Lecture Notes in

Ordered Sets

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Good order is the foundation of all things.
Edmund Burke (1729 – 1797)

Contents

Notation

- $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of nonnegative integers, integers, rational numbers, real numbers and complex numbers.
- For positive integers k and n, $k|n$ means that k divides n.
- For a finite set X, $|X|$ denotes its size, that is, the number of elements in X.
- 2^X denotes the **power set** of X, that is, the family of all subsets of X including the empty set $\emptyset$, and X, itself.
- The **identity relation** on a set X is defined by

  $$\iota_X = \{(x, x) \mid x \in X\}$$

  and the **universal relation** on X is defined by

  $$\omega_X = \{(x, y) \mid x, y \in X\}\,.$$

  We often omit the subscript X from these notations.
- For a family $\mathscr{A} \subseteq 2^X$ of subsets,

  $$\bigcup \mathscr{A} = \bigcup_{A \in \mathscr{A}} A \quad \text{and} \quad \bigcap \mathscr{A} = \bigcap_{A \in \mathscr{A}} A\,.$$

- A **partition** of a set X is a family $\pi = \{X_i \mid i \in I\}$ of nonempty subsets of X such $X_i \cap X_j = \emptyset$ whenever $i \neq j$ and $X = \bigcup_{i \in I} X_i$. The elements X_i are **blocks** of the partition.
- For a mapping $\alpha \colon X \to Y$, and a subset $A \subseteq X$, let $\alpha(A) = \{\alpha(x) \mid x \in A\}$.

Some early history

Partially ordered sets and lattices can be said to have their origin in the work of G. Boole when he tried to axiomatize propositional logic. Lattices were introduced at the end of the 19th century by C.S. Peirce and E. Schröder ("Die Algebra der Logik") in the context of Boolean algebras. In 1890's R. Dedekind considered lattices (under the name 'dualgruppen') in the context of ideals generated by algebraic numbers, and subgroups abelian groups. Modularity of lattices is Dedekind's product and this notion comes from subgroup considerations. After Dedekind's pioneering studies nothing much happened in lattice theory until 1930's when G. Birkhoff and O. Ore revived the theory. Modern lattice theory owes much to Birkhoff who systematized their study in the 1930s. He also showed the connection of lattice theory to universal algebra.

1 Posets

1.1. First notions

Ordered sets

Let X and Y be sets, and $R \subseteq X \times Y$ a relation between the elements of X and Y, i.e., R consists of some pairs (x, y) with $x \in X$ and $y \in Y$. We often write xRy instead of $(x, y) \in R$. We let

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

be the **inverse relation** of R. For two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, their **composition** is the relation $R \circ S \subseteq X \times Z$ defined by

$$R \circ S = \{(x, z) \mid \exists y \in Y : (x, y) \in R \text{ and } (y, z) \in S\}.$$

In the order theory we usually consider relations $R \subseteq X \times X$ **on the set** X. Such a relation R is called

- **reflexive** if $(x, x) \in R$ for all $x \in X$;
- **symmetric** if $(x, y) \in R$ implies $(y, x) \in R$;
- **antisymmetric** if $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$;
- **transitive** if $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$.

Let $R^0 = \iota_X$, $R^1 = R$, $R^2 = R \circ R$, and inductively $R^{n+1} = R^n \circ R$ for $n \geq 1$.

The above definitions have the following equivalent formulations:

Lemma 1.1. *Let $R \subseteq X \times X$ be a relation. Then*

(1) *R is reflexive if and only if $\iota_X \subseteq R$.*
(2) *R is symmetric if and only if $R = R^{-1}$.*
(3) *R is transitive if and only if $R^2 \subseteq R$.*

- For a relation R, let R^+ be its **transitive closure**:

$$R^+ = \bigcup_{n=1}^{\infty} R^n.$$

Hence $(x, y) \in R^+$ if and only if there exists a finite sequence $z_0, z_1, \ldots, z_n$ for some n such that $x = z_0$, $z_i R z_{i+1}$ for $i = 0, 1, \ldots, n-1$, and $z_n = y$.

A relation $R \subseteq X \times X$ is called

- a **quasi-order** (also known as a **preorder**) if it is reflexive and transitive;
- a **partial order** if it is reflexive, transitive and antisymmetric;
- a **linear order** (or a **chain** or a **total order**) if it is a partial order and

$$xRy \quad \text{or} \quad yRx \quad \text{for all } x, y \in X;$$

- an **equivalence relation** if it is reflexive, transitive and symmetric.

If R is an equivalence relation on the set X, then the **equivalence classes**

$$xR = \{y \mid xRy\} \quad \text{for } x \in X$$

form a partition of X, that is,

$$X = \bigcup_{x \in X} xR \quad \text{and} \quad xR \cap yR = \emptyset \text{ for } xR \neq yR\,.$$

Theorem 1.2. *Let R be a quasi-order on a set X, and let $\sim$ be defined by*

$$x \sim y \iff xRy \quad and \quad yRx\,.$$

Then $\sim$ is an equivalence relation that satisfies: if $x_1 \sim y_1$ and $x_2 \sim y_2$, then x_1Rx_2 if and only if y_1Ry_2.

Proof. Exercise. □

Example 1.3. Let $\alpha \colon X \to Y$ be a function. Then the preimages $\alpha^{-1}(y) = \{x \mid \alpha(x) = y\}$ form a partition of X, and the **kernel** defined by

$$\ker(\alpha) = \{(x, y) \mid \alpha(x) = \alpha(y)\}$$

is an equivalence relation on X. □

Posets

If R is a partial order on the set X, then $P = (X, R)$ is a **partially ordered set**, or a **poset** for short. Later we write mostly $x \in P$ and $A \subseteq P$ instead of $x \in X$ and $A \subseteq X$, respectively. Also, we denote by $\leq_P$ the order of the poset P. Let us write $x <_P y$ if $x \leq_P y$ and $x \neq y$. The relation $<_P$ is the **strict partial order** associated to $\leq_P$.

We let

$$[x, y]_P = \{z \mid x \leq_P z \quad \text{and} \quad z \leq_P y\}$$

be the **interval** (or **segment**) of the elements $x, y \in X$. If $x \not\leq_P y$ then the interval is empty.

A poset P is said to be **locally finite** if every interval $[x, y]_P$ is finite.

Example 1.4. The poset $(\mathbb{N}, \leq)$ of natural numbers is infinite, but locally finite. On the other hand, the poset $(\mathbb{Q}, \leq)$ is not locally finite. □

If $\leq_P$ is a partial order, then we write $x \bowtie_P y$, or simply $x \bowtie y$, if x and y are **comparable** elements, that is, if $x \leq_P y$ or $y \leq_P x$. If x and y are not comparable, denoted by $x \| y$, then they are **incomparable**. The relation $\bowtie$ is reflexive and symmetric, but it need not be transitive. To see this, consider the 3-element poset where $x \leq_P y$ and $x \leq_P z$, but y and z are incomparable.

A poset P is said to be **connected** if for each pair of elements $x, y \in P$ there is a sequence $x = z_0 \bowtie z_1 \bowtie z_2 \ldots \bowtie y$ of comparable elements.

Let P be a poset, and let $A \subseteq P$ be a nonempty subset.

- An element $x \in A$ is **minimal** in A, if, for all $y \in A$ with $y \leq_P x$, we have $y = x$. An element x satisfying the condition $x \leq y$ for all $y \in A$ is called the **minimum element** of A. If the full poset P has a minimum element, it is called the **bottom element** of P, and it is often denoted by 0_P.
- An element $x \in A$ is **maximal** in A, if for all $y \in A$ with $x \leq_P y$, we have $y = x$. An element x satisfying the condition $y \leq x$ for all $y \in A$ is called the **maximum element** of A. If the full poset P has a maximum element, it is called the **top element** of P, and it is often denoted by 1_P.

We say that an element $y \in P$ **covers** $x \in P$, denoted by $x \prec_P y$, if $[x, y] = \{x, y\}$, that is, $x \neq y$, $x \leq_P y$ and there exists no element $z \neq x, y$ such that $x \leq_P z \leq_P y$.

Example 1.5. (1) Note that the covering relation may be empty. Indeed, consider the poset $(\mathbb{Q}, \leq)$ of rational numbers with their usual linear ordering. Then no element covers any other element. Notice that the poset of rational numbers is not locally finite.

(2) A locally finite poset P is completely determined by its cover relations. In P if $x <_P y$ then there exists an element z such that $x \prec z$ and an element z' such that $z' \prec y$. Of course, it can be that $z = z'$ or that $z = y$ and $z' = x$. $\square$

Hasse diagrams

When P is finite, one can represent it conveniently using the covering relation (which determines P). In the **Hasse diagram** of P if $x \prec_P y$ then we draw a line connecting x and y so that y is above x in the picture.

Each locally finite poset P can be represented by a $\{0, 1\}$-matrix M, where

$$M_{xy} = 1 \iff x = y \text{ or } x \prec_P y .$$

Here the powers $M, M^2, \ldots$ tell how many chains there are from each x to each y:

$$\begin{aligned} &M^k_{xy} \text{ is the number of ascending chains (with repetitions):} \\ &\quad x = x_0 \leq_P x_1 \leq_P \ldots \leq_P x_k = y \\ &\quad \text{where } x_i = x_{i+1} \text{ or } x_i \prec_P x_{i+1} . \end{aligned}$$

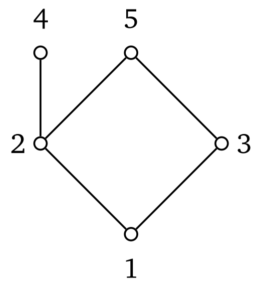
In particular, for a finite P, there is a power M^k, with $k \leq |P|$, such that

$$M^k_{xy} > 0 \iff x \leq_P y .$$

Example 1.6. Consider the 5-element poset on $P = \{1,2,3,4,5\}$, where the order is defined by the following matrix M:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The covering relation is represented by the Hasse diagram on the right. $\square$



Example 1.7 (Subsets)**.** Let X be a set, and consider the family 2^X of its subsets with respect to inclusion. Then $(2^X, \subseteq)$ is a poset that is called the **subset poset** (or the **Boolean poset**) on X.

For the eight subsets of the finite set $X = \{1,2,3\}$, the Hasse diagram of 2^X is given in Figure 1.1. $\square$

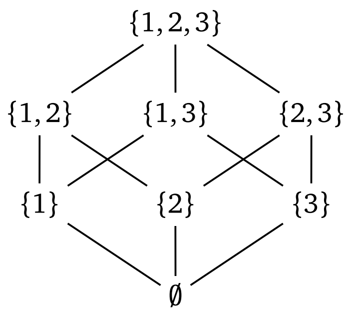


FIG. 1.1. The subset poset of $\{1,2,3\}$.

Subposets

Let P be a poset on a set X, and $A \subseteq X$ a subset of the underlying set X. We define the **suborder** $\leq_A$ as the restriction of $\leq_P$ to pairs of A:

$$x \leq_A y \iff x \leq_P y \text{ and } x, y \in A.$$

Then $(A, \leq_A)$ is a **subposet** of P.

- A subposet C is called a **chain** in P, if C is a linearly ordered set. A chain C is **maximal** if for all $z \notin C$, $C \cup \{z\}$ is not a chain, i.e., there exists an element $x \in C$ such that $x || z$.
- A subposet A of P consisting of only incomparable elements is called an **antichain**. An antichain A of P is **maximal** if for all $x \in P$, x is comparable to an element in A (including the possibility that $x \in A$).

Example 1.8. Let P_i, $i \in I$, be a set of posets on a common set X. Then also their intersection $P = \bigcap_{i \in I} P_i$ is a poset. Indeed, xPy holds if and only if xP_iy holds for all $i \in I$, and the poset conditions for P follow from the conditions for each individual poset P_i.

On the other hand, a union $Q = \bigcup_{i \in I} P_i$ need *not* be a poset, since it can be that xP_iy and yP_jx for some $i \neq j$ and $x \neq y$, and in Q we would have to have both xQy and yQx. Also, Q need not be transitive and thus not even a quasi-order. □

For a subset $A \subseteq P$, let

$$\begin{aligned} \uparrow A &= \{y \in P \mid \exists x \in A\colon x \leq_P y\}\,, \\ \downarrow A &= \{y \in P \mid \exists x \in A\colon y \leq_P x\} \end{aligned}$$

be the **up-set** (or the **order filter**) and the **down-set** (or the **order ideal**) of P generated by A. If $A = \{x\}$ is a singleton set, then we write $\uparrow x$ and $\downarrow x$ instead of $\uparrow\{x\}$ and $\downarrow\{x\}$. These are the **principal up-set** and the **principal down-set** of P generated by the element x.

Bounds

Let $A \subseteq P$ be a subset of the poset P.

- An element $x \in P$ is an **upper bound** of A if $x \in A^{\mathrm{u}}$, i.e., if $y \leq_P x$ for all $y \in A$. Let
$$A^{\mathrm{u}} = \{x \in P \mid y \leq_P x \text{ for all } y \in A\}$$
be the set of all upper bounds of A. We say that $x \in A^{\mathrm{u}}$ is a **least upper bound** (or a **supremum**) of A, if $x \leq_P y$ for all upper bounds $y \in A^{\mathrm{u}}$. It will be denoted by $\bigvee A$, if it exists.
- An element $x \in P$ is a **lower bound** of A if $x \in A^{\mathrm{l}}$, i.e., if $x \leq_P y$ for all $y \in A$. Let
$$A^{\mathrm{l}} = \{x \in P \mid x \leq_P y \text{ for all } y \in A\}$$
be the set of all lower bounds of A. Also, $x \in A^{\mathrm{l}}$ is a **greatest lower bound** (or an **infimum**) of A, if $y \leq_P x$ for all lower bounds $y \in A^{\mathrm{l}}$. It will be denoted by $\bigwedge A$, if it exists.

Afterwards, for instance, we let A^{ul} denote $\left(A^{\mathrm{u}}\right)^{\mathrm{l}}$. (It is the set of all lower bounds of the upper bounds of A.) Note that usually $A^{\mathrm{ul}} \neq A^{\mathrm{lu}}$.

The sets A^{u} and A^{l} can be empty. In general,

$$A^{\mathrm{u}} = \bigcap_{x \in A} \uparrow x \quad \text{and} \quad A^{\mathrm{l}} = \bigcap_{x \in A} \downarrow x\,.$$

If $A = \{x\}$ is a singleton, then clearly $\uparrow x = \{x\}^{\mathrm{u}}$ and $\downarrow x = \{x\}^{\mathrm{l}}$.

Example 1.9. Recall the poset of Example 1.6. There $\uparrow 2 = \{2, 4, 5\} = \uparrow\{2, 4\}$, and $\downarrow 5 = \{1, 2, 3, 5\}$. Note that $\{2, 4\}^{\mathrm{u}} = \{4\} \neq \uparrow\{2, 4\}$. □

Lemma 1.10. *Let $A \subseteq P$ be a nonempty subset of a poset P. If A does have a least upper bound, then it is unique. Similarly, if A has a greatest lower bound, then it is unique.*

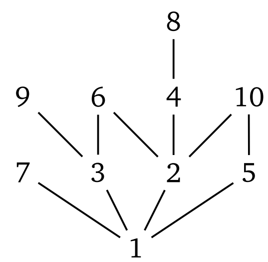
Proof. Let x be a least upper bound of A, and let $y \in A^{\mathrm{u}}$, then $x \leq_P y$. Hence if also y is a least upper bound then $x = y$. The proof for the greatest lower bound is similar. □

Examples

Example 1.11 (Subgroups)**.** For a group G, its subgroups form a poset under the subgroup relation. Notice that the *normal subgroups* of G need *not* form a poset under the 'normal subgroup'-relation $\lhd$, since a normal subgroup $H \lhd N$ of a normal subgroup $N \lhd G$ need not be a normal subgroup of G, and so transitivity condition fails for this relation. □

Example 1.12 (Projective geometry)**.** Let V be a vector space over a field F. The **projective geometry** $\mathrm{PG}(V)$ is the poset on the subspaces of V with respect to the inclusion relation. □

Example 1.13 (Divisibility poset)**.** The positive natural numbers $\mathbb{N} \setminus \{0\}$ form an infinite poset under the divisibility relation: $n \leq_{\mathbb{N}} m$ if and only if n divides m. Among the finite subposets of this are the **divisibility posets** $\mathbb{N}_n = (\{1, 2, \ldots, n\}, |)$. On the right there is the Hasse diagram of $\mathbb{N}_{10}$. □



Example 1.14 (Divisor poset)**.** On the other hand, the **divisor poset** T_n of a postive integer n consists of the divisors of n, and the relation is the divisibility relation. Thus T_n is a subposet of $\mathbb{N}_n$. □

Example 1.15 (Partitions)**.** The set $\Pi(X)$ of all partitions of a set X forms a poset under the **partition order**: $\pi_1 \leq \pi_2$, if each block Y of π_2 is a union of blocks of π_1. Equivalently, each block of π_1 is contained in some block of π_2.

The poset of partitions goes hand-in-hand with the poset of **equivalence relations**. The set $\mathrm{Eq}(X)$ of equivalence relations on a set X forms a poset under inclusion. □

Example 1.16 (Words)**.** Let Σ be a set of symbols, called an **alphabet**. Each sequence of symbols $a_1 a_2 \ldots a_n$, for $n \geq 1$, is a **word** over Σ. The empty word, which has no symbols, is denoted by ε. The set of all words over Σ is denoted by Σ^*. A word u is a **factor** of v, denoted $u \lhd v$ if $v = v_1 u v_2$, where v_1 and v_2 can be empty. Then $(\Sigma^*, \lhd)$ is a poset, the **factor poset** of Σ^*. □

Example 1.17. For a poset P, consider the set P^* of all words over P. The **lexicographic order** $\leq_P^*$ on P^* is defined as follows:

$$\begin{aligned} u \leq_P^\ell v \iff & v = uw \text{ for some } w, \quad \text{or} \\ & u = wau' \text{ and } v = wbv' \text{ where } a <_P b\,. \end{aligned}$$

Moreover, we put $\varepsilon \leq_P^\ell u$ for all $u \in P^*$.

With respect to the lexicographic order P^* is a poset. Also, if P is linearly ordered, so is P^*. Note that although P can be finite, P^* is always infinite, and it has no maximal elements. The empty word is the bottom element. □

Example 1.18 (Subsequences)**.** Also, Σ^* is a poset under the **subsequence order**:

$$\left.\begin{aligned} x &= x_1x_2\cdots x_n \\ y &= y_1x_1y_2x_2\cdots y_nx_ny_{n+1} \end{aligned}\right\} \Longrightarrow x \leq y\,.$$

Words can be used in many contexts to describe (or code) sequences. For instance, let $\Sigma = \{-,+\}$ be a binary alphabet. If P is a poset then each sequence $s = (x_0,\dots,x_n)$, where $x_i \bowtie x_{i+1}$, can be associated with a word $\sigma(s) = (\sigma_0,\dots,\sigma_{n-1})$ where

$$\sigma_i = \begin{cases} + & \text{if } x_i \leq_P x_{i+1}\,, \\ - & \text{if } x_i \geq_P x_{i+1}\,. \end{cases}$$

□

Isotone mappings

Let P and Q be two posets.

- A function $\alpha\colon P \to Q$ is an **isotone mapping** (also called **monotone** or **order preserving**) if for all $x, y \in P$,

$$x \leq_P y \implies \alpha(x) \leq_Q \alpha(y)\,.$$

- An injective isotone mapping is called an **order embedding**.
- Moreover, α is an **order isomorphism** if it is bijective and also its inverse $\alpha^{-1}: Q \to P$ is isotone. Two posets P and Q are **isomorphic**, denoted by $P \cong Q$, if there exists an order isomorphism between them.

Example 1.19. For a poset P, let P^P be the set of all mappings $\alpha\colon P \to P$. Then P^P forms a poset under the operation

$$\alpha \leq_{P^P} \beta \iff \alpha(x) \leq_P \beta(x) \text{ for all } x \in P\,.$$

It is straightforward to check the poset conditions for P^P. Also, the isotone mappings $P \to P$ form a poset with respect to the same ordering. □

Lemma 1.20. *Two posets P and Q are isomorphic if and only if there exists a surjective mapping $\alpha: P \to Q$ such that, for all $x, y \in P$,*

$$x \leq_P y \iff \alpha(x) \leq_Q \alpha(y)\,. \tag{1.1}$$

Proof. Notice first that if α satisfies (1.1), then it is necessarily a bijection from P onto Q. Indeed, if $\alpha(x) = \alpha(y)$ then $\alpha(x) \leq_Q \alpha(y)$ gives $x \leq_P y$, and symmetrically we obtain $y \leq_P x$, and thus $x = y$. Hence α is injective. It is surjective by hypothesis.

Now by (1.1), α is isotone, and so is α^{-1}, because (1.1) is equivalent to the condition

$$\alpha^{-1}(u) \leq_P \alpha^{-1}(v) \iff u \leq_Q v\,.$$

Hence α is an order isomorphism.

Conversely, assume then that $\alpha: P \to Q$ is an order isomorphism. Then both α and α^{-1} are isotone, and hence (1.1) holds. □

Example 1.21. One has to be careful in the above, since an isotone bijection $\alpha: P \to Q$ need not be an order isomorphism. Indeed, consider the two posets of Figure 1.2. There $\alpha: P \to Q$ is an isotone bijection, but it is not an order isomorphism. In fact, P and Q are not isomorphic.

For isomorphic posets the Hasse diagrams look the same except for the names of the elements. □

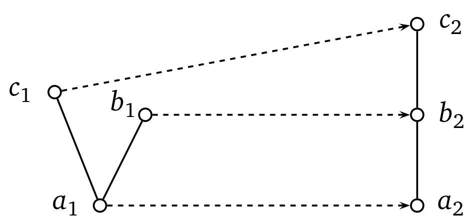


FIG. 1.2. Two posets with an isotone mapping.

Example 1.22 (Abian)**.** Consider the posets P and Q with Hasse diagrams of Figure 1.3. The mapping $\varphi: P \to Q$ defined by $\varphi(x_i) = y_i$ is isotone and bijective. Also, the mapping $\psi: Q \to P$ defined by

$$\begin{aligned} &\psi(y_1) = x_2, \quad \psi(y_3) = x_0\,, \\ &\psi(y_i) = x_{i+4}, \quad i \geq 0 \text{ even}\,, \\ &\psi(y_i) = x_{i-4}, \quad i \geq 5 \text{ odd} \end{aligned}$$

is isotone and bijective. However, these posets are not isomorphic. □

We shall now prove that *all posets can be represented as set systems*.

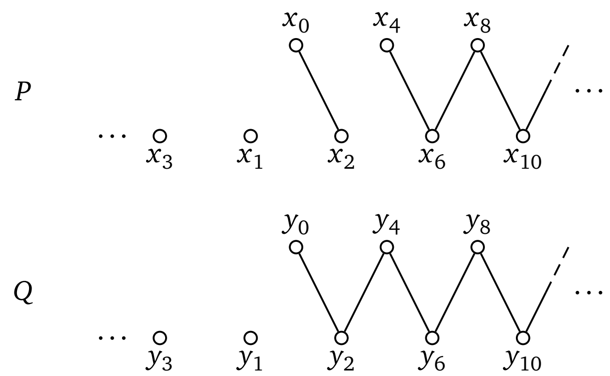


FIG. 1.3. Nonisomorphic posets with bijective isotone mapping $P \to Q$ and $Q \to P$.

Theorem 1.23. *Each poset P is isomorphic to the dual poset of its principal down-sets under set inclusion. To be more precise, let $\varphi\colon P \to 2^P$ be defined by*

$$\varphi(x) = \downarrow x\,.$$

Then φ is an order isomorphism from P onto the set of all principal down-sets of P.

Proof. First, φ is a bijection to the principal down-sets:

$$\varphi(x) = \varphi(y) \iff \downarrow x = \downarrow y \iff x \leq_P y \text{ and } y \leq_P x \iff x = y\,.$$

To show that φ is an order isomorphism, observe that if $x \leq_P y$, then also $\varphi(x) \subseteq \varphi(y)$. Also, since $x \in \downarrow x$, $\varphi(x) \subseteq \varphi(y)$ implies $x \leq_P y$. Therefore $x \leq_P y$ if and only if $\varphi(x) \subseteq \varphi(y)$, and the claim follows. □

Direct product

Let P and Q be two posets. Define their **direct product** $P \times Q$ as the cartesian product $\{(x, y) \mid x \in P, y \in Q\}$ together with the partial order $\leq_{P\times Q}$:

$$(x_1, y_1) \leq_{P\times Q} (x_2, y_2) \iff x_1 \leq_P x_2 \text{ and } y_1 \leq_Q y_2\,.$$

Theorem 1.24. *The direct product of two posets is a poset.*

Proof. Exercise. □

Example 1.25. The Hasse diagram of the direct product of two chains of length two is the square as depicted in Figure 1.4. □

As usual, the direct products

$$\Pi_{i=1}^{k} P_i = P_1 \times P_2 \times \cdots \times P_k \tag{1.2}$$

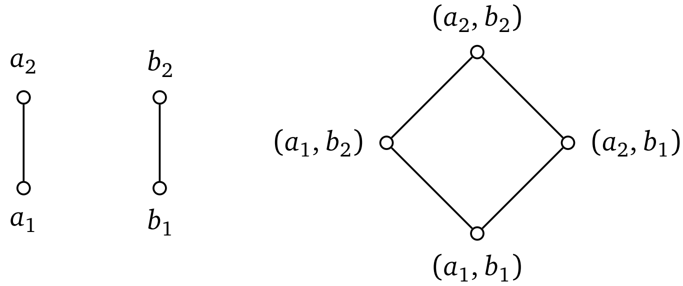


FIG. 1.4. Direct product of two posets.

are defined inductively so that $\Pi_{i=1}^k P_i = \left(\Pi_{i=1}^{k-1} P_i\right) \times P_k$. This poset is isomorphic to $P_1 \times \left(\Pi_{i=2}^k P_i\right)$, and this allows the notation (1.2).

Example 1.26 (Subset poset)**.** Let X be a finite set of n elements, and consider the subset poset 2^X. We show that 2^X is isomorphic to the n-fold direct product

$$\Pi^n \mathbf{C}_2 = \mathbf{C}_2 \times \mathbf{C}_2 \times \cdots \times \mathbf{C}_2$$

of the 2-element chain $\mathbf{C}_2 = \{0, 1\}$ where $0 < 1$. Indeed, let $X = \{x_1, x_2, \ldots, x_n\}$, and let $\alpha\colon 2^X \to \Pi^n \mathbf{C}_2$ be such that the ith component of $\alpha(A)$ is 1 just in case $x_i \in A$. For instance, if $n = 5$, then $\alpha(\{x_2, x_3, x_5\}) = (0, 1, 1, 0, 1)$. In $\Pi^n \mathbf{C}_2$,

$$(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n) \iff a_i \leq b_i \text{ for all } i\,,$$

and this corresponds to the subset relation in 2^X. □

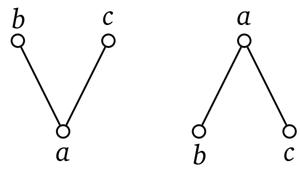
Dual posets

For a poset P, its **dual poset** P^d reverses the order:

$$x \leq_{P^d} y \iff y \leq_P x\,.$$

It is immediate that P^d is a poset, and that $(P^d)^d = P$.

Example 1.27. The Hasse diagram of a finite dual poset P^d is obtained from the diagram for P by turning it over. □



Many notions and results concerning a poset P have dual statements for the poset P^d. For instance, we immediately have the following dual representation theorem:

Theorem 1.28. *Each poset P is isomorphic to the poset of its principal up-sets under inclusion.*

Finite posets

In Figure 1.5 all connected posets are represented with four elements.

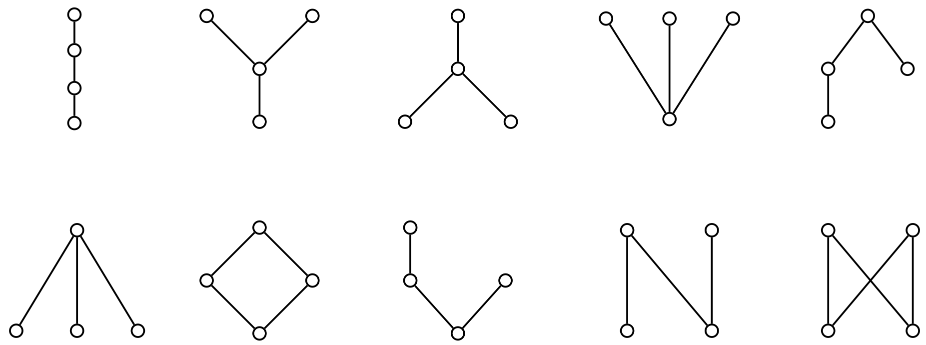


FIG. 1.5. Nonisomorphic connected posets of 4 elements. There are also 6 disconnected posets of four elements.

Table 1 gives the number of all nonisomorphic posets of n elements for $n = 1, 2, \ldots, 14$. The values are from Sequence A000112 in Sloane's integer sequence web-page: `http://www.research.att.com/ njas/sequences/Seis.html`.

n	posets	n	posets
1	1	8	16 999
2	2	9	183 231
3	5	10	2 567 284
4	16	11	46 749 427
5	63	12	1 104 891 746
6	318	13	33 823 827 452
7	2 045	14	1 338 193 159 771

TABLE 1. The number of nonisomorphic posets.

No formula is known for the number $p(n)$ of finite posets, but it is known (Kleitman and Rothschild 1970) that we have the following asymptotic bound for the number $p(n)$:

$$\lim_{n\to\infty} \frac{\log p(n)}{\frac{n^2}{4}\log 2} = 1\,.$$

The following is a finite version of the so called Whaley's theorem. We say that a subset $A \subseteq P$ of a poset is a **super-antichain** if for all different $x, y \in A$, $\downarrow x \cap \downarrow y = \emptyset$ and $\uparrow x \cap \uparrow y = \emptyset$, i.e., distinct elements of A do not have a common upper bound or a common lower bound.

Theorem 1.29 (Freese, Hyndman, Nation). *Let P be a finite poset of n elements. Then P has a subset A such that $|A| \geq \lceil n^{1/3} \rceil$, where $A = \downarrow x$ or $A = \uparrow x$ for some $x \in P$ or A is a super-antichain of P.*

Proof. Let m be the maximum size of a principal down-set or a principal up-set of P, and assume that $m \leq \lceil n^{1/3} \rceil$.

For each element $x \in P$, let

$$N_x = \bigcup_{u \geq_P x} \downarrow u \cup \bigcup_{v \leq_P x} \uparrow v \,.$$

Denote $a = |\downarrow x \setminus \{x\}|$ and $b = |\uparrow x \setminus \{x\}|$. Then

$$\begin{aligned} |N_x| &= \Big| \bigcup_{u >_P x} (\downarrow u \setminus \downarrow x) \cup \bigcup_{v <_P x} (\uparrow v \setminus \uparrow x) \Big| + 1 \\ &\leq b(m - (a+1)) + a(m - (b+1)) + 1 = (m-1)(a+b) - 2ab + 1 \\ &= (m-1)^2 + 1 - (m-1-a)(m-1-b) - ab \leq (m-1)^2 + 1 \,. \end{aligned}$$

We now define a super-antichain A: Let $x_1 \in P$ be arbitrary, and suppose $x_1, \ldots, x_i$ form a super-antichain. Then let $x_{i+1} \notin N_{x_1} \cup \cdots \cup N_{x_i}$, if the union is a proper subset of P. Finally, we obtain an index k such that $P = \bigcup_{i=1}^{k} N_{x_i}$, from which it follows that $n \leq k((m-1)^2 + 1)$, and hence $m \geq \lceil n^{1/3} \rceil$ or $k \geq \lceil n^{1/3} \rceil$, since $(m-1)^2 + 1 \leq m^2$ and so $n \leq km^2$. □

Order-preserving functions behave nicely on finite posets.

Theorem 1.30 (Abian). *Let P be a finite poset, and $\varphi\colon P \to P$ an order embedding. Then φ is an isomorphism.*

Proof. Exercise. □

1.2. Chains

Zorn's lemma

Let $\mathscr{X}$ be a family of nonempty sets. A **choice function** on $\mathscr{X}$ is a mapping $\alpha\colon \mathscr{X} \to \bigcup \mathscr{X}$ such that for each $A \in \mathscr{X}$, $\alpha(A) \in A$. For infinite families $\mathscr{X}$, the existence of a choice function is not self-evident.

Axiom of Choice. *For any family of nonempty sets there exists at least one choice function.*

This principle was first stated and used (in a different form) by Zermelo in 1904 in his proof that any set can be well-ordered.

This axiom has many important equivalent formulations; more than 200 of these are recorded in the book of Rubin and Rubin. For instance, the following results are equivalent to the Axiom of Choice:

- Every vector space has a basis. Proved by Blass in 1984 to be equivalent to the Axiom of Choice.
- Every field has an algebraic closure (Steinitz 1910).
- Every commutative ring with identity has a maximal ideal (Hodges 1979).
- Tychonov's Theorem stating that the product of compact spaces is compact (Kelley 1950).
- Krein-Milman Theorem which states that the unit ball B of the dual of a real normed linear space has a point which is not an interior point of any line segment in B (Bell and Fremlin 1972).

The following theorem is considered to be an axiom for us. Zorn's Lemma plays an important role in the foundations of mathematics. It is equivalent to the Axiom of Choice.

Theorem 1.31 (Zorn's Lemma)**.** *Let P be a poset, where every chain has an upper bound. Then P contains a maximal element.*

Proof. Omitted. □

We have the following result using Zorn's lemma.

Theorem 1.32 (Kuratowski)**.** *Let P be a poset, and C be a chain in P. Then there exists a maximal chain M of P such that $C \subseteq M$.*

Proof. Let $\mathscr{C}$ be the set of all chains containing C. It is nonempty since $C \in \mathscr{C}$. We consider $\mathscr{C}$ as a poset with respect to the set inclusion, and we show that $\mathscr{C}$ has a maximal element, which then proves the claim.

Consider a chain $\mathscr{B}$ of sets in $\mathscr{C}$, and regard $B = \cup\mathscr{B}$ as a subposet of P. We show that B is a chain in P. To this end, let $x, y \in B$, and let $B_x, B_y \in \mathscr{B}$ be such that $x \in B_x$ and $y \in B_y$. Then either $B_x \subseteq B_y$ or $B_y \subseteq B_x$, since $\mathscr{B}$ is a chain. Without loss of generality, we can assume that $B_x \subseteq B_y$, and so $x, y \in B_y$, which means that x and y are comparable in P, since B_y is a chain in P. Therefore $B \in \mathscr{C}$.

Clearly, B is an upper bound of $\mathscr{B}$, and hence $\mathscr{C}$ has a maximal element by Zorn's lemma. □

Well-orders

Later in this section we study *partial* well-orders. We start with the 'total' notion of this order.

A poset P is said to be **well-ordered** if every nonempty subset $X \subseteq P$ has a minimum element, i.e., an element $x \in X$ such that $x <_P y$ for all $y \in X \setminus \{x\}$.

Lemma 1.33. *Every well-ordered poset P is a chain.*

Proof. If $x, y \in P$ are any two different elements, then the subset $\{x, y\}$ has a minimum element, and therefore either $x <_P y$ or $y <_P x$, and thus P is linearly ordered. □

The converse of this lemma does not hold, since, for instance, $P = (\mathbb{Z}, \leq)$ is a chain but it has no minimum element. Note, however, that every finite chain is well-ordered.

Theorem 1.34. *Let P be well-ordered. Then each element $x \in P$ is either maximum or there exists a unique $y \in P$ such that $x \prec_P y$.*

Proof. If x is not a maximum element of P (i.e., the unique top element of P, if it exists), the set $\Uparrow(x) \setminus \{x\}$ has a unique minimum element y, which is as claimed. □

The following theorem states that one can define an ordering of each set such that the result is a well-order. For instance, for the integers we can take the unorthodox order $0 \leq' -1 \leq' +1 \leq' -2 \leq' +2 \leq' \ldots$.

Theorem 1.35 (Well-Ordering)**.** *Every set X can be given a well-ordering.*

Proof. Omitted. □

Chain conditions

Let P be a poset on X. Then P satisfies

- the **finite chain condition** or **FCC** if every chain in P is finite.
- the **ascending chain condition** or **ACC** if every strictly increasing chain $x_1 <_P x_2 <_P \ldots$ is finite.
- the **descending chain condition** or **DCC** if every strictly descending chain $x_1 >_P x_2 >_P \ldots$ is finite. A poset that satisfies DCC is **well-founded**.
- the **finite antichain condition** or **FAC** if every antichain of P is finite.

Note that a poset P satisfying the finite chain condition can still have unbounded height, since there can be arbitrarily long chains in P. Obviously the FCC implies both ascending and descending chain conditions.

Lemma 1.36. *A poset P satisfies the ascending chain condition if and only if every nonempty subset $X \subseteq P$ has a maximal element.*

Proof. Let $A \subseteq P$ be nonempty such that A has no maximal elements. Fix an element $x_0 \in A$. Since x_0 is not maximal, there exists an $x_1 \in A$ such that $x_0 <_P x_1$, and so forth, $x_0 <_P x_1 <_P x_2 <_P \ldots$, which gives a infinite ascending chain. (Note that the 'so forth' part needs Axiom of Choice: one can always pick x_i.)

Assume then that every nonempty subset has a maximal element, and let $x_1 \leq_P x_2 \leq_P \ldots$ be an ascending chain in P. Write $X = \{x_i \mid i = 1, 2, \ldots\}$, and let $x_k \in X$ be a maximal element in X. Hence $x_i = x_k$ for all $i \geq k$, and therefore X is finite. It follows that P satisfies the ACC. □

A dual argument shows that a poset P satisfies the descending chain condition if and only if every nonempty subset of P has a minimal element. In other words,

Lemma 1.37. *A poset P is well-founded if and only if every nonempty subset of P has a minimal element.*

Theorem 1.38 (König)**.** *Let P be a poset satisfying both the finite chain and antichain conditions (FCC and FAC). Then P is finite.*

Proof. Notice first that FCC implies both ACC and DCC. For each $A \subseteq P$, let

$$\min(A) = \{x \mid x \text{ minimal in } A\}.$$

The set $\min(A)$ is an antichain and thus it is always finite by FAC. By DCC, for each $y \in A$, there exists an $x \in A$ such that $x \leq_P y$, i.e., $A \subseteq \uparrow\min(A)$. (Indeed, for $x <_P y$, consider the half open interval $[x, y]_P \setminus \{x\}$.)

Suppose contrary to the claim that P is infinite, and let

$$S(x) = \min(\uparrow x \setminus \{x\}).$$

Then there exists an element $x_1 \in \min(P)$ such that $\uparrow x_1$ is infinite. Inductively, we obtain a sequence $x_1, x_2, \ldots$ such that $x_i \in \min(S(x_{i-1}))$ and $\uparrow x_i$ is infinite. However, now $x_0 <_P x_1 <_P x_2 < \ldots$ contradicts the fact that P satisfies ACC. □

Example 1.39. Consider the cartesian product $\mathbb{Z} \times \mathbb{Z}$ of integers with the partial order $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. This poset is clearly locally finite. However, it does not satisfy the chain conditions ACC, DCC, nor FAC. Indeed, the sequence $\ldots < (-2, 0) < (-1, 0) < (0, 0) < (1, 0) < \ldots$ gives an counter-example to both ACC and DCC, and $\{(-n, n) \mid n = 0, 1, \ldots\}$ is an infinite set of incomparable elements in the poset. □

Higman's Theorem

We consider a special case of Higman's theorem for posets. The general theorem is stated for quasi-orders in algebras. A well-founded poset P satisfying the finite antichain condition is called **partially well-ordered**. Thus a poset P is partially well-ordered if and only if it satisfies the descending chain condition and the finite antichain condition.

Theorem 1.40. *The following are equivalent for a poset P.*

(1) *P is partially well-ordered.*
(2) *If $x_1, x_2, \ldots$ is an infinite sequence of elements in P, then $x_i \leq_P x_j$ for some $i < j$.*
(3) *Every infinite sequence of elements of P has an infinite ascending subsequence.*
(4) *P is well-founded and any subset has only finitely many minimal elements.*

Proof. Exercises □

We show a particular case of the above result.

Lemma 1.41. *Let P be a partially well-ordered set, and let $x_1, x_2, \ldots$ be an infinite sequence in P. Then there exists an infinite ascending subsequence $x_{i_1} \leq_P x_{i_2} \leq_P \ldots$ such that $i_1 < i_2 < \ldots$.*

Proof. Let $A = \{x_1, x_2, \ldots\} \subseteq P$, and let

$$M = \{m \mid x_m \nleq_P x_i \text{ for all } i > m\}.$$

Then M is finite, since P is partially well-ordered, say $m \leq k$ for all $m \in M$. For an index $i_1 > k$ such that $i_1 \notin M$ define recursively i_{j+1} such that $i_{j+1} > i_j$ and $x_{i_{j+1}} \geq_P x_{i_j}$. Then this sequence is an infinite ascending sequence as required by the claim. □

Next we extend P to P^*. The **subsequence order** on P^* is defined by

$$\begin{aligned} u \leq_P^* v \iff & v = y_1 y_2 \cdots y_n \text{ and } u = x_{i_1} x_{i_2} \cdots x_{i_m} \\ & \text{where } 1 \leq i_1 < i_2 < \ldots < i_m \leq n, \text{ and} \\ & x_{i_k} \leq_P y_{i_k} \text{ for all } k = 1, 2, \ldots, m. \end{aligned}$$

This generalizes the subsequence relation defined in Example 1.18.

Lemma 1.42. *Assume P is well-founded. Then also P^* is a well-founded poset under the relation subsequence order $\leq_P^*$.*

Proof. That $\leq_P^*$ is a partial order is an exercise. For well-foundedness, assume that there exists an infinite strictly decreasing chain $v_1 >_{P^*} v_2 >_{P^*} \ldots$ in P^*. Now since $0 < |v_{i+1}| \leq |v_i|$ for all i, we can assume that, for all $i \geq m$, they are all of the same length n, and so

$$v_i = x_{i1} x_{i2} \cdots x_{in}.$$

for each $i \geq m$, where all x_{ij} are in P. Since $v_{i+1} <_{P^*} v_i$, we have $x_{i+1,k} \leq_P x_{ik}$ for all $k = 1, 2, \ldots, n$, and there exists an index $k_i \leq n$ with $x_{i+1,k_i} <_P x_{ik_i}$. Let k_i be the first one with this property. The sequence v_i is infinite, and hence there exists an index k for which $k = k_i$ for infinitely many i. Thus the sequence x_{ik}, $i = 1, 2, \ldots$, contains a strictly decreasing subsequence. This contradicts the well-foundedness of P. □

For a poset P, a **bad sequence** is an infinite sequence $x_1, x_2, \ldots$ with $x_i \nleq_P x_j$ for all $i < j$. A bad sequence is **minimal** if there is no bad sequence

$$x_1, x_2, \ldots, x_{i-1}, y_i, y_{i+1}, \ldots \quad \text{where} \quad y_i <_P x_i$$

for some i. Notice that an infinite subsequence of a bad sequence is bad.

Lemma 1.43. *Let P be a well-founded poset that is not partially well-ordered. Then P has a minimal bad sequence.*

Proof. The proof is by induction on the positions in sequences. First of all there exists at least one bad sequence, since P is not partially well-ordered. Since each subset of P has a minimal element, there exists an element $x_1 \in P$ which is minimal among the first elements of bad sequences. Inductively, we can choose $x_i \in P$ such that it is a minimal element extending the sequence $x_1, x_2, \ldots, x_{i-1}$ to a bad sequence. The resulting sequence is a minimal bad sequence. □

Lemma 1.44. *Let $x_1, x_2, \ldots$ be a minimal bad sequence of a poset P. Then $\leq_P$ is a partial well-order on the subposet*

$$A = \{x \mid x <_P x_i \text{ for some } i\}.$$

Proof. Assume contrary to the claim that A is not partially well-ordered and let $y_1, y_2, \ldots$ be a bad sequence in A. We have that $y_i <_P x_{j_i}$ for some x_{j_i} from the bad sequence. Let m be such that the index j_m is the smallest. The sequence $y_m, y_{m+1}, \ldots$ is bad as a subsequence of a bad sequence.

We show that

$$x_1, \ldots, x_{j_m-1}, y_m, y_{m+1}, \ldots \tag{1.3}$$

is bad. This proves the lemma, since the sequence x_i was assumed to be minimal, but now $y_m <_P x_{j_m}$.

If (1.3) is not bad then $x_r <_P y_k$ for some $r < j_m$ and $k \geq m$. Also, $y_k <_P x_{j_k}$, and $j_k \geq j_m$ by the choice of j_m. Therefore $x_r <_P y_k <_P x_{j_k}$ and $r < j_m$. This contradicts the assumption that the sequence x_i is bad. □

The present proof of Higman's theorem is due to Nash–Williams.

Theorem 1.45 (Higman)**.** *Let P be partially well-ordered. Then the subsequence order is a partial well-order on P^*.*

Proof. The proof is by contradiction. Let $v_1, v_2, \ldots$ be a minimal bad sequence given by Lemma 1.43. Write

$$v_i = x_i u_i,$$

where $x_i \in P$. Now $|u_i| < |v_i|$ and so $u_i <_{P^*} v_i$. By Lemma 1.44, the set $\{u_i \mid i = 1, 2, \ldots\}$ is partially well-ordered. By Lemma 1.40, we can assume that $u_1 \leq_{P^*} u_2 \leq_{P^*} \ldots$. Since P is partially well-ordered, there are indices $i < j$ such that $x_i \leq_P x_j$. Hence $v_i \leq_{P^*} v_j$; a contradiction. □

Example 1.46 (Robertson–Seymour)**.** One of the most impressive results on discrete mathematics during the last decades is the Robertson–Seymour Theorem that solved the famous Wagner Conjecture. For two graphs, define the **minor relation** by $G \preceq H$ if G can be obtained from H by contracting (i.e., identifying the ends) or deleting zero or more edges. Then $\preceq$ is a well quasi-order on the family of graphs. The present proof of this result requires more than 500 pages. □

1.3. Extensions and dimension

Linear extensions

Let P and Q be two posets on a common domain X. Then Q is an **extension** of P if $x \leq_P y$ implies $x \leq_Q y$ for all $x, y \in X$. That is, if the relation $\leq_P$ is contained in $\leq_Q$. An extension of P that is a linear order is a **linear extension** of P.

Lemma 1.47. *Let P be a poset, and let $x \| y$ be incomparable elements in P. Then there exists a poset Q extending P such that $x \leq_Q y$.*

Proof. Let $Q = P \cup (\downarrow x \times \uparrow y)$, i.e., Q is obtained from P by adding all pairs (u, v) for which $u \leq_P x$ and $y \leq_P v$. Since $x \leq_Q y$, we need to show that Q is a poset.

First of all, $\downarrow x \cap \uparrow y = \emptyset$, since if z were a common element, $y \leq_P z$ and $z \leq_P x$ would yield $y \leq_P x$; a contradiction.

That Q is reflexive is clear, since $P \subseteq Q$.

For antisymmetry, suppose that $u \leq_Q v$ and $v \leq_Q u$ for some elements $u \neq v$. By the definition of Q, either $u \leq_P v$ or $u \leq_P x$ and $y \leq_P v$. Similarly either $v \leq_P u$ or $v \leq_P x$ and $y \leq_P u$. There are the following cases to be considered:

Case $u \leq_P v$. In this case $v \nleq_P u$, since otherwise $u = v$. Hence $v \leq_P x$ and $y \leq_P u$, However, now $y \leq_P u \leq_P v \leq_P x$ gives a contradiction: $y \leq_P x$.

Case $v \leq_P u$. This is symmetric to the previous case.

Case $u \leq_P x$ and $y \leq_P v$. Now also $v \leq_P x$ and $y \leq_P u$, which contradicts the fact that $\downarrow x$ and $\uparrow y$ are disjoint.

For transitivity, suppose $u \leq_Q v$ and $v \leq_Q w$. Now either $u \leq_P v$ or $u \leq_P x$ and $y \leq_P v$, and either $v \leq_P w$ or $v \leq_P x$ and $y \leq_P w$. Again there are cases to be considered.

Case $u \leq_P v$. If also $v \leq_P w$, then the claim follows from transitivity of P. Suppose thus that $v \leq_P x$ and $y \leq_P w$. Now also $u \leq_P x$, and hence $u \in \downarrow x$ and $w \in \uparrow y$, and so $u \leq_Q w$ as required.

Case $u \leq_P x$ and $y \leq_P v$. Now if $v \leq_P w$, then also $y \leq_P w$, and so $u \in \downarrow x$ and $w \in \uparrow y$, and hence $u \leq_Q w$. The case $v \leq_P x$ and $y \leq_P w$ is not possible, since $\downarrow x$ and $\uparrow y$ are disjoint.

This concludes the proof. □

Theorem 1.48 (Szpilrajn)**.** *Every finite partially ordered set has a linear extension.*

Proof. The claim follows by applying Lemma 1.47 inductively to incomparable elements. □

Theorem 1.49. *Every poset P on a finite set X is the intersection of a set of linear orders on X.*

Proof. Consider the intersection

$$\widehat{P} = \bigcap_{P \subseteq R} R$$

of all linear extensions R of P. By Theorem 1.48, the family of such extensions is nonempty, and thus $\widehat{P}$ is well defined. It is clear that $\widehat{P}$ is a partial order and $P \subseteq \widehat{P}$. Assume that x and y are incomparable in P. By Lemma 1.47, there is a linear order P_x for which $P \subseteq P_x$ and $x \leq_{P_x} y$, and a linear order P_y for which $P \subseteq P_y$ and $y \leq_{P_y} x$. Therefore the elements x and y are incomparable in $\widehat{P}$. It follows that $\widehat{P} = P$. □

The above proof is constructive for finite posets, and it does not generalize to infinite partial orders. In the infinite case we need Zorn's Lemma.

Theorem 1.50 (Dushnik–Miller)**.** *Every poset P on a set X is the intersection of a set of linear orders on X.*

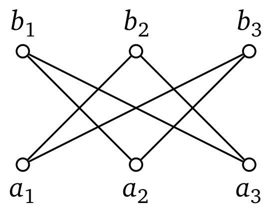
Proof. Consider the poset of all partial orders R that are extensions of P. If $R_1 \subseteq R_2 \subseteq \ldots$ is an ascending chain of such posets then the union $\bigcup_{i \geq 1} R_i$ is also a poset and an extension of P. Hence every chain of such posets has an upper bound, and by Zorn's Lemma, there exists a maximal poset R extending P. Now R is a linear order by Lemma 1.47. □

The **dimension** of a poset P, denoted by $\dim(P)$, is the least number of linear orders for which P is the intersection. We can have that $\dim(P) = \infty$. If $\dim(P) = 1$, then P, itself, is a linear order.

Example 1.51 (Crowns)**.** The crown $P = \mathrm{Cr}_{2n}$ on $2n$ elements

$$X = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}.$$

is the poset, where for all $i \neq j$, we have $a_i \leq_P b_j$ and the other pairs are incomparable. We show that $\dim(P) = n$. Indeed, define a linear order $\leq_i$ by setting



$$a_1 \leq_i \ldots \leq_i a_{i-1} \leq_i a_{i+1} \leq_i \ldots \leq_i a_n \leq_i$$
$$b_i \leq_i a_i \leq_i b_1 \leq_i \ldots \leq_i b_{i-1} \leq_i b_{i+1} \leq_i b_n .$$

Let $L_i = (X, \leq_i)$ be the poset for $\leq_i$. Each of these is a linear extension of the crown order P, and P is the intersection $\bigcap_{i=1}^n L_i$. Hence $\dim(P) \leq n$.

On the other hand, assume that $P = \bigcap_{i=1}^k L_i$ for some linear extensions L_i. For each i, there exists an L_j such that $b_i \leq_j a_i$ in L_j. (Otherwise, $a_i \leq_P b_i$.) Suppose there exists an L_j for which also $b_r \leq_j a_r$ for another index r, say $i < r$. Now, however, $b_i \leq_j a_i \leq_P b_r \leq_j a_r$ implies $b_i \leq_P a_r$; a contradiction. □

Example 1.52. An **alternating cycle** of a poset P is a subset $S = \{(x_i, y_i) \mid i = 1, 2, \ldots, k\}$ of pairs such that

$$x_1 \| y_1 \leq_P x_2 \| y_2 \leq_P \ldots \leq_P x_k \| y_k \leq_P x_1 ,$$

where $k \geq 1$. As an exercise we state: Let S be a subset of pairs of incomparable elements. *If S contains no alternating cycles, then the transitive closure $(P \cup S)^+$ is a poset*. In fact this is true also in converse. □

Theorem 1.53 (Hiraguchi). *Let C be a chain in a poset P. Then there are linear extensions D and U of P such that, for all $x \| y$ with $x \in C$ and $y \in P$,*

$$y \leq_D x \ \text{ and } \ x \leq_U y\,.$$

Proof. Let $P_D = (P \cup S_D)^+$ be the transitive closure of $P \cup S_D$, where $S_D = \{(y, x) \mid x \in C \text{ and } x \| y\}$. It is clear that S_D contains no alternating cycles, and therefore P_D is a poset by Example 1.52. Let D be a linear extension of P_D. Similarly, let $P_U = (P \cup S_U)^*$ for $S_U = \{(x, y) \mid x \in C \text{ and } x \| y\}$. Also P_U is a poset, and let U be a linear extension of P_U. □

Theorem 1.54. *Let P be a poset. If for every finite subposet R of P we have* $\dim(R) \leq n$*, then also* $\dim(P) \leq n$.

Proof. Omitted. □

Dilworth's Theorem

In this section we consider finite posets P.

- The **height** of P, denoted by $h(P)$, is the number of elements in a longest chain in P.
- The **width** of P, denoted by $w(P)$, is the largest number of elements in an antichain of P.

Clearly, if C is a chain in P and A is an antichain, then $|C \cap A| \leq 1$. Therefore in every set of antichains, whose union is the full poset P, there are at least $h(P)$ antichains.

Theorem 1.55. *Let P be a finite poset. Then P is partitioned into $h(P)$ antichains.*

Proof. We prove the claim by induction on $h(P)$. For $h(P) = 1$ the claim is obvious, since then P consists of incomparable elements. Assume then that the claim holds for posets of height less than that of P.

Let M be the set of all maximal elements of P. Then M forms an antichain in P, and every maximal chain contains an element from M (as an end point). Therefore the poset $P \setminus M$ has height $h(P) - 1$, and by the induction hypothesis, $P \setminus M$ is partitioned into $h(P) - 1$ antichains. Therefore together with M, the poset P is partitioned into $h(P)$ antichains. □

The following result is a dual statement of Theorem 1.55. The present proof is due to Tverberg.

Theorem 1.56 (Dilworth)**.** *Let P be a finite poset. Then there is a partition of P into exactly $w(P)$ chains.*

Proof. Again, if C is a chain in P and A is an antichain, then $|C \cap A| \leq 1$. Therefore in every set of chains partitioning P there are at least $w(P)$ chains. Thus we need to prove that there exists a set of $w(P)$ chains such that every element of P is in exactly one of these chains.

We proceed by induction on $|P|$. The case $|P| = 1$ is trivial. Suppose then that the claim holds for posets of size at most n, and let P be a poset of size $n+1$.

If $w(P) = 1$, then the claim obviously holds. Assume thus that $w(P) > 1$. Let C be a maximal chain in P. It is clear that

$$w(P) - 1 \leq w(P \setminus C) \leq w(P).$$

If the subposet $P \setminus C$ has width $w(P) - 1$, then, by the induction hypothesis, $P \setminus C$ can be partitioned into $w(P) - 1$ chains, and together with C we have a partition of P into $w(P)$ chains.

Assume now that $w(P \setminus C) = w(P)$, and let $A = \{a_1, a_2, \ldots, a_{w(P)}\}$ be an antichain in $P \setminus C$. Notice that $\downarrow A \cup \uparrow A = P$, since $|A| = w(P)$ and thus all elements of P are comparable with an element from A. Since C is a maximal chain, its maximum element is not in $\downarrow A$, for, otherwise the chain C could be extended by one element of A. By the induction hypothesis, $\downarrow A$ can be partitioned into $w(P)$ chains, $C_1^-, C_2^-, \ldots, C_{w(P)}^-$, where $a_i \in C_i^-$ for each i.

For each $x \in \downarrow A \setminus A$, there exists an index i such that $x \leq_P a_i$, and hence $a_j \not\leq_P x$ for all j. This means that a_j is the maximal element of the chain C_j^-.

Similarly, the minimal element of C is not in $\uparrow A$, which can then be partitioned into $w(P)$ chains $C_1^+, C_2^+, \ldots, C_{w(P)}^+$, where, moreover, a_j is the minimal element of C_j^+. Combining the chains C_j^- and C_j^+, we obtain a partition of P into $w(P)$ chains as required. □

Theorem 1.57 (Dilworth)**.** *For each poset P,* $\dim(P) \leq w(P)$.

Proof. By Theorem 1.56, P can be partitioned into $w(P)$ chains $C_1, C_2, \ldots, C_{w(P)}$. Let L_i $(= D)$ be a linear extension of P provided by Hiraguchi's Theorem 1.53 for the chain C_i. Then $P = \bigcap_i L_i$ (and so $\dim(P) \leq w(P)$). Indeed, if $x \| y$ in P, then $y \leq_{L_i} x$, where i is such that $x \in C_i$, and $x \leq_{L_j} y$, where $y \in C_j$. Hence also $x \| y$ in the intersection. □

Applications

Example 1.58 (Monotone sequences)**.** Consider the set $N = \{1, 2, \ldots, n^2 + 1\}$ of integers, and let $\sigma = (k_1, k_2, \ldots, k_{n^2+1})$ be a linear order on N. We show that this sequence contains a monotonic subsequence of length $n+1$.

Consider the set $P = \{(i, k_i) \mid i = 1, 2, \ldots, n^2 + 1\}$ together with the partial order

$$(i, k_i) \leq_P (j, k_j) \iff i \leq j \text{ and } k_i \leq k_j\,.$$

A subsequence σ' of σ is ascending if and only if σ' corresponds to a chain in the poset P. By Theorem 1.56, P can be partitioned into $w(P)$ chains, and thus σ can be partitioned into $w(P)$ ascending subsequences. Thus σ has an ascending subsequence of length at least $|N|/w(P)$.

Assume that $(i_1, k_{i_1}), (i_2, k_{i_2}), \ldots, (i_t, k_{i_t})$ is an antichain in P ordered so that $i_1 < i_2 < \ldots < i_t$. Then $k_{i_1} > k_{i_2} > \ldots > k_{i_t}$ is a descending subsequence. Now, if $|N|/w(P) \leq n$ (i.e., σ does not have an ascending chain of length $n+1$), then $w(P) \geq (n^2+1)/n \geq n + 1/n$, and so $w(P) \geq n+1$, as required. □

Example 1.59 (Sperner). Consider the poset 2^N of the subsets of $N = \{1, \ldots, n\}$ under inclusion. We show that if $A_1, A_2, \ldots, A_m \in 2^N$ are such that $A_i \nsubseteq A_j$ for all $i \neq j$, then

$$m \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Such a family of sets is an antichain of 2^N. The proof is due to Lubell.

Each maximal chain $\mathscr{C}$ of 2^N consists of sets $B_0, B_1, \ldots, B_n$, where $B_0 = \emptyset$ and $B_{i+1} = B_i \cup \{x_i\}$ for some $x_i \in N$. There are $n!$ maximal chains, because at stage $i+1$ we can choose x_i from a set of $n-i$ remaining elements. Also, there are $k!(n-k)!$ maximal chains that contain a fixed subset A with $|A| = k$. Indeed, first you can choose the elements of A in $k!$ different orders and then the rest of the elements in $(n-k)!$ different ways.

Let t be the number of pairs $(i, \mathscr{C})$, where $\mathscr{C}$ is a maximal chain containing A_i, and let r_k denote the number of the sets A_i of size k. Then

$$t = \sum_{k=0}^{n} r_k k!(n-k)!\,.$$

Counting with respect to the maximal chains, we obtain that $t \leq n!$, since each maximal chain contains at most one element from an antichain. Therefore

$$\sum_{k=0}^{n} \frac{r_k}{\binom{n}{k}} \leq 1\,,$$

where the binomial coefficient obtains its maximum at $k = \lfloor n/2 \rfloor$, and so

$$1 \geq \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \sum_{k=0}^{n} r_k = \frac{m}{\binom{n}{\lfloor n/2 \rfloor}}\,.$$

which proves the claim. □

Example 1.60 (Hall's Marriage Theorem). Let $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ be a family of subsets of a finite set X such that $X = \bigcup_{i=1}^{n} S_i$. (The sets S_i can intersect with each other.) A function $\sigma\colon \{1, 2, \ldots, n\} \to X$ is called a **disjoin representative function**

for $\mathscr{S}$ if it is injective and, for all $1 \leq i \leq n$, there exists an integer j such that $\sigma(j) \in S_i$. Here $\sigma(i)$ **represents** S_i and only S_i. For an index set $I \subseteq \{1, 2, \ldots, n\}$, denote

$$S(I) = \bigcup_{i \in I} S_i \,.$$

We show that the family $\mathscr{S}$ has a distinct representative function if and only if the following **Hall's condition** holds

$$|S(I)| \geq |I|$$

for all index sets $I \subseteq \{1, 2, \ldots, n\}$.

If $|I| < |S(I)|$ then clearly no distinct representative function can exist. To prove the sufficiency of Hall's condition, consider the relation $\leq$ on the set $P = \{1, 2, \ldots, n\} \cup X$ defined by

$$x \leq i \iff x \in S_i \,.$$

Then P is is a partial order (with height $h(P) = 2$).

Let A be an antichain of P with $|A| = w(P)$. Let $A = I \cup A'$, where $I \subseteq \{1, 2, \ldots, n\}$ and $A' \subseteq X$. Now $S(I) \subseteq X \setminus A'$ by the definition of P, and hence $|X| - |A'| \geq |S(I)| \geq |I|$ (by Hall's condition), i.e., $|A| \leq |X|$. Hence $w(P) = |X|$, since X is an antichain.

By Dilworth's theorem, P can be partitioned into $w(P)$ chains. This gives a matching of the elements in $\{1, 2, \ldots, n\}$ with those of X. This proves the sufficiency of Hall's condition. □

1.4. Möbius function

The incidence algebra of a poset

For a *locally finite* poset P denote by

$$I(P) = \{f : P \times P \to \mathbb{R} \mid f(x, y) = 0 \text{ if } x \not\leq_P y\}$$

the set of all real-valued functions for which $f(x, y) = 0$ if $x \notin \downarrow y$. The **sum** and the **scalar product** in $I(P)$ are defined in the usual way,

$$(f + g)(x, y) = f(x, y) + g(x, y)\,,$$
$$(cf)(x, y) = c \cdot f(x, y)$$

for $c \in \mathbb{R}$.

Lemma 1.61. *If $f, g \in I(P)$ and $c \in \mathbb{R}$, also $f + g \in I(P)$ and $cf \in I(P)$.*

Proof. Exercise. □

In the summations we usually leave out the index P, and write $[x,y]$ instead of $[x,y]_P$.

The **convolution** (or **matrix product**) of two functions $f,g \in I(P)$ is defined by

$$(f*g)(x,y) = \begin{cases} \sum_{z\in[x,y]} f(x,z)g(z,y) & \text{if } x \leq_P y\,, \\ 0 & \text{if } x \not\leq_P y\,. \end{cases}$$

This product is well defined, since P is locally finite and hence the sum is over a finite interval $z \in [x,y]_P$.

The **incidence algebra** of the locally finite poset P is the set $I(P)$ together with the operations $+$, $*$ and the scalar product.

Theorem 1.62. *Let P be a locally finite poset. The operation of convolution is associative on $I(P)$, that is, $(f*g)*h = f*(g*h)$.*

Proof. The claim follows from

$$\begin{aligned}
((f*g)*h)(x,y) &= \sum_{z\in[x,y]} (f*g)(x,z)h(z,y) \\
&= \sum_{z\in[x,y]} \Big(\sum_{t\in[x,z]} f(x,t)g(t,z)\Big)h(z,y) \\
&= \sum_{t\in[x,y]} f(x,t)\Big(\sum_{z\in[t,y]} g(t,z)h(z,y)\Big) \\
&= \sum_{t\in[x,y]} f(x,t)(g*h)(t,y) \\
&= (f*(g*h))(x,y)\,.
\end{aligned}$$

□

Let P be a locally finite poset, and let δ be the **identity** element of the incidence algebra $I(P)$:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y\,, \\ 0 & \text{if } x \neq y\,. \end{cases}$$

The function δ is also known as the **delta function** and the **Kronecker function** of the poset P. It satisfies the condition $f*\delta = f = \delta*f$ for all f. A function $f \in I(P)$ has an **inverse** $f^{-1} \in I(P)$ if

$$f*f^{-1} = \delta = f^{-1}*f\,.$$

The inversion formula

The **zeta function** of the poset P is the characteristic function of the poset P, i.e., it is defined by

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \le_P y\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

As shown in the next lemma, the zeta function has an inverse, which is called the **Möbius function** of P, denoted by μ $(= \zeta^{-1})$: $\mu(x,y) = 0$ if $x \not\le y$, and

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y\,, \\ -\sum_{x \le_P z <_P y} \mu(x,z) & \text{if } x <_P y\,. \end{cases} \tag{1.4}$$

In the above the summation is over the half open interval. Note that, by (1.4), we always have that if $x <_P y$, then

$$\sum_{z \in [x,y]} \mu(x,z) = 0\,. \tag{1.5}$$

Lemma 1.63. *Let P be a locally finite poset. The Möbius function μ of P is the inverse of the zeta function ζ.*

Proof. For the cases $x <_P y$, we have, by (1.5),

$$(\mu * \zeta)(x,y) = \sum_{z \in [x,y]} \mu(x,z)\zeta(z,y) = \sum_{z \in [x,y]} \mu(x,z) \cdot 1 = 0\,.$$

Also $\mu(x,x)\zeta(x,x) = 1$, and hence $\mu * \zeta = \delta$. Similarly, $\zeta * \mu = \delta$, and hence $\mu = \zeta^{-1}$. □

Theorem 1.64 (Möbius inversion formula)**.** *Let P be a locally finite poset having a bottom element* 0*, and let $f, g \colon P \to \mathbb{R}$ be functions. Then*

$$g(x) = \sum_{z \in [0,x]} f(z) \tag{1.6}$$

if and only if

$$f(x) = \sum_{z \in [0,x]} g(z)\mu(z,x)\,. \tag{1.7}$$

Proof. Let g be as in (1.6). Then

$$\begin{aligned} f(x) &= \sum_{t\in[0,x]} f(t)\delta(t,x) = \sum_{t\in[0,x]} f(t)(\zeta * \mu)(t,x) \\ &= \sum_{t\in[0,x]} \Big(f(t) \sum_{z\in[t,x]} \zeta(t,z)\mu(z,x) \Big) \\ &= \sum_{t\in[0,x]} \sum_{z\in[t,x]} f(t)\zeta(t,z)\mu(z,x) \\ &= \sum_{z\in[0,x]} \Big(\sum_{t\in[0,z]} f(t)\zeta(t,z) \Big)\mu(z,x) \\ &= \sum_{z\in[0,x]} \Big(\sum_{t\in[0,z]} f(t) \Big)\mu(z,x) = \sum_{z\in[0,x]} g(z)\mu(z,x), \end{aligned}$$

which proves the claim. □

When we apply Theorem 1.64 for the dual poset P^d, we have

Theorem 1.65 (Dual Möbius inversion formula)**.** *Let P be a locally finite poset with a top element 1, and let $f, g\colon P \to \mathbb{R}$ be functions. Then*

$$g(x) = \sum_{z\in[x,1]} f(z)$$

if and only if

$$f(x) = \sum_{z\in[x,1]} \mu(x,z)g(z).$$

Poset of subsets

For the chain $(\mathbb{N}, \leq)$ of integers, we have the following characterization of its Möbius function. It follows directly from (1.4).

Theorem 1.66 (Chains)**.** *For the chain $(\mathbb{N}, \leq)$, we have*

$$\mu(k,n) = \begin{cases} 1 & \text{if } k = n\,, \\ -1 & \text{if } k+1 = n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Proof. Assume that $k < n$. Then $\mu(k,n) = -\sum_{i=k}^{n-1} \mu(k,i)$. Hence, by (1.5), if $k < n-1$, then $\mu(k,n) = 0$. If $k = n-1$, then the claim follows from $\mu(n-1,n)+\mu(n,n) = 0$. □

In this case, the Möbius inversion formula states a rather unsurprising result: for $n > 0$,

$$g(n) = \sum_{i=0}^{n} f(i) \iff f(n) = g(n) - g(n-1).$$

Theorem 1.67. *The Möbius function $\mu_{P\times Q}$ of the direct product $P \times Q$ is the product of the Möbius functions μ_P and μ_Q of P and Q, that is,*

$$\mu_{P\times Q}((x_1,y_1),(x_2,y_2)) = \mu_P(x_1,x_2)\cdot\mu_Q(y_1,y_2)\,.$$

Proof. Exercise □

For the poset of subsets, we obtain

Theorem 1.68. *Consider the poset $P = (2^X, \subseteq)$ for a finite set X. The Möbius function for P is*

$$\mu(Z,Y) = \begin{cases} (-1)^{|Y|-|Z|} & \text{if } Z \subseteq Y\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

Proof. Recall from Example 1.26, that the poset 2^X of subsets of X with $|X| = n$ is isomorphic to the n-fold direct product $\Pi^n\mathbf{C}_2 = \mathbf{C}_2 \times \mathbf{C}_2 \times \cdots \times \mathbf{C}_2$ of the 2-element chains $\mathbf{C}_2 = \{0,1\}$ where $0 < 1$.

The poset $\mathbf{C}_2$ is a chain, and its Möbius function is

$$\mu_{\mathbf{C}_2}(x,y) = (-1)^{y-x} \;\text{ for }\; x < y\,.$$

(There is only one such pair: $(0,1)$.)

Let then A and B be subsets of X, and let $u = (a_1,a_2,\ldots,a_n) = \alpha(A)$ and $v = (b_1,b_2,\ldots,b_n) = \alpha(B)$ be their corresponding n-tuples in $\Pi^n\mathbf{C}_2$. Then, by Theorem 1.67, we have the claim: for $A \subseteq B$,

$$\mu(A,B) = \mu_{\Pi^n\mathbf{C}_2}(u,v) = \prod_{i=1}^{n}\mu_{\mathbf{C}_2}(a_i,b_i) = (-1)^{\sum b_i - \sum a_i} = (-1)^{|B|-|A|}\,.$$

□

From Theorem 1.64 we have

Theorem 1.69. *Let $g, f : 2^X \to \mathbb{R}$ be mappings from a finite set X such that*

$$f(Y) = \sum_{Z\subseteq Y} g(Z)\,. \tag{1.8}$$

Then for all $Y \subseteq X$,

$$g(Y) = \sum_{Z\subseteq Y}(-1)^{|Y|-|Z|} f(Z)\,. \tag{1.9}$$

Example 1.70 (The divisor poset)**.** Consider the divisor poset $(\mathbb{N}_+, |)$ of positive integers (with the bottom element 1). In this case, we browse through intervals $[k,n]$ w.r.t. divisibility. An element z is in this interval if and only if $z|k$ and $k|n$.

Assume first that $n = p^i$ for a prime number p. Then the poset D_{p^i} is a chain of the $i+1$ elements $1, p, \dots, p^i$. Hence the corresponding Möbius function μ_{p^i} is

$$\mu_{p^i}(p^k, p^j) = \begin{cases} 1 & \text{if } k = j\,, \\ -1 & \text{if } k+1 = j\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Now let

$$n = p_1^{i_1} p_2^{i_2} \cdots p_m^{i_m}$$

be the factorization of $n \geq 2$ into prime numbers. The poset D_n is isomorphic to

$$D_{p_1^{i_1}} \times D_{p_2^{i_2}} \times \cdots \times D_{p_m^{i_m}}$$

in a natural way. By Theorem 1.67, the Möbius function of the divisor poset is given by

$$\mu(k,n) = \begin{cases} 1 & \text{if } k = n \\ (-1)^t & \text{if } n = k \cdot p_1 p_2 \cdots p_t \text{ for distinct primes } p_i \\ 0 & \text{otherwise.} \end{cases}$$

The number theoretic Möbius function is obtained, since $\mu(n/k) = \mu(1, n/k) = \mu(k,n)$, if $k|n$. □

Structure of the Möbius function

Theorem 1.71. *Let P be a locally finite poset. Then $\zeta^k(x,y)$ is the number of chains $x = x_0 \leq_P x_1 \leq_P x_2 \leq_P \cdots \leq_P x_k = y$, where equalities $x_i = x_{i+1}$ are allowed.*

Proof. Exercise. □

We have

$$(\zeta - \delta)(x,y) = \begin{cases} 1 & \text{if } \ x <_P y\,, \\ 0 & \text{if } \ x = y\,, \end{cases}$$

Theorem 1.72. *Let P be a locally finite poset. Then $(\zeta - \delta)^k(x,y)$ is the number of chains $x = x_0 <_P x_1 <_P x_2 <_P \ldots <_P x_k = y$ in P.*

Proof. Exercise. □

By the following Hall's theorem, the value $\mu(x,y)$ is a local property of the poset: it depends only on the interval $[x,y]_P$.

Theorem 1.73 (Hall)**.** *Let P be a locally finite poset, and denote by $c_i(x,y)$ the number of chains $x = x_0 <_P x_1 <_P \ldots <_P x_i = y$ of length i. Then*

$$\mu(x,y) = \sum_{i=0}^{\infty} (-1)^i c_i(x,y)\,.$$

Proof. We use the fact that, in general, $(1+z)^{-1} = \sum_{i=0}^{\infty}(-1)^i z^i$. Now

$$\begin{aligned}\mu(x,y) &= \zeta^{-1}(x,y) = (\delta + (\zeta-\delta))^{-1}(x,y)\\ &= (\delta - (\zeta-\delta) + (\zeta-\delta)^2 - \cdots)(x,y)\\ &= \delta(x,y) - (\zeta-\delta)(x,y) + (\zeta-\delta)^2(x,y) - \cdots\\ &= c_0(x,y) - c_1(x,y) + c_2(x,y) - \cdots,\end{aligned}$$

as required. □

Theorem 1.73 can be restated as follows.

Corollary 1.74. *Let $x, y \in P$ for a locally finite poset P. Then*

$$\mu(x,y) = \sum_C (-1)^{|C|},$$

where the summation is over all chains C from x to y, and where $|C|$ denotes the length of the chain C.

2 Lattices

2.1. Definition

Recall that if the least upper bound of A exists in a poset, it is denoted by $\bigvee A$, and if the greatest lower bound of A exists, it is denoted by $\bigwedge A$. If $A = \{x_1, x_2, \ldots, x_n\}$ is a finite subset, then we can also write $\bigvee A = x_1 \vee x_2 \vee \cdots \vee x_n$, and $\bigwedge A = x_1 \wedge x_2 \wedge \cdots \wedge x_n$. We also adopt the notation $\bigvee_{x \in A} x = \bigvee A$ and $\bigwedge_{x \in A} x = \bigwedge A$, if these exist.

A poset L is a **lattice** if $x \vee y$ and $x \wedge y$ exist for all elements $x, y \in L$. In a lattice $\vee$ and $\wedge$ can be regarded as binary operations, called **join** and **meet**, $\vee\colon L \times L \to L$ and $\wedge\colon L \times L \to L$.

Example 2.1. All chains (linearly ordered posets) such as $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$, are lattices. In these $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. $\square$

Example 2.2. The divisibility poset $(\mathbb{N} \setminus \{0\}, \mid)$ is a lattice, where $x \vee y = \mathrm{lcm}(x, y)$, i.e., the least common multiple and $x \wedge y = \gcd(x, y)$, i.e., the greatest common divisor. $\square$

Example 2.3. The poset of subsets 2^X is a lattice, where $x \vee y = x \cup y$ and $x \wedge y = x \cap y$. $\square$

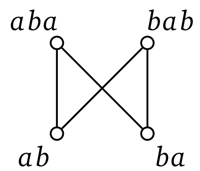
Example 2.4. The continuous functions $f : [0, 1] \to \mathbb{R}$ form a lattice, when the order relation is defined pointwise:

$$f \leq g \iff f(x) \leq g(x) \ \text{ for all } \ 0 \leq x \leq 1\,.$$

In this case $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$. $\square$

Example 2.5. The poset $\mathrm{Sub}(G)$ of subgroups of a group forms a lattice. Here $G_1 \vee G_2$ is the subgroup generated by $G_1 \cup G_2$, and $G_1 \wedge G_2 = G_1 \cap G_2$. $\square$

Example 2.6. The factor poset of the set of words Σ^* is not a lattice. Indeed, for instance, the incomparable words ab and ba are both lower bounds of $A = \{aba, bab\}$, and A does not have a greatest lower bound. $\square$



In the next lemma it is essential that the subset A is assumed to be finite. Indeed, in the lattice $(\mathbb{N}, \leq)$ the subset $A = \{1, 3, 5, \ldots\}$ does not have a least upper bound.

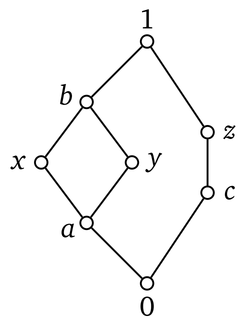
Lemma 2.7. *Let L be a lattice, and $A \subseteq L$ a finite nonempty subset. Then the least upper bound $\bigvee A$ and the greatest lower bound $\bigwedge A$ both exist.*

Proof. We proceed by induction on the size $n = |A|$. The claim holds for $n \leq 2$ by the definition. Assume it holds for all subsets of n elements, and let $A = \{x_1, x_2, \ldots, x_{n+1}\}$. By the induction hypothesis, the least upper bound $z = \bigvee(A \setminus$

$\{x_{n+1}\}$) exists, and so does $x = z \vee x_{n+1}$. It is now easy to show that x is the least upper bound of A.

The case for the greatest lower bound is dual to the above case. □

Example 2.8. The Hasse diagram on the right represents a poset on eight elements. Since it is finite, the bottom element $0 = O_L$ and the top element $1 = 1_L$ exist in it. On can check that it is a lattice by observing that every two elements x, y have a meet $x \wedge y$ and a join $x \vee y$. Also, for instance,



$$x \wedge (y \wedge z) = x \wedge 0 = 0\,, \quad (x \wedge y) \wedge z = a \wedge z = 0\,,$$

and so $x \wedge (y \wedge z) = (x \wedge y) \wedge z$. □

In the next lemma some important special cases are mentioned.

Lemma 2.9. *Let L be a lattice. If $\bigvee\emptyset$ exists in L, then $\bigvee\emptyset = \bigwedge L$ is the bottom element $\bigwedge L$ of L. If $\bigwedge\emptyset$ exists in L, then $\bigwedge\emptyset = \bigvee L$ is the top element $\bigvee L$ of L.*

Proof. Suppose $z = \bigvee\emptyset$ exists. Then, by definition, $z \le_L x$ for all $x \in L$, and thus $\uparrow z = L$. Therefore z is the bottom element of L, which is, again by definition, $\bigwedge L$. The case for $\bigwedge\emptyset$ is similar. □

Example 2.10 (Equivalence relations)**.** Let $\mathrm{Eq}(X)$ denote the set of all equivalence relations on the set X. It is clear that $\mathrm{Eq}(X)$ forms a poset under inclusion. The set $\mathrm{Eq}(X)$ is a lattice where

$$\begin{aligned} R_1 \wedge R_2 &= R_1 \cap R_2\,, \\ R_1 \vee R_2 &= (R_1 \cup R_2)^+ \end{aligned}$$

the transitive closure of the union. To see this, one needs to observe that if $R_1, R_2 \in \mathrm{Eq}(X)$ then also $R_1 \cap R_2$ and $(R_1 \cup R_2)^+$ are equivalence relations, and the claim easily follows from this.

In particular,

$$\begin{aligned} x(R_1 \vee R_2)y \iff &\text{ there is a sequence } x_1, x_2, \ldots, x_n \text{ such that} \\ &\ x_i(R_1 \cup R_2)x_{i+1} \text{ for all } i = 1, 2, \ldots, n-1 \\ &\text{ with } x = x_1,\ y = x_n \\ \iff &\text{ there is a sequence } x_1, x_2, \ldots, x_n \text{ such that} \\ &\ x_{2i-1} R_1 x_{2i} \text{ and } x_{2i} R_2 x_{2i+1} \text{ for all } i \\ &\text{ with } x = x_1,\ y = x_n\,. \end{aligned}$$

The latter equivalence follows because both R_1 and R_2 are equivalence relations, and so one can shorten an instance $x_i R_k x_{i+1}$ and $x_{i+1} R_k x_{i+2}$ to $x_i R_k x_{i+2}$ (for both $k = 1, 2$). □

Dual lattices

Lemma 2.11. *Let L be a lattice. Then its dual poset L^d is also a lattice.*

Proof. The definition of a lattice is symmetric with respect to the dual order, and this gives the claim. □

In the dual lattice L^d of a lattice L we denote the join and meet by $\vee^d$ and $\wedge^d$, respectively. Then

$$x \vee^d y = x \wedge y \quad \text{and} \quad x \wedge^d y = x \vee y$$

(where $\vee$ and $\wedge$ are the join and meet of L).

Duality is an important tool for lattices. Indeed, for every proposition there corresponds a dual proposition, which is obtained by changing the order and the operations $\wedge$ and $\vee$. For instance, the dual proposition of $x \wedge y \leq_L y$ is $x \vee y \geq_L y$.

Complete lattices

A lattice L is **complete** if both $\bigvee A$ and $\bigwedge A$ exist for all subsets $A \subseteq L$. Thus if L is a complete lattice, it does have a top and a bottom element:

$$1_L = \bigvee L \quad \text{and} \quad 0_L = \bigwedge L\,.$$

In literature, these are denoted also by $\top$ and $\bot$.

Theorem 2.12. *Let P be a poset where every subset has a least upper bound. Then P is a complete lattice.*

Proof. Notice first that, by assumption, $\bigvee \emptyset$ exists, and it is the bottom element 0_P of P as in Lemma 2.9. For a nonempty subset $A \subseteq P$, define

$$A_* = \{x \in P \mid x \leq_P y \quad \text{for all} \quad y \in A\}\,.$$

Since 0_P exists, it is in A_*, and so $A_* \neq \emptyset$. By assumption, $\bigvee A_*$ exists and it is a lower bound of A, since if $y \in A$, then, by definition, $\bigvee A_* \leq_P y$. Also, $\bigvee A_*$ is the greatest lower bound of A by its definition. Therefore the greatest lower bound $\bigwedge A$ exists for each subset A, and consequently P is a complete lattice. □

Therefore in the definition of a complete lattice we might have dropped off the requirement that $\bigwedge A$ should always exist.

The following result is a dual version of Theorem 2.12.

Theorem 2.13. *Let P be a poset where every subset has a greatest lower bound. Then P is a complete lattice.*

Lattices as algebras

The join and meet are both binary operations in a lattice, and thus we can consider the lattice $(L, \leq_L)$ also as an algebra $(L, \vee, \wedge)$ with respect to these operations.

Theorem 2.14. *The following identities are valid in every lattice L:*

(L1)	$x \wedge x = x$,	$x \vee x = x$,
(L2)	$x \wedge y = y \wedge x$,	$x \vee y = y \vee x$,
(L3)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$,	$x \vee (y \vee z) = (x \vee y) \vee z$,
(L4)	$x \wedge (x \vee y) = x$,	$x \vee (x \wedge y) = x$.

Proof. Most of the claims are obvious by the definitions of the operations. We prove only the first part of (L4).

First, we have $x \leq_L x$ and $x \leq_L x \vee y$, and hence $x \leq_L x \wedge (x \vee y)$. Also, if $z \leq_L x \wedge (x \vee y)$, then obviously $z \leq_L x$, and therefore x is the greatest lower bound of x and $x \vee y$. This shows (L4). □

In Theorem 2.14 the properties on the same line are dual to each other. By (L1), the elements of L are **idempotent**; by (L2), L is **commutative**; by (L3), L is **associative** (a **semigroup**) with respect to both operations. The conditions (L4) are the **absorption laws** that weave the two operations together.

The associative laws (L3) allows us to remove the brackets from meets (and joins) so that we can write $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ instead of $(\cdots(x_1 \wedge x_2) \wedge \cdots) \wedge x_n$.

Example 2.15. The set of laws (L1) – (L4) is not minimal. Indeed, (L1) follows from the absorption law: First, $x = x \wedge (x \vee (x \wedge x))$, when we put $y = x \wedge x$ in (L4). Here $x \vee (x \wedge x) = x$ by (L4), and thus $x = x \wedge x$. □

Example 2.16 (One law)**.** One can even reduce the laws to one law, which is a bit more complicated than (L1) – (L4). There are several such laws. The first one was given by R. McKenzie in 1970. The length of the law was 300 000 involving 34 variables. The following is due to McCune, Padmanabhan, and Veroff:

$$(((y \vee x) \wedge x) \vee (((z \wedge (x \vee x)) \vee (u \wedge x)) \wedge v)) \wedge (w \vee ((s \vee x) \wedge (x \vee t))) = x\,.$$

Therefore an algebra $(X, \vee, \wedge)$ satisfies this law if and only if it is a lattice. □

Lemma 2.17. *In a lattice L we have, for all $x, y \in L$,*

$$x \leq_L y \iff x \wedge y = x \iff x \vee y = y\,.$$

Proof. These are obvious by the definitions of meet and join. □

For two binary operations $\odot, \oplus \colon L \times L \to L$ state the following conditions from Theorem 2.14:

(A1)	$x \odot x = x$,	$x \oplus x = x$,
(A2)	$x \odot y = y \odot x$,	$x \oplus y = y \oplus x$,
(A3)	$x \odot (y \odot z) = (x \odot y) \odot z$,	$x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(A4)	$x \odot (x \oplus y) = x$,	$x \oplus (x \odot y) = x$.

Theorem 2.18. *Let L be a set with two binary operations $\oplus$ and $\odot$ that satisfy the conditions (A1) – (A4). Then L is a lattice with respect to the order relation defined by*

$$x \le y \iff x = x \odot y\,.$$

Proof. It is an exercise to show that $\le$ is a partial order. We show that $x \odot y = x \wedge y$ under the relation $\le$.

Now,

$$x \odot (x \odot y) \overset{(A3)}{=} (x \odot x) \odot y \overset{(A1)}{=} x \odot y\,,$$

and hence $x \odot y \le x$. Similarly, using (A2), we obtain that $x \odot y \le y$, and so $x \odot y$ is a lower bound of $\{x, y\}$. Assume then that for an element $z \in L$ with $z \le x$ and $z \le y$. Then $z = z \odot x$ and $z = z \odot y$, and we have

$$z = z \odot y = (z \odot x) \odot y = z \odot (x \odot y)$$

and so $z \le x \odot y$. This shows that $x \odot y$ is the greatest lower bound of x and y.

For the least upper bound, we need (A4). We show that $x \oplus y = x \vee y$. For this, we observe that, for any $x, z \in L$,

$$\begin{aligned} x \le z &\implies x = x \odot z \\ &\implies z \overset{(A4)}{=} z \oplus (x \odot z) = z \oplus x \\ &\implies z = z \oplus x\,. \end{aligned}$$

Then a dual argument to the above shows that $x \oplus y$ is the least upper bound of x and y. □

Notice that, by the preceding proof, the conditions (A1), (A2) and (A3) for the operation $\odot$ imply that the defined order $\le$ is a partial order for which the meet $x \wedge y$ always exists. Such posets are called (**meet**) **semilattices**. The dual conditions (A1), (A2) and (A3) for $\oplus$ imply that $\le$ is also a poset, a (**join**) **semilattice**. Thus a semilattice is a commutative semigroup $(A, \odot)$, where all elements are idempotent.

2.2. Isomorphism and sublattices

Isomorphism

Two lattices are isomorphic if they are order isomorphic as posets.

Lemma 2.19. *Let $\alpha\colon L \to K$ be a mapping between the lattices L and K. Then the following conditions are equivalent for α:*

(a) *α is an isotone mapping.*
(b) *$\alpha(x \vee y) \geq_K \alpha(x) \vee \alpha(y)$ for all $x, y \in L$.*
(c) *$\alpha(x \wedge y) \leq_K \alpha(x) \wedge \alpha(y)$ for all $x, y \in L$.*

Proof. We always have that $x \leq_L y$ implies $x \leq_L x \vee y$. Hence if α is isotone, then $\alpha(x) \leq_K \alpha(x \vee y)$ and similarly, $\alpha(y) \leq_K \alpha(x \vee y)$. These prove the implication (a) $\Rightarrow$ (b).

On the other hand suppose that (b) holds, and let $x \leq_L y$. Now, $y = x \vee y$, and by (b), $\alpha(y) = \alpha(x \vee y) \geq_K \alpha(x) \vee \alpha(y)$, which gives that $\alpha(x) \geq_K \alpha(y)$. Hence we have also (b) $\Rightarrow$ (a).

The equivalence of the cases (a) and (c) is dual to the above case. □

A mapping $\alpha\colon L \to K$ between two lattices is

- **join preserving** if $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$;
- **meet preserving** if $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$;
- a (lattice) **homomorphism** if it is both join and meet preserving;
- a (lattice) **isomorphism** if it is a bijective (lattice) homomorphism;
- an **automorphism** if $L = K$ and α is an isomorphism.

Theorem 2.20. *Let L and K be lattices. Then $L \cong K$ (as posets) if and only if there exists a lattice isomorphism $\alpha\colon L \to K$.*

Proof. Suppose first that L and K are order isomorphic, and let $\alpha\colon L \to K$ be such that, as in Theorem 1.20,

$$x \leq_L y \iff \alpha(x) \leq_K \alpha(y). \tag{2.1}$$

Let $x, y \in L$. Then $x \leq_L x \vee y$ and $y \leq_L x \vee y$ give $\alpha(x) \vee \alpha(y) \leq_K \alpha(x \vee y)$. On the other hand, since α maps L onto K, there exists a $z \in L$ such that $\alpha(z) = \alpha(x) \vee \alpha(y)$. We have $\alpha(x) \leq_K \alpha(z)$ and by (2.1), also $x \leq_L z$. Similarly, $y \leq_L z$, and so $x \vee y \leq_L z$. Finally, by (2.1), $\alpha(x \vee y) \leq_K \alpha(z) = \alpha(x) \vee \alpha(y)$, and thus $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$ as required. A dual argument shows that $\alpha(x \wedge y) = \alpha(x) \wedge \alpha(y)$, and hence α is a lattice homomorphism. Since it is bijective, it is an isomorphism.

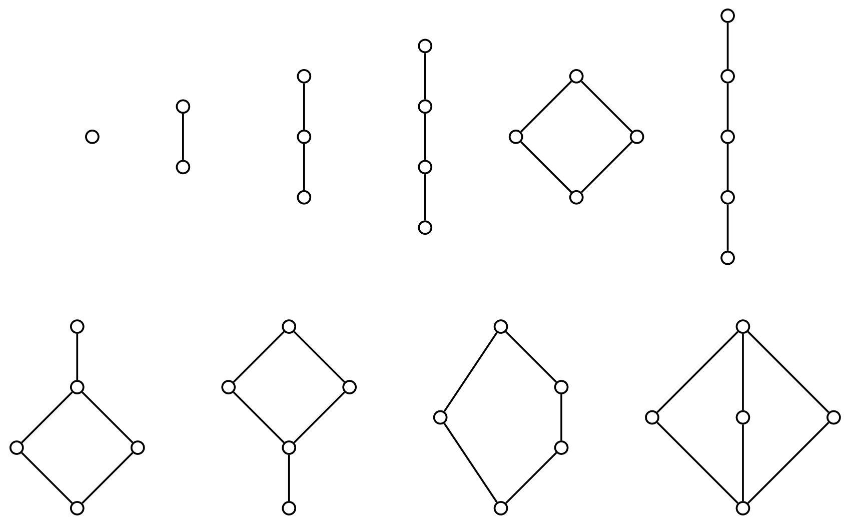
On the other hand, if α is a lattice isomorphism, then

$$\begin{aligned} x \leq_L y &\iff x \vee y = y \iff \alpha(x \vee y) = \alpha(y) \\ &\iff \alpha(x) \vee \alpha(y) = \alpha(y) \iff \alpha(x) \leq_K \alpha(y), \end{aligned}$$

and hence α is an order isomorphism by Theorem 1.20. □

If $\alpha\colon L \to K$ is an isomorphism, then also its inverse $\alpha^{-1}\colon K \to L$ is an isomorphism. The relation $\cong$ is an equivalence relation among lattices. As usual in algebra, isomorphic lattices are often identified with each other.

Example 2.21. In the following figures there are the Hasse diagrams of the finite lattices, up to isomorphism, of at most five elements. □



Sublattices

Let L be a lattice. Then a nonempty subset $K \subseteq L$ is a **sublattice** of L if K is closed under the join and meet operations of L:

$$x \vee y \in K \text{ and } x \wedge y \in K \text{ whenever } x, y \in K .$$

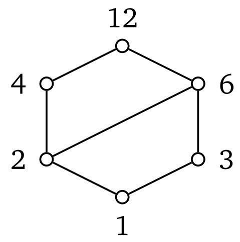
By the definition of a sublattice, for all $x, y \in K$, $x \leq_K y$ if and only if $x \leq_L y$. We denote by $\mathrm{Sub}(L)$ the set of all sublattices of the lattice L. We also say that a lattice K **embeds** into L, if K is isomorphic to a sublattice of L. In this case we often say just that K is a sublattice of L.

Example 2.22. Each chain $\mathbf{C}_n$ of height n is a sublattice of $\mathbf{C}_m$ for $m \geq n$. Note that $\mathbf{C}_n$ can be embedded in many different ways into $\mathbf{C}_m$ for large $n < m$. □

Example 2.23. If L is not a chain then it has a square as a sublattice: there are incomparable elements $x, y \in L$ such that $x \wedge y, x, y, x \vee y$ form a square. □

Example 2.24. Let L be a lattice and $x \leq_L y$, then the (closed) interval $[x, y]_L = \{z \mid x \leq_L z \leq_L y\}$ is a sublattice of L. The open intervals need not be sublattices as can be seen from the four element square lattice. □

Example 2.25. Consider the lattice $(T_{12}, |)$ of divisors of 12. Then T_{12} has, among others, the following sublattices: all singleton sets $\{d\}$ with $d|12$; all sets $\{1, d\}$ with $d|12$; the sets $\{1, 3, 4, 12\}$ and $\{2, 4, 6, 12\}$. □



Example 2.26. A sublattice of a lattice is always a lattice, but a lattice that is a subset of a lattice is not necessarily a sublattice! To see this, consider the lattice $L = (2^{\{1,2,3\}}, \subseteq)$ and the chain K consisting of the sets $\emptyset, \{1\}, \{2\}, \{1, 2, 3\}$ under inclusion. The lattice K is a subset of L, but

$$\{1\} \vee \{2\} = \{1, 2\} \text{ in } L \text{ and } \{1\} \vee \{2\} = \{1, 2, 3\} \text{ in } K.$$

In this example, the lattice operations $\vee$ and $\wedge$ are not the same in K and L. However, the lattice K can be embedded into L by the homomorphism α defined by $\alpha(\emptyset) = \emptyset$, $\alpha(\{1\}) = \{1\}$, $\alpha(\{2\}) = \{2\}$, $\alpha(\{1, 2, 3\}) = \{1, 2\}$. □

As seen the equivalence relations $\mathrm{Eq}(X)$ and the subgroups $\mathrm{Sub}(G)$ of a group G form a lattice. In fact, these lattices have the general form:

Theorem 2.27 (Whitman). *Every lattice L can be embedded into the lattice of equivalence relations* $\mathrm{Eq}(X)$ *on some set X.*

Proof. Omitted. □

Theorem 2.28 (Birkhoff). *Every lattice L can be embedded into the lattice of subgroups* $\mathrm{Sub}(G)$ *of some group.*

Proof (Idea). Consider L as a lattice of equivalence relations with the embedding $\psi \colon L \to \mathrm{Eq}(X)$ with $\psi(x) = \Psi_x$. The group G in the claim can be chosen to consist of the permutations π on L that move, $\pi(x) \neq x$, only finitely many elements of L. Then

$$\varphi(x) = \{\pi \in G \mid (y, \pi(y)) \in \Psi_x \text{ for all } y \in L\}$$

is a required embedding. □

Theorem 2.29. *For each lattice L, the set* $\mathrm{Sub}_0(L) = \mathrm{Sub}(L) \cup \{\emptyset\}$ *forms a complete lattice under inclusion.*

Proof. Let $\mathscr{S} = \{K_i \mid i \in I\}$ be a family of sublattices of L such that $K = \bigcap_{i \in I} K_i$. Then, if $K \neq \emptyset$, it is a sublattice of L. (This can be quickly shown relying only on the definition of a sublattice.) In this case, $\bigwedge \mathscr{S} = K$, and the claim follows from Theorem 2.13. □

In particular, if $A \subseteq L$ is any subset of the lattice L, then there exists the smallest sublattice of L containing A:

$$\mathrm{Sg}(A) = \bigcap \{K \mid K \in \mathrm{Sub}(L) \text{ and } A \subseteq K\}.$$

$\mathrm{Sg}(A)$ is called the sublattice of L **generated by** A.

Example 2.30. Consider the divisor lattice T_{12}. Here $\mathrm{Sg}(\{4,6\}) = \{2,4,6,12\}$. It is the smallest subset that is closed under both operations $\vee$ and $\wedge$. □

2.3. Congruences and ideals

Congruences

Let L be a lattice, and $\theta \subseteq L \times L$ an equivalence relation of L. Then θ is a **congruence** if for all $x, y, z \in L$,

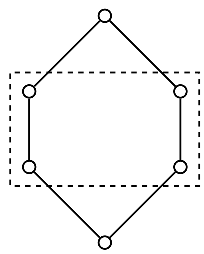
$$x\theta y \implies (x \vee z)\theta(y \vee z) \text{ and } (x \wedge z)\theta(y \wedge z).$$

The set of all congruences of L is denoted by $\mathrm{Con}(L)$. The equivalence class $x\theta = \{y \mid x\theta y\}$ is called the **congruence class** of the element x.

For a subset $A \subseteq L$ is **convex** if

$$x, y \in A \implies [x, y]_L \subseteq A.$$

Notice that a convex subset need not be a sublattice.



Theorem 2.31. *Let $\theta \in \mathrm{Con}(L)$ for a lattice L. Then the congruence class $x\theta$ is a convex sublattice for all $x \in L$. Moreover, if $x\theta y$ then $[x \wedge y, x \vee y]_L \subseteq x\theta$.*

Proof. First of all, $x\theta$ is a sublattice, since if $a, b \in x\theta$ then

$$(a \vee b)\theta(x \vee b) \text{ and } (x \vee b) = (b \vee x)\theta(x \vee x) = x$$

and so $(a \vee b)\theta x$, which gives that $a \vee b \in x\theta$. Similarly, we obtain that $a \wedge b \in x\theta$.

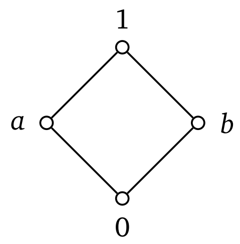
For convexity, suppose that $a, b \in x\theta$ and $a \leq_L z \leq_L b$. Then $a\theta b$ and thus also $(a \wedge z)\theta(b \wedge z)$. Here $a \wedge z = a$ and $b \wedge z = z$, and so $a\theta z$, which means that $z \in a\theta = x\theta$. Hence $[a, b]_L \subseteq x\theta$. This shows that $x\theta$ is convex.

For the second claim, since $x\theta$ is a sublattice, if $y \in x\theta$ then also $x \wedge y \in x\theta$ and $x \vee y \in x\theta$, and so $[x \wedge y, x \vee y]_L \subseteq x\theta$ as required. □

Example 2.32. Consider the 4-element square lattice L.

We can represent L as a table, where the element in the crossing of (x,y) is $x \wedge y / x \vee y$.

$\wedge/\vee$	0	a	b	1
0	$0/0$	$0/a$	$0/b$	$0/1$
a	$0/a$	a/a	$0/1$	$a/1$
b	$0/b$	$0/1$	b/b	$b/1$
1	$0/1$	$a/1$	$b/1$	$1/1$



Let θ be a congruence of L. Then the following cases hold:

$$0\theta a \iff b\theta 1,\ 0\theta b \iff a\theta 1, \tag{2.2}$$

$$a\theta b \implies 0\theta a \text{ and so } \theta = \omega_L, \tag{2.3}$$

$$0\theta 1 \implies \theta = \omega_L. \tag{2.4}$$

Using these properties we deduce that there are four congruences in $\mathrm{Con}(L)$: the identity congruence ι_L, the universal congruence ω_L, and the congruences θ_1 and θ_2 that have the following congruence classes:

$$\theta_1 \colon \{0,a\},\{b,1\} \text{ and } \theta_2 \colon \{0,b\},\{a,1\}.$$

□

Theorem 2.33. *Let $\{\theta_i \mid i \in I\}$ be a set of congruences of a lattice. Then the intersection $\bigcap_{i\in I} \theta_i$ is also a congruence of L. In particular, if R is an equivalence relation on L, then there there exists the smallest congruence in* $\mathrm{Con}(L)$ *containing R.*

Proof. Notice that $\iota_L \subseteq \theta_i$ for all i, and hence also the intersection contains ι_L, and so the intersection is nonempty. The claim reduces then to the individual congruences. □

For an equivalence relation R, let $\langle R\rangle$ denote the smallest congruence containing R, that is,

$$\langle R\rangle = \bigcap\{\theta \mid \theta \in \mathrm{Con}(L), R \subseteq \theta\}.$$

Theorem 2.34. *The set* $\mathrm{Con}(L)$ *of congruences forms a complete lattice under inclusion. The operations of this lattice are*

$$\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2 \quad \textit{and} \quad \theta_2 \vee \theta_2 = \langle \theta_1 \cup \theta_2\rangle.$$

Proof. The claim follows from Theorem 2.13. □

Theorem 2.35. *The congruence lattice* $\mathrm{Con}(L)$ *is a sublattice of the lattice* $\mathrm{Eq}(L)$ *of all equivalence relations on L.*

Proof. Exercise. □

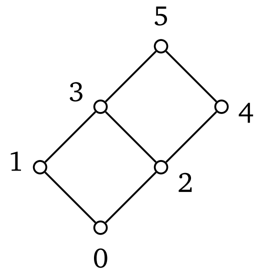
Theorem 2.36. *Let $\theta_1, \theta_2 \in \mathrm{Con}(L)$ for a lattice L. Then $\theta_1 \vee \theta_2 = \Psi$, where*

$$\begin{aligned} (x,y) \in \Psi \iff & \textit{there is a finite sequence} \\ & x \wedge y = x_0 \leq_L x_1 \leq_L \ldots \leq_L x_n = x \vee y \\ & \textit{where } (x_i, x_{i+1}) \in \theta_1 \cup \theta_2 \textit{ for all } i\,. \end{aligned}$$

Proof. Exercise. □

Example 2.37. Consider the given 6-element lattice L together with its table for operations.

$\wedge/\vee$	0	1	2	3	4	5
0	0/0	1/0	2/0	3/0	4/0	5/0
1	1/0	1/1	3/0	3/1	5/0	5/1
2	2/0	3/0	2/2	3/2	4/2	5/2
3	3/0	3/1	3/2	3/3	5/2	5/3
4	4/0	5/0	4/2	5/2	4/4	5/4
5	5/0	5/1	5/2	5/3	5/4	5/5



The congruences of L are the following in addition to the identity ι_L and the universal congruence ω_L:

$$\begin{aligned} &\{\{0,1\},\ \{2,3\},\ \{4,5\}\}, \quad \{\{0,2\},\ \{1,3\},\ \{4\}\ \{5\}\} \\ &\{\{0,1,2,3\},\ \{4,5\}\}, \quad \{\{0,2,4\},\ \{1,3,5\}\}, \\ &\{\{2,4\},\ \{3,5\},\ ,\{0\},\{1\}\}, \quad \{\{0,1\},\ \{2,3,4,5\}\}\,. \end{aligned}$$

If θ is a congruence, then we denote its congruence classes by $\overline{i \ldots j}$, where $i, \ldots, j$ are the elements in the class. Then the congruence lattice $\mathrm{Con}(L)$ is given in Figure 2.1.

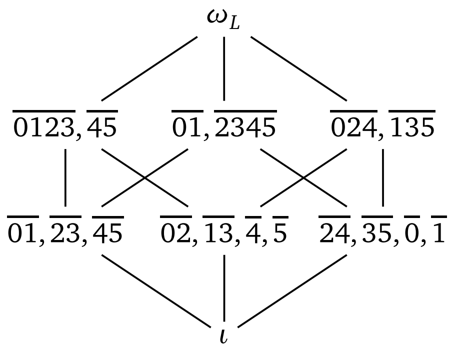


FIG. 2.1. The congruence lattice of the example.

□

The congruence classes $x\theta$, $x \in L$, partition the lattice L. The **quotient lattice** (modulo θ) is defined as the set

$$Q = L/\theta = \{x\theta \mid x \in L\}$$

of all congruence classes together with the operations

$$x\theta \wedge y\theta = (x \wedge y)\theta \quad \text{and} \quad x\theta \vee y\theta = (x \vee y)\theta\,.$$

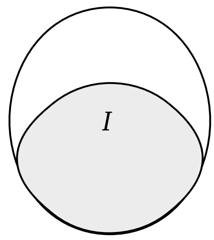
These operations are well defined, and hence L/θ is a lattice.

Ideals

A nonempty subset $I \subseteq L$ of a lattice L is an **ideal** if, for all $x, y \in I$,

$$x \vee y \in I \quad \text{and} \quad \downarrow x \subseteq I\,.$$

The set of all ideals of a lattice L is denoted by $\mathrm{Id}(L)$.



Notice that each ideal is a down-set and a sublattice. Indeed, $I = \downarrow I$ and for all $x, y \in I$, $x \vee y \in I$ and $x \wedge y \in I$.

The lattice L is its own ideal, $L \in \mathrm{Id}(L)$. If L has other ideals, they are called **proper**. Note that if the lattice L has the bottom element 0_L then $0_L \in I$ for every ideal of L.

We notice that always

$$\downarrow x = \{x \wedge y \mid y \in L\}\,,$$

and therefore

Lemma 2.38. *Let $I \neq \emptyset$ be a subset of a lattice L. Then I is an ideal of L if and only if I is closed under finite joins and, moreover,*

$$x \in I \text{ and } y \in L \implies x \wedge y \in I\,.$$

Lemma 2.39. *Let I_i be a set of ideals of a lattice L for all $i \in A$. Then also $\bigcap_{i \in A} I_i$ is an ideal of L if it is nonempty. In particular, every subset $X \subseteq L$ has the smallest ideal containing X:*

$$(X] = \bigcap \{I \mid I \in \mathrm{Id}(L) \text{ and } X \subseteq I\}\,.$$

Proof. Exercise. □

The ideal $(X]$ is called the **ideal generated by** X. The ideal generated by a singleton set $\{x\}$ is a **principal ideal**, and it is denoted by $(x]$ (i.e., without set brackets). Hence $(x] = \downarrow x$, since if $y \leq_L x$ and $z \leq_L x$ then also $y \vee z \leq_L x$.

Example 2.40. In the subset lattice 2^X, the ideal generated by a subset $Y \subseteq X$ is simply $(Y] = 2^Y$. □

Lemma 2.41. *Let X be a nonempty subset of a lattice L. Then for all $x \in L$, $x \in (X]$ if and only if there exists a* finite *subset $Y \subseteq X$ such that $x \leq_L \bigvee Y$.*

Proof. If $Y \subseteq X$ is a finite subset then $\bigvee Y$ exists and $\bigvee Y \in (X]$, since ideals are closed under finite joins. Hence if $x \leq_L \bigvee Y$ then also $x \in (X]$. Now, the set

$$I = \{x \in L \mid x \leq_L \bigvee F \text{ for some finite } F \subseteq X\}$$

is an ideal, since if $x, y \in I$, say $x \leq_L \bigvee F_x$ and $y \leq_L \bigvee F_y$ for some finite subsets $F_x, F_y \subseteq X$, then $x \vee y \leq_L \bigvee (F_x \cup F_y)$, and hence $x \vee y \in I$. It is clear that if $x \in I$ and $y \leq_L x$, then also $y \in I$. Now $X \subseteq I$, and therefore $I = (X]$. □

Theorem 2.42 (Ideal Lattice)**.** *The set* $\mathrm{Id}(L)$ *of ideals of a lattice L is a lattice under set inclusion (called the* ***ideal lattice*** *of L). The operations in* $\mathrm{Id}(L)$ *are the following:*

$$I_1 \wedge I_2 = I_1 \cap I_2 = \{x \wedge y \mid x \in I_1,\ y \in I_2\},$$
$$I_1 \vee I_2 = (I_1 \cup I_2] = \{z \in L \mid z \leq x \vee y \ \textit{ for some } \ x \in I_1,\ y \in I_2\}.$$

Proof. First of all, always $I_1 \cap I_2 \neq \emptyset$, since $x \wedge y \in I_1 \cap I_2$ whenever $x \in I_1$ and $y \in I_2$. Moreover, if $x \in I_1 \cap I_2$ then $x = x \wedge x$ where $x \in I_1$ and $x \in I_2$. Hence $\{x \wedge y \mid x \in I_1,\ y \in I_2\} = I_1 \cap I_2$. By Lemma 2.39, $I_1 \cap I_2 \in \mathrm{Id}(L)$, and, surely then $I_1 \cap I_2 = I_1 \wedge I_2$.

For the join, let

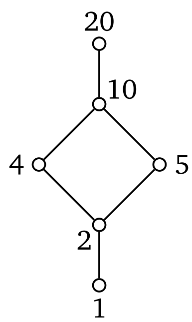
$$J = \{z \in L \mid z \leq x \vee y \ \text{ for some } \ x \in I_1,\ y \in I_2\}.$$

The J is obviously an ideal, and $I_1 \cup I_2 \subseteq J$. Therefore also $(I_1 \cup I_2] \subseteq J$. Now, if $x \vee y \in J$ for some $x \in I_1$ and $y \in I_2$ then $x, y \in I_1 \cup I_2$ and so $x \vee y \in (I_1 \cup I_2]$. This shows that $J = (I_1 \cup I_2]$ and hence $J = I_1 \vee I_2$. □

Theorem 2.43. *If L is a finite lattice then every ideal of L is principal.*

Proof. If $X \subseteq L$, then $(X] = (\bigvee X]$ by the above results. □

Example 2.44. If X is an infinite set then its finite subsets form an ideal of the subset lattice 2^X that is not a principal ideal. □



Example 2.45. Consider the lattice L on the right. There $(\{2,4\}] = \{1,2,4\} = (4]$, and $(4,5] = \{1,2,4,5,10\} = (10]$. Of course, $(20] = L$ the full lattice, since 20 is the top element in L. □

Recall that a lattice L **embeds** into a lattice K if there exists an injective homomorphism $\varphi \colon L \to K$.

Theorem 2.46 (Embedding result)**.** *Every lattice L can be embedded into its ideal lattice. Moreover, if L is a finite lattice then $L \cong \mathrm{Id}(L)$.*

Proof. Let, for all $x \in L$,

$$\varphi(x) = (x]\,.$$

The mapping $\varphi\colon L \to \mathrm{Id}(L)$ is clearly injective, and it is a homomorphism, since, by Theorem 2.42, $(x] \wedge (y] = (x \wedge y]$ and $(x] \vee (y] = (x \vee y]$.

The second claim follows from Theorem 2.43. □

Prime ideals

A proper ideal I, $I \neq L$, is said to be a **prime ideal** if

$$x \wedge y \in I \implies x \in I \ \text{ or } \ y \in I\,. \tag{2.5}$$

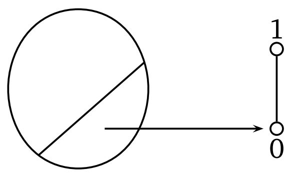
In the following (2.5) is changed to equivalence.

Lemma 2.47. *A proper subset I of a lattice L is a prime ideal if I is closed under finite joins and*

$$x \wedge y \in I \iff x \in I \ \text{ or } \ y \in I\,. \tag{2.5'}$$

Proof. Exercise. □

Let $\mathbf{C}_2$ be the chain on two elements 0 and 1.



Theorem 2.48 (Prime Embedding)**.** *Let $I \in \mathrm{Id}(L)$ be an ideal of a lattice L. Then I is a prime ideal if and only if there exists a surjective homomorphism $\varphi\colon L \to \mathbf{C}_2$ such that $\varphi^{-1}(0) = I$.*

Proof. Suppose first that I is a prime ideal, and define

$$\varphi(x) = \begin{cases} 0 & x \in I\,, \\ 1 & x \notin I\,, \end{cases}$$

for all $x \in L$. Then φ is surjective, since every prime ideal is proper and nonempty.

That φ is a homomorphism, follows from

$$\begin{aligned} \varphi(x \vee y) = 0 &\iff x \vee y \in I \iff x, y \in I \\ &\iff \varphi(x) = 0 = \varphi(y) \\ &\iff \varphi(x) \vee \varphi(y) = 0\,, \end{aligned}$$

and

$$\begin{aligned}\varphi(x \wedge y) = 0 &\iff x \wedge y \in I \\ &\iff x \in I \text{ or } y \in I \\ &\iff \varphi(x) = 0 \text{ or } \varphi(y) = 0 \\ &\iff \varphi(x) \wedge \varphi(y) = 0\,.\end{aligned}$$

Conversely, suppose that $\varphi\colon L \to \mathbf{C}_2$ is a surjective homomorphism. Then $\varphi^{-1}(0)$ is a nonempty proper subset of L. If $\varphi(x) = 0 = \varphi(y)$ then $\varphi(x \vee y) = \varphi(x) \vee \varphi(y) = 0$, and so $x \vee y \in \varphi^{-1}(0)$. Furthermore,

$$\begin{aligned}x \wedge y \in \varphi^{-1}(0) &\iff \varphi(x \wedge y) = 0 \\ &\iff \varphi(x) \wedge \varphi(y) = 0 \\ &\iff \varphi(x) = 0 \text{ or } \varphi(y) = 0\,.\end{aligned}$$

Hence $\varphi^{-1}(0)$ is a prime ideal. □

Filters

Dually, we say that a nonempty subset $F \subseteq L$ of a lattice L is a **filter** if, for all $x, y \in F$,

$$x \wedge y \in F \text{ and } \uparrow x \subseteq F\,.$$

We define **proper filters** and **prime filters** accordingly.

A filter is always an up-set of L, $F = \uparrow F$. The filter generated by a subset $X \subseteq L$ is denoted by $[X)$. It is the smallest filter containing X.

The results for ideals can be easily modified for filters by taking dual statements of the claims. Indeed, a filter F of L is an ideal of the dual lattice L^d.

Theorem 2.49. *Let A be a subset of a lattice L. Then A is a convex sublattice if and only if $A = I \cap F$ for some ideal I and filter F (with nonempty intersection).*

Proof. Assume that A is a convex sublattice, and let $I = (A]$ and $F = [A)$. Then $A \subseteq I \cap F$. On the other hand, if $x \in I \cap F$ then there are finite subsets $Y \subseteq A$ and $X \subseteq A$ such that $\bigvee Y \leq_L x \leq_L \bigvee X$. Since A is a sublattice, both joins are in A, and hence $A = I \cap F$.

Suppose then that $A = I \cap F$ for an ideal I and filter F. Note first that A is a sublattice. Indeed, if $x, y \in A$ then $x \wedge y \in I$ and $x \wedge y \in F$, and so $x \wedge y \in I \cap F = A$. Similarly, $x \vee y \in A$. Now if $x, y \in A$ and $x \leq_L z \leq_L y$ then $z \in I \cap F$, and we conclude that A is convex. □

2.4. Fixed points

Continuous functions

Let $\alpha\colon L \to L$ be a function on the lattice L. An element $x \in L$ is a **fixed point** of α if $\alpha(x) = x$. The set of all fixed points of α is denoted by $\mathrm{Fix}(\alpha)$.

Let $\alpha\colon L \to L$ be a function on L. Then α is **continuous** if α preserves existing least upper bounds: if $A \subseteq L$ is such that $\bigvee A$ exists, then also $\bigvee \alpha(A)$ exists and $\bigvee \alpha(A) = \alpha\big(\bigvee A\big)$.

Especially if L is a complete lattice, then $\bigvee A$ exists for all subsets A, and then a continuous function satisfies $\bigvee \alpha(A) = \alpha\big(\bigvee A\big)$ for all A.

Lemma 2.50. *Every continuous function on a lattice L is isotone.*

Proof. Let α be continuous on L, and let $x \le_L y$. Denote $A = \{x, y\}$. Then

$$\alpha(x) \le_L \bigvee \alpha(A) = \alpha\big(\bigvee A\big) = \alpha(y)\,.$$

Here, of course, the joins do exist, since A is finite. □

Knaster–Tarski

Theorem 2.51 (Knaster–Tarski). *Let L be a complete lattice, and $\alpha\colon L \to L$ be an isotone mapping. Then α has a fixed point. In fact, α has a largest fixed point $z_{\max}$ and a least fixed point $z_{\min}$ given by:*

$$z_{\max} = \bigvee \{x \in L \mid x \le_L \alpha(x)\}\,,$$
$$z_{\min} = \bigwedge \{x \in L \mid \alpha(x) \le_L x\}\,.$$

Proof. We show that $z_{\max}$ is the largest fixed point of α. The case for z_{min} can be proved dually. Let

$$A = \{x \in L \mid x \le_L \alpha(x)\}\,.$$

So that $z_{\max} = \bigvee A$. Observe at this point that $A \neq \emptyset$, since at least $0_L \in A$.

We show that $z_{\max}$ is a fixed point of α.

First of all, for every $x \in A$, we have $x \le_L z_{\max}$. As α is an isotone mapping, it follows that $\alpha(x) \le_L \alpha(z_{\max})$, an so, for every $x \in A$, $x \le_L \alpha(x) \le_L \alpha(z_{\max})$. Thus $\alpha(z_{\max})$ is an upper bound for A, and so $z_{\max} \le_L \alpha(z_{\max})$. From this it follows that $\alpha(z_{\max}) \le_L \alpha(\alpha(z_{\max}))$, and hence $\alpha(z_{\max}) \in A$. Therefore $\alpha(z_{\max}) \le_L z_{\max}$, as $z_{\max}$ is an upper bound for A. We conclude that $z_{\max} = \alpha(z_{\max})$.

To show that $z_{\max}$ is the largest fixed point of α, let $x \in \mathrm{Fix}(\alpha)$. Then $x \le_L \alpha(x)$, and thus $x \in A$ and therefore $x \le_L \bigvee A = z_{\max}$. □

Example 2.52. In the above proof we need the existence of the bottom element 0_L. Indeed, consider the chain $\{\dots, -1, 0, 1, 2, \dots, m\}$. Here $\bigvee A$ exists for all subsets A, but the isotone mapping defined by $\alpha(k) = k - 1$ does not have any fixed points. □

Example 2.53 (Schröder–Berstein)**.** Let X and Y be any sets such that there are injective mappings $\alpha\colon X \to Y$ and $\beta\colon Y \to X$. Then there exists a bijection $\gamma\colon X \to Y$.

We prove this using the fixed-point theorem. Let $\varphi\colon 2^X \to 2^X$ be the mapping defined by

$$\varphi(A) = X \setminus \beta\big(Y \setminus \alpha(A)\big).$$

Then φ is an isotone mapping: if $A \subseteq B$ then $\varphi(A) \subseteq \varphi(B)$. Since 2^X is a complete lattice with respect to union and intersection, φ has fixed point A:

$$A = X \setminus \beta\big(Y \setminus \alpha(A)\big).$$

Now, $X = A \cup B$ for $B = X \setminus A$, and $Y = \alpha(A) \cup C$, where $C = Y \setminus \alpha(A)$, and so $\beta(C) = B$, since A was a fixed point. Now, β maps C bijectively onto B. Define then $\gamma\colon X \to Y$ as follows:

$$\gamma(x) = \begin{cases} \alpha(x) & \text{if } x \in A, \\ \beta^{-1}(x) & \text{if } x \in B. \end{cases}$$

Then γ is a required bijection. □

Theorem 2.54 (Davis–Tarski)**.** *A lattice L is complete if and only if every isotone mapping $\alpha\colon L \to L$ has a fixed point.*

In the other direction this is Theorem 2.51. We omit the proof of the converse.

Abian–Brown

A mapping $\alpha\colon P \to P$ is said to be **increasing** if $x \le_P \alpha(x)$ for all $x \in P$. A poset P is **chain-complete** if $\bigvee C$ exists for every chain C in P. The following theorem is also known as the Bourbaki–Witt theorem.

Theorem 2.55 (Abian–Brown)**.** *If P is a chain-complete poset then every increasing function α has a fixed point.*

Proof. We say that a subset $A \subseteq P$ is (α–)**closed** if $\alpha(A) \subseteq A$ and $\bigvee C \in A$ for all chains $C \subseteq A$ inside A. Let $z \in P$ be any element. Clearly every intersection of closed sets is closed, and so

$$Z = \bigcap \{A \mid z \in A \text{ and } A \text{ is closed}\}$$

is also closed. The full poset P is closed by definition, and therefore $Z \neq \emptyset$. Also, if A is closed and $z \in A$, then the set $A \cap {\uparrow} z$ satisfies these conditions, since α is increasing. Hence $Z \subseteq {\uparrow} z$, i.e., $z \le_P y$ for all $y \in Z$.

An element $x \in Z$ is **normal**, if for all $y \in Z$

$$y <_P x \implies \alpha(y) \le_P x. \tag{2.6}$$

Claim 1. Let x be normal. Then for all $y \in Z$, either $y \le_P x$ or $y \ge_P \alpha(x)$.

Proof of Claim 1. We show that the set

$$A = \{y \in Z \mid y \leq_P x \ \text{ or } \ y \geq_P \alpha(x)\}$$

is closed and it contains the element z, and thus that $A = Z$, by the definition of Z. This will prove the present claim.

It is clear that $z \in A$, since $z \leq_P y$ for all $y \in Z$.

Let then $y \in A$. We show that also $\alpha(y) \in A$. If $y <_P x$ then $\alpha(y) \leq_P x$ by (2.6), and hence $\alpha(y) \in A$. If $y = x$ then $\alpha(y) \geq_P \alpha(x)$, and so $\alpha(y) \in A$. If $y >_P x$ then $y \geq_P \alpha(x)$, since $y \in A$, and $\alpha(y) \geq_P y \geq_P \alpha(x)$, since α is increasing. Hence in all cases, $\alpha(y) \in A$, and so A is closed.

Let then C be a chain in A. We show that $\bigvee C \in A$. If $y \leq_P x$ for all $y \in C$ then $\bigvee C \leq_P x$, and so $\bigvee C \in A$. Suppose thus that there exists an element $y \in C$ such that $x <_P y$. Then $y \geq_P \alpha(x)$, since $y \in A$. Hence $\bigvee C \geq_P y \geq_P \alpha(x)$ and therefore $\bigvee C \in A$ as required.

Claim 2. All elements in Z are normal.

Proof of Claim 2. Denote by $N = \{x \mid x \in Z \text{ is normal}\}$. We show that N is closed and it contains z.

First of all, $z \in N$, since $z \leq_P y$ for all $y \in Z$.

Let then $x \in N$. We show that also $\alpha(x) \in N$. Now $y \leq_P \alpha(x)$ implies $y \leq_P x$ by Claim 1. Therefore $\alpha(y) = \alpha(x)$ or $\alpha(y) \leq_P x \leq_P \alpha(x)$ since $x \in N$. This implies that $\alpha(x) \in N$.

Let C be a chain in N. Now $y <_P \bigvee C$ implies that $y <_P x$ for some $x \in C$, and so $\alpha(y) <_P x$ by normality of x. Hence $\alpha(y) \leq_P \bigvee C$. Since the set N is closed, $N = Z$.

We proceed the proof of the theorem. For all $x, y \in Z$, we have that $y \leq_P x$ or $y \geq_P \alpha(x) \geq_P x$. Therefore Z is a chain. As Z is closed, $\bigvee Z \in Z$, which implies that also $\alpha(\bigvee Z) \in Z$ and consequently $\alpha(\bigvee Z) \leq_P \bigvee Z$, that is, $\alpha(\bigvee Z) = \bigvee Z$ since α is increasing. $\square$

3 Special Lattices

3.1. Distributive and modular lattices

Distributive law

Lemma 3.1. *Let L be any lattice. Then*

$$x \wedge (y \vee z) \geq_L (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in L$.

Proof. Exercise. □

A lattice L is said to be **distributive** if there is an equality in the above formula, i.e., if L satisfies the **distributive law**: for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\,. \tag{D1}$$

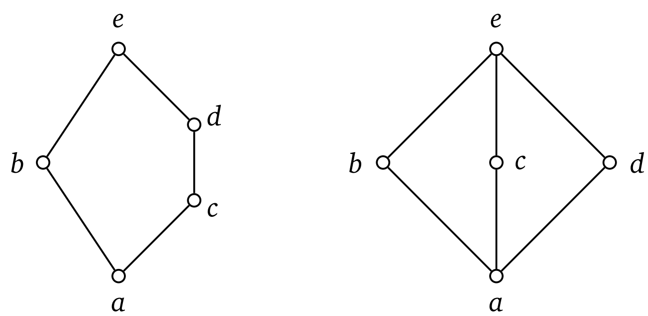
Example 3.2. Not all lattices are distributive. The following two basic lattices N_5 (**pentagon**) and M_5 (**diamond**) are not distributive. In N_5 we have

$$d \wedge (b \vee c) = d \neq c = (d \wedge b) \vee (d \wedge c)$$

and in M_5,

$$b \wedge (c \vee d) = b \neq a = (b \wedge c) \vee (b \wedge d)\,.$$

□



N_5 M_5

Example 3.3. The subset lattice 2^X of a set X is distributive. The law (D1) is well known to hold for the subsets with respect to the operations $\cap$ and $\cup$. Also, all chains are trivially distributive. □

Lemma 3.4. *A lattice L is distributive if and only if it satisfies*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\,. \tag{D2}$$

for all $x, y, z \in L$.

Proof. Note that (D2) is the dual of (D1). Assume (D1). We have

$$\begin{aligned}
(x \vee y) \wedge (x \vee z) &\overset{(D1)}{=} \big((x \vee y) \wedge x\big) \vee \big((x \vee y) \wedge z\big) \\
&\overset{(L4)}{=} x \vee \big((x \vee y) \wedge z\big) \overset{(L2)}{=} x \vee \big(z \wedge (x \vee y)\big) \\
&\overset{(D1)}{=} x \vee \big((z \wedge x) \vee (z \wedge y)\big) \overset{(L3)}{=} \big(x \vee (z \wedge x)\big) \vee (z \wedge y) \\
&\overset{(L4)}{=} x \vee (z \wedge y) \overset{(L2)}{=} x \vee (y \wedge z).
\end{aligned}$$

The converse is proved similarly. □

Theorem 3.5 (Funayama–Nakayama)**.** *The congruence lattice* Con(L) *of a lattice is distributive.*

Proof. Exercise. □

Modular lattices

Lemma 3.6. *Let L be any lattice. Then*

$$x \vee (y \wedge z) \leq_L y \wedge (x \vee z)$$

for all $x, y, z \in L$ with $x \leq_L y$.

Proof. Exercise. □

If in (D2) we assume that $x \leq_L y$ then $x \vee y = y$ and there is an equality in Lemma 3.6: $x \vee (y \wedge z) \overset{(D2)}{=} (x \vee y) \wedge (x \vee z) = y \wedge (x \vee z)$.

We say that the lattice L is **modular**, if it satisfies

(M) $\qquad$ if $x \leq_L y$ then $x \vee (y \wedge z) = y \wedge (x \vee z)$

for all $x, y, z \in L$.

Example 3.7. The lattice $L = N_5$ of Example 3.2 is not modular, since $c \leq_L d$ but

$$c \vee (d \wedge b) = c \neq d = d \wedge (c \vee b).$$

All distributive lattices are modular, and so is the nondistributive lattice M_5. □

Lemma 3.8. *A lattice L is modular if and only if for all $x, y, z \in L$:*

(M') $$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z)).$$

Proof. Exercise. □

Theorem 3.9. *A lattice L is modular if and only if it has no sublattices (isomorphic to) N_5.*

Proof. If N_5 is a sublattice of L then clearly the condition (M) is not satisfied in L.

Assume then that L does not satisfy (M), and let $x, y, z \in L$ be such that

$$x \leq_L y \quad \text{and} \quad x \vee (y \wedge z) <_L y \wedge (x \vee z).$$

(Recall that, by Lemma 3.6, for all lattices we do have $\leq_L$ in the above.)

Denote

$$a = x \vee (y \wedge z) \quad \text{and} \quad b = y \wedge (x \vee z).$$

Then $a <_L b$, and

$$\begin{aligned} z \vee a &= z \vee (x \vee (y \wedge z)) \overset{(L2,3)}{=} (z \vee (y \wedge z)) \vee x \overset{(L4)}{=} z \vee x\,, \\ z \wedge b &= z \wedge (y \wedge (x \vee z)) \overset{(L2,3)}{=} (z \wedge (y \vee z)) \wedge y \overset{(L4)}{=} z \wedge y\,. \end{aligned}$$

Moreover $z \wedge y \leq_L x \vee (z \wedge y) = a <_L b$, and so

$$z \wedge y \leq_L z \wedge a \leq_L z \wedge b = z \wedge y\,,$$

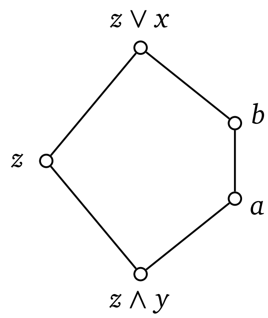
that is,

$$z \wedge y = z \wedge a = z \wedge b\,.$$

Similarly we obtain that

$$z \vee x = z \vee a = z \vee b\,,$$

and therefore we have an N_5 in L. □



Theorem 3.10. *A lattice L is distributive if and only if it has no sublattices N_5 and M_5.*

Proof. It is clear that if either N_5 or M_5 is a sublattice then L is not distributive.

Assume then that L is not distributive, and suppose that L does not contain N_5. Thus, by Theorem 3.9, L is modular. By Lemma 3.1 and the definition of a distributive lattice, there are elements $x, y, z \in L$ such that

$$(x \wedge y) \vee (x \wedge z) <_L x \wedge (y \vee z).$$

Define then the elements

$$\begin{aligned} u &= (x \wedge y) \vee (x \wedge z) \vee (y \wedge z), \\ v &= (x \vee y) \wedge (x \vee z) \wedge (y \vee z), \\ a &= (x \wedge v) \vee u\,, \\ b &= (y \wedge v) \vee u\,, \\ c &= (z \wedge v) \vee u\,. \end{aligned}$$

These elements generate an M_5 in L as can be seen from the following claims the proofs of which are exercises.

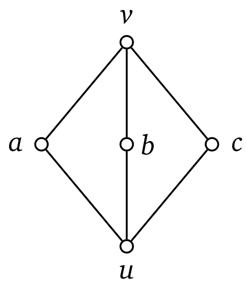
Claim. $u \leq_L a \leq_L v$.

Similar claims hold for b and c.

Claim. $u <_L v$.

Claim. $a \wedge b = u$ and $a \vee b = v$.

Similar claims hold for the other cases of a, b, c.



□

Example 3.11. It can be shown that a lattice L is distributive if and only if it satisfies the law:

$$\text{(D3)} \qquad (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x).$$

It is rather obvious that the special lattices M_5 and N_5 do not satisfy (D3), and therefore it remains to be shown that every distributive lattice does satisfy (D3). We omit this proof. □

Modularity of normal subgroups

Let G be a group. Then a subgroup H is a **normal subgroup**, denoted by $H \lhd G$, if for all $g \in G$ and $h \in H$, also $g^{-1}hg \in H$. Denote by $\mathcal{N}(G)$ the set of the normal subgroups of G.

Theorem 3.12. *$\mathcal{N}(G)$ is a modular lattice under inclusion.*

Proof. Let $H_1, H_2 \in \mathcal{N}(G)$. Then $H_1 \cap H_2 \lhd G$ and hence $H_1 \wedge H_2 = H_1 \cap H_2$. Also, $H_1H_2 \lhd G$ and $H_1 \cup H_2 \subseteq H_1H_2$, since the identity element is in both H_1 and H_2. This gives

$$H_1 \vee H_2 = H_1H_2 .$$

(Notice that if G_1 and G_2 are just subgroups of G then G_1G_2 might not be a subgroup. The case is favourable when these are normal subgroups: if $h_1, g_1 \in H_1$ and $h_2, g_2 \in H_2$ then $h_1h_2 \cdot g_1g_2 = h_1g_1(g_1^{-1}h_2g_1)g_2 \in H_1H_2$.)

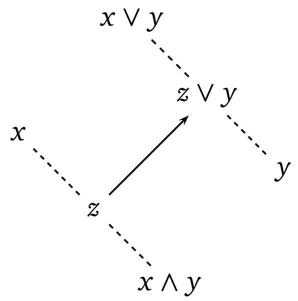
Recall now Dedekind's law which states that if H_1, H_2, H_3 are subgroups of G and $H_1 \subseteq H_2$ then $H_1(H_2 \cap H_3) = H_2 \cap H_1H_3$. (Here, in general, $H_1(H_2 \cap H_3)$ and H_1H_3 need not be subgroups.)

So for normal subgroups H_1, H_2, H_3 with $H_1 \subseteq H_2$, we have

$$H_1 \vee (H_2 \wedge H_3) = H_2 \wedge (H_1 \vee H_3),$$

and so $\mathcal{N}(G)$ is modular. □

Jordan–Dedekind condition



Theorem 3.13. *Let L be a modular lattice and $x, y \in L$. Then*

$$[x \wedge y, x]_L \cong [y, x \vee y]_L\,,$$

where the isomorphism is given by $\varphi(z) = z \vee y$.

Proof. Consider the mapping in the reverse direction:

$$\alpha\colon [y, x \vee y]_L \to [x \wedge y, x]_L \ \text{ with } \alpha(z) = z \wedge x\,.$$

Then we have $\alpha\varphi(z) = z$ for all $z \in [x \wedge y, x]_L$, since

$$\alpha\varphi(z) = \alpha(z \vee y) = (z \vee y) \wedge x \overset{(M)}{=} (x \wedge y) \vee z = z\,.$$

Similarly, $\varphi\alpha(z) = z$ for all $z \in [y, x \vee y]_L$, and hence $\alpha = \varphi^{-1}$, which shows that φ is a bijection. Clearly, both φ and φ^{-1} are isotone, and so the claim follows. □

We have then the following corollary:

Theorem 3.14. *If L is a modular lattice and $x, y \in L$ then, for the cover relation,*

$$x \wedge y \prec x \iff y \prec x \vee y\,.$$

Proof. If $x \wedge y \prec x$ then $[x \wedge y, x]_L$ is a chain of two elements, and thus, by Theorem 3.13, so must be $[y, x \vee y]_L$, that is, $y \prec x \vee y$. The converse goes similarly. □

For a lattice L, a sequence

$$x = x_0 \prec x_1 \prec \ldots \prec x_n = y$$

is a **cover chain** from x to y of length n.

Theorem 3.15 (Jordan–Dedekind)**.** *Let L be a modular lattice. Then any two cover chains from x to y have the same length.*

Proof. We prove the claim by induction on the length of cover chains. If $x \prec y$ then this is the only cover chain from x to y. Assume then that the claim holds for chains of length at most $n-1$, and let

$$x = x_0 \prec x_1 \prec \ldots \prec x_n = y\,,$$
$$x = y_0 \prec y_1 \prec \ldots \prec y_m = y$$

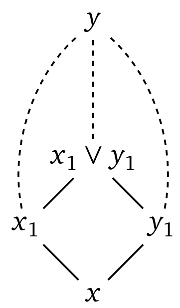
be two cover chains from x to y. If $x_1 = y_1$, then the claim follows by the induction hypothesis. Suppose then that $x_1 \neq y_1$. Now $x = x_1 \wedge y_1$ and so $x_1 \wedge y_1 \prec x_1$, which gives by Theorem 3.14 that $y_1 \prec x_1 \vee y_1$. Symmetrically, we have that $x_1 \prec x_1 \vee y_1$.

Again if $x_2 = y_2$, then the claim follows by the induction hypothesis, and thus assume that $x_2 \neq y_2$. By symmetry, we can suppose that $x_2 \neq x_1 \vee y_1$. (Otherwise, $y_2 \neq x_1 \vee y_1$.) Then $x_2 \notin [x_1 \vee y_1, y]_L$, since $x_1 \prec x_2$ and $x_1 \prec x_1 \vee y_1$ both hold.

Consider the sequence

$$x_1 \vee y_1 \leq_L x_2 \vee y_1 \leq_L \ldots \leq_L x_{n-1} \vee y_1 \,. \tag{3.1}$$

Here $(x_1 =)\ x_2 \wedge (x_1 \vee y_1) \prec x_2$, and hence, by Theorem 3.14, $x_1 \vee y_1 \prec x_2 \vee (x_1 \vee y_1) = x_2 \vee y_1$. Inductively, we obtain that $x_i \vee y_1 \prec x_{i+1} \vee y_1$ for all $i = 1, 2, \ldots, n-2$, and hence (3.1) is a cover chain of length $n-2$ from $x_1 \vee y_1$ to y. This gives a cover chain of length $n-1$ from y_1 to y, and by the induction hypothesis, we have $m = n$, as was required. □



Ideals of distributive lattices

Lemma 3.16. *Let L be a distributive lattice. Then*

$$x \vee y = x \vee z \ \textit{ and } \ x \wedge y = x \wedge z \implies y = z \,.$$

Proof. Indeed, suppose that $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$. Then

$$\begin{aligned} y &= y \wedge (x \vee y) = y \wedge (x \vee z) \\ &\overset{(D1)}{=} (y \wedge x) \vee (y \wedge z) \\ &= (x \wedge z) \vee (y \wedge z) = z \wedge (x \vee y) \\ &= z \wedge (x \vee z) = z \,. \end{aligned}$$

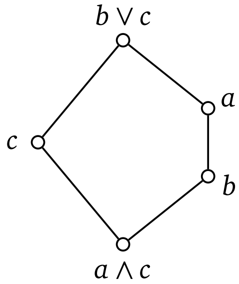
□

Theorem 3.17. *A lattice L is distributive if and only if for all ideals $I, J \in \mathrm{Id}(L)$,*

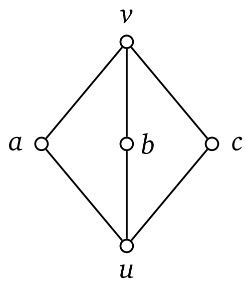
$$I \vee J = \{x \vee y \mid x \in I,\ y \in J\} \,.$$

Proof. Assume first that L is distributive, and let $x \in I \vee J$. By Theorem 2.42, there are elements $a \in I$ and $b \in J$ such that $x \leq_L a \vee b$. Therefore $x = x \wedge (a \vee b) \overset{(D1)}{=} (x \wedge a) \vee (x \wedge b)$, where $x \wedge a \in I$ and $x \wedge b \in J$ by the definition of an ideal. Hence x has the required form.

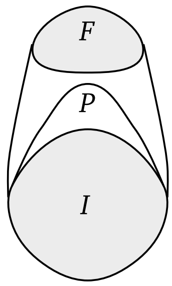
Suppose now that L is not distributive. Then L contains the sublattice N_5 or M_5.



Case N_5. Let $I = (b]$ and $J = (c]$. Then $a \leq_L b \vee c$, and hence $a \in I \vee J$. Suppose that there are elements $x \in I$ and $y \in J$ such that $a = x \vee y$. In this case, $y \leq_L a$. Since $y \in J = (c]$, also $y \leq_L c$, and so $y \leq_L a \wedge c <_L b$. Therefore $y \in I = (b]$. Now $x, y \in I$ imply that $a = x \vee y \in I$, that is, $a \leq_L b$; a contradiction.



Case M_5. Let $I = (b]$ and $J = (c]$. Then $a \leq_L b \vee c$ and hence $a \in I \vee J$. As in the previous case, we obtain a contradiction from the assumption that $a = x \vee y$ for some $x \in I$ and $y \in J$. □



Theorem 3.18. *Let L be a distributive lattice, I an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let

$$\mathscr{S} = \{J \in \mathrm{Id}(L) \mid I \subseteq J,\ J \cap F = \emptyset\}.$$

Then $\mathscr{S}$ is nonempty, because $I \in \mathscr{S}$. Let $\mathscr{C}$ be any ascending chain in $\mathscr{S}$, and let $M = \bigcup \mathscr{C}$. Then M is an ideal of L, $I \subseteq M$ and $M \cap F = \emptyset$, that is, $M \in \mathscr{S}$. Therefore every ascending chain in $\mathscr{S}$ has an upper bound in $\mathscr{S}$. By Zorn's lemma, there exists a maximal element P in $\mathscr{S}$.

Of course, P is an ideal. Suppose that P is not a prime ideal. Then there are elements $x, y \notin P$ such that $x \wedge y \in P$, and hence $P \subset P \vee (x]$ properly. By maximality of P, we have that $(P \vee (x]) \cap F \neq \emptyset$. Let $z \in F$ be in $P \vee (x]$. Then by Theorem 3.17, there exist $a \in P$ and $x' \leq_P x$ such that $z = a \vee x'$. Consequently, also $a \vee x \in F$. Similarly $(P \vee (y]) \cap F \neq \emptyset$, and there exists a $b \in P$ such that $b \vee y \in F$. Since F is a filter, $(a \vee x) \wedge (b \vee y) \in F$. However,

$$(a \vee x) \wedge (b \vee y) \overset{(D1)}{=} (a \wedge b) \vee (a \wedge y) \vee (x \wedge b) \vee (x \wedge y) \in P,$$

gives that $P \cap F \neq \emptyset$; a contradiction. □

Corollary 3.19. *Let I be an ideal in a distributive lattice L, and let $x \notin I$. Then there is a prime ideal P of L such that $I \subseteq P$ and $x \notin P$.*

Proof. Consider the principal filter $F = [x)$. Then $I \cap [x) = \emptyset$, and the claim follows from Theorem 3.18. □

Corollary 3.20. *Let L be a distributive lattice. Then every ideal I of L is the intersection of some prime ideals.*

Proof. We have

$$I = \bigcap \{P \mid P \text{ prime ideal, } I \subseteq P\}.$$

□

Corollary 3.21. *Let L be a distributive lattice, and let $x, y \in L$ be different elements. Then there exists a prime ideal P that contains exactly one of them.*

Proof. Apply the previous corollary to the principal ideals $(x]$ and $(y]$. □

Birkhoff–Stone Theorem

Let $\mathscr{R} \subseteq 2^X$ be a set of subsets of X. Then $\mathscr{R}$ is a **set ring**, if $A \cup B \in \mathscr{R}$ and $A \cap B \in \mathscr{R}$ for all $A, B \in \mathscr{R}$. Clearly, every set ring is a distributive lattice.

Theorem 3.22 (Birkhoff–Stone)**.** *Each distributive lattice L is isomorphic to a set ring.*

Proof. For each element $x \in L$, define

$$\varphi(x) = \{P \mid P \text{ a prime ideal, } x \notin P\}.$$

Let $\mathscr{R} = \{\varphi(x) \mid x \in L\}$.

Claim. The mapping $\varphi\colon L \to \mathscr{R}$ is a homomorphism and $\mathscr{R}$ is a set ring.

First, $\varphi(x \wedge y) = \varphi(x) \cap \varphi(y)$:

$$P \in \varphi(x \wedge y) \iff x \wedge y \notin P \iff x \notin P \text{ and } y \notin P \iff P \in \varphi(x) \cap \varphi(y).$$

Secondly, $\varphi(x \vee y) = \varphi(x) \cup \varphi(y)$:

$$P \in \varphi(x \vee y) \iff x \vee y \notin P \iff x \notin P \text{ or } y \notin P \iff P \in \varphi(x) \cup \varphi(y).$$

Claim. The mapping $\varphi\colon L \to \mathscr{R}$ is an isomorphism.

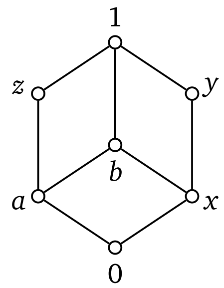
By its definition, φ is surjective, and by the first claim it is a homomorphism. For injectivity, let $x \neq y$. By Corollary 3.21, there exists a prime ideal P that separates x and y, and thus $\varphi(x) \neq \varphi(y)$. □

Geometric lattices

Recall that for a modular lattice L, the intervals $[x \wedge y, x]_L$ and $[y, x \vee y]_L$ are isomorphic, and so are the intervals $[x \wedge y, y]_L$ and $[x, x \vee y]_L$. We say that a lattice L is **semimodular**, if

$$\text{(S)} \qquad x \wedge y \prec x \implies y \prec x \vee y\,,$$

for all elements $x, y \in L$.



Example 3.23. The lattice on the right is the smallest semimodular lattice that is not modular. Notice that the elements $0, x, y, z$ and 1 form an N_5 which is forbidden in a modular lattice. By Theorem 3.14, all modular lattices are also semimodular. □

Let L be a lattice that has the bottom element 0_L. Then an element $a \in L$ is an **atom**, if $0_L \prec a$. Let Atom(L) be the set of the atoms of a lattice L. We say that L is

- an **atomic lattice** if for all $x \in L$ with $x \neq 0_L$, there exists an atom a such that $a \leq_L x$.
- a **point lattice** if for all $x \in L$ with $x \neq 0_L$,

$$x = \bigvee \{a \in L \mid a \leq_L x,\ a \text{ an atom}\}\,.$$

- a **geometric lattice** if it is semimodular point lattice that satisfies the finite chain condition (FCC).

Example 3.24. Every finite lattice is atomic. The atoms of the lattice L in Example 3.23 are a and x. However, L is not a point lattice, since, for instance, the element z is not a join of atoms. □

Example 3.25. The subset lattice 2^X of a set X is always a point lattice. Its atoms are the singleton sets $\{x\}$, and every set $A \subseteq X$ is the join $\bigvee_{x \in A}\{x\}$. In 2^X we have $A \prec B$ if and only if $|B \setminus A| = 1$, and $A \cap B = A \wedge B \prec A$ implies that $B \prec A \vee B = A \cup B$. Thus 2^X is semimodular. If X is infinite, 2^X does not satisfy the FCC. Every finite lattice does satisfy FCC, and so for finite sets X, 2^X is geometric. □

Lemma 3.26. *Every geometric lattice is complete.*

Proof. If $A = \{x_1, x_2, \ldots\} \subseteq L$ is infinite, then

$$x_1,\ x_1 \vee x_2, \ldots, x_1 \vee x_2 \vee \cdots \vee x_n, \ldots$$

forms an ascending chain, which by FCC can contain only finitely many terms. □

Lemma 3.27. *Let L be a geometric lattice. If $a \in \mathrm{Atom}(L)$ and $x \in L$ are such that $a \nleq_L x$ then $x \prec x \vee a$.*

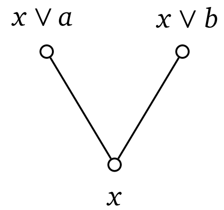
Proof. Since $a \nleq_L x$, we have $x \wedge a <_L a$ and, since a is an atom, it follows that $x \wedge a = 0_L \prec a$. From the condition (S) we obtain that $x \prec x \vee a$ which proves the present claim. □

Theorem 3.28 (Exchange law)**.** *Let L be a geometric lattice, and $a, b \in \mathrm{Atom}(L)$ and $x \in L$. Then*

$$a \nleq_L x \ \text{ and } \ a \leq_L x \vee b \implies b \leq_L x \vee a\,.$$

Proof. Let $a \leq_L x \vee b$ and $a \nleq_L x$. Now $a \nleq_L x$ implies that $x \prec x \vee a$ by Lemma 3.27. If $b \leq_L x$ then $b \leq_L x \vee a$.

Let us assume thus that $b \nleq_L x$. In this case, by Lemma 3.27, $x \prec x \vee b$. On the other hand, $a \leq_L x \vee b$ and therefore also $x \vee a \leq_L x \vee b$, and there must be equality here. Consequently,



$$x \prec x \vee a = x \vee b$$

which shows that $b \leq_L x \vee a$. □

3.2. Algebraic lattices

Compact elements

An element x of a complete lattice L is called **compact** if whenever $x \leq_L \bigvee A$ for a set $A \subseteq L$ then there exists a *finite* subset $F \subseteq A$ such that $x \leq_L \bigvee F$.

Theorem 3.29. *Let L be a complete lattice. Then all elements of L are compact if and only if L satisfies the ascending chain condition.*

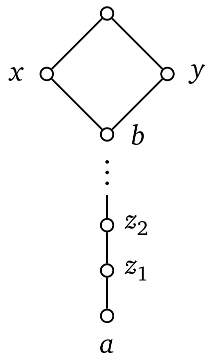
Proof. Suppose that the ACC holds for L, and thus every nonempty subset of L has a maximal element; see Theorem 1.36. Let $x \leq_L \bigvee A$ for a subset $A \subseteq L$. The set $\{\bigvee F \mid F \subseteq A,\ |F| < \infty\}$ has a maximal element, say $\bigvee M$. Since $y \leq_L \bigvee M$ for all $y \in A$, also $\bigvee M = \bigvee A$ and hence $x \leq_L \bigvee M$ as was required.

In the other direction, assume that every element is compact, and let A be an ascending chain $a_1 <_L a_2 <_L \dots$ of elements. Now $\bigvee A \leq_L \bigvee A$, and hence there exists a finite subset $F \subseteq A$ such that $\bigvee A \leq_L \bigvee F$. Since A is an ascending chain, $\bigvee F \leq a_m$ for some index m, and thus also $\bigvee A \leq a_m$, implying that a_m is a maximal element of A. □

Lemma 3.30. *If x and y are compact elements in L, then so is their join $x \vee y$.*

Proof. Exercise. □

Example 3.31. The meet of two compact elements need not be compact. Consider the poset on the right where between a and b there are infinite chain of elements $a \prec z_1 \prec z_2 \prec \dots$ with $z_i \leq_L b$ for all i. Then both x and y are compact, but $b = x \wedge y$ is not. Indeed, let $Z = \{z_i \mid i = 1, 2, \dots\}$. Then $b \leq_L \bigvee Z$, but if $F \subseteq Z$ is a finite subset, then $\bigvee F \leq_L z_m$ for the maximum element in F. □



A complete L is **algebraic** (or **compactly generated**) if all its elements are joins of compact elements, that is, if $x \in L$ then there exists a set A of compact elements of L such that $x = \bigvee A$. In particular, in an algebraic lattice

$$x = \bigvee \{c \mid c \leq_L x,\ c \ \text{ compact}\}$$

for all $x \in L$.

Example 3.32. The subset lattice 2^X is algebraic. The compact elements of 2^X are the finite subsets of X, and each subset $A \subseteq X$ is the union of the finite subsets of A. □

Example 3.33. The subgroup lattice of a group G is a compact lattice. The compact elements in this lattice are the finitely generated subgroups. □

Theorem 3.34. *Let L be an algebraic lattice, and let $x <_L y$. Then there exists a compact element c such that $c \leq_L y$ but $c \not\leq_L x$.*

Proof. Let A_c be a set of compact elements such that $y = \bigvee A_c$. If all $c \in A_c$ were such that $c \leq_L x$ then $\bigvee A_c \leq_L x$ would imply that $x = y$; a contradiction. Thus at least one $c \in A_c$ satisfies $c \not\leq_L x$. □

We say that a subset D of a lattice L is (**upward**) **directed** if

$$x, y \in D \implies \exists z \in D \colon\ x \vee y \leq_L z\,.$$

It follows that if D is a directed subset and $F \subseteq D$ is a finite subset, then there exists an element $z \in D$ such that $\bigvee F \leq_L z$.

Theorem 3.35 (Generalized distributive law). *Let L be an algebraic lattice and D its directed set. Then, for all $x \in L$,*

$$x \wedge \bigvee D = \bigvee_{y \in D} (x \wedge y)\,.$$

Proof. Clearly,

$$\bigvee_{y \in D} (x \wedge y) \leq_L x \quad \text{and} \quad \bigvee_{y \in D} (x \wedge y) \leq_L \bigvee D\,,$$

and hence

$$\bigvee_{y \in D} (x \wedge y) \leq_L x \wedge \bigvee D\,.$$

In converse, let c be a compact element such that $c \leq_L x \wedge \bigvee D$. Then $c \leq_L x$ and $c \leq_L \bigvee D$. Let $F \subseteq D$ be a finite subset such that $c \leq_L \bigvee F$. Since F is finite and D is directed, there exists an element $a \in D$ such that $\bigvee F \leq_L a$. Therefore

$$c \leq_L x \wedge a \leq_L \bigvee_{y \in D} (x \wedge y)\,. \tag{3.2}$$

Hence all compact $c \leq_L x \wedge \bigvee D$ satisfy $c \leq_L \bigvee_{y \in D}(x \wedge y)$. By Theorem 3.34, these elements must be equal. ☐

Suppose, moreover, that L is distributive in the above, and let $\bigvee F = \bigvee_{i=1}^{n} x_i$ for some x_i. Then $x \wedge \bigvee F \overset{(D1)}{=} \bigvee_{i=1}^{n}(x \wedge x_i)$, and (3.2) can be rewritten in the form

$$c \leq_L x \wedge \bigvee F \leq_L \bigvee_{y \in D} (x \wedge y)\,.$$

where one did not need the fact that D is directed. Therefore we have shown

Theorem 3.36. *Let L be a distributive and algebraic lattice. Then*

$$x \wedge \bigvee A = \bigvee_{y \in A} (x \wedge y)$$

for all $A \subseteq L$ and $x \in L$.

Closure operations

Let X be a set. A mapping $C\colon 2^X \to 2^X$ is a **closure operation** if for all $A, B \subseteq X$,

$$A \subseteq C(A)\,, \tag{C1}$$

$$C^2(A) = C(A)\,, \tag{C2}$$

$$A \subseteq B \implies C(A) \subseteq C(B)\,. \tag{C3}$$

If $C\colon 2^X \to 2^X$ is a closure operation then $C(A)$ is the **closure** of the subset $A \subseteq X$. We say that a subset $A \subseteq X$ is **closed** w.r.t. the closure operation C if $A = C(A)$. The condition (C1) is **extension condition**; (C2) is **idempotency condition**; (C3) is **isotonicity condition**.

Example 3.37. The operations C that maps a subset of an algebra to its generated subalgebra is a closure operation. For a group/lattice, C maps a subset A to the subgroup/sublattice Sg(A). ☐

Example 3.38. The topological closure operations satisfy the above conditions. Moreover, the topological closure concerns union which is not required in the general case. ☐

Example 3.39. The operation that adjoins the smallest equivalence relation to a relation is a closure operation. □

Lemma 3.40. *A mapping $C\colon 2^X \to 2^X$ is a closure operation if and only if it satisfies the conditions (C1) and*

$$A \subseteq C(B) \implies C(A) \subseteq C(B)\,. \tag{C2+3}$$

Proof. Exercise. □

Lemma 3.41. *Let $C\colon 2^X \to 2^X$ be a closure operation, and $\{A_i \mid i \in I\}$ a family of sets. Then*

$$\bigcup_{i\in I} C(A_i) \subseteq C\Big(\bigcup_{i\in I} A_i\Big)\,.$$

Proof. Since $U = C\big(\bigcup_{i\in I} A_i\big)$ is closed and $A_i \subseteq U$ for all i, also $C(A_i) \subseteq U$ for all i, and thus the claim follows. □

The closed subsets form a poset $\mathrm{Clo}(C)$ under inclusion. Such a poset is called a **closure system**.

Lemma 3.42. *Let C be a closure operation and let $\{A_i \mid i \in I\} \subseteq \mathrm{Clo}(C)$ be a family of closed sets. Then also the intersection $\bigcap_{i\in I} A_i$ is closed.*

Proof. Write $A = \cap_{i\in I} A_i$ $(=\bigcap \mathscr{A})$. Now, $A \subseteq A_i$ for all i, and hence by (C3)

$$C(A) \subseteq C(A_i) = A_i$$

for all $i \in I$. Therefore, $C(A) \subseteq A \subseteq C(A)$, and so $C(A) = A$, which proves the claim. □

By Theorem 2.13, we have

Theorem 3.43. *Let C be a closure operation. Then $\mathrm{Clo}(C)$ is a complete lattice with*

$$\bigwedge_{i\in I} C(A_i) = \bigcap_{i\in I} C(A_i) \quad \textit{and} \quad \bigvee_{i\in I} C(A_i) = C\Big(\bigcup_{i\in I} A_i\Big)\,.$$

Also the converse holds:

Theorem 3.44. *Let L be a complete lattice. Then L is isomorphic to a closure system $\mathrm{Clo}(C)$ for some closure operation C.*

Proof. Define

$$C(A) = (\textstyle\bigvee A]\,.$$

Then, as is easy to show, $C\colon 2^L \to 2^L$ is a closure operation, and a subset is closed if and only if its is an ideal. (Notice that in a complete lattice every ideal is principal. Indeed, $I = (\bigvee I]$.) By (the proof of) Theorem 2.46 the mapping $\varphi\colon L \to \mathrm{Clo}(C)$ defined by $\varphi(x) = (x]$ is a required isomorphism. □

A closure operation $C\colon 2^X \to 2^X$ is **algebraic** if for all subsets $A \subseteq X$,

$$C(A) = \bigcup \{C(F) \mid F \subseteq A,\ |F| < \infty\}. \tag{C4}$$

Lemma 3.45. *A closure operation $C\colon 2^X \to 2^X$ is algebraic if and only if it is* ***locally finite****: if $x \in C(A)$ then there exists a finite subset $F \subseteq A$ such that $x \in C(F)$.*

Proof. If C is algebraic and $x \in C(A)$ then the claim follows from the condition (C4), since $C(A)$ is a union of closed sets generated by finite sets. On the other hand, if C is locally finite then the claim follows from the definition. □

The next lemma is a reformulation of the definition.

Lemma 3.46. *Let $C\colon 2^X \to 2^X$ be an algebraic closure operation. Then*

$$C(A) = \bigvee \{C(F) \mid F \subseteq A,\ |F| < \infty\}$$

for all $A \subseteq X$.

Proof. Indeed,

$$\bigvee \{C(F) \mid F \subseteq A,\ |F| < \infty\} = C\Big(\bigcup_{\substack{F \subseteq A \\ |F| < \infty}} C(F)\Big) = \bigcup_{\substack{F \subseteq A \\ |F| < \infty}} C(F).$$

□

Example 3.47. Let G be a group and $\mathrm{Sg}(A)$ be the subgroup generated by A. Then $\mathrm{Sg}\colon 2^G \to 2^G$ is an algebraic closure operation. Indeed, $g \in \mathrm{Sg}(A)$ if and only if it is a finite product $g = a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n}$ of elements, where $a_i \in A$ and $e_i = \pm 1$. Therefore $g \in \mathrm{Sg}(\{a_1, a_2, \dots, g_n\})$.

Similarly, if R is a ring, then the operation $I\colon 2^R \to 2^R$ that maps a subset A to the ideal $I(A)$ generated by A is an algebraic closure operation. □

Theorem 3.48. *Let $C\colon 2^X \to 2^X$ be an algebraic closure operation. Then the closure system* $\mathrm{Clo}(C)$ *is an algebraic lattice. The compact elements of* $\mathrm{Clo}(C)$ *are the closed sets $C(F)$ for finite subsets $F \subseteq X$.*

Proof. We show first that a closed set A is compact if and only if there exists a finite set F such that $A = C(F)$.

Let $F \subseteq X$ be a finite subset, and suppose that $C(F) \subseteq \bigvee_{i \in I} A_i$, where each A_i is a closed set, $C(A_i) = A_i$ for all $i \in I$. Denote $U = \bigcup_{i \in I} A_i$. Then $C(U) = \bigvee_{i \in I} A_i$, and since C is algebraic,

$$\begin{aligned} C(F) \subseteq C(U) &= \bigcup \{C(B) \mid\ |B| < \infty,\ B \subseteq U\} \\ &= \bigcup \{C(B) \mid\ |B| < \infty,\ B \subseteq \bigcup_{i \in J} A_i,\ |J| < \infty\}. \end{aligned}$$

Since F is finite and $F \subseteq C(F)$, there are finite sets $B_1, B_2, \dots, B_k \subseteq U$ such that $F \subseteq \bigcup_{i=1}^{k} C(B_i)$. Each B_i is finite, and so

$$B_i \subseteq \bigcup_{j \in I_i} A_j \quad \text{for some index set } |I_i| < \infty$$

for all $i = 1, 2, \dots, k$. Therefore

$$\begin{aligned} F \subseteq \bigcup_{i=1}^{k} C(B_i) &\subseteq \bigvee_{i=1}^{k} C\Big(\bigcup_{j \in I_i} A_j\Big) \\ &= \bigvee \{C(A_j) \mid j \in I_i,\ i = 1, 2, \dots, k\}. \end{aligned}$$

We have now that $C(F) \subseteq \bigvee_{j \in J} C(A_j)$ for the finite index set

$$J = \bigcup_{i=1}^{k} I_i \,.$$

Thus $C(F)$ is compact.

Suppose then that the closed subset $A = C(A)$ is compact. Now

$$\begin{aligned} C(A) &= \bigcup \{C(B) \mid\ |B| < \infty,\ B \subseteq A\} \\ &= \bigvee \{C(B) \mid\ |B| < \infty,\ B \subseteq A\} \\ &\subseteq \bigvee_{i=1}^{k} C(B_i), \end{aligned}$$

for some finite $B_1, B_2, \dots, B_k \subseteq A$, by compactness of A. Moreover,

$$C(A) \subseteq \bigvee_{i=1}^{k} C(B_i) = C\Big(\bigcup_{i=1}^{k} B_i\Big) \subseteq C(A),$$

where there must be equalities. The claim follows since $\bigcup_{i=1}^{k} B_i$ is finite.

That the closure system $\mathrm{Clo}(C)$ is algebraic follows directly from the fact that C is algebraic, that is, from the condition (C4). □

We say that a subset A **generates** a closed set Y if $C(A) = Y$. Moreover, Y is said to be **finitely generated** if there exists a finite set F such that $C(F) = Y$.

We leave the following corollaries as exercises.

Corollary 3.49. *Let $C\colon 2^X \to 2^X$ be an algebraic closure operation. Then the finitely generated subsets of X form the compact elements of the lattice* $\mathrm{Clo}(C)$*.*

Corollary 3.50. *Every algebraic lattice is isomorphic to some closure system* $\mathrm{Clo}(C)$ *where C is an algebraic closure operation.*

Recall that in a directed family of sets $\mathscr{D}$ for every two sets $A, B \in \mathscr{A}$ there exists a set $C \in \mathscr{D}$ such that $A \cup B \subseteq C$.

Theorem 3.51. *Let $C : 2^X \to 2^X$ be a closure operation. Then C is algebraic if and only if for every directed family $\mathscr{A} = \{A_i \mid i \in I\}$ of subsets of X,*

$$C\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} C(A_i).$$

Proof. Assume that C is algebraic and let $\{A_i \mid i \in I\}$ be a directed family. Let $U = \bigcup_{i\in I} A_i$. By (C4),

$$C(U) = \bigcup \{C(F) \mid F \subseteq U,\ |F| < \infty\}.$$

Every finite subset $F \subseteq U$ is included in some element A_F of the directed family $\{A_i \mid i \in I\}$. Now, $C(F) \subseteq C(A_F)$, and hence

$$C(U) = \bigcup \{C(F) \mid F \subseteq U,\ |F| < \infty\} \subseteq \bigcup_{i\in I} C(A_i).$$

On the other hand, by Lemma 3.41, the inclusion is always valid in the other direction.

Conversely, suppose that C satisfies the condition for directed families. For each subset A, let

$$\mathscr{M}(A) = \{C(F) \mid F \subseteq A,\ |F| < \infty\}.$$

Then $\bigcup \mathscr{M}(A) \subseteq C(A)$. The family $\mathscr{M}(A)$ is directed, and hence $\bigcup \mathscr{M}(A)$ is closed. On the other hand, $A \subseteq \bigcup \mathscr{M}(A)$ and therefore

$$C(A) \subseteq C(\bigcup \mathscr{M}(A)) = \bigcup \mathscr{M}(A),$$

which was the claim. □

Example 3.52 (Exchange law)**.** Theorem 3.28 can be reformulated using closure operations as follows. Let L be a geometric lattice L, and define

$$C(A) = \{a \in \mathrm{Atom}(L) \mid a \le_L \bigvee A\}$$

for all $A \subseteq \mathrm{Atom}(L)$. Then C is a closure operation on the atoms of L that satisfies the following **exchange property**:

$$a \in C(A \cup \{b\}) \text{ with } a \notin C(A) \implies b \in C(A \cup \{a\}).$$

□

Dedekind–MacNeille completions

Completions of posets were motivated by Dedekind's method to construct the real numbers from rational numbers. This approach was generalized by MacNeille for arbitrary posets.

Let P be a poset, and L a complete lattice. An order embedding $\varphi\colon P \to L$ is called a **completion** of P.

By Theorem 1.23, every poset P is isomorphic to the poset of its principal down-sets under inclusion. Hence the mapping $\varphi\colon P \to 2^P$ defined by $\varphi(x) = {\downarrow}x$ is an order embedding of P into the complete lattice 2^P (under inclusion). The disadvantage of this completion is in the large size of the power set 2^P compared to the size of P.

Recall that, for a subset $A \subseteq P$, let

$$A^{\mathrm{l}} = \{x \in P \mid x \leq_P A\} \quad\text{and}\quad A^{\mathrm{u}} = \{x \in P \mid x \geq_P A\}$$

be the sets of lower and upper bounds in the poset P.

Lemma 3.53. *The following hold for subsets $A, B \subseteq P$:*

$$A \subseteq A^{\mathrm{ul}} \ \text{ and } \ A \subseteq A^{\mathrm{lu}}, \tag{3.3}$$

$$\text{if } A \subseteq B \ \text{ then } \ B^{\mathrm{u}} \subseteq A^{\mathrm{u}} \ \text{ and } \ B^{\mathrm{l}} \subseteq A^{\mathrm{l}}, \tag{3.4}$$

$$A^{\mathrm{l}} = A^{\mathrm{lul}} \text{ and } \ A^{\mathrm{u}} = A^{\mathrm{ulu}}. \tag{3.5}$$

Also, A^{l} is a down-set and A^{u} is an up-set.

Proof. Exercise. □

The **Dedekind–MacNeille completion** of a poset P is defined to be

$$\mathrm{DM}(P) = \{A \mid A \subseteq P,\ A = A^{\mathrm{ul}}\}$$

together with inclusion. Here $x \in A^{\mathrm{ul}}$ if and only if it is a lower bound of all upper bounds of A.

Example 3.54. Let $P = (\mathbb{N}, \leq)$. If A is finite, then $A^{\mathrm{u}} = \{n \mid n \geq \max(A)\}$, and $A^{\mathrm{ul}} = \{0, 1, \ldots, \max(A)\}$. On the other hand, if A is infinite, then $A^{\mathrm{u}} = \emptyset$, and then $A^{\mathrm{ul}} = \mathbb{N}$. Also, $\emptyset^{\mathrm{ul}} = \{0\}$. Therefore, $\mathrm{DM}(P)$ consists of the sets $\mathbb{N}$ and $\{0, 1, \ldots, n\}$ for all $n \geq 0$. □

Theorem 3.55. *Let P be a poset. Then* $\mathrm{DM}(P)$ *is a complete lattice.*

Proof. First of all, $\mathrm{DM}(P)$ has the minimum element $\emptyset^{\mathrm{ul}}$. By Theorem 2.12, it suffices to show that each nonempty subset $\mathscr{A}$ of $\mathrm{DM}(P)$ has a least upper bound. Let

$$\mathscr{A} = \{A_i \mid i \in I\} \quad\text{and}\quad A = \bigcup_{i \in I} A_i\,.$$

By Lemma 3.53, we have that $A^{\mathrm{ul}} \in \mathrm{DM}(P)$. Also, A^{ul} is an upper bound of $\mathscr{A}$ in $\mathrm{DM}(P)$. Consider then a subset $B \in DM(P)$ that is also an upper bound of $\mathscr{A}$.

Then $A \subseteq B$, and therefore, by Lemma 3.53, $B^{\mathrm{u}} \subseteq A^{\mathrm{u}}$, and, furthermore, $A^{\mathrm{ul}} \subseteq B^{\mathrm{ul}}$. Hence A^{ul} is the least upper bound of $\mathscr{A}$, and this proves that $\mathrm{DM}(P)$ is complete. □

Lemma 3.56. *Let $x \in P$ for the poset P. Then $\downarrow x \in \mathrm{DM}(P)$.*

Proof. One needs to show that $\downarrow x = (\downarrow x)^{\mathrm{ul}}$. This is an exercise. □

Lemma 3.57. *Let P be a poset and $A \subseteq P$ such that $\bigvee A$ exists in P. Then $A^{\mathrm{ul}} = (\bigvee A)^{\mathrm{l}}$.*

Proof. Exercise. □

Theorem 3.58. *Let L be a complete lattice. Then L and $\mathrm{DM}(L)$ are isomorphic.*

Proof. By Theorem 3.55, $\mathrm{DM}(L)$ is a complete lattice. Define the mapping $\varphi \colon L \to \mathrm{DM}(L)$ by

$$\varphi(x) = \downarrow x\,. \tag{3.6}$$

(Notice that $\downarrow x = x^{\mathrm{l}}$ for all elements $x \in L$.)

Let $A \subseteq L$. We show that $\varphi\big(\bigvee A\big) = \bigvee \varphi(A)$, and $\varphi\big(\bigwedge A\big) = \bigwedge \varphi(A)$.

Using Lemma 3.57, we obtain

$$A^{\mathrm{ul}} = \big(\bigvee A\big)^{\mathrm{l}} = \downarrow\big(\bigvee A\big) = \varphi\big(\bigvee A\big).$$

It is clear that A^{ul} is an upper bound in $DM(L)$ for $\{\varphi(x) \mid x \in A\}$. Let $B \in DM(L)$ be another upper bound. Since $x \in \varphi(x) \subseteq B$ for all $x \in A$, we have $A \subseteq B$, and thus $A^{\mathrm{ul}} \subseteq B^{\mathrm{ul}} = B$. This shows that $A^{\mathrm{ul}} = \bigvee \varphi(A)$, and so $\varphi\big(\bigvee A\big) = \bigvee \varphi(A)$ as required.

For the meet, we observe that $\bigwedge A \in \downarrow x$ for all $x \in A$, and hence $\downarrow(\bigwedge A) \subseteq \cap\{\downarrow x \mid x \in A\}$. On the other hand, if $y \in \cap\{\downarrow x \mid x \in A\}$, then $y \leq_L \bigwedge A$, and so

$$\begin{aligned} \bigwedge \varphi(A) &= \cap\{\varphi(x) \mid x \in A\} = \cap\{\downarrow x \mid x \in A\} \\ &= \downarrow(\bigwedge A) = \varphi(\bigwedge A) \end{aligned}$$

as required.

Also, $A^{\mathrm{ul}} = \downarrow\big(\bigvee A\big) = \varphi\big(\bigvee A\big)$, and hence φ is surjective. Surely it is injective, and thus it is an isomorphism. □

3.3. Complemented lattices

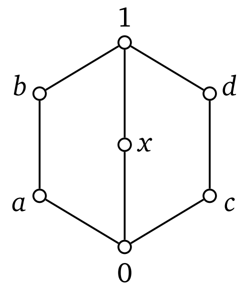
Complements

Let L be a lattice with the bottom and top elements 0_L and 1_L. An element $\overline{x} \in L$ is a **complement** of $x \in L$ if

$$x \wedge \overline{x} = 0_L \quad \text{and} \quad x \vee \overline{x} = 1_L\,. \tag{C}$$

The lattice L is **complemented** if each of its elements has a complement. Clearly, if x has a complement $\overline{x}$ then x is a complement of $\overline{x}$.

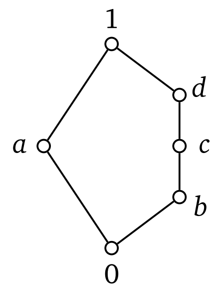
Example 3.59. The lattice L on the right is complemented. We always have that 0_L and 1_L are complements of each other. Notice that an element may have several complements. For instance, in the given lattice, x has the complements a, b, c and d. □



Example 3.60. The subset lattice 2^X is complemented in a natural way. On the other hand, a divisor lattice T_n need not be complemented. For instance, the integer 2 does not have a complement in $L = T_{12}$: We do have that $2 \wedge 3 = 1$ $(= 0_L)$, but $2 \vee 3 = 6$ $(\neq 1_L)$. □

A lattice L (which possibly does not have a bottom nor a top element) is **relatively complemented**, if every interval $[x, y]_L$ with $x \leq_L y$ is a complemented sublattice.

Example 3.61. It is clear that every relatively complemented lattice is complemented if it has a top and a bottom element. The converse is not true as shown in the lattice L on the right. This lattice is complemented but the sublattice $[b, d]_L$ is not complemented. □



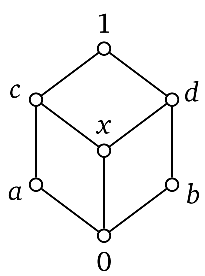
Theorem 3.62. *Let L be a distributive lattice with 0_L and 1_L. Then every element can have at most one complement in L.*

Proof. Assume both y and z are complements of $x \in L$. Then

$$
\begin{aligned}
y &= y \wedge 1_L = y \wedge (z \vee x) \\
&\overset{(D1)}{=} (y \wedge z) \vee (y \wedge x) \\
&= (y \wedge z) \vee 0_L = y \wedge z\,,
\end{aligned}
$$

and hence $y \leq_L z$. Symmetrically, we have $z \leq_L y$, and therefore $y = z$. □

Example 3.63. A point lattice (where all elements are joins of atoms) need not be complemented. The element x in the point lattice L on the right has no complement. □



Theorem 3.64. *Each geometric lattice is relatively complemented, and thus also complemented.*

Proof. Recall that every geometric lattice L is complete, and thus it has the bottom and the top elements.

Let $x \leq_L y$ for $x, y \in L$, and let $z \in [x, y]_L$ and choose an element $z_1 \in [x, y]_L$ such that $z \wedge z_1 = x$. (Such an elements exists, e.g., $z_1 = x$.) If $z \vee z_1 = y$, we are done: z_1 is a complement of z in $[x, y]_L$.

Suppose thus that $z \vee z_1 <_L y$. Since each geometric lattice is a point lattice, there exists an atom $a \in L$ such that $a \leq_L y$ and $a \not\leq_L z \vee z_1$. (Since y is the join of the atoms $b \leq_L y$ and $z \vee z_1$ is the join of the atoms $b \leq_L z \vee z_1$.) Let $z_2 = z_1 \vee a$. Clearly, $z_2 \in [x, y]_L$ and so $z \vee z_2 \leq_L y$. Moreover,

$$z \vee z_1 <_L z \vee z_2 .$$

We show now that $z \wedge z_2 = x$.

Suppose contrary to this claim that $x <_L z \wedge z_2$. Then there exists an atom b of L such that $b \leq_L z \wedge z_2$ but $b \not\leq_L x$. We have

$$x = z \wedge z_1 <_L (z \wedge z_1) \vee b \leq_L z \wedge z_2 = z \wedge (z_1 \vee a) .$$

Since $b \leq_L z$ and $b \not\leq_L x = z \wedge z_1$, it follows that $b \not\leq_L z_1$. Also, $b \leq_L z_2 = z_1 \vee a$, and Theorem 3.28 gives that $a \leq_L z_1 \vee b$, and so $z_1 \vee a \leq_L z_1 \vee b$. Also $z_1 \vee b \leq_L z_1 \vee a$, since $b \leq_L z_1 \vee a$, and hence $z_1 \vee b = z_1 \vee a = z_2$, which then implies that

$$z \vee z_1 \vee b = z \vee z_2 .$$

However, $b \leq_L z$ and $b \leq_L z_1 \vee a$, and hence $z \vee z_2 = z \vee z_1 \vee b = z \vee z_1$, which is a contradiction. Thus, indeed, $z \wedge z_2 = x$.

Now we have found an element z_2 for which

$$z \wedge z_2 = x \quad \text{and} \quad z \vee z_1 <_L z \vee z_2 \leq_L y .$$

Again, if $z \vee z_2 = y$, we have found a complement z_2 of z. Otherwise we repeat the above process for z_2, and obtain in this fashion a sequence $z_1, z_2, \dots$ such that

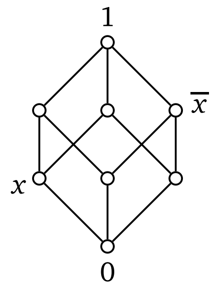
$$z \wedge z_i = x \quad \text{and} \quad z \vee z_{i-1} <_L z \vee z_i \leq_L y .$$

Since every chain in L is finite, the above construction will end in an element z_i for which $z \vee z_i = y$, and thus z_i is a complement of z in the interval $[x, y]_L$. □

Boolean lattices

A complemented distributive lattice B with $0_B \neq 1_B$ is called **Boolean lattice**.

Example 3.65. The 2-element chain $\mathbf{C}_2$, also often denoted by $\mathbb{B}_2$, is a Boolean lattice, and so are the direct products $\mathbb{B}_{2^n}$ of $\mathbb{B}_2$ that have 2^n elements. □



Example 3.66. A nonempty family $\mathscr{K}$ of subsets of X is a **set algebra**

$$A, B \in \mathscr{K} \implies A \cup B,\ A \cap B,\ X \setminus A \in \mathscr{K}\,.$$

Every set field is a Boolean lattice. □

Theorem 3.67. *Let B be a Boolean lattice. Then every element $x \in B$ has a unique complement $\overline{x}$. Moreover,*

(1) $\overline{0}_B = 1_B$ *and* $\overline{1}_B = 0_B$;
(2) $\overline{\overline{x}} = x$ *for all* $x \in B$;
(3) $\overline{x \vee y} = \overline{x} \wedge \overline{y}$ *and* $\overline{x \wedge y} = \overline{x} \vee \overline{y}$;
(4) $x \leq_B y \iff \overline{y} \leq_B \overline{x}$.

Proof. The uniqueness of complements follows from Theorem 3.62. The rest of the cases are exercises. □

In addition to the lattice laws (L1) – (L4), the Boolean lattices B satisfy the distributive laws (D1) and (D2), and the law for complements.

Note that from these laws it can be derived that 0_B is the bottom element of B and 1_B is the top element of B. Indeed, for all $x \in B$,

$$x \wedge 0_B \overset{(C)}{=} x \wedge (x \wedge \overline{x}) \overset{(L3)}{=} (x \wedge x) \wedge \overline{x} \overset{(L1)}{=} x \wedge \overline{x} \overset{(C)}{=} 0_B\,,$$

and similarly $x \vee 1_B = 1_B$.

Example 3.68 (Huntington)**.** The following three axioms determine Boolean lattices, that is, they are equivalent to the previous set of axioms:

(H1) $x \vee (y \vee z) = (x \vee y) \vee z$

(H2) $x \vee y = y \vee x$

(H3) $\overline{\overline{x} \vee \overline{y}} \vee \overline{\overline{x} \vee y} = x$

The first two of these are (L3) and (L2). The last one is the **Huntington equation**. Note that these axioms do not use $\wedge$. □

Example 3.69 (Robbins Conjecture)**.** It was conjectured by Robbins that the following three axioms determine Boolean lattices:

$$x \vee (y \vee z) = (x \vee y) \vee z \tag{R1}$$

$$x \vee y = y \vee x \tag{R2}$$

$$\overline{\overline{x \vee y} \vee \overline{x \vee \overline{y}}} = x \tag{R3}$$

The first two are again (L3) and (L2). The last one is the **Robbins equation**. This conjecture was proved by McCune in 1997 by an extensive usage of computers. Note that these axioms do not use $\wedge$. □

Example 3.70 (Single axiom)**.** The following single axiom determines the Boolean lattices:

$$\overline{\overline{x \vee y} \vee \overline{z}} \vee \overline{\overline{\overline{u} \vee u} \vee \overline{z} \vee x} = z\,. \tag{3.7}$$

This is due to McCune, Veroff, Fitelson, Harris, Feist and Wos (2000). □

In the finite case we have the following characterizations of Boolean lattices.

Theorem 3.71. *Let L be a finite distributive lattice. Then the following conditions are equivalent:*

(1) *L is a boolean algebra,*
(2) *L is complemented,*
(3) *L is relatively complemented,*
(4) *L is atomic,*
(5) *1_L is the join of the atoms of L,*
(6) *L is geometric.*

Proof. Exercise. □

3.4. Möbius function for lattices

Theorems of Hall and Weisner

Recall that the Möbius function of a locally finite poset P is the inverse of the zeta function:

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq_L y\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then

$$\mu(x,y) = \begin{cases} 0 & \text{if } x \not\leq_P y\,, \\ 1 & \text{if } x = y\,, \\ -\sum\limits_{x \leq_P z <_P y} \mu(x,z) & \text{if } x <_P y\,. \end{cases} \tag{3.8}$$

In this section we study the Möbius function for lattices.

The following tool is needed in the proof of Theorem 3.73. It states that each nonempty finite set X has equally many subsets of even size and of odd size.

Lemma 3.72. *Let X be a nonempty finite set. Then*

$$\sum_{A\subseteq X}(-1)^{|A|}=0\,.$$

Proof. Let $n=|X|\geq 1$. For each $i=0,1,\dots,n$ there are $\binom{n}{i}$ subsets $A\subseteq X$ such that $|A|=i$. Hence

$$\sum_{A\subseteq X}(-1)^{|A|}=\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}=\sum_{i=0}^{n}(-1)^{i}1^{n-i}\binom{n}{i}=(1-1)^{n}=0$$

by the binomial theorem. □

Second proof. The second proof goes by induction on $|X|$. The claim is true for singleton sets, since $\emptyset$ is has even size and X has odd size. Let then $x\in X$, where $|X|=n+1$. By the induction hypothesis on $X\setminus\{x\}$ there are equally many even and odd subsets of X that do not contain x. To obtain all subsets, add x to the previous subsets; even subsets turn to odd, and odd to even, and so there are equally many of subsets of even and odd elements that contain x. □

Theorem 3.73 (Hall)**.** *Let L be a finite lattice. Then*

$$\mu(0,x)=\sum_{A\in Y(x)}(-1)^{|A|}\,,$$

where

$$Y(x)=\{A\subseteq \operatorname{Atom}(L)\mid x=\bigvee A\}\,.$$

Proof. Let $x\in L$, and denote $\alpha(x)=\sum_{A\in Y(x)}(-1)^{|A|}$. Let also $A(x)=\{a\in \operatorname{Atom}(L)\mid a\leq_L x\}$, and $\beta(x)=\sum_{y\leq_L x}\alpha(y)$. Then

$$\begin{aligned}\beta(x)&=\sum_{y\leq_L x}\alpha(y)=\sum_{A\subseteq A(x)}(-1)^{|A|}\\&=\begin{cases}1 & \text{if } A(x)=\emptyset\,,\\ 0 & \text{if } A(x)\neq\emptyset\,,\end{cases}\\&=\begin{cases}1 & \text{if } x=0\,,\\ 0 & \text{if } x\neq 0\,.\end{cases}\end{aligned}$$

By the Möbius inversion formula, we have the claim:

$$\alpha(x)=\sum_{y\leq_L x}\beta(y)\mu(y,x)=\mu(0,x)\,.$$

□

Corollary 3.74. *Let L be a finite lattice. If $x \in L$ is not a join of atoms, then $\mu(0,x) = 0$.*

Applying the previous corollary to the sublattice $[x,y]_L$ we obtain

Corollary 3.75. *Let L be a finite lattice, and let $A_x = \{z \mid x \prec z\}$. Then $\mu(x,y) = 0$ if y is not a join of elements from A_z.*

Theorem 3.76 (Weisner)**.** *Let L be a finite lattice, and let $x \in L$ and $a \in L$ with $a \neq 0$. Then*

$$\sum_{z \vee a = x} \mu(0,z) = 0\,.$$

Proof. Let $\alpha(x) = \sum_{z\vee a = x} \mu(0,z)$, and let $\beta(x) = \sum_{y \leq_L x} \alpha(y)$. Then

$$\beta(x) = \sum_{y \leq_L x} \alpha(y) = \sum_{z \vee a \leq_L x} \mu(0,z)\,.$$

If $a \not\leq_L x$ then obviously $\beta(x) = 0$. Suppose that $a \leq_L x$. Since $a \neq 0$, $\beta(x) = \sum_{z \leq_L x} \mu(0,z) = 0$. By the inversion formula, also $\alpha(x) = 0$ as required. □

Closure operations

We generalize the notion of a closure operation to all lattices as follows. A function $c\colon L \to L$ is a **closure operation** if, for all $x, y \in L$,

$$\begin{aligned}
&x \leq_L c(x)\,,\\
&x \leq_L y \implies c(x) \leq_L c(y)\,,\\
&c(x) = c(c(x))\,.
\end{aligned}$$

The set of all closed elements, i.e., fixed points of c, is denoted by

$$\mathrm{Clo}(c) = \{x \mid c(x) = x\}\,.$$

Example 3.77. Let a be a fixed element of the lattice L. Then the function $c\colon L \to L$ defined by $c(x) = x \vee a$ is a closure operation. □

Lemma 3.78. *Let L be a lattice and $c\colon L \to L$ a closure operation on L. Then* $\mathrm{Clo}(c)$ *is a complete lattice under inclusion.*

Proof. It is easy to show that $\mathrm{Clo}(c)$ is closed under arbitrary intersections, and thus the claim follows from Theorem 2.13. □

Theorem 3.79 (Rota)**.** *Let L be a locally finite lattice and $c\colon L \to L$ a closure operation on L, and let $K =$ $\mathrm{Clo}(c)$ be the lattice of closed elements for c. Then for all $x, y \in L$,*

$$\sum_{\substack{z \in L\\ c(z) = c(y)}} \mu(x,z) = \begin{cases} \mu_K(c(x), c(y)) & \text{if } x = c(x)\,,\\ 0 & \text{if } x <_L c(x)\,.\end{cases}$$

Proof. For brevity we write $\overline{x}$ instead of $c(x)$ for all x. Also μ and ζ refer to the lattice L, and μ_K, δ_K and ζ_K refer to the lattice K. Then

$$\begin{aligned}
\sum_{\substack{z\in L\\ \overline{z}=\overline{y}}}\mu(x,z) &= \sum_{z\in L}\mu(x,z)\delta_K(\overline{z},\overline{y})\\
&= \sum_{z\in L}\mu(x,z)\Big(\sum_{\overline{u}\in K}\zeta_K(\overline{z},\overline{u})\mu_K(\overline{u},\overline{y})\Big)\\
&= \sum_{z\in L}\sum_{\overline{u}\in K}\mu(x,z)\zeta_K(\overline{z},\overline{u})\mu_K(\overline{u},\overline{y})\\
&\overset{(*)}{=} \sum_{z\in L}\sum_{\overline{u}\in K}\mu(x,z)\zeta(z,\overline{u})\mu_K(\overline{u},\overline{y})\\
&= \sum_{\overline{u}\in K}\Big(\sum_{z\in L}\mu(x,z)\zeta(z,\overline{u})\Big)\mu_K(\overline{u},\overline{y}) = \sum_{\overline{u}\in K}\delta(x,\overline{u})\mu_K(\overline{u},\overline{y}),
\end{aligned}$$

where in (*) we used the fact that $z \leq_L \overline{u}$ if and only if $\overline{z} \leq_K \overline{u}$. □

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