EM field as a Hamiltonian System

Let $\rho = 0$, $\vec{j} = 0 \implies \varphi = 0$. Such that we get

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \Delta\vec{A}$$

and

$$\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{E}$$

hence

$$-\frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \Delta\vec{A}$$

We now Claim that these two are the canonical equations of motion of a hamiltonian system.
We recall

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha} = \{H,q^\alpha\}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha} = \{H,p_\alpha\}$$

With $\{f, g\}$ the poisson bracket.

We now need to find the phase space Γ of infinite dimension given as

$$\Gamma = \{\left(\vec{A}\left(\vec{x}\right), -\vec{E}\left(\vec{x}\right)\right)_{\vec{x}\in\mathbb{R}^3} | div\vec{A} = 0, div\vec{E} = 0\}$$

Our phase space thus consists of pairs of functions in $\mathbb{R}^3$, the partial deriviatives in the poission bracket then gets replaced by functional deriviatives. We first consider the scalar function $\varphi\left(\vec{x}\right)$ and consider the functional $F = F[\varphi]$ then the functional deriviative is given as

$$\int d^3x \frac{\delta F}{\delta\varphi\left(\vec{x}\right)} \cdot \psi\left(\vec{x}\right) = \frac{d}{d\lambda}F[\phi + \lambda\psi]_{\lambda=0}$$

How will the functional deriviative look for a vector field such as $\vec{E}$. If we just copy the definition for scalar functions and draw vector arrows we get the wrong result. We need some test function $\vec{v}$ that needs to be divergence free $div\vec{v} = 0$. This then imlpies $rot\vec{w} = \vec{v}$. The definition is then completed by requiring

$$div\frac{\delta F}{\delta\vec{A}} = 0$$

If we consider a general vector field $\vec{u}$, it is determined by $rot\vec{u}$ and $div\vec{u}$; since $\Delta\vec{u} = \vec{\nabla}div\vec{u} - rotrot\vec{u}$ and $\vec{u} = -\frac{1}{4\pi}\int d^3y \frac{(\Delta\vec{u})(\vec{y})}{|\vec{x}-\vec{y}|}$.

Example for $div\vec{v} = 0$.

$$\int d^3x' \frac{\delta A_i\left(\vec{x}\right)}{\delta\vec{A}\left(\vec{x'}\right)}\vec{v}\left(\vec{x'}\right) = \frac{d}{d\lambda}\left(A_i\left(\vec{x}\right) + \lambda v_i\left(\vec{x}\right)\right)|_{\lambda=0} = v_i\left(\vec{x}\right)$$

$$\frac{\delta A_i\left(\vec{x}\right)}{\delta A_j\left(\vec{x'}\right)} = \left(\delta_{ij}, -\partial_i\partial_j\Delta^{-1}\right)$$

The poisson bracket betwen functions is then given as

$$\{F, G\} = c\int d^3x \left(\frac{\delta F}{\delta\vec{A}\left(\vec{x}\right)} \cdot \frac{\delta G}{\delta\vec{E}\left(\vec{x}\right)} - \frac{\delta F}{\delta\vec{E}\left(\vec{x}\right)}\frac{\delta G}{\delta\vec{A}\left(\vec{x}\right)}\right)$$

In particulare we get

$$\{F, \vec{A}\left(\vec{x}\right)\} = -c\frac{\delta F}{\delta\vec{E}\left(\vec{x}\right)}$$

$$\{F, \vec{E}\left(\vec{x}\right)\} = c\frac{\delta F}{\delta\vec{A}\left(\vec{x}\right)}$$

Which then yields:

- $$-\{\int\vec{E}\left(\vec{x}\right)\vec{u}\left(\vec{x}\right)d^3x, \int\vec{A}\left(\vec{x}\right)\vec{v}\left(\vec{x}\right)d^3x\} = c\int\vec{u}\cdot\vec{v}d^3x$$

- $$-\{E_i\left(\vec{x}\right), A_j\left(\vec{y}\right)\} = c\left(\delta_i j - \partial_i\partial_j\Delta^{-1}\right)\delta\left(\vec{x}-\vec{y}\right)$$

- $$\{E_i\left(\vec{x}\right), E_j\left(\vec{x}\right)\} = 0$$

- $$\{A_i\left(\vec{x}\right), A_j\left(\vec{x}\right)\} = 0$$

2.2.1 Hamiltonian

With this we can define the hamiltonian functon on the phase space Γ, which is just the energy

$$H = \frac{1}{2} \int \left(\vec{E}^2 \left(\vec{x} \right) + \vec{B}^2 \left(\vec{x} \right) \right) d^3 x$$

With $\vec{B} = rot \vec{A}$ which then yields:

$$H = \frac{1}{2} \int \left(\vec{E}^2 \left(\vec{x} \right) - \vec{A} \left(\vec{x} \right) \cdot \Delta \vec{A} \left(\vec{x} \right) \right) d^3 x$$

We can now see that the canonical equations of motion are given by

$$\frac{\partial \vec{A}}{\partial t} = \{ H, \vec{A} \left(\vec{x} \right) \}$$
$$= -c \frac{\delta H}{\delta \vec{E} \left(\vec{x} \right)}$$
$$= -c \vec{E} \left(\vec{x}, t \right)$$
$$\frac{\partial \vec{E}}{\partial t} = \{ H, \vec{E} \left(\vec{x} \right) \}$$
$$= +c \frac{\delta H}{\delta \vec{A} \left(\vec{x} \right)}$$
$$= -c \Delta \vec{A} \left(\vec{x}, t \right)$$

We now want to get rid of the constraints that $div \vec{A} = 0 = div \vec{E}$. To archieve this we apply a spacial fourier transformation

$$\vec{A} \left(\vec{x} \right) = c \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \vec{q} \left(k \right) e^{i \vec{k} \vec{x}}$$

$$\vec{E} \left(\vec{x} \right) = \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \vec{p} \left(k \right) e^{i \vec{k} \vec{x}}$$

from which we get $\vec{q} \left(-\vec{k} \right) = \overline{\vec{q} \left(\vec{k} \right)}$.

To sattisfy the conditions we have $\vec{k} \vec{q} \left(\vec{k} \right) = 0$, and $\vec{B} \left(\vec{x} \right) = rot \vec{A} = c \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \left(i \vec{k} \times \vec{q} \left(\vec{k} \right) \right) e^{u \vec{k} \vec{x}}$.

The poission brackets are then

$$\{ \int \vec{p} \vec{u} d^3 k, \int \vec{q} \vec{v} d^3 k \} = \int \overline{\vec{u}} \left(k \right) \vec{v} \left(k \right) d^3 k$$

and all other poisson brackets are just 0.

The Hamiltonian in the momemtum representation is then given as

$$H = \frac{1}{2} \int d^3 k \left(\left| \vec{p} \left(\vec{k} \right) \right|^2 + \left(c \vec{k} \right)^2 \left(\vec{q} \vec{k} \right)^2 \right)$$

The energy of the EM field has then been described as a collection of harmonic oscilators each of the form

$$H_\alpha = \frac{1}{2} \left(p_\alpha^2 + \omega_\alpha^2 \left(q^\alpha \right)^2 \right)$$

but the modes at $\vec{k}$ and $-\vec{k}$ are not independent (which reflects the fact that we've taken a real field but have represented in complex modes).

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2.3   Canonical Quantization Missing

2.4   Non-relativistic QED Missing

2.5   Perturbation Theory and Thomson Scattering Missing

Lecture 28.10. Missing

2.6   Lamb Shift

For a free particle and in the dipole approximation we found

$$\delta E\left(\vec{p}\right) = -\frac{1}{6\pi^2}\left(\frac{e}{mc}\right)^2 \vec{p}^2 \int_0^\infty dk$$

We can interpret that as the energy now being

$$E = \frac{\vec{p}^2}{2m} + \delta E\left(\vec{p}\right) = \frac{\vec{p}^2}{2m_{\text{physical}}}$$

with $m_{\text{physical}} > m_{\text{bare}}$.

2.6.1   Bound state

We now want to consider a bound state $|\alpha\rangle$ of the H-atom. We now have the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{4\pi r}$$

With $|\alpha\rangle$ picked from the ONB $|\beta\rangle$ of the bound, or not, eigenstates.
Then

$$\delta E\left(\alpha\right) = -\left(2\pi\right)^3\left(\frac{e}{mc}\right)^2 \hbar c \sum_\beta \int \frac{d^3k}{2\left|\vec{k}\right|} \frac{\sum_\lambda \left|\vec{\epsilon}_\lambda\left(\vec{k}\right)\langle\beta|\,e^{-i\vec{k}\vec{x}}\vec{p}\,|\alpha\rangle\right|^2}{E_\beta + \hbar c\left|\vec{k}\right| - E_\alpha}$$

Remarks:

- Dipole approximation: $e^{-i\vec{k}\vec{x}} \to 1$

- Denominator is rotationally invariant in $\vec{k}$
  $\implies$ average on numerator
  $$\delta E\left(\alpha\right) - \frac{1}{6\pi^2}\left(\frac{e}{mc}\right)^2 \sum_\beta |\langle\beta|\,\vec{p}\,|\alpha\rangle|^2 \int_0^\infty dk \frac{\hbar k}{E_\beta + \hbar k - E_\alpha}$$

- $|\alpha\rangle$ has a wave function $\Psi_\alpha\left(\vec{x}\right)$ spreading ovec the bohrradius a_0. It's Fourier conjugation $\hat{\Psi}_\alpha\left(\vec{p}\right)$ then spreads over $\frac{\hbar}{a_0}$

- The Continium states $|\beta\rangle$ go over into plane waves $|\vec{q}\rangle$ at high energies.

- If we forego the dipole approximation: Both integrals $(\vec{p}, |\beta\rangle$) have the same log divergence at $k \geq \lambda_c^{-1}$

We now renormalize. The correct shift is neither $\delta E\left(\alpha\right)$ nor $\delta E\left(\vec{p}\right)$, instead it's the difference

$$\Delta E\left(\alpha\right) = \delta E\left(\alpha\right) - \langle\alpha|\,\delta E\left(\vec{p}\right)|\alpha\rangle$$

Because:

Informally

$$\frac{\vec{p}^2}{2m} \to \delta E\left(\vec{p}\right)$$
$$\frac{\vec{p}^2}{2m} - \frac{e^2}{4\pi r} \to \delta E\left(\alpha\right)$$

In the atomic case we already include the correction of the free particle! We therefore "over correct" and need to substract

$$\int d^3p|\langle\vec{p}|\alpha\rangle|^2\delta E\left(\vec{p}\right) = \langle\alpha|\,\delta E\left(\vec{p}\right)|\alpha\rangle$$

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Formally With the field hamiltonian $H_{\mathrm{f}} = \int d^3k \sum_\lambda \hbar c \left|\vec{k}\right| a_\lambda^* \left(\vec{k}\right) a_\lambda(\vec{k})$ we have

$$H_{\mathrm{tot}} = H + H_{\mathrm{f}} + H_{\mathrm{I}}$$

$$= \underbrace{\left(\frac{\vec{p}^2}{2m_{\mathrm{phys}}} - \frac{e^2}{4\pi r} + H_{\mathrm{f}}\right)}_{H_0'} + \underbrace{H_{\mathrm{I}} + \left(\frac{1}{2m} - \frac{1}{2m_{\mathrm{phys}}}\right)\vec{p}^2}_{H_I'}$$

$$= H_0' + H_{\mathrm{I}}'$$

With

$$H_{\mathrm{I}}' = H_{\mathrm{I}}' - \delta E\left(\vec{p}\right)$$

resulting in

$$\Delta E\left(\alpha\right) = \delta E\left(\alpha\right) - \langle\alpha| E\left(\vec{p}\right)|\alpha\rangle$$

Then we get

- The log divergencies cancel each other

- The linear divergencies gets "demoted" to a log divergence, with a natural cutoff at λ_c^{-1} (with an uncertainty factor of ≈ 2).

- Dipole aprroximation can be used

Using $\langle\alpha| \vec{p}^2 |\alpha\rangle = \sum_\beta |\langle\beta| \vec{p}|\alpha\rangle|^2$ we get

$$-\frac{E_\beta - E_\alpha}{E_\beta + \hbar k - E_\alpha}$$

$$\implies \Delta E\left(\alpha\right) = \frac{1}{6\pi^2}\left(\frac{e}{mc}\right)^2 \sum_\beta |\langle\beta| \vec{p}|\alpha\rangle|^2 \left(E_\beta\ E_\alpha\right) \int_0^{\lambda_c^{-1}} \frac{dk}{E_\beta + \hbar k - E_\alpha}$$

Remarks

- The matrix element $\langle\beta| \vec{p}|\alpha\rangle$ mediates between state with differing energies $|E_\beta - E_\alpha| \leq R_y = \frac{1}{2}\alpha^2 mc^2 \ll mc^2$. We can estimate the order of magnitude as $\langle\beta| [H, \vec{p}] |\alpha\rangle = (E_\beta - E_\alpha) \langle\beta| \vec{p}|\alpha\rangle \approx (E_\beta - E_\alpha) \frac{\hbar}{a_0} \approx \frac{R_y}{a_0}$. Therefore $\langle\beta| \vec{p}|\alpha\rangle \approx \frac{\hbar}{a_0} \frac{R_y}{|E_\beta - E\alpha|}$

- The denominator in our $\Delta E\left(\alpha\right)$ can vanish if $|\alpha\rangle$ is not the ground state. $\implies$ compute the integral as a principal value which then yields $\frac{1}{\hbar c} \log \left|\frac{E_\beta - E_\alpha + \hbar c\lambda_c^{-1}}{E_\beta - E_\alpha}\right| \approx \frac{1}{\hbar c} \log \frac{mc^2}{|E_\beta - E_\alpha|}$.

Thus

$$\Delta E\left(\alpha\right) = \frac{1}{6\pi^2}\left(\frac{e}{mc}\right)^2 \frac{1}{\hbar c} \sum_\beta |\langle\beta| \vec{p}|\alpha\rangle|^2 \left(E_\beta - E_\alpha\right) \log \frac{mc^2}{|E_\beta - E_\alpha|}$$

- Any reference to the field has vanished and we are left with an expression that only depends on the atom.

- Without the the log factor we can solve the sum with the sum rule:
  for $\alpha = (n, l)$ we have

$$C_\alpha = \sum_n |\langle\beta| \vec{p}|\alpha\rangle|^2 \left(E_\beta - E_\alpha\right) = \frac{e^2 \hbar^2}{2\pi a_0^3 m^2}\delta_{l,0}$$

The delta function showing that only s waves matter, it also shows that the sum is dominated by C_α.

- We can split the logarithm to $\log \frac{mc^2}{|E_\beta - E_\alpha|} = \log \frac{mc^2}{R_y} - \log \frac{|E_\beta - E_\alpha|}{R_y}$ This allows us to define the bette logarithm $\log \langle E - E_\alpha\rangle$ as

$$\text{missing}$$

$$\implies \Delta E\left(\alpha\right) = C_\alpha' \left(\log \frac{mc^2}{R_y} - \log \frac{\langle E - E_\alpha\rangle}{R_y}\right)$$

With $C_\alpha' = C_\alpha \frac{1}{6\pi^2}\left(\frac{e}{mc}\right)^2 \frac{1}{\hbar c} = \frac{8}{3\pi}\frac{\alpha^3}{n^3}\delta_{l,0} R_y$

- The larger of the two terms gives

$$\frac{mc^2}{R_y} = 2\alpha^{-2}, \ \log 2\alpha^{-2} \approx 10.53$$

- and the smaller term

$$\frac{\langle E - E_\alpha \rangle}{R_y} = 17.8, \ \log\ldots = 2.88$$

Which finally yields

$$\Delta E\left(\alpha\right) = \frac{8}{3\pi}\frac{\alpha^3}{n^3}\delta_{l,0}R_y \log \frac{mc^2}{\langle E - E_\alpha \rangle}$$

With it's application

$$E\left(2S_{\frac{1}{2}}\right) - E\left(2p_{\frac{1}{2}}\right) = 1040 \ \text{MHz}$$

Which compares really well with the experimental value of 1057 MHz.

3
Classical Field Theory

3.1 Remembering classical Mechanics

- Euler-Lagrange Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$

are equivalent to

$$\frac{d}{dt}\langle p, \delta q\rangle = \delta L \ , \ \ \langle p, \delta q\rangle = \sum_\alpha p_\alpha \delta q^\alpha$$

Proof

$$\text{LHS} = \sum_\alpha \dot{p}_\alpha \delta q^\alpha + p_\alpha \delta \dot{q}^\alpha$$

$$\text{RHS} = \sum_\alpha \frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L}{\partial \dot{q}_\alpha}\delta \dot{q}^\alpha$$

We only need to have $\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}$ which is true iff $q(t)$ is a motion.

- Claim: The variation of the action $\delta S = \left(\langle p, \Delta q^i\rangle - (\langle p, \dot{q}\rangle - L)\,\Delta t^i\right)^{i=1,2}$ Proof

$$S[q_\lambda] = \int_{t_1}^{t_2} L\left(q\left(t, \lambda\right), \dot{q}\left(t, \lambda\right)\right) dt$$

$$\frac{d}{d\lambda}S[q_\lambda] = L\Delta t^i |_1^2 + \int_{t_1}^{t_2} \underbrace{\delta L}_{\frac{d}{dt}\langle p, \delta q\rangle}\ dt$$

$$= \left(\langle p, \delta q\rangle + L\Delta t^i\right)_1^2$$

$$= \left(\langle p, \Delta q^i - \dot{q}\Delta t^i\rangle + L\Delta t^i\right)_1^2$$

$$= \left(\langle p, \Delta q^i\rangle - (\langle p, \dot{q}\rangle - L)^i\right)_1^2$$

.

Extended Configuration Space $(q, t) \in \mathbb{R}^f \times \mathbb{R}$.
We can define a flow ψ^λ

$$(q, t) \to \psi^\lambda\left(q, t\right) = \left(\phi^\lambda\left(q, t\right), \tau^\lambda\left(q, t\right)\right)$$

and we can look at the generating vector fields

$$\frac{d\psi^\lambda}{d\lambda}|_{\lambda=0} = \left(v\left(q, t\right), s\left(q, t\right)\right)$$

Which of course also works with curves $q\left(t\right)$, with start/end point $t^1 < t < t^2$, which is then mapped to $q^\lambda\left(t\right)$ as

$$q^\lambda\left(\tau^\lambda\left(q, t\right)\right) = \phi^\lambda\left(q\left(t\right), t\right)$$

The end points are maped to

$$\phi^\lambda\left(q\left(t^1\right), t^1\right)\ , \ \tau^\lambda\left(q\left(t^1\right), t^1\right)$$

$$\phi^\lambda \left(q\left(t^2\right), t^2\right) \;,\; \tau^\lambda \left(q\left(t^2\right), t^2\right)$$

With variations

$$\Delta q^i = v\left(q\left(t^i\right), t^i\right) \;,\; \Delta t^i = s\left(q\left(t^i\right), t^i\right)$$

We define a continious *symmetry* of the action S as a flow ψ^λ such that

$$S[q^\lambda] = S[q] + F(q^i, \lambda)|_1^2$$

for some function $F\left(q^i, \lambda\right)$ and for all curves.

This choice is motivated by

$$\delta S[q^\lambda] = \delta S[q] + \underbrace{\delta F\left(q^i, \lambda\right)|_1^2}_{=0}$$

In particular if q is a motion, meaning $\delta S[q]$, so is q^λ.

We now want to consider $q\left(t, \lambda\right) := q^\lambda\left(t\right)$ (a variation of $q\left(t\right)$, but one without fixed endpoints). This can be done in two different ways

1. by using the variation formula:

$$\delta S = \left(\langle p, v\rangle - \left(\langle p, \dot q\rangle - L\right) S\right)_1^2$$

2. by the definition of a symmetry:

$$\delta S = \delta F|_1^2$$

with $\delta F = \frac{\partial}{\partial \lambda} F\left(q\left(t\right), \lambda\right)$.

Therefor we get

$$\langle p, v\rangle - \left(\langle p, \dot q\rangle - L\right) S - \delta F = \text{const.}$$

at every point of the curve. In other words $\langle p, v\rangle - \left(\langle p, \dot q\rangle - L\right) S - \delta F$ is a conserved quantiy for a motion $q\left(t\right)$. This is called *Noether Theorem*.

We consider a few special cases

- Symmetry of $\phi^\lambda\left(q\right)$ of the lagrangian

$$L\left(q^\lambda\right) = L\left(q, \dot q\right)$$

with $q^\lambda = \phi^\lambda\left(q\left(t\right)\right)$ leading to an $v\left(q, t\right) = v\left(q\right)$, and $\tau^\lambda\left(q, t\right) = t$ (meaning $s = 0$). Leading to $F = 0$.

All that is left of Noethers theorem is then that

$$\langle p, v\rangle = \text{const.}$$

- Conservation of Energy:
For $\phi^\lambda\left(q\right) = q \implies v = 0$, $\tau^\lambda\left(q, t\right) = t - \lambda \implies s = -1$ and $F = 0$ the symmetry condition is met. Leading to $<p, \dot q> - L = E$ is conserved.

3.2 Classical Fields

A classical field associates to any space-time point $(t, \vec x) \in \mathbb{R}^4$ a value $\varphi = (\varphi^\alpha)_{\alpha \in I}$ in field space F (eg. $F = \mathbb{R}, \mathbb{C}$ some scalar field, or $F = V$ some linear field (vector field), a manifold).

A classical field theory is therefore just the classical mechanics of a system of ∞-many DOF. By analogy we got

$$q^\alpha(t) \to \varphi^\alpha(t, \vec x)$$

Where we have two different possibilities of reading this

- $t \to (t, \vec x)$, $\alpha \to \alpha$

- $t \to t$, $\alpha \to (\vec x, \alpha)$

It turns out that the first interpretation is particularly suited for a Lagrangian approach and makes Lornz covariance easy; while the second approach is suited for the Hamiltonian approach and canonical quantization.

We now look at the Lagrangian

$$L\left[\varphi, \partial\varphi\right] = \int d^3 x \mathcal{L}\left(\varphi^\alpha(t, \vec x), \partial\varphi^\alpha(t, \vec x)\right)$$

where $\partial\varphi^\alpha = (\partial_0\varphi^\alpha, \ldots, \partial_3\varphi^\alpha)$ and $\mathcal{L}$ the Lagrangian density. The Action is then is then given as

$$S\left[\varphi\right] = \int dt L = \int d^4x \mathcal{L}\left(\varphi^\alpha, \partial\varphi^\alpha\right)$$

Where we have Lorentz invariance as $x \to x'$ with $d^4x = d^4x'$ and $\varphi(x) \to \varphi'(x')$. Meaning Lorenz invariance means

$$\mathcal{L}\left(\varphi'(x'), \partial'\varphi(x')\right) = \mathcal{L}\left(\varphi(x), \partial\varphi(x)\right)$$

For the principle of least action we get

$$\delta S\left[\varphi\right] = 0$$

for any variation $S\varphi^\alpha(x)$ on G such that $\delta\varphi^\alpha|_{\partial G} = 0$ (ie. variation with fixed boundary values). The principle of least action is Lorentz invariant, if $\mathcal{L}$ is. From the principle of least action follow the Eurler-Lagrange equations, which are now given (in terms of the Lagrangian density)

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)} - \frac{\partial\mathcal{L}}{\partial\varphi^\alpha} = 0$$

We proof this:

$$\delta S = \int_G d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi^\alpha}\delta\varphi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)} \underbrace{\delta(\partial_\mu\varphi^\alpha)}_{\partial_\mu(\delta\varphi^\alpha)} \right)$$

$$= \int_{\partial G} d\sigma_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha - \int_G d^4x \left(\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)} - \frac{\partial\mathcal{L}}{\partial\varphi^\alpha} \right)\delta\varphi^\alpha$$

We have the canonical momenta

$$\Pi^\mu_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\left(\varphi, \partial\varphi\right)$$

The variation with varying domain G, not vanishing on ∂G (but variation around a motion). A point $x \in \partial G \to y = y(x, \lambda) \in \partial G_\lambda$, $\delta v^\mu = \frac{\partial y^\mu}{\partial\lambda}|_{\lambda=0}$, with a varied action $S\left[\varphi_\lambda\right] = \int d^4x \mathcal{L}\left(\varphi_\lambda, \partial\varphi_\lambda\right)$. We claim

$$\delta S = \int_{\partial G} d\sigma_\mu \left(\mathcal{L}\delta v^\mu + \Pi^\mu_\alpha\delta\varphi^\alpha\right)$$

Examples

- φ real scalar field with

$$\mathcal{L}\left(\varphi, \partial\varphi\right) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V\left(\varphi\right)$$
$$= \frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V\left(\varphi\right)$$

Leading to

$$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{1}{2}2\partial^\mu\varphi = \partial^\mu\varphi$$
$$\frac{\partial\mathcal{L}}{\partial\varphi} = V'\left(\varphi\right)$$

The Euler-Lagrange equations are then given as

$$\underbrace{\partial_\mu\partial^\mu}_{\Box}\varphi + V'\left(\varphi\right) = 0$$

Which for $V = \frac{1}{2}m^2\varphi^2$ yields

$$\Box\varphi + m^2\varphi = 0$$

which is the Klein-Gordan equation (but here it's for a classical, real field).

- We now generalize to N components $\varphi = (\varphi_i)_{i=1}^N$ yielding

$$\mathcal{L} = \frac{1}{2}\left(\sum_{i=1}^N \partial_\mu \varphi_i \partial^\mu \varphi_i - m^2 \sum_i \varphi_i^2\right)$$

which yields a KG equation for each component φ_i.

- for $N = 2$ we can read it as a complex field as

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}} \in \mathbb{C}$$

Then we get

$$\mathcal{L} = \partial^\mu \overline{\varphi} \partial_\mu \varphi - m^2 \overline{\varphi}\varphi$$

Using $(f = f(\varphi))$

$$df = \frac{\partial f}{\partial \varphi_1}d\varphi_1 + \frac{\partial f}{\partial \varphi_2}d\varphi_2 = \frac{1}{\sqrt{2}}\left(\frac{\partial f}{\partial \varphi_1} - \frac{\partial f}{\partial \varphi_2}\right)\frac{d\varphi_1 + id\varphi_2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{\partial f}{\partial \varphi_1} + i\frac{\partial f}{\partial \varphi_2}\right)\frac{d\varphi_1 - id\varphi_2}{\sqrt{2}}$$

Which we rewrite as

$$df = \frac{\partial f}{\partial \varphi}d\varphi + \frac{\partial f}{\partial \overline{\varphi}}d\overline{\varphi}$$

Despite φ and $\overline{\varphi}$ technically not being independet, we can treat them as such! We get

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(d_\mu \varphi)} = \partial^\mu \overline{\varphi}$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \overline{\varphi}$$

and

$$\overline{\Pi}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\varphi})} = \partial^\mu \varphi$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\varphi}} = -m^2 \varphi$$

giving us the EL equations

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$$
$$\partial_\mu \partial^\mu \overline{\varphi} + m^2 \overline{\varphi} = 0$$

which are the KG equation and it's complex conjugate.

- Minimal coupling to external EM field

$$\mathcal{L}(\varphi, \partial\varphi, x) = (-i\partial^\mu - eA^\mu)\overline{\varphi}(i\partial_\mu - eA_\mu)\varphi - m^2\overline{\varphi}\varphi$$

Note: there's an explicit x dependence in the term $A^\mu(x)$. For the canonical momenta we find

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = (\partial^\mu - ieA^\mu)\overline{\varphi}$$

$$\overline{\pi}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\varphi})} = (\partial^\mu + ieA^\mu)\varphi$$

and

$$\frac{\partial \mathcal{L}}{\partial \overline{\varphi}} = -eA_\mu(i\partial_\mu - eA_\mu)\varphi - m^2\varphi$$

which then leads to the EL eqs

$$(-\partial^\mu - eA^\mu)(i\partial_\mu - eA_\mu)\varphi + m^2\varphi = 0$$

and it's complex conjugate for $\overline{\varphi}$.

Note: the miminmal coupling does not work for the real scalar field (as the Lagrangian would not be real).

- EM Field coupeled to external sources $j^\mu = j^\mu(x)$.

$$\mathcal{L}(A, \partial A, x) = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - j_\alpha A^\alpha$$

with

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$\pi_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} = -\frac{1}{4} 4 F_{\mu\nu} = F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -j_\nu$$

Leading to EL equ.

$$\partial^\mu \pi_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0$$

$$\implies -F_{\mu\nu}^{;\mu} + j_\nu = 0$$

$\implies j^\nu_{,\nu} = 0$ or else S has no stationary points.

3.3   Symmetries and conservation laws

$\Psi^\lambda[\varphi]$ is a flow on configuration $\varphi = (\varphi^\alpha(x))_{x \in \mathbb{R}^4}$, of the following (local) form:

$$\implies \Psi^\lambda[\varphi](x) = T_\lambda(\varphi(t_{-\lambda} x))$$

With t_λ a flow in $\mathbb{R}^4$, T_λ a flow in field space F. We define

$$v = \frac{dt_\lambda}{d\lambda}\Big|_{\lambda=0}$$

$$V = \frac{dT_\lambda}{d\lambda}\Big|_{\lambda=0}$$

Then $\varphi_\lambda = \Psi^\lambda[\varphi]$ has variation

$$(\delta\varphi)^\alpha(x) = (V\varphi(x))^\alpha - v^\nu{}_\nu \varphi^\alpha(x)$$

Which is a Lie-deriviative.
We call Ψ^λ a symmetry of the action if

$$S_{t_\lambda[G]}\left[\Psi^\lambda[\varphi]\right] = S_G[\varphi] + \int_{\partial G} d\sigma_\mu F^\mu(\varphi(x), \lambda)$$

for some field $F^\mu = F^\mu(\varphi, \lambda)$, if φ is a motion, so is $\Psi^\lambda[\varphi]$. We now draw two conclusions

1. $\delta S = \int_{\partial G} d\sigma_\mu \delta F^\mu$, with $\delta F^\mu = \frac{\partial F^\mu}{\partial \lambda}\big|_{\lambda=0}$.

2. $\delta S = \int_{\partial G} d\sigma_\mu \left(\mathcal{L} \underbrace{\delta v^\mu}_{=v^\mu} + \pi^\mu_\alpha \delta\varphi^\alpha \right)$

Hence we get

$$\int_{\partial G} (\mathcal{L} v^\mu + \pi^\mu_\alpha \delta\varphi^\alpha - \delta F^\mu) = 0$$

Since G is arbitrary we can instead say that

$$\partial_\mu (\mathcal{L} v^\mu + \pi^\mu_\alpha d\varphi^\alpha - \delta F^\mu) = 0$$

$$\partial_\mu J^\mu = 0$$

with J^μ the Noether current whitch leads to a conserved Noether charge

$$Q = \int d^3x J^0\left(t, \vec{x}\right)$$

We can restate the Noether current as

$$J^\mu = \pi_\alpha^\mu \left(V\varphi\right)^\alpha - \underbrace{\left(\pi_\alpha^\mu \partial_\nu \varphi^\alpha - \mathcal{L}\delta_\nu^\mu\right)}_{T_\nu^\mu \text{ the Energy momentum tensor}} v^\nu - \delta F^\mu$$

Remarks:

- J^μ looks like a 4-vector, but this in not quite true as it might carry additional indices "hidden" in v.

- We can do a replacement
$$J^\mu \to J^\mu + \partial_\lambda f^{\lambda\mu}$$
with $f^{\lambda\mu} = -f^{\mu\lambda}$. Such a replacement does not effect $\partial_\mu J^\mu$ since $\partial_\mu \partial_\lambda f^{\lambda\mu} = 0$. The contribution to the Noether charge $\int d^3x \partial_\lambda f^{\lambda 0} = \sum_{i=1}^3 \int d^3x \partial_i f^{i0} = 0$ (if $f^{\lambda\mu}$ decays sufficiently fast).

- Likewise we can rewrite the energy-momentum tensor T_α^μ for $v = v(x)$ constant as
$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu} \ , \ f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$$
Which changes J^μ by $\left(\partial_\lambda f^{\lambda\mu\nu}\right) v_\nu = \partial_\lambda f^{\lambda\mu}$.

- A symmetry of the lagrangian density means
$$\mathcal{L}\left(\Psi^\lambda\left[\varphi\right], \partial\Psi^\lambda\left[\varphi\right]\right)|_{t_\lambda x} = \mathcal{L}\left(\varphi, \partial\varphi\right)|_x + \partial_\mu F^\mu\left(\varphi, \lambda\right)|_x$$
with t_λ volume preserving.
(just a special case of the above sometimes more usefull in applications).

A for the examlpes below we look at symmetries that fit the special case just introduced, sometimes even with $F = 0$ (we say the lagrangian $\mathcal{L}$ is invariant).

Special cases

Internal Symmetries: $t_\lambda = \mathbb{I}, \to v = 0$. Then
$$J^\mu = \pi_\alpha^\mu \left(V\varphi\right)^\alpha - \delta F^\mu$$

Examples:

- Complex scalar field coupled to external EM field (see above), where $\mathcal{L}$ is invariant under *global gauge transformations* $\varphi \to e^{-i\lambda e}\varphi, \overline{\varphi} \to \overline{\varphi}e^{i\lambda e} \implies V\varphi = -ie\varphi$; giving us the Noether current
$$J^\mu = e\left[\overline{\varphi}\left(i\partial^\mu - eA^\mu\right)\varphi + \varphi\left(-i\partial^\mu - eA^\mu\right)\overline{\varphi}\right]$$
$$\partial_\mu J^\mu = 0$$
Yielding the KG equation.

- EM field A^μ coupled to external source (likewise see above). As a symmetry we take *local gauge transformations*:
$$A_\mu \to A_\mu + \lambda\partial_\mu\chi$$
with $\chi(x)$ arbitriry but real. Under such a transformation the Lagrangian is not invariant, but with change in the $-j_\alpha A^\alpha$ term, by
$$-\lambda\left(\partial_\mu\chi\right)j^\mu = -\lambda\partial_\mu\left(\chi j^\mu\right)$$
i.e. $F^\mu = -\lambda\chi j^\mu$, $\delta F^\mu = -\chi j^\mu$. By
$$(VA)_\alpha = \partial_\alpha\chi$$
$$\implies J^\mu = -F^{\mu\alpha}\partial_\alpha\chi + \chi j^\mu$$
$$\to \partial_\mu J^\mu = -\underbrace{F^{\mu\alpha}\partial_\mu\partial_\alpha\chi}_{=0} - \left(F_{,\mu}^{\mu\alpha} - j^\alpha\right)\partial_\alpha\chi + \underbrace{\chi j_{,\mu}^\mu}_{0}$$
$$\implies \partial_\mu J^\mu = -\left(F_{,\mu}^{\mu\alpha} - j^\alpha\right)\partial_\alpha\chi$$
Which is 0 if and only if $F_{,\mu}^{\mu\alpha} = j^\alpha$ which implies the maxwell equations.

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- The previous tow coupeled, no external fields or sourses:

$$\mathcal{L} = \mathcal{L}\left(\varphi, \overline{\varphi}, A, \partial\varphi, \partial\overline{\varphi}, \partial A\right) = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \left(i\partial^\mu - eA^\mu\right)\overline{\varphi} + \left(i\partial_\mu - eA_\mu\right)\varphi - m^2\overline{\varphi}\varphi$$

Which then yields the KG equations for $\varphi, \overline{\varphi}$ and the maxwell equations for A.
We consider the local gauge symmetry with:

$$\varphi(x) \to \varphi(x)\exp\left(-i\lambda e\chi(x)\right)$$
$$A_\mu \to A_\mu + \lambda\partial_\mu\chi$$

leading to

$$\left(i\partial_\mu - e\left(A_\mu - \partial_\mu\chi\right)\right)\exp\left(ie\chi\right)\varphi = \exp\left(-ie\chi\right)\left(i\partial_\mu - eA_\mu\right)\varphi$$

which is indeed a symmetry. Let j^μ be

$$j^\mu = e\left[\overline{\varphi}\left(i\partial^\mu - eA^\mu\right)\varphi + \varphi\left(-i\partial^\mu - eA^\mu\right)\overline{\varphi}\right]$$

as before (but here it's not the Noether current). Yielding $V\varphi = -ie\chi\varphi$ and $(VA)_\alpha = \partial_\alpha\chi$. Giving us

$$J^\mu = -F^{\mu\alpha}\partial_\alpha\chi + \chi j^\mu$$

$$\implies \partial_\mu J^\mu = -\left(F^{\mu\alpha}_{,\alpha} - j^\alpha\right)\partial_\alpha\chi + j^\mu_{,\mu}\chi = 0$$

Which is only true iff either

- for global gauge transformation $\chi(x) = \text{const.} \implies j^\mu_{,\alpha} = 0$
- for local gauge transformations $\implies F^{\mu\alpha}_{,\alpha} = j^\mu$.

Spacetime Translations:

$$\Psi^\lambda\left[\varphi^\alpha\right](x) = \varphi^\alpha\left(x - \lambda a\right) \ , \ a \in \mathbb{R}^4$$

Hence we get $T_\lambda = \mathbb{I}$, $V = 0$, $v = a$, and $\mathcal{L}$ invariant, then

$$J^\mu = -T^{\mu\nu}a_\nu$$

Using the Noether Theorem we get

$$T^{\mu\nu}_{,\mu} = 0$$

The Noether charge comming with it is the total momentum

$$P^\nu = \int d^3x T^{0\nu}\left(t, \vec{x}\right)$$

We consider examples:

- Real scalar field with lagrangian $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right) - V\left(\varphi\right)$, giving us

$$\pi^\mu = \partial^\mu\varphi$$

$$T^\mu_{\nu} = \partial^\mu\varphi\partial_\nu\varphi - \mathcal{L}\delta^\mu_{\nu}$$

we look at individual parts

$$T^{00} = \frac{1}{2}\left(\left(\partial_0\varphi\right)^2 + \left(\vec{\nabla}\varphi\right)^2\right) + V\left(\varphi\right)$$

$$T^{0i} = -\partial_0\varphi\partial_i\varphi$$

- EM Field with source $j = 0$.

$$T^\mu_{\nu} = -F^\mu_{\alpha}\partial_\nu A^\alpha - \mathcal{L}\delta^\mu_{\nu}$$

$$T^{\mu\nu} = -F^{\mu\alpha}\partial^\nu A_\alpha - \frac{1}{4}\left(F_{\alpha\beta}F^{\beta\alpha}\right)g^{\mu\nu}$$

Which is not symmetric under exchanging $\mu \leftrightarrow \nu$ and also not gauge invariant. We compare with the result from electro dynamics

$$\hat{T}^{\mu\nu} = F^\mu_{\alpha}F^{\alpha\nu} - \frac{1}{4}\left(F_{\alpha\beta}F^{\beta\alpha}\right)g^{\mu\nu}$$

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We compare the ED and the canonical energy momentum tensor

$$\hat{T}^{\mu\nu} - T^{\mu\nu} = F^{\mu\nu}\partial_\alpha A^\nu = \partial_\alpha \left(F^{\mu\alpha}A^\nu\right) - \underbrace{F^{\mu\alpha}_{,\alpha}}_{=0 \text{ for motions}} A^\nu$$

Therefore the difference between $\hat{T}$ and T is only a divergence and thus admissible since it's antisymmetric in μ, α, and P^μ is the same as in ED.

Lorentz Transformations:

$$t_\lambda(x)^\mu = \Lambda^\mu{}_\nu x^\nu \;,\; \epsilon = \frac{d\Lambda}{d\lambda}|_{\lambda=0}$$

$$v^\mu(x) = \epsilon^\mu{}_\nu x^\nu \;,\; \epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$$

Transformation for the field is then

$$\varphi'(x') = (T_\Lambda \varphi)\left(\Lambda^{-1}x'\right)$$

With T_Λ a representation of L_+ on the field space F, therefore T_Λ scohuld be linear in φ.

$$\rightarrow V = \frac{dT_\Lambda}{d\lambda}|_{\lambda=0} \text{ linear in } \varphi, \epsilon$$

$$\implies V = \frac{1}{2}\epsilon_{\lambda\rho}S^{\lambda\rho}$$

Where $S^{\lambda\rho}$ is an antisymmetric linear map.
Examples

- Scalar Field: $T_\Lambda = \mathbb{I}, \implies V = 0, S^{\rho\lambda} = 0$.
- EM Field: $A'(x') = (\Lambda A) A\left(\Lambda^{-1}x'\right)$

$$\implies V = \epsilon = \frac{1}{2}\epsilon_{\lambda\rho}S^{\lambda\rho}$$

$$\left(S^{\lambda\rho}\right)_{\mu\nu} = \delta^\lambda_\mu \delta^\rho_\nu - \delta^\rho_\mu \delta^\lambda_\nu$$

A back to the general setting we find

$$J^\mu = \pi^\mu_\alpha \frac{1}{2}\epsilon_{\lambda\rho}\left(S^{\lambda\rho}\varphi\right)^\alpha - T^\mu_\nu \epsilon^\nu_\lambda x^\lambda$$

$$= \frac{1}{2}\left(\pi^\mu_\alpha \left(S^{\lambda\nu}\varphi\right)^\alpha + x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda}\right)\epsilon_{\lambda\nu}$$

$$= \frac{1}{2}\Theta^{\mu\lambda\nu}\epsilon_{\lambda\nu}$$

With $\Theta^{\mu\lambda\nu}$ the Angular momentum tensor.

$$\Theta^{\mu\lambda\nu} = \underbrace{\pi^\mu_\alpha(S\dots)}_{\text{spin part}} + \underbrace{\dots}_{\text{orbital part}}$$

The conserved charge is then

$$L^{\lambda\nu} = \int d^3x\,\Theta^{0\lambda\nu}(t, \vec{x})$$

For $\lambda, \nu = i$ spacial parts we get the ordinary angular momentum, however for $\lambda, \nu = 0$ we get

$$L^{0i} = tP^i - X^i P^0$$

Where X^i is the center of the energy density. This is the law of constant inertia for $\vec{X}$.

3.3.1 Hamiltonian Field Theory

$$L\left[\varphi, \partial_0\varphi\right] = \int d^3x \mathcal{L}\left(\varphi^\alpha, \partial\varphi^\alpha\right)$$

$$\implies H\left(\varphi, \pi\right) = \int\int d^3x \mathcal{H}\left(\varphi^\alpha, \vec{\nabla}\varphi^\alpha, \pi_\alpha\right)$$

with

$$\mathcal{H}\left(\varphi, \vec{\nabla}\varphi, \pi\right) = \pi_\alpha \dot{\varphi}^\alpha - \mathcal{L}\left(\varphi, \vec{\nabla}\varphi, \partial_0\varphi\right)$$

and $\dot{\varphi}^\alpha = \partial_0\varphi^\alpha$ and $\pi_{0\alpha} = \frac{\partial\mathcal{L}}{\partial\varphi^\alpha(x)}\left(\varphi, \vec{\nabla}\varphi, \dot{\varphi}\right)$.

The poisson brackets

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta\pi_\alpha}\frac{\delta G}{\delta\varphi^\alpha} - \frac{\delta F}{\delta\varphi^\alpha}\frac{\delta G}{\delta\pi_\alpha}\right)$$

$$\implies \{\pi_\alpha\left(\vec{x}\right), \varphi_\beta\left(\vec{y}\right)\} = \delta_\alpha^\beta \delta\left(\vec{x} - \vec{y}\right)$$

The equation of motion are then given

$$\dot{\pi}_\alpha\left(t, \vec{x}\right) = \{H, \pi_\alpha\left(t, \vec{x}\right)\}$$
$$\dot{\varphi}\left(t, \vec{x}\right) = \{H, \varphi^\alpha\left(t, \vec{x}\right)\}$$

Examples

- Real scalar field $\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right) - V\left(\varphi\right)$ giving us

$$\mathcal{H} = \pi^2 - \mathcal{L} = \frac{1}{2}\left(\pi^2 + \left(\vec{\nabla}\varphi\right)^2\right) + V\left(\varphi\right)$$

- EM field (with $j = 0$) $\mathcal{L} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$.

$$F_{\alpha\beta} =$$

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)$$

$$\pi_\nu = \pi_{0\nu} = -F_{0\nu} = \partial_\nu A_0 - \partial_0 A_\nu$$

Which we now have to solve for $\partial_0 A_\nu$: which is impossible for $\nu = 0$, meaning π_0 cannot be used as a variable. To fix this problem we have to impose a gauge condition.

- Coulemb gauge:

$$div\vec{A} = 0, A^0 = 0$$

from this we still have A^k ($k = 1, 2, 3$) romain as field degrees of freedom and we get

$$\pi_k = -\dot{A}_k = \dot{A}^k = -E_k$$

with the constraint that $div\vec{\pi} = 0$. The Hamiltonian is then

$$\mathcal{H} = -\vec{E}\cdot\dot{\vec{A}} - \mathcal{L} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$$

4
Dirac Equation

4.1 The Quantum Mechanical Lorentz Group

Lecture of 11.11 Missing

Basis of $V_0 \otimes V$ as

$$\{\sigma_\mu\} = (\sigma_0 = \mathbb{I}, \sigma_i), \sigma_\mu^\dagger = \sigma_\mu$$

giving

$$\tilde{x} = x^\mu \sigma_\mu = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}$$

ie $\tilde{x}_{\alpha\dot{\beta}} = x^\mu \sigma_{\mu,\alpha\dot{\beta}}$. with $\tilde{x} = \tilde{x}^\dagger$ iff $x^\mu \in \mathbb{R}$.
We consider the basis for the other representation $\dot{V} \otimes V^0 : \{\hat{\sigma}_\mu\}$

$$\hat{\sigma}_\mu = \left(\hat{\sigma}_\mu^{\dot{\alpha}\beta}\right) = \epsilon\sigma_\mu^\dagger\epsilon^\dagger = (\det\sigma_\mu) \underbrace{\sigma_\mu^{-1}}_{=\sigma_\mu} = \begin{cases} +1\sigma_\mu & \mu = 0 \\ -1\sigma_\mu & \mu = 1,2,3 \end{cases} = (P\sigma)_\mu$$

With $P : \mathbb{R}^4 \to \mathbb{R}^4, \chi \to P\chi$ parity. The previously given map between the $\hat{x}$ and $\tilde{x}$ says that they have the same x_μ iff $\hat{x} = \epsilon\tilde{x}^T\epsilon^T$. Further we get $\sigma_\nu\hat{\sigma}_\mu + \sigma_\mu\hat{\sigma}_\nu = 2g_{\mu\nu}\mathbb{I}$, and the trace as $\text{Tr}\,(\hat{\sigma}_\mu\sigma_\nu) = 2g_{\mu\nu}$. This leads to $\implies x_\mu = \frac{1}{2}\text{Tr}\,(\hat{\sigma}_\mu\tilde{x})$. The last remark is that $\det\tilde{x} = \left(x^0\right)^2 - \vec{x}^2 = x_\mu x^\mu$ which is obviously lorentz invariant.
We return to the first representation and conclude that $\det\tilde{x}' = \det\tilde{x}$, which induces a linear map $x = x^\mu \to x' = x'^\mu$, with the nice property that $x'_\mu x'^\mu = x_\mu x^\mu$. Which means that this map is a Lorenz transformation $\Lambda(A) \in L_+^\uparrow$ giving us $A\tilde{x}A^* = \Lambda\,(\tilde{A})\,x$. Giving us

$$\implies \Lambda(A)_\nu^\mu = \frac{1}{2}\text{Tr}\,(\hat{\sigma}^\mu A\sigma_\nu A^*)$$

We can write anp $\Lambda \in L_+^\uparrow$ is of the form $\Lambda(A)$ (in more than one way in fact), $\Lambda(A) = \mathbb{I}$ iff $A = \pm 1$. Hence

$$L_+^\uparrow \hat{=} SL\,(2,\mathbb{C})\,/\,\{\pm 1\}$$

We now look at the second representation where we have $\hat{x} \to (A^*)^{-1}\,\hat{x}A^{-1} = \Lambda\,(\hat{A})\,x$ but by $\hat{x} = \epsilon\tilde{x}^T\epsilon^T$ this defines the same $\Lambda(A)$ as in the other representation. By $\hat{x} = \tilde{P}x$ we therefore get $(A^*)^{-1}\,\tilde{P}xA^{-1} = P\Lambda\,(\tilde{A})\,x$ leading to

$$\implies \Lambda\left(A^{*-1}\right)P = P\Lambda(A)$$

Remarks

- For $U \in SU(2)$ we have

$$\Lambda(U)_0^\mu = \frac{1}{2}\text{Tr}\left(\hat{\sigma}_\mu \underbrace{U\mathbb{I}U^*}_{\mathbb{I}}\right) = \begin{cases} 1 & \mu = 0 \\ 0 & \mu \neq 0 \end{cases}$$

$$\implies \Lambda(U) = \begin{bmatrix} 1 & 0 \\ 0 & R(U) \end{bmatrix}$$

- The infinitessimal version of $A\,(x^\nu\sigma_\nu)\,A^* = \Lambda_\nu^\mu x^\nu\sigma_\mu$ is given as

$$a = \frac{dA(\lambda)}{d\lambda}\Big|_{\lambda=0}$$

leading to infinitessimal lorentz transformations

$$\epsilon = \frac{d\Lambda\left(\lambda\right)}{d\lambda}|_{\lambda=0}$$

by

$$a = \frac{1}{2}\epsilon_{\mu\nu}\underbrace{S^{\mu\nu}}_{\in SL(2,\mathbb{C})}, \epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$$

We now claim that

$$S^{0j} = \frac{\sigma_j}{2}, S^{j+1,j+2} = -\frac{i\sigma_j}{2}$$

we prove

$$(a\sigma_\nu + \sigma_\nu a^*) x^\nu = \left(\epsilon^\mu_{\ \nu} x^\nu \sigma_\mu\right)$$
$$= \epsilon_{\mu\nu} x^\nu \sigma^\mu$$
$$\implies (a\sigma_\nu + \sigma_\nu a^*) = \epsilon_{\mu\nu}\sigma^\mu$$
$$\implies \begin{cases} a + a^* = \sigma^j & \nu = 0 \\ a\sigma_j + \sigma_j a^* = -\sigma^0 & \nu = j \\ a\sigma_k + \sigma_j a^* = 0 & k \neq j \end{cases}$$

Leading to the conclusion that

$$a = \frac{\sigma^j}{2} = -\frac{\sigma_j}{2} = S^{j0}$$

4.2   The Weyl Equations

We consider the Lagrangian of a massless spin (or helicity since it's massless) $\frac{1}{2}$ field:

$$\mathcal{L}_L = \frac{i}{2}\varphi^\dagger \hat{\sigma}_\mu \partial^{\leftrightarrow,\mu}\varphi = \frac{i}{2}\left(\overline{\varphi}_{\dot{\alpha}}\hat{\sigma}_\mu^{\dot{\alpha}\beta}\partial^\mu\varphi_\beta - \partial^\mu\overline{\varphi}_{\dot{\alpha}}\hat{\sigma}_\mu^{\dot{\alpha}\beta}\varphi_\beta\right)$$

The transformation law of the field $\varphi = \varphi\left(x\right)$ under $L_+^\uparrow$ is

$$\varphi'\left(x'\right) = \varphi'\left(\Lambda\left(A\right)x\right) = A\varphi\left(x\right)$$

ie. $\varphi = A^{-1}\varphi'$ or $\varphi_\beta\left(x\right) = \alpha^\gamma_{\ \beta}\varphi'_\gamma\left(x'\right) = \varphi'_\gamma\left(x'\right)A^\gamma_{\ \beta}$.
We now claim that $\mathcal{L}_L$ is invariont under these transformations and write

$$\overline{\varphi}_{\dot{\alpha}}\left(x\right)\hat{\sigma}_\mu^{\dot{\alpha}\beta}\partial^\mu\varphi_\beta = \varphi'_{\dot{\delta}}\left(x'\right)\overline{A}^{\dot{\delta}}_{\ \dot{\alpha}}\left(\hat{\sigma}_\mu^{\dot{\alpha}\beta}\partial^\mu\right)A^\gamma_{\ \beta}\varphi'_\gamma\left(x'\right)$$
$$= \overline{\varphi}'_{\dot{\delta}}\hat{\sigma}_\mu^{\dot{\delta}\gamma}\left(\Lambda\left(A\right)\partial\right)^\mu\varphi'_\gamma\left(x'\right)$$
$$= \overline{\varphi}'_{\dot{\delta}}\hat{\sigma}_\mu^{\dot{\delta}\gamma}\partial'^\mu\varphi'_\gamma\left(x'\right)$$

We now compute the canonical momenta which are given as

$$\pi_\mu^{\dot{\alpha}} = \frac{\partial\mathcal{L}}{\partial\left(\partial^\mu\overline{\varphi}_{\dot{\alpha}}\right)}$$
$$= -\frac{i}{2}\hat{\sigma}_\mu^{\dot{\alpha}\beta}\varphi_\beta$$
$$\frac{\partial\mathcal{L}}{\partial\overline{\varphi}_{\dot{\alpha}}} = \frac{i}{2}\hat{\sigma}_\mu^{\dot{\alpha}\beta}\partial^\mu\varphi_\beta$$

Giving us the Euler-Lagrange equations as

$$i\hat{\sigma}_\mu^{\dot{\alpha}\beta}\partial^\mu\varphi_\beta = 0$$

Which is the first Weyl equations, whose solutions satisfy the (massless!) Klein-Gordon equations. Which we can see by taking

$$o_{\nu,\gamma\dot{\alpha}}\partial^\nu\hat{\sigma}_\mu^{\dot{\alpha}\beta}\partial^\mu = \varsigma_{\mu\nu}\delta^\beta_\gamma\partial^\nu\partial^\mu = \delta^\beta_\gamma\Box$$

A few remarks:

- Gauge symmetry (global)

$$\varphi_\beta \to e^{-i\lambda}\varphi_\beta$$
$$\overline{\varphi}_{\dot{\alpha}} \to e^{i\lambda}\overline{\varphi}_{\dot{\alpha}}$$

Giving us the Noether, with $(V\varphi_\beta = -i\varphi_\beta)$, as

$$J_\mu = \overline{\varphi}_{\dot{\alpha}}\sigma_\mu^{\dot{\alpha}\beta}\varphi_\beta$$

- Infinitessimal Lorentz transformations $\delta\varphi_\alpha = -\epsilon^\mu{}_\nu x^\nu \partial_\mu \varphi_\alpha + a_\alpha^\beta \varphi_\beta = \frac{1}{2}\epsilon^{\mu\nu}\left((x_\mu\partial_\nu - \partial_\nu x_\mu) + S_{\mu\nu}\right)\varphi_\alpha$

$$\implies M_{\mu\nu} = x_\mu\partial_\nu - \partial_\nu x_\mu + S_{\mu\nu}$$

giving us the angular momentum, by multiplying with $-i$ we get the self-adjoint operator

$$M_{\mu\nu} = -i\left(x_\mu\partial_\nu - \partial_\nu x_\mu\right) - iS_{\mu\nu}$$

With the spin being $-iS_{i+1}^{i+2} = \frac{\sigma_i}{2}$. Using the EL with a plane wave with $p^\mu = i\partial^\mu$ says that

$$\vec{p}\cdot\vec{\sigma}\varphi = -p^0\varphi, p^0 > 0$$

Meaning that momentum $\vec{p}$ and spin $\vec{\sigma}$ are anti-parallel, meaning that the particle/field are left-handed. The (first) Weyl equation only describes these left handed fields and is not parity invariant. There is no transformation implementing parity P ie. $\varphi'(x') = S\varphi(x)$ with $x' = Px$ such that φ' still satisfies the Weyl equation does not exist.

To treat right handed fields we can now write simmilarly

$$\mathcal{L}_R = \frac{i}{2}\chi^\dagger \sigma_\mu \partial^{\leftrightarrow,\mu}\chi = \frac{i}{2}\left(\overline{\chi}^\alpha \sigma_{\mu,\alpha\dot{\beta}}\partial^\mu \chi^{\dot{\beta}} - \partial^\mu\overline{\chi}^\alpha \sigma_{\mu,\alpha\dot{\beta}}\chi^{\dot{\beta}}\right)$$

Which is the second Weyl equation, we won't go through the calculation again, but it turns out it describes right handed particles/fields.

4.3 The Dirac Equation

A field having parity symmetry can be be constructed by combining the two Weyl fields as

$$\Psi = \begin{pmatrix} \varphi_\beta \\ \chi^{\dot{\alpha}} \end{pmatrix}$$

with the lagrangian

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R = \frac{i}{2}\left(\varphi^\dagger \hat{\sigma}_\mu \partial^{\leftrightarrow,\mu}\varphi + \chi^\dagger \sigma_\mu \partial^{\leftrightarrow,\mu}\chi\right)$$

which we can rewrite as

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \partial^{\leftrightarrow,\mu}\Psi$$

with $\overline{\Psi} = \Psi^\dagger\gamma^0$, the Dirac conjugate. By comparison we get

$$\gamma^0\gamma^\mu = \begin{bmatrix} \hat{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{bmatrix}$$

Which is solved by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies \overline{\Psi} = \left(\chi^{*,\beta}, \varphi^*_{\dot{\alpha}}\right)$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \hat{\sigma}^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

Which are the so called chiral, or Weyl, representation of the Dirac algebra.
We look at the transformations of Ψ under Lorenz-transformations $\Lambda \in L_+^\uparrow$

$$\Psi'(x') = S(A)\Psi(x), x' = \Lambda(A)x$$

with

$$S\left(A\right)=\begin{pmatrix} A & 0 \\ 0 & \left(A^{\dagger}\right)^{-1} \end{pmatrix}$$

Which leaves $\mathcal{L}_{L/R}$ seperately invariant.
We now look at the parity transformation

$$\Psi'\left(x'\right)=\gamma^{0}\Psi\left(x\right),x'=Px$$

which just exchanges the Weyl spinors against each other $\varphi\leftrightarrow\chi$, $\mathcal{L}_{L}\leftrightarrow\mathcal{L}_{R}$. Giving us

$$\gamma^{0}S\left(A\right)\gamma^{0}=\begin{pmatrix} \left(A^{\dagger}\right)^{-1} & 0 \\ 0 & A \end{pmatrix}=S\left(\left(A^{\dagger}\right)^{-1}\right)$$

which matches our expectations. This extends the double valued reprsentation of $L_{+}^{\uparrow}$ on $V_{0}\oplus V$ to $L^{\uparrow}=L_{+}^{\uparrow}\cup PL_{+}^{\uparrow}$.
A few remarks

- We had

$$A\left(x^{\mu\sigma_{\mu}}\right)A^{\dagger}=\left(\Lambda\left(A\right)x\right)^{\mu}\sigma\mu$$

$$\left(A^{\dagger}\right)^{-1}\left(x^{\mu}\hat{\sigma}_{\mu}\right)A^{-1}=\left(\Lambda\left(A\right)x\right)^{\mu}\hat{\sigma}_{\mu}.$$

  joining them we get

$$S\left(A\right)x_{\nu}\gamma^{\nu}S\left(A\right)^{-1}=\left(\Lambda\left(A\right)x\right)_{\nu}\gamma^{\nu}$$

- With a mass torm added to $\mathcal{L}$ we get

$$\mathcal{L}=\frac{i}{2}\overline{\Psi}\gamma^{\mu}\overset{\leftrightarrow}{\partial_{\mu}}\Psi-m\overline{\Psi}\Psi$$

  giving us the invariance of the mass term:

$$\overline{\Psi}\Psi=\chi^{\dagger}\varphi+\varphi^{\dagger}\chi=\overline{\chi}^{\alpha}\varphi_{\alpha}+\overline{\varphi}_{\dot{\alpha}}\chi^{\dot{\alpha}}$$

  Where both terms of the sum are invariant under Lorentz transformations and the whole sum is invariant under parity.

We now look at the canonical momenta of this theory (with $\overline{PSi}$ and Ψ independent)

$$\pi^{\mu}=\frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\overline{\Psi}\right)}=-\frac{i}{2}\gamma^{\mu}\Psi$$

$$\frac{\partial\mathcal{L}}{\partial\overline{\Psi}}=\frac{i}{2}\gamma^{\mu}\partial_{\mu}\Psi-m\Psi$$

Giving us the Euler-Lagrange Equations:

$$-i\gamma^{\mu}\partial_{\mu}\Psi+m\Psi=0$$

ie.

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\Psi=0$$

The EL equation with respect to $\overline{\Psi}$ gives

$$\overline{\Psi}\left(i\gamma^{\mu}\overset{\leftarrow}{\partial_{\mu}}+m\right)=0$$

In the bispinor notation we get:

$$i\hat{\sigma}_{\mu}^{\dot{\alpha}\beta}d^{\mu}\varphi=m\chi^{\dot{\alpha}}$$

$$i\sigma_{\mu,\alpha\dot{\beta}}\partial^{\mu}\chi^{\dot{\beta}}=m\varphi_{\alpha}$$

Remark:

- Global gauge symmetry

$$\Psi\rightarrow e^{i\lambda}\Psi$$

$$\overline{\Psi}\rightarrow e^{-i\lambda}\overline{\Psi}$$

  Giving us the Noether current

$$J^{\mu}=i\overline{\Psi}\gamma^{\mu}\Psi$$

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- Infinitessimal Lorentz transformations give us $A \to S(A)$, $a \to S(a)$

$$S(a) = \frac{1}{2}\frac{1}{2}\epsilon_{\mu\nu}\mu\nu$$

with

$$\varsigma^{\mu\nu} = \frac{1}{4}[\gamma^{\mu}, \gamma^{\nu}] = \sigma^{\mu\nu}$$

ie

$$\varsigma^{0j} = \frac{1}{2}\begin{pmatrix} \sigma^{j} & 0 \\ 0 & -\sigma^{j} \end{pmatrix}, \varsigma^{j+1,j+2} = -\frac{i}{2}\begin{pmatrix} \sigma_{j} & 0 \\ 0 & \sigma_{j} \end{pmatrix}$$

Lecture 18.11. missing

5
Canonical Quantization

Corollary to Wicks' Lemma: (Vacuum expectation values)

$$\langle 0 | A_1 \ldots A_{2n+1} | 0 \rangle = 0$$

$$\langle 0 | A_1 \ldots A_{2n} | 0 \rangle = \sum_{P:(i_1 < k_1)\ldots(i_n < k_n)} \varsigma(\sigma_p) \prod_{l=1}^{n} \langle 0 | A_{i_l} A_{k_l} | 0 \rangle$$

There are several variants of this result:

- $: A_1 \ldots A_k : \cdot : A_{k+1} \ldots A_n :$

$$= \sum_{P:(i_1 < k_1)\ldots(i_p < k_p)} {}' \left(\prod_{l=1}^{n} \langle 0 | A_{i_l} A_{k_l} | 0 \rangle \right) : A_1 \ldots \hat{A}_{i_1} \ldots \hat{A}_{k_p} \ldots A_n :$$

With $\sum'$ the sum over pairings P with no pair from a same normal ordered product.

$$\implies \langle 0 | : A_1 \ldots A_k : \cdot : A_{k+1} \ldots A_n : | 0 \rangle = \sum_{P} \varsigma(\sigma_P) \left(\prod_{l=1}^{n} \langle 0 | A_{i_l} A_{k_l} | 0 \rangle \right)$$

- Suppose some order (not neccesarily that of the labeling) is given among $A_1, \ldots, A_n$. E.g. each of the operators could come with a time t_i and we order them according to their "time stamp". This is called a time ordered product:

$$T(A_1, \ldots, A_n) = \varsigma(\sigma_P) A_{\sigma(1)} \ldots A_{\sigma(n)}$$

Then we can write such a time ordered product as a sum over normal ordered products

$$T(A_1, \ldots, A_n) = \sum_{P} \varsigma(\sigma_P) \prod_{l=1}^{n} \langle 0 | T(A_{i_l}, A_{k_l}) | 0 \rangle : A_1, \ldots \hat{A}_{i_1} \ldots \hat{A}_{k_p} \ldots A_n :$$

$$\implies \langle 0 | T(A_1, \ldots, A_n) | 0 \rangle = \sum_{P} \varsigma(\sigma_P) \prod_{l=1}^{n} \langle 0 | T(A_{i_l} A_{k_l}) | 0 \rangle$$

- The previous two combine to give us

$$T(: A_1 \ldots A_k : \cdot : A_{k+1} \ldots A_n :) = = \sum_{P:(i_1 < k_1)\ldots(i_p < k_p)} {}' \left(\prod_{l=1}^{n} \langle 0 | T(A_{i_l}, A_{k_l}) | 0 \rangle \right) : A_1 \ldots \hat{A}_{i_1} \ldots \hat{A}_{k_p} \ldots A_n :$$

With again the $\sum'$. Analogous for the vacuum expectation.

5.1 The Real Scalar Field

We use the cannonical quantisation with $\hbar = c = 1$. We replace $\varphi(\vec{x}), \pi(\vec{x}), H$ by operators (at first they are objects on some algebra, later then operators on a concrete Hilbertspace) with

$$[\varphi(x), \pi(\vec{y})] = i\delta(\vec{x} - \vec{y})$$

With all other commutators vanishing. The dynamics are then given as

$$\dot{\varphi}(t,\vec{x}) = i\,[H, \varphi(t,\vec{x})]$$

$$\dot{\pi}(t,\vec{x}) = i\,[H, \pi(t,\vec{x})]$$

The mode expansion is given as (Fourier in position)

$$\varphi(\vec{x}) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2p^0} \left(e^{i\vec{p}\cdot\vec{x}} a(\vec{p}) + e^{-i\vec{p}\cdot\vec{x}} a^*(\vec{p}) \right)$$

$$= (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2p^0} e^{i\vec{p}\cdot\vec{x}} (a(\vec{p}) + a^*(-\vec{p}))$$

$$\pi(\vec{x}) = -i\,(2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2} \left(e^{i\vec{p}\cdot\vec{x}} a(\vec{p}) - e^{-i\vec{p}\cdot\vec{x}} a^*(\vec{p}) \right)$$

$$= -i\,(2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2} e^{i\vec{p}\cdot\vec{x}} (a(\vec{p}) - a^*(-\vec{p}))$$

In analogy to ED, where we had two real fields $\varphi(\vec{x}), \pi(\vec{x})$ which we then replaced by one single complex field $a(\vec{p})$. For ED we had the pattern: $q_+ \to e^{i\vec{p}\cdot\vec{x}}, \omega = p^0 \implies q_+ = \frac{1}{\sqrt{2\omega}}(a_+\bar{a}_-), p_4 = i\sqrt{\frac{\omega}{2}}(a_+ - \bar{a}_-)$.

We now have $a(\vec{p}) \leftrightarrow \sqrt{2\omega}a_+$. In the CCR case we have

$$\left[a(\vec{p}), a^\dagger(\vec{p}'),\right] = 2p^0\delta(\vec{p} - \vec{p}')$$

With all others vanishing. The Hamiltonian is given as

$$H = \frac{1}{2}\int d^3x : \pi^2(\vec{x}) + \left(\vec{\nabla}\varphi(\vec{x})\right)^2 + m^2\varphi^2(\vec{x}) : \underbrace{=}_{\text{ccr}} \int \frac{d^3p}{2p^0} \times p^0 a^\dagger(\vec{p}) a(\vec{p})$$

See notes for the derivation of above identity.
We can similarly look at the momentum

$$\vec{P} = -\int d^3x : \pi(\vec{x})\vec{\nabla}\varphi(\vec{x}) :$$

$$\underbrace{=}_{\text{ccr}} \int \frac{d^3p}{2p^0} \times \vec{p}\,a^\dagger(\vec{p}) a(\vec{p})$$

Giving us the Heisenberg equation of motion as

$$\dot{a}(\vec{p}) = i\,[H, a(\vec{p})]$$
$$= -ip^0 a(\vec{p})$$
$$\dot{a}^\dagger(\vec{p}) = i\,[H, a^\dagger(\vec{p})]$$
$$= ip^0 a^\dagger(\vec{p})$$

Giving us

$$a(\vec{p}, t) = e^{-ip^0 t} a(\vec{p})$$
$$a^\dagger(\vec{p}, t) = e^{ip^0 t} a^\dagger(\vec{p})$$

Giving us for the field:

$$\partial_t \varphi(x) = i\,[H, \varphi(x)]$$

$$\implies \varphi(\vec{x}, t) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2p^0} \left(e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} a^\dagger(\vec{p}) \right)$$

With $p^0 = \sqrt{\vec{p}^2 + m^2}$ and $p \cdot x = p^0 t - \vec{p}\cdot\vec{x}$.
All operators are represented on a Fock space $\mathcal{F}$, which is spanned by

$$|\vec{p}_1, \ldots, \vec{p}_n\rangle = a^\dagger(\vec{p}_1)\ldots a^\dagger(\vec{p}_n)\,|0\rangle$$

which we can interpret as follows

- $n = 0 \implies |0\rangle$ the vacuum with $P^\mu |0\rangle = 0$, meaning it's invariant under space-time translation.

- $n = 1 \implies a^\dagger (\vec{p}) |0\rangle$ giving us $P^\mu |\vec{p}\rangle = p^\mu |p\rangle$ which is the state of a single particle of mass m.

- $n \geq 2$ is an n-particle state with $P^\mu |\vec{p}_1, \ldots\rangle = \sum_{i=1}^n p^{\mu,i} |\vec{p}_1, \ldots\rangle$

We can now get the spectrum of P_μ, as they all commute. The joint EV are give as

$$\sigma(P^\mu) = \left\{ \sum_{i=1}^n p^{(i)\mu} | p^{(i)} \in V_m, n = 0, 1, \ldots \right\} = \{0\} \cup \left\{ p \in \mathbb{R}^4 | p^0 \geq 0 \text{ and } (p^2 = m^2 \text{ or } p^2 \geq 4m^2) \right\}$$

We have $H = P^0 \geq 0$ giving us stability all "by itself". In contrast to the problems with negative energy solutions we encountered in the Klein-Gordan equation, which we had to cross out "by hand", we now get stability (positive energy) simply by quantizing a field theory.

Two point function (aka Green's function) is given as

$$\langle 0| \varphi(x) \varphi(y) |0\rangle = (2\pi)^{-3} \int \int \frac{d^3p}{2p^0} \frac{d^3p'}{2p'^0} e^{-ipx} e^{ip'y} \underbrace{\langle 0| a(\vec{p}) a^\dagger(\vec{p}') |0\rangle}_{[a(\vec{p}), a^\dagger(\vec{p}')]}$$

Giving us

$$\langle 0| \varphi(x) \varphi(y) |0\rangle = (2\pi)^{-3} \int \int \frac{d^3p}{2p^0} \frac{d^3p'}{2p'^0} 2p^0 \delta(\vec{p} - \vec{p}') = i\Delta_+(x - y)$$

with

$$i\Delta_+(z) = (2\pi)^{-3} \int \frac{d^3p}{2p^0} e^{-ipz}$$

which is an L-invariant distirbution. To see this we write

$$i\Delta_+(\Lambda z) = (2\pi)^{-3} \int_{\mathbb{R}^4} d\mu(p) e^{-ip\cdot\Lambda z}$$

$$= (2\pi)^{-3} \int d\mu(p) e^{-i(\Lambda^{-1}p)z}$$

$$= (2\pi)^{-3} \int d\mu(p') e^{-ip'z}$$

$$= (2\pi)^{-3} \int d\mu(p) e^{-ipz}$$

$$= i\Delta_+(z)$$

In particular for z spacelike we get $z^2 = z_\mu z^\mu < 0$ there is a lorenztransformation $\Lambda \in L_+^\uparrow$ such that $\Lambda z = -z$, hence we get

$$i\Delta_+(-z) = i\Delta_+(z) \text{ for } (z^2 < 0)$$

The commutator between two space-time points in the fields $[\varphi(x), \varphi(y)]$ is a multiple of $\mathbb{I}$, by CCR. They are given as

$$[\varphi(x), \varphi(y)] = \langle 0| [\varphi(x), \varphi(y)] |0\rangle$$
$$= i(\Delta_+(x - y) - \Delta_+(y - x))$$
$$:= i\Delta(x - y)$$

In particular, if $(x - y)^2 < 0$, meaning x, y are space-like separated

$$[\varphi(x), \varphi(y)] = 0 \text{ for } x, y \text{ space-like seperated}$$

This implies locality, ie. there can be no influence between events that are space-like separated. Giving us the compatibility of the theory with special relativity.

To be used in perturbation theory we introduce the *Time-ordered Green's function* (also called the Feynman propagator).

$$i\Delta_F(x - y) = \langle 0| T(\varphi(x), \varphi(y)) |0\rangle$$
$$= \Theta(x^0 - y^0) \langle 0| \varphi(x) \varphi(y) |0\rangle + \Theta(y^0 - x^0) \langle 0| \varphi(y) \varphi(x) |0\rangle$$
$$\implies \Delta_F = \Theta(z^0) \Delta_+(z) + \Theta(-z^0) \Delta_+(-z)$$

It's important to remember;

- $\Theta\left(z^0\right)$ is not $L^\uparrow$ invariant, in contrast $\Delta_+\left(z\right)$ is iff $z^2 < 0$. But then we get $\Delta_F\left(z\right) = \Delta_+\left(z\right)$ anyways. So in the end the entire Δ_F is $L^\uparrow$-invariant (in fact it is L-invariant).

The fourier representation of this Feynman propagator is given as

$$\Delta_F\left(z\right) = \int \frac{d^4 p}{\left(2\pi\right)^4} \frac{1}{p^2 - m^2 + i0} e^{-ipz} =: I$$

We prove for $z^0 > 0$

$$I = \frac{i}{\left(2\pi\right)^3} \int d^3 p\, e^{i\vec{p}\vec{z}} \frac{1}{2\pi} \int dp^0 \frac{e^{-ip^0 z^0}}{\left(p^0\right)^2 - \left(p^2 + m^2\right)^2 + i0}$$

The poles of this integral is given as: $p^0 = \pm\sqrt{\vec{p}^2 + m^2 - i0} = \pm\left(\sqrt{\vec{p}^2 + m^2} - i0\right)$

$$\implies I = \text{Residue over the positive pole}$$

$$= -\frac{i}{\left(2\pi\right)^3} \int \frac{d^3 p}{2p^0} e^{-i\left(p^0 z^0 - \vec{p}\vec{z}\right)}$$

$$\implies iI = i\Delta_+\left(z\right)$$

For $z^0 < 0$ the argument is simmilar.
A remark that should have been made earlier is

- $\partial_\mu \varphi\left(x\right) = i\left[P_\mu, \varphi\left(x\right)\right]$. If CAR had been used on the classical expression for P_μ then we would have gotten $P_\mu = 0$. However if $P_\mu = \int \frac{d^3 p}{2p^0} p^\mu a^\dagger\left(\vec{p}\right) a\left(\vec{p}\right)$ is accepted as is then $\partial_\mu \varphi\left(x\right)$ still yields the same.

5.1.1 Spin-Statistics

For all of the calculations above we relied heavily on the CCR (ie. the bosonic case) to quantize $\varphi\left(\vec{x}\right)$. We now explore what would happen when we use CAR instead, then we have:

1. either $\left[\varphi\left(x\right), \varphi\left(y\right)\right] \not\propto \mathbb{I}$.

2. or $\left\{\varphi\left(x\right), \varphi\left(y\right)\right\} \propto \mathbb{I}$.

We explore case 1. first and get

$$\left[\varphi\left(x\right), \varphi\left(y\right)\right] = \varphi\left(x\right)\varphi\left(y\right) - \varphi\left(y\right)\varphi\left(x\right)$$
$$=: \varphi\left(x\right)\varphi\left(y\right): + \langle 0| \varphi\left(x\right)\varphi\left(y\right) |0\rangle - :\varphi\left(y\right)\varphi\left(x\right): - \langle 0| \varphi\left(y\right)\varphi\left(x\right) |0\rangle$$
$$= 2 :\varphi\left(x\right)\varphi\left(y\right): + i\Delta\left(x - y\right)$$

In this case, when $\left(x - y\right)^2 < 0$, ie they are spacially seperated we have

$$= \underbrace{2 :\varphi\left(x\right)\varphi\left(y\right):}_{\neq 0} + \underbrace{i\Delta\left(x - y\right)}_{= 0}$$

We check that the first case indeed isn't 0

$$\langle \vec{p}| :\varphi\left(x\right)\varphi\left(y\right): |\vec{p}\rangle = 2i\left(2\pi\right)^3 \sin\left(p\left(x - y\right)\right)$$
$$= \langle 0| a\left(\vec{p}\right) :\varphi\left(x\right)\varphi\left(y\right): a^\dagger\left(\vec{p}\right) |0\rangle$$
$$= \langle 0| a\left(\vec{p}\right) \varphi\left(x\right) |0\rangle \langle 0| \varphi\left(y\right) a^\dagger\left(\vec{p}\right) |0\rangle - \left(x \leftrightarrow y\right)$$

by

$$\langle 0| a\left(\vec{p}\right) \varphi\left(x\right) |0\rangle = \left(2\pi\right)^{-\frac{3}{2}} e^{ipx}$$

we get

$$\langle \vec{p}| :\varphi\left(x\right)\varphi\left(y\right): |\vec{p}\rangle = \left(2\pi\right)^3 \left(e^{ip\left(x-y\right)} - \left(x \leftrightarrow y\right)\right)$$
$$\neq 0$$

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Meaning locality is violated for option one.
We therefore take a look at the second option:

$$\{\varphi(x), \varphi(y)\} = \langle 0| \{\varphi(x), \varphi(y)\} |0\rangle$$
$$= i\Delta_+(x-y) + i\Delta_+(y-x)$$
$$\text{for } (x-y)^2 < 0 \text{ we have } = 2i\Delta_+(x-y)$$
$$\neq 0$$

Again we found that locality is violated. Meaning that for free scalar field we have to use Bosonic quantization.

5.2   Complex Scalar Field

We consider two real scalar (quantum) fields $\varphi_i(x) = \varphi_i(x)^*$ $(i = 1, 2)$ realized on $\mathcal{F}_+ \otimes \mathcal{F}_+$, as $\varphi_1(x) = \varphi_1(x) \otimes \mathbb{I}$, $\varphi_2(x) = \mathbb{I} \otimes \varphi_2(x)$. Giving us $[\varphi_1(x), \varphi_2(y)] = 0$
We combine the two fields as

$$\varphi(x) = \frac{1}{\sqrt{2}}(\varphi_1(x) + i\varphi_2(x)) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3p}{2p^0} \left(e^{-ipx}a(\vec{p}) + e^{ipx}b^*(\vec{p})\right)$$

With

$$a(\vec{p}) = \frac{1}{2}(a_1(\vec{p}) + ia_2(\vec{p}))$$
$$b^*(\vec{p}) = \frac{1}{\sqrt{2}}(a_1^*(\vec{p}) + ia_2^*(\vec{p}))$$

The only non-zero commutators are:

$$[a(\vec{p}), a^*(\vec{p}')] = [b(\vec{p}), b^*(\vec{p}')] = 2p^0\delta(\vec{p} - \vec{p}')$$

4-Momentum   (generator of translations)

$$p^\mu = \int \frac{d^3p}{2p^0}p^\mu \left(a_1^*(\vec{p})a_1(\vec{p} + a_2^*(\vec{p})a_2(\vec{p}))\right) = \int \frac{d^3p}{2p^0}p^\mu \left(a^*(\vec{p})a(\vec{p}) + b^*(\vec{p})b(\vec{p})\right)$$

Which is only definitively positive because we have not used fermionic relations (stability would not be given).

Global gauge symmetry   Clasically we have $\varphi(x) \to e^{-i\lambda}\varphi(x)$ giving us a Noether charge $Q = i \int_{x^0=t} d^3x \left(\overline{\varphi}(x)(\partial_0\varphi(x)) - (\partial_0\overline{\varphi}(x))\right)$
giving us the Hamiltonian as a generator of symmetry.
In a quantum case we get

$$Q = \int \frac{d^3p}{2p^0}\left(a^*(\vec{p})a(\vec{p}) - b^*(\vec{p})b(\vec{p})\right)$$

This Q also generates a symmetry as

$$e^{i\lambda Q}\varphi(x)e^{-i\lambda Q} = e^{-i\lambda}\varphi(x)$$

this follows from it's infinitessimal version

$$[Q, \varphi(x)] = -\varphi(x)$$

and from this in turn follows

$$[Q, a^*(\vec{p})] = a^*(\vec{p})$$
$$[Q, a(\vec{p})] = -a(\vec{p})$$
$$[Q, b^*(\vec{p})] = -b^*(p)$$

If we expand the first commutator we get

$$Qa^*(\vec{p}) = a^*(\vec{p})(Q+1)$$
$$Qb^*(\vec{p}) = b^*(\vec{p})(Q-1)$$

We therefore have that $a^*(\vec{p})$ raises the charge by 1, while $b^*(\vec{p})$ lowers it by 1. Which lends itself to the interpretation that $a^*(\vec{p})$ creates a particle, while $b^*(\vec{p})$ creates an anti-particle.

The Green's functions are given as

$$\langle 0| \varphi(x) \varphi^*(y) |0\rangle = i\Delta_+(x-y)$$
$$[\varphi(x), \varphi^*(y)] = i\Delta(x-y)$$
$$\langle 0| T(\varphi(x)\varphi(y)) |0\rangle = i\Delta_F(x-y)$$

The same relations hold if the * is on the respective first factor. All others (φ, φ and φ^*, φ^*) are 0.

5.3 Dirac Field

We have

$$(i\gamma^\mu - m)\psi = 0$$

with solutions (plane waves)

$$\psi(x) = u e^{-ipx}$$

with $\not{p} = p_\mu \gamma^\mu$ we get

$$(\not{p} - m)u = 0$$

and $p \in \pm V_m$. When $p \in -V_m$ we write instead $v e^{ipx}$ with $p \in V_m$ but

$$\psi(x) = v e^{ipx}$$

is then

$$(\not{p} + m)v = 0$$

The solution space of the two equations is each two dimensional. We have to choose a basis in each solution space as

$$u^{(\alpha)}(p), v^{(\alpha)}(p) \quad [\alpha = 1, 2 \text{ and } p \in V_m]$$

We want to take the general solution of the classical dirac field

$$\psi(x) = (2\pi)^{-\frac{3}{2}} \sum_\alpha \int \frac{d^3p}{2p^0} \left(b_\alpha(p) u^{(\alpha)}(p) e^{-ipx} + \overline{d_\alpha(p)} v^{(\alpha)}(p) e^{ipx} \right)$$

with $b_\alpha(p), d_\alpha(p) \in P$ arbitrary.

The choice of polarization is not unique, but such that $\left(\overline{u} = u^* \gamma^0 \right)$

$$\overline{u}^{(\alpha)}(p) u^{(\beta)}(p) = 2m\delta_{\alpha\beta}$$
$$\overline{v}^{(\alpha)}(p) v^{(\beta)}(p) = -2m\delta_{\alpha\beta}$$
$$\overline{u}^{(\alpha)}(p) v^{(\beta)}(p) = \overline{v}^{(\alpha)}(p) u^{(\beta)}(p) = 0$$

We justify this choice

- for $p = \left(m, \vec{0} \right)$ the two dirac equations state that

$$(\gamma^0 - 1)u = 0$$
$$(\gamma^0 + 1)v = 0$$

And since $\gamma^0 = \gamma^{0*}$ the eigenvectors can be chosen orthogonal (with respect to the standard innerproduct on $\mathbb{C}^4$). As given above.

- for other p we get, in the same sense that if it holds at $p \in V_m$ then it also holds at $p' = \Lambda(A)p$ sice

$$u^{(\alpha)}(p') = S(A) u^{(\alpha)}(p)$$

$$\implies u^{(\alpha)*}(p') = u^{(\alpha)*}(p) S^*(A)$$

$$\overline{u}^{(\alpha)}(p') u^{(buta)}(p') = \overline{u}^{(\alpha)}(p) \underbrace{\gamma^0 S(A)^* \gamma^0}_{S(A^{-1})=S(A)^{-1}} S(A) u^{(\beta)}(p) = \overline{u}^{(\alpha)}(p) u^{(\beta)}(p)$$

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We also have a completeness realation

$$\sum_{\alpha=1,2} u^{(\alpha)}(p) \otimes \bar{u}^{(\alpha)}(p) = \not{p} + m$$

$$\sum_{\alpha=1,2} v^{(\alpha)}(p) \otimes \bar{v}^{(\alpha)}(p) = \not{p} - m$$

5.3.1 Quantisation of the Dirac field

Quantizing the Lagrangian is not possible, as we can't do the Legendre transformations needed to find the Hamliltonian. We therefore choose a different approach.

Toy Model We start with a toy model with two degrees of freedom $(q, \bar{q})$. Let

$$L = \frac{1}{2}\left(\bar{q}\dot{q} - \dot{\bar{q}}q\right) - m\bar{q}q$$

We observe that the first term is homogenious of degree 1 in $\dot{q}, \dot{\bar{q}}$.

$$\implies p = \frac{\partial L}{\partial \dot{q}} = \frac{1}{2}\bar{q}$$

$$\bar{p} = \frac{\partial L}{\partial \dot{\bar{q}}} = -\frac{1}{2}q$$

Since $\sum_i^k x_i \frac{\partial f}{\partial x_i} = nf$ for $f(\lambda \vec{x}) = \lambda^n f(\vec{x})$

$$\implies H(q, \dot{q}) = p\dot{q} + \bar{p}\dot{\bar{q}} - L$$
$$= m\bar{q}q$$

We now realize that the legendre transform is not possible as we cannot express $\dot{q}$ as $\dot{q}(q, p)$. However this is not needed because $H(q, \dot{q})$ is independent of $\dot{q}$. To quantize such a theory we can chose the $q, \bar{q}$ as the conjugate variables.

$$\bar{q} \to q^*$$
$$[q, q^*]_{\pm} = 1$$
$$[q, q]_{\pm} = 0$$

Using

$$[AB, C] = A[B, C] + [A, C]B$$
$$[AB, C] = A\{B, C\} - \{A, C\}B$$

we get

$$\implies [q^*q, q] = -q$$
$$\implies \{q^*q, q\} = -q$$

This then allows us to write the Heisenberg equation of motion

$$\dot{q} = i[H, q]$$
$$= im[q^*q, q]$$
$$= -imq$$
$$\implies q(t) = e^{-imt}q(0)$$
$$q^*(t) = e^{imt}q^*(0)$$

Back to the Dirac-Field We promote the $b_\alpha(p)$ and $d_\alpha(p)$ to operators and $\psi(x)$ likewise to $\Psi(x)$. We do the quantization at $t = 0$ and worry about the time evolution later.

$$\Psi(0, \vec{x}) = (2\pi)^{-\frac{3}{2}} \sum_\alpha \int \frac{d^3p}{2p^0} \left(b_\alpha(p) u^{(\alpha)}(p) e^{-i\vec{p}\vec{x}} + \overline{a_\alpha(p)} v^{(\alpha)}(p) e^{i\vec{p}\vec{x}} \right)$$

Giving us the Heisenberg equation of motion

$$\partial_t \Psi(t, \vec{x}) = i[H, \Psi(t, \vec{x})]$$

ought to be a solution of the Dirac equation for the field $\Psi(x)$. This amounts to show:

$$b_\alpha(p, t) = b_\alpha(p) e^{-ip^0 t}$$
$$d_\alpha^*(p, t) = d_\alpha^*(p) e^{ip^0 t}$$

We show this by taking a more general approach of the one outlined in the toy model above:

$$\partial_\mu \Psi(x) = i[P_\mu, \Psi(x)]$$

With $H = P^0$. We take P_μ from Noether, and impose normal ordering. This then yields the result: (using a Parseval in the spacial FT, and the normalizations of $u^{(\alpha)}, v^{(\alpha)}$)

$$P^\nu = \sum_{\alpha=1,2} \int \frac{d^3p}{2p^0} p^\nu : b_\alpha^*(p) b_\alpha(p) - d_\alpha(p) d_\alpha^*(p) :$$

Here given without the calculation, the full derivation can be found in the lecture materials. This result then gives

$$P^\nu = \sum_\alpha \int \frac{d^3p}{2p^0} p^\nu \begin{cases} b_\alpha^*(p) b_\alpha(p) - d_\alpha(p) d_\alpha^*(p) & \text{Bosonic} \\ b_\alpha^*(p) b_\alpha(p) + d_\alpha(p) d_\alpha^*(p) & \text{Fermionic} \end{cases}$$

For the fermionic case we find $H \geq 0$ (stability). But for the bosonic case H is unbounded above and below.
To give the Bosons a 2nd chance, we can fix the sign in the result to a + "by hand". This appears to be dodgy at first but we do it and worry about it later. (this is actualy reminicient of the "hole theory" we had earlier on).
In both cases we therefore get:

$$[b_\alpha^*(p) \phi_\alpha(p), b_\beta(p')] = -2p^0 \delta(\vec{p} - \vec{p}') \delta_{\alpha\beta} \phi_\beta(p)$$
$$[b_\alpha^*(p) \phi_\alpha(p), b_\beta^*(p')] = 2p^0 \delta(\vec{p} - \vec{p}') \delta_{\alpha\beta} \phi_\beta(p)$$

Thus

$$[P^\mu, b_\beta(p)] = -p^\mu b_\beta(p)$$
$$[P^\mu, d_\alpha^*(p)] = +p^\mu d_\alpha^*(p)$$

For the special case of the spacial cases we therefore get

$$i[P_i, \Psi(0, \vec{x})] = \partial_i \Psi(0, \vec{x})$$
$$\implies \dot{b}_\alpha = i[H, b_\alpha(p)]$$
$$= -ip^0 b_\alpha(p)$$
$$\implies b_\alpha(p, t) = e^{-ip^0 t} b_\alpha(p)$$
$$\implies d_\beta^*(p, t) = e^{+ip^0 t} d_\beta^*(p)$$

The quantum field $\Psi(x)$ satisfies the dirac equation

$$(i\gamma^\mu \partial_\mu - m) \Psi(x) = 0$$

Also we have

$$\overline{\Psi}(x) \left(i\gamma^\mu \overleftarrow{\partial_\mu} + m \right) = 0$$

for

$$\overline{\Psi}(x) = \Psi^*(x) \gamma^0$$
$$= (2\pi)^{-\frac{3}{2}} \sum_\alpha \int \frac{d^3p}{2p^0} \left(b_\alpha^*(p) \overline{u}^{(\alpha)}(p) e^{ipx} + d_\alpha(p) \overline{v}^{(\alpha)}(p) e^{-ipx} \right)$$

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Green's function The two point function is given by

$$\langle 0| \Psi_i(x) \overline{\Psi}_j(y) |0\rangle = (2\pi)^{-3} \int \int \frac{d^3p}{2p^0} \frac{d^3p'}{2p'^0} e^{-ipx} e^{ip'y} \underbrace{\sum_{\alpha,\beta} u_i^{(\alpha)}(p) \overline{u}_j^{(\beta)}(p')}_{=(\not{p}+m)_{ij}} \underbrace{\langle 0| [b_\alpha(p), b_\beta^*(p')] |0\rangle}_{=2p^0\delta(\vec{p}-\vec{p}')\delta_{\alpha\beta}}$$

$$= (2\pi)^{-3} \int \frac{d^3p}{2p^0} e^{-ip(x-y)} (\not{p}+m)_{ij}$$

$$= (i\not{\partial}_x + m)_{ij} \, i\Delta_+(x-y)$$

Where we have used the completeness relation of the $u^{(\alpha)}$'s and $v^{(\alpha)}$'s.
Similarly we have

$$\langle 0| \overline{\Psi}_i(x) \Psi_j(y) |0\rangle = (2\pi)^{-3} \int \frac{d^3p}{2p^0} e^{-ip[x-y]} (\not{p}-m)_{ji}$$

$$= (i\not{\partial}_x - m)_{ji} \, i\Delta_+(x-y)$$

This holds for both Bosons and Fermions (no choice made yet!).
We look at the time ordered product and get

$$T\left(\Psi_i(x), \overline{\Psi}_j(y)\right) = \Theta\left(x^0 - y^0\right) \Psi_i(x) \overline{\Psi}_j(y) - \Theta\left(y^0 - x^0\right) \overline{\Psi}_j(y) \Psi_i(x)$$

And we write the Feynman propagator as

$$iS_F(x-y)_{ij} = \langle 0| T\left(\Psi_i(x), \overline{\Psi}_j(y)\right) |0\rangle$$

$$iS_F(z) = \Theta(z^0)(i\not{\partial}_z + m) i\Delta_+(z) - \Theta(-z^0)(-i\not{\partial}_z + m) i\Delta_+(-z)$$

$$= (i\not{\partial}_z + m) i\Delta_F(z)$$

The Fourier representation of the Feynman propagator is often more usefull:

$$S_F(z) = \int \frac{d^4p}{(2\pi)^4} \frac{\not{p}+m}{p^2 - m^2 + i0} e^{-ipz}$$

We now consider only the Fermionic case

$$\mathbb{I} \propto \left\{\Psi_i, \overline{\Psi}_j(y)\right\} = \langle 0| \left\{\Psi_i(x), \overline{\Psi}_j(y)\right\} |0\rangle$$

$$= (i\not{\partial} - m)_{ij} \, i\Delta_+(x-y) + \underbrace{(i\not{\partial}_y - m)_{ij}}_{=-(i\not{\partial}_x+m)_{ij}} i\Delta_+(y-x)$$

$$= (i\not{\partial}_x + m)_{ij} \underbrace{(i\Delta_+(x-y) - i\Delta_+(y-x))}_{i\Delta(x-y)}$$

$$\implies \left\{\Psi_i(x), \overline{\Psi}_j(y)\right\} = 0 \text{ for space like seperation } (x-y)^2 < 0$$

This is anonther form of locality.

$$\implies \left[\Psi_i(x) \overline{\Psi}_k(x), \Psi_j(y)\right] = 0 \text{ for } (x-y)^2 < 0$$

This form of locality is acceptable because the obselvables are bilinear in $\Psi(x)$. For example the current is given as

$$J^\mu(x) =: \overline{\Psi}(x) \gamma^\mu \Psi(x) :$$

Remark on Spin and Statistics

- In the Bosonic case we have two options:

  1. $\left\{\Psi_i(x), \overline{\Psi}_j(y)\right\} \not\propto \mathbb{I}$ which is not good

  2. $\left[\Psi_i(x), \overline{\Psi}_j(y)\right] \propto \mathbb{I}$ in this case we can evaluate it as an expectation value of the vacuum, giving us

  $$\left[\Psi_i(x), \overline{\Psi}_j(y)\right] = (i\not{\partial}_x + m)_{ij} \left(i\Delta_+(x-y) + i\Delta_{(y-x)}\right)$$

  Which is not $= 0$ for $(x-y)^2 < 0$.
  We can therefore conclude that the locality of the free Dirac Field requires Fermionic quantization.

Generalization   In general for arbitrary fields (arbitrary Spin S, not neccesarily free fields), with the following two assumptions

1. Locality for $(x - y)^2 < 0$

   - Either $[\varphi(x), \varphi(y)] = 0$
   - or $\{\varphi(x), \varphi(y)\} = 0$

2. Stability $H \geq 0$

Then

- S an integer, we have to pick the first variant of the first assumption

- S half-integer, we have to take the second variant of the first assumption

Gauge Transformations   (global) $\Psi \to e^{-i\lambda}\Psi$ giving us

$$j^\mu(x) =: \overline{\Psi}(x)\gamma^\nu\Psi(x):$$

$$Q = \int_{x^0=t} d^3x : \Psi^*(x)\Psi(x):$$

$$= \sum_{\alpha=1,2} \int \frac{d^3p}{2p^0}\left(b_\alpha^*(p)b_\alpha(p) - d_\alpha^*(p)d_\alpha(p)\right)$$

This Noether Charge implements the gauge transformations

$$e^{i\lambda Q}\Psi(x)e^{-i\lambda Q} = e^{-i\lambda}\Psi(x)$$

5.4   Electro Magnetic Field - Covariant Quantization

We consider plane wave solutions of $\Box A_\mu = 0$

$$A_\mu(x) = \epsilon_\mu e^{\mp ikx}$$

with $k^2 = 0, k^0 > 0$. With polarizations

$$\epsilon_\mu^{(\lambda)} = \epsilon_\mu^{(\lambda)}(k) \text{ real, } \lambda = 0, \ldots, 3$$

such that

$$\epsilon^{(\lambda)} \cdot \epsilon^{(\lambda')} = g^{\lambda\lambda'}$$

A more specific choice can be made by first picking a time-like four vector $n \in \mathbb{R}^4, n \cdot n = 1, n^0 > 0$ (a frame of reference). We then pick

- $\lambda = 0$: $\epsilon^{(0)}(k) = n$

- $\lambda = 1, 2$: $\epsilon^{(i)}(k)^2 = -1, n \cdot \epsilon^{(i)}(k) = 0, k \cdot \epsilon^{(i)}(k) = 0$

- $\lambda = 3$: $\epsilon^{(3)}(k) = \frac{k-(k\cdot n)n}{k\cdot n}$

$$\implies \epsilon^{(0)}(k) + \epsilon^{(3)}(k) = \frac{k}{k\cdot n}$$

$$k \cdot \left(\epsilon^{(0)} + \epsilon^{(3)}\right) = 0$$

An example:

$$n = \left(1, \vec{0}\right) \qquad k \cdot n \qquad\qquad = k^0 = \left|\vec{k}\right|$$

$$\epsilon^{(0)} = \left(1, \vec{0}\right)$$

$$\epsilon^{(1,2)} = \left(0, \vec{\epsilon}^{(1,2)}\right)$$

$$\vec{\epsilon}^{(i=1,2)} \cdot \vec{\epsilon}^{(j=1,2)} = \delta_{ij}$$

$$\vec{\epsilon}^{(1,2)} \cdot \vec{k} = 0$$

$$\epsilon^{(3)} = \frac{\vec{k}}{\left|\vec{k}\right|}$$

Giving us the interpretation as

- $\epsilon^{(0)}$ scalar polerisation

- $\epsilon^{(1,2)}$ transverse polerisation

- $\epsilon^{(3)}$ longitudinal polerisation

Where the first and third option are unphysical - the price we pay for a covariant quantization.
We first write the classical general solution of the wave equation $\Box A_\mu = 0$

$$A_\mu (x) = (2\pi)^{-3} \sum_{\lambda=0}^{3} \int \frac{d^3 k}{2k^0} \epsilon_\mu^{(\lambda)} (k) \left(a_\lambda (k) e^{-ikx} + \overline{a_\lambda (k)} e^{ikx} \right)$$

$$= A_\mu^+ (x) + A_\mu^- (x)$$

$$\pi_\mu (x) = i (2\pi)^{-3} \sum_{\lambda=0}^{3} \int \frac{d^3 k}{2k^0} k^0 \epsilon_\mu^{(\lambda)} (k) \left(a_\lambda (k) e^{-ikx} - \overline{a_\lambda (k)} e^{ikx} \right)$$

With a_λ arbitraty.

Quantisazion   (Gupta, Bleuler, 1950)
At first we procede like usual (ignoring the gauge constraint) and impose CCR at $t = 0$.

$$[A_\mu (\vec{x}), \pi_\nu (\vec{y})] = ig_{\mu\nu} \delta (\vec{x} - \vec{y})$$
$$[A^\mu (\vec{x}), \pi_\nu (\vec{y})] = i\delta^\mu_\nu \delta (\vec{x} - \vec{y})$$

At variance with $[q, p] = i$ for $\mu = \nu = 0$. Equivalent to that is

$$[a_\lambda (x), a_{\lambda'} (k')] = -2k^0 g_{\lambda\lambda'} \delta \left(\vec{k} - \vec{k}' \right)$$

At variance with $[a, a^*] = +1$ for $\lambda = \lambda' = 0$.
Note: if $[a, a^*] = -1$ and $a |0\rangle = 0$ leads to $N = a^* a$ has spectrum $0, -1, -2, \dots$.
The Fieldmomentum (for bosonic statistics and normal ordered results)

$$P^\nu = \int \frac{d^3 k}{2k^0} k^\nu \left(\sum_{i=1}^{3} a_i^* (k) a_i (k) - a_0^* (k) a_0 (k) \right)$$

The negative spectrum of the $\lambda = 0$ case acctualy is correct since it appears after a minus. one finds

- $H = P^0 \geq 0$

- $\partial_t A (x) = i [H, A (x)]$ is $\Box A^\mu = 0$

Construction of Fock Space $\mathcal{F}$
We call $|0\rangle$ the vacuum ($\langle 0|0\rangle = 1$) and $a_\lambda (k) |0\rangle = 0$ for all λ, k. $\mathcal{F}$ generated by $a_\lambda^* (k)$ eg. for 1-particle states

$$|f\rangle = \int \frac{d^3 k}{2k^0} \sum_{\lambda=0}^{3} f_\lambda (k) a_\lambda^* (k) |0\rangle$$

$$\langle f|f\rangle = \sum_{\lambda,\lambda'} \int \int \frac{d^3 k}{2k^0} \frac{d^3 k'}{2k'^0} \underbrace{\langle 0| [a_\lambda (k), a_\lambda^* (k')] |0\rangle}_{-2k^0 g_{\lambda\lambda'} \delta(\vec{k} - \vec{k}')} \overline{f_\lambda (k)} f_{\lambda'} (k')$$

$$= \int \frac{d^3 k}{2k^0} \left(-|f_0 (k)|^2 + \sum_{i=1}^{3} |f_i (k)|^2 \right)$$

There are states $|f\rangle$ with $\langle f|f\rangle \leq 0$ (so called Ghost states).
With the interpretation of quantum mechanics being probabilistic. Suppose we have a system prepared in state $|\psi\rangle$ and we measure whether it's in state $|\varphi\rangle$. The probability to find the system in state $|\varphi\rangle$ is $p = \frac{|\langle \varphi|\psi\rangle|^2}{\langle \varphi|\varphi\rangle \langle \psi|\psi\rangle}$. This now obviously leads to a problem with the above result of ghost states where $\langle f|f\rangle \leq 0$.
It is no longer possible to give a interpretation if the denominator is smaller than 0.
But we still need to impose gauge constraints (Lorentz). We now look for a subspace (a physical subspace $\mathcal{F}_{\text{Phys}}$) such that

- $\partial_\mu A^\mu = 0$ is somehow realized

- $\langle f | f \rangle$ is positive in $\mathcal{F}_{\text{Phys}}$

We could first attemt to define $\psi \in \mathcal{F}_{\text{Phys}}$ as

$$\partial_\mu A^\mu (x) |\psi\rangle = 0$$

But this would fail as for $x = (0, \vec{x})$ we get

$$0 = \langle\psi| [\partial_\mu A^\mu (x), A_\nu (y)] |\psi\rangle$$
$$\text{by CCR only } \mu = 0 \text{ contributes:} \quad = - \langle\psi| [\psi^0 (x), A_\nu (y)] |\psi\rangle$$
$$= i\delta^0{}_\nu \delta (\vec{x} - \vec{y}) \langle\psi|\psi\rangle$$
$$= 0 \forall \psi$$

Which violates the second criterion.
We make a second attempt by first defining a subspace of the Fockspace $\tilde{\mathcal{F}}$ as $\psi \in \tilde{\mathcal{F}}$

$$\partial_\mu A^{+\mu} (x) |\psi\rangle = 0$$

Leading to

$$\langle\psi| \partial_\mu A^\mu |\psi\rangle = 0$$

Using the mode expansion for A we get

$$\partial_\mu A^{+\mu} (x) = -i (2\pi)^{-\frac{3}{2}} \sum_{\lambda=0}^{3} \int \frac{d^3 k}{2k^0} k_\mu \epsilon^{\lambda\mu} a_\lambda (x) e^{-ikx}$$

$$= -i (2\pi)^{-\frac{3}{2}} \int \frac{d^3 k}{2k^0} k \cdot \epsilon^{(3)} (k) (a_3 (k) - a_0 (k)) e^{-ikx}$$

$$\implies (a_3 (k) - a_0 (k)) |\psi\rangle = 0 \forall k \in V_0$$

We claim (proof in the lecture materials): For $\psi \in \tilde{\mathcal{F}}$ we have that $\langle\psi|\psi\rangle \geq 0$ ie. it is positive semi-definite. However this is enought for the couchy-schwarz inequality

$$|\langle\psi|\varphi\rangle|^2 \leq \langle\psi|\psi\rangle \langle\varphi|\varphi\rangle$$

5.4.1 Part of lecture 2.12. missing

6
Interacting Fields

6.1 Introduction (missing)

6.2 The LZS asymptotic condition (partially missing, beginning missing)

One sheet of handwritten notes (both sides) is allowed at the exam.
We consider the idea of scattering theory: For large times $t \to \pm\infty$ (some) states look like independent free particles. Such a situation should be describable by a free quantum free theory associated to asymtptotically free particles.

$$\varphi(t, \vec{x}) \underset{t \to -\infty}{\longrightarrow} \varphi_{\text{in}}(t, \vec{x}) \, \tilde{z}^{\frac{1}{2}}$$

$$\varphi(t, \vec{x}) \underset{t \to +\infty}{\longrightarrow} \varphi_{\text{out}}(t, \vec{x}) \, z^{\frac{1}{2}}$$

Where $\varphi_{as}(x)$ (as = in/out) are free fields having vacuum $|\Omega\rangle$ and 1-particle states $|p\rangle$.

- Why is $\tilde{z} = z$?
  Let $x = (x^0 + t, \vec{x})$, $y = (y^0 + t, \vec{y})$ let $t \to \pm\infty$:

$$z i \underbrace{\Delta_+ (x - y; m^2)}_{\text{independent of } t} = \langle\Omega| \varphi(x) P(M^2 = m^2) \varphi(y) |\Omega\rangle$$

$$= \int \frac{d^3 p}{2p^0} \langle\Omega| \varphi(x) |p\rangle \langle p| \varphi(y) |\Omega\rangle$$

$$\underset{\text{LSZ}}{\longrightarrow} \tilde{z} \int \frac{d^3 p}{2p^0} \langle\Omega| \varphi_{\text{as}} |p\rangle \langle p| \varphi_{\text{as}}(y) |\Omega\rangle$$

$$= \tilde{z} \cdot i \Delta_+ (x - y; m^2)$$

Note: $(\Box + m^2) \varphi_{\text{as}}(x) = 0$.

- The interpretation of the LSZ condition given above: $\varphi_{\text{as}(x)}$ is a free field of mass m

$$\varphi_{\text{as}}(x) = (2\pi)^{-\frac{3}{2}} \int \frac{d^3 p}{2\omega(p)} \left(a_{\text{as}}(p) e^{-ipx} + \text{h.c.} \right)$$

$$\text{with: } \omega(p) = \sqrt{\vec{p}^2 + m^2}$$

By consruction we have

$$|p\rangle = a_{\text{as}}^*(p) |\Omega\rangle$$

is the free state. For a free field we therefore get the same asymptote for the past and the future (in/out).
We write

$$|p_1, \ldots, p_n\rangle_{\text{as}} := a_{\text{as}}^*(p_1) \ldots a_{\text{as}}^*(p_n) |\Omega\rangle$$

which describes states which for $t \to \pm\infty$ consist of n outgoing/incomming free particles. We interpret $a_{\text{as}}^*(p)$ as adding a particle (non-free!) of *asymptotic* momentum $p \in V_m$.

A more precise form of LSZ is:

Let $f = f(t, \vec{x})$ be a solution of the KG equation $(\Box + m^2) f = 0$ with positive energy. Then

$$i \int_{x^0 = t} d^3 x\, \varphi(t, \vec{x})\, \overset{\leftrightarrow}{\partial_0} f(t, \vec{x}) \underset{t \to \pm\infty}{\to} z^{\frac{1}{2}} i \underbrace{\int_{x^0 = t} d^3 x\, \varphi_{\text{as}}(t, \vec{x})\, \overset{\leftrightarrow}{\partial_0} f(t, \vec{x})}_{\text{independent of } t}$$

$$= z^{\frac{1}{2}} \int \frac{d^3 p}{2\omega(\vec{p})} f(p)\, a_{\text{as}}^*(p)$$

$$= z^{\frac{1}{2}} a_{\text{as}}^*(f)$$

For plane waves $f(x) = (2\pi)^{-\frac{3}{2}} e^{-iqx}$ with $(q \in V_m)$, $f(p) = 2\omega(p) \delta(\vec{p} - \vec{q})$, therefore the limit above gives

$$\implies (2\pi)^{-\frac{3}{2}} i \int_{x^0 = t} d^3 x\, \varphi(x)\, \overset{\leftrightarrow}{\partial_0} e^{-iqx} \underset{t \to \pm\infty}{\to} z^{\frac{1}{2}} a_{\text{as}}^*(q)$$

6.3 The LSZ reduction formula

The goal of this section is to express scattering amplitudes $_{\text{out}}\langle p_1, \ldots, p_k | q_1, \ldots, q_n \rangle_{\text{in}}$ in terms of time ordered vacuum expectation values (VEVs), also called n-point functions (in the example above $n + k$-point function) or Green's functions.

Let $p_i = (\omega(\vec{p}_i), \vec{p}_i)$, $q_i = (\omega(\vec{q}_i), \vec{q}_i)$ with $p_i \neq q_j$ for all i, j. Then

$$_{\text{out}}\langle p_1, \ldots, p_k | q_1, \ldots, q_n \rangle_{\text{in}} = \left(\frac{iZ^{-\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \right)^{k+n} \int \prod_{i=1}^{k} \left(d^4 y_i\, e^{ip_i y_i} (\Box_{y_i} + m^2) \right) \cdots$$

$$\cdots \int \prod_{j=1}^{n} \left(d^4 x_j\, e^{-iq_j x_j} (\Box_{x_j} + m^2) \right) \langle \Omega | T \left(\prod_i \varphi(y_i) \prod_j \varphi(x_j) \right) | \Omega \rangle$$

Remarks:

- As a recipe: in $_{\text{out}}\langle p_1, \ldots, p_k | q_1, \ldots, q_n \rangle$ as

  - replace $|q_j\rangle \to e^{-iq_j x_j} (\Box_{x_j} + m^2) \varphi(x_j)$ and $|p_x\rangle \to e^{-ip_i x_i} (\Box_{x_i} + m^2) \varphi(x_i)$
  - take $\langle \Omega | T \,(\text{all fields}) | \Omega \rangle$
  - Inegrate over all $d^4 y_j, d^4 x_i$

- If $p_i = q_j$, add terms $\propto \delta(\vec{p}_i - \vec{q}_j) |p_1, \ldots, \hat{p}_i, \ldots, p_k\rangle |q_1, \ldots, \hat{q}_j, \ldots, q_n\rangle$.

- Let $G(p_1, \ldots, p_n) = \int d^4 x_1 \ldots d^4 x_n\, e^{-i \sum_{j=1}^{n} p_j x_j} \langle \Omega | T \,(\text{fields}) | \Omega \rangle$ the fourier transform of the VEVs. Then

  $$_{\text{out}}\langle p_1, \ldots, p_k | q_1, \ldots, q_n \rangle = \left(\frac{-iZ^{-\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} \right) \prod_{i=1}^{k} (p_i^2 - m^2) \prod_{j=1}^{n} (q_j^2 - m^2)\, G(-p_1, \ldots, -p_k, q_1, \ldots, q_n)$$

  $$\neq 0 \text{ at } p_i^2 = m^2, q_j^2 = m^2$$

  Example: free field

  $$\langle \Omega | T (\varphi(x) \varphi(y)) | \Omega \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2 + i0} \quad \text{Feynman propagator}$$

  $$\implies G(p, q) = \frac{\delta(p + q)}{p^2 - m^2 + i0}$$

- If we substitute $x_i \to x_i + a$, $y_j \to y_j + a$ in the big formula above, we get

  $$\left(1 - e^{i(\sum_i p_i - \sum_j q_j) a} \right) {}_{\text{out}}\langle p_1, \ldots, p_k | q_1, \ldots, q_n \rangle = 0$$

  $$\implies \delta^{(4)} \left(\sum_i p_i - \sum_j q_j \right) \text{ can be factored out from } {}_{\text{out}}\langle \ldots | \ldots \rangle_{\text{in}}$$

- Idea of proof: from the fundamental theorem of calculus we have

$$\lim_{t_2 \to \infty, t_1 \to -\infty} \int d^3x\, f\left(t, \vec{x}\right) \big|_{t=t_1}^{t=t_2} = \int d^4x \frac{\partial}{\partial t} f\left(t, \vec{x}\right)$$

$$\partial_0 \left(f \partial_0^{\leftrightarrow} g\right) = f\left(\partial_0^2 g\right) - \left(\partial_0^2 f\right) g$$

The discussion here is valid for the real scalar field, the discussion for fermions and photons was done on the slides and is not copied here.

6.3.1 Application: Compton scattering

In the following we use the lable i for initial and f for final. For compton scattering we consider an electron and a photon scattering of each other. We therefore have p_i, α_i and p_f, α_f (momentum, polarization) of the electron and k_i, ϵ_i / k_f, ϵ_f of the photon. We therefore have $|i\rangle_{\text{in}} = |(p_i, \alpha_i), (k_i, \epsilon_i)\rangle$ and $\langle f|_{\text{out}} = \langle (p_f, \alpha_f), (k_f, \epsilon_f)|$. We calculate

$$\langle f|i\rangle = \frac{Z_2^{-1} Z_3^{-1}}{(2\pi)^6} \int dx_1 dx_2 dz_1 dz_2 \exp\left(-i\left(p_i x_1 + k_i z_1 - p_f x_2 - k_f z_2\right)\right) \overline{u}^{(\alpha_f)}(p_f) \left(i\overrightarrow{\partial}_{x_2} - m\right)$$

$$\langle \Omega | T \dots \Psi(x_2) \epsilon^{(f)}(k_f) A(z_2) \overline{\Psi}(x_1) \epsilon^{(i)}(k) A(z_1) |\Omega\rangle$$

$$\dots \left(-i\overleftarrow{\partial}_{x_1} - m\right) u^{(\alpha_i)}(p_i)$$

6.4 S-matrix and scattering cross section

The S-matrix is an operator (better S-operator) acts on in-states and is defined as

$$_{\text{out}}\langle p_1, \dots, p_n | q_1, \dots, q_k\rangle_{\text{in}} =_{\text{in}} \langle p_1, \dots, p_n | S | q_1, \dots, q_k\rangle_{\text{in}}$$

The idea is to connect the two states carrying label α which are $|\alpha\rangle_{\text{in}}$ and $|\alpha\rangle_{\text{out}}$. We want to understand the scattering as a map between labels $\alpha \to \beta$ not between states. This is then $S|\alpha\rangle_i = |\beta\rangle$ but $|\beta\rangle_{\text{in}} = |\alpha\rangle_{\text{in}}$ meaning $S|\beta\rangle_{\text{out}} = |\beta\rangle_{\text{in}}$. Conversely $\langle\beta|_{\text{in}} = \langle\beta|_{\text{out}} S^*$. We now just use in states so we drop the lable i for in.
Let's set

$$S = \mathbb{I} + iT$$

We write

$$\langle p_1, \dots, p_k | T | q_1, \dots, q_n\rangle = (2\pi)^4 \delta\left(\sum_i p_i - \sum_j q_j\right) \cdot (2\pi)^{-\frac{3}{2}(k+n)} T(p_1, \dots, p_k; q_1, \dots, q_n)$$

We now consider $n = 2$ and write the scattering cross section $d\sigma$, it is defined as the number of scattering events of $p_1, \dots, p_k$ into $d^3 p_1 \dots d^3 p_k$, per particle in the beam and per particle in the target.

$$d\sigma = \frac{(2\pi)^{-3}}{4\left((q_1 q_2)^2 - m_1^2 m_2^2\right)^{\frac{1}{2}}} (2\pi)^4 \delta\left(P_f - q_1 - q_2\right) \cdot |T(p_1, \dots, p_k, q_1, q_2)|^2 \frac{d^3 p_1}{1 p_1^0} \dots \frac{d^3 p_k}{2 p_k^0}$$

with $P_f = \sum_i^k p_i$.
Remarks:

- We have to integrate over unresolved momenta (or polerazation, or other unobserved quantities)

- $d\sigma$ i Lorentz invariant.

- In the frame of the target (typically the lab frame), $q_2 = \left(m_2, \vec{0}\right)$ giving us $q_1 q_2 = q_1^0 q_2^0 \vec{q}_1 \vec{q}_2 = q_1^0 m_2$ meaning $(q_1 q_2)^2 - m_1^2 m_2^2 = \left(q_1^{02} - m_1^2\right) m_2^2 = |\vec{q}_1|^2 m^2$. The denominator in $d\sigma$ is then $4|\vec{q}_1| m_2$.

- This then allows for the derivation of Fermi Golden Rule (see lecture materials).

Increasingly more special cases:

- $n = k = 2$:
  The differential cross section of $1, 2 \rightarrow 1', 2'$ $(q_1, q_2) \rightarrow (p_1, p_2)$ where we do not observe $2'$ and only observe the direction of $1'$ $\vec{p}_1 = |\vec{p}_1| \vec{e}_1$. In the rest frame of particle 2 we then get

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 |\vec{q}_1| m_2} \frac{|\vec{p}_1|^2}{|\vec{p}_1| p_2^0 - p_1^0 \vec{p}_2 \cdot \vec{e}_1} |T(p_1, p_2; q_1, q_2)|^2$$

- $m_1 = 0$ meaning $\vec{p}_1 = p_1^0 \vec{e}_1$ giving us

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 |\vec{q}_1| m_2} \frac{|\vec{p}_1|^2}{p_1 p_2} |T(p_1, p_2; q_1, q_2)|^2$$

- Elastic scattering (same species of particles before and after). This means $p_1 p_2 = q_1 q_2 = |\vec{q}_1| m_2$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_2^2} \frac{|\vec{p}_1|^2}{|\vec{q}_2|^2} |T(p_1, p_2; q_1, q_2)|^2$$

7
Perturbation Theory

7.1 Missing

7.2 φ^4 Feynman rules

Parts missing.
No line can connect a vertex to itself (as per wicks lemma for products of normal ordered factors).

7.2.1 Feynmanrules for the Numerator

1. Draw all topologically disticnt diagrams wtih label $\begin{cases} x_1, \ldots, x_n & \text{for external} \\ y_1, \ldots, y_p & \text{for internal} \end{cases}$ verticies.
   Then for each diagram you drew this way do steps 2 - 5.

2. For each internal vertex write a factor of $-i\lambda$

3. For each line between two verticies z_i, z_j write a factor of $\langle 0 | T \left(\varphi_0 \left(z_i \right) \varphi_0 \left(z_j \right) \right) | 0 \rangle = i\Delta_F \left(z_i - z_j \right)$.

4. Integrate over internal vertecies:
$$\int d^4 y_1 \ldots d^4 y_p$$

   Remark: we should lable the valences (half-line comming from a vertex), but we don't do so. Each diagram we draw contributes with a certain weight w :
$$w = \left(\frac{1}{4!} \right)^p m$$

   with m the number of diagrams that coincide upon forgetting the labeling of valences, m is also the number of ways to connect the valences. Note: $m \neq (4!)^p$ since not all permutations of valences lead to new Wick-pairings.
   We can also write
$$w = \frac{1}{S}$$

   Where S is the order of the group of permutations of lines of the diagram that leave it invariant. S is called the symmetry factor.

5. Divide by S and by $p!$. (multiply by $\frac{1}{Sp!}$)

6. Sum over all diagrams you got from rule 1.

Remarks

- For n odd external veticies no diagram exists. In particular $G \left(x_1, \ldots, x_{2l+1} \right) = 0$

- Diagrams differing in the labels of the internal verticies contribute the same. This leads to a version of the Feynman rules where we have to adjust the weight of each diagram. We'd get:
  Rule 1': Only lable external veticies but leave internal internal vetrilcies unlabelled.
  Rule 5': New weight given by
$$w' = \frac{1}{p!} \left(\frac{1}{4!} \right)^p m_1 m_2 = \frac{1}{S \cdot S'}$$

where m_1 is the number of ways to connect external verticies and m_2 is the number of ways to connect internal verticies. And where S' is the order of the group of the permutations of the internal verticies that leave the diagram invariant (invariant in the sense of rule 1, not of rule 1').

- Disconnected components of a diagram contribute multiplicatively (they factorize).

To get back to $G(x_1, \ldots, x_n) = \frac{N}{D}$ with huge numerator N and denominator D. We now from before that $D = N$ for $n = 0$ ie. the denomitanor only contains vacuum diagrams or bubbles. This includes $p = 0$ the empty bubble which contributes 1.
On the other hand we hae N with n external verticies, in this case we can distinguish two types of diagrams:

1. Contain a (non-empty) bubble among it's connected components

2. does not contain a non-empty bubble, this category we called linked diagrams.

We then get (symbolically)

$$\implies \frac{N}{D} = \frac{(\text{linked diagrams})(1 + (\text{bubbles}))}{\underbrace{1}_{\text{empty bubbles}} + (\text{bubblus})} = (\text{linked diagrams})$$

We therefore get:

$$\implies G(x_1, \ldots, x_n) = \sum_{n=0}^{\infty} \frac{(-i)^k}{k!} \int d^4 y_1^4 y_k \, \langle 0| \, T \, [\varphi_0(x_1) \ldots \varphi_0(x_n) : \mathcal{H}_I(\varphi_0(y_1)) : \ldots : \mathcal{H}_H(\varphi_0(y_k)) :] \, |0\rangle_0$$

Where we noted that we only consider connected diagrams with the $\langle 0| \ldots |0\rangle_0$.

Remarks

- The derivation is done more rigerously ni the notes

- Integrand deas decay as any of the $y_i \to \infty$

- No bubbles: no ∞-valume divergences

7.2.2 Greens functions in Fourierspace

$$G(p_1, \ldots, p_n) = \int d^4 x_1^4 x_n e^{-i \sum_i^n p_i x_i} G(x_1, \ldots, x_n)$$

We'd now like to get Feynman rules for the fourier transformed grenes functions. We recall that previously we had factors of the form:

$$i\Delta_F(z_i - z_j) = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i0} e^{-ik(z_i - z_j)}$$

This then gives us the

Feynman rules in momentum space

1. Draw all topographically distinct linked diagrams with

   - n labelled external verticies with momenta $p_1, \ldots, p_n$ oriented inwards
   - any number of internal verticies
   - internal lines labelled for the application of further rules, not for topological distigtion, carrying momenta $k_1, \ldots, k_m$.

2. For the i-th external line we write a factor $i \left(p_i^2 - m^2 + i0\right)^{-1}$

   (Exception: for a so called short-circuited line the factor is $i \left(p_i^2 - m^2 + i0\right)^{-1} (2\pi)^4 \delta(p_i + p_j)$. A short circuited line is a line that directly connects two external verticies directly)

3. For the ℓ-th internal line we write a factor $i \left(k_\ell^2 - m^2 + i0\right)^{-1}$

4. For each internal vertex we write a factor $-i\lambda (2\pi)^4 \delta\left(\sum_\ell \sigma_\ell k_\ell\right)$ (where σ_ℓ gives the sign of the momentum of line at vertex ℓ, ± 1 for incommeing/outgoing momenta)

5. Integrate over internal momenta $k_1, \ldots, k_m$.

6. Divide by $S \cdot S'$

7. Sum over all diagrams.

Note: the resulting integrals over the internal momenta are divergent for $k_\ell \to \infty$ requiring renormalization.

Remark

- A factor of $(2\pi)^4 \delta \left(\sum_i p_i \right)$ can always be facored out.

- Rules for $\langle p_1, \ldots, p_k | q_1, \ldots, q_n \rangle$ are given in terms of the greens functions in momentumspace just as we now calculated them. $G(-p_1, \ldots, -p_k, q_1, \ldots, q_n)$ we have to modify rule 2 silghtly to include the prefactors of the LSZ reduction formula: an external line yields: $(2\pi)^{-\frac{3}{2}} Z^{-\frac{1}{2}}$ (external propagator essentially dropped - truncated diagrams).

- The reduced matrix elements $iT(p_1, \ldots, p_k, q_1, \ldots, q_n)$ also drop:

  - $(2\pi)^{-\frac{3}{2}}$
  - $(2\pi)^4 \delta \left(\sum_i^k p_i - \sum_j^n q_j \right)$

7.3 Feynman rules for QED

The lagrangian of the QED field is given as

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu A_\nu \right) \left(\partial^\mu A^\nu \right) + \overline{\Psi} \gamma^\mu \left(i \partial_\mu - e A_\mu \right) \Psi - m \overline{\Psi} \Psi$$

We write

$$\mathcal{L}_0 = \mathcal{L}(e = 0)$$
$$\mathcal{L}_I = -e \overline{\Psi} \gamma^\mu A_\mu \Psi$$
$$\implies H_I = e \int_{x^0 = 0} d^3 x : \overline{\Psi}_b(x) A_\mu(x) (\gamma^\mu)_{ab} \Psi_a(x):$$

The Greens fuction is give as

$$G(x_1, \ldots, x_n; x_{n+1}, \ldots, x_{2n}; y_1, \ldots, y_p) = \langle \Omega | T \left[\Psi_{a_1}(x_1) \ldots \Psi_{a_n}(x_n) \overline{\Psi}_{b_1}(x_{n+'}) \ldots \overline{\Psi}_{b_n}(x_{2n}) A_{\mu_1}(y_1) \ldots A_{\mu_p}(y_p) \right] | \Omega \rangle$$

Remarks

- On the lhs inices a_1, b_1, μ are subsumed in x_i, y_j.

- By global gauge invariance the number of Ψ must equal the number of $\overline{\Psi}$, or else $G = 0$. Since $|\Omega\rangle \to |\Omega\rangle$ but $\Psi \to e^{-i\lambda} \Psi$ and $\overline{\Psi} \to e^{i\lambda} \overline{\Psi}$, meaning $G \to E^{-i\lambda D} G$ for D the difference in number between PSi and $\overline{\Psi}$, giving us that either G or D must be 0.

- Diagarmatic expression for G gives us the VEV of the free fields in a free vacuum.

The ingredients for our diagrams are

- for $A\mu$ gives us an external vertex

- for $\overline{\Psi}_b(x)$ gives out an external vertex with a second line type with outgoing orientation

- for $\Psi_a(x)$ we get an extaral vertex with a second line type and an incomming orinentation.

- For $\overline{\Psi}_b(z) A_\mu(z) \Psi_a(z)$ we get an internal vertex with two type two lines and one type one line.

We then consider all possible contractions as per Wicks lemma, for example

$$\langle 0| T A_\mu (z_i) A_\nu (z_j) |0\rangle = -i g_{\mu\nu} \Delta_F (z_i - z_j)$$

$$\langle 0| T \Psi_a (z_i) \overline{\Psi}_b (z_j) |0\rangle = i S_F (z_i - z_j)$$

$$\langle 0| T \Psi_a (z_i) \Psi_b (z_j) |0\rangle = 0$$

$$\langle 0| T \overline{\Psi}_a (z_i) \Psi_b (z_j) |0\rangle = 0$$

$$\langle 0| T \Psi_a (z_i) A_\mu (z_j) |0\rangle = 0$$

We now draw diagrams with complete pairing sand with given external veticies. Note:

- Ferimon lines will tie up to oriented chains

- a chain of such fermion liens can either

  - be a loop
  - be a open line connecting some external veticies $\overline{\Psi}_b (x_{n+i})$ to $\Psi (x_i)$. This then defines a map from $\sigma : i \to j$ which is a permutation.

7.3.1   Feynman rules for QED in momentum space

The fourier transform of the greens function is

$$G (p_1, \ldots, p_n; p_{n+1}, \ldots, p_{2n}; q_1, \ldots, q_p)$$

with the additional indicies a, b, μ subsumed.

Feynman rules

1. Draw all topologically disticnt, linked diagrams with n labelled incomming fermionic lines and n labelled outgoing fermionic lines with momenta $p_1, \ldots, p_n$ respectively $p_{n+1}, \ldots, p_{2n}$. and p photon lines with momenta $q_1, \ldots, q_p$. Here again we orient all momenta inwards.

2. Factors

   - $-\frac{i}{k^2 + i0} g_{\mu\nu}$ for any internal photon lines of momentum k

   - $\frac{i(\not{k} + m)_{ab}}{k^2 - m^2 + i0}$ for internal fermion lines of momentum k

   - $-ie (\gamma^\mu)_{ba} (2\pi)^4 \delta (\sum \text{"incomming momenta"})$ for verticies with two ferimon and one photon line

   Sum over all the indicies a, b, μ.

3. Integrate over all internal momenta k_ℓ :
$$\int \prod_\ell \frac{dk_\ell}{(2\pi)^4}$$

   and sum over all indicies a, b, μ.

4. Multiply by -1 for each loop in the diagram and by the sign of the permutation σ; ie add a factor of $(-1)^{\frac{n(n-1)}{2}} \operatorname{sgn}(\sigma)$

5. Sum over all diagrams with loops of even lenght.

Remark

- In contrast to the φ^4 theory in QED we have no symmetry factors.

- Furry's Lemma: Two topologically distinct loops with identicall external labels contribute oppositely. In rule 5 we therefore only need to consider loops of even length, as the odd ones cancel each other.

7.4 Compton Scattering

We are going to compute Feynman diagrams for reduced T-matrix elements $c \to \frac{d\sigma}{d\Omega}$, given by the summ of all linked diagrams of the form of a electron and a photon scattering off each other. The incomming electron has momentum and polarisation p_i, η^i and the incomming photon has momentum and polarisation k_i, ϵ^i. The outgoing particles have corresponding momenta k_f, ϵ^f, p_f, η^f. We consider this scenario to lowest order. The lowest possible order are diagrams with two internal verticies (order $O\left(e^2\right)$); there are two such diagrams (see lecture materials), we'll call them the crossed version and the uncrossed version. Both of these diagrams are so called tree diagrams, ie. they have no loops.

The two diagrams also have common factors $e^2 z_2^{-1} z_3^{-1}$. For the free theory these z 's would just be $= 1$, for the interacting theory these will be of the form $z_i = 1 + O\left(e^2\right)$, since we decided to only consider up to first order can ignore the correction and write $z_i \to 1$.

We now apply the feynman rules and get: The integration over $d^4 p$ between the internal verticies and the delta function we can write that in the uncrossed diagram we have $p = p_i + k_i$, while in the crossed diagram we get $p = p_i - k_i$.

We consider the left internal vertex of the uncrossed diagram and get

$$-ie\bar{u}_b^{(f)}\left(\gamma^\mu\right)_{ba} \epsilon_\mu^f = -ie\left(\bar{u}\slashed{\epsilon}^f\right)_a$$

Hence: $iT\left(p_f, k_f; p_i, k_i\right) = (\text{crossed}) + (\text{uncrossed})$

$$= (-ie)^2 \bar{u}^f \left(\slashed{\epsilon}^f \frac{\slashed{p}_i + \slashed{k}_i + m}{2p_i k_i + i0} \slashed{\epsilon}^i + \slashed{\epsilon}^i \frac{\slashed{p}_i - \slashed{k}_f + m}{-2p_i k_f + i0} \slashed{\epsilon}^f \right) u^i$$

Note:

- The contribution of the two diagrams (first and second term) are related by $\left(\epsilon^f, k_i\right) \leftrightarrow \left(\epsilon^i, k_f\right)$.

In a typical compton experiment the electron is not observed, only the outcomming photon (direction, polarisation) of the outgoing photon is observed. Simmilarly for the incomming particles, in particular the incomming electron is not polarized. This is now enough to compute $|T|^2$, but replace it by $\frac{1}{2} \sum_{\alpha,\beta=1}^{2} |T|^2$, where we averaged over the incomming electron polarization and sum over the outgoing ones.

We write $M = \slashed{\epsilon}^f \frac{\slashed{p}_i + \slashed{k}_i + m}{2p_i k_i} \slashed{\epsilon}^i + \slashed{\epsilon}^i \ldots$:

$$\frac{1}{2}\sum_{\alpha,\beta} |T|^2 = \frac{1}{2}e^4 \sum_{\alpha,\beta=1}^{2} \left| \underbrace{u^\beta\left(p_f\right) M u^\alpha\left(p_i\right)}_{=c\in\mathbb{C}} \right|^2$$

We can now ask about $\bar{c}$ and get

$$\bar{c} = u^\alpha\left(p_i\right)^* \underbrace{\quad}_{\gamma^0\gamma^0} M^* \underbrace{\quad}_{\gamma^0\gamma^0} \bar{u}\left(\beta\right)\left(p_f\right)^*$$

$$\bar{c} = \bar{u}^\alpha\left(p_i\right) \overline{M} u^\beta\left(p^f\right) \text{'}$$

$$\overline{M} := \gamma^0 M^* \gamma^0 = \slashed{\epsilon}^i \frac{\slashed{p}_i + \slashed{k}_i + m}{2p_i k_i} \slashed{\epsilon}^f + \slashed{\epsilon}^f \frac{\slashed{p}_i + \slashed{k}_f + m}{-2p_i k_f} \slashed{\epsilon}^i$$

With this we can rewrite:

$$\frac{1}{2}e^4 \sum_{\alpha,\beta=1}^{2} \left| \underbrace{u^\beta\left(p_f\right) M u^\alpha\left(p_i\right)}_{=c\in\mathbb{C}} \right|^2 = \frac{1}{2}e^4 \sum_{\alpha,\beta} \bar{u}^\beta\left(p_f\right) M u^\alpha\left(p_i\right) \bar{u}^\alpha\left(p_i\right) \overline{M} u^\beta\left(p_f\right)$$

$$= \frac{1}{2}e^4 \text{tr}\left(\left(\slashed{p} + m\right) M \left(\slashed{p} + m\right) \overline{M}\right)$$

$$= \underbrace{\cdots}_{\text{tedious calculation}}$$

$$= e^4 \left(\frac{p_i k_f}{p_i k_i} + \frac{p_i k_i}{p_i k_f} + 4 \underbrace{\left(\epsilon^f \epsilon^i\right)^2}_{=0\cdot0 - \bar{\epsilon}^f \cdot \bar{\epsilon}^i} - 2 \right)$$

Where we've used that $\{\not{a}, \not{b}\} = 2ab$.

We can now compute $\frac{d\sigma}{d\Omega}$ in the rest frame of the incomming electron (ie. the lab frame).

$$p_i = \left(m, \vec{0}\right)$$

$$\implies p_i k_x = m \left|\vec{k}_x\right|$$

$$\implies \frac{d\sigma}{d\Omega} = \underbrace{\frac{1}{64\pi^2 m^2}}_{\frac{r_0^2}{4}} \frac{\left|\vec{k}^f\right|^2}{\left|\vec{k}^i\right|^2} \left(\frac{\left|\vec{k}^f\right|}{\left|\vec{k}^i\right|} + \frac{\left|\vec{k}^i\right|}{\left|\vec{k}^f\right|} + 4\left(\vec{\epsilon}^f \cdot \vec{\epsilon}^i\right)^2 - 2\right)$$

With $r_0 = \frac{e^2}{4\pi m}$ the classical electron radius. Which is the so called klein-Nishina formula, which was already found before field theory in 1929 (validation of field theory).

Remarks:

- In the low energy limit one regains the formula from Thomson-Scattering

-

8
Renormalization

8.1 Two point function of a scalar φ^3 theory

We start with the Kaellen-Lehmann representation of the 2 pt function which then carries over to the time ordered 2 point function

$$G(x) = \langle \Omega | T (\varphi(x) \varphi(0)) | \Omega \rangle$$

as

$$G(x) = zi\Delta_{\mathrm{F}}\left(x; m^2\right) + \int_0^\infty d\rho^2 \left(m'^2\right) i\Delta_{\mathrm{F}}\left(x; m'^2\right)$$

We consider it's fourier transform:

$$G(p) = \int d^4x\, e^{-ipx} G(x)$$

$$\implies G(p) = z\frac{i}{p^2 - m^2 + i0} + \int_0^\infty d\rho^2 \left(m'^2\right) \frac{i}{p^2 - m'^2 + i0} = g\left(p^2\right)$$

.

We expect the spectrum of $M^2 = P^\mu P_\mu$ qualitatively stable under perturbation of free theory, in particular the free theory has mass m_0, therefore we expect that the perturbed theory still has an isolated, maybe different, eigenvalue $m \neq m_0$.

In particular we expect $g\left(p^2\right)$ is analytic in p^2 in a domain given by the picture in the lecture material (the complement of $\{m^2\} \cup \left[(2m)^2, \infty\right)$).

We do have that we have a cut singularity for $p^2 > (2m)^2$ since

$$g\left(p^2 + i\epsilon\right) - g\left(p^2 - i\epsilon\right) \underbrace{=}_{\epsilon \to 0} i \int d\tilde{\rho}\left(m'^2\right) \underbrace{\left(\frac{i}{p^2 - m'^2 + i\epsilon} - \frac{i}{p^2 - m'^2 - i\epsilon}\right)}_{\to -2\pi i \delta(p^2 - m'^2)}$$

$$= 2\pi \tilde{p}'\left(p^2\right)$$

where we found a discontinuity at the cut.

We'd now like to ask the question: Can perturbation theory confirm this picture? And what are m^2 and z?

These two quantities can be computed in perturbation theory by finding the pole m^2 and it's residual z of $g\left(p^2\right)$.

φ^3 Theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - \frac{m_0^2}{2}\varphi^2 - \frac{\lambda}{3!}\varphi^3$$

$$\implies \mathcal{L} = \mathcal{L}_0 - \mathcal{L}_I$$

With the bare mass m_0.

We consider the time ordered two point function

$$G(p, p') = (2\pi)^4 \delta(p + p') g\left(p^2\right)$$

The contributing diagrams are given in the lecture material (one with 0 internal verticies, one with two, three with four, etc.).
Giving us:

$$g\left(p^2\right) = \frac{i}{p^2 - m_0^2 + i0} + \frac{i}{p^2 - m_0^2 + i0}\left(-i\lambda^2 \Sigma_2\left(p^2, m_0^2\right)\right)\frac{i}{p^2 - m_0^2 + i0} + O\left(\lambda^4\right)$$

$$-i\Sigma_2\left(p^2, m_0^2\right) = \underbrace{\frac{1}{2}}_{\frac{1}{SS'}}\left(-i\lambda\right)^2 \int \frac{dk}{(2\pi)^4}\frac{i}{k^2 - m_0^2 + i0}\frac{i}{(p-k)^2 - m_0^2 + i0}$$

Which is now up to $O\left(\lambda^2\right)$ we can rewrite $g\left(p^2\right)$ as

$$g\left(p^2\right) = \frac{i}{p^2 - m_0^2 + i0}\left(1 - \lambda^2\frac{\Sigma_2\left(p^2, m_0^2\right)}{p^2 - m_0^2 + i0}\right)^{-1}$$

$$\text{because: } (1-q)^{-1} = 1 + q + q^2 + q^3 + \dots$$

Where we have "resummed" all diagrams of the form with individual blobs (see graphics in lecture material). We can then even rewrite this nicer as

$$g\left(p^2\right) = \frac{i}{p^2 - m_0^2 - \lambda^2 \Sigma_2\left(p^2, m_0^2\right)}$$

While the free theory had a pole at m_0^2, but this new $g\left(p^2\right)$ now has a pole at a different m. Σ is called the selfs-energy, and Σ_2 is the self-energy at order $O\left(\lambda^2\right)$.
This all looks nice but: Σ_2 is logarithmically divergent. We write

$$\Sigma_2\left(p^2, m_0^2\right) = \frac{i}{32\pi^4}\left(\int d^4k \frac{1}{(k^2 - m_0^2 + i0)\left((p-k)^2 - m_0^2 + i0\right)} - \frac{1}{(k^2 - m_0^2 + i0)^2} + \int d^4k \frac{1}{(k^2 - m_0^2 + i0)^2}\right)$$

$$= \underbrace{\Sigma'\left(p^2, m_0^2\right)}_{\text{convergent}} + \underbrace{\Sigma''\left(m_0^2\right)}_{\text{divergent, but independent of } p^2}$$

Feynman diagrams can be made finite by something called regularization, there are various methods of which we'll see two: Pauli-Villars where we replace the propagator $\frac{i}{k^2 - m_0^2 + i0} \to \frac{i}{k^2 - m_0^2 + i0} - \frac{i}{k^2 - M^2 + i0} = \frac{-i\left(M^2 - m_0^2\right)}{\left(k^2 - m_0^2 + i0\right)\left(k^2 - M^2 + i0\right)}$ where M^2 is called the cut-off as this is then

$$= \begin{cases} O\left(\left(k^2\right)^{-2}\right) & \left(k^2 \to \infty\right) \text{ and } M^2 \text{ fixed} \\ \frac{i}{k^2 - m_0^2 + i0} & \left(M^2 \to \infty\right) \text{ and } k^2 \text{ fixed} \end{cases}$$

This substitution can now be applied in Σ' which is irrelevant since Σ' is already convergent, but it makes Σ'' convergent since

$$\Sigma'' \to i\left(M^2 - m_0^2\right)^2 \cdot \int d^4k \frac{1}{(k^2 - m_0^2 + i0)(k^2 - M^2 + i0)}$$

is obviously convergent. But the divergence is back for $M^2 \to \infty$.
Louper comutation:

$$\begin{cases} \Sigma'\left(p^2, m_0^2\right) & = \frac{1}{32\pi^2}\int_0^1 dx \log\left(\frac{m_0^2 - p^2 x(1-x)}{m_0^2}\right) \\ \Sigma''\left(m_0^2; M^2\right) & = -\frac{1}{16\pi^2}\log\left(\frac{M}{m}\right) + \text{convergent terms} \end{cases} \implies \Sigma = \Sigma\left(p^2, m_0^2; M^2\right)$$

Tools used for the computation:

- $\Sigma'\left(p^2, m_0^2\right) = \Sigma'\left(0, m_0^2\right) + \int_0^{p^2} d\left(p'^2\right)\frac{\partial\Sigma'}{\partial(p')^2}$

- $\frac{\partial\Sigma'}{\partial p^2} = -\frac{i}{32\pi^4 p^2}\int d^4k \frac{p(p-k)}{\underbrace{\left((p-k)^2 - m_0^2 + i0\right)^2}_{A^2}\underbrace{(k^2 - m_0^2 + i0)}_{B}}$ and we use: $\frac{1}{A^2 B} = 2\int_0^1 \frac{x}{(Ax - B(1-x))^3}$ (Feynman parametriza-

  tion).

After which we end up with integrals of the type:

$$I = \int d^4k \frac{1}{(K - k^2 - i0)^3}$$

with $k^2 = k_0^2 - \vec{k}^2$ and $K > 0$. We integrate

- over dk_0 and see where we have poles at $k_0^2 = K + \vec{k}^2 - i0$ ie. $k_0 = \pm\left(\sqrt{K + \vec{k}^2} - i0\right)$. Integrating over the path given in the lecture material gives 0, we can then introduce a Wick-rotation with $k_0 = ik_4 \implies k^2 = -(k_4)^2 - \vec{k}^2 = -\underline{k}^2$ with $\underline{k} = \left(\vec{k}, k_4\right)$

$$\implies I = \int d^4\underline{k} \frac{1}{\left(K + \underline{k}^2\right)^3}$$

Which is a special case of

$$\int d^d\underline{k} \frac{1}{\left(K + \underline{k}^2\right)^\alpha} = \pi^{\frac{d}{2}} \frac{\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)} K^{\frac{d}{2} - \alpha}$$

We can now interpret this divergence:

1. "Unrenormalized" perturbation theory:
   The Particle without self-interaction is not observable in an interacting theory, likewise the bare mass m_0 is not observable. Meaning it is adjustable, even adjustable in a divergent way.
   Conversely the physical mass $m \neq m_0$ is obselvable, it's the mass of the particle which can be measured. This physical mass m includes all interaction and is experimentally given and theoretically fixed. It is this mass that corresponds to the pole of $g\left(p^2\right)$.
   The requirement now is that:

   $$p^2 - m_0^2 - \lambda^2 \Sigma_2\left(p^2, m_0^2\right) = 0 \text{ for } p^2 = m^2$$

   The bare mass m_0^2 can then be seen as a function (ie a powerseries in λ^2)

   $$m_0^2 = m_0^2\left(\lambda, m^2\right)$$

   $$m_0^2 = m_{0,0}^2\left(M^2\right) + m_{0,2}^2\left(M^2\right) \lambda^2 + \ldots$$

   We now choose all coeficients $m_{0,n}^2\left(M^2\right)$ such that the requirement above holds at each order in λ. In particular the LHS of the requirement will converge at $p^2 = m^2$ but since we already now that the p^2 part is already convergent, this means that it will converge at any p^2.

   $$\implies m_0^2 + \lambda^2 \Sigma_2\left(p^2, m_0^2; M^2\right) \to m^2 + \lambda^2 \Sigma_R\left(p^2\right)$$

   we can now consider this at different orders of λ for $M^2 \to \infty$:

   - λ^0 : $m_{0,0}^2\left(M^2\right) \to m^2$
   - λ^2 : $m_{0,2}^2\left(M^2\right) + \Sigma_2\left(p^2, m_{0,0}^2; M^2\right) \to \Sigma_R\left(p^2\right)$
     This fixes the limit up to an additive constant.

     $$m_0^2\left(\lambda, M^2\right) = m^2 + \lambda^2 \left(\text{const.} + \frac{1}{16\pi^2} \log\left(\frac{M}{m}\right)\right) + O\left(\lambda^4\right)$$

     While this is ambigous, but it will be fixed by the requirement when $p^2 = m^2$ which gives

     $$\Sigma_R\left(p^2 = m^2\right) = 0$$

   We therefore get the result:

   $$\implies \Sigma_R\left(p^2\right) = \frac{1}{32\pi^2} \int_0^1 dx \log\left(\frac{m^2 - p^2 x(1-x)}{m^2(1 - x(1-x))}\right)$$

one lecture missing

8.1.1 Dimensional regularization

THe second regularization scheme we look at is tthe dimensional regularization. We now consider the Sigma function not in 4 but in d dimensions and u

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Organisational Remarks

- There will be a list of exam topics on the moodle

- The Exam will have two parts a theory part and a problem solving part

- Rules for the exam will also be posted on the moodle