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Invariant Theory, Young Bitableaux, and Combinatorics

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1. INTRODUCTION

Since its emergence in the middle of the last century, invariant theory has oscillated between two clearly distinguishable poles. The first, and the one that was later to survive the temporary "death" of the field, is geometry. Invariants were identified with the invariants of surfaces. Their study, the aim of which was to give information about the solution of systems of polynomial equations, was to lead to the rise of commutative algebra. From this standpoint, projective invariants were eventually seen as poor relations of the richer algebraic invariants.

A casualty of this trend was the study of the projective generation of surfaces, a problem which was condemned by Cremona as "too difficult," and which has never quite recovered from the blow, despite the recent excitement over finite fields. In contrast, other heretical schools survived the *Fata Morgana* of algebra with the promise, not always fulfilled, that sooner or later they would be brought back into the commutative fold. Thus, the genial computations of the high school teacher Hermann Schubert were proclaimed a "problem" by Hilbert, who was articulating the general feeling at the time that enumerative geometry required a justification in terms of the dominant concepts of the day, namely, rings and fields.

Similarly, the mystical vision of Hermann Grassmann, another high school teacher, was only appreciated by other oddballs like Peano, Study, and several inevitable English gentleman-mathematicians. It took the advocacy of someone

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of the stature of Elie Cartan to get Grassmann's techniques accepted by a public by then avid for simplications, but reluctant to acknowledge embarrassing oversights; and then, only at the cost of putting them to a use for which they were not intended, though magically suited. The recognition that anticommutativity is a sibling, with an equally noble genealogy, of commutativity is only now beginning, under the prodding of the particle physicists, who with exquisite salesmanship have proclaimed it a law of nature.

The second pole of invariant theory was algorithmic. To be sure, all invariant theory is ultimately concerned with one problem. In crude, oversimplified, off-putting language, this problem is to generalize to tensors the eigenvalue theory of matrices, and all invariant theorists from Boole to Mumford have been, tacitly or otherwise, concerned with it. The algorithmic school, however, saw this problem as one of "explicit computation," an expression which was to smack of mathematical bad taste in the 1930s. In a century which prefers existence to construction, structure to algorithm, algebra to combinatorics, such a school could not thrive, and it did not, supported as it was more by the English and Italians than by the Germans and French. There were, however, weightier reasons for this defeat of the algorithmic school. Their most striking productions, the expansions that go under the names of Capelli, Clebsch, Gordan, and Young, were hopelessly tethered to characteristic zero, and seemed to belie the avowed combinatorial ideal of doing away with all numbers that are not integers, and preferably positive ones at that. To top it all, Igusa showed that, with the massive machinery of algebraic geometry, some of the results of classical invariant theory could be extended to fields of positive characteristic.

In this environment, the 1974 paper of Doubilet, Rota, and Stein [3], which for the first time succeeded in extending to arbitrary infinite fields, by constructive algorithmic methods, the two "fundamental theorems" of invariant theory, could only appear as an intrusion. To make things worse, the authors' sympathy for the nineteenth century went as far as to embrace matters of style, thus alienating many readers in a less romantic century. In 1976, de Concini and Procesi [1] charitably rewrote parts of that paper and developed some of the suggestions made therein, thus showing that the authors' claims were indeed well-founded.

In this paper we give a self-contained combinatorial presentation—the first one, to be sure—of vector invariant theory over an arbitrary infinite field. We begin by proving the *Straightening Formula*, which is probably one of the fundamental algorithms of multilinear algebra. This formula is the culmination of a trend of thought that can be traced back to Capelli, and was developed most notably by Alfred Young and the Scottish invariant theorists. Had it not been for the disrepute into which algorithmic methods had fallen in the thirties, the full proof of this formula would have appeared earlier than in 1974, and might have anticipated the current revival of classical invariant theory.

In comparison with other classical expansions, the straightening formula offers two advantages. First, it holds over the ring of integers. Second, it recognizes the crucial role played by the notion of a bitableau in obtaining a characteristic-free proof of the first fundamental theorem. In fact, we give two proofs of this result. Both of these proofs are based on new ideas, first presented in 1974. Even in characteristic zero, either of these proofs differs from any previously given, and is, in addition, much simpler as it only relies on elementary linear algebra and some combinatorics. The success of the notion of a bitableau also shows why previous attempts to prove the first fundamental theorem by expansion into *single* Young tableaux were bound to fail. Strangely, Alfred Young himself was the first to consider bitableaux in his study of the representations of the octahedral group, but it did not occur to him that they would be useful in the study of the projective group.

Less surprisingly, the straightening formula is also used to give a simple proof of the second fundamental theorem, in a version that has been proved by van der Waerden in characteristic zero. The present proof shows that the straightening formula is indeed the characteristic-free replacement of the Gordan–Capelli expansion.

The second fundamental theorem has lived in a limbo ever since Weyl's fumbling justification in "The Classical Groups" [4]. Some invariant theorists have taken the easy way out and claimed it as a result in algebraic geometry, stating certain facts about the coordinate rings of Grassmannians or flag manifolds. We believe on the contrary that the second fundamental theorem plays a crucial role in invariant theory which can perhaps be best understood by analogy with the predicate calculus. Here, two aspects have long been recognized as complementary: a syntactical aspect, where the subject is presented as a purely algebraic system subject to formal rules; and a semantical aspect, where the possible set–theoretic interpretations, or models, are classified. These two aspects are connected by the Gödel completeness theorem.

A corresponding situation obtains in invariant theory. Here, what we call the letter place algebra is the syntactic counterpart to the semantics of representing abstract brackets by actual inner products of vectors and covectors in a vector space. The second fundamental theorem is the invariant–theoretic analog of the Gödel completeness theorem. This suggests a host of questions on invariants which can be gleaned from analogous questions in the predicate calculus.

Other applications of the straightening formula, some of which were adumbrated in 1974, will be given elsewhere. We mention, as examples, a characteristic-free theory of symmetric functions, the study of polynomial identities in an associative algebra, the classification of transvectants, and connections with the algebra of second quantization. The present work is merely the first in what is hoped to be a far-reaching extension of the research program of projective invariant theory.

2. Young Tableaux

The fundamental combinatorial notion in this study is that of a *Young tableau*. Let $(\lambda) = (\lambda_1, \ldots, \lambda_p)$ be a *partition* of the integer n: that is, (λ) is a finite sequence of positive integers such that

$$\lambda_1 + \cdots + \lambda_p = n$$

and

$$\lambda_1 \geqslant \cdots \geqslant \lambda_p > 0.$$

If (λ) is a partition of n, its *shape*, also denoted by (λ), is the set of integer points $(i, -j)$ in the plane, with $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant \lambda_j$. The shape $(\lambda) = (\lambda_1, \ldots, \lambda_s)$ is said to be *longer* than the shape $(\mu) = (\mu_1, \ldots, \mu_t)$ if, considered as a finite sequence, (λ) is greater than (μ) in the lexicographic order from left to right.

A *Young tableau* on the shape (λ) with values in the set E is an assignment of an element of E to each point in the shape (λ). For example, T_1 and T_2 are Young tableaux of shape $(\lambda) = (5, 4, 2, 2, 1, 1)$ with values in the integers:

$$T_1 = \begin{array}{l} 31264 \\ 2212 \\ 43 \\ 45 \\ 7 \\ 8 \end{array} \qquad T_2 = \begin{array}{l} 11223 \\ 2234 \\ 44 \\ 56 \\ 7 \\ 8 \end{array}$$

In this paper, E is always a totally ordered set. A Young tableau is said to be *standard* if the entries in each row are increasing from left to right, and the entries in each column are nondecreasing downward. In our previous example, T_2 is standard but T_1 is not. This definition, though unconventional, is the natural one for dealing with bitableaux (which are introduced in the sequel).

A word on notation: $\mathfrak{S}_p$ denotes the symmetric group on p symbols, and for a permutation $\sigma \in \mathfrak{S}_p$, its signature is denoted $\operatorname{sgn}(\sigma)$.

3. The Straightening Formula

Let $\mathscr{X} = \{x_1, \ldots, x_n\}$ and $\mathscr{U} = \{u_1, \ldots, u_k\}$ be two alphabets, and let P be the algebra of polynomials over the field K in the indeterminates $(x_i \mid u_j)$; P is called the *letter place algebra*. Suppose $(x_{i_1}, \ldots, x_{i_p})$ and $(u_{i_1}, \ldots, u_{i_p})$ are two finite sequences with the same length of letters from $\mathscr{X}$ and $\mathscr{U}$. Their inner product $(x_{i_1} \cdots x_{i_p} \mid u_{j_1}, \ldots, u_{j_p})$ is the polynomial in P defined by

$$(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}) = \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}(\sigma)(x_{i_{\sigma 1}} \mid u_{j_1}) \cdots (x_{i_{\sigma p}} \mid u_{j_p}).$$

The inner product is an antisymmetric function in x_i and u_j. Thus, we may suppose, up to a change in sign, that in any inner product, the indices of x and u are increasing. Moreover, an inner product is nonzero if and only if no letter is repeated.

The *content* of a monomial in P is the pair of vectors

$$(\alpha, \beta) = ((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_k)),$$

where α_s (resp. β_t) is the total degree of the factors in the monomial of the form $(x_s \mid u_j)$, $1 \leqslant j \leqslant k$ (resp. $(x_i \mid u_t)$, $1 \leqslant i \leqslant n$). The monomial of content (α, β) generates a subspace of P, denoted by $P(\alpha, \beta)$. The elements of $P(\alpha, \beta)$ are homogeneous polynomials, in which each monomial has the same content; we say that a polynomial in $P(\alpha, \beta)$ has content (α, β). It is clear that the product of a polynomial of content (α, β), and a polynomial of content (α', β') is a polynomial of content $(\alpha + \alpha', \beta + \beta')$. For example, the inner product $(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p})$ has content (α, β) where α_i (resp. β_i) is 1 if x_i is in the sequence $x_{i_1}, \dots, x_{i_p}$ (resp. u_i is in the sequence $u_{j_1}, \dots, u_{j_p}$) and 0 otherwise.

A *bitableau* is a pair $[T, T']$ of Young tableaux of the same shape (λ), where the tableau T has entries from $\mathscr{X}$ and the tableau T' has entries from $\mathscr{U}$. The *content* of the bitableau $[T, T']$ is the pair of vectors (α, β) where α_i (resp. β_j) is the number of occurrences of x_i in T (resp. u_j in T'). With a bitableau $[T, T']$ of content (α, β), we associate the polynomial, denoted by $(T \mid T')$, obtained by taking the product of the inner products of each row of T with the corresponding row in T'. The polynomial $(T \mid T')$, which is in $P(\alpha, \beta)$, is called the *bideterminant* of the bitableau $[T, T']$, or simply, the bideterminant $(T \mid T')$.

EXAMPLE.

$$\left(\begin{array}{l|l} x_1x_2x_3 & u_1u_3u_4 \\ x_2x_3 & u_1u_2 \\ x_1 & u_3 \end{array}\right) = (x_1x_2x_3 \mid u_1u_3u_4)(x_2x_3 \mid u_1u_2)(x_1 \mid u_3).$$

As for inner products, the bideterminant $(T \mid T')$ is nonzero if and only if no letter is repeated in any row of T or T'. Moreover, we can suppose, up to a change of sign, that the entries in each row of T and T' in the bideterminant are increasing.

A bitableau $[T, T']$ is *standard* if both T and T' are standard. For example, the bitableau

$$\left[\begin{array}{lcl} x_1x_2x_3 & & u_1u_2u_4 \\ x_1x_3 & , & u_1u_3 \\ x_2 & & u_3 \end{array}\right]$$

is standard.

We can now state the main result of this section.

THEOREM (the straightening formula). *Suppose $[T, T']$ is a bitableau of shape (λ) and content (α, β). Then, its bideterminant $(T \mid T')$ is a linear combination, with integer coefficients, of bideterminants of standard tableaux of the same content and of the same or longer shape.*

EXAMPLE.

$$\begin{pmatrix} x_2 & u_1 \\ x_1 & u_2 \end{pmatrix} = \begin{pmatrix} x_1 & u_1 \\ x_2 & u_2 \end{pmatrix} - (x_1 x_2 \mid u_1 u_2).$$

COROLLARY. *The vector space $P(\alpha, \beta)$ is generated by the bideterminants of standard tableaux of content (α, β).*

Proof of corollary. We only need to observe that the monomial $(x_{i_1} \mid u_{j_1}) \cdots (x_{i_p} \mid u_{j_p})$ is the bideterminant of the bitableau:

$$\begin{bmatrix} x_{i_1} & & u_{j_1} \\ \vdots & , & \vdots \\ x_{i_p} & & u_{j_p} \end{bmatrix}.$$

To facilitate the proof of Theorem 1, we introduce the notion of a *shuffle product*. Let

$$(i_1, ..., i_p, l_1, ..., l_q)$$

be an increasing sequence of integers, and

$$A = (x_{i_1} \cdots x_{i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s}),$$

$$B = (x_{l_1} \cdots x_{l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t})$$

be two inner products. The *shuffle product AB supported by the variables $x_{i_1}, ..., x_{i_p}$, $x_{l_1}, ..., x_{l_q}$* is defined by

$$(\dot{x}_{i_1} \cdots \dot{x}_{i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})(\dot{x}_{l_1} \cdots \dot{x}_{l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t})$$

$$= \sum_{\sigma}{}' \operatorname{sgn}(\sigma)(x_{\sigma i_1} \cdots x_{\sigma i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})(x_{\sigma l_1} \cdots x_{\sigma l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t}),$$

where the summation is over all permutations σ of the set $\{i_1, ..., i_p, l_1, ..., l_q\}$ for which $\sigma i_1 < \cdots < \sigma i_p$ and $\sigma l_1 < \cdots < \sigma l_q$. This restricted summation is indicated by the notation $\sum'$. Another notational device is: A *dot* over a letter indicates that the letter is in the support of the shuffle product. The notion of a shuffle product supported by letters in $\mathscr{U}$ is similar.

EXAMPLE. The shuffle product $(x_1x_2x_3 \mid \mathbf{u})(x_3x_4x_1 \mid \mathbf{u}')$ supported by x_1, x_2 in the first term and x_3, x_4 in the second is given by

$$\begin{aligned}
&(\dot{x}_1\dot{x}_2x_3 \mid \mathbf{u})(\dot{x}_3\dot{x}_4x_1 \mid \mathbf{u}') \\
&= (x_1x_2x_3 \mid \mathbf{u})(x_3x_4x_1 \mid \mathbf{u}') - (x_1x_3x_3 \mid \mathbf{u})(x_2x_4x_1 \mid \mathbf{u}') + (x_1x_4x_3 \mid \mathbf{u})(x_2x_3x_1 \mid \mathbf{u}') \\
&\quad + (x_2x_3x_3 \mid \mathbf{u})(x_1x_4x_1 \mid \mathbf{u}') - (x_2x_4x_3 \mid \mathbf{u})(x_1x_3x_1 \mid \mathbf{u}') + (x_3x_4x_3 \mid \mathbf{u})(x_1x_2x_1 \mid \mathbf{u}').
\end{aligned}$$

Only two of the terms in the expansion are nonzero, and after an appropriate reordering, we have

$$(\dot{x}_1\dot{x}_2x_3 \mid \mathbf{u})(x_1\dot{x}_3\dot{x}_4 \mid \mathbf{u}') = (x_1x_2x_3 \mid \mathbf{u})(x_1x_3x_4 \mid \mathbf{u}') - (x_1x_3x_4 \mid \mathbf{u})(x_1x_2x_3 \mid \mathbf{u}').$$

Now, observe that, by definition,

$$(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}) = \begin{vmatrix} (x_{i_1} \mid u_{j_1}) \cdots (x_{i_1} \mid u_{j_p}) \\ \vdots \\ (x_{i_p} \mid u_{j_1}) \cdots (x_{i_p} \mid u_{j_p}) \end{vmatrix}.$$

We can expand the determinant by the first column to obtain the identity

$$(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}) = (\dot{x}_{i_1} \mid u_{j_1})(\dot{x}_{i_2} \cdots \dot{x}_{i_p} \mid u_{j_2} \cdots u_{j_p}).$$

Similarly, using Laplace's expansion, we have

$$(x_{i_1} \cdots x_{i_p} \mid u_{j_1} \cdots u_{j_p}) = (\dot{x}_{i_1} \cdots \dot{x}_{i_s} \mid u_{j_1} \cdots u_{j_s})(\dot{x}_{i_{s+1}} \cdots \dot{x}_{i_p} \mid u_{j_{s+1}} \cdots u_{j_p}).$$

These two identities are examples of the fact that, under certain assumptions, the shuffle product of two inner products is equal to an inner product of length greater than that of each of the two original inner products.

LEMMA. *Let* $(x_{i_1} \cdots x_{i_p}x_{i_{p+1}} \cdots x_{i_1} \mid u_{j_1} \cdots u_{j_s})$ *and* $(x_{l_1} \cdots x_{l_q}x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t})$ *be two inner products satisfying*: $i_1 < \cdots < i_p < l_1 < \cdots < l_q$, $j_1 < \cdots < j_s$, $m_1 < \cdots < m_t$, $s < p + q$, *and* $t < p + q$. *Then the shuffle product*

$$C = (\dot{x}_{i_1} \cdots \dot{x}_{i_p}x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})(\dot{x}_{l_1} \cdots \dot{x}_{l_q}x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t})$$

is a linear combination, with integer coefficients, of bideterminants of bitableaux of shape strictly longer than each of the partitions (s) *and* (t) *of the integers* s *and* t.

The *proof* is a computation with four steps. First, expand the shuffle product C:

$$\begin{aligned}
C = \sum_\sigma{}' \operatorname{sgn}(\sigma)&(x_{\sigma i_1} \cdots x_{\sigma i_p}x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})(x_{\sigma l_1} \cdots x_{\sigma l_q} x_{l_{q+1}} \\
&\cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t}).
\end{aligned}$$

Now apply Laplace's identity to the letters in U;

$$C = \sum_{\sigma}{}' \operatorname{sgn}(\sigma)(x_{\sigma i_1} \cdots x_{\sigma i_p} \mid \dot{u}_{j_1} \cdots \dot{u}_{j_p})(x_{i_{p+1}} \cdots x_{i_s} \mid \dot{u}_{j_{p+1}} \cdots \dot{u}_{j_s})$$
$$\times\ (x_{\sigma l_1} \cdots x_{\sigma l_q} \mid \bar{u}_{m_1} \cdots \bar{u}_{m_q})(x_{l_{q+1}} \cdots x_{l_t} \mid \bar{u}_{m_{q+1}} \cdots \bar{u}_{m_t}).$$

To distinguish between the two shuffle products, a bar instead of a dot is used in the second. We next group together the first and third factor:

$$C = \sum_{\sigma,\tau,\mu}{}' \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \operatorname{sgn}(\mu)[(x_{\sigma i_1} \cdots x_{\sigma i_p} \mid u_{\tau j_1} \cdots u_{\tau j_p})(x_{\sigma l_1} \cdots x_{\sigma l_q} \mid u_{\mu m_1} \cdots u_{\mu m_q})]$$
$$\times\ (x_{i_{p+1}} \cdots x_{i_s} \mid u_{\tau j_{p+1}} \cdots u_{\tau j_s})(x_{l_{q+1}} \cdots x_{l_t} \mid u_{\mu m_{q+1}} \cdots u_{\mu m_t}).$$

Finally, we apply Laplace's identity, this time on the letters $x_{i_1}, \ldots, x_{i_p}, x_{l_1}, \ldots, x_{l_q}$:

$$C = \sum_{\tau,\mu}{}' \operatorname{sgn}(\tau) \operatorname{sgn}(\mu)(x_{i_1} \cdots x_{i_p} x_{l_1} \cdots x_{l_q} \mid u_{\tau j_1} \cdots u_{\tau j_p} u_{\mu m_1} \cdots u_{\mu m_q})$$
$$\times\ (x_{i_{p+1}} \cdots x_{i_s} \mid u_{\tau j_{p+1}} \cdots u_{\tau j_s})(x_{l_{q+1}} \cdots x_{l_t} \mid u_{\mu m_{q+1}} \cdots u_{\mu m_t}).$$

Each term in this last expansion for C is a bideterminant with three rows, with the first row of length $p + q > s, t$. This concludes the proof of the lemma.

Since the summation in the shuffle product always includes the identity permutation, we can restate the previous lemma in the following equivalent form:

LEMMA. *Let* $(x_{i_1} \cdots x_{i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})$ *and* $(x_{l_1} \cdots x_{l_q} x_{l_{q+1}} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t})$ *be two inner products satisfying* $i_1 < \cdots < i_p < l_1 < \cdots < l_q$, $j_1 < \cdots < j_s$, $m_1 < \cdots < m_t$, $s < p + q$, *and* $t < p + q$. *Then,*

$$(x_{i_1} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})(x_{l_1} \cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t})$$
$$= - \sum_{\sigma, \sigma \neq \mathrm{id}}{}' \operatorname{sgn}(\sigma)(x_{\sigma i_1} \cdots x_{\sigma i_p} x_{i_{p+1}} \cdots x_{i_s} \mid u_{j_1} \cdots u_{j_s})(x_{\sigma l_1} \cdots x_{\sigma l_q} x_{l_{q+1}}$$
$$\cdots x_{l_t} \mid u_{m_1} \cdots u_{m_t}) + D,$$

where the summation is over all the nonidentical permutations σ *satisfying* $\sigma i_1 < \cdots < \sigma i_p$ *and* $\sigma l_1 < \cdots < \sigma l_q$, *and the term* D *is a linear combination with integer coefficients of bideterminants of bitableaux of shape strictly longer than* (s) *and* (t).

Remarking that all we have done remains valid if we exchange the roles of the alphabets $\mathscr{X}$ and $\mathscr{U}$, we are now ready to prove Theorem 1.

Proof of Theorem 1. We begin by defining a total order on bitableaux of the same shape. Let $[T, T']$ be a bitableau of form $(\lambda) = (\lambda_1, \ldots, \lambda_p)$. The entry in T (resp. T') on row s and column t is denoted by $x_{i(s,t)}$ (resp. $u_{j(s,t)}$). With the bitableau $[T, T']$, we associate the sequence $(i(1, 1), \ldots, i(1, \lambda_1), i(2, 1), \ldots, i(2, \lambda_2), \ldots, i(p, 1), \ldots, i(p, \lambda_p), j(1, 1), \ldots, j(1, \lambda_1), j(2, 1), \ldots, j(2, \lambda_2), \ldots, j(p, 1), \ldots, j(p, \lambda_p))$, which is the sequence obtained by reading off the bitableau row by row. The bitableaux are now ordered according to the lexicographic order on their associated sequences.

Now, suppose the theorem is false. Let (λ) be the longest shape with a bitableau not satisfying the theorem. Among the bitableaux of shape (λ), let $[T, T']$ be the smallest bitableau not satisfying the theorem:

$$[T, T'] = \begin{bmatrix} x_{i(1,1)} \cdots x_{i(1,\lambda_1)} & u_{j(1,1)} \cdots u_{j(1,\lambda_1)} \\ \vdots \qquad \vdots & , \quad \vdots \qquad \vdots \\ x_{i(p,1)} \cdots x_{i(p,\lambda_p)} & u_{j(p,1)} \cdots u_{j(p,\lambda_p)} \end{bmatrix}.$$

Suppose that $[T_1, T_1']$ is obtained from $[T, T']$ by putting each row in increasing order. Then, $[T_1, T_1']$ has a lexicographically smaller associated sequence than $[T, T']$. But $(T_1 \mid T_1') = \pm(T \mid T')$, and hence, if $[T, T']$ is a counterexample, so is $[T_1, T_1']$. We conclude that all the rows in $[T, T']$ are in increasing order.

Clearly, $[T, T']$ is not standard; let us suppose that T is nonstandard. Then, there exist integers l and m, $1 \leqslant l \leqslant p$, $1 \leqslant m \leqslant \lambda_l$, such that $i(l, m) > i(l + 1, m)$. That is, we have the following situation:

$$\begin{array}{llll} x_{i(l,1)} \cdots x_{i(l,m-1)} & x_{i(l,m)} & x_{i(l,m+1)} \cdots \cdots x_{i(l,\lambda_l)}, \\ x_{i(l+1,1)} \cdots x_{i(l+1,m-1)} & x_{i(l+1,m)} & x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} . \end{array}$$

We call such a situation a *violation*.

In the bideterminant $(T \mid T')$, consider the shuffle product of the two inner products corresponding to rows l and $l + 1$, which support the letters $x_{i(l+1,1)}, \ldots, x_{i(l+1,m)}, x_{i(l,m)}, \ldots, x_{i(l,\lambda_l)}$. Since the support contains $\lambda_l + 1$ letters, and the length of each of the inner products is at most λ_l, we can apply the lemma to obtain

$$(T \mid T') = \sum_{\sigma:\sigma\neq\mathrm{id}}' \pm (T_\sigma \mid T') + D,$$

where D is a linear combination of bideterminants of shape greater than (λ). By our choice of (λ), D is also a linear combination of bideterminants of standard tableaux of shape greater than (λ).

Now, each tableau T_σ differs from T only in rows l and $l + 1$:

$$\begin{array}{llll} x_{i(l,1)} \cdots x_{i(l,m-1)} & x_{\sigma i(l,m)} & x_{\sigma i(l,m+1)} \cdots \cdots x_{\sigma i(l,\lambda_l)} \\ x_{\sigma i(l+1,1)} \cdots x_{\sigma i(l,m-1)} & x_{\sigma i(l+1,m)} & x_{i(l+1,m+1)} \cdots x_{i(l+1,\lambda_{l+1})} \end{array}$$

In the tableau T, however, we have the inequalities

$$\begin{array}{rcl} & & i(l,m) < i(l,m+1) < \cdots < i(l,\lambda_l) \\ & & \vee \\ i(l+1,1) < \cdots < i(l+1,m-1) < i(l+1,m) & & \end{array}$$

For any nonidentical permutation σ in the shuffle product, the index $\sigma i(l, m)$ must equal one of the indices $i(l+1, 1),\ldots, i(l+1, m)$, in particular, $\sigma i(l, m) < i(l, m)$. Thus, the tableau T_σ has a lexicographically smaller associated sequence than T. By our choice of $[T, T']$, however, each of the tableaux $[T_\sigma, T']$ satisfies the theorem, and hence, by substitution, we can write $(T \mid T')$ as a sum of bideterminants of standard tableaux of shape equal to or longer than (λ). This contradicts our initial assumption.

It remains to observe that if T were standard, then T' would have to be nonstandard; the same reasoning can then be applied to T' to yield a contradiction. This concludes the proof of the theorem.

The proof contains implicitly an algorithm for expressing any bitableau as a linear combination of standard bitableaux by successive corrections of violations. This is inefficient for practical computations, as the number of bitableaux introduced during a correction is, in general, very large.

As an exercise, apply the algorithm to obtain the following identity (only the subscripts are shown):

$$\begin{pmatrix} 23 & 12 \\ 14 & 13 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 12 \\ 24 & 13 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 12 & 12 \\ 24 & 13 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 12 & 12 \\ 23 & 13 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 123 & 123 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}.$$

4. The Basis Theorem

As we have seen, the standard bideterminants (i.e., the bideterminants of standard tableaux) of content (α, β) spans the vector space $P(\alpha, \beta)$. In this section, using the technique of set polarization operators, we show that, in fact, they form a basis.

We augment the alphabets $\mathscr{X}$ and $\mathscr{U}$ by adding new letters from the sets $\mathscr{S}$ and $\mathscr{T}$, respectively. The sets $\mathscr{S}$ and $\mathscr{T}$ are supposed finite, but large enough that the ensuing constructions can be performed. This enlarges the algebra P, even though the vector space $P(\alpha, \beta)$ remains unchanged.

Let x_i, u_i, s_j, and t_j be letters from the alphabets $\mathscr{X}$, $\mathscr{U}$, $\mathscr{S}$, and $\mathscr{T}$. The *set polarization operators* $D^l(s_j, x_i)$ are defined as follows: Let $M = (x_{i_1} \mid u_{m_1}) \cdots (x_{i_p} \mid u_{m_p})$ be a monomial of content (α, β). Then,

(i) if $\alpha_i < l$, we set $D^l(s_j, x_i)M = 0$,

(ii) if $\alpha_i \geqslant l$, we set $D^l(s_j, x_i)M = \sum_r M_r$, where $M_1, \ldots, M_r, \ldots, M_{\binom{\alpha_i}{l}}$

are all the $\binom{\alpha_i}{l}$ distinct monomials obtained from M by replacing each subset of l letters x_i by l letters s_j. (In particular, each of the monomials M_r contains the letter x_i $(\alpha_i - l)$ times and the letter s_j l times.) The operator $D^l(s_j, x_i)$ is now extended to all of $P(\alpha, \beta)$ by linearity.

The operator $D^0(s_j, x_i)$ is the identity operator, and for $1 \leqslant l \leqslant \alpha_i$, the set polarization operator $D^l(s_j, x_i)$ maps a polynomial in $P(\alpha, \beta)$ to a polynomial lying outside $P(\alpha, \beta)$.

For bideterminants, the set polarization operators act in the following simple way.

Lemma. *Let $(T \mid T')$ be a bideterminant of content (α, β). Then,*

(i) *if $\alpha_i < l$, $D^l(s_j, x_i)(T \mid T') = 0$,*

(ii) *if $\alpha_i \geqslant l$, $D^l(s_j, x_i)(T \mid T') = \sum_r (T_r \mid T')$, where $T_1, \ldots, T_r, \ldots, T_{\binom{\alpha_i}{l}}$*

are all the distinct $\binom{\alpha_i}{l}$ tableaux obtained from T by replacing each subset of l letters x_i by l letters s_j.

Proof. Expand $(T \mid T')$ into a sum of monomials M^t of content (α, β):

$$(T \mid T') = \sum_t M^t.$$

Now, if $\alpha_i < l$, $D^l(s_j, x_i)M^t = 0$ for all the monomials M^t. Hence (i). Now, suppose that $\alpha_i \geqslant l$. Then,

$$D^l(s_j, x_i)(T \mid T') = \sum_t D^l(s_j, x_i)\, M^t = \sum_t \sum_r M_r{}^t,$$

where $M_r{}^t$ are the $\binom{\alpha_i}{l}$ monomials obtained from M^t according to rule (ii). Interchanging the order of summation, we have

$$D^l(s_j, x_i)(T \mid T') = \sum_r \sum_t M_r{}^t.$$

But,

$$\sum_t M_r{}^t = (T_r \mid T'),$$

where the same set of l letters x_i are replaced by l letters s_j on both sides of the equation. Hence,

$$D^l(s_j, x_i)(T \mid T') = \sum_r (T_r \mid T').$$

This proves the lemma.

The set polarization operators $D^l(t_j, u_i)$ are defined in an analogous manner, and the analog of the previous lemma is true for these operators.

EXAMPLES

$$D^2(s_1, x_2)\left(\begin{matrix} x_1x_2x_3 \\ x_1x_2 \\ x_2 \end{matrix}\;\middle|\; T'\right) = \left(\begin{matrix} x_1s_1x_3 \\ x_1s_1 \\ x_2 \end{matrix}\;\middle|\; T'\right) + \left(\begin{matrix} x_1s_1x_3 \\ x_1x_2 \\ s_1 \end{matrix}\;\middle|\; T'\right) + \left(\begin{matrix} x_1x_2x_3 \\ x_1s_1 \\ s_1 \end{matrix}\;\middle|\; T'\right),$$

while

$$D^2(s_1, x_3)\left(\begin{matrix} x_1x_2x_3 \\ x_1x_2 \\ x_2 \end{matrix}\;\middle|\; T'\right) = 0.$$

Note that since the alphabets $\mathscr{X}$, $\mathscr{U}$, $\mathscr{S}$, and $\mathscr{T}$ are disjoint, the set polarization operators commute.

The set polarization operators are the building blocks of the *Capelli operator*, which is defined for each bitableau $[T, T']$ of shape (λ) as follows: Let $\alpha_i(q)$ (resp. $\beta_j(q)$) be the number of occurrences of x_j (resp. u_j) in the qth column of T (resp. T'). The Capelli operator is defined by the following fomula:

$$C(T, T') = \prod_{1 \leqslant q \leqslant \lambda_1} \Big[\prod_{1 \leqslant i \leqslant n} D^{\alpha_i(q)}(s_q, x_i) \Big] \Big[\prod_{1 \leqslant j \leqslant k} D^{\beta_l(q)}(t_q, u_j) \Big].$$

EXAMPLE. Suppose

$$[T, T'] = \begin{bmatrix} x_1x_2x_3 & u_1u_2u_3 \\ x_1x_2 & , \; u_1u_3 \\ x_2 & u_1 \end{bmatrix}.$$

Then

$$\begin{aligned} C(T, T') = {} & D^2(s_1, x_1)\, D^1(s_1, x_2)\, D^2(s_2, x_2)\, D^1(s_3, x_3) \\ & \times D^3(t_1, u_1)\, D^1(t_2, u_2)\, D^1(t_2, u_3)\, D^1(t_3, u_3). \end{aligned}$$

We now impose a new total order on bitableaux of the same shape. Associate with each bitableau the sequence formed by reading off the indices down each column, successively, first in T and then in T'. The bitableaux are then ordered according to the lexicographic order of their associated column sequences.

EXAMPLE. For the bitableau in the preceding example, the associated column sequence is

$$(1, 1, 2, 2, 2, 3, 1, 1, 1, 2, 3, 3).$$

If $[T, T']$ is standard, the associated column sequence can be written

$$(1^{\alpha_1(1)} \cdots n^{\alpha_n(1)} 2^{\alpha_2(2)} \cdots n^{\alpha_n(2)} \cdots n^{\alpha_n(\lambda_1)} 1^{\beta_1(1)} \cdots k^{\beta_k(1)} \cdots k^{\beta_k(\lambda_1)}).$$

We have used the fact that for the bideterminant $(T \mid T')$ to be nonzero, we must have $\alpha_i(q) = \beta_i(q) = 0$ for $i < q$.

THEOREM 2.1. *Let $(T \mid T')$ and $(V \mid V')$ be two standard bideterminants of shape (λ) and (μ) with the same content. Then,*

(i) $C(T, T')(T \mid T') \neq 0$;

(ii) *if (μ) is longer than (λ), then $C(T, T')(V \mid V') = 0$;*

(iii) *if $(\lambda) = (\mu)$, and if $C(T, T')(V \mid V') \neq 0$, then $[V, V']$ is smaller than $[T, T']$ in the lexicographic order of their associated column sequences.*

Proof. (i) We calculate $C(T, T')(T \mid T')$ explicitly; we have

$$(T \mid T') = \left[\begin{array}{r|c} \alpha_1(1)\left\{\begin{matrix} x_1 \cdots \\ \vdots \\ x_1 \end{matrix}\right. & \\ \vdots & \\ \alpha_l(1)\left\{\begin{matrix} x_l \cdots \\ \vdots \\ x_l \cdots \end{matrix}\right. & T' \\ \vdots & \\ \alpha_n(a)\left\{\begin{matrix} x_n \cdots \\ \vdots \\ x_n \cdots \end{matrix}\right. & \end{array}\right].$$

Now, the letter x_1 can only be found in the first column of T. Hence, $D^{\alpha_1(1)}(s_1, x_i)(T \mid T')$ consists of a single term, obtained by substituting s_1 for all the x_1 in T.

Assume that $\prod_{1 \leqslant i \leqslant l-1} D^{\alpha_1(1)}(s_1, x_i)(T \mid T')$ consists of a single term, obtained by substituting all the letters x_i, $1 \leqslant i \leqslant l-1$, in the first column of T by the letter s_1. Then,

$$\prod_{1 \leqslant i \leqslant l} D^{\alpha_i(1)}(s_1, x_i)(T \mid T') = D^{\alpha_l(1)}(s_1, x_l) \prod_{1 \leqslant i \leqslant l-1} D^{\alpha_i(1)}(s_1, x_i)(T \mid T')$$

$$= D^{\alpha_l(1)}(s_1, x_l) \left[\begin{array}{r|c} \sum\limits_{1 \leqslant i \leqslant l-1} \alpha_i(1)\left\{\begin{matrix} s_1 \cdots \\ \vdots \\ s_1 \cdots \end{matrix}\right. & \\ \alpha_l(1)\left\{\begin{matrix} x_l \cdots \\ \vdots \\ x_l \cdots \end{matrix}\right. & T' \\ \vdots & \\ \alpha_n(1)\left\{\begin{matrix} x_n \cdots \\ \vdots \\ x_n \cdots \end{matrix}\right. & \end{array}\right].$$

Since tableau T is standard, any occurrence of x_l in other than the first column must be in the first $\sum_{1\leqslant i\leqslant l-1} \alpha_i(1)$ rows. If any of these x_l are chosen for substitution during the polarization, the letter s_1 would be repeated within a row, and the resulting bideterminant would be zero. Hence, the only nonzero term in the above expression is the term obtained by substituting s_1 for all the $\alpha_l(1)$ letters x_l in the first column of T. By induction, we have shown that the expression $\prod_{1\leqslant i\leqslant n} D^{\alpha_i(1)}(s_1, x_i)(T \mid T')$ consists of a single nonzero term, obtained by substituting s_1 for all the letters in the first column of T.

Repeating this argument for the other columns, we can easily see that $C(T, T')(T \mid T')$ is obtained by substituting s_q (resp. t_q) for all the letters in the qth column of T (resp. T'); that is,

$$C(T, T')(T \mid T') = \left(\begin{array}{cccc|cccc} s_1 & s_2 & \cdots & s_{\lambda_1} & t_1 & t_2 & \cdots & t_{\lambda_1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & & & s_{\lambda_1} & & & & t_{\lambda_1} \\ & & & \cdot & & & & \cdot \\ & & \cdot & & & & \cdot & \\ & \cdot & & & & \cdot & & \\ & s_2 & & & & t_2 & & \\ s_1 & & & & t_1 & & & \end{array}\right).$$

(ii) The expression $C(T, T')(V \mid V')$ consists of a sum of bideterminants of the same shape (μ). If it is nonzero, then one of the bideterminants, say, $(W \mid W')$, is nonzero. The content of $(W \mid W')$ is the same as that of $(U \mid U') = C(T, T')(T \mid T')$; that is, for $1 \leqslant l \leqslant \lambda_1$, the letters s_l and t_l occur in $(W \mid W')$ $\tilde{\lambda}_l$ times, where

$$\tilde{\lambda}_l = \sum_{1\leqslant i\leqslant n} \alpha_i(l)$$

$$= \text{the height of the } l\text{th column.}$$

We are required to show that (μ) is shorter than (λ). If $(\mu) \neq (\lambda)$, let m be the smallest integer such that $\lambda_m \neq \mu_m$. If $m = 1$, it must be the case that $\mu_1 < \lambda_1$, for the first row of W contains μ_1 *distinct* letters chosen from the set $\{s_1, \ldots, s_{\lambda_1}\}$.

Now, suppose that $m \geqslant 2$. We claim that: For $1 \leqslant i \leqslant m - 1$, the contents of the ith row in U and in W are identical.

The proof is by induction on i. If $i = 1$, then $\mu_1 = \lambda_1$; by our preceding observations, the first row in both W and U consists of the set $\{s_1, \ldots, s_{\lambda_1}\}$ arranged in some order. Now assume that the proposition is true up to the ith row. The letters s_l, for $l \geqslant \lambda_i + 1$, have all been used in the first $i - 1$ rows in U, hence in W. For the ith row in W, we have to choose μ_i distinct letters from $\{s_1, \ldots, s_{\lambda_i}\}$. But $\lambda_i = \mu_i$; hence the contents of the ith row in U and W are identical.

Now, consider the mth row. As the first $m - 1$ rows are identical, the mth row in W contains μ_m distinct letters from the set $\{s_1, \ldots, s_{\lambda_m}\}$. Since $\mu_m \neq \lambda_m$, we must have $\mu_m < \lambda_m$.

(iii) We can now suppose that $(\mu) = (\lambda)$, and $C(T, T')(V \mid V') \neq 0$. Recall that the associated column sequence of $(T \mid T')$ is

$$(1^{\alpha_1(1)} \cdots n^{\alpha_n(1)} \cdots n^{\alpha_n(\lambda_1)} 1^{\beta_1(1)} \cdots k^{\beta_k(1)} \cdots k^{\beta_k(\lambda_1)}).$$

We shall denote by $\gamma_i(q)$ (resp. $\delta_j(q)$) the number of occurrences of x_i (resp. u_i) in the qth column of V (resp. V'). The associated column sequence of $(V \mid V')$ is

$$(1^{\gamma_1(1)} \cdots n^{\gamma_n(1)} \cdots n^{\gamma_n(\lambda_1)} 1^{\delta_1(1)} \cdots k^{\delta_k(3)} \cdots k^{\delta_k(\lambda_1)}).$$

Suppose now that $[T, T']$ and $[V, V']$ differ in the left tableau; the reasoning is similar if the difference lies in the right tableau.

Let p be the first column where T and V differ, and in the pth column, let x_l be the smallest index that is different. That is, we have

for $1 \leqslant i \leqslant n$ and $1 \leqslant q \leqslant p-1$, $\alpha_i(q) = \gamma_i(q)$;
for $1 \leqslant i \leqslant l-1$, $\alpha_i(p) = \delta_i(p)$;
but $\alpha_l(p) \neq \gamma_l(p)$.

Now, assume that $(V \mid V')$ is lexicographically *greater* than $(T \mid T')$; that is, $\alpha_l(p) > \gamma_l(p)$. Consider the action of the Capelli operator $C(T, T')$ on $(V \mid V')$. The polarizations $D^{\alpha_1(1)}(s_1, x_1), D^{\alpha_2(1)}(s_1, x_2), \ldots, D^{\alpha_{l-1}(p)}(s_p, x_{l-1})$ act on $(V \mid V')$ exactly as they do on $(T \mid T')$. At this instant, the expression

$$\prod_{1 \leqslant i \leqslant l-1} D^{\alpha_i(p)}(s_p, x_i) \cdot \prod_{1 \leqslant \alpha \leqslant p-1} \left[\prod_{1 \leqslant i \leqslant n} D^{\alpha_i(q)}(s_q, x_i)\right](V \mid V')$$

is a bideterminant of the form

$$\left[\begin{array}{rcccc|c|}
\sum\limits_{1 \leqslant i \leqslant l-1} \alpha_i(p) \left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c} s_1 \\ \vdots \\ s_1 \end{array} & \begin{array}{c} s_2 \\ \vdots \\ s_2 \end{array} & \begin{array}{c} s_{p-1} \\ \vdots \\ s_{p-1} \end{array} & \begin{array}{c} s_p \\ \vdots \\ s_p \end{array} & \\
\gamma_l(p) \left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c} s_1 \\ \vdots \\ s_1 \end{array} & \begin{array}{c} s_2 \\ \vdots \\ s_2 \end{array} & \begin{array}{c} s_{p-1} \\ \vdots \\ s_{p-1} \end{array} & \begin{array}{c} x_l \\ \vdots \\ x_l \end{array} & V' \\
 & \vdots & \vdots & \vdots & \vdots & \\
 & s_1 & s_2 & s_{p-1} & & \\
 & \vdots & \vdots & \ddots & & \\
 & s_1 & s_2 & & &
\end{array}\right].$$

That is, the first $p-1$ columns are replaced by the appropriate letters s_q; in the pth column, the first $\sum_{1 \leqslant i \leqslant l-1} \alpha_i(p)$ letters are replaced by s_p, and the remainder of the tableau is unchanged.

Since $\alpha_l(p) > \gamma_l(p)$, and V is standard, any choice of $\gamma_l(p)$ letters x_l must involve a letter x_l lying in the first $\sum_{1 \leqslant i \leqslant l-1} a_i(p)$ rows of V. The set polarization operator $D^{\alpha_i(p)}(s_p, x_l)$ substitutes s_p for this particular x_l. The resulting bideterminant is zero, since there are two letters s_p in a single row. This contradicts the assumption; we have therefore proved that $(V \mid V')$ is lexicographically smaller than $(T \mid T')$.

With Theorem 2.1 proved, we can now proceed to the main result of this section.

THEOREM 2.2. *The standard bideterminant of content* (α, β) *form a basis of the vector space* $P(\alpha, \beta)$.

Proof. By Theorem 1, the standard bideterminants span the vector space $P(\alpha, \beta)$. Suppose we have a nontrivial linear relation between these bideterminants. We can write this linear relation as follows:

$$a(T \mid T') + A + B = 0,$$

where a is a nonzero scalar in K, (λ) is the shortest shape occurring in the linear relation, $(T \mid T')$ is the bitableau of shape (λ) with the lexicographically smallest associated *column* sequence, A is the linear combination of the remaining tableaux of shape (λ), and B is the linear combination of the remainder of the bitableaux, which are necessarily of shape longer than (λ).

Applying the Capelli operator $C(T, T')$ to the relation we have, by Theorem 2.1,

$$C(T, T')A = 0,$$

$$C(T, T')B = 0,$$

but

$$C(T, T')(T \mid T') \neq 0.$$

This implies $aC(T, T')(T \mid T') = 0$, which is a contradiction.

The proof of this theorem contains another algorithm for expressing any bitableau as a linear combination of standard bitableaux. Suppose

$$(T \mid T') = \sum_i a_i(T_i \mid T_i')$$

is the unique decomposition of $(T \mid T')$ into standard bitableaux, written so that if $i < j$, then either T_i is of shape shorter than T_j, or T_j have the same shape, and T_i has a lexicographically smaller associated column sequence than T_j. The coefficients a_i are called the ***straightening coefficients***. They can be computed by applying the Capelli operators $C(T_j, T_j')$ to both sides of the linear relation; by Theorem 2.1, we obtain a triangular array of equations between bideterminants with entries from the alphabets $\mathscr{S}$ and $\mathscr{T}$. From this, we can extract a triangular system of linear equations for the coefficients a_i, which can then be solved.

EXAMPLE. Consider the bideterminant (where, for simplicity, all but the subscripts are suppressed):

$$\begin{pmatrix} 23 & 12 \\ 14 & 13 \\ 2 & 1 \end{pmatrix}.$$

The standard bitableaux of the same or longer shape of the same content are

$$\begin{bmatrix} 13 & 12 \\ 24 \;, & 13 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 12 & 12 \\ 24 & 13 \\ 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 12 & 12 \\ 23 & 13 \\ 4 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 134 & 123 \\ 2 \;, & 1 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 124 & 123 \\ 2 \;, & 1 \\ 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 123 & 123 \\ 2 \;, & 1 \\ 4 & 1 \end{bmatrix}.$$

Let $a_1, \ldots, a_6$ be the corresponding straightening coefficients. We obtain, through the Capelli operators, the equations:

$$\begin{array}{llllllll} a_1 & & & & & & = & 1 \\ & a_2 & & & & & = & -1 \\ -a_1 & & +a_3 & & & & = & 0 \\ & & & a_4 & & & = & 0 \\ & & & & a_5 & & = & 0 \\ & & a_3 & & & +a_6 & = & 0 \end{array}$$

Therefore, we obtain

$$\begin{pmatrix} 23 & 12 \\ 14 & 13 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 12 \\ 24 & 13 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 12 & 12 \\ 24 & 13 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 12 & 12 \\ 23 & 13 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 123 & 123 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}.$$

5. INVARIANT THEORY

Classical invariant theory is concerned with the behavior of forms under the action of linear transformations. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ be a dual basis for the vector space V_d (of dimension d). A *form* $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ on m vectors in V_d is a polynomial in the md scalar products of the m vectors $\mathbf{x}_i$ with the d covectors $\mathbf{u}_i$ in the dual basis. More pedantically, consider the polynomial algebra $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$ in the indeterminates x_{ir}, u_{jr}, s_{pr}, t_{qr}, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$, $1 \leqslant p$, $1 \leqslant q$, and $1 \leqslant r \leqslant d$. In this algebra, we distinguish the following polynomials:

$$\langle x_i \mid u_j \rangle = \sum_{1 \leqslant r \leqslant d} x_{ir} u_{jr} ,$$

$$\langle s_p \mid u_j \rangle = \sum_{1 \leqslant r \leqslant d} s_{pr} u_{jr} ,$$

$$\langle x_i \mid t_q \rangle = \sum_{1 \leqslant r \leqslant d} x_{ir} t_{qr} ,$$

$$\langle s_p \mid t_q \rangle = \sum_{1 \leqslant r \leqslant d} s_{pr} t_{qr} .$$

Let $\hat{P}$ be the subalgebra of $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$ generated by these polynomials; $\hat{P}$ is called the *algebra of forms.*

There exists a homomorphism ϕ from P (constructed as in the previous section from the alphabets $\mathscr{X}$, $\mathscr{U}$, $\mathscr{S}$, and $\mathscr{T}$) to $\hat{P}$ defined by

$$\phi: \quad \begin{aligned} (x_i \mid u_j) &\mapsto \langle x_i \mid u_j \rangle, \\ (s_p \mid u_j) &\mapsto \langle s_p \mid u_j \rangle, \\ (x_i \mid t_q) &\mapsto \langle x_i \mid t_q \rangle, \\ (s_p \mid t_q) &\mapsto \langle s_p \mid t_q \rangle. \end{aligned}$$

Consider a monomial $m = (x_{i_1} \mid u_{j_1}) \cdots (x_{i_a} \mid u_{j_a})$ in P. Its image in $\hat{P}$ under ϕ is given by

$$\phi m = \sum_f x_{i_1 f_1} u_{j_1 f_1} \cdots x_{i_a f_a} u_{j_a f_a} ,$$

where the summation is over the set of all functions $f\colon i \mapsto f_i$ from $\{1,..., a\}$ to $\{1,..., d\}$. We shall use the simpler notation

$$\phi m = \sum_f m_f .$$

The restriction of the homomorphism ϕ to $P(\alpha, \beta)$ is called the *Pascal homomorphism.*

THEOREM 3.1 (the second fundamental theorem of invariant theory). *The kernel of the Pascal homomorphism is the subspace of $P(\alpha, \beta)$ spanned by the standard bideterminants of shape strictly longer than (d).*

Some preliminary observations are in order.

In the same fashion as for P, we define set polarization operators for $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$. Let $v = x_{ir_1} \cdots x_{ir_a}$ be a monomial containing only variables of the form

x_{ir}, where i is fixed and r is arbitrary. Suppose p is a positive integer, and E a subset of $\{1,..., a\}$. Then, $V^{E,p}$ is the monomial obtained from v by replacing the variable x_{ir_b} by the variable s_{pr_b} whenever $b \in E$. Now, for a given positive integer l, the *set polarization operator* $\hat{D}^l(s_p, x_i)$ acts on the monomial v as follows:

$$\hat{D}^l(s_p, x_i)v = \sum_E v^{E,p},$$

where the summation is over all the l-subsets of $\{1,..., a\}$. Consider now an arbitrary monomial w. We can write w as the product of two monomials w' and w'', where w' is the product of all the variables in w of the form x_{ir}, and w'' is the product of the remaining variables. Then, we set

$$\hat{D}^l(s_p, x_i)w = (\hat{D}^l(s_p, x_i)w')w''.$$

The operator $\hat{D}^l(s_p, x_i)$ is extended to all of $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$ by linearity.

The operators $\hat{D}^l(t_q, u_j)$ are defined analogously. It is clear that, as in the case of the operators D, the operators $\hat{D}$ commute.

We have the following identity:

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$$\hat{D}^l(s_p, x_i)\phi = \phi D^l(s_p, x_i).$$

Proof. It suffices to verify the identity for monomials of the form

$$m = (x_i \mid u_{j_1}) \cdots (x_i \mid u_{j_a}),$$

where all the letters x have index i, Let $m^{E,p}$ denote the monomial obtained from m by replacing each variable $(x_i \mid u_{j_b})$ by $(s_p \mid u_{j_b})$ whenever $b \in E$. Then, we observe (notation as earlier):

$$(m^{E,p})_f = (m_f)^{E,p}.$$

We can now finish the proof through the following computation:

$$\begin{aligned}
\phi D^l(s_p, x_i)m &= \phi \sum_E m^{E,p} = \sum_E \phi m^{E,p} \\
&= \sum_E \sum_f (m^{E,p})_f = \sum_E \sum_f (m_f)^{E,p} = \sum_f \hat{D}^l(s_p, x_i)\, m_f \\
&= \hat{D}^l(s_p, x_i) \left(\sum_f m_f\right) = \hat{D}^l(s_p, x_i)\, \phi m.
\end{aligned}$$

Consider now a bitableau $[T, T']$. The *Capelli operator* $\hat{C}(T, T')$ are defined on $K[\mathbf{x}, \mathbf{s}, \mathbf{u}, \mathbf{t}]$ by mimicking the definition of $C(T, T')$ with $\hat{D}$ instead of D. The lemma yields as a *corollary* the identity

$$\hat{C}(T, T')\phi = \phi C(T, T').$$

With these tools in hand, we can begin the proof of the theorem.

The image under ϕ of an inner product $(x_{i_1} \cdots x_{i_l} \mid u_{j_1} \cdots u_{j_l})$ is the determinant

$$\begin{vmatrix} \langle x_{i_1} \mid x_{j_1} \rangle & \cdots & \langle x_{i_1} \mid u_{j_l} \rangle \\ \vdots & & \vdots \\ \langle x_{i_l} \mid u_{j_1} \rangle & \cdots & \langle x_{i_l} \mid u_{j_l} \rangle \end{vmatrix}.$$

This is the determinant of the matrix

$$\begin{bmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & & \vdots \\ x_{i_l,1} & \cdots & x_{i_l,d} \end{bmatrix} \cdot \begin{bmatrix} u_{j_1,1} & \cdots & u_{j_l,1} \\ \vdots & & \vdots \\ u_{j_1,d} & \cdots & u_{j_l,d} \end{bmatrix}.$$

As the variables x_{ir}, u_{jr} are algebraically independent, the above matrix is of maximum possible rank; that is to say, its determinant is zero iff $l > d$.

Now, consider a bideterminant $(T \mid T')$. It is a product of inner products of lengths $\lambda_1, \ldots, \lambda_l$. As P (being a subring of $K[\mathbf{x}, \mathbf{u}, \mathbf{s}, \mathbf{t}]$) is an integral domain, $\phi(T \mid T')$ is zero iff one of its constituent inner products has zero image. This happens iff $\lambda_1 > d$, or (λ) is strictly longer than (d).

Finally, consider an element M in the kernel of ϕ. Using the straightening formula, write M as a linear combination of standard bideterminants:

$$M = a(T \mid T') + N,$$

where (λ) is the shortest shape occurring in the expansion, and, of all the bideterminants of shape (λ) occurring in the expansion, $(T \mid T')$ is the one with the lexicographically smallest column sequence. Applying the Capelli operator $C(T, T')$, and observing that, by Theorem 2.1, $C(T, T')N = 0$, we obtain

$$C(T, T')M = aC(T, T')(T \mid T').$$

Applying ϕ and using the identity in the previous corollary, we have

$$\begin{aligned} a\phi C(T, T')(T \mid T') &= \phi C(T, T')M \\ &= \hat{C}(T, T')\phi M. \end{aligned}$$

But $\phi M = 0$; hence, $C(T, T')(T \mid T')$ must be a bideterminant of shape strictly longer than (d). As $C(T, T')(T \mid T')$ and $(T \mid T')$ have the same shape, we conclude that (λ) is strictly longer than (d). But we have chosen (λ) to be the shortest shape occurring in the expansion of M. Therefore, M is a linear combination of standard bideterminants of shape strictly longer than (d). This concludes the proof of the theorem.

Consider the algebra P, defined on the alphabets $\mathscr{X} = \{x_1, ..., x_n\}$ and $\mathscr{U} = \{u_1, ..., u_d\}$. In P, all the bideterminants of shape strictly longer than (d) are identically zero; hence, by Theorem 3.1, ϕ is an injection. This allows us to transfer questions about forms to questions about elements of P.

Let L be a linear transformation from the vector space V_d to itself. If F is a form on V_d, L acts on F by

$$LF(\mathbf{x}_1, ..., \mathbf{x}_m) = F(L\mathbf{x}_1, ..., L\mathbf{x}_m).$$

A form is *invariant* if, for all invertible linear transformations L, there exists a scalar $a(L)$ such that $LF = a(L)F$.

Transferring to the algebra P, a *form* $F(\mathbf{x}_{i_1}, ..., \mathbf{x}_{i_m})$ is a polynomial in P in the variables $(x_i \mid u_j)$, $i \in \{i_1, ..., i_m\}$, $1 \leqslant j \leqslant d$. An invertible linear transformation L, given by an invertible $d \times d$ square matrix (l_{jk}) acts upon P as an algebra homomorphism as follows:

$$L(x_i \mid u_j) = \sum_{1 \leqslant k \leqslant d} l_{jk}(x_i \mid u_k), \qquad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant d.$$

A form F is *invariant* if, for all invertible linear transformations L, there exists a scalar $a(L)$ such that $LF = a(L)F$.

An example of an invariant form is the inner product $(x_{i_1} \cdots d_{i_d} \mid u_1 \cdots u_d)$; it is the determinant $|(x_{i_j} \mid u_k)|_{1 \leqslant j,k \leqslant d}$, and in this case $a(L) = \det(l_{jk})$. Similarly, the bideterminants of shape $(d, ..., d)$ are invariant forms; these bideterminants are called *rectangular*. For a rectangular bideterminant with g rows, $a(L) = (\det(l_{jk}))^g$. Note that since any bideterminant of shape longer than (d) is zero, a rectangular bideterminant is a linear combination of standard rectangular bideterminants (with the same number of rows).

These examples are in fact paradigmatic.

THEOREM 3.2 (the first fundamental theorem of invariant theory). *Over an infinite field K, a form in P is invariant iff it is a linear combination of standard rectangular bideterminants, all of which has the same shape* $(d, ..., d)$.

First proof. Suppose F is invariant. Using the straightening formula, we can express F as a linear combination of standard bideterminants:

$$F = \sum_s \alpha_s(T_s \mid T_s').$$

We shall probe F with appropriate linear transformations.

Consider first the linear transformation L defined by

$$L(x_i \mid u_j) = c(x_i \mid u_j),$$
$$L(x_i \mid u_k) = (x_i \mid u_k), \qquad \text{for} \quad k \neq j,$$

where c is a nonzero scalar. If $b_j^{(s)}$ is the number of occurrences of u_j in T_s', then

$$L(T_s \mid T_s') = c^{b_j^{(s)}}(T_s \mid T_s')$$

and

$$LF = \sum_s \alpha_s c^{b_j^{(s)}}(T_s \mid T_s').$$

As F is invariant, we also have

$$LF = a(L)F = \sum_s \alpha_s a(L)(T_s \mid T_s').$$

But the expansion into standard bideterminants is unique. Hence, we must have, for all s and t,

$$c^{b_j^{(s)}} = c^{b_j^{(t)}} = a(L).$$

This equality holds for all the scalars c in the infinite field k. Therefore, $b_j^{(s)} = b_j^{(t)}$ for all s and t. We shall write b_j for the common value of the integers $b_j^{(s)}$.

Now, let L be the linear transformation defined by

$$L(x_i \mid u_j) = (x_i \mid u_k),$$
$$L(x_i \mid u_k) = (x_i \mid u_j),$$
$$L(x_i \mid u_p) = (x_i \mid u_p), \qquad \text{for } p \neq j \text{ and } p \neq k.$$

Each of the bideterminants $L(T_s \mid T_s')$ contain the letter u_k b_j times. As the content of a bideterminant is preserved under straightening, LF is a linear combination of standard bideterminants each containing the letter u_k b_j times. But F is invariant, and $LF = \sum_s \alpha_s a(L)(T_s \mid T_s')$, where each of the bideterminants in this expansion contain the letter u_k b_k times. Applying the basis theorem, we conclude that $b_j = b_k$. We shall denote by b the common value of the integers b_j.

Since each letter u_j is repeated b times, the minimum number of rows in T_s' is b. The number of rows is exactly b if and only if T_s' is rectangular.

Now, suppose that T_s' is not rectangular; that is, the number of rows in T_s' is strictly greater than b. In T_s', all the letters u_1 are in the first column. Let u_l be the first letter in the first column following the run of letters u_1. Then, all the letters from u_1 to u_{l-1} occur in the first b rows. Moreover, if q is the number of occurrences of u_l in the first column, the remaining $b - q$ letters u_l must all occur in the first $b - q$ rows. The situation is summarized by:

$$T' = \begin{array}{l} b\left\{\begin{array}{l} u_1 \cdots u_{l-1} u_l \cdots \\ \vdots \qquad \vdots \;\; \vdots \\ u_1 \cdots u_{l-1} u_l \cdots \\ q\left\{\begin{array}{l} u_1 \cdots u_{l-1} u_m \cdots \\ \vdots \qquad \vdots \;\; \vdots \\ u_1 \cdots u_{l-1} \end{array}\right. \end{array}\right. \\ \\ \quad q\left\{\begin{array}{l} u_l \cdots \\ \vdots \\ u_l \cdots \end{array}\right. \\ \qquad \vdots \end{array}$$

where $m \geqslant l + 1$. Such a tableau is called an *l-critical tableau*, and its *parameter* is q.

Let j be the smallest index such that there exists a j-critical tableau in the expansion of F. We break up the expansion of F into

$$F = \sum_s \alpha_s(T_s \mid T_s') + \sum_t \alpha_t(T_t \mid T_t') + G,$$

where the first summation is over all the indices s such that T_s' is j-critical, the second summation is over all t such that T_t' is l-critical, for some $l > q$, and G is the linear combination of all the $b \times d$ rectangular standard bideterminants.

Let L be the linear transformation defined by

$$\begin{aligned} L(x_i \mid u_{j-1}) &= (x_i \mid u_{j-1}) + (x_i \mid u_j) \\ L(x_i \mid u_k) &= (x_i \mid u_k), \qquad \text{for} \quad k \neq j - 1. \end{aligned}$$

Under L, those bideterminants in which all the letters u_{j-1} and u_j occur in the first b rows are unchanged; in particular, the rectangular bideterminants in G and the l-critical bideterminants in the second summation remain unaltered. For the j-critical bideterminants,

$$L(T_s \mid T_s') = (T_s \mid T_s') + \sum_r (T_s \mid T_s^r),$$

where T_s^r is a tableau of the following form:

$$b\left\{\begin{array}{l} u_1 \cdots u_{j-1} u_j \cdots \\ \vdots \qquad \vdots \\ u_1 \cdots u_{j-1} u_j \cdots \\ q\left\{\begin{array}{l} u_1 \cdots \;\; * \;\; u_m \cdots \\ \vdots \qquad\;\; \vdots \quad \vdots \\ u_1 \cdots \;\; * \end{array}\right. \end{array}\right.$$

$$q\begin{cases} u_j \cdots \\ \vdots \\ u_j \cdots \end{cases}$$
$$\vdots$$

where $*$ may be u_{j-1} or u_j, and $m \geqslant j+1$.

Let T_s'' be the tableau such that all the $*$'s are u_j; thus, T_s'' is a standard tableau containing u_j $b+q$ times. As F is invariant, we obtain the equality

$$LF = a(L)F = \sum_s \alpha_s(T_s \mid T_s') + \sum_{s,r} \alpha_s(T_s \mid T_s{}^r) + \sum_t \alpha_t(T_t \mid T_t') + G.$$

Each bideterminant $(T_s \mid T_s{}^r)$ contains the letter u_j at least $b+1$ times. Let q' be the largest parameter for the j-critical tableaux. The only bideterminants in the above equality containing u_j exactly $b+q'$ times are the bideterminants $(T_v \mid T_v'')$, where T_v' is a j-critical tableau with parameter q'. Therefore, the projection of this equality onto the subspace spanned by the bideterminants containing u_j exactly $b+q'$ times yields

$$0 = \sum_v \alpha_v(T_v \mid T_v'').$$

All the bideterminants $(T_v \mid T_v')$, and hence $(T_v \mid T_v'')$, are distinct and standard. By the basis theorem, all the coefficients α_v must be zero. We conclude that there cannot be a j-critical bideterminant in the expansion of F; in particular, there cannot be a nonrectangular bideterminant in the expansion. This completes the first proof of Theorem 3.2.

Second proof. We begin with some classical results.

LEMMA A. *Considered as a polynomial in the indeterminates l_{ij}, $1 \leqslant i, j \leqslant d$, the determinant $\Delta = \mid l_{ij} \mid$ is irreducible.*

Proof. Suppose that $\Delta = AB$. Since Δ is linear in each variable l_{ij}, l_{11} cannot occur in both A and B. Suppose that l_{11} occurs in A. In the expansion of Δ into monomials, each monomial contains exactly one variable from each row and column. Hence, none of the variables l_{1r}, l_{s1}, $1 < r, s \leqslant d$ can occur in B.

If B is not a constant polynomial, then B contains a variable l_{pq}, where, a fortiori, $p, q > 1$. By a similar argument, A cannot contain the variables l_{pr}, l_{sq}, $r \neq q$, $s \neq p$. But this implies that neither A nor B contain the variables l_{p1} and l_{1q}, both of which appear in the expansion of Δ. Hence, B must be a constant polynomial, and our lemma is proved.

Recall that a form is invariant if, for all linear transformations L, there exists a scalar $a(L)$ such that $LF = a(L)F$. Let $F = F_0 + \cdots + F_t$ be the decomposition of the polynomial F into homogeneous components with respect to the *total* degree. Then, if F is invariant,

$$LF = LF_0 + \cdots + LF_t = a(L)F_0 + \cdots + a(L)F_t$$

Since the action of L on F preserves the total degree, we must have

$$LF_i = a(L)F_i, \qquad 0 \leqslant i \leqslant t.$$

It suffices, therefore, to consider only invariant forms that are homogeneous.

Consider the linear transformation $L = cI$, where I is the identity matrix. Under this action, $(x_i \mid u_j) \mapsto c(x_i \mid u_j)$. If F is a homogeneous invariant form of degree t, $LF = c^t F$; hence, for F,

$$a(cI) = c^t.$$

Moreover, the function $a(L)$ is multiplicative, in the sense that for any two invertible linear transformations L_1 and L_2,

$$a(L_1 L_2) = a(L_1)\, a(L_2).$$

The proof is a simple computation. Further, as F is a polynomial, $a(L)$ is also a polynomial in the entries l_{jk} of the matrix of L.

Given a linear transformation L with matrix (l_{jk}), its adjugate L^* is the linear transformation with matrix (l^*_{jk}), where

$$\begin{aligned} l^*_{jk} &= \text{the } jk\text{th cofactor of } (l_{jk}) \\ &= (-1)^{j+k} \mid l_{pq} \mid_{p \neq j, q \neq k}. \end{aligned}$$

The adjugate is characterized by the property

$$LL^* = \mid l_{ij} \mid I.$$

Indeed, the stth entry of the matrix of LL^* is given by

$$\sum_m l_{sm} l^*_{mt} = \sum_m (-1)^{m+t} l_{sm} \mid l_{pq} \mid_{p \neq m, q \neq t}.$$

By the Laplace expansion, the right-hand side is the determinant of the matrix (l_{jk}) with the tth column replaced by the column vector $(l_{ms})_{1 \leqslant m \leqslant d}$. This determinant is zero if $s \neq t$ and equals $\mid l_{ij} \mid$ if $s = t$. Hence the assertion.

As an immediate consequence, we obtain

$$a(L)\, a(L^*) = \mid l_{ij} \mid^t,$$

where t is the total degree of F. As the determinant is irreducible, each of the factors on the left must be a power of the determinant. We have thus proved

LEMMA B. *$a(L) = [\det L]^g$, for some nonnegative integer g.*

A technical result we shall use time and again is *Weyl's principle of the irrelevance of algebraic inequalities*: Let K be an infinite field, $\{z_1, \ldots, z_p\}$ a finite set of indeterminates, and $f, g_1, \ldots, g_r$ polynomials in $K[z_1, \ldots, z_p]$. Suppose that $f(s_1, \ldots, s_p) = 0$ for all $s_1, \ldots, s_p \in K$ such that $g_i(s_1, \ldots, s_p) \neq 0$, $1 \leqslant i \leqslant r$. Then, f is identically zero. The proof is routine commutative algebra (see Weyl [4]).

We also introduce the notation: Let L be a linear transformation with matrix (l_{jk}). The evaluation ϵ_L is the homomorphism from P to K given by

$$(x_j \mid u_k) \mapsto l_{kj}, \qquad \text{if } 1 \leqslant j, k \leqslant d$$

and

$$0, \qquad \text{otherwise.}$$

An easy computation shows that the evaluation satisfies

$$\epsilon_L F = \epsilon_I L F,$$

where I is the identity matrix.

LEMMA C. *If $F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is a nonzero homogeneous invariant form with $m < d$, then F is constant.*

Proof. The argument consists of three main steps. First, we prove that for the identity matrix I, $\epsilon_I F \neq 0$. Suppose the contrary. Then, as F is invariant, for any invertible linear transformation $L = (l_{jk})$,

$$\epsilon_L F = \epsilon_I L F = (\det L)^g \, \epsilon_I F = 0$$

Therefore, by Weyl's principle, F is identically zero, contradicting the hypothesis.

Now, let L be the linear transformation with matrix

$$\begin{pmatrix} I_{d-1} & 0 \\ 0 & l_{dd} \end{pmatrix},$$

where I_{d-1} is the $(d-1) \times (d-1)$ identity matrix. Since the variable $(x_d \mid u_d)$ does not occur in F, $\epsilon_L F = \epsilon_I F$. Invoking invariance again, we obtain

$$\epsilon_I F = \epsilon_L F = \epsilon_I L F = (\det L)^g \, \epsilon_I F = (l_{dd})^g \, \epsilon_I F,$$

which holds for any nonzero $l_{dd} \in K$. As K is infinite, we conclude that $g = 0$.

But this implies, for any invertible linear transformation L,

$$\epsilon_L F = \epsilon_I L F = \epsilon_I F.$$

Applying Weyl's principle once again, we conclude that F must be a constant.

LEMMA D. *If $F(\mathbf{x}_1, ..., \mathbf{x}_d)$ is a nonzero homogeneous invariant, then*

$$F(\mathbf{x}_1, ..., \mathbf{x}_d) = c(x_1 \cdots x_d \mid u_1 \cdots u_d)^g$$

for some nonnegative integer g, and scalar c.

Proof. For an invertible linear transformation L,

$$\epsilon_L F = \epsilon_I LF = (\det L)^g \, \epsilon_I F.$$

Applying Weyl's principle, we conclude that

$$F(\mathbf{x}_1, ..., \mathbf{x}_d) = c(x_1 \cdots x_d \mid u_1 \cdots u_d)^g,$$

where $c = \epsilon_I F$.

We adopt the following *bracket* notation:

$$[x_{i_1}, ..., x_{i_d}] = (x_{i_1} \cdots x_{i_d} \mid u_1 \cdots u_d).$$

LEMMA E. *In the letter place algebra P,*

$$[x_1, ..., x_d](x_j \mid u_m) = \sum_{k=1}^{d} [x_1, ..., x_{k-1}, x_j, x_{k+1}, ..., x_d](x_k \mid u_m)$$

for $1 \leqslant j \leqslant n$, and $1 \leqslant m \leqslant d$.

Proof. The identity follows from expanding, by Laplace's rule, the inner product $(x_1 \cdots x_d x_j \mid u_1 \cdots u_d u_m)$, which is identically zero in P.

The last lemma is a simple variation on the multinomial theorem. Let $F(\mathbf{x}_1, ..., \mathbf{x}_m)$ be a homogeneous form of degree g; let $F(\sum_{i=1}^d \lambda_{1i}\mathbf{x}_i, ..., \sum_{i=1}^d \lambda_{mi}\mathbf{x}_i)$ be the form obtained from F by substituting $\sum_{i=1}^d \lambda_{ji}(x_i \mid u_r)$ for $(x_j \mid u_r)$, $1 \leqslant j \leqslant m$, $1 \leqslant r \leqslant d$. Then,

LEMMA F.

$$\begin{aligned} &F\left(\sum \lambda_{1i}\mathbf{x}_i \;....\sum \lambda_{mi}\mathbf{x}_i\right) \\ &\qquad = \sum \lambda_{11}^{i(1,1)} \cdots \lambda_{1d}^{i(1,d)} \cdots \lambda_{md}^{i(m,d)} \cdot F_{i(1,1),...\, i(m,d)}(\mathbf{x}_1, ..., \mathbf{x}_d), \end{aligned}$$

where the summation is over all $i(1, 1), ..., i(m, d)$ such that

$$\sum_{q=1}^{d} \sum_{p=1}^{m} i(p, q) = g$$

and the polynomials $F_{i(1,1),...,i(m,d)}$ are homogeneous of degree g.

The preliminary lemmas are now disposed of. Let $F(\mathbf{x}_1, ..., \mathbf{x}_m)$ be a homogeneous nonconstant invariant form of degree g. By Lemma C, $m \geqslant d$. We claim that the polynomial

$$[x_1, \ldots, x_d]^g F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$$

equals a polynomial in the brackets $[x_{i_1}, \ldots, x_{i_d}]$. Indeed, as F is homogeneous of degree g,

$$[x_1, \ldots, x_d]^g F(\mathbf{x}_1, \ldots, \mathbf{x}_m) = F([x_1, \ldots, x_d]\mathbf{x}_1, \ldots, [x_1, \ldots, x_d]\mathbf{x}_m).$$

Applying the identity in Lemma E to each of the arguments, and expanding as in Lemma F, we obtain

$$[x_1, \ldots, x_d]^g F(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_M C_M F_M(\mathbf{x}_1, \ldots, \mathbf{x}_d), \qquad (*)$$

where M ranges over a set of multi-indices given in Lemma F, and the coefficients $C_M(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ are products of brackets of the form

$$B_{jk} = [x_1, \ldots, x_{k-1}, x_j, x_{k+1}, \ldots, x_d].$$

Note that we have used implicitly Weyl's principle throughout the computation.

To justify our claim, it suffices, by Lemma D, to show that each of the forms F_M is invariant.

First, observe that the brackets B_{jk}, $j > d$, are algebraically independent, since any nontrivial algebraic relation $f(B_{jk}) = 0$ specializes, under the partial evaluation

$$\begin{aligned} (x_j \mid u_k) &\mapsto \delta_{jk} && \text{if } 1 \leqslant j, k \leqslant d, \\ &(x_j \mid u_k) && \text{otherwise,} \end{aligned}$$

to a nontrivial algebraic relation $f((x_j \mid u_k)) = 0$, which, if it exists, would contradict the fact that $(x_j \mid u_k)$ are indeterminates.

Now, $[x_1, \ldots, x_n]^g F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is invariant, being the product of two invariants. Applying this fact to (*), we obtain, for any invertible linear transformation L,

$$\sum_M (LC_M)(LF_M) = a(L) \sum_M C_M F_M.$$

But the coefficients C_M, being product of brackets, are invariant, and satisfy $LC_M = a_M(L) C_M$. Hence,

$$\sum_M a_M(L)\, C_M(LF_M) = \sum_M a(L)\, C_M F_M,$$

where both sides are polynomials in the algebraically independent expressions $B_{jk}, j > d$. We conclude, therefore, that

$$LF_M = (a(L)/a_M(L))F_M .$$

This justifies our claim that $[x_1, \ldots, x_d]^g F(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ is a polynomial in the brackets $[x_{i_1}, \ldots, x_{i_d}]$; in the language of bideterminants, we have shown that

$$[x_1, \ldots, x_d]^g F(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_i a_i(T_i \mid T),$$

where T is the standard tableau

$$\begin{matrix} u_1 & \cdots & u_d \\ \vdots & & \vdots \\ u_1 & \cdots & u_d \end{matrix}.$$

To finish the proof of the theorem, we have to show that we can "cancel" the factor $[x_1, \ldots, x_d]^g$ without changing the rectangular shape of the right-hand tableau T.

By the straightening formula, we can write F as a linear combination of standard tableaux:

$$F = \sum_i b_i(U_i \mid U_i').$$

Hence,

$$[x_1, \ldots, x_d]^g F = \sum_i b_i(\hat{U}_i \mid \hat{U}_i'),$$

where $\hat{U}_i$ (and analogously for $\hat{U}_i'$) is the tableau

$$\begin{matrix} x_1 & \cdots & x_d \\ \vdots & & \vdots \\ x_k & \cdots & x_d \\ & U_i & \end{matrix}.$$

But both $\hat{U}_i$ and $\hat{U}_i'$ are standard. By the basis theorem, the two linear combinations for $[x_1, \ldots, x_d]^g F$ must be equal. In particular,

$$\hat{U}_i' = T.$$

It follows that U_i must be rectangular, with each row having the form

$$u_1 \cdots u_d .$$

This concludes the second proof of the theorem.

REFERENCES[1]

1. C. DE CONCINI AND C. PROCESI, A characteristic free approach to invariant theory, *Advances in Math.* **21** (1976), 330–354.
2. P. DOUBILET AND G.-C. ROTA, Skew-symmetric invariant theory, *Advances in Math.* **21** (1976), 196–201.
3. P. DOUBILET, G.-C. ROTA, AND J. STEIN, On the foundations of combinatorial theory: IX. Combinatorial methods in invariant theory, *Stud. Appl. Math.* **53** (1974), 185–216.
4. H. WEYL, "The Classical Groups," Princeton Univ. Press, Princeton, N.J., 1946.
5. W. WHITELEY, Logic and invariant theory. I : Invariant theory of projective properties, *Trans. Amer. Math. Soc.* **177** (1973), 121–139.
6. J. DÉSARMÉNIEN AND G.-C. ROTA, Théorie Combinatoire des Invariants Classiques, Series de Mathématique Pures et Appliquées, IRMA, Strasbourg, 1977.
7. DAVID G. MEAD, Determinantal ideals, identities and the Wronskian, *Pacific J. Math.* **42** (1972), 167–175.
8. W. SPECHT, Die irreduciblen Darstellungen der symmetrische Gruppe, *Math. Zeit.* **39** (1935), 696–711.

[1] Papers mentioned in bibliographies of papers above are not repeated here.