

11287 Advanced Quantum Mechanics and Quantum Field Theory

Summary

The_Reto

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Introduction

On this document

This document are my personal notes on the lecture *11287 Advanced Quantum Mechanics and Quantum Field Theory* at Uni Basel in the spring semester 2021. I type part of this document during the lectures (while the Professor speaks), and another part as a write up when I personally read the associated literature. All of this happens with minimal editing - there are A LOT of spelling mistakes, typos and errors of various severity.

Information about this Course

To Pass participants need 50% of points from the exercises as well as 50% of the points in the final exam. Exercise sheets need to be handed in to ADAM by the times specified on the sheets.

The lecture is based on the book "Quantum Field Theory for the Gifted Amateur" by Lancaster and Blundell. The wirtten test

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Chapter 1

Introduction & Repetition of important concepts

1.1 Special Relativity

In special relativity two inertial reference frames are related by the Lorentz transformation. For two inertial frames S and \bar{S} moving along the x -axis with velocity v .

$$\begin{aligned}\bar{t} &= \gamma \left(t - \frac{vx}{c^2} \right) \\ \bar{x} &= \gamma (x - vt) \\ \bar{y} &= y \\ \bar{z} &= z\end{aligned}$$

with $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ and $\beta = \frac{v}{c}$.

In special relativity points in space time are given by four-vectors, with one "Time-like" and three "space-like" components.

$$a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

1.2 Lagrange Formalism in Classical Mechanics

1.2.1 Transition to fields - Lagrange Densities

1.2.2 Generalisation to special relativity

We consider a field $\phi(x)$ where $x = x^\mu$ is a point in space time. The lagrangian is then given by

$$L = L(\phi, \partial_\mu \phi)$$

The action is then

todo

and the Euler-Lagrange equation becomes

todo

.

Let's consider an example:

$$L(\phi, \partial_\mu \phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

We now consider the Euler-Lagrange Equation:

$$\begin{aligned}
 \frac{\partial L}{\partial \phi} &= -m^2 \phi \\
 \frac{\partial L}{\partial (\partial_\mu \phi)} &= \frac{\partial}{\partial (\partial_\mu \phi)} \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \\
 &= \frac{\partial}{\partial (\partial_\mu \phi)} \partial_\mu \phi g^{\mu\nu} \partial_\nu \phi = \dots \\
 &= \partial^\mu \phi
 \end{aligned}$$

In the end we find:

$$\implies 0 = -m^2 \phi - \partial_\mu \partial^\mu \phi = (\partial_\mu \partial^\mu + m^2) \phi$$

Chapter 2

Recapitulation: Simple harmonic oscillators

The harmonic oscillator hamiltonian (in one dimension) is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

We introduce operators a and a^\dagger which give us:

$$\begin{aligned}x &= x_0 (a^\dagger + a) \\ p &= -i\sqrt{\frac{\hbar m\omega}{2}} (a - a^\dagger)\end{aligned}$$

or equivalent

$$\begin{aligned}a &= \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p \right) \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p \right)\end{aligned}$$

With $x_0 = \sqrt{\frac{\hbar}{2m\omega}}$.

$$\implies H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

In OFT \hbar is often assumed to be 1 and left out. For the creation and annihilation operators a, a^\dagger we have

$$\begin{aligned}[a, a^\dagger] &= 1 \\ a^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\ a |n\rangle &= \sqrt{n} |n-1\rangle\end{aligned}$$

For N uncoupled oscillators, $\omega_k, k = 1, \dots, N$ we get

$$H = \sum_{k=1}^N \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$$

For each oscillator we have eigenstates $|n\rangle$ so our combined eigenstates are given by $|n_1, n_2, \dots, n_N\rangle$.

2.0.1 Linear Chain of harmonic oscillators

We imagine a linear chain of coupled harmonic oscillators (masses connected by springs).

The rest position of mass j is given by $R_j = ja$, the deviation from it's rest position is called x_j . The hamiltonian of this system is given by

$$H = \sum_j \frac{1}{2m} p_j^2 + \frac{1}{2} K (x_{j+1} - x_j)^2$$

We assume periodic boundary conditions (we assume the last mass is coupled to the first mass); $x_{N+1} = x_1$

We solve this problem by fourier transform:

$$x_j = \frac{1}{\sqrt{N}} \sum_k x_k \exp(ikja)$$

With $k = \frac{2\pi n}{Na}$, $-\frac{N}{2} < n \leq \frac{N}{2}$. What you find when doing the transformation is

$$\begin{aligned} \sum_j p_j^2 &= \sum_k p_k p_{-k} \\ \sum_j (x_{j+1} - x_j)^2 &= \sum_k x_k x_{-k} \left(4 \sin^2\left(\frac{ka}{2}\right) \right) \end{aligned}$$

With $\omega_k^2 = \frac{4K}{m} \sin^2\left(\frac{ka}{2}\right)$ the hamiltonian thus becomes

$$\Rightarrow H = \sum_k \left(\frac{1}{2m} p_k p_{-k} + \frac{1}{2} m \omega_k^2 x_k x_{-k} \right)$$

All of this was classical, to now go into the "quantum case", we quantize this model. We "promote" the x_k and p_k to operators \hat{x}_k and \hat{p}_k and then define a_k^\dagger , a_k as before. We get a slight complication as x_k and p_k are not hermitian. We find $x_k^\dagger = x_{-k}$ and $p_k^\dagger = p_{-k}$. The hamiltonian now takes the form

$$\begin{aligned} H &= \sum_{k=1}^N \hbar \omega_k \frac{1}{2} \left(a_k a_k^\dagger + \underbrace{a_{-k}^\dagger a_{-k}}_{a_k^\dagger a_k} \right) \\ \Rightarrow H &= \sum_{k=1}^N \hbar \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right) \end{aligned}$$

This is an interesting result. We started with a chain of coupled, interactiong oscillators, but end up with a sum over non-interacting harmonic oscillators! In Solid State physics the modes, labeled by k , are called phonons. For each of these phonon-modes we can add/substract integer multiples of ω_k , this gives rise to the interpretation of phonons as discrete particles.

Chapter 3

Occupation Number Representation

We consider a particle in a one dimensional "box" of length L and periodic boundary conditions (a bit of a strange box). The states of this particle will be plane waves of the form

$$e^{ipx} = e^{ip(x+L)}, \implies p_m = \frac{2\pi m}{L}, m = 0, \pm 1, \pm 2, \dots$$

The eigenstates can be labeled by this p_m .

We now suppose that instead of just one, we want to put n particles in this Box. The idea of the occupation number representation is to characterise the N -particle state by specifying how many particles are in each state.

$$|\psi\rangle = |n_1, n_2, \dots\rangle$$

where n_2 is the number of particles in state $|p_2\rangle$, etc.

Example: $|\psi\rangle = |2, 1, 0, 0, \dots\rangle = |2, 1\rangle$ means that we have two particles in state $|p_1\rangle$ and one particles in $|p_2\rangle$, in this case $N = 3$.

We now assume that these states are constructed by applying creation operators $a_{p_i}^\dagger$ to the vacuum state $|0\rangle$.

Example:

$$a_{p_1}^\dagger |0\rangle = |1, 0\rangle$$

$$a_{p_2}^\dagger |0\rangle = |0, 1\rangle$$

$$a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle = |1, 1\rangle$$

The order in which we apply these operators may matter, but in both cases the resulting state will be $|1, 1\rangle$ - it could only differ by phase. We therefore require

$$a_{p_1}^\dagger a_{p_2}^\dagger = \lambda a_{p_2}^\dagger a_{p_1}^\dagger$$

With λ a constant phase factor. There are two cases

1. For Bosons $\lambda = 1$.

$$[a_{p_1}^\dagger, a_{p_2}^\dagger] = 0$$

$$[a_{p_i}^\dagger, a_{p_i}^\dagger] = 1$$

We therefore get:

$$|\psi\rangle = |n_1, n_2, \dots\rangle = \dots \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} |0\rangle$$

2. For Fermions $\lambda = -1$.

$$\{c_i^\dagger, c_j^\dagger\} = 0$$

$i = j \implies (c_i^\dagger)^2 = 0$ the occupation of a state cannot be > 1 . This is called the Pauli-Exclusion principle (two Fermions cannot occupy the same state).

$$\{c_i, c_j^\dagger\} = \delta_{ij}$$

We therefor get:

$$|\psi\rangle = |n_1, n_2, \dots\rangle = (c_1^\dagger)^{n_1} (c_2^\dagger)^{n_2} \dots |0\rangle$$

We consider a an example: $c_i^\dagger |n_1, \dots, n_i, \dots 0\rangle$ with $n_i = 0$ or $= 1$. If $n_i = 0$:

$$c_i^\dagger |n_1, \dots, n_i, \dots\rangle = (-1)^{\sum_{j=1}^{i-1} n_j} |n_1, \dots, n_i + 1, \dots\rangle$$

In the opposite case we get

$$c_i |n_1, \dots, n_i, \dots\rangle = (-1)^{\sum_{j=1}^{i-1} n_j} |n_1, \dots, n_i - 1, \dots\rangle$$

For both, bosons and fermions, the symmetry requirement is taken care of by the algebraic (commutation/anticommutation) properties of the operators (a_i for bosons, c_i for fermions). We will see that all the observables can be expressed in terms of the a_i 's or c_i 's - this reformulation of quantum mechanics goes under the name of *Second Quantisation*.

Chapter 4

Making second quantisation work

Consider a system in a closed box of volume V . We can then define so called field-operators, which are the analogues of the a_i 's c_i 's in the position representation.

The position representation of the creation/annihilation operators is given by

$$\Psi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} a_{\vec{p}} e^{i\vec{p}\vec{x}}$$

The properties of these $\Psi(x)$ can be looked up in Lancaster, we will only give the commutation relations.

$$\begin{aligned} [\Psi(x), \Psi^\dagger(x')] &= \delta(x - x') && \text{for bosons} \\ \{\Psi_\sigma(x), \Psi^\dagger_{\sigma'}(x')\} &= \delta(x - x') \delta_{\sigma\sigma'} && \text{for fermions} \end{aligned}$$

For one particle operators we can write

$$A_i = \sum_{\alpha\beta} |\alpha\rangle_i \langle\alpha|_i A |\beta\rangle_i \langle\beta|_i = \sum_{\alpha\beta} A_{\alpha\beta} |\alpha\rangle_i \langle\beta|_i$$

In the second quantized form we can write:

$$A = \sum_{\alpha\beta} A_{\alpha\beta} a_\alpha^\dagger a_\beta$$

This acts on the whole group of particles. As an example let's look at an single particle momentum operator:

$$\begin{aligned} \hat{p}_i &= \sum_{\vec{p}} \vec{p} |p\rangle_i \langle p|_i \\ \implies \hat{P} &= \sum_{\vec{p}} \vec{p} a_p^\dagger a_p \end{aligned}$$

It's important to notice that $a_p^\dagger a_p$ counts the number of particles in state $|p\rangle$, second quantisation thus gives us an extremely natural way of writing the total momentum - it's just the sum of the individual momenta.

A second example is the free hamiltonian of non-interacting particles

$$H = \sum_p \frac{p^2}{2m} a_p^\dagger a_p$$

In the position representation we get for a single particle operator $A_i(x)$

$$A = \int d^3x \Psi^\dagger(x) A(x) \Psi(x)$$

Example: free hamiltonian

$$A_i(x) = \frac{\hbar^2}{2m} \nabla_i^2$$

$$\implies A = \int d^3x \Psi(x) \left[\frac{\hbar^2}{2m} \nabla^2 \right] \Psi(x)$$

For two particle operators (eg. coulomb interaction) the procedure is similar

$$\hat{U} = \frac{1}{2} \int d^3x d^3y \Psi^\dagger(x) \Psi^\dagger(y) U(x, y) \Psi(x) \Psi(y)$$

It's important to note that the order of $\Psi(x)$ and $\Psi(y)$ matters (in the fermionic case, with bosons it doesn't matter).

To switch to momentum representation we just insert the definition of the field operators:

$$\Psi(x)^\dagger = \frac{1}{\sqrt{V}} \sum_p a_p^\dagger e^{-ipx}$$

and insert this definition into the operator. By using that $\frac{1}{V} \int d^3y e^{ipy} = \delta_{p,0}$ we get

$$U = \frac{1}{2} \sum_{p_1, p_2, q} U_q a_{p_1+q}^\dagger a_{p_2-q}^\dagger a_{p_2} a_{p_1}$$

If we assume that $U(x, y) = U(x - y)$ we have

$$U_q = \frac{1}{V} \int d^3z U(z) e^{-iqz}$$

Example of a famous model in condensed matter is the Hubbard model, it has the hamiltonian

$$H = -t \sum_{i, \tau, \sigma} c_{i, \sigma}^\dagger c_{i+\tau, \sigma} + U \sum_i n_{i, \uparrow} n_{i, \downarrow}$$

The first term describes the kinetic energy, like in the tightbinding model. The second term describes a potential resulting from electrons sitting on the same site. On exercise sheet 1 there is an exercise on the so called two site hubbard model.

Chapter 5

Continuos Systems

How are the Lagrangian and Hamiltonian related to each other? We know that in classical mechanics

$$L(\vec{q}, \dot{\vec{q}}, t)$$

$$H(\vec{q}, \vec{p}, t) = \vec{p} \dot{\vec{q}} - L(\vec{q}, \dot{\vec{q}}, t)$$

We need to express $\dot{\vec{q}}$ in terms of \vec{q} , \vec{p} and t , using $\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}$.

We also remember Hamiltons equation and poisson bracket from classical mechanics.

The relativistic lagrangian of a free particle is given by: with $dt = \gamma d\tau$, τ the eigenzeit

$$\text{action: } S = \int_{t_1}^{t_2} L dt = \int_{\tau_1}^{\tau_2} L \gamma d\tau$$

$$L\gamma = \text{const.} = -\alpha$$

To get a Lorentz-invariant action. To determine α we set it to mc^2 to obtain the correct low energy limit.

$$L(x, \dot{x}, t) = -mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}}$$

Which for $\dot{x} \rightarrow 0$ goes to $-mc^2 + \frac{1}{2}m\dot{x}^2 + \dots$ which is the correct low energy limit.

We now look at the case where we add an electromagnetic field

$$A^\mu = \left[\frac{V\vec{x}}{\vec{A}(\vec{x})} \right]$$

The action then becommes

$$S = \int_{t_i}^{t_f} dt \left(mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} + q\vec{A}(\vec{x})\dot{\vec{x}} - qV(\vec{x}) \right)$$

We read of our Lagrangian as $L = \left(-mc^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} + q\vec{A}\dot{\vec{x}} - qV \right)$. The fields are given by:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$$

The relativistic field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & & & \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & & 0 & -B_1 \\ -E_3 & & & 0 \end{bmatrix}$$

$$F^{\mu\nu} = g^{\mu\eta} g^{\nu\alpha} F_{\eta\alpha} = \begin{bmatrix} 0 & & & \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & & 0 & -B_1 \\ E_3 & & & 0 \end{bmatrix}$$

5.1 Lagrangian and Hamiltonian Densities

In many cases the Lagrangian and Hamiltonian can be written as:

$$H = \int d^3x \mathcal{H}$$

$$L = \int d^3x \mathcal{L}$$

With the densities \mathcal{H} and \mathcal{L} that are functions of ϕ , $\dot{\phi}$, ϕ' and x . We remember that the conjugate momentum was given by $\vec{p} = \frac{\partial L}{\partial \vec{x}}$, we now define a similar quantity as

$$\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\delta L}{\delta \dot{\phi}}$$

We call this $\pi(x)$ the conjugate momentum to ϕ . If there are several field components ϕ_i there will also be several components of the conjugate momentum $\pi^i(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$.

We now look at the hamiltonian density, from the definition of the Hamiltonian we can read off that

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

As an example we now look at the electromagnetic field:

The field is given by the A^μ , we find

$$\mathcal{L}(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\vec{E}^2 - \vec{B}^2)$$

$$A^\mu = \begin{bmatrix} V \\ \vec{A} \end{bmatrix}, \quad A_\mu = \begin{bmatrix} V \\ -\vec{A} \end{bmatrix}$$

We assume we're using a gauge in which $V = 0$. We now calculate the conjugate momentum

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} \text{ for } i = 1, 2, 3 \rightarrow \vec{\pi} = -\dot{\vec{A}} = \vec{E}$$

The hamiltonian density is then

$$\mathcal{H} = \pi^i \dot{A}_i - \mathcal{L} = \frac{1}{2} (\vec{E}^2 + \vec{B}^2)$$

which is what we would expect for the energy density of the electro magnetic field (note: we use Heaviside units to not have to write all of the prefactors ϵ_0 and μ_0 , in SI units we'd have $\frac{1}{2} (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2)$).

We know the Euler-Lagrange equation is given by

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial (\partial_\lambda A_\mu)} \right) = 0$$

Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ the first term is 0. For the second term we look at

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\lambda A_\eta)} = -\frac{1}{4} 4 (\partial^\lambda A^\eta - \partial^\eta A^\lambda)$$

$$= -F^{\lambda\eta}$$

Our Euler-Lagrange equation therefore yields

$$\partial_\lambda F^{\lambda\eta} = 0$$

Which is just the source free maxwell equations!
To write \mathcal{L} in the presence of sources we get

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu$$

The Euler-Lagrange equation then yields

$$\partial_\lambda F^{\lambda\eta} = J^\eta$$

Again we find the maxwell's equations. Closing remarks on this chapter: We have seen a continuous classical system that we described by \mathcal{L} , we saw that the Euler-Lagrange Equations led to the correct equations of motions.

Chapter 6

A first stab at relativistic Quantum Mechanics

One obtains the free Schrödinger-Equation by replacing the classical energy E by an 'energy operator' $\hat{E} = i\hbar \frac{\partial}{\partial t}$, etc.

$$\begin{aligned} E &\rightarrow \hat{E} = i\hbar \frac{\partial}{\partial t} \\ p &\rightarrow \hat{p} = -i\hbar \frac{\partial}{\partial x} \\ \Rightarrow i\hbar \frac{\partial}{\partial t} \phi(\vec{x}, t) &= -\frac{\hbar^2}{2m} \vec{\nabla}^2 \phi(\vec{x}, t) \end{aligned}$$

The solution is known as

$$\phi(\vec{x}, t) = N e^{-i(\omega t - \vec{k} \cdot \vec{x})}$$

With N being a normalisation constant.

We now try to write this solution using 4-vectors.

$$\begin{aligned} \phi &= N e^{-i\vec{p} \cdot \vec{x}} \\ p^\mu &= \begin{bmatrix} \frac{E}{c} \\ \vec{p} \end{bmatrix} \\ p^\mu p_\mu &= \frac{E^2}{c^2} - \vec{p}^2 = m^2 c^2 \\ p^\mu x_\mu &= \frac{E}{c} ct - \vec{p} \cdot \vec{x} = \omega t - \vec{k} \cdot \vec{x} \end{aligned}$$

How do we get a relativistic wave equation?

The first attempt would be to use $E = (p^2 c^2 + m^2 c^4)^{\frac{1}{2}}$.

We now try to construct a wave equation in the same way as we did before

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \phi = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4} \phi$$

This naive approach has many problems:

1. It's not Lorentz covariant, to solve the square-root of an operator on the right-hand side we expand it and get spacial derivatives of infinite order. The left hand side only contains time derivative of the first order. Time and Space dimensions are therefore not on equal footing

The second attempt was to use $E^2 = p^2 c^2 + m^2 c^4$ and plug it into the 'squared' equation.

$$\Rightarrow -\hbar^2 \frac{\partial^2}{\partial t^2} \phi = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \phi$$

We now have time and spacial derivatives of the same order. We can write this equation as

$$(\partial_\mu \partial^\mu + m^2) \phi(x) = 0$$

With $\phi(x)$ a function of the 4-vector x , and $\hbar = c = 1$. This equation is called the Klein-Gordon equation. This form is obviously Lorentz-covariant. The solutions are given as

$$\Phi(x) = N e^{-ipx}$$

which leads to $E^2 = \vec{p}^2 + m^2$. Taking the squareroot now yields $E = \pm \sqrt{\vec{p}^2 + m^2}$, which leads to two problems.

1. Problem: What is the meaning of the negative energy solution?!
2. Problem: probability density should fullfill a continuity equation. Does such an equation exist?

Let's deal with the second problem. We derive such a continuity equation in a simmlar way as we did in the introductory quantum mechanics class.

We first multiply the Klein-Gordan equation from the left with Φ^\dagger and substract the hermitian conjugate.

$$\begin{aligned} \Phi^\dagger \partial_\mu \partial^\mu \Phi - \Phi \partial_\mu \partial^\mu \Phi^\dagger &= 0 \\ \implies \partial_\mu (\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger) &= 0 \\ \partial_\mu j^\mu &= 0 \end{aligned}$$

with $j^\mu = \phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger$. One can think of the density j as

$$j^\mu = \begin{bmatrix} \rho \\ \vec{j} \end{bmatrix}$$

for some ρ and \vec{j} . In our case we find that $\rho = \frac{\hbar i}{2m} \left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right)$ and $\vec{j} = -\frac{i\hbar}{2m} \left(\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^* \right)$. The factor $\frac{i\hbar}{2m}$ is only introduced to make the equation look like in non-relativistic quantum mechanics.

The problem now arrises from comparison with the Schroedinger equation. In the Schroedinger case we found $\rho = |\phi|^2 \geq 0$. But now in the Klein-Gordan case our ρ is not neccesarily positive, $j^0 = \rho = 2|N|E$, but we saw that E can be positive or negative.

The interpretation of j^μ is unclear.

In the 1920ties Stueckelberg and Feynman tried to interpret the negative energy solution as anti-particles, more details can be found in Lancaster-Blundell.

Chapter 7

Examples of Lagrangians

Is mainly reading material and will only briefly be covered in the lecture.

- "Massless scalar field"

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi$$

- "massive scalar field"

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2$$

We can think of the second term in terms of a potential energy, as "the cost of having a field". The Euler-Lagrange equation of this field is given as $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$, which is the Klein-Gordon equation.

- "External source"

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 + J(x) \phi$$

With $J(x)$ an external current, a so called source current. For the E-L equation we now get $\partial_\mu \partial^\mu \phi + m^2 \phi = J(x)$.

- " ϕ^4 - Theory.

- more than one scalar fields. Eg. ϕ_1, ϕ_2 with the same mass and potential energy $U(\phi_1, \phi_2) = g(\phi_1^2 + \phi_2^2)^2$.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} m^2 \phi_2^2 - g(\phi_1^2 + \phi_2^2)^2$$

If we apply an rotation matrix to the ϕ_i 's

$$\begin{bmatrix} \phi'_1 \\ \phi'_2 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

The lagrangian remains invariant.

$$\mathcal{L}(\phi'_1, \phi'_2) = \mathcal{L}(\phi_1, \phi_2)$$

In fancy language we can say that " \mathcal{L} has an SO(2) symmetr" ie. remains unchanged under two dimensional rotations

- "complex scalar field"

It turns out that this is equivalent to the previous case. More details can be found in Lancaster-Blundell.

$$\mathcal{L} = \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi - g(\psi^\dagger \psi)^2$$

In this case the symmetry is that " \mathcal{L} has a U(1) symmetry" ie. remains invariant under multiplication with a phase factor.

Chapter 8

The passage of Time

This chapter too will only be covered briefly in the lecture, as the topics should already be known. The topics are as follows:

- time evolution operators
- Heisenberg representation

$$\langle \psi(t) | A | \psi(t) \rangle = \langle \psi | U^\dagger(t) A U(t) | \psi \rangle = \langle \psi | A(t) | \psi \rangle$$

We have the Heisenberg equation of motion

$$i\hbar \frac{d}{dt} H(t) = [A(t), H]$$

Lancaster-Blundell that "single-particle QM is inconsistent":

$$\langle \vec{x} | e^{-iHt} | 0 \rangle$$

$$H | \vec{p} \rangle = E_{\vec{p}} | \vec{p} \rangle = \sqrt{\vec{p}^2 + m^2} | \vec{p} \rangle$$

LB then show that there is a probability density at points different from $\vec{x} = 0$ at arbitrary small timesteps, which violates relativity. $\langle \vec{x} | e^{-iHt} | 0 \rangle$ is finite ie. ≥ 0 outside the lightcone.

Chapter 9

Quantum Mechanical Transformations

We write translations as $T(\vec{a})$ with $\vec{x}' = T(\vec{a})\vec{x} = \vec{x} + \vec{a}$. We now want to study the situation quantum mechanically

$$U(\vec{a})|\vec{x}\rangle = |\vec{x} + \vec{a}\rangle$$

We therefore have two different notations; T which acts on geometric vectors and U which acts on states in a hilbertspace. We now want to look at the properties of this unitary operator U .

- $U^\dagger(\vec{a})\hat{x}U(\vec{a}) = \hat{x} + \vec{a}$

$$\hat{x}U(\vec{a})|\vec{x}\rangle = (\vec{x} + \vec{a})|\vec{x} + \vec{a}\rangle$$

Now we multiply from the left with $\langle\vec{x} + \vec{a}| = \langle\vec{x}|U^\dagger(\vec{a})$ we get the desired result.

- $U(\vec{a}) = e^{-i\vec{p}\vec{a}}$

Since for wavefunctions we know that: $\psi(\vec{x} + \vec{a}) = e^{i\vec{p}\vec{a}}\psi(\vec{x})$. We can say that the momentum generates the translations.

We will now do the same for rotations.

$$\vec{x}' = R(\vec{\theta})\vec{x}$$

for the geometric operator and

$$|\vec{p}'\rangle = U(\vec{\theta})|\vec{p}\rangle = |R(\vec{\theta})\vec{p}\rangle$$

The properties of this operator are

- $U^\dagger(\vec{\theta})\hat{p}U(\vec{\theta})$
- $U(\vec{\theta}) = e^{-i\vec{J}\vec{\theta}}$, with \vec{J} the Angular momentum operator.

The question now is how do quantum fields transform under these transformations?

We look at a scalar field and a translation by \vec{a}

$$\begin{aligned}\langle\vec{y}|\hat{\phi}(\vec{x})|\vec{y}\rangle &= \langle\vec{y} + \vec{a}|U\hat{\phi}(\vec{x})U^\dagger|\vec{y} + \vec{a}\rangle = \langle\vec{y} + \vec{a}|\hat{\phi}(\vec{x})|\vec{y} + \vec{a}\rangle \\ \implies U(\vec{a})\hat{\phi}(\vec{x})U^\dagger(\vec{a}) &= \hat{\phi}(\vec{x} + \vec{a})\end{aligned}$$

Note the order of U and U^\dagger is reversed from what it was when we looked at a operator.

Another example is to look at a creation operator in momentum space a_k^\dagger . We find

$$U^\dagger(\vec{a})a_k^\dagger U(\vec{a})|\vec{q}\rangle$$

With the momentum eigenstate $\hat{p}|\vec{q}\rangle = \vec{q}|\vec{q}\rangle$.

$$\begin{aligned}e^{i\vec{p}\vec{a}}a_k^\dagger e^{-i\vec{p}\vec{a}}|\vec{q}\rangle &= e^{i\vec{p}\vec{a}}a_k^\dagger|\vec{q}\rangle e^{-i\vec{q}\vec{a}} \\ e^{i\vec{p}\vec{a}}|\vec{k}, \vec{q}\rangle e^{-i\vec{q}\vec{a}} &= e^{i\vec{k}\vec{a}}|\vec{k}, \vec{q}\rangle\end{aligned}$$

$$\implies U^\dagger(\vec{a})a_k^\dagger U(\vec{a}) = e^{i\vec{k}\vec{a}}a_k^\dagger$$

For scalar fields we have

$$U^\dagger(\vec{a})\hat{\phi}(\vec{x})U(\vec{a}) = \hat{\phi}(\vec{x} - \vec{a})$$

For vector fields we have something more complex

$$U^\dagger(\theta)\hat{\vec{\phi}}(\vec{x})U(\theta) = D(\theta)\hat{\vec{\phi}}(R^{-1}(\theta)\vec{x})$$

With $U(\theta)$ the unitary that corresponds to the geometric transformation $R(\theta)$ and $D(\theta)$ the (matrix) representation of $R(\theta)$ in the appropriate vector space. This ofcourse also applies to the scalar case if we choose $D(\theta) = \mathbb{1}$.

In the textbook there is a section (9.5) on Lorentz group transformations, we will not discuss it in class but there are exercises on it. (See problem 2 on sheet 3)

The only thing we are discussing in the lecture from this section is the following:

$$\Lambda(\beta_1) = \begin{bmatrix} \gamma_1 & \beta_1\gamma_1 & & \\ \beta_1\gamma_1 & \gamma_1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

If we want to combine such a boost in the x -direction with parameters β_1 and another boost in the same direction with parameter β_2 , is equivalent to a third boost, but $\beta_3 \neq \beta_1 + \beta_2$. For ease of use we would like to have an additive law, we could write the lorentz transformations as $\Lambda(\phi) = e^{-i\vec{K}\vec{\phi}}$.

To get an additive law, introduce the "rapidities" ϕ_i by $\cosh(\phi_i) = \gamma_i$ and $\sinh(\phi_i) = \gamma_i\beta_i$ ($\tanh(\phi_i) = \beta_i$).

What we get in this notation is that a total boost consisting of a boost described by ϕ_1 followed by a boost described by ϕ_2 is described by a boost with $\phi_3 = \phi_1 + \phi_2$.

Chapter 10

Symmetry

Noethers Theorem: "Symmetries lead to conservation laws". Eg. Translational symmetry \rightarrow conservation of momentum, rotational symmetry \rightarrow conservation of angular momentum.

Assume a field $\phi(x^\mu)$ changes under a transformation that is parametrized by λ . We define the change in the field $D\phi = \frac{\partial\phi}{\partial\lambda}$ at $\lambda = 0$, ie an infinitesimal change can be written as $\delta\phi = D\phi\delta\lambda$. We consider as an example a translation:

$$\begin{aligned}\phi(x^\mu) &\rightarrow \phi(x^\mu + \lambda a^\mu) \\ D\phi &= \frac{\partial\phi}{\partial y^\mu} \frac{\partial y^\mu}{\partial \lambda} = a^\mu \partial_\mu \phi\end{aligned}$$

We now look at the Lagrangian

$$\begin{aligned}\phi(x) &\rightarrow \phi(x) + \delta\phi(x) \\ \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\end{aligned}$$

with the momentum density

$$\begin{aligned}\Pi^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \\ \delta\mathcal{L} &= \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\Pi^\mu\right)\delta\phi + \partial_\mu(\Pi^\mu\delta\phi)\end{aligned}$$

The first term is just the Euler-Lagrange expression. Iff ϕ fulfills the EL expression the first term is equal to 0 and disappears.

$$\delta\mathcal{L} = \partial_\mu(\Pi^\mu D\phi)\delta\lambda$$

If the transformation we talk about is a symmetry, it should leave \mathcal{L} invariant. What we really need for a symmetry is that the action $S = \int d^4x \mathcal{L}$ remains invariant. We have a symmetry iff $\delta\mathcal{L} = 0$ up to a 4-divergence. ie.

$$\delta\mathcal{L} = \partial_\mu W^\mu$$

Hence $\partial_\mu(\Pi^\mu D\phi - W^\mu) = 0$. If the transformation is a symmetry, there exists a W^μ such that this equation is fulfilled, since the equation has the form of a conservation law we can say that symmetries generate conserved quantity.

The conserved quantity is given by

$$J_n^\mu = \Pi^\mu D\phi - W^\mu$$

Remarks:

- For internal symmetries we get $W = 0$
- conserved current leads to a conserved charge. $Q_n = \int d^3x J_n^0$
- in the quantum case: symmetry \rightarrow conserved 'charge' operator \hat{Q}_n

Example of Noethers Theorem:

We consider a theory that is symmetric under spacetime translations $x^\mu \rightarrow x^\mu + \lambda a^\mu$. Under this transformation we get $\phi(x^\mu) \rightarrow \phi(x^\mu + \lambda a^\mu)$

$$\begin{aligned} D\phi &= \frac{\partial \phi}{\partial \lambda} \text{ at } \lambda = 0 \\ \implies D\phi &= a_\mu \partial^\mu \phi \end{aligned}$$

We now look at the lagrangian under this transformation

$$\begin{aligned} \delta \mathcal{L} &= D\mathcal{L} \delta \lambda = a^\mu d_\mu \mathcal{L} \delta \lambda \\ D\mathcal{L} &= \partial_\mu (a^\mu \mathcal{L}) \end{aligned}$$

This last expression has the form of a four divergence and therefore leads to the Noether current

$$\begin{aligned} J_N^\mu &= \Pi^\mu D\phi - W^\mu \\ &= a_\nu (\Pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L}) \end{aligned}$$

We call the quantity in the brackets as the Energy-momentum-Tensor or Stresstensor $T^{\mu\nu}$.

$$T^{\mu\nu} = \Pi^\mu \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

The therefore get the result

$$J_N^\mu = a_\nu T^{\mu\nu}$$

From Noethers-Theorem we know that $\partial_\mu J^\mu = 0$ we see that

$$\partial_\mu T^{\mu\nu} = 0$$

Which expresses the conservation of energy and momentum.

We now also look at the conserved charges. We define

$$P^\alpha = \int d^3x T^{0\alpha}$$

we now look at the four componetns of P^α

$$\begin{aligned} P^0 &= \int d^3x T^{00} = \int d^3x \left(\Pi(x) \dot{\phi}(x) - \mathcal{L} \right) \\ &= \int d^3x \mathcal{H} \\ &= E \implies \text{We've found energy conservation!} \\ P^k &= \int d^3x T^{0k} = \int d^3x \Pi(x) \partial^k \phi(x), \forall k = 1, 2, 3 \end{aligned}$$

From Lorentzinvariance tells us that this must be the physical momentum carried by the field (since P^0 is the energy, and we know that the energy-momentum-fourvector has the structure $P^\alpha = \begin{bmatrix} E \\ \vec{p} \end{bmatrix}$).

Note: the physical momentum carried by the field \vec{p} is different to the canonical momentum Π^μ !

In the definition above $T^{\mu\nu}$ is not symmetric, it can be made symmetric by adding a term of the form $\partial_\lambda X^{\lambda\mu\nu}$, where $X^{\lambda\mu\nu} = -X^{\mu\lambda\nu}$. Such a term will not disturbe the conserved quantities, but will symmetrize the energy-momentum-Tensor (which will of course simplify calculations).

Chapter 11

Canonical quantization of fields

Cononical quantization is a method to turn a classical fiold theory to a quantum field theory (QFT). We will not derive it, but only describe it in 5 steps.

1. Define a classical field theory by writing down it's Lagrangian-density $\mathcal{L}(x)$.
2. Calculate the conjugate momentum density $\Pi^\mu(x)$ and the Hamiltonian-density $\mathcal{H}(x)$.
3. Treat the field(s) and momentum as operators by imposing *equal time* commutation relations. Meaning that the commutators will only be different from 0 if considered at the same time (this is a generalization of $[x_j, p_k] = i\hbar\delta_{jk}$).
4. Expand the fields in terms of creation and annihilation operators. Then reintroduce this expansion to the Hamiltonian to get the quantum hamiltonian.
5. Introduce normal ordering (i.e. reorder all the operatorproducts such that annihilation operators act first, followed by creation operators).

11.1 Examples

11.1.1 massive scalar field

We already saw the classical example of this case, this is the Lagrangian that leads to the Klein-Gordan equation. We now follow our five steps

1. $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$
2. $\Pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$, the timelike component is then $\Pi^0(x) = \pi(x) = \partial^0\phi = \dot{\phi}$.
 $\mathcal{H} = \pi(x)\dot{\phi} - \mathcal{L} = \frac{1}{2}\dot{\phi}^2(x) + \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 + \frac{1}{2}m^2\phi^2(x)$
3. $\phi \rightarrow \hat{\phi}$, $\Pi^0(x) \rightarrow \hat{\Pi}^0(x)$.

$$\begin{aligned} [\hat{\phi}(t, \vec{x}), \hat{\Pi}^0(t, \vec{y})] &= i\delta^{(3)}(\vec{x} - \vec{y}) \\ [\hat{\phi}(t_1, \vec{x}), \hat{\Pi}^0(t_2, \vec{y})] &= 0 \end{aligned}$$

The next question is: how does $\hat{\phi}(x)$ act on occupation number states $|n_1, n_2, \dots\rangle$?

4. Analoguos procedure to the coupeled oscillator problem, there we had operators \hat{x}_j (possition operator for the j -th operator). And we wrote this as

$$\hat{x}_j = \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} \sum_k \frac{1}{(2\omega_k N)^{\frac{1}{2}}} \left(a_k e^{ijk a} + a_k^\dagger e^{-ijk a}\right)$$

We now just boldly proclaim that we expand the $\hat{\phi}(x)$ in a similar way (because we now have a continuous system the sum turns into an integral):

$$\hat{\phi}(\vec{x}) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{i\vec{p}\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\vec{x}} \right)$$

With E_p the KG energy given by $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ (we only consider the positive solution). However, we ignored the time dependence of $\hat{\phi}$ until now, how do we get time-dependent field operators?

$$\hat{\phi}(x) = \hat{\phi}(t, \vec{x}) = e^{iHt} \hat{\phi}(\vec{x}) e^{-iHt}$$

When we do this, $a_{\vec{p}}$ picks up a phase factor $e^{-iE_p t}$ (and correspondingly for $a_{\vec{p}}^\dagger$). Our field operator is then given by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx} \right)$$

Where p and x are now fourvectors.

Remark: We see that the coefficients of the $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ are the solutions of the Klein-Gordon equation ($e^{\pm ipx}$), i.e. the equation of motion of the corresponding classical field theory. We will see that this generalizes. We skip remarks on normalization factors (see Lancaster-Blundell for details).

The quantum hamiltonian operator \hat{H} is given by: (replace ϕ by the expansion we just calculated)

$$\begin{aligned} H &= \int d^3x \mathcal{H} = \int d^3x \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) \\ &= \frac{1}{2} \int d^3p E_{\vec{p}} \left(a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}} \right) \end{aligned}$$

With $\dot{\phi}$ and $\vec{\nabla} \phi$ given by

$$\begin{aligned} \Pi_\mu(x) &= \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} = \partial_\mu \phi = \left[\begin{array}{c} \dot{\phi} \\ \vec{\nabla} \phi \end{array} \right] \\ &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_{\vec{p}}}} (-ip_\mu) \left(a_{\vec{p}} e^{-ipx} - a_{\vec{p}}^\dagger e^{ipx} \right) \end{aligned}$$

With $p_\mu = \begin{bmatrix} E_{\vec{p}} \\ -\vec{p} \end{bmatrix}$. Up to now we have not used any commutator relations etc.

5. By using the commutation relations to impose normal ordering.

$$[a_{\vec{q}}, a_{\vec{p}}^\dagger] = \delta(\vec{q} - \vec{p})$$

We now use this to impose normal ordering

$$N(\hat{H}) = \int d^3p E_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}} = \hat{H}$$

Remarks:

- The normal ordering we impose seems to be an ad hoc procedure without justification. The problems stem from the fact that the classical ϕ in the classical hamiltonian commute, we therefore get an ambiguity in the quantum hamiltonian. To get a unique result we have to impose an ordering on the quantum hamiltonian. *The normal ordering removes the ambiguity of order in the classical Lagrangian/Hamiltonian.* Eg. for a classical, complex field theory we'd have that $\Psi^\dagger \Psi = \Psi \Psi^\dagger$. For the quantum case we have

$$\hat{H} = \int d^3p E_{\vec{p}} \hat{n}_{\vec{p}}, \text{ with } \hat{n}_{\vec{p}} = a_{\vec{p}}^\dagger a_{\vec{p}}$$

the particles we introduced here are bosons (because we used bosonic commutation relations) with spin $S = 0$.

- This canonical quantisation procedure is a nice recipe that works for non-interacting theories (i.e. real and complex scalar theories, Dirac theory), but for more interesting, interacting situations we have to find different methods. (Spoiler: we will have to treat interactions in terms of perturbation theory).
- A topic we will not pursue further in this lecture is the Casimir-Effect, which stems from the vacuum energy (the infinite constant we discarded when we imposed normal ordering). The term we discarded therefore can have measurable effects.

The Casimir force leads to a force between uncharged metallic plates. It turns out that for two plates at distance d the force per area resulting from the Electro-Magnetic field is given by (instead of the EM field, Lancaster-Blundel uses a one component toy model, so the results don't match)

$$F = -\frac{\pi^2 \hbar c}{240 d^4}$$

The vacuum energy is infinite, but for any considered volume it's just a constant (but still infinite) term that can be discarded.

Chapter 12

Examples of canonical quantization

We will now consider a complex scalar field: ψ, ψ^\dagger

1. $\mathcal{L} = \partial^\mu \psi^\dagger \partial_\mu \psi - m^2 \psi^\dagger \psi$.
It can be helpful to think of $\psi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x))$.

2. $\Pi_\Psi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = \partial^0 \psi^\dagger$
 $\Pi_{\psi^\dagger}^0 = \dots = \partial^0 \psi$

We write both as Π_σ^0 with $\sigma = \psi, \psi^\dagger$. The hamiltonian density is given as

$$\mathcal{H} = \sum_\sigma \Pi_\sigma^0 \partial_0 \sigma^\dagger - \mathcal{L} = \partial^0 \psi^\dagger \partial_0 \psi + \vec{\nabla} \psi^\dagger \vec{\nabla} \psi + m^2 \psi^\dagger \psi$$

3. We switch to operators. $\psi, \psi^\dagger \rightarrow \hat{\psi}, \hat{\psi}^\dagger$.
4. mode expansion: since the classical field ψ is complex, the quantum field operators $\hat{\psi}$ are not hermitian. The solution to this problem is

$$\hat{\psi}(x) = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ipx} + b_{\vec{p}}^\dagger e^{ipx} \right)$$

where $[a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta(\vec{p} - \vec{q}) = [b_{\vec{p}}, b_{\vec{q}}^\dagger]$ all the other commutators vanish.

5. We now plug this into the Hamiltonian and impose normal ordering, lengthy but straight forward calculation yields:

$$\hat{H} = \int d^3 p E_p \left(a_{\vec{p}}^\dagger a_{\vec{p}} + b_{\vec{p}}^\dagger b_{\vec{p}} \right) = \int d^3 p E_p \left(n_{\vec{p}}^{(a)} + n_{\vec{p}}^{(b)} \right)$$

The a and b particles have the same energy, but are still "somehow" different from each other. The interpretation of these a and b particles is that they are anti-particles of each other.

$a_{\vec{p}}^\dagger$ creates a particle with momentum \vec{p}

$b_{\vec{p}}^\dagger$ creates an anti-particle with momentum \vec{p} .

The complex scalar field theory considered here has an internal symmetry ($U(1)$): $\psi \rightarrow e^{i\alpha} \psi$ leaves \mathcal{L} invariant. We can therefore ask what the conserved Noether current is. Since $\mathcal{L} \rightarrow \mathcal{L} \implies W^\mu = 0$, and $D\psi = i\psi$. The Noether current is therefore

$$J_N^\mu = \sum_\sigma \Pi_\sigma^\mu D\sigma = i \left((\partial^\mu \psi^\dagger) \psi - (\partial^\mu \psi) \psi^\dagger \right)$$

We quantize this current, by using the mode expansion and normal ordering

$$J_N^\mu \rightarrow \hat{J}_N^\mu$$

The Noether charge then is given by

$$\hat{Q}_N = \int d^3 x \hat{J}_N^0 = \int d^3 x i \left((\partial^0 \hat{\psi}^\dagger) \hat{\psi} - (\partial^0 \hat{\psi}) \hat{\psi}^\dagger \right) = \int d^3 p \left(b_{\vec{p}}^\dagger b_{\vec{p}} - a_{\vec{p}}^\dagger a_{\vec{p}} \right)$$

The antiparticles, and particles appear with opposite charge, as we'd expect for antiparticles. The lecture will skip the rest of the chapter. In particular section 12.3 is recommended, as the non-relativistic limit is discussed - it turns out that this leads to the Schrödinger equation. Nice! When starting from the Schrödinger theory lagrangian from Exercise 4 sheet 2 and go through this canonical quantization procedure, this will lead to the second quantized hamiltonian with field operators.

$$H = \int d^3x \hat{\psi}(x) \left(\frac{\hbar^2}{2m} \nabla^2 \right) \hat{\psi}(x)$$

With $[\psi(x), \psi^\dagger(y)] = \delta(x - y)$, note the relation to the commutation in step 3 of our recipe.

Chapter 13

Fields with many (> 2) Components and massive Electro-Magnetism

As an example we consider a three componetn field

$$\vec{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

with $\phi_i \in \mathbb{R}$ and assume we have a rotation symmetry (3-dimensional rotation group $SO(3)$).

$$\mathcal{L} = \frac{1}{2} \left(\partial^\mu \vec{\phi} \right) \left(\partial_\mu \vec{\phi} \right) - \frac{m^2}{2} \vec{\phi} \cdot \vec{\phi}$$

Note: $\vec{\phi}$ does not 'live' in Minkowski Space, meaning $\phi^\mu = \phi_\mu$ and $\vec{\phi} \cdot \vec{\phi} = \sum_i \phi_i \phi_i$.

We now walk through our quantisation procedure for steps 1 ... 3. Only in step 4 we have some complications.

$$\vec{\phi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} \sum_{n=1}^3 \vec{h}_n \left(a_{\vec{p}n} e^{-ipx} + a_{\vec{p}n}^\dagger e^{ipx} \right)$$

with \vec{h}_a a polarisation vector. Any basis will work, for simplicities sake we take the example as $\vec{h}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{h}_2 =$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{h}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

13.1 massive electromagnetism

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

We now consider the Euler-Lagrange equations for this Lagrangian:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} &= 0 \\ \implies m^2 A^\nu + \partial_\mu F^{\mu\nu} &= 0 \end{aligned}$$

We now want to take the divergence of this ∂_ν

$$m^2 \partial_\nu A^\nu + \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

The second term is zero (because $F^{\mu\nu}$ is antisymmetric and $\partial_\nu \partial_\mu$ is symmetric). We get:

$$\implies \partial_\nu A^\nu = 0$$

We are "forced" to choose a particular gauge. We now use this result to rewrite the euler-lagrange equation from above. Using

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \partial_\mu \partial^\mu A^\nu \\ \implies (\partial_\mu \partial^\mu + m^2) A^\nu &= 0 \end{aligned}$$

We find the form of the Klein-Gordand equation for every component. We find therefore $E_p = \sqrt{\vec{p}^2 + m^2}$.

13.1.1 Quantisation

1. Already complete (see above)

$$\begin{aligned} 2. \Pi^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = -F^{\mu\nu} \\ \implies \Pi^{00} &= 0, \Pi^{0i} = -F^{0i} = E^i \end{aligned}$$

$$\begin{aligned} \mathcal{H} &= \sum_\alpha \Pi^{0\alpha} \dot{A}_\alpha - \mathcal{L} \\ &= -\vec{E} \cdot \dot{\vec{A}} - \frac{1}{2} (E^2 - B^2) - \frac{1}{2} m^2 ((A^0)^2 - \vec{A}^2) \\ &= \frac{1}{2} (E^2 + B^2) + \frac{1}{2} m^2 \vec{A}^2 + \vec{E} \vec{\nabla} A^0 - \frac{1}{2} m^2 (A^0)^2 \end{aligned}$$

Using that $A^0 = -\frac{1}{m^2} \vec{\nabla} \cdot \vec{E}$ we find

$$\mathcal{H} = \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 + m^2 \vec{A}^2 + \frac{1}{m^2} (\vec{\nabla} \cdot \vec{E})^2 \right)$$

3. Equal time commutation relations:

$$[\hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})] = i\delta(\vec{x} - \vec{y})\delta_{kj} = -i\delta(\vec{x} - \vec{y})g^{kj}$$

4. Mode expansion: expect something like $A = \int d^3p \dots (a_p e^{-ipx} \dots)$. From before we know that $\partial_\mu A^\mu = 0$, this means that there are only three polarization directions since $p_\mu A^\mu = 0$.

$$\hat{A}^\mu(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{\lambda=1}^3 \left(\epsilon_\lambda^\mu(p) a_{\vec{p}\lambda} e^{-ipx} + \epsilon_\nu^{\mu*} a_{\vec{p}\lambda}^\dagger e^{ipx} \right)$$

with the ϵ 's being three distinct four-vectors. Ideally they should be chosen to be orthogonal: $\epsilon_\lambda^*(p) \epsilon_{\lambda'}(p) = g_{\mu\nu} \epsilon_\lambda^{\mu*} \epsilon_{\lambda'}^\nu = -\delta_{\lambda\lambda'}$.

Example: (Proca-)particle in its rest frame: $p^\mu = [m \ 0 \ 0 \ 0]^T$, we now require that $p^\mu \epsilon_{\lambda\mu}(p) = 0$. One possible choice of polarization vectors in the rest frame are now just the 3d unit vectors. In any other frame the polarization vectors are given by the lorentzboost into that different frame. ϵ_λ in an arbitrary frame is obtained by a lorentz boost.

5. Plug everything into the hamiltonian, and impose normal ordering. We find:

$$\hat{H} = \int d^3p E_{\vec{p}} \sum_{\lambda=1}^3 a_{\vec{p}\lambda}^\dagger a_{\vec{p}\lambda}$$

Chapter 14

Gauge Fields & Gauge Theory

As we discussed, the complex scalar field theory $\mathcal{L} = (\partial^\mu \Psi^\dagger) (\partial_\mu \Psi) - m^2 \Psi^\dagger \Psi$ is $U(1)$ -symmetric under global transformations $\Psi(X) \rightarrow e^{i\alpha} \Psi(X)$, $\Psi^\dagger(X) \rightarrow e^{-i\alpha} \Psi^\dagger(X)$ with α independent of X .

14.1 Local Transformations

What about *local* transformations $\Psi \rightarrow e^{i\alpha(X)} \Psi$?

Obviously this has no effect on the "mass term". But for the kinetic term we get $\partial_\mu \Psi \rightarrow e^{i\alpha(x)} \partial_\mu \Psi + \Psi e^{i\alpha(x)} i \partial_\mu \alpha(x) = e^{i\alpha(x)} (\partial_\mu + i \partial_\mu \alpha(x)) \Psi$. For Ψ^\dagger we find the equivalent result. This transformation definitely does not leave the kinetic term invariant in general. There is no local symmetry.

Local symmetry can be obtained by introducing a *new field* A^μ and defining a new, covariant derivative:

$$D_\mu := \partial_\mu + iqA_\mu(x)$$

with the *gauge field* A_μ . We can now claim that $(D^\mu \Psi)^\dagger (D_\mu \Psi) = (\partial^\mu \Psi^\dagger) (\partial_\mu \Psi)$ iff we require that $A_\mu \rightarrow A_\mu - \frac{1}{q} \partial_\mu \alpha(x)$.

Gauge theory: a gauge field $A^\mu(x)$ is introduced to produce an invariance with respect to the local transformation. It looks like just a mathematical trick, but we will see later that it has physical consequences.

14.2 Electromagnetic Field

$$A^\mu(x) = \begin{bmatrix} V(x) \\ \vec{A}(x) \end{bmatrix}$$

The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_{\text{em}}^\mu A_\mu$$

with a fourvector current $J_{\text{em}}^\mu = [\rho \quad \vec{J}]^T$. (Attention typo in Lancaster Blundell). We find the equation of motion as

$$\begin{aligned} \partial_\lambda F^{\lambda\nu} &= J_{\text{em}}^\nu \\ \partial^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) &= J_{\text{em}}^\nu \end{aligned}$$

The Gauge invariance of this field is

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \chi(x)$$

For an arbitrary $\chi(x)$. Under this transformation $F^{\mu\nu}$, \mathcal{L} and the equations of motions are unchanged. We have two popular gauges:

- Lorentz gauge: $\partial_w u A^\mu(x) = 0$

$$\implies \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = \rho$$

$$\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \vec{J}_{\text{em}}$$

The Lorentz gauge is not a tight gauge, as we still have some choice in how we choose A^μ (specifically we can set one of the parts to 0)

- Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0$

In the Coulomb gauge a plane wave is given by

$$\vec{A} = \vec{\epsilon} e^{-i\vec{p}\vec{x}}$$

$$\vec{\nabla} \vec{A} = 0 \implies \vec{p} \vec{A} = \vec{p} \vec{\epsilon} = 0$$

For $q^\mu = [\vec{q} \quad 0 \quad 0 \quad |\vec{q}|]^T$ we can choose the polarisation vectors as a linear polarisation:

$$\vec{\epsilon}_1(q) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{\epsilon}_2(q) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

, circular polarisation is also possible.

14.3 Minimal Coupling

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + iqA_\mu$$

In our example we look at the lagrangian

$$\mathcal{L}_0 = (\partial^\mu \Psi)^\dagger (\partial_\mu \Psi) - m^2 \Psi^\dagger \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

To couple the two fields we write

$$\mathcal{L} = (D^\mu \Psi)^\dagger (D_\mu \Psi) - m^2 \Psi^\dagger \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \mathcal{L}_0 + \text{coupling term}$$

14.3.1 Cannonical Quantisation of EM Field

We write our Lagrangian without the source field

1.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

2.

$$\Pi^{\mu\nu} = -(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\Pi^{0i} = E^i, H = \frac{1}{2} (\vec{E}^2 + \vec{B}^2)$$

3.

$$[\hat{A}^i(\vec{x}), \hat{E}^j(\vec{y})] = -i \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} \left(\delta_{ij} - \frac{p^i p^j}{\vec{p}^2} \right)$$

4.

$$\hat{A}^\mu(x) = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} \sum_{\lambda=1}^2 \left(\epsilon_\lambda^\mu(p) \hat{a}_{p\lambda} e^{-ipx} + \epsilon_\lambda^{\mu*}(p) \hat{a}_{p\lambda}^\dagger e^{ipx} \right)$$

5.

$$\hat{H} = \int d^3 p \sum_{\lambda=1}^2 E_p \hat{a}_{p\lambda}^\dagger \hat{a}_{p\lambda}$$

with $E_p = |p|$. This is the photon field. It has no longitudinal polarization (thus the sum runs only from $\lambda = 1$ to $\lambda = 2$).

Chapter 15

Discrete Transformation

15.1 Charge conjugation

The charge conjugation allows us to transform from particle $|p\rangle$ to antiparticle $|\bar{p}\rangle$ and vice versa

$$c|p\rangle = |\bar{p}\rangle$$

The action on the charge operator \hat{Q} is

$$\begin{aligned} c^\dagger \hat{Q} c &= -\hat{Q} \\ \implies c \hat{Q} &= -\hat{Q} c \end{aligned}$$

15.2 Parity operator

$$P^{-1} \hat{x} P = -\hat{x}$$

$$P^{-1} \hat{p} P = -\hat{p}$$

The parity operator acts in the following way:

$$P(\text{scalar}) = \text{scalar}$$

$$P(\text{vector}) = -\text{vector}$$

$$P(\text{pseudoscalar}) = -\text{pseudoscalar}$$

$$P(\text{pseudovector}) = \text{pseudovector}$$

A pseudo scalar is a scalar triple product $(\vec{a} \times \vec{b}) \cdot \vec{c}$

A pseudo vector is a cross product of vectors $\vec{a} \times \vec{b}$.

15.3 Time reversal

$$T : t \rightarrow -t$$

$$T^{-1} \hat{x} T = \hat{x}$$

$$T^{-1} \hat{p} T = -\hat{p}$$

The time reversal operator does not change the position, but it does change the momentum.

$$T^{-1} (i\hbar) T = T^{-1} [\hat{k}, \hat{p}] T = T^{-1} (\hat{x} \hat{p} - \hat{p} \hat{x}) T = -[\hat{x}, \hat{p}] = -i\hbar$$

The time reversal operator in general is proportional to the operator of complex conjugation. We now try to find T .

$$\begin{aligned} T &= UK, \text{ with } U \text{ a unitary operator and } K \text{ the complex conjugate} \\ T^2 &= UKUK \\ &= U(KUK) = UU^* \\ &= U(UT)^{-1} = \Phi \end{aligned}$$

Where Φ is a diagonal matrix of phase. The time reversal operator applied twice must get back to the original state.

$$\begin{aligned} \implies U &= \Phi U^T, U^T = U\Phi \\ U &= \Phi U\Phi \\ \implies T^2 &= \pm 1 \end{aligned}$$

We call T antiunitary (operator with any phase).

For spinless particles we have

$$T = K$$

For particles with spin we find

$$T^i - 1 \hat{S} T = -\hat{S}$$

A spin flips it's direction on time reversal. Looking at the componets of \hat{S} we find

$$\begin{aligned} K^{-1} \hat{S}_x K &= \hat{S}_x \\ K^{-1} \hat{S}_y K &= -\hat{S}_y \\ K^{-1} \hat{S}_z K &= \hat{S}_z \end{aligned}$$

We therefore get

$$\hat{T} = e^{-i\pi \hat{s}_y} \hat{K}$$

In general (for systems with many particles) we find

$$T = \Pi_i \exp\{-i\pi \hat{S}_{i,y}\} \hat{K} \implies \begin{cases} \text{for integer spin: } \hat{T}\hat{T} = 1 \text{ (even number of electrons)} \\ \text{for half integer spin: } \hat{T}\hat{T} = -1 \text{ (odd number of electrons)} \end{cases}$$

Kramer's theorem: the energy levels of a T invariant system with an odd number of electrons are even-fold degenerate.

Proof:

Let's assume a Hamiltonian, which is invariant under T (ie. no magnetic field or spin-orbit coupling). $\implies |\psi\rangle$ and $T|\psi\rangle$ are degenerate. The question now is, are they different?

Assume $T|\psi\rangle = \alpha|\psi\rangle$

$$T^2|\psi\rangle = T(\alpha|\psi\rangle) = \alpha^* T|\psi\rangle = |\alpha|^2 |\psi\rangle$$

but, we know that for an odd number of spin $\frac{1}{2}$ particles $T^2 = 1$.

$$\implies |\psi\rangle \neq T|\psi\rangle$$

Therefore each state is at least two-fold degenerate. *QED*.

Example:

- The trivial example is one spin $\frac{1}{2}$ particle.
- three spin $\frac{1}{2}$ particles

$$H = J \left(\vec{S}_1 \vec{S}_2 + \vec{S}_2 \vec{S}_3 + \vec{S}_3 \vec{S}_1 \right)$$

Three particles on a ring, each one interacting with it's nearest neighbour.

Chapter 16

Propagators and Green's functions

Motivation: Schroedinger Eq:

$$H\phi(x, t) = i \frac{\partial}{\partial t} \phi(x, t)$$

Wouldn't it be nice to have a function G with

$$\phi(x, t_x) = \int dy G^+(x, t_x; y, t_y) \phi(y, t_y)$$

Such a function G is called a propagator (the superscript $+$ means, that we only allow $t_x > t_y$). How do we find this function? Using the time evolution operator $U(t)$

$$G^+(x, t_x; y, t_y) = \theta(t_x - t_y) \langle x | U(t_x, t_y) | y \rangle$$

Is this the correct function?

$$\begin{aligned} \phi(y, t_y) &= \langle y | \phi(t) \rangle \\ \implies \phi(x, t_x) &= \int dy G^+(x, t_x; y, t_y) \phi(y, t_y) \\ &= \int dy \theta(t_x - t_y) \langle x | U(t_x, t_y) | y \rangle \langle y | \phi(t) \rangle \\ &= \dots \end{aligned}$$

For a time independent hamiltonian H , with eigenstates $|\phi_n\rangle$ we find

$$G^+ = \theta(t_x, t_y) \sum_n \phi_n(x) \phi_n^*(y) e^{-iE_n(t_x - t_y)}$$

If we fourier transform this in the variable $t = t_x - t_y$ (ie. $\int dt e^{iEt} \dots$) we get:

$$G^+(x, y; E) = \lim_{\epsilon \rightarrow 0} \sum_n \frac{i \phi_n(x) \phi_n^*(y)}{E - E_n + i\epsilon}$$

This form of the propagator is especially usefull, as it contains the Eigen-Energies (Poles at teh Eigen energies) and the eigenfunction (residues at the poles).

It turns out that G^+ is a (mathematical) Green's function in the following sense:

$$(H - i \frac{\partial}{\partial t_x}) G^+(x, t_x; y, t_y) = -i \delta(x - y) \delta(t_x - t_y)$$

The rest of chapter 16 will not be discussed in the lecture.

Chapter 17

Propagators and Fields

We assume we have a hamiltonian

$$H_0 = \sum E_{\vec{p}} a_{\vec{p}}^\dagger a_{\vec{p}}, \text{ with ground state: } |0\rangle$$

$$H = \dots, \text{ with ground state: } |\Omega\rangle$$

We now ask: what is the amplitude $G^+(x, y)$ (x, y four vectors) for the following process:

1. Start with ground state
2. create particle at $y = (y^0, \vec{y})$
3. anihilate it at a different time and position x
4. and end up in the groundstate again.

$$G^+(x, y) = \theta(x^0 - y^0) \langle \Omega | \hat{\phi}(x) \hat{\phi}^\dagger(y) | \Omega \rangle$$

With $\hat{\phi}(x)$ heisenberg operators $\hat{\phi}(x) = e^{iHx^0} \hat{\phi}(\vec{x}) e^{-iHx^0}$.

From now on we use the short hand notation

$$\int_{\vec{p}} \rightarrow \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}}$$

Plugging in the complex scalar field $\Psi^\dagger(x) = \int_{\vec{p}} a_{\vec{p}}^\dagger e^{ipx} + b_{\vec{p}} e^{-ipx}$ the "antiparticle" contribution from $b_{\vec{p}}$ drops out. This is a problem, as we want to treat particles and antiparticles on an equal footing.

Feynman realized that to treat (anti-) particles the same way, we should generalize our G^+ to

$$G(x, y) = \langle \Omega | T \hat{\phi}(x) \hat{\phi}^\dagger(y) | \Omega \rangle$$

With the *Time ordering symbol* T (this is *NOT* the time reversal operator). The time ordering symbol is defined as

$$T \hat{\phi}(x^0) \hat{\phi}(y^0) = \begin{cases} \hat{\phi}(x^0) \hat{\phi}(y^0) & \text{for } x^0 > y^0 \\ \hat{\phi}(y^0) \hat{\phi}(x^0) & \text{for } x^0 < y^0 \end{cases}$$

We can write this explicitly as

$$G(x, y) = \theta(x^0 - y^0) \langle \Omega | \phi(x) \phi^\dagger(y) | \Omega \rangle + \theta(y^0 - x^0) \langle \Omega | \phi^\dagger(y) \phi(x) | \Omega \rangle$$

In the first term we create a particle at y which then propagates to x where it is annihilated. In the second term we create an antiparticle at x which then propagates to y where it is annihilated.

The simplest examlpe of such a propagator is the so called *free propagator* G_0 , which is frequently called Δ :

$$\Delta(x, y) = \langle 0 | T \phi(x) \phi^\dagger(y) | 0 \rangle$$

the explicit form is given by:

$$\phi^\dagger(y) | 0 \rangle = \int_{\vec{p}} a_{\vec{p}}^\dagger | 0 \rangle e^{ipy} = \int_{\vec{p}} e^{ipy} |\vec{p}\rangle$$

$$\langle 0 | \phi(x) = \dots = \int_{\vec{q}} \langle \vec{q} | e^{-iqx}$$

In total we get

$$\langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3 2E_{\vec{p}}} e^{-ip(x-y)}$$

$$\langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3 2E_{\vec{p}}} e^{ip(x-y)}$$

To get the total propagator we need to add these two terms together (multiplied by the appropriate θ function).

$$\Delta(x, y) = \int \frac{d^3 p}{(2\pi)^3 2E_{\vec{p}}} \theta(x^0 - y^0) e^{-ip(x-y)} + \int \frac{d^3 p}{(2\pi)^3 2E_{\vec{p}}} \theta(y^0 - x^0) e^{ip(x-y)} = \Delta_1 + \Delta_2$$

It turns out that the free propagators Δ , Δ_1 and Δ_2 are Green's functions (of the Klein Gordan equation)

$$(\partial_\mu \partial^\mu + m^2) \Delta(x - y) = -i\delta(x - y)$$

Chapter 18

missing lecture - 13.04.21

Chapter 19

expanding the S -matrix - Feynman diagrams

Beginning of chapter missing - 13.04.21.

Feynmans brilliance comes from the insight that we can draw a diagram for all of these terms. There is a 1:1 correspondence with the terms of the expansion for \hat{S} . The so called Feynman rules for translating a diagram (for the ϕ^4 -theory)

- each vertex contributes a factor $-i\lambda$
- each internal line gives a propagator $\Delta(x - y)$, where x and y are the start and endpoints.
- external lines contribute $\begin{cases} e^{ipx} & \text{if its an incoming line} \\ e^{iqx} & \text{if its an outgoing line} \end{cases}$
- integrate the position of the vertices over spacetime
- divide by the symmetry factor $D \in \mathbb{N}$ (combinatorial factor)
 - every loop gives a factor $D_i = 2$
 - every pair of vertices joined by n lines gives a factor $D_i = n!$

The total D is given by

$$D = \prod D_i$$

In practise it's more convenient to work in momentum space. Feynman rules in momentum space

- Each vertex gives a factor $-i\lambda$
- each internal line has a directed momentum label q_i and leads to a propagator $\frac{i}{q^2 - m^2 + i\epsilon}$
- momentum is conserved in each vertex $\sum_{i \text{ at vertex}} q_i = 0$
- integrate over unconstrained internal momenta
- divide by the symmetry factor
- include an overall 4-momentum conserving δ -function $((2\pi)^4 \delta(p - q))$

Chapter 20

Scattering theory

This chapter will not be discussed in detail in the lecture, it is mainly self reading material. We will look at an example "Yukana's" $\Psi^\dagger \Psi \phi$ theory (that resembles QED).

$$\mathcal{L} = \partial^\mu \Psi^\dagger \partial_\mu \Psi - m^2 \Psi^\dagger \Psi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \mu \phi^2 - g \Psi^\dagger \Psi \phi$$

We have "Psions" with mass m and "Phions" with mass μ .

$$\mathcal{H}_I = +g \Psi_I^\dagger \Psi_I \phi_I ; \text{ from now on we omit the index } I$$

mode expansion

$$\begin{aligned} \Psi(x) &= \int_{\vec{p}} \frac{d^3 p}{(2\pi)^{\frac{3}{4}} \sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + b_{\vec{p}}^\dagger e^{ipx} \right) \\ \phi(x) &= \int_{\vec{q}} \frac{d^3 q}{(2\pi)^{\frac{3}{2}} \sqrt{eE_{\vec{q}}}} \left(c_{\vec{q}} e^{-iqx} + c_{\vec{q}}^\dagger e^{iqx} \right) \end{aligned}$$